PARTIALLY ORDERED TOPOLOGICAL SPACES .

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BY

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SCOPE AND CONTENTS : In this thesis we consider categories of partially ordered topological spaces, reflexions and coreflexions, and in particular compactifications and real compactifications .

We study projective and injective objects in some subcategories .

The category of partially ordered topological spaces being an extension of the category of topological spaces, we generalize theorems on CX of Stone and Shirota, by the introduction of the appropriate functors .

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Last but not least I dedicate this thesis to my parents whose advise to pursue my studies has fructified here.

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CHAPTER

INTRODUCTION

We consider the category of topological spaces on which a partial order has been defined : PTop .

If we collect those objects of PTop whose partial order is discrete (i.e. no two elements are comparable), we have a copy of the category Top . However if we collect the objects of PTop with a discrete topology, a copy of the category of partially ordered sets is obtained .

Because of the above reasons PTop appealed to the author as a frame to generalize results of Top . This hope was strengthened by reading " Topology and Order " by L. Nachbin [17] , and " Partially ordered Topological Spaces " by L.E. Ward Jr. [24] which contain clean generalizations of the Urysohn Theorem .

Having been exposed recently to generalizations of injective and projective - with respect to classes of maps-, and to the well-behaved nature of P-projectivity in Top, (see B. Banaschewski [2] and [3]) it was inevitable for the author to wonder about PTop and/or some of its subcategories .

In order to apply some of the previously known results, and because of the very nature of this thesis, it has been necessary to keep introducing new subcategories of PTop , for which the author hopes to have found a notation which will not be too confusing .

It has been a pleasant surprise for the author that so many strong results of Top could not only be generalized in a simple way but also into surprisingly simple statements .

Chapter 0 contains some basic material for quick reference .

Chapter I surveys the objects, subobjects, quotients, initial and final structures, limits and colimits of subcategories of the category we shall call HPOTS - (Hausdorff spaces of PTop with a condition on the order stated in [17]) . We obtain all the essential results to enable us to apply some of the strongest statements of [10] as aids in finding reflective and coreflective subcategories of HPOTS and in establishing two different adjoint situations between subcategories of HPOTS and the corresponding subcategories of H .

Chapter II concerns itself with "complete regularization" in the sense given in [10] for Top, and with the related ideas of "compactification" and "real-compactification" . A subcategory of HPOTS is found which bears an analogous relation to HPOTS as Cr does to H .

Two statements compare the underlying topological spaces of the "compactification" and "real-compactification" of $(X,\leq)$ with βX and υX .

In Chapter III, a successful effort is made to extend the results of [2], to a "perfect projectivity" in subcategories of <u>HPOTS</u> . With the results of Chapter I and direct applications of methods in [2] and [3] the <u>P</u>-projective objects and <u>P</u>-projective covers are found of a properly behaved <u>P</u>-projectivity . In a similar way a special <u>E</u>-injectivity is studied and partial results of a "local" proper behavior are obtained .

Chapter IV concludes this work by showing that the Hom (. , R) functor has no simple generalization to the well known results about CX in <u>Top</u> .

Counterexamples are provided, followed by more involved generalizations of results by Stone and Shirota .

CHAPTER 0

PRELIMINARIES.

The purpose of this chapter is to make this thesis more readable by including in it some of the well known results which are referred to more often in later chapters. At the same time we take the opportunity to introduce notation.

0.1 <u>REMARK</u>: In all of the categories that we shall introduce, unless otherwise stated, the morphisms will be the structure preserving functions between the underlying sets of two objects, and we shall mean by a subcategory a full subcategory.

0.2 <u>EXAMPLES</u>:

1. We denote by <u>PTop</u> the category whose objects are topological spaces on which a partial order has been defined and whose morphisms are the continuous isotone maps.

2. We describe the category of ℓ-rings as the one having as objects algebras of the type $(X,+,.,\wedge,\vee)$ such that $(X,+,.)$ is a ring and $(X,\wedge,\vee)$ a lattice.

   The morphisms will then be those maps which are ring-homomorphisms with respect to $+,.$, and lattice-homomorphisms with respect to $\wedge,\vee$.

0.3 NOTATION: If we represent a certain subcategory of Top by K, we shall mean by KPTop the subcategory of PTop whose underlying topological spaces belong to K.

All symbols denoting categories will be underlined, and if K is any category, we shall mean by K(A,B) the set of K-morphisms between the K-objects A and B.

0.4 EXAMPLES: We shall denote by H the Hausdorff spaces in Top

by Cr the completely regular spaces in H

by C the compact spaces in H

by Rc the realcompact spaces in H

Accordingly, CPTop will mean the compact spaces in HPTop, and PTop(X,R) the set of continuous isotone functions with the partially ordered topological space X as their domain, and RcPTop as their codomain.

0.5 DEFINITION: (Ward Jr. [24]). Let X be a topological space with a partial order $\leq$, then $\leq$ is called

1. lower semicontinuous if, whenever $a \nleq b$ in X, there exists an open neighbourhood U of a, such that, if $x \in U$, then $x \nleq b$

2. upper semicontinuous if for $a \nleq b$, there exists V, an open neighbourhood of b, such that $x \in V$ implies $a \nleq x$

3. semicontinuous if it is both upper and lower semicontinuous.

4. continuous if, whenever $a \nleq b$. there exists U open neighbourhood of a and V open neighbourhood of b, such that if x is in U and y is in V, $x \nleq y$.

0.6 NOTATION: If we represent a certain subcategory of $\underline{H}$ by $\underline{K}$, we shall mean by KPOTS the category of all spaces in HPTop, whose underlying topological space belongs to $\underline{K}$, and whose partial order is semicontinuous. Similarly, we shall denote by KOTS the subcategory of all the spaces in KPOTS, whose partial order is continuous.

0.7 LEMMA: (Ward Jr. [24]). If $(X,\leq)$ belongs to PTop and $\leq$ is continuous, then X is a Hausdorff space and the graph of $\leq$ in $X \times X$ is closed.

0.8 DEFINITION: (Nachbin [17]). Let $(X,\leq)$ be a partially ordered set and $S \subset X$. We call S decreasing if, whenever $a \leq b$ and $b \in S$, $a \in S$. Similarly S will be increasing if, from $a \leq b$ and $a \in S$, $b \in S$ follows.

We denote by LS the smallest decreasing subset of X containing S, and by MS the corresponding smallest increasing set. Since decreasing and increasing have the intersection property, and X is both decreasing and increasing, LS and MS always exist.

0.9 DEFINITION: (Nachbin [17]) $(X,\leq) \in$ PTop will be said to be normally ordered if for every two disjoint closed subsets A, B of X such that A is decreasing and B increasing, there exist two disjoint open sets U, V such that U contains A and is decreasing, and V contains B and is increasing.

0.10 NOTATION: We denote by NOR the category of normally ordered spaces in PTop, and by NORC the intersection of NOR with HOTS.

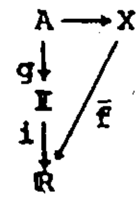
0.11 THEOREM 1: (Nachbin [17]). In order that $(X, \leq) \in$ PTop be normally ordered it is necessary and sufficient that, for any two disjoint closed subsets A, B of X where A is decreasing and B increasing, there exists a continuous isotone function $f: X \to \mathbb{R}$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in B$ and $\operatorname{Im} f \subset [0,1]$.

0.12 THEOREM 6: (Nachbin [17]). If $(X, \leq) \in$ NORC and A is a compact subset of X, every continuous isotone function $f: A \to \mathbb{R}$ can be extended to $\bar{f}$ so as to make the following HOTS-diagram commutative:

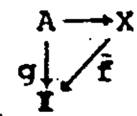
$$\begin{array}{ccc} A & \rightarrow & X \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\ \mathbb{R} & & \end{array}$$

0.13 REMARK: By analyzing the proof of the above theorem, it follows directly that given $g: A \to \mathbb{I}$, a function f will exist for which the HOTS-diagram below commutes.

However, the map $\bar{f}$ so obtained was constructed in the proof of Theorem 2 [17] pags. 36-42, where, specifically, if $\operatorname{Im} ig \subset \mathbb{I}$, then $\operatorname{Im} \bar{f} \subset \mathbb{I}$.



We can therefore interpret $\bar{f}$ as a map $X \to \mathbb{I}$ and for this particular case obtain the commutativity of



0.14 <u>REMARK</u>: We have already mentioned that the category <u>PTop</u> is "larger" than <u>Top</u> .

Even for <u>Top</u> the study of the problems we are concerned with in this thesis required the precise definition of subobjects and quotient-objects of a category given in [10] page 60 and [19] page 20 . For a convenient reference we include those definitions here .

0.15 <u>DEFINITION</u>: (Herrlich [10]) . Let $\underline{K}$ be a category, X an object of $\underline{K}$, m_1 and m_2 morphisms in $\underline{K}$. We say that, <u>m_1 is equivalent to m_2</u> if and only if an isomorphism h exists such that the following $\underline{K}$-diagramm commutes:

The above defined relation is clearly an equivalence relation .

If $m:Y \to X$ is a $\underline{K}$-monomorphism, the equivalence class of (Y,m) under the above equivalence relation is called a subobject of X.

One defines dually the quotient objects .

0.16 REMARK: In the categories Set, Group and A-modul , the subobjects are completely determined by the underlying set of a subset, a subgroup and a submodule, respectively, and their inclusion maps . For other categories like Top, H and Partially ordered Sets a subset of the underlying set of X does not uniquely determine a subobject . For those categories the concept of "extremal subobject" is of interest, as it will be for PTop .

0.17 DEFINITION: (Herrlich [10]) . A monomorphism $f:A \to B$ in $\underline{K}$ is called an extremal monomorphism if whenever g and h exist such that g is an epimorphism and $f=h \cdot g$, then g is already an isomorphism .

The subobject corresponding to an extremal monomorphism, is called an extremal subobject .

Extremal epimorphisms and extremal quotients are defined dually .

0.18 DEFINITION: (Herrlich [10]) . If $\underline{S}$ is a subcategory of $\underline{K}$ and $E:\underline{S} \to \underline{K}$ the inclusion functor we call $\underline{S}$ reflective in $\underline{K}$ if E has a left-adjoint R, and coreflective if E has a coadjoint C.

If $\underline{S}$ is reflective in $\underline{K}$, the functor R is called a reflector and it satisfies the Right Universal Problem. The situation for C is symmetric. $\underline{S}$ is reflective in $\underline{K}$ if and only if for every $X \in \underline{K}$ the Right Universal Problem has a solution $r_X : X \to rX$ in $\underline{K}$ with $rX \in \underline{S}$, for all $f : X \to Y$ in $\underline{K}$ with $Y \in \underline{S}$.

0.19 DEFINITION: (Herrlich [10]). Let $\underline{A}$ be a subcategory of $\underline{B}$. A diagramm $D : \underline{S} \to \underline{B}$ is said to be partially in $\underline{A}$, if for every $X \in \underline{S}$, there exists a $Y \in \underline{S}$ such that $D(Y) \in \underline{A}$ and $\underline{S}(X,Y) \neq \phi$. The subcategory $\underline{A}$ is said to be strongly closed with respect to $\underline{S}$-limits in $\underline{B}$, if whenever a diagramm $D : \underline{S} \to \underline{B}$ partially in $\underline{A}$ has a limit (L, l_i), then $L \in \underline{A}$.

One defines dually copartially in $\underline{A}$ and strongly closed with respect to $\underline{S}$-colimits.

0.20 PROPOSITION: (Herrlich [10]). If $\underline{K}$ is complete, locally small and colocally small, the following statements are equivalent :

1) $\underline{S}$ is epireflective in $\underline{K}$

2) $\underline{S}$ is strongly closed with respect to products and equalizers in $\underline{K}$.

0.21 DEFINITION: (Mitchell [16]). If $\underline{K}$ is a category and G an object of $\underline{K}$, then G is called a generator for

K if for every pair of distinct morphisms $m,n:A\to B$ there is a morphism $h:G\to A$ such that $mh \neq nh$.

0.22 PROPOSITION: (Mitchell [16]) . If K has coproducts, then G is a generator for K if and only if for each $A\in K$ there is an epimorphism $e_G:\coprod_I G\to A$ for some set I . Furthermore, in this case we can take $I = K(G,A)$ with e_G the morphism whose u-th coordinate is u for all $u\in K(G,A)$.

0.23 PROPOSITION: (Herrlich [10]) . Let S be a subcategory of K . If S contains a generator of K, the following statements are equivalent :

1) S is coreflective in K
2) S is bicoreflective in K

0.24 PROPOSITION: (Herrlich [10]) . If K is a cocomplete, locally small and colocally small category such that every K-object is initial or a generator, the following statements are equivalent for every subcategory S which does not consist enterely of initial objects .

1) S is coreflective
2) S is bicoreflective
3) S is closed with respect to coproducts and coequalizers in K
4) S is closed with respect to coproducts and extremal quotients in K .

0.25 REMARK: Our notation in General Topology will include IA ΓA and ∁A for interior, closure and complement of A, respectively.

We include the following definition from General Topology, in order to have it ready for its generalization in Chapter II.

0.26 DEFINITION: Let E, X∈Top. We say that the space X is E-regular if it is homeomorphic to a subspace of some power of E. The space X will be called E-compact if it is homeomorphic to a closed subspace of some power of E.

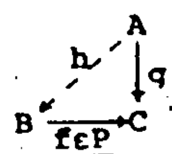
A definition from Category Theory will be useful:

0.27 DEFINITION: The subcategory S of K will be called right-fitting (with respect to a class M of maps in K if f: X→Y in M and X∈S implies Y∈S. Similarly if Y∈S implies X∈S, S will be called left-fitting.

For the study of Projectivity and Injectivity in Chapter III, some further material is included:

0.28 DEFINITION: Let P be a class of epimorphisms of a category K. Then f∈P is called coessential if whenever g∈K and f·g∈P, g∈P. We denote by P* the class of coessential elements of P . In a dual way we denote by E a class of monomorphisms of K, we define essential elements of E and we denote by E* the class of essential elements of E .

0.29 DEFINITION: Let K be a category and P a class of its epimorphisms . An object A of K is called P-projective if, whenever we have the following diagram in K , a morphism $h \in K(A,B)$ exists such that $f \circ h = q$:



Given $f: B \to A$, f (or B) is called a P-projective cover of A if f is coessential and B P-projective.

E-injective and the E-injective hull are defined in the dual way .

0.30 DEFINITION: (Banaschewski [2]) . P-projectivity in the category K is said to be properly behaved if (I), (II) and (III) below are satisfied :

I) For every $A \in K$, the following statements are equivalent:

A1): A is P-projective .

A2): Every $f: B \to A$ such that $f \in P$, has a right inverse .

A3): Every coessential $f: B \to A$ is an isomorphism .

II) Each object of $\underline{K}$ has an essentially unique $\underline{P}$-projective cover .

III) For every $f: B \to A$ in $\underline{P}$, the following statements are equivalent .

C1): f is a $\underline{P}$-projective cover of A .

C2): f is coessential and whenever $g \in \underline{K}$ and fg is coessential, g is an isomorphism .

C3): B is $\underline{P}$-projective and if f factors as follows, with $g, h \in \underline{P}$ and C $\underline{P}$-projective , h is an isomorphism :

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ & {}_{h}\searrow \quad \nearrow_{g} & \\ & C & \end{array}$$

Properly behaved $\underline{E}$-injectivity is introduced dually.

0.31 <u>CONDITIONS WHICH ENSURE PROPER BEHAVIOR:</u>

(Banaschewski [2]) :

P1): $\underline{P}$ is closed under composition and all the isomorphisms belong to $\underline{P}$.

P2): If $f, g \in \underline{P}$ and $gf = f$, g is an identity .

P3): For every $f \in \underline{P}$, there exists a $g \in \underline{K}$ such that $f \cdot g \in \underline{P}$* .

P4): Every diagram (I) in $\underline{K}$ can be completed to a commutative diagram (II) :

(I) $$\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f \in \underline{P}]{} & B \end{array}$$, (II) $$\begin{array}{ccc} D & \xrightarrow{u \in \underline{P}} & C \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f \in \underline{P}]{} & B \end{array}$$

P5): Each well ordered inverse system of $\underline{P}$ has a lower bound in $\underline{P}$.

P6): For every object A in K, the class of all coessential f:B→A has, up to isomorphism, a representative set .

Dual conditions to these ensure also proper behavior of E-injectivity .

0.32 PROPOSITION 3: (Banaschewski [2]) . Let P denote the class of perfect surjective maps in H . In a subcategory K of H , P-projectivity is properly behaved if :

i) K is closed hereditary, closed with respect to pull backs in H , and projective limits in H of well ordered inverse systems with P-maps; or

ii) K is a full subcategory of H which is left-fitting with respect to coessential P-maps; or

iii) K consists of all objects and all perfect mappings from a category L which satisfies one of these conditions .

0.33 COROLLARY 3: (Banaschewski [2]) . In any full subcategory K of H which is left-fitting with respect to coessential P-mappings, the P-projectives are exactly the extremally disconnected spaces belonging to K, and the same holds for the subcategory of K with the same objects, but only the perfect mappings from K .

0.34 REMARK: (Banaschewski [2]) . Some subcategories of H, to which all the considerations given above apply, are given by the following classes of spaces together with either <u>all</u> their continuous mappings, or all their <u>perfect</u> mappings :

1) compact spaces
2) locally compact spaces
3) paracompact spaces
4) σ-compact spaces
5) Lindelöf spaces
6) regular spaces
7) completely regular spaces
8) zero-dimensional spaces
9) real-compact spaces
10) k-compact spaces

For these categories $\underline{P}$-projectivity is properly behaved, and the $\underline{P}$-projectives are exactly the extremely disconnected spaces .

0.35 <u>REMARK</u>: In section 3 of [2] the space $\Lambda(X)$ of convergent ultrafilters of the topology $T(X)$ of X is given . One reads in addition :

0.36 <u>PROPOSITION 8</u>: (Banaschewski [2]) . Let $\underline{K}$ be a replete subcategory of $\underline{H}$. If all spaces belonging to $\underline{K}$ are semiregular then, for any $X \in \underline{K}$, $\Lambda(X)$ and $\lim_X$ belong to $\underline{K}$, and $\lim_X : \Lambda(X) \to X$ is a $\underline{P}$-projective cover of X in $\underline{K}$. In general, a projective cover of X is given by the mapping determined by $\lim_X$ on the space $\Lambda'(X)$ whose underlying set is the same as that of $\Lambda(X)$ and whose topology is generated by that of $\Lambda(X)$ together with $\lim_X^{-1}(T(X))$.

0.37 DEFINITION: (Shirota [2]]) . By a translation lattice L we mean a lattice where for every $a \varepsilon L$ and for real numbers α, a sum $a+\alpha$ is defined, and which satisfies the following conditions :

1) $a+o = a$.
2) $(a+\alpha)+\beta = a+(\alpha+\beta)$.
3) If $\alpha \geqslant o$ then $a+\alpha \geqslant a$.
4) If $a \geqslant b$ then $a+\alpha \geqslant b+\alpha$.

0.38 REMARK: If L is a translation lattice, every real number r induces on L an unary operation $\tilde{r}:L \to L$, given by $\tilde{r}(a) := a+r$.

0.39 REMARK: (Shirota [21]) . $C(X,R)$ can be considered as a translation lattice by setting $(f+\alpha)(x) := f(x)+\alpha$ for a real number α and for a function $f \varepsilon C(X,R)$.

0.40 THEOREM 8: (Shirota [21]) . Let X be real-compact and ϕ a homomorphism of the translation lattice $C(X,R)$ into the reals . Then there exists a point $x \varepsilon X$ and a real number α such that for any $f \varepsilon C(X,R)$, $\phi(f) = f(x)+\alpha$ where $\alpha = \phi(\bar{o})$.

0.41 THEOREM 9: (Shirota [21]) . Any two real-compact spaces X and Y are homeomorphic if and only if the translation lattice $C(X,R)$ is isomorphic to the translation lattice $C(Y,R)$.

CHAPTER I

STRUCTURE OF PTop AND SOME SUBCATEGORIES .

In this chapter we survey the subobjects and quotients (in the sense of [10]), the initial and final structures, limits and colimits in subcategories of PTop .

The existence of initial and final structures, which is guaranteed in restricted cases, is sufficient to show that PTop is complete, cocomplete, locally small and colocally small . This allows the use of Propositions of [10] to show that HPOTS is an epireflective subcategory of PTop . For a subcategory K of H we show that K is epireflective in H if and only if KPTop is epireflective in HPTop , if and only if KPOTS is epireflective in HPOTS and if and only if KOTS is epireflective in HOTS .

For subcategories K of H let U denote the natural functor which associates $X \in K$ with $UX = (X,d) \in$ KOTS (d for discrete order) . Then U is left-adjoint to the forgetful functor $F: \text{KOTS} \to K$ and has in turn, for a list of subcategories K, a left-adjoint which is not F but G, where $G(X,\leq)$ is obtained from $(X,\leq)$, essentially, by identifying points a , b of X which can be extrema of finite chains $a = a, a_1 \ldots, a_n = b$ where a_i and a_{i+1} are comparable .

In the last section of this chapter, we establish that every nonempty object of PTop is a generator, and by using results of the previous sections and of [10], we show the dual to our main proposition about epireflective subcategories.

1. Introduction to PTop and subcategories.

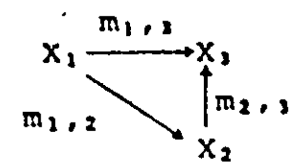
Considering the results of [17], we became interested in HPOTS. However, considerations in HPOTS directed our search into the more general category PTop, which has already been mentioned in Chapter 0.

1.1 LEMMA: For the following full subcategories of PTop subsets of underlying sets do not determine subobjects uniquely, and there are subobjects which are not extremal:

1) HPOTS
2) POTS
3) Compact-POTS
4) Lindelöf-POTS
5) Countably compact-POTS
6) Paracompact-POTS
7) Regular-POTS
8) CrPOTS
9) Zero dimensional-POTS
10) Realcompact-POTS
11) COTS
12) Boolean space -OTS

PROOF: Let $S := \{0,1,2\}$ and τ_S be the discrete topology on S. Let $\leq_1 := \Delta_S$, $\leq_2 := \Delta_S \cup \{(0,1)\}$,

$\leq_3 := \Delta_S \cup \{(0,1),\ (0,2),\ (1,2)\}$, $X_1 := (S,T_S,\leq_1)$, $X_2 := (S,T_S,\leq_2)$ and $X_3 := (S,T_S,\leq_3)$. The subobject represented by $m_{1,3}: X_1 \to X_3$ where $m_{1,3}(s) = s$ is clearly different from 1_{X_3} . If, for every element s of S, we define $m_{1,2}(s) = m_{2,3}(s) = s$, we obtain in all the listed categories the commutative diagram :



However, $m_{1,2}$ is an epimorphism but not an isomorphism .

1.2 INITIAL STRUCTURES: We define initial structures for some families of functions . Let X be a set, $(Y_i)_{i\in I}$ a family of PTop, $f_i: X \to Y_i$ a family of functions which separates the points of X . Let X have the initial topology with respect to $(f_i)_{i\in I}$. We define a partial order $\leq_X$ on X by :

$a \leq_X b$ in X if and only if $f_i(a) \leq f_i(b)$ for all $i \in I$

Then $(X,\leq_X)$ has the initial PTop-structure with respect to $(f_i)_{i\in I}$

PROOF: From $a \leq_X b$ and $b \leq_X a$ follows that $f_i(a) = f_i(b)$ for all $i \in I$. Since $(f_i)_{i\in I}$ separate points of X, $a = b$. The reflexive and transitive properties being trivially satisfied by $\leq_X$, $\leq_X$ is a partial order and $(X,\leq_X)$ an object of PTop .

We show that $(X,\leq_X)$ has the initial structure with respect to $(f_i)_{i\varepsilon I}$. Let $Z\varepsilon$PTop be arbitrary . If $g:Z\rightarrow(X,\leq_X)$ is a PTop-morphism, so, clearly, is $f_i \circ g$, for every $i\varepsilon I$. Conversely, suppose that $f_i \circ g$ is continuous and isotone for all $i\varepsilon I$. Since X has the initial topological structure with respect to $(f_i)_{i\varepsilon I}$, g is continuous . Let $a,b\varepsilon Z$ and $a \leq b$. By hypothesis

$$f_i(g(a)) = (f_i \circ g)(a) \leq (f_i \circ g)(b) = f_i g(b)) \quad \text{for all } i\varepsilon I.$$

By the definition of $\leq_X$, $g(a)\leq_X g(b)$. Thus we have shown that g , in addition to being continuous, is isotone .

1.3 REMARK: For families of functions which do not separate the points, a PTop-structure can still be defined, but it is not always an initial structure .

1.4 PRODUCTS: Let $(X_i)_{i\varepsilon I}$ be a family in PTop, and P the cartesian product of the underlying sets. P, with the initial PTop-structure with respect to the projections, is the product $\Pi_{i\varepsilon I}X_i$ in PTop .

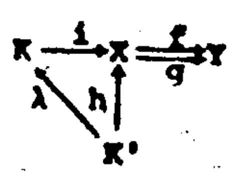
PROOF: From the definition of a cartesian product, we conclude that the family of its projections separates points. Therefore, P has an initial structure . With this structure, P satisfies the Universal Property for products. Let $f_i:N\rightarrow X_i$ be a family of continuous isotone functions .

Since P has initial topological structure with respect to the projections, there exists a unique continuous function f such that $p_i \circ f = f_i$ for all $i \in I$. But, then, since P has initial <u>PTop</u>-structure and $p_i \circ f$ is continuous and isotone for all $i \in I$, f has to be both continuous and isotone .

1.5 <u>SUBSPACES</u>: Let $(Y,\leq) \in$ <u>PTop</u> and X be a subset of Y . The inclusion function of X into Y certainly separates the points of X and therefore induces an initial structure on X . This kind of subspace is especially useful .

1.6 <u>LEMMA</u>: <u>PTop</u> has equalizers .

<u>PROOF</u>: Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be given in <u>PTop</u> .
Let $K := \{x \in X \mid f(x) = g(x)\}$ and induce on K the initial structure with respect to its inclusion i in X . To see that (K,i) is an equalizer of f,g , suppose that (K', h) is given such that $f \circ h = g \circ h$. It follows that for every $x \in K'$, $h(x) \in K$, and by defining $\lambda : K' \to K$ by $\lambda(x) := h(x)$ we obtain the unique map which makes following diagram commutative :



This map is clearly continuous and isotone .

1.7 <u>COROLLARY</u>: <u>PTop</u> is a complete category .

<u>PROOF</u>: Since we have just shown that <u>PTop</u> has products and equalizers, we need now only apply Proposition 5.8.1 [10] , page 39, to obtain the result .

1.8 <u>REMARK</u>: As in the case of the initial structures, the final structures do not seem to exist for every family of maps. Having not found a simple condition for their existence, we turn to the particular cases of interest: the coproducts , the quotients and the coequalizers .

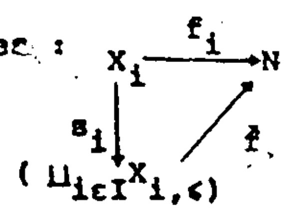
1.9 <u>COPRODUCTS</u>: Let $(X_i,\leq_i)_{i\in I}$ be a family in <u>PTop</u> . On the underlying set of the space $\coprod_{i\in I}X_i$ of <u>Top</u> we define following partial order :
$(x,i) \leq (y,j)$ if and only if $x \leq_i y$ and $i = j$. Then $(\coprod_{i\in I}X_i,\leq)$ is the coproduct $\coprod_{i\in I}(X_i,\leq_i)$ in <u>PTop</u> .

<u>PROOF</u>: Let $(s_i)_{i\in I}$ be the family of natural injections, $s_j:X_j\to\coprod_{i\in I}X_i$ and $s_j(x) = (x,j)$. We show next that $(\coprod_{i\in I}X_i,\leq)$ has the final <u>PTop</u>-structure with respect to $(s_i)_{i\in I}$. Let $Z\in$<u>PTop</u> be arbitrary . If $(\coprod_{i\in I}X_i,\leq)\xrightarrow{g}Z$ is continuous and isotone, so, clearly, is $g\circ s_i$ for each $i\in I$. Conversely, if $g\circ s_i$ is continuous and isotone for each $i\in I$, g is continuous, since $\coprod_{i\in I}X_i$ has the final <u>Top</u>-structure with respect to

$(s_i)_{i\in I}$. Let $(x,i) \leq (y,j)$ in $(\coprod_{i\in I} X_i, \leq)$. Then $i = j$ and $x \leq y$. Since $g \circ s_i$ is isotone, by hypothesis, $g(x,i) = g\circ s_i(x) \leq g\circ s_i(y) = g(y,i) = g(y,j)$.

This shows that g is isotone and $(\coprod_{i\in I} X_i, \leq)$ indeed has the final structure with respect to $(s_i)_{i\in I}$.

The Universal Property for Coproducts is satisfied: From the coproduct property of $\coprod_{i\in I} X_i$ in Top, we know that given a family of PTop-morphisms $f_i : X_i \longrightarrow N$, there exists a unique continuous function $\bar{f}$ such that for all $i \in I$ the following diagram commutes :



All we need now is to see that this unique continuous $\bar{f}$ is also isotone, a fact which is obvious from the definition of $\leq$.

1.10 REMARK: If every $\leq_i$ in above family of spaces had been semicontinuous (or continuos), the partial order $\leq$ would also have been semicontinuous (continuous) .

From this remark, it is clear that if $(X_i, \leq_i)_{i\in I}$ is a family in POTS (or in HOTS), so is $\coprod_{i\in I}(X_i, \leq_i)$.

1.11 REMARK: As is well known, every equivalence relation π in a topological space X uniquely determines a quotient X/π in Top .

The corresponding situation in PTop is not as simple; even if we define a congruence R on $(X,\leq)\in$PTop as an equivalence relation for which a PTop-morphism $f:(X,\leq)\to Y$ exists such that $R = \ker f$, the pair $((X,\leq),R)$ still does not determine a unique quotient.

1.12 EXAMPLE: Let S be the topological space $\{0,1\}$ with the discrete topology, $X := (S,d)$ and $Y := (S, 0 \leq 1)$. Let $f:X\to Y$ have Δ_S as its graph. Then $\ker 1_X = \Delta_S = \ker f$, but, since $X \not\cong Y$, 1_X and f represent different quotients.

1.13 DEFINITION: Let $f:(X,\leq)\to Y$ be a PTop-epimorphism and R an equivalence relation on X. If $R = \ker f$, we call (R,f) a congruence on $(X,\leq)$.

For brevity we describe (R,f) as "ker f"

1.14 LEMMA: Let $(X,\leq)\in$PTop, and R be an equivalence relation on X. If a partial order $\leq_R$ on X/R is well defined by:

$$a_R \leq_R b_R \quad \text{if and only if} \quad a_R = b_R \text{ or } a \leq b$$

then $(X/R, {}_R)$ has the final structure with respect to the natural map ν_R given by $a \mapsto a_R$, and $R = \ker \nu_R$.

PROOF: By the hypothesis on R and $\leq_R$, $a \leq b$ implies $a_R \leq_R b_R$ and, consequently, ν_R, in addition to being continuous, is isotone. Let $Z\in$PTop be arbitrary.

those epimorphisms $f:X \to Y$ such that $\pi \subset \ker f$, and let $R(\pi)$ denote the intersection of all the kernels of maps in $S(\pi)$. We call an epimorphism $f:X \to Y$ in <u>PTop</u> a <u>special quotient</u> if there exists an equivalence relation π on X such that $f:X \to Y$ is equivalent to $\nu:X \to X/R(\pi)$ as a quotient, where ν is the natural map and the order on $X/R(\pi)$ is the one induced by the set of <u>PTop</u>-epimorphisms, $S(\pi)$.

We call $f:(X,\leq) \to (Y,\leq)$ a t-quotient if $f:X \to Y$ is already a quotient in <u>Top</u>.

1.17 <u>LEMMA</u>: <u>PTop</u> has coequalizers.

<u>PROOF</u>: Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be given in <u>PTop</u>.

Let $\pi := \{(f(x),g(x)) \mid x \in X\} \cup \{(g(x),f(x)) \mid x \in X\} \cup \Delta_Y$.

The set $S(\pi)$ of 1.16 is nonempty as one map of the type $Y \to \{p\}$, belongs to it. We denote by $(\psi_i)_{i\in I}$ the family of <u>PTop</u>-epimorphisms in $S(\pi)$. Let $R := \bigcap_{i\in I} \ker \psi_i$, and let $(Y/R, \leq_R)$ be the special quotient induced by the family $(\psi_i)_{i\in I}$. We now show that $\nu:Y \to Y/R$ is the coequalizer of the given maps f,g.

Let $x \in X$. By the definition of π, $f(x)\pi g(x)$. Since $\pi \subset R$, $f(x) R g(x)$, which we can write as $\nu \circ f(x) = \nu \circ g(x)$. Therefore $\nu \circ f = \nu \circ g$.

Suppose $h:Y \to Z$ has been given such that $h \circ f = h \circ g$. Then $\pi \subset \ker h$, and since $\ker h$ is clearly a congruence

If $g:X/R \to Z$ is a PTop-morphism $g \circ \nu_R$ will also be . Conversely, suppose that $g \circ \nu_R$ is continuous and isotone . Since X/R has the final topological structure with respect to ν_R , g is continuous . Let $a_R \leq_R b_R$ in X/R . If $a_R = b_R$, $g(a_R) = g(b_R)$ and therefore $g(a_R) \leq g(b_R)$. If $a_R \neq b_R$, it follows from the definition of $\leq_R$ that $a \leq b$; by hypothesis

$$g(a_R) = g \circ \nu_R(a) \leq g \circ \nu_R(b) = g(b_R) .$$

Thus we have obtained that g is isotone .

1.15 REMARK: Let $f_i : X \to Y_i$ be a family of PTop-morphisms indexed by I. Let R be the intersection of all $\ker f_i$, $i \in I$. Define $\leq_R$ in X/R as follows :

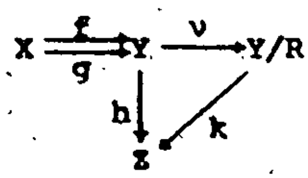
$a_R \leq_R b_R$ if and only if $f_i(a) \leq f_i(b)$ for all $i \in I$.

Then $\leq_R$ is a well defined partial order on X/R .

PROOF: The reflexivity of $\leq_R$ is obvious. Let $a_R \leq_R b_R$ and $b_R \leq_R a_R$, then for every $i \in I$, $f_i(a) \leq f_i(b) \leq f_i(a)$. Therefore $(a,b) \in \ker f_i$ for all $i \in I$ and accordingly $a_R = b_R$. Finally if $a_R \leq_R b_R$ and $b_R \leq_R c_R$, we obtain $f_i(a) \leq f_i(c)$ for all $i \in I$, from which $a_R \leq_R c_R$ follows .

1.16 DEFINITION: Let $X \in$ PTop . For every equivalence relation π on X, let $S(\pi)$ be a representative set of on Y, $R \subset \ker h$. Without loss of generality, let h be an epimorphism in $(\psi_i)_{i\varepsilon I}$. Define $k: Y/R \to Z$ by $k(\nu(y)) := h(y)$. If $\nu(y) = \nu(y')$, yRy' and since $R \subset \ker h$, $h(y) = h(y')$. The function k is therefore well defined . Suppose now that $\nu(y) \leq_R \nu(y')$, and rewrite it as $y_R \leq_R y'_R$. By the definition of $\leq_R$, $h(y) \leq h(y')$, which means $k(\nu(y)) \leq k(\nu(y'))$. Let U be an open set in Z . Since h is continuous, $h^{-1}U = (k \circ \nu)^{-1}U = \nu^{-1}k^{-1}U$ is open in Y . But Y/R has the quotient topology, whence $k^{-1}U$ is open in Y/R .

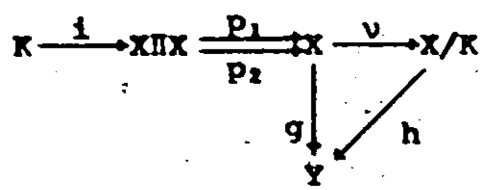
Having shown that k is a continuous isotone map, we should remark that it is the unique such which makes the following diagram commutative :



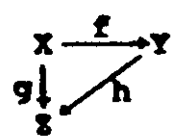
The uniqueness is obvious from the fact that ν is surjective .

1.18 PROPOSITION: In PTop the special quotients are exactly the coequalizers .

PROOF: From the proof of 1.17 we have seen that every coequalizer is an special quotient . Without loss of generality let $\nu: X \to X/K$ be an special quotient where $K = \ker \nu$, ν is the natural map, and the order on X/K is the one induced by a family of all the types of PTop-morphisms m with domain X and $K \subset \ker m$. Let $i: K \to X \Pi X$ be the inclusion map . We show now that ν is the coequalizer of $p_1 \circ i$, $p_2 \circ i$. It is clear that $\nu \circ p_1 \circ i = \nu \circ p_2 \circ i$. Suppose $g: X \to Y$ has been given such that $g \circ p_1 \circ i = g \circ p_2 \circ i$. Then $K \subset \ker g$ and a unique continuous function h exists such that $h \circ \nu = g$. To show that h is isotone , let $\nu(x) \leq \nu(y)$ in X/K . By the definition of the order in X/K , $g(x) \leq g(y)$. This shows that h is the unique continuous isotone function which makes the following diagram commutative :



1.19 PROPOSITION: An epimorphism $f: X \to Y$ is an special quotient in PTop if and only if for every PTop-morphism $g: X \to Z$ such that $\ker f \subset \ker g$ there exists a unique PTop-morphism $h: Y \to Z$ which makes the following diagram commutative :



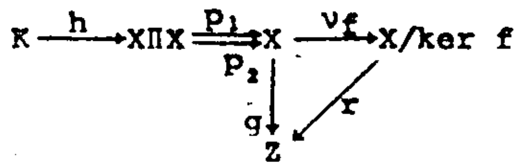
PROOF: Let $f: X \to Y$ be a special quotient and let $g: X \to Z$ be continuous and isotone such that $\ker f \subset \ker g$. As in 1.18 we define h and show that it is the unique continuous, isotone map which makes the diagram commutative . Let us assume the converse hypothesis. Call $\mathcal{S}$ a representative family of all PTop-epimorphisms m with domain X and $\ker f \subset \ker m$ and let $\nu : X \to X/R$ be the special quotient induced by $\mathcal{J}$. By hypothesis there is a unique PTop-morphism $h : Y \to X/R$ such that $h \circ f = \nu$. We apply the first part of this proposition to the special quotient $\nu : X \to X/R$ and f , and obtain a unique map $k : X/R \to Y$ such that $k \circ \nu = f$. Since both f and ν are surjective and therefore epimorphisms, from $h \circ k \circ \nu = \nu$ and $k \circ h \circ f = f$, $h \circ k = 1_{X/R}$ and $k \circ h = 1_Y$ follow . We conclude that f is equivalent to ν and is therefore a special quotient .

1.20 LEMMA: Let $f : X \to Y$ be a surjective PTop-morphism where $f(a) \leq f(b)$ implies $a \leq b$. If we define $\leq$ in $X/\ker f$ by $\nu_f(a) \leq \nu_f(b)$ if and only if $f(a) \leq f(b)$, then $\nu_f : X \to X/\ker f$ is a coequalizer .

PROOF: From 1.6 we find h the equalizer of $\nu_f \circ p_1$, $\nu_f \circ p_2$ and we claim that ν_f is the coequalizer of $p_1 \circ h$, $p_2 \circ h$. Since h is an equalizer :

$$\nu_f \circ (p_1 \circ h) = (\nu_f \circ p\ ) \circ h = (\nu_f \circ p_2) \circ h = \nu_f \circ (p_2 \circ h) \quad .$$

Suppose $q : X \to Z$ is given, such that $q \circ p_1 \circ h = q \circ p_2 \circ h$ in PTop . Then $\ker \nu_f = \ker f \subset \ker q$. Therefore there exists a unique continuous map $r : X/\ker f \to Z$ given by $r(\nu_f(x)) := q(x)$. We show next that r is also an isotone map and therefore the unique PTop-morphism which makes the following diagram commutative :



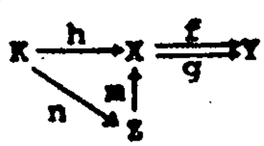
Let $\nu_f(x) \leq \nu_f(y)$. If $\nu_f(x) = \nu_f(y)$, then $g(x) = g(y)$, since r is well defined (in Top) . If $\nu_f(x) \leq \nu_f(y)$, $f(x) \leq f(y)$, and by hypothesis $x \leq y$. Therefore $g(x) \leq g(y)$.

1.21 LEMMA: Every special quotient in PTop is an extremal quotient .

PROOF: Let $f:X \to Y$ be an special quotient, $f = h \circ g$ and h a monomorphism in PTop . Then h is injective and $\ker f \subset \ker g$. Since f is a special quotient, there exists k such that $k \circ f = g$. We obtain $h \circ k \circ f = h \circ g = f$ and $k \circ h \circ g = k \circ f = g$. Since f is surjective , so is g and therefore $h \circ k$ and $k \circ h$ are identities .

1.22 LEMMA: Let S be a subcategory of PTop closed with respect to finite coproducts and t-quotients . Then the extremal subobjects of S are exactly the equalizers.

PROOF: Let $h:K \to X$ be the equalizer of f and g . Suppose $h = m \circ n$ where n is an epimorphism :



Since $f \circ h = g \circ h$, then $f \circ m \circ n = g \circ m \circ n$. Because n is an epimorphism $f \circ m = g \circ m$. Since h is the equalizer of f and g, there exists a unique PTop-morphism $k: Z \to K$ such that $h \circ k = m$. We thus obtain that $h \circ k \circ n = m \circ n = h = h \circ 1_K$ and, using the fact that h is a monomorphism, $k \circ n = 1_K$. Moreover $n \circ k \circ n = n$ and, since n is an epimorphism, $n \circ k = 1_Z$. Since n is therefore an isomorphism, we have shown that h is an extremal subobject. To prove the converse, let $h: K \to X$ be an extremal subobject, and define R in $X \amalg X$ as follows:

$$R := \{((h(k),1),(h(k),2)) \mid k \in K\} \cup \{((h(k),2),(h(k),1)) \mid k \in K\} \cup \Delta_{X \amalg X}$$

R is an equivalence relation, and we define on $(X \amalg X)/R$ a partial order $\leq_R$ by: $(x,i)_R \leq_R (y,j)_R$ if and only if $(x,i) \leq (y,j)$ or there exists $k \in K$ such that $x \leq h(k)$ and $h(k) \leq y$.

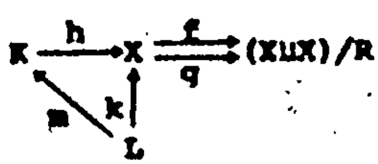
To see that $\leq_R$ is well defined, we assume that $(h(a),i)_R \leq_R (b,i)_R$. From the definition of $\leq_R$ it follows that $h(a) \leq b$. Since $h(a) \leq h(a)$ and $h(a) \leq b$, we obtain that $(h(a),j)_R \leq_R (b,i)_R$. Similarly, if $(b,i)_R \leq_R (h(a),i)_R$, then $(b,i)_R \leq_R (h(a),j)_R$.

It is obvious that $\leq_R$ is reflexive. Let $(x,i)_R \leq_R (y,j)_R$ and $(y,j)_R \leq_R (x,i)_R$. If $i = j$, by the definition of $\leq_R$, $x \leq y$ and $y \leq x$. Therefore $x = y$. If $i \neq j$ $h(a)$ and $h(b)$ exist such that

$x \leq h(a) \leq y \leq h(b) \leq x$. Therefore $x = y \in \mathrm{Im}\, h$ and $(x,i)_R = (y,j)_R$ follows .

To show the transitivity of $\leq_R$ suppose $(x,i)_R \leq_R (y,j)_R$ and $(y,j)_R \leq_R (z,k)_R$. If $i = j = k$, $x \leq y \leq z$ and $(x,i)_R \leq_R (z,k)_R$ follows . If $i = j$ but $j \neq k$, from the definition of $\leq_R$ we obtain $x \leq y$ and $y \leq h(a) \leq z$, from which $x \leq h(a) \leq z$ and $(x,i)_R \leq_R (z,k)_R$ follows . An analogous argument is used when $i \neq j$ and $j = k$, while if $i \neq j$ and $j \neq k$, one obtains $x \leq z$ and $i = k$, and from it directly, $(x,i)_R \leq_R (z,k)_R$.

Once we know that $\leq_R$ is a well defined partial order on $(X \sqcup X)/R$, we call $f := \nu_R \circ \sigma_1$ and $g := \nu_R \circ \sigma_2$ and begin to prove that h is the equalizer of f and g . We remark first that h, being an extremal monomorphism, must be an embedding and, by definition of R , $f \circ h = g \circ h$. Suppose $k : L \to X$ has been given such that $f \circ k = g \circ k$. Then for every $x \in L$, $\nu_R \circ \sigma_1 \circ k(h) = \nu_R \circ \sigma_2 \circ k(h)$. This means that $(k(x),1)R(k(x),2)$ and therefore $k(x) \in hK$. From this and because h is a monomorphism (injective in PTop), we can define a map $m : L \to K$ by $h(m(x)) = k(x)$, and we can conclude the proof by showing next that m is the unique continuous isotone map which makes the following diagram commutative :



Let $x \leq y$ in L. Then $k(x) \leq k(y)$, i.e. $h(m(x)) \leq h(m(y))$. Since h is an embedding, $m(x) \leq m(y)$. On the other hand, if U is open in K, since h is an embedding, an open set V of X exists such that $U = h^{-1}V$. From this we see that $m^{-1}U = m^{-1}h^{-1}V = (hm)^{-1}V = k^{-1}V$ which is open by the continuity of k. The map m is clearly unique.

1.23 LEMMA: Let $\underline{S}$ be a subcategory of HPOTS closed with respect to finite coproducts and t-quotients in HPOTS. If $h: K \to X$ is an equalizer in $\underline{S}$, then h is an extremal subobject.

PROOF: The same argument as in 1.20 may be used.

2. Epireflective subcategories of PTop, HPTop, HPOTS and HOTS.

In order to make use of 0.20 we begin this section by considering the properties "complete", "locally small" and "colocally small" in connection with PTop.

2.1 PROPOSITION: The category PTop is complete, cocomplete, locally small and colocally small.

PROOF: We have already obtained in 1.7 that PTop is complete and in 1.9 and 1.17 that it has coproducts and coequalizers, being therefore cocomplete . To see that it is locally small we remark first that every monomorphism in it is injective . For let $f:Y\to X$ be a monomorphism and suppose $a,b\in Y$, $a \neq b$ and $f(a) = f(b)$. We define $g,h:Y\to Y$ by $g(y) = a$ and $h(y) = b$ for all $y\in Y$. Since g,h are both continuous, isotone maps such that $f\circ g = f\circ h$ but $g \neq h$, we obtain a contradiction and conclude that f is injective . Therefore the cardinality $\bar{\bar{Y}}$ of Y is less than or equal to the cardinality $\bar{\bar{X}}$ of X. Now, consider the PTop-spaces whose underlying sets are subsets of X . Let $S\subset X$. Its PTop-structure is an element of $\mathcal{P}\mathcal{P}S\subset\mathcal{P}\mathcal{P}X$, where $\mathcal{P}$ means "power set of" . Its partial order belongs to $\mathcal{P}(S\times S)\times\mathcal{P}(X\times X)$. Therefore the PTop-structure of S belongs to $\mathcal{P}\mathcal{P}X\times\mathcal{P}(X\times X)$, which is a set . Hence PTop is locally small .

In order to show that PTop is colocally small we shall first show that every epimorphism $f:X\to Y$ in PTop is surjective . Let R be the equivalence relation defined on $Y\amalg Y$ by :

$$R:=\{((f(x),1),(f(x),2))\,|\,x\in X\}\cup\{((f(x),2),(f(x),1))\,|\,x\in X\}\cup\Delta_{Y\amalg Y}.$$

Define in $(Y\amalg Y)/R$: $(a,i)_R \leq_R (b,j)_R$ if and only if $(a,i) \leq (b,j)$ or there exists $c\in X$ such that $a \leq f(c)$ and $f(c) \leq b$.

As in 1.22 we may convince ourselves that $\leq_R$ is a well defined partial order, and that $\nu_R:Y\amalg Y\to(Y\amalg Y)/R$ is continuous and isotone . If we denote by σ_1 and σ_2 the canonical injections $Y \to Y \amalg Y$, we have

$\nu_R \circ \sigma_1 \circ f(x) = (f(x),1)_R = (f(x),2)_R = \nu_R \circ \sigma_2 \circ f(x)$ for all $x \in X$. It follows that $\nu_R \circ \sigma_1 \circ f = \nu_R \circ \sigma_2 \circ f$, and, since f is an epimorphism, $\nu_R \circ \sigma_1 = \nu_R \circ \sigma_2$. This means that , for any arbitrary $y \in Y$, we have $(y,1)_R = (y,2)_R$; it then follows by the definition of R, that $y \in \mathrm{Im}\, f$.

We have shown that if $f: X \to Y$ is an epimorphism then $\bar{\bar{Y}} \leq \bar{\bar{X}}$. Accordingly, for every such space Y we can induce an isomorphic copy on a subset of X . There exists only a set of spaces, each of which has as its underlying set a subset of X and a <u>PTop</u>-structure . Therefore <u>PTop</u> is colocally small .

2.2 <u>PROPOSITION</u>: The category <u>HPTop</u> is colocally small.

<u>PROOF</u>: We first prove that if $Y \in$ <u>HPTop</u>, every proper closed subspace U of Y is an equalizer . Given one such $U \subset Y \in$ <u>HPTop</u> , define :

$$R := \{((u,1),(u,2)) \mid u \in U\} \cup \{((u,2),(u,1)) \mid u \in U\} \cup \Delta_{Y \amalg Y} .$$

As in 2.1 , we convince ourselves that R is an equivalence relation in $Y \amalg Y$ and that the relation $\leq_R$ defined by " $(x,i)_R \leq_R (y,j)$ if and only if $(x,i) \leq (y,j)$ or there exists $u \in U$ such that $x \leq u$ and $u \leq y$ " is a partial order in $(Y \amalg Y)/R$. We call $Z := ((Y \amalg Y)/R, \leq_R)$, and show that it is Hausdorff . Let $(x,i)_R$, $(y,j)_R$ be two distinct points of Z. If $x = y$, then $i \neq j$ and $x,y \in \complement U$, from which two disjoint saturated neighbourhoods $\complement U \times \{i\}$ and $\complement U \times \{j\}$ of (x,i) and (y,j), respectively, are found . If $x \neq y$, there exist V, W , disjoint open neighbourhoods of x and y, respectively, since Y is Hausdorff, and we obtain with them the disjoint saturated open sets $V \times \{i,j\}$ and $W \times \{i,j\}$, which are neighbourhoods of (x,i) and (y,j), respectively . From Bourbaki [5] Chap. 1, it follows that Z is Hausdorff .

Call $f := \nu_R \circ \sigma_1$ and $g := \nu_R \circ \sigma_2$. By considering $U \xrightarrow{i} Y \underset{\sigma_2}{\overset{\sigma_1}{\rightrightarrows}} Y \sqcup Y \xrightarrow{\nu_R} Z$

we see readily that $U \xrightarrow{i} Y$ is an equalizer of f,g . Having shown that every proper closed subspace of an space in HPTop is an equalizer, we consider an HPTop-epimorphism $e: X \to Y$. The closure of its image $\Gamma \mathrm{Im}\, e$ can not be a proper subset of Y, or e would be an equalizer .■

Having obtained $\Gamma \mathrm{Im}\, e = Y$, i.e. $\bar{\bar{Y}} \leq 2^{2^{\bar{\bar{X}}}}$ by considerations identical to those at the end of 2.1, we see that HPTop is colocally small .

2.3 REMARK: In what follows we adopt for limits and diagrams the terminology of [10] § 5 and § 9 .

2.4 PROPOSITION: The categories HPTop, HPOTS and HOTS are strongly closed with respect to products and equalizers in PTop .

PROOF: A product is the limit of a diagram $F: \underline{S} \to \underline{PTop}$ where $\underline{S} = \{\{s\} | s \in S\}$ and there are no maps between $\{s\} \neq \{t\}$. Therefore , if F is partially in HPTop , for example, it has to be in HPTop . See 0.19 . Given a family $(X_s)_{s \in S}$ in HPTop, its PTop-product belongs to HPTop, since H is productive . If it is a family in HPOTS we show that $\Pi_{s \in S} X_s \in$ HPOTS by proving that its partial order, given in 1.4, is semicontinuous . Let $a, b \in \Pi_{s \in S} X_s$ and $a \nleq b$. Let $s \in S$, such that $p_s(a) \nleq p_s(b)$. Since X_s is in HPOTS , there exists an open neighbourhood U of $p_s(a)$ such that for every $u \in U$, $u \nleq p_s(b)$. We consider the open neighbourhood $p_s^{-1}U$ of a and see that for every $v \in p_s^{-1}U$, then $p_s(v) \in U$ and $p_s(v) \nleq p_s(b)$ and $v \nleq b$. This shows that the partial order of $\Pi_{s \in S} X_s$ is lower semicontinuous . A symmetric argument shows that it is upper semicontinuous, too, and therefore semicontinuous . If the family $(X_s)_{s \in S}$ had been given in HOTS a similar proof would show that its product belongs to HOTS .

Now, we consider the equalizers. An equalizer is the limit of a diagram $F: \underline{S} \to \underline{PTop}$ where $\underline{S}$ can be described by

$$S \underset{m_2}{\overset{m_1}{\rightrightarrows}} T$$

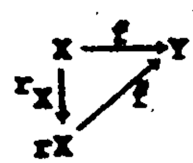
If P is partially in some subcategory SC of PTop, FS has to belong to this subcategory. We show that in our cases the equalizer found in 1.6 is in HPTop, HPOTS and HOTS respectively. Let $X_S \underset{g}{\overset{f}{\rightrightarrows}} X_T$ be given in PTop such that $X_S \in$ HPTop, (HPOTS and HOTS). We know that the PTop-equalizer is given by $K := \{x \in X_S \mid f(x) = g(x)\}$ and its inclusion morphism. Since H is a hereditary subcategory of Top, $K \xrightarrow{i} X_S$ belongs to HPTop. Since the semicontinuity (continuity) of a partial order is trivially hereditary, $K \xrightarrow{i} X_S$ is also an equalizer in the HPOTS and HOTS cases.

2.5 PROPOSITION: The categories HPTop, HPOTS and HOTS are epireflective in PTop.

PROOF: Follows from 2.4 and 0.20

2.6 LEMMA: Let S be a subcategory of B and A a subcategory of S. If A is epireflective in B, A is epireflective in S.

PROOF: Since A is epireflective in B, for every $X \in$ B there exists $rX \in$ A and $r_X : X \to rX$ a B-epimorphism such that whenever $f : X \to Y$ is in A, there exists a unique $\bar{f}$ in A which makes the following diagram commutative:

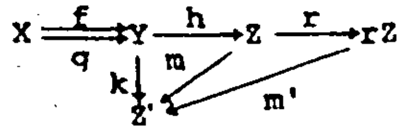


Since $\underline{S} \subset \underline{B}$, this is also true for every $X \in \underline{S}$, and since r_X is a $\underline{B}$-epimorphism, it is a $\underline{S}$-epimorphism also. By [10] § 8, see 0.18, $\underline{A}$ is epireflective in $\underline{S}$.

2.7 COROLLARY: The categories HPOTS and HOTS are epireflective in HPTop .

2.8 PROPOSITION: The categories HPTop, HPOTS and HOTS are complete, cocomplete and locally small .

PROOF: From 2.4, it follows directly that these three categories are complete . By the same method used in 2.1, one shows that the monomorphisms are injective, and concludes that our categories are locally small . One sees trivially from 1.9 and 1.10 that they are closed with respect to coproducts . All we need to show then, is that they are also closed with respect to coequalizers . For convenience, let us denote any one of the categories HPTop, HPOTS and HOTS by SC . Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be two different SC-morphisms . Using the same ideas as in 1.17, we find $h: Y \to Z$ their PTop-coequalizer . Since Z may not be Hausdorff or may not have semicontinuous (continuous) partial order, we take the SC-reflection rZ of Z, which exists by 2.7 . Let $r: Z \to rZ$ be the reflection map. The following diagram will be useful in the course of the proof (all maps will be introduced) :



Since h is the coequalizer of f and g, $h \circ f = h \circ g$. Therefore $(r \circ h) \circ f = r \circ (h \circ f) = r \circ (h \circ g) = (r \circ h) \circ g$. Suppose $k : Y \to Z'$ has been given in SC such that $k \circ f = k \circ g$. Since $k \in$ PTop and h is a PTop-coequalizer, there exists a unique continuous isotone map m such that $m \circ h = k$. Since r is the SC-reflection, there exists a unique $m' : rZ \to Z'$ in SC, such that $m' \circ r = m$. Therefore $k = m \circ h = (m' \circ r) \circ h = m' \circ (r \circ h)$. We complete the proof by checking that m' is the unique map such that $k = m' \circ (r \circ h)$. Suppose we had another, say m''. By the uniqueness of m such that $m \circ h = k$ we obtain $m'' \circ r = m$, and by the uniqueness of m' such that $m' \circ r = m$, it follows that $m'' = m'$.

2.9 COROLLARY: If K is a productive, closed hereditary subcategory of H, KPTop is complete and locally small.

2.10 LEMMA: A subcategory K of H is epireflective in H if and only if KPTop is epireflective in HPTop.

PROOF: Let K be epireflective in H. Since K is productive and closed hereditary, by 2.9, KPTop is complete. By arguments similar to those in the proof of 2.4, KPTop is strongly closed with respect to products and equalizers in <u>HPTop</u> , and by 0.20, <u>KPTop</u> is epireflective in <u>HPTop</u> . Conversely, let $(X_i)_{i\epsilon I}$ be a family in <u>K</u> . By considering $UX_i := (X_i, d)$ where d is the discrete order, since <u>KPTop</u> is productive, then $\Pi_{i\epsilon I} UX_i \epsilon$ <u>KPTop</u> . By the definition in 1.4 , $\Pi_{i\epsilon I} UX_i$ has the discrete order and $\Pi_{i\epsilon I} X_i$ as its underlying topological space . The conclusion that $\Pi_{i\epsilon I} X_i \epsilon \underline{K}$ follows . As for equalizers, given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in <u>H</u> with $X\epsilon\underline{K}$, we consider $(X,d) \overset{f}{\underset{g}{\rightrightarrows}} (Y,d)$ and its equalizer K in <u>KPTop</u>. K inherits from (X,d) the discrete order, and its underlying topological space with its inclusion map is the equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in <u>K</u> . By 0.20 <u>K</u> is epireflective in <u>H</u> .

2.11 <u>PROPOSITION</u> : If <u>K</u> is a subcategory of <u>H</u> , the following statements are equivalent :

1) <u>K</u> is epireflective in <u>H</u>
2) <u>KPTop</u> is epireflective in <u>HPTop</u>
3) <u>KPOTS</u> is epireflective in <u>HPOTS</u>
4) <u>KOTS</u> is epireflective in <u>HOTS</u> .

<u>PROOF</u>: We already know from 2.10 that 1) and 2) are equivalent . Suppose <u>KPTop</u> is epireflective in <u>HPTop</u> . By 2.7 <u>HPOTS</u> is epireflective in <u>HPTop</u>. Therefore <u>KPOTS</u>, the intersection of <u>KPTop</u> and <u>HPOTS</u> , is epireflective in <u>HPTop</u> . By 2.6 <u>KPOTS</u> is epi-

reflective in HPOTS . Similarly, since HOTS is also epireflective in HPTop, KOTS is epireflective in HPTop and in HOTS . If KPOTS is epireflective in HPOTS, by [10] § 9, KPOTS is strongly closed with respect to products and equalizers in HPOTS . By analogous considerations to the ones given in the second part of the proof of 2.10 , K is strongly closed with respect to products and equalizers in H , and therefore epireflective in H . A parallel argument shows that 4) implies 1), and our proof concludes .

2.12 COROLLARY: CPOTS is epireflective in HPOTS and COTS is epireflective in HOTS .

2.13 REMARK: We have seen in 2.9 that if K is an epireflective subcategory of H, KPTop is complete . From 2.11 we obtain that KPOTS and KOTS are also complete .

3. Leftadjoints of the Inclusion and Forgetful functors

3.1 LEMMA: Let K be a subcategory of H, $F: \text{KOTS} \to K$ the order-forgetful functor, and $U: K \to \text{KOTS}$ where $UX := (X,d)$ and where d is the discrete order . Then U is left-adjoint of F .

PROOF: Let $f:A\to B$ in $\underline{K}$ and $g:C\to D$ in $\underline{KOTS}$ be given. Define $\eta_{B,C}:\underline{KOTS}(UB,C)\to \underline{K}(B,FC)$ by $\eta_{B,C}(h)(b):=h(b)$. Since $\eta_{B,C}$ is clearly a bijection, we only need to show that the following diagram commutes :

$$\begin{array}{ccc} \underline{KOTS}(UB,C) & \xrightarrow{\eta_{B,C}} & \underline{K}(B,FC) \\ {\scriptstyle \underline{KOTS}(Uf,g)}\downarrow & & \downarrow{\scriptstyle \underline{K}(f,Fg)} \\ \underline{KOTS}(UA,D) & \xrightarrow[\eta_{A,D}]{} & \underline{K}(A,FD) \end{array}$$

Let $h\in\underline{KOTS}(UB,C)$ and $a\in A$ be arbitrary .

Then

$$(\underline{K}(f,Fg)\circ\eta_{B,C})(h)(a) = \underline{K}(f,Fg)(\eta_{B,C}(h))(a) =$$

$$= (Fg\circ\eta_{B,C}(h)\circ f)(a) = Fg(\eta_{B,C}(h)(f(a))) = Fg(h(f(a)))$$

$$= g(h(f(a))) = (g\circ h\circ Uf)(a) = \eta_{A,D}(g\circ h\circ Uf)(a) =$$

$$= (\eta_{A,D}\circ\underline{KOTS}(Uf,g))(h)(a) .$$

Since a is arbitrary , $(\underline{K}(f,Fg)\circ\eta_{B,C})(h) = (\eta_{A,D}\circ\underline{KOTS}(Uf,g))(h)$ and since h is arbitrary , $\underline{K}(f,Fg)\circ\eta_{B,C} = \eta_{A,D}\circ\underline{KOTS}(Uf,g)$. This shows that η is a natural equivalence as required .

3.2 REMARK: $\underline{K}$ is a coreflective subcategory of $\underline{KOTS}$.

PROOF: See [10] § 8 .

3.3 REMARK: We show next that even for $\underline{C}$ (compact spaces) , F is not left-adjoint of U .

PROOF: Suppose F is a left-adjoint of U . Then, for every $X \in \underline{C}$ and $Y \in \underline{COTS}$, there exists a bijection $b_{X,Y}: \underline{C}(FY,X) \rightarrow \underline{COTS}(Y,UX)$.

Define $X := (\{0,1\},d)$ and $Y := (\{a,b,c\}, a \leq b \leq c)$, both with the discrete topology .

Then $\underline{C}(FY,X) = X^{FY}$ has $2^3 = 8$ elements, and $\underline{COTS}(Y,UX)$ has only two elements, since all isotone functions coming from a chain into a discrete partial order are constant .

3.4 __LEMMA:__ Let $U: \underline{K} \rightarrow \underline{KOTS}$ be defined by $UX := (X,d)$. Then U has a left-adjoint for the following subcategories $\underline{K}$ of $\underline{H}$:

1) Hausdorff
2) completely regular
3) compact
4) real-compact
5) zero-dimensional
6) boolean spaces .

__PROOF:__ Since all these subcategories of $\underline{H}$ are productive and closed hereditary, they have equalizers and products, and are therefore complete . As an embedding, U clearly preserves limits . By [19] Theorem 2 page 110 (Pareigis), we need only show that for every $D \in \underline{KOTS}$, there exists a set $\mathcal{S}_D$ of $\underline{K}$-objects, which is a solution set of D with respect to U . Choose a set D' such that $\overline{\overline{D'}} = 2^{2^{\overline{\overline{D}}}}$. Define

$$\mathcal{S}_D := \{(S,t) \mid S \subset D' \text{ and } t \text{ is a topology on } S\}$$

We shall show that $\mathcal{S}_D$ is a solution set . Let $D \xrightarrow{h} UC$ be given . Then $\overline{\overline{\mathrm{Im}\,h}} \leq \overline{\overline{D}}$, whence $\overline{\overline{\Gamma\mathrm{Im}\,h}} \leq \overline{\overline{D'}}$. Choose a subset S_h of D' such that $\overline{\overline{S_h}} = \overline{\overline{\Gamma\mathrm{Im}\,h}}$, find a bijection $b: S_h \to \Gamma\mathrm{Im}\,h$ and induce on S_h the topological structure from $\Gamma\mathrm{Im}\,h$. We obtain in this way that $S_h \in \mathcal{S}_D$ and b is an homeomorphism . Call h' the map $D \to U(\Gamma\mathrm{Im}\,h)$ defined by $h'(d) = h(d)$, and define $k, f,$ so that the following diagrams commute :

$$\begin{array}{ccc} D & \xrightarrow{k} & US_h \\ & {\scriptstyle h'}\searrow \quad \nearrow{\scriptstyle U(b^{-1})} & \\ & U(\Gamma\mathrm{Im}\,h) & \end{array} \qquad\qquad \begin{array}{ccc} S_h & \xrightarrow{f} & C \\ & {\scriptstyle b}\searrow \quad \nearrow{\scriptstyle i} & \\ & \Gamma\mathrm{Im}\,h & \end{array}$$

Now $Uf \circ k = U(i \circ b) \circ k = U(i \circ b) \circ U(b^{-1}) \circ h' = U(i) \circ h' = h$, $S_h \in \mathcal{S}_D$ and the following diagram commutes :

$$\begin{array}{ccc} D & \xrightarrow{h} & UC \\ & {\scriptstyle k}\searrow \quad \nearrow{\scriptstyle U(f)} & \\ & U(S_h) & \end{array}$$

Therefore $\mathcal{S}_D$ is a solution set as required .

3.5 <u>REMARK</u>: From 3.4 it follows that the subcategories $\underline{K}$ listed therein are reflective in <u>KOTS</u> .

Next we shall obtain a description of the left adjoint of U . The proof of our first lemma is trivial .

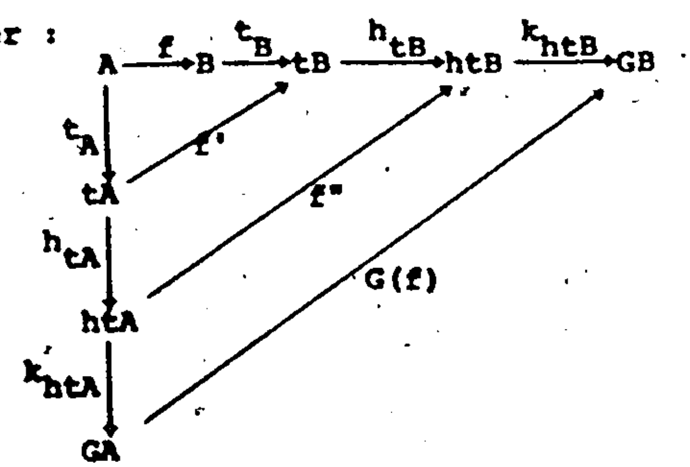
3.6 <u>LEMMA</u>: Let $Y \in$ <u>KOTS</u> . Define in Y $a \sim b$ if and only if $\{a,b\}$ has an upper or a lower bound, and $a \approx_Y b$ if and only if there exist $a_1, \ldots, a_n \in Y$ such that $a \sim a_1$, $a_1 \sim a_2, \ldots, a_n \sim b$. Then $\approx_Y$ is an equivalence relation on Y .

3.7 NOTATION: Let K be an epireflective subcategory of H . Let $Y \varepsilon$ KOTS . Let tY denote the topological space Y/π_Y , and h:Top→H and k:H→K the epireflexions. We see that the natural map $Y \to Y/\pi_Y$ is isotone, as tY has been obtained by identifying all the points which can be compared or are extrema of chains $c_1,\ldots,c_n$ where c_i can be compared to c_{i+1} .

We denote khtY by GY defining in this fashion a functor G:KOTS→K , as will be shown in the next proposition .

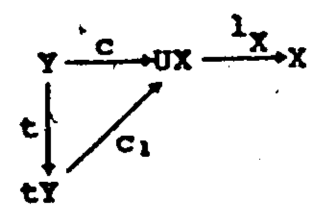
3.8 PROPOSITION: Let K be an epireflective subcategory of H . Let G:KOTS→K as defined in 3.7 . Then G is the left-adjoint of U .

PROOF: G is certainly a functor . Given $A \xrightarrow{f} B$ in KOTS , we consider :

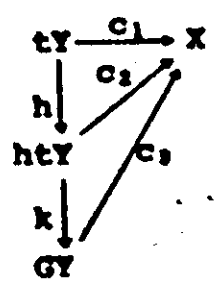


For $t_B \circ f$ and t_A , f' is unique . For $h_{tB} \circ f'$ and h_{tA} , f" is unique and for $k_{htB} \circ f"$ and k_{htA} , G(f) is well defined . It is an easy routine to check all the functor properties of G . For every $Y \in \underline{KOTS}$, call $q_Y := k_{htY} \circ h_{tY} \circ t_Y$. Let $X \in K$ and $Y \in \underline{KOTS}$ and define $\lambda_{Y,X}$ as follows: $\lambda_{Y,X} : \underline{K}(GY,X) \to \underline{KOTS}(Y,UX)$ and $\lambda_{Y,X}(f) := f \circ q_Y$. Since t_Y is a natural map and h and k are epireflections, q_Y is continuous and isotone . Therefore $f \circ q_Y \in \underline{KOTS}(Y,UX)$. We claim that λ is a natural equivalence .

Let $\lambda_{Y,X}(a) = \lambda_{Y,X}(b)$. Then $a \circ q_Y = b \circ q_Y$. Since t_Y is surjective and h,k epireflexions, we obtain that $a = b$. Therefore $\lambda_{Y,X}$ is injective . Let $Y \xrightarrow{c} UX$. To show that $\pi_Y \subset \ker c$, let $a,b \in Y$ and $a \pi_Y b$. If $a \sim b$, since UX has discrete order , $c(a) = c(b)$. If $a \sim a_1$, $a_1 \sim a_2$,...,$a_n \sim b$, similarly $c(a) = c(b)$. Therefore there exists c_1 such that the following diagram commutes :



Moreover, since h and k are epireflectors, there exists c_2 and c_3 , such that $c_2 \circ h = c_1$ and $c_3 \circ k = c_2$.



Therefore $c_3 \circ k \circ h \circ t = c_2 \circ h \circ t = c_1 \circ t = c$, which can be rewritten as $c = c_1 \circ g_Y = \lambda_{Y,X}(c_1)$, and shows that $\lambda_{Y,X}$ is surjective .

To show that λ is natural, let $A \xrightarrow{c} B$ in $\underline{K}$ and $C \xrightarrow{i} D$ in $\underline{KOTS}$. Consider the following diagrams :

$$\begin{array}{ccc} \underline{K}(GD,A) & \xrightarrow{\lambda_{D,A}} & \underline{KOTS}(D,UA) \\ {\scriptstyle \underline{K}(Gi,c)}\downarrow & & \downarrow{\scriptstyle \underline{KOTS}(i,Uc)} \\ \underline{K}(GC,B) & \xrightarrow[\lambda_{C,B}]{} & \underline{KOTS}(C,UB) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} a & \longmapsto & a \circ g_D \\ \big| & & \big\downarrow {\scriptstyle Uc \circ a \circ g_D i} \\ c \circ a \circ Gi & \longmapsto & c \circ a \circ Gi \circ g_C \end{array}$$

By the definition of Gi, we know that $Gi \circ g_C = g_D \circ i$.

Since Uc is the same map as c, we obtain $Uc \circ a \circ g_D \circ i = c \circ a \circ Gi \circ g_C$ and the diagram commutes . Thus λ is a natural equivalence and the proof is complete .

3.9 __LEMMA:__ Let $\underline{K}$ be a reflective subcategory of $\underline{H}$, $R:\underline{H} \to \underline{K}$ the reflector and $E:\underline{K} \to \underline{H}$ the inclusion functor . Then URG is left adjoint to UEF .

__PROOF:__ Consider the adjoint situations :

$$\underline{KOTS} \underset{U}{\overset{F}{\rightleftarrows}} \underline{K} \underset{R}{\overset{E}{\rightleftarrows}} \underline{H} \underset{G}{\overset{U}{\rightleftarrows}} \underline{HOTS}$$

Let η ,μ ,λ be the corresponding natural equivalences . The composition of them will clearly be a natural equivalence as $\eta_{X,EFY} \circ \mu_{GX,FY} \circ \lambda_{RGX,Y}$ is a bijection and, since every one of the interior squares in following diagram is commutative, so is the exterior one .

$$\begin{array}{ccccccc}
\underline{KOTS}(URGX,Y) & \xrightarrow{\lambda_{RGX,Y}} & \underline{K}(RGX,FY) & \xrightarrow{\mu_{GX,FY}} & \underline{H}(GX,EFY) & \xrightarrow{\eta_{X,EFY}} & \underline{HOTS}(X,UEFY) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{KOTS}(URGA,B) & \xrightarrow[\lambda_{RGA,B}]{} & \underline{K}(RGA,FB) & \xrightarrow[\mu_{GA,FB}]{} & \underline{H}(GA,EFB) & \xrightarrow[\eta_{A,EFB}]{} & \underline{HOTS}(A,UEFB)
\end{array}$$

3.10 LEMMA: Let $\underline{K}$ be a coreflective subcategory of $\underline{H}$, and $C:\underline{H}\to\underline{K}$ the coreflector. Then UEG is left adjoint to UCF.

PROOF: Similar to 3.9.

4. Coreflective Subcategories.

4.1 LEMMA: Every nonempty object of $\underline{PTop}$ is a generator.

PROOF: We use 0.22. Let G be a nonempty object of $\underline{PTop}$ and $A\in\underline{PTop}$ arbitrary. Define $e_G:\underline{PTop}(G,A)\times G\to A$ by $e_G(v,x):=v(x)$. Let U be an open neighbourhood of $v(x)$. Since v, in particular is continuous, $v^{-1}U$ is open in G. Therefore $(v,v^{-1}U)$ is open in $\underline{PTop}(G,A)\times G$ which is the coproduct $\coprod_{\underline{PTop}(G,A)} G$ of as many copies of G as $\underline{PTop}(G,A)$ has elements.

Since $e_G(v,v^{-1}U) = vv^{-1}U\subset U$, e_G is continuous. If $(v,x)\leq(v',x')$ in $\underline{PTop}(G,A)\times G$, $v=v'$ and $x\leq x'$. Therefore $e_G(v,x)=v(x)\leq v(x')=v'(x')=e_G(v',x')$ and thus we see that e_G is isotone.

Let $a \varepsilon A$. If $\bar{a}:G \to A$ is defined by $\bar{a}(g) = a$ for all $g \varepsilon G$, $\bar{a} \varepsilon$ PTop(G,A) and $e_G(\bar{a},g) = \bar{a}(g) = a$. Since G is nonempty, it follows that e_G is surjective .

4.2 REMARK: Since HPTop and HPOTS are cocomplete, and therefore have coproducts, every nonempty object in these categories is a generator .

4.3 REMARK: If K is a subcategory of Top , KPTop is closed with respect to limits in PTop if and only if K is closed with respect to limits in Top . Moreover if KPTop is closed with respect to colimits in PTop, K is also closed with respect to colimits in Top .

PROOF: This follows directly from the definition of products, coproducts, equalizers and coequalizers given in section 1, and the observation that K is a subcategory of KPTop and Top of PTop .

4.4 PROPOSITION: Let K be a subcategory of Top(H) . Then KPTop is coreflective in PTop(HPTop) if and only if K is coreflective in Top(H) .

PROOF: The necessity follows directly from the previous remark and 0.24 . Let K be coreflective in Top . We show that KPTop is closed with respect to coproducts and coequalizers in PTop . If $(X_i, \xi_i)_{i \varepsilon I}$ is a family in

KPTop, we know from 1.9 that the underlying topological space of $\coprod_{i\epsilon I}(X_i, \leq_i)$ is $\coprod_{i\epsilon I}X_i$. Since $\underline{K}$ is coreflective in Top, $\coprod_{i\epsilon I}X_i \epsilon \underline{K}$. Therefore $\coprod_{i\epsilon I}(X_i, \leq_i) \epsilon$ KPTop. Similarly, given $(X, \leq) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leq)$ in KPTop, we found in 1.17 that the coequalizer in PTop had as its underlying topological space a quotient of Y. Since $Y \epsilon \underline{K}$, since $\underline{K}$ is coreflective in Top, and since every quotient in $\underline{K}$ is a coequalizer in $\underline{K}$, then the coequalizer of $(X, \leq) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leq)$ in PTop belongs to KPTop.

If $\underline{K}$ is a coreflective subcategory of $\underline{H}$, we show exactly as above that KPTop is closed with respect to coproducts. Given $(X, \leq) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leq)$ in KPTop, we construct its coequalizer in HPTop by obtaining first its coequalizer in PTop and then its HPTop-reflection. (See the proof of 2.8). But the PTop-coequalizer already belongs to KPTop $\subset$ HPTop and its reflection is therefore itself.

4.5 REMARK: It should be noted that the underlying topological space Z, of the coequalizer $h: (Y, \leq) \rightarrow (Z, \leq)$ of $(X, \leq) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \leq)$ in PTop, may not be the coequalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in Top

4.6 LEMMA: Let $\underline{S}$ be a subcategory of $\underline{B}$ and $\underline{A}$ a subcategory of $\underline{S}$. If $\underline{A}$ is coreflective in $\underline{B}$, $\underline{A}$ is coreflective in $\underline{S}$.

PROOF: Dual to 2.5 .

4.7 LEMMA: Let $\underline{K}$ be a subcategory of $\underline{Top}(\underline{H})$. Then $\underline{K}$ is coreflective in $\underline{PTop}(\underline{HPTop})$ if and only if $\underline{K}$ is coreflective in $\underline{Top}(\underline{H})$.

PROOF: The necessity follows by 4.6 . Conversely if we consider $\underline{K}$ as a subcategory of $\underline{PTop}$ the coproducts and coequalizers of $\underline{K}$ in $\underline{PTop}$ will have the discrete order and therefore belong to $\underline{K}$. Similarly for $\underline{HPTop}$.

4.8 PROPOSITION: If $\underline{K}$ is a subcategory of $\underline{H}$, the following statements are equivalent :

1) $\underline{K}$ is coreflective in $\underline{H}$
2) $\underline{KPTop}$ is coreflective in $\underline{HPTop}$
3) $\underline{KPOTS}$ is coreflective in $\underline{HPOTS}$
4) $\underline{KOTS}$ is coreflective in $\underline{HOTS}$.

PROOF: We already know from 4.4 that 1) and 2) are equivalent . Suppose $\underline{KPTop}$ is coreflective in $\underline{HPTop}$. Let $(X, \leq_1) \in \underline{HPOTS}$ and $c_X : c(X, \leq_1) \to (X, \leq_1)$ be its $\underline{KPTop}$-coreflexion . We shall show that $c(X, \leq)$ has a semicontinuous partial order . Let $d_X : dX \to X$ be the $\underline{K}$-coreflexion of X . By 0.23 we can assume without loss of generality that the underlying sets of X, dX and $c(X, \leq_1)$ are all the same, and that the graph of the maps c_X and d_X is the diagonal . Since $(dX, \leq_1) \in \underline{KPTop}$, and the map $d_X : (dX, \leq_1) \to (X, \leq_1)$ is in HPTop, there exists a unique continuous and isotone map f which makes the following diagram commutative :

$$\begin{array}{ccc} (dX, \leq_1) & \xrightarrow{d_X} & (X, \leq_1) \\ & \searrow^{f} & \uparrow{\scriptstyle c_X} \\ & & c(X, \leq_1) \end{array}$$

Let $a \nleq b$ in $c(X, \leq_1)$. If $c_X(a) \leq_1 c_X(b)$, then $d_X^{-1}c_X(a) \leq_1 d_X^{-1}c_X(b)$ and $a = fd_X^{-1}c_X(a) \leq fd_X^{-1}c_X(b) = b$ which is a contradiction . Therefore $c_X(a) \nleq_1 c_X(b)$. Since $\leq_1$ is semicontinuous in $(X, \leq_1)$, we find two open neighbourhoods, U of $c_X(a)$ and V of $c_X(b)$ such that $c_X(a) \nleq v$ and $u \nleq c_X(b)$ for all $u \in U$ and $v \in V$. This shows that $U = c_X^{-1}U$ and $V = c_X^{-1}V$ are two open neighbourhoods in $c(X, \leq_1)$ such that $a \nleq v$ and $u \nleq b$ for all $u \in U$ and $v \in V$. Accordingly the partial order of $c(X, \leq_1)$ is semicontinuous .

We have shown that if $(X, \leq_1) \in$ HPOTS , given an arbitrary $f : Y \to (X, \leq_1)$ in HPTop such that $Y \in$ KPTop , there exists $c(X, \leq_1) \in$ KPOTS , $c_X : c(X, \leq_1) \to (X, \leq_1)$ in HPTop and $\bar{f}$ continuous and isotone such that the following diagram commutes :

$$\begin{array}{ccc} Y & \xrightarrow{f} & (X, \leq_1) \\ & \searrow^{\bar{f}} & \uparrow{\scriptstyle c_X} \\ & & c(X, \leq_1) \end{array}$$

This will be true in particular whenever $Y \in$ KPOTS $\subset$ KPTop .

The same argument shows that 2) implies 4) .

Let KPOTS be coreflective in HPOTS . If $X \in$ HPOTS has the discrete order, it is clear from 0.23, that its KPOTS-coreflection has the discrete order . Therefore K is coreflective in H . Similarly 4) implies 1).

CHAPTER II

SPECIAL EPIREFLEXIONS IN HOTS

This Chapter is devoted to generalizations of complete regularization, compactification and real-compactification in HOTS, and to comparing those spaces with the corresponding for the underlying topological spaces . Having been unable to generalize some of the characterizations of completely regular spaces in Top , to CrPOTS or to CrOTS , we consider, as did Nachbin [17] , subcategories of these as more suitable generalizations of complete regularity . Using the obvious generalization of the concept of E-regular - see 0.26 , we see that $\mathbb{I}$-regular , in PTop sense, is equivalent to $\mathbb{R}$-regular and both are characterized by the evaluation maps being embeddings (as in Top) . We call CrORR the category of $\mathbb{I}$-regular spaces, and this is a subcategory of CrOR, the class of spaces defined as completely regular ordered by Nachbin [17]. The category CrORR is epireflective in HOTS . In a very similar way, we introduce the $\mathbb{I}$-compact and $\mathbb{R}$-compact spaces and study the corresponding epireflexions . The category of $\mathbb{I}$-compact partially ordered topological spaces coincides with COTS, and the category of such spaces which are $\mathbb{R}$-compact includes the realcompact spaces of Top .

A comparison in the last section between the functors β_1 and υ_1 introduced here, and β and υ, leads to the fact that $\beta_1(X,d) \cong (\beta X,d)$ and $\upsilon_1(X,d) \cong (\upsilon X,d)$. However we introduce κ-separable topological spaces for arbitrary infinite cardinals κ, and by using these spaces we are able to exhibit a space $(X,\leq)$ in CrORR for which $F\beta_1(X,\leq) \not\cong \beta X$.

5. Complete regularization

In this section we study the category of $\mathbb{I}$-regular and $\mathbb{R}$-regular spaces in the sense taken from 0.26, and adapted to partially ordered topological spaces just by substituting "PTop-isomorphic" for "homeomorphic". We compare these with the class of "completely regular ordered spaces" introduced by Nachbin and with CrOTS.

5.1 NOTATION: We denote by CrOR the category of "completely regular ordered spaces" as introduced by Nachbin ([17], Chap. II, pages 52 and 54), and we denote by IrOT and RrOT the categories of $\mathbb{I}$-regular spaces and $\mathbb{R}$-regular spaces in PTop respectively. As we shall use the sets HOTS$(X,\mathbb{R})$ and HOTS$(X,\mathbb{I})$ often, we shall abbreviate them by C_1X and I_1X respectively.

5.2 <u>LEMMA</u>: $\underline{\text{IrOT}} \subset \underline{\text{CrOR}} \subset \underline{\text{CrOTS}}$

<u>PROOF</u>: Let $X \in \text{IrOT}$, and S a set such that X is a subspace of I^S. By [17] Theorem 7 page 55, we have $X \in \underline{\text{CrOR}}$. If $X \in \underline{\text{CrOR}}$, it follows by [17] Prop.8 page 59 and [24] (Ward Jr.) Lemma 1 page 145, that X has a continuous partial order, and by [17] Prop.6 page 53, that the underlying topological space of X is completely regular.

5.3 <u>LEMMA</u>: Let $X \in \underline{\text{CrOR}}$ and let $\rho : X \to \mathbb{R}^{C_1 X}$ and $j : X \to I^{I_1 X}$ be defined by $\rho(x)(f) := f(x)$, $j(x)(f) := f(x)$, respectively. Then ρ and j are monomorphisms.

<u>PROOF</u>: We consider only ρ as the arguments for j are completely analogous. Since $\mathbb{R}^{C_1 X}$ has the product structure and since, for every $f \in C_1 X$, the f-projection of ρ is f, a continuous and isotone map, ρ is continuous and isotone. Let $x, y \in X$ and $x \not\leq y$. Since $X \in \underline{\text{CrOR}}$, by [17] 1) page 52, there exist two continuous functions f, g where f is isotone, g decreasing, $\text{Im } f \subset I$, $\text{Im } g \subset I$, $f(x) = 1$, $g(x) = 1$ and $\inf\{f(y), g(y)\} = o$. If $f(y) = o$, then $\rho(x)(f) = f(x) = 1 \neq o = f(y) = \rho(y)(f)$; if $g(y) = o$ then $\rho(x)(1-g) = (1-g)(x) = o \neq 1 = (1-g)(y) = \rho(y)(1-g)$. Therefore $\rho(x) \neq \rho(y)$ and ρ in injective.

5.4 <u>REMARK</u>: If $X \in \underline{\text{CrOR}}$, and $\rho : X \to \mathbb{R}^{C_1 X}$ is the evaluation, then ρ is an order-embedding.

PROOF: We know from 5.3 that ρ is a monomorphism. Let $\rho(x) \leq \rho(y)$ and suppose that $x \nleq y$. By [17] 2) page 53, an $f \in C_1 X$ exists such that $f(x) > f(y)$ i.e. : $\rho(x)(f) > \rho(y)(f)$ which is a contradiction. Therefore $x \leq y$.

5.5 PROPOSITION: Let X be an space in PTop. X is $\mathbb{I}$-regular if and only if the evaluation map $j: X \to \mathbb{I}^{I_1 X}$ given by $j(x)(f) := f(x)$ is an embedding.

PROOF: Suppose X is $\mathbb{I}$-regular. By 5.2 and 5.3, j is an injective monomorphism. We show now that j is an order embedding. By the definition of $\mathbb{I}$-regular, there exists a set S and a PTop-embedding $e: X \to \mathbb{I}^S$. If $j(x) \leq j(y)$, then for every $f \in I_1 X$, $f(x) = j(x)(f) \leq j(y)(f) = f(y)$. Hence, for every $s \in S$, $e(x)(s) = (p_s \circ e)(x) \leq (p_s \circ e)(y) = e(y)(s)$ since $p_s \circ e \in I_1 X$. Therefore $e(x) \leq e(y)$, and e being an embedding, $x \leq y$. To complete the proof that j is an embedding, we show that it is open in its image. Since $e: X \to \mathbb{I}^S$ is an embedding, the topology on X is generated by

$$\mathcal{B} := \{e^{-1} p_s^{-1} U \mid s \in S,\ U \text{ open in } \mathbb{I}\}$$

Call $f_s := p_s \circ e$. Then $p_{f_s} \circ j(x) = j(x)(f_s) = f_s(x)$. Given a set $(p_s \circ e)^{-1} U$ of $\mathcal{B}$, we have : $j[(p_s \circ e)^{-1} U] = j[f_s^{-1} U] = j[\{x \in X \mid f_s(x) \in U\}] = \{j(x) \in jX \mid p_{f_s}(j(x)) \in U\} = p_{f_s}^{-1} U \cap jX$ which is open in jX. The converse is obvious from the definition of $\mathbb{I}$-regular .

5.6 <u>COROLLARY</u>: Let X be an space in <u>PTop</u> . X is $\mathbb{R}$-regular if and only if the evaluation map $\rho: X \to \mathbb{R}^{C_1 X}$ given by $\rho(x)(f) := f(x)$ is an embedding .

<u>PROOF</u>: The proof is like that given in 5.5 .

5.7 <u>LEMMA</u>: Every completely regular topological space (with discrete order) is $\mathbb{I}$-regular .

<u>PROOF</u>: Let X be a completely regular space. As is well known the evaluation $j: X \to \mathbb{I}^{C(X,\mathbb{I})}$ is a <u>Top</u>-embedding, and since the order in X is assumed discrete, $I_1 X = C(X,\mathbb{I})$. Therefore in order to obtain $X \cong jX$ in <u>PTop</u> , all we need is that the order in jX be discrete . To this end, let $x,y \in X$ and $x \neq y$. Since X has discrete order $x \nleq y$ and $y \nleq x$. Since X is completely regular, there exists $f \in C(X,\mathbb{I})$ such that $f(x) = o$ and $f(y) = 1$.

We have $j(x)(f) = f(x) = o \nleq 1 = f(y) = j(y)(f)$

and $j(x)(1-f) = (1-f)(x) = 1 \nleq o = (1-f)(y) = j(y)(1-f)$.

Therefore $j(x) \nleq j(y)$ and $j(y) \nleq j(x)$.

5.8 <u>PROPOSITION</u>: <u>$\mathbb{I}$rOT</u> = <u>$\mathbb{R}$rOT</u> .

<u>PROOF</u>: Since for an arbitrary set S, $\mathbb{I}^S \subset \mathbb{R}^S$, we see that <u>$\mathbb{I}$rOT</u> $\subset$ <u>$\mathbb{R}$rOT</u> . To show the converse, we first prove

that $\mathbb{R}\in\underline{\text{IrOT}}$, and in particular $\mathbb{R} \cong \,]o,1[$.

Define : $f:\mathbb{R}\to]o,1[$ by $f(r):= \frac{1}{2} + \frac{r}{2(|r|+1)}$.

If $o < r < s$, $2r < 2s$, $2rs+2r < 2rs+2s$, $\frac{r}{2(r+1)} < \frac{s}{2(s+1)}$.

If $r < o < s$, $\frac{r}{2(|r|+1)} < o < \frac{s}{2(|s|+1)} = \frac{s}{2(s+1)}$.

If $r < s < o$, $-2rs+2r < -2rs+2s$, whence

$2(|s|+1)r < 2(|r|+1)s$, and $\frac{r}{2(|r|+1)} < \frac{s}{2(|s|+1)}$.

Therefore f is continuous, isotone and injective .

Let us now define: $\bar{f}:]o,1[\to\mathbb{R}$ by $\bar{f}(y)= \frac{2y-1}{2-2y}$ if $y \geqslant \frac{1}{2}$ and $\bar{f}(y) = \frac{2y-1}{2y}$ if $y < \frac{1}{2}$, $\bar{f}$ is clearly continuous and is the inverse of f . Moreover $\bar{f}$ is isotone since, if $f(x) < f(y)$, $y \nleq x$ and therefore $x < y$.

Having shown that $\mathbb{R} \cong \,]o,1[$, let $X\in\underline{\text{RrOT}}$ and $X \cong Y\subset\mathbb{R}^S$. Then $X \cong Y\subset\mathbb{R}^S \cong \,]o,1[^S \subset \mathbb{I}^S$. Therefore $X\in\underline{\text{IrOT}}$.

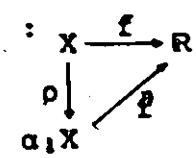
5.9 <u>REMARK</u>: Because of Proposition 5.8, in order to avoid emphasizing the properties of spaces in $\underline{\text{IrOT}} = \underline{\text{RrOT}}$ as I-regular (or R-regular), we choose a neutral name for this category : $\underline{\text{CrORR}}$.

$\underline{\text{CrORR}}$ is obviously productive and hereditary, and we shall give a further interesting characterization of it after we have shown in next section that the category of

K-compact spaces is precisely COTS .

5.10 PROPOSITION: CrORR is an epireflective subcategory of HOTS .

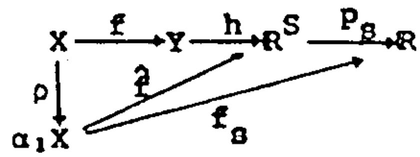
PROOF: Given $X \in$ HOTS , we can always define $\rho : X \to R^{C_1X}$, by $\rho(x)(f) := f(x)$. This is a continuous, isotone map. (Although it may not be injective) . We call $\alpha_1 X := \rho X$, and in order to show that $\alpha_1 :$ HOTS $\to$ CrORR is the desired epireflector , we first show that for every $f \in C_1 X$ a unique $\bar{f} : \alpha_1 X \to R$ exists such that $\bar{f} \circ \rho = f$; i.e., such that the following diagram commutes :



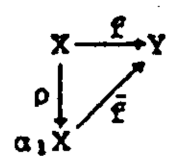
To obtain this, let $\bar{f} := p_f \circ i$ where i is the inclusion and p_f the f-projection : $\alpha_1 X \xrightarrow{i} R^{C_1X} \xrightarrow{p_f} R$. $\bar{f}$ is clearly continuous, isotone and $\bar{f} \circ \rho = f$. Moreover since ρ is surjective on $\alpha_1 X$, $\bar{f}$ is unique with respect to these properties .

Consider now the more general case where $f : X \to Y$ is an arbitrary HOTS-morphism such that $Y \in$ CrORR and let $h : Y \to R^S$ be an embedding . From the result just proved, we find for every $s \in S$ a unique continuous, isotone map f_s such that $f_s \circ \rho = p_s \circ h \circ f$. From the Universal Property of the product R^S again, there exists a unique continuous, isotone map $\hat{f}$ such that for every $s \in S$, $p_s \circ \hat{f} = f_s$.

We include the corresponding diagram for the benefit of the reader :



Having obtained for all $s \in S$ that $p_s \circ \hat{f} \circ \rho = p_s \circ h \circ f$, we see that, since the family $(p_s)_{s \in S}$ is universal, $\hat{f} \circ \rho = h \circ f$. Therefore, $\hat{f}\alpha_1 X = \hat{f} \circ \rho X = h \circ f X \subset \mathrm{Im}\, h$. Let $a \in \alpha_1 X$ and $\hat{f}(a) = h(y)$; since h is an embedding, y is unique and we can define a map $\bar{f}: \alpha_1 X \to Y$ by $\bar{f}(a) = y$. Consider $f^*: \alpha_1 X \to hY$ given by $f^*(a) = \hat{f}(a)$, and $h^1: hY \to Y$ by $h^1(h(y)) = y$. Both f^* and h^1 are continuous and isotone, and so then will be the map $\bar{f} = h^1 \circ f^*$. We observe in addition that, since $\hat{f}(\rho(x)) = h(f(x))$, we have $(\bar{f} \circ \rho)(x) = \bar{f}(\rho(x)) = f(x)$. The following diagram having been shown to commute for $\bar{f}$,



it follows that, since ρ is surjective, $\bar{f}$ is unique .

5.11 <u>REMARK</u>: We could have given a parallel proof using the evaluation $j: X \to I^{I_1 X}$ instead of ρ . Suppose we had defined $\alpha_2 X := jX$. Since $(\rho, \alpha_1 X)$ and $(j, \alpha_2 X)$ would then be solutions to the same Universal Problem represented by the Reflexion Property, we would have obtained $\alpha_1 X \cong \alpha_2 X$.

6. Compactification and Realcompactification

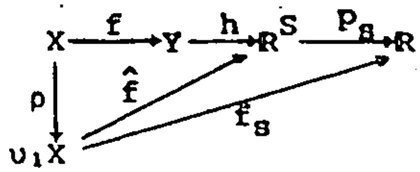
6.1 NOTATION: If $X \in$ CrORR, we denote by $\beta_1 X$ the closure in $\mathbb{I}^{I_1 X}$ of jX, and by $\upsilon_1 X$ the closure in $\mathbb{R}^{C_1 X}$ of ρX. $\mathbb{I}$cOT and $\mathbb{R}$cOT will mean respectively the categories of $\mathbb{I}$-compact and of $\mathbb{R}$-compact partially ordered topological spaces.

6.2 REMARK:

1) $X \cong jX$ and jX is a dense subspace of $\beta_1 X$.
2) $X \cong \rho X$ and ρX is a dense subspace of $\upsilon_1 X$.

6.3 PROPOSITION: The categories of $\mathbb{I}$-compact and of $\mathbb{R}$-compact spaces are epireflective in CrORR. The epireflectors are β_1 and υ_1 respectively.

PROOF: We consider first υ_1, since β_1 could only be easier. By practically the same steps as in 5.10 we obtain that for every $f \in C_1 X$, there exists a unique $\bar{f}: \upsilon_1 X \to \mathbb{R}$ such that $\bar{f} \circ \rho = f$. In this case, the uniqueness is assured because ρ is now dense in $\upsilon_1 X$. We generalize here to the case where $f: X \to Y$ is a CrORR-morphism and $Y \in \mathbb{R}$cOT. As in 5.10, we obtain an embedding $h: Y \to \mathbb{R}^S$ (with hY closed), an f_s in $\mathbb{R}$cOT such that $f_s \circ \rho = p_s \circ h \circ f$, for every $s \in S$, and, from the family $(f_s)_{s \in S}$, the map $\hat{f}$ which makes the following diagram commutative for all $s \in S$:



For this case :

$$\hat{f}\upsilon_1 X = \hat{f}\Gamma\rho X \subset \Gamma\hat{f}\rho X \subset \Gamma \mathrm{Im}\ h = \mathrm{Im}\ h \quad ,$$

and we can therefore complete the proof by defining $\bar{f}$, f^* and h^1 as in 5.10 .

If we consider β_1 we pass through exactly the same steps using j instead of ρ , and I_1X instead of C_1X .

6.4 LEMMA: $\mathbb{I}$COT = COTS $\subset$ $\mathbb{R}$COT.

PROOF: It is obvious that $\mathbb{I}$COT $\subset$ COTS. Let $X \in$ COTS As in previous cases the evaluation map $j: X \to \mathbb{I}^{I_1X}$ is continuous and isotone . We show that j is injective . By [17], Theorem 4,page 48, we know that X is a normally ordered space . Let x,y be two distinct points of X, and, without loss of generality, let $x \nleq y$. Since the partial order on X is continuous, the sets $Ly = \{z \in X | z \leq y\}$ and $Mx = \{z \in X | x \leq z\}$ are disjoint and closed, and we conclude from [17] that a continuous isotone function $f \in C_1X$ exists such that $\mathrm{Im}\ f \subset \mathbb{I}$, $f(y) = o$ and $f(x) = 1$. We obtain essentially the same function on I_1X and $j(x)(f) = f(x) = 1 \neq o = f(y) = j(y)(f)$. Therefore $j(x) \neq j(y)$. Since X is compact, $\mathbb{I}^{I_1X}$ Hausdorff and j continuous, we conclude that j is a closed map, and all that is left to finish our proof is to show that if $j(x) \leq j(y)$, then $x \leq y$. Suppose it were not so; say $x \nleq y$. As in our considerations to establish that j is injective, there exists an $f \in I_1 X$ such that $j(y)(f) = f(y) = o < 1 = f(x) = j(x)(f)$, a contradiction.

We have shown that j is an embedding and jX closed. Therefore $X \in \mathbb{I}$cOT and $\mathbb{I}$cOT = COTS.

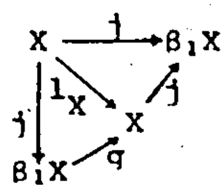
By the same method we show that the evaluation $\rho : X \to \mathbb{R}^{C_1 X}$ is an embedding and ρX closed in $\mathbb{R}^{C_1 X}$. Accordingly, COTS $\subset \mathbb{R}$cOT.

6.5 PROPOSITION: Let Y be a space in HOTS. Then Y is in CrORR if and only if Y is a subspace of some space X in COTS.

PROOF If $Y \in$ CrORR, $Y \subset \mathbb{I}^S$ for some set S. Conversely, let $Y \subset X$ and $X \in$ COTS. By 6.4, X is in $\mathbb{I}$cOT and therefore in $\mathbb{I}$rOT. Since $\mathbb{I}$rOT is hereditary $Y \in \mathbb{I}$rOT = CrORR.

6.6 LEMMA: An space X in CrORR is in COTS if and only if $\beta_1 X \cong X$, and it belongs to $\mathbb{R}$cOT if and only if $\upsilon_1 X \cong X$.

PROOF: Let $X \in$ COTS. By the Universal Property shown for β_1 in 6.3, there exists a unique map $g : \beta_1 X \to X$ such that $g \circ j = 1_X$, and since $j \circ g$ makes the following diagram commutative :



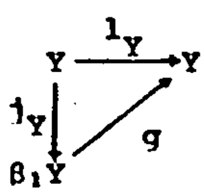
it follows, again by the uniqueness of the Universal Property, that $j \circ q = 1_{\beta_1 X}$. Therefore $X \cong \beta_1 X$. Since $\beta_1 X$ is clearly in COTS, the converse is obvious . The statement about υ_1 is proved in exactly the same way .

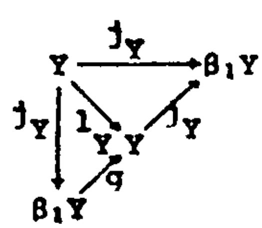
6.7 REMARK: Let Cl be the class of all spaces $Y \varepsilon$ CrORR which satisfy the following Universal Property : "For every $X \varepsilon$ CrORR and continuous, isotone map $f: X \to Y$, there exists a unique continuous, isotone map $\bar{f}: \beta_1 X \to Y$ such that $\bar{f} \circ j = f$."

In the proof of 6.3, we have shown that COTS $\subset$ Cl .

The converse is easy to prove, and a parallel remark can be made for υ_1 and RcOT .

PROOF: Let $Y \varepsilon$ Cl , then a unique continuous, isotone map $q: \beta_1 Y \to Y$ exists such that the following diagrams commute :





We obtain as in 6.6 that $q \circ j_Y = 1_Y$, $j_Y \circ q = 1_{\beta_1 Y}$, $Y \cong \beta_1 Y$ and therefore $Y \varepsilon$ COTS . The remark for υ_1 and RcOT is now obvious .

6.8 REMARK: Since COTS is epireflective in CrORR, and CrORR epireflective in HOTS, we obtain that COTS is reflective in HOTS and by the corresponding considerations RcOT is reflective in HOTS. We had already obtained those results in 2.11.

6.9 LEMMA: All real-compact spaces (with discrete partial order) belong to RcOT.

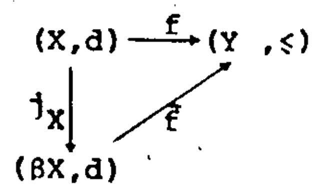
PROOF: Let $X \in \underline{Rc}$. Then CX coincides with C_1X, $\rho : X \to R^{CX}$ is a PTop-embedding, $X \cong \rho X = \beta X$, and one shows as in 5.7 that the partial order in ρX is discrete. Since $X \cong \beta X$ is in HOTS and $\beta X = \beta_1 X$, we obtain $X \in \underline{RcOT}$.

7. Connections between β and β_1, υ and υ_1.

Let F be the forgetful functor $\underline{HOTS} \to \underline{H}$ such that $F(X, \leq) = X$. If $(X, \leq) \in \underline{COTS}$, $X \in \underline{C}$ and therefore $\beta X \cong X$. On the other hand $\beta_1(X, \leq) \cong (X, \leq)$ and this shows that $F\beta_1(X, \leq) \cong \beta X$. Similarly, we obtain that if $(X, \leq) \in \underline{RcOT}$, $F\upsilon_1(X, \leq) \cong \upsilon X$. In this section, we show that $\beta_1(X,d) \cong (\beta X,d)$ and exhibit a space $(X, \leq)$ in CrORR such that $F\beta_1(X, \leq) \not\cong \beta X$.

7.1 PROPOSITION: For every $(X, \leq)$ in CrORR, $\beta_1(X,d) \cong (\beta X,d)$ and there exists a perfect, isotone surjective map $p:(\beta X,d)\rightarrow\beta_1(X\ ,\leq)$.

PROOF: Let $(X\ ,\leq)\in$ CrORR . By 5.2 and 5.7, we have $(X,d)\in$ CrORR . Since X is completely regular, for every continuous, isotone $f:(X,d)\rightarrow(Y\ ,\leq)$ with $(Y\ ,\leq)\in$ COTS , there exists a unique continuous, isotone map $\bar{f}$ such that following diagram commutes :



This shows that $(\beta X,d)$ and $\beta_1(X,d)$ are both solutions to the same Universal Problem, and therefore

$$(\beta X,d)\cong\beta_1(X,d)\quad .$$

Let $h:(X,d)\rightarrow(X\ ,\leq)$ be given by $h(x)=x$ for all $x\in X$. We show that $\beta_1 h$ is a perfect, isotone and surjective map . Since β_1:CrORR$\rightarrow$COTS is a functor, $\beta_1 h$ is a COTS-morphism , and therefore perfect and isotone . Since $j_{X\leq}(X\ ,\leq)=j_{X\leq}\circ h(X,d)=(\beta_1 h)\circ j_{Xd}(X,d)\subset \mathrm{Im}\ \beta_1 h$, we obtain $\beta_1(X\ ,\leq)=\Gamma j_{X\leq}(X\ ,\leq)\subset\Gamma\mathrm{Im}\ \beta_1 h=\mathrm{Im}\ \beta_1 h$.

We set p for the composition of $\beta_1 h$ and the isomorphism $\beta_1(X,d)\cong(\beta X,d)$.

7.2 COROLLARY: For every $(X\ ,\leq)$ in CrORR , $\upsilon_1(X,d)\cong(\upsilon X,d)$ and there exists a continuous, isotone dense map $q:(\upsilon X,d)\rightarrow\upsilon_1(X,\leq)$.

PROOF: As in 7.1 we obtain $\upsilon_1(X,d) \cong (\upsilon X,d)$, and $\rho_{X\leq}(X,\leq) = \rho_{X\leq}\circ h(X,d) = (\upsilon_1 h)\circ\rho_{Xd}(X,d) \subset \mathrm{Im}\ \upsilon_1 h \subset \upsilon_1(X,\leq)$.

7.3 REMARK: Let $U:\underline{H}\to\underline{HOTS}$ be given by $UX := (X,d)$. From 7.1 and 7.2 we obtain a commutativity condition of functors :

$$\begin{array}{ccc} \underline{Cr} & \xrightarrow{U} & \underline{CrORR} \\ \beta\downarrow & & \downarrow\beta_1 \\ \underline{C} & \xrightarrow[U]{} & \underline{COTS} \end{array} \quad\text{and}\quad \begin{array}{ccc} \underline{Cr} & \xrightarrow{U} & \underline{CrORR} \\ \upsilon\downarrow & & \downarrow\upsilon_1 \\ \underline{Rc} & \xrightarrow[U]{} & \underline{RcOT} \end{array}$$

i.e.:

$$\beta_1 U = U\beta \qquad \text{and} \qquad \upsilon_1 U = U\upsilon$$

The space $\beta_1(X,d)$ is the Stone-Cech compactification of X with the discrete partial order, and similarly for $\upsilon_1(X,d)$.

7.4 REMARK: In order to show that the underlying topological space of $\beta_1(X,\leq)$ is not necessarily βX , we first introduce the concept of κ-separable topological spaces for an infinite cardinal number κ . We also introduce distinguished intervals on the set $\mathcal{P}(X)\setminus\{X,\phi\}$ with inclusion as the partial order . We denote $2^{\aleph_0}$ by c .

The proof of 7.6 below is a generalization of the proof given in [6] VIII 7.2 .

7.5 DEFINITION: Let κ be an infinite cardinal number. We call a topological space κ-separable if it contains a dense subset of cardinality at most κ .

Let $P := \mathcal{P}(X)\setminus\{X,\phi\}$ and $J\subset P$. We call J a <u>distinguished interval</u> of P if there exist two finite subsets S_1, S_2 of X, such that $J = \{T\subset X | S_1\subset T\subset X\setminus S_2\}$. We denote J by $[S_1;-S_2]$.

7.6 <u>PROPOSITION</u>: Let A be a set of cardinality at most 2^κ, and $(Y_a)_{a\in A}$ a family of κ-separable spaces. Then $\Pi_{a\in A}Y_a$ is κ-separable.

<u>PROOF</u>: Without loss of generality, let $\bar{\bar{A}} = 2^\kappa$. Let S be a set of cardinality κ and $s_0\in S$

Let $P := \mathcal{P}(S)\setminus\{S,\phi\}$ and $\phi: A\to P$ be a bijective function. For every $a\in A$ let $y_a: S\to Y_a$ be a function such that $D_a := y_a S$ is dense in Y_a.

For every finite pairwise disjoint family of distinguished intervals $J_1,\dots, J_k$ of P and $s_1,\dots,s_k\in S$, we define a point $p(J_1,\dots,J_k;s_1,\dots,s_k)\in\Pi_{a\in A}Y_a$ by

$$p(J_1,\dots,J_k;s_1,\dots,s_k)(a) = \begin{cases} y_a(s_i) & \text{if } \phi(a)\in J_i \\ y_a(s_0) & \text{otherwise}\end{cases}$$

Let $D := \{p(J_1,\dots,J_k;s_1,\dots,s_k) | \text{all}(J_1,\dots,J_k;s_1,\dots,s_k),\text{all } k\}$

Clearly $\bar{\bar{D}} = \bar{\bar{S}} = \kappa$.

To prove that D is a dense subset of $\Pi_{a\in A}Y_a$, we shall show first that given $U_{a_1},\dots,U_{a_n}$ open sets of $Y_{a_1},\dots,Y_{a_n}$ respectively, there exists a family $J_1,\dots,$ J_n of pairwise disjoint distinguished intervals of P such that $\psi(a_i)\varepsilon J_i$.

If $n = 1$ and $\psi(a) = S_3$ where $b\varepsilon S_3$ and $d\notin S_3$ we set $J_1 := [\{b\};-\{d\}]$, and $\{J_1\}$ is the required family . Given $U_{a_1},\dots,U_{a_k},U_{a_{k+1}}$ open sets in $Y_{a_1},\dots,Y_{a_k},Y_{a_{k+1}}$ respectively, suppose that the family $[S_1;-T_1]$, $[S_2;-T_2]$, ..., $[S_k,-T_k]$ of pairwise disjoint distinguished intervals of P has been given such that $\psi(a_i)\varepsilon[S_i,-T_i]$.

Let $h\varepsilon\psi(a_{k+1})$ and $q\notin\psi(a_{k+1})$ and denote $[\{h\};-\{q\}]$ by $I_{k+1,o}$. If $h\varepsilon T_1$ or $q\varepsilon S_1$, we set $J_1 := [S_1;-T_1]$. If $h\notin T_1$ and $q\notin S_{1_0}$, since $\psi(a_1) \neq \psi(a_{k+1})$ we have two posibilities : There exists $x\varepsilon\psi(a_1)$ such that $x\notin\psi(a_{k+1})$, or there exists $y\varepsilon\psi(a_{k+1})$ such that $y\notin\psi(a_1)$. In the first case we set $J_1 := [S_1\cup\{x\};-T_1]$ and $I_{k+1,1} := [\{h\};-\{q,x\}]$. In the second case $J_1 := [S_1;-T_1\cup\{y\}]$ and $I_{k+1,1} := [\{h,y\};-\{q\}]$. By using the same method sucessively with $\psi(a_2) \neq \psi(a_{k+1}),\dots,\psi(a_k) \neq \psi(a_{k+1})$ we define $J_2,\dots,J_k$ and $I_{k+1,2},\dots,I_{k+1,k}$. By setting $J_{k+1} := I_{k+1,k}$ the required family is obtained .

Given arbitrary open sets $U_{a_1},\dots,U_{a_n}$ of $Y_{a_1},\dots,Y_{a_n}$ and a corresponding family $J_1,\dots,J_n$ of distinguished intervals of P as above, since for each i, $Y_{a_i}S$ is dense in Y_{a_i} , there exists $s_i\varepsilon S$ such that $y_{a_i}(s_i)\varepsilon U_{a_i}$.

This shows that $p(J_1,\dots,J_n;s_1,\dots,s_n)\varepsilon \bigcap p_{a_i}^{-1}U_{a_i}$

and D is dense in $\Pi_{a\in A}Y_a$.

7.7 REMARK: We denote by A^B as costumary, the set of functions $f:B\to A$, and by A^{B^C} , $A^{(B^C)}$. We denote by $\bar{\bar{A}}$ the cardinality of the set A . If a,b and c are cardinal numbers, we shall mean by a^{b^c} , $a^{(b^c)}$.

7.8 COROLLARY: If S is an infinite set , $\mathbb{I}^{\mathbb{I}^S}$ is $\bar{\bar{S}}$-separable .

PROOF: If we denote $A:=\mathbb{I}^S$ and $Y_a=\mathbb{I}$ for every $a\in A$, $\mathbb{I}^{\mathbb{I}^S}$, is $\Pi_{a\in A}Y_a$. We apply 7.6 because $\bar{\bar{A}}=\overline{\overline{\mathbb{I}^S}}=$ $=c^{\bar{\bar{S}}}=(2^{\aleph_0})^{\bar{\bar{S}}}=2^{\aleph_0\cdot\bar{\bar{S}}}=2^{\bar{\bar{S}}}$.

7.9 COROLLARY: If S is an infinite set, and we denote the discrete topological space on S also by S, then $\overline{\overline{\beta S}}=2^{2^{\bar{\bar{S}}}}$.

PROOF: Since βS is a closed subset of $\mathbb{I}^{\mathbb{I}^S}$, $\overline{\overline{\beta S}}\leqslant\overline{\overline{\mathbb{I}^{\mathbb{I}^S}}}=2^{2^{\bar{\bar{S}}}}$. By 7.8 $\mathbb{I}^{\mathbb{I}^S}$ is $\bar{\bar{S}}$-separable . Let $D\subset\mathbb{I}^{\mathbb{I}^S}$ be a dense subset and $b:S\to D$ a bijection . If $i:D\to\mathbb{I}^{\mathbb{I}^S}$ is the inclusion, the map $\hat{b}:\beta S\to\mathbb{I}^{\mathbb{I}^S}$ which extends $i\circ b$ is surjective and therefore $2^{2^{\bar{\bar{S}}}}=\overline{\overline{\mathbb{I}^{\mathbb{I}^S}}}\leqslant\overline{\overline{\beta S}}$.

7.10 PROPOSITION: For $(X,\leqslant)\in$CrORR , the underlying topological space of $\beta_1(X,\leqslant)$ is not necessarily βX .

PROOF: Let S be a set of ordinals of cardinality at least c. Let X be the topological space with S as its underlying set and the discrete topology. Let $\leq$ be the natural order on S.

By 7.9 we know that $\overline{\overline{\beta X}} = 2^{2^{\overline{\overline{S}}}}$.

We show that $(X, \leq) \in \underline{\text{CrORR}}$ and $\overline{\overline{\beta_1(X, \leq)}} \leq 2^{\overline{\overline{S}}}$.

The evaluation $(X, \leq) \to \mathbb{I}^{I_1(X, \leq)}$ is clearly continuous and isotone. Let $j(s) \leq j(t)$. If $s \nleq t$, $t \leq s$ and we can define $g: S \to \mathbb{I}$ by $g(x) = 0$ if $x \leq t$, $g(x) = 1$ if $x > t$. Then $g(t) = 0 < 1 = g(s)$ is a contradiction. To show that j is open in jX, given $s \in S$, we define $h: S \to \mathbb{I}$ by $h(x) = 0$ if $x < s$, $h(s) = \frac{1}{2}$ and $h(x) = 1$ if $x > s$. Since $h \in I_1(X, \leq)$ and $p_h^{-1}]0,1[\cap jX = \{j(s)\}$, $\{j(s)\}$ is open in jX. Therefore j is an embedding and $(X, \leq) \in \underline{\text{CrORR}}$.

To see that $\overline{\overline{\beta_1(X, \leq)}} \leq 2^{\overline{\overline{S}}}$, we notice that since S is well ordered, every isotone function $f: X \to \mathbb{I}$ can be described by the subset of graph $f \subset S \times \mathbb{I}$ where f is stricly increasing. Therefore the cardinality of $I_1(X, \leq)$ is at most that of the set of countable subsets of $S \times \mathbb{I}$, which is $\overline{\overline{(S \times \mathbb{I})}}^{\aleph_0} = \overline{\overline{S}}^{\aleph_0} = \overline{\overline{S}}$. Since $\beta_1(X, \leq)$ can be embedded into $\mathbb{I}^{I_1(X, \leq)}$, and $\overline{\overline{\mathbb{I}^{I_1(X, \leq)}}} \leq c^{\overline{\overline{S}}} = 2^{\overline{\overline{S}}}$, we obtain $\overline{\overline{\beta_1(X, \leq)}} \leq 2^{\overline{\overline{S}}}$. Having obtained $\overline{\overline{\beta_1(X, \leq)}} \leq 2^{\overline{\overline{S}}} < 2^{(2^{\overline{\overline{S}}})} = \overline{\overline{\beta X}}$ it is clear that $\beta_1(X, \leq) \ncong \beta X$.

CHAPTER III

PROJECTIVITY AND INJECTIVITY

In this Chapter, we use a result of § 3, see 3.2 , to connect the projectivity in subcategories K of H with the projectivity in KOTS and KPTop . If we have a class P of epimorphisms in K which contains all isomorphisms, and we denote the class of continuous isotone maps m of KPTop such that $F(m) \varepsilon P$ by Pi, we can show that an object Y is Pi-projective in KPTop if and only if Y = UX for a P-projective X in K . The corresponding statement also holds for KOTS . If P is the class of all the epimorphisms in H the projectivity is too trivial, the projectives being exactly the discrete spaces. Banaschewski [2], has studied with very interesting results the P-projectivity in various subcategories of Top when P means the class of perfect surjective maps. In this Chapter, we extend some of his results to our partially ordered spaces, by showing that for the same list of subcategories K mentioned in [2], the Pi-projectivity is properly behaved in both KPTop and KOTS and by characterizing the free and the Pi-projective objects and the Pi-projective covers. The author found these results particularly interesting by their very nature (the same objects which are P-projective in K happen to be Pi-projective in the much larger category KPTop 1) and by the rather simple way in which they could be derived by application of the ideas contained in [2] .

As for injectivity, if E contains all the monomorphisms, the E-injectivity is trivial in most subcategories of PTop, leaving as E-injective only the one-point spaces. We therefore select a more appropriate class of maps to replace the monomorphisms, namely the embeddings, and show that the E-injective objects of CrORR are connected spaces with greatest and lowest elements . A space is E-injective in COTS if and only if it is a retract of a power of $\mathbb{I}$, where $\mathbb{I} = [o,1] \subset \mathbb{R}$. In the category of $\mathbb{2}$-compact spaces (where $\mathbb{2}$ denotes the set $\{o,1\}$ endowed with the discrete topology and partial order $o \leq 1$), a space is E-injective if and only if it is a retract of a power of $\mathbb{2}$. The finite E-injectives are lattices . We examine E-injectivity for proper behavior and for $\mathbb{2}$-compact spaces, find it is properly behaved at X whenever X is a finite space .

Chapter 0 contains some of the definitions and results of Banaschewski [2] , for convenience of reference .

8. Pi-projectivity

In this section we relate projectivity in KPTop and in KOTS with the already known projectivity in K , where K is a suitable subcategory of H . After remarking that the projectivity in the general sense is not very interesting in these categories, we label the class of perfect, surjective, isotone maps as Pi and study Pi-projectivity, extending results for K from [2] to KPTop and KOTS in this way .

8.1 PROPOSITION: Let K be a subcategory of H, P a class of epimorphisms in K which contains all the isomorphisms, Pi the class of continuous isotone maps m in KPTop such that $P(m) \in P$. Then $(X, \leq) \in$ KPTop is Pi-projective if and only if X is P-projective in K and $\leq$ is the discrete partial order .

PROOF: Let $(X, \leq)$ be Pi-projective in KPTop . Suppose $\leq$ is non discrete , i.e. there exist $a, b \in X$, $a \neq b$ and $a \leq b$. Consider (X,d) where d is the discrete order . Then $f:(X,d) \to (X, \leq)$ given by $f(x) := x$ is in Pi but there is no continuous isotone map $(X, \leq) \to (X,d)$ which completes the following diagram to a commutative one :

$$\begin{array}{ccc} & & (X, \leq) \\ & & \downarrow 1_{(X, \leq)} \\ (X,d) & \xrightarrow[f]{} & (X, \leq) \end{array}$$

This contradiction shows that $\leq$ has to be the discrete partial order . To see that X is P-projective, we consider the diagrams (I) in K and (II) in KPTop .

$$\text{(I)}\quad \begin{array}{ccc} & & X \\ & & \big\downarrow g \\ A & \xrightarrow[f\in \underline{P}]{} & B \end{array} \qquad\qquad \text{(II)}\quad \begin{array}{ccc} & & (X,d) \\ & & \big\downarrow g \\ (A,d) & \xrightarrow[f\in \underline{Pi}]{} & (B,d) \end{array}$$

Since (X,d) is $\underline{Pi}$-projective in $\underline{KPTop}$, there exists $\bar{g}:(X,d)\rightarrow(A,d)$ in $\underline{KPTop}$ such that $f\circ\bar{g} = g$. This $\bar{g}$ is, in particular, a continuous map $X\rightarrow A$. To prove the converse, suppose that X is $\underline{P}$-projective and consider f and $p\in\underline{Pi}$ in the following diagram :

$$\begin{array}{ccc} & & (X,d) \\ & & \big\downarrow f \\ (Y,\leq) & \xrightarrow[p]{} & (Z,\leq) \end{array}$$

Since X is $\underline{P}$-projective, f,p are continuous and $p = Fp\in\underline{P}$, there exists a continuous map $\bar{f}:X\rightarrow Y$ such that $p\circ\bar{f} = f$. But since (X,d) has the discrete order $\bar{f}$ is also isotone.

8.2 REMARK: An examination of the proof of 8.1 yields that, for $\underline{P}$ as in 8.1, $(X,\leq)\in\underline{KOTS}$ is $\underline{Pi}$-projective if and only if X is $\underline{P}$-projective in $\underline{K}$ and $\leq$ is the discrete partial order.

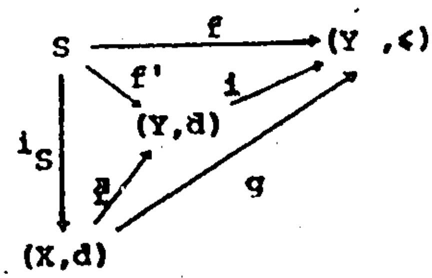
8.3 LEMMA: If $\underline{KPTop}$ ($\underline{KOTS}$) has an object $(X,\leq)$ which is free over a set S, then $\leq$ is the discrete order.

PROOF: Let $\underline{P}$ be the class of all epimorphisms and apply 8.1. Every free object is then $\underline{P}$-projective.

8.4 PROPOSITION: Let S be a set and $\underline{K}$ a subcategory of $\underline{H}$. $(X, \leq)$ is free on S over KPTop (KOTS) if and only if X is free on S over $\underline{K}$ and $\leq$ is discrete.

PROOF: If $(X, \leq)$ is free on S, by 8.3, $\leq$ is discrete. Let i_S be the universal map $S \to (X,d)$. We also denote by i_S the map $S \to X$ having the same graph. Let $f: S \to Y$ be an arbitrary map where $Y \in \underline{K}$. Also denote by f the map $S \to (Y,d)$ having the same graph. Since (X,d) is free on S over KPTop (KOTS), there exists a unique morphism $\bar{f}$ such that $\bar{f} \circ i_S = f$. Since $\bar{f}$ is defined as a continuous $X \to Y$ map, (X, i_S) is free on S over $\underline{K}$.

Conversely, let (X, i_S) be free on S over $\underline{K}$; given $f: S \to (Y, \leq)$, we define $f': S \to (Y,d)$ and $i: (Y,d) \to (Y, \leq)$ by $f'(s) := f(s)$ and $i(y) := y$. Since X is free on S, there exists a unique continuous map $\bar{f}: X \to Y$ such that $\bar{f} \circ i_S = f'$. Setting $g := i \circ \bar{f}$ we find the unique continuous, isotone map which makes the following diagram commutative:



Therefore (X,d) is free on S over KPTop (KOTS).

8.5 COROLLARY: The free object on S over CPTop (COTS) is obtained by considering S as a discrete space and taking (βS,d) .

8.6 <u>REMARK</u>: The first choice for the class $\underline{P}$ in the study of projectivity in any category, is to take all the epimorphisms . However this projectivity in $\underline{H}$ and some of its subcategories is trivial . A space is projective if and only if it is discrete .

A suitable choice of $\underline{P}$ has been the class of all the perfect surjective maps, which, in the important case of compact spaces, coincides with the class of all epimorphisms. Since we have some interesting results on this kind of $\underline{P}$-projectivity (see 0.29 to 0.36), from now on we shall reserve the symbol $\underline{P}$ for the class of "perfect surjective maps" .

8.7 <u>COROLLARY</u>: For the following subcategories $\underline{K}$ of $\underline{H}$, (X , $\leq$) is $\underline{Pi}$-projective in <u>KPTop</u> (in <u>KOTS</u>) if and only if X is extremely disconnected and the partial order $\leq$ is discrete :

1) Hausdorff
2) regular
3) completely regular
4) paracompact
5) locally compact
6) σ-compact
7) Lindelöf
8) real-compact
9) k-compact
10) compact
11) zero-dimensional

PROOF: Since the class P of perfect surjective maps satisfies the hypotheses of 8.1, we simply apply 0.34 .

8.8 REMARK: We obtain directly from 3.8 that, if Y is projective in KOTS, then GY is projective in K . However if Y is projective in KOTS , its order is discrete and $GY \cong Y$.

8.9 REMARK: As in H we call a surjective morphism $f:X \to Y$ of HPTop minimal , if the image of every proper closed subset of X is a proper subset of Y .

8.10 LEMMA: A morphism f of Pi is coessential if and only if it is minimal .

PROOF: Let $f:(A, \leq) \to (B, \leq)$ be coessential . We shall show that $f:A \to B$ is coessential in H as well, because we can then apply [2] page 69 and conclude that f is minimal . Let $g:X \to A$ be continuous and $f \circ g \in P$. Then $g:(X,d) \to (A, \leq)$ is continuous, isotone and $f \circ g \in Pi$. Since $f \in Pi_*$, then $g \in Pi$, whence $g:X \to A$ belongs to P , and therefore $f \in P_*$. Conversely, let $f:(A, \leq) \to (B, \leq)$ be minimal , $h:(Z, \leq) \to (A, \leq)$ continuous and isotone and $f \circ h \in Pi$. By [2] page 69, since f is minimal, f is coessential in H , and since $f \circ h \in P$, $h \in P$. Considering that h is isotone by assumption , $h \in Pi$ and we have proved that $f \in Pi_*$.

8.11 COROLLARY: If $f:(A, \leq) \to (B, \leq)$ belongs to Pi*

$\bar{A} < 2^{2^{\bar{B}}}$.

PROOF: By 8.10 f is minimal, and the result is obtained from [2] Lemma 3, page 70 .

8.12 PROPOSITION: Pi-projectivity in HPTOP (HPOTS, HOTS) is properly behaved .

PROOF: We show that the axioms P1) to P6) of [2] , see 0.31, are satisfied . P1) and P2) are obviously satisfied by the perfect, isotone surjective maps. Considering 8.10, P3) follows by the same arguments which are used in the case of P-projectivity in H (see [2] page 69 and 70) . Similarly using 2.4, we can consider essentially the same explicit descriptions of the pullbacks and projective limits in HPTop (HPOTS , HOTS) as the ones given for H in [2] page 70, thus obtaining P4) and P5) . Finally, P6) is obvious from 8.11 .

8.13 REMARK: The statements 0.32 to 0.34 have been proved in [2] by analysis of the proof of [2] Lemma 2 page 69, which corresponds in our case to 8.12 . Since our proof of 8.12 follows all the details of Lemma 2, as was just mentioned, the corresponding statements for HPTop (HPOTS, HOTS) follow .

8.14 COROLLARY: If S is a subcategory of HPTop (HPOTS, HOTS), Pi-projectivity is properly behaved in S whenever:

i) S is closed hereditary, closed with respect to pullbacks in HPTop (HPOTS, HOTS), and projective limits in HPTop (HPOTS, HOTS) of well ordered inverse systems with Pi-maps; or

ii) S is a full subcategory of HPTop (HPOTS, HOTS) which is left fitting with respect to coessential Pi-maps; or

iii) S consists of all objects and all perfect isotone mappings from a category L which satisfies one of these conditions.

8.15 REMARK: Every $(X, \leq)$ in HPTop (HPOTS, HOTS), is the homomorphic image of a minimal Pi-map from a discretely ordered, extremely disconnected space.

PROOF: Follows from 8.12, 8.1 and 0.34.

8.16 COROLLARY: In any full subcategory S of HPTop (HPOTS, HOTS) which is left-fitting with respect to coessential Pi-mappings, the Pi-projectives are exactly the discretely ordered, extremely disconnected spaces belonging to S and the same holds for the subcategory of S with the same class of objects, but whose morphisms are only the perfect, isotone mappings from S.

8.17 REMARK: The result of 8.16 follows also if S is productive and closed hereditary .

These statements follow directly from the proof of 8.12 and also from 8.1

8.18 PROPOSITION: The mapping $f:(Y,d) \to (X, \leq)$ is a Pi-projective cover if and only if $f:Y \to X$ is a P-projective cover .

PROOF: By 8.1 , (Y,d) is Pi-projective if and only if Y is P-projective; by 8.10, f is Pi-coessential if and only if f is minimal i.e. P-coessential .

8.19 REMARK: As we have explicit descriptions of the P-projective covers in H and some of its subcategories K , Proposition 8.18 provides an explicit description of Pi-projective covers in HPTop, HPOTS, HOTS and KPTop , KPOTS, KOTS , for those subcategories K .

9. E-injectivity

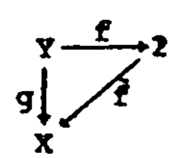
As in section 8, injectivity is not interesting, in the general sense and we select the class E of all embeddings . E-injectives in COTS are the retracts of powers of I, while in 2-compact spaces they are the retracts of powers of 2 . E-injectivity for 2-compact spaces is locally properly behaved at the finite spaces .

9.1 LEMMA: In the following subcategories of PTop, if E is the class of all monomorphisms, every E-injective space is trivial (has only one point) :

1) HPTop	6) ZerodimOTS
2) HPOTS	7) 2cOT
3) HOTS	8) ℝcOT
4) CrORR	9) BsOTS
5) COTS	

PROOF: Because the product of two non trivial chains is not a chain, it is enough to show that every E-injective is a chain .

Let S be one of the above mentioned subcategories of PTop, and let $X \in S$ be an E-injective space such that $a, b \in X$ and $a \nleq b \nleq a$. Let Y be the subspace $(\{a,b\},d)$ of X , $q:Y \to X$ the inclusion and $f:Y \to 2$ defined by $f(a) = o$ and $f(b) = 1$. Since $f \in E$ and X is E-injective , there exists a continuous, isotone map $\bar{f}$ such that the following diagram commutes :

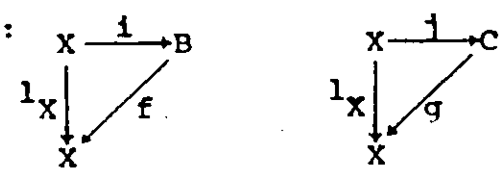


Then $a = q(a) = \bar{f} \circ f(a) = \bar{f}(o) \leq \bar{f}(1) = b$, a contradiction .

9.2 NOTATION: For the rest of this section, we shall denote the class of all embeddings by E .

9.3 <u>LEMMA</u>: If X is $\underline{E}$-injective in one of the categories mentioned in 9.1, X has a greatest and a lowest element.

<u>PROOF</u>: Define $B := X \amalg \{x\} = X \times \{1\} \cup \{(x,2)\}$ and $C := \{x\} \amalg X = \{(x,1)\} \cup X \times \{2\}$ with coproduct topology and lexicographic partial order. Consider the following two commutative diagrams :



We note that f and g exist because X is $\underline{E}$-injective. It is clear that for every $a \in X$, $a \leq_B x$ and accordingly $a \leq_X f(x)$. Similarly $g(x) \leq_X a$.

9.4 <u>LEMMA</u>: Every $\underline{E}$-injective space in <u>CrORR</u> is compact and connected.

<u>PROOF</u>: Let X be $\underline{E}$-injective in <u>CrORR</u>. By 5.5 the evaluation $j: X \to \mathbb{I}^{I_1 X}$ belongs to $\underline{E}$, and since X is $\underline{E}$-injective, there exists a $f: \mathbb{I}^{I_1 X} \to X$ continuous, isotone map such that $f \circ j = 1_X$. As f is surjective and continuous, and $\mathbb{I}^{I_1 X}$ is compact and connected, then X is also compact and connected.

9.5 <u>PROPOSITION</u>: The $\underline{E}$-injective spaces in <u>COTS</u> are exactly the retracts of powers of $\mathbb{I}$.

PROOF: To see that every retract of a power of $\mathbb{I}$ is $\underline{E}$-injective, it suffices to show that $\mathbb{I}$ is $\underline{E}$-injective. But this follows from 0.13 considering [17] Theorem 4, page 48 . Conversely, if X is $\underline{E}$-injective in COTS , then the evaluation $j:X\rightarrow\mathbb{I}^{I_1X}$ belongs to $\underline{E}$ by 6.4, and there exists a continuous isotone function $f:\mathbb{I}^{I_1X}\rightarrow X$ such that $f\circ j = 1_X$. This shows that X is a retract of $\mathbb{I}^{I_1X}$.

9.6 LEMMA: If U is a closed subset of a space X in COTS , LU and MU are also closed .

PROOF: Let y be a point which is not in LU . Then for every $x\in U$ we have $y \nleq x$. By [24], Lemma 1 page 145, for every $x\in U$, there exist two open neighbourhoods U_x of x and V_x of y such that $v \nleq u$ whenever $u\in U_x$ and $v\in V_x$. Therefore $V_x \cap LU_x = \phi$. Since X is compact, there exists a finite open cover of U given by $U_{x_1}, \dots, U_{x_n}$. Let $V:= \bigcap_{i=1}^{n} V_{x_i}$, $N:= \bigcup_{i=1}^{n} LU_{x_i}$. Since $LU \subset N$, and $V \cap N = \phi$, we have $V \cap LU = \phi$; it then follows that $V \subset \complement LU$, and LU is closed . One shows similarly that MU is closed .

9.7 LEMMA: Let S be a set and U a closed and open subset of 2^S . Then LU is closed and open .

PROOF: Since 2 is in COTS, so is 2^S, and, by 9.6 LU is closed. To show that LU is also open, let $x \in LU$ and $x \leq u$, $u \in U$. Since U is open, and 2^S has the product topology, there exist $s_1, \ldots s_n$, $t_1, \ldots, t_m$ in S such that

$$u \in \left(\bigcap_{i=1}^{n} p_{s_i}^{-1}(o) \right) \cap \left(\bigcap_{j=1}^{m} p_{t_j}^{-1}(1) \right) \subset U .$$

Therefore :

$$x \in \bigcap_{i=1}^{n} p_{s_i}^{-1}(o) = L\left[\left(\bigcap_{i=1}^{n} p_{s_i}^{-1}(o) \right) \cap \left(\bigcap_{j=1}^{m} p_{t_j}^{-1}(1) \right) \right] \subset LU .$$

9.8 LEMMA: Let S be a set, A, B two disjoint closed subsets of 2^S, A increasing and B decreasing. There exists an increasing open and closed neighbourhood U of A such that $U \cap B = \varnothing$.

PROOF: By [17] Theorem 4, page 46 there exists an open increasing neighbourhood U' of A such that $U' \cap B = \varnothing$. Since the topology on 2^S is zero-dimensional, there exists a family of closed and open sets $(C_i)_{i \in I}$ such that $U' = \bigcup_{i \in I} C_i$. A finite subfamily $C_{i_1}, \ldots C_{i_n}$ then covers the compact set A. We let

$U := \bigcup_{j=1}^{n} MC_{i_j}$. By 9.7 every MC_{i_j} is closed and open and so, therefore, is U. Moreover, since $U \subset MU' = U'$, $U \cap B = \varnothing$.

9.9 PROPOSITION: If X is a space in COTS, the following statements are equivalent :

1) X is 2-compact.

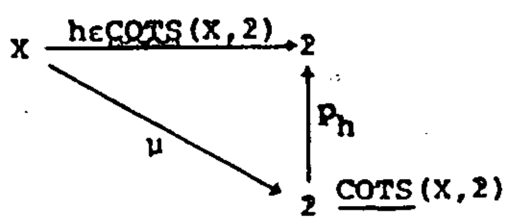
2) For any two disjoint closed subsets A,B of X, such that A is increasing and B decreasing, there exists an increasing closed and open neighbourhood U of A such that $U \cap B = \emptyset$.

3) For any two disjoint closed subsets A,B of X, such that A is increasing and B decreasing, there exists an increasing closed and open neighbourhood U of A and a decreasing closed and open neighbourhood V of B, such that $U \cap V = \emptyset$.

4) For any two points a,b of X, such that $a \not\leq b$, there exists a continuous isotone function $f:X \to 2$ such that $f(b) = o$ and $f(a) = 1$.

<u>PROOF</u> : To show that 1) implies 2), we assume, without loss of generality, that $X \subset 2^S$. Let A,B be closed disjoint subsets of X, A increasing and B decreasing. By 9.6, considering that X is compact, MA is closed increasing and LB closed decreasing in 2^S. Since $LB \cap MA = \emptyset$, it follows by 9.8 that there exists an increasing, closed and open neighbourhood U' of MA such that $U' \cap LB = \emptyset$. Let $U := U' \cap X$. U is clearly increasing and closed and open in X. Since $A \subset MA \cap X \subset U$ and $U \cap B = \emptyset$, 2) is proved. To show that 2) implies 3) one just denotes by V the complement of U. Suppose 3) is satisfied and let $a,b \in X$ and $a \not\leq b$. Apply 3) for

Lb and Ma , and let U be the increasing, closed and open neighbourhood of Ma such that $U \cap Lb = \emptyset$. Define $f: X \to 2$ by $f(x) = 1$ if $x \in U$ and $f(x) = o$ otherwise . Since U is closed and open , f is clearly continuous, and all we have to show to obtain 4) is that f is isotone, $f(b) = o$ and $f(a) = 1$. Now if $x \leq y$ and $y \in U$, then $f(x) \leq f(y) = 1$, while, if $y \in \complement U$, since U is increasing, it follows that $x \in \complement U$ and $f(x) = o \leq o = f(y)$. Moreover $a \in Ma \subset U$ and $b \in Lb \subset \complement U$, whence $f(a) = 1$ and $f(b) = o$.
To finish the proof we must now prove that 4) implies 1) . Consider the evaluation $\mu: X \to 2^{\underline{COTS}(X,2)}$ defined by $\mu(x)(f) := f(x)$. Since it is a solution to the Universal Problem for the Product ,

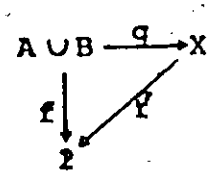


μ is continuous and isotone . Let $a, b \in X$ and $a \neq b$. Without loss of generality $a \nleq b$. By 4) there exists a continuous, isotone map $f: X \to 2$ such that $f(a) = 1 \neq o = f(b)$. Therefore $\mu(a)(f) = 1 \neq o = \mu(b)(f)$, $\mu(a) \neq \mu(b)$ and μ is injective . Since X and $2^{\underline{COTS}(X,2)}$ are both compact, μ is a closed map . By 4), if $\mu(a) \leq \mu(b)$, one obtains $a \leq b$. Therefore μ is an embedding, X is 2-compact and the proposition is proved .

9.10 NOTATION: We denote by Bs the subcategory of H which consists of the boolean spaces, i.e. compact Hausdorff and zero-dimensional .

9.11 PROPOSITION: Let B be a closed hereditary full subcategory of BsOTS, such that $2 \in B$. Then 2 is E-injective in B if and only if $B \subset 2cOT$.

PROOF: Suppose that 2 is E-injective in B . Let $X \in B$. Let A, B be closed, disjoint subsets of X, with A increasing and B decreasing . By [17] Theorem 4, page 46, there exist two disjoint open neighbourhoods U' of A and V' of B . Accordingly A and B are closed and open in $A \cup B$ and we can define a continuous isotone function $f: A \cup B \to 2$, by $f(a) = 1$ for $a \in A$, and $f(b) = o$ for $b \in B$. Since 2 is E-injective, there exists a continuous isotone funcion $\bar{f}: X \to 2$ such that if $g: A \cup B \to X$ is the inclusion, the following diagram commutes :



Now, $fA = \{1\}$ and $fB = \{o\}$; therefore, we obtain $A \subset \bar{f}^{-1}(1)$ which is increasing, closed and open, while $B \subset \bar{f}^{-1}(o)$ which is decreasing, closed and open . By 9.9 $X \in 2cOT$.

To show the converse, suppose that $B \subset 2cOT$ and consider diagram (I) in B where, without loss of generality, j is assumed to be an inclusion, and where μ is an embedding .

$$\text{(I)}\quad \begin{array}{ccc} A & \xrightarrow{j} & X \\ {\scriptstyle f}\downarrow & & \\ 2 & & \end{array} \qquad\qquad \text{(II)}\quad \begin{array}{ccccc} A & \xrightarrow{j} & X & \xrightarrow{\mu} & 2^S \\ {\scriptstyle f}\downarrow & & & & \\ 2 & & & & \end{array}$$

The sets $\mu\circ j[f^{-1}(o)]$ and $\mu\circ j[f^{-1}(1)]$ are clearly closed and disjoint and $L\mu\circ j[f^{-1}(o)]\cap M\mu\circ j[f^{-1}(1)] = \emptyset$. By 9.8 there exists an increasing open and closed neighbourhood U of $\mu\circ j[f^{-1}(1)]$ such that $U\cap L\mu\circ j[f^{-1}(o)] = \emptyset$.

We define $\bar{f}:X\to 2$ by $\bar{f}(x) = 1$ if $x\varepsilon\mu^{-1}U$ and $\bar{f}(x) = o$ otherwise . Since $\bar{f}$ is continuous and isotone and $\bar{f}\circ j = f$, we have shown that 2 is E-injective .

9.12 <u>COROLLARY</u>: 2 is E-injective in <u>2cOT</u> .

9.13 <u>PROPOSITION</u>: The E-injectives in <u>2cOT</u> are exactly the retracts of powers of 2 .

<u>PROOF</u>: By 9.12 the retracts of powers of 2 are E-injective . Conversely, let $X\varepsilon$<u>2cOT</u> be E-injective . As we observed in the proof of 9.9 the evaluation map $\mu:X\to 2^{\underline{COTS}(X,2)}$ is an embedding and since X is E-injective , there exists λ continuous and isotone such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & 2^{\underline{COTS}(X,2)} \\ {\scriptstyle 1_X}\downarrow & \swarrow{\scriptstyle \lambda} & \\ X & & \end{array}$$

Therefore X is a retract of $2^{\underline{COTS}(X,2)}$.

9.14 COROLLARY: $2 := (\{o,1\}, d)$ is not $\underline{E}$-injective in <u>2cOT</u> .

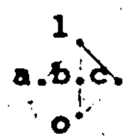
<u>PROOF:</u> Suppose 2 $\underline{E}$-injective, and consider the diagram ,

$$\begin{array}{ccc} 2 & \xrightarrow{j} & 2^2 \\ {\scriptstyle 1_2}\downarrow & & \\ 2 & & \end{array} \qquad \begin{array}{l} j(o) = (o,1) \\ j(1) = (1,o) \end{array}$$

This diagram cannot be extended to a commutative one by any continuous isotone map $\lambda : 2^2 \to 2$ since every such map would be constant .

9.15 <u>REMARK:</u> The observation of the proofs of 9.5 and 9.13 leads to the following general statement : If $X \in$ <u>PTop</u>, all $\underline{E}$-injectives in the category of X-regular or X-compact spaces are retracts of powers of X. To obtain statements similar to 9.5 and 9.13 one has to test X for $\underline{E}$-injectivity . This is for example the case for <u>RcOT</u> . We consider again <u>2cOT</u> .

9.16 <u>LEMMA:</u> Not every retract of a power of 2 is itself a power of 2 .

<u>PROOF:</u> Let A:= ({o,a,b,c,1} ) ; since A is finite and we want it to be Hausdorff, we endow it with the discrete topology . Define $f : A \to 2^3$ by $f(o) := (o,o,o)$, $f(a) := (o,o,1)$, $f(b) := (o,1,o)$, $f(c) := (1,o,o)$,

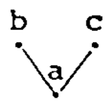
$f(1) := (1,1,1)$ and $g: 2^3 \to A$ by $g(o,o,o) := o$, $g(o,o,1) := a$, $g(o,1,o) := b$, $g(1,o,o) := c$ and $g\{(o,1,1), (1,o,1), (1,1,o), (1,1,1)\} = \{1\}$. Since f is an embedding, g is continuous, isotone and $f \circ g = 1_A$, we have shown that A is a retract of 2^3 which is not a power of 2.

9.17 LEMMA: Every finite lattice is E-injective in 2cOT.

PROOF: We show that if X is a finite lattice, it is a retract of $2^{\underline{COTS}(X,2)}$. Let μ_X be the evaluation $X \to 2^{\underline{COTS}(X,2)}$ and $j: \mu_X X \to X$ given by $j(\mu_X(a)) := a$. For every $a \in 2^{\underline{COTS}(X,2)}$ define the finite set $S_a := \{\mu_X(x) \in \mu_X X \mid a \leq \mu_X(x)\}$ and $g: 2^{\underline{COTS}(X,2)} \to \mu_X X$ by $g(a) := 1_{\mu_X X}$ if $S_a = \emptyset$ and $g(a) := \wedge S_a$ otherwise. Since $2^{\underline{COTS}(X,2)}$ is finite and Hausdorff, it has the discrete topology and g is therefore continuous. Let $a \leq b$ in $2^{\underline{COTS}(X,2)}$, then $S_b \subset S_a$, $\wedge S_a \leq \wedge S_b$ and $g(a) \leq g(b)$. Let $h := j \circ g$. Then h is continuous and isotone, and $h \circ \mu_X = 1_X$. Therefore X is a retract of $2^{\underline{COTS}(X,2)}$ and accordingly E-injective.

9.18 LEMMA: Let S be a subcategory of PTop which contains the finite spaces of 2cOT. If we denote by I the class of its isomorphisms, we have $\underline{I} \subsetneq \underline{E}^* \subsetneq \underline{E}$.

PROOF: Let $B \in \underline{S}$ such that $\bar{\bar{B}} > 1$, and consider the maps $h: B \to B$ given by $h(b) = o$ for all $b \in B$, $g := 1_B \sqcup h$, $f: B \to B \sqcup B$ given by $f(b) = (b,1)$. It is clear that $f \in \underline{E}$, and $g \circ f = f$, but $g \notin \underline{E}$. Therefore $f \notin E^*$. To find an essential map which is not an isomorphism, let A be a space in $\underline{S}$ without a greatest element (for example $(\{a,b,c\}$,



)) . Let $B := A \sqcup_t \{p\}$ with the coproduct topology and lexicographic partial order : $(a,1) \leq (p,2)$ for all $a \in A$. Define $f: A \to B$ by $f(a) = (a,1)$ for all $a \in A$. Now, f is obviously not an isomorphism, so it remains only to show that it is essential . Clearly, f is an embedding . Let $g \circ f \in \underline{E}$. Suppose there is $a \in A$ such that $g(a,1) = g(p,2)$. Then $g(a',1) \leq g(a,1)$ for all $a' \in A$ and since $g(x,1) = g \circ f(x)$ and $g \circ f \in \underline{E}$, we obtain $a' \leq a$ for all $a' \in A$, a contradiction . Therefore $g \circ f)(a') =$ $= g(a',1) \lneq g(p,2)$ for all $a' \in A$, and $g(A \sqcup_t \{p\}) =$ $= gfA \sqcup_t \{g(p,2)\} \cong A \sqcup_t \{p\}$. Since $g \in \underline{E}$, $f \in \underline{E}^*$.

9.19 REMARK: Some examples of subcategories of PTop , which satisfy the hypothesis of 9.18 are : HPTop, HPOTS, HOTS, CrORR, COTS, Zero-dimOTS, 2cOT, RcOT and BsOTS .

9.20 LEMMA: $\underline{E}$-injectivity in COTS does not satisfy the axiom E3 (Dual of P3 - see 0.31 -) .

PROOF: Let $f: 2 \to I$ be the inclusion map . To show that, for every $g: \mathbb{I} \to C$ such that $g \circ f \in \underline{E}$, then $g \circ f \notin \underline{E}^*$; we first show that $g \circ f \in \underline{E}$ implies $g\mathbb{I} \cong \mathbb{I}$. Since $g \circ f \in \underline{E}$, $g(o) \neq g(1)$. Let $a, b \in g\mathbb{I}$, and $x, y \in \mathbb{I}$ such that $a = g(x)$ and $b = g(y)$. Without loss of generality, we may assume $x \leq y$ and therefore $a \leq b$. Since $g\mathbb{I}$, which is a subset of $\mathbb{I}^S$ for some S, is a connected chain, we have $g\mathbb{I} \cong \mathbb{I}$. Since $g(o), g(1) \in g\mathbb{I} \subset \mathbb{I}^S$ and $g(o) \neq g(1)$, there exists $s \in S$ such that $p_s(g(o)) \neq p_s(g(1))$. Consider $2 \xrightarrow{f} \mathbb{I} \xrightarrow{g} C \xrightarrow{i} \mathbb{I}^S \xrightarrow{p_s} \mathbb{I}$, and call $h := p_s \circ i$. It is clear that $h \circ g \circ f \in \underline{E}$. If h is not injective, we would obtain that $g \circ f$ is not essential and our proof would be complete.

Suppose therefore that h is injective, and let $c, d \in g\mathbb{I} \subset C$ such that $g(o) < c < d < g(1)$. Then $o < hg(o) \leq h(c) \leq h(d) \leq hg(1) < 1$. Define $k: \mathbb{I} \to \mathbb{I}$ by:

$k(x) = x$ if $x < h(c)$,

$k(x) = h(c)$ if $h(c) < x < h(d)$,

$k[h(d) + \lambda(1-h(d))] = h(c) + \lambda(1-h(c))$ for $o < \lambda < 1$.

Then k is continuous and isotone. Let $\ell := k \circ h$. Then $\ell \circ g \circ f(o) = \ell \circ g(o) = k \circ h \circ g(o) = h(g(o)) \leq h(c) = k \circ h(d) \leq k \circ h(g(1)) = \ell \circ g \circ f(1)$, and, since $\ell \circ g \circ f(o) \leq \ell \circ g \circ f(1)$, we have $\ell \circ g \circ f \in \underline{E}$. But $\ell(c) = k \circ h(c) = k \circ h(d) = \ell(d)$; i.e. ℓ is not injective. Therefore $g \circ f \notin \underline{E}^*$.

9.21 <u>REMARK</u>: As the conditions E1-E6 are sufficient for proper behavior, it remains still open whether $\underline{E}$-injectivity in COTS is or is not properly behaved .

9.22 LEMMA: Every finite space in 2cOT has an E-injective hull .

PROOF: Let X be a finite space in 2cOT , DX the Dedekind McNeille Completion of the underlying partially ordered set and U_X the natural POSet-embedding $U_X: X \to DX$.

Since X is finite, DX is also finite and we consider the discrete topology on it so that $DX \in$ 2cOT . From [3] we know that U_X is embeddings-essential in POSet .

Suppose $h: DX \to Y$ given in 2cOT such that $h \circ U_X \in E$. Since $h \circ U_X$ is also an embedding in POSet, h is an embedding in POSet . Moreover since DX is compact Hausdorf and finite , so is hDX and accordingly hDX has the discrete topology. This shows that $h: DX \to hDX$ is an homeomorphism and therefore $h \in E$ and U_X is essential .

By 9.17 DX is E-injective .

9.23 COROLLARY: Let X be a finite 2cOT space . Then, for every $f: X \to Y$ in E , there exists $g: Y \to Z$ in 2cOT such that $g \circ f \in E^*$.

PROOF: Consider $h: X \to Z$ the E-injective hull of X.

9.24 LEMMA: E-injectivity in 2cOT (and in COTS) satisfies E4 and E6 .

PROOF: This is a consequence of the fact that both categories are locally small and have enough E-injectives. See [2] and [3].

9.25 PROPOSITION: E-injectivity in 2cOT is properly behaved at every finite X.

PROOF: This follows from a proof given by B.Banaschewski to show that the conditions E1 to E6 are sufficient for proper behavior. B.Banaschewski uses E5 only to show that there exists an injective hull, a fact which we have shown in 9.22.

CHAPTER IV

GENERALIZATIONS OF STONE AND SHIROTA THEOREMS.

The work covered by this Chapter started with our attempt to extend to $\bar{R}cOT$ the theorem given in [22] page 127, by T. Shirota, which, for real-compact topological spaces X, states that the lattice CX determines the space. This result supercedes earlier results of Kaplansky [13] about the lattice CX ; of M.H. Stone [23] about the ring CX ; and of A.N. Milgram [15] about the multiplicative semigroup CX, for X compact Hausdorff, and of T. Shirota [21] for the translation lattice and for the semigroup CX , and of E. Hewitt [12] for the ring CX where X is real-compact.

We call C_1X the set of continuous isotone real valued functions on a partially ordered topological space X. This set is a subset of CX and therefore contains less information than CX . If the generalization had been successful, this small set would have provided the information not only on the topology of X , but on its partial order as well .

Having been unable to generalize the above mentioned Theorem, we restricted ourselves to compact Hausdorff X and to rings, ℓ-rings, ℓ-groups and translation lattices, generating them with C_1X when necessary . This led to counterexamples for rings, ℓ-rings and for pointed ℓ-groups .

However, we do define categories of pairs with first component a ring, an ℓ-ring, a pointed ℓ-group, or a pointed translation lattice, and display new objects which actually characterize compact ordered topological spaces. In some of these cases (CX, C_1X) characterizes real-compact ordered topological spaces . Our results, when specialized for spaces with discrete partial order, deliver the corresponding results on CX in <u>Top</u> .

10. <u>General statements about C_1X.</u>

We include here an statement which we feel will be of interest for the rest of this Chapter . $\mathcal{F}X$ (or C_1X) may characterize X, but we are interested in whether the characterization happens in such a way that $\phi : \mathcal{F}X \cong \mathcal{F}Y$, implies the existence of $f : Y \cong X$ such that $\phi = \mathcal{F}(f)$ and $\mathcal{F}(f)(h) := h \circ f$.

10.1 <u>NOTATION</u>: For the rest of this thesis we shall denote by C_1X the set of all continuous, isotone, real valued functions defined on the partially ordered topological space X .

10.2 LEMMA: Every space X in CrORR has the initial PTop-structure with respect to C_1X .

PROOF: By 5.6 and 5.8 the evaluation map $\rho: X \to R^{C_1X}$ is an embedding . This shows that C_1X separates points of X , and, by 1.2 , there exists a PTop-initial structure X^i on X with respect to C_1X . Now consider $\rho': X^i \to R^{C_1X}$ given by $\rho'(x) := \rho(x)$. since $C_1X = C_1X^i$, then ρ' is also an embedding, and we have $X \cong \rho X = \rho' X^i \cong X^i$.

10.3 PROPOSITION: Let A be a category such that

C_1:CrORR $\to$ A and $C_1(f)(g) = g \circ f$ defines a functor.

$X \mapsto C_1X$

Let $\psi: C_1Y \cong C_1X$ in A . Then the following statements are equivalent :

1) There exists a bijection $f: X \to Y$ such that $p_x \circ \psi = p_{f(x)}$ for all $x \in X$.

2) There exists a CrORR-isomorphism $f: X \to Y$ such that $C_1(f) = \psi$.

PROOF: Let $f: X \to Y$ be a bijection as in 1) . Let $h \in C_1Y$, and $x \in X$ be arbitrary . Then $\psi(h)(x) = (p_x \circ \psi)(h) = p_{f(x)}(h) = (h \circ f)(x)$. Therefore $\psi(h) = h \circ f$. To prove 2) it is then sufficient to show that f is continuous and isotone, since the same argument will give f^{-1} continuous and isotone . Since ψ is bijective , we have $C_1X = \{h \circ f | h \in C_1Y\}$. Since Y has initial structure with respect to C_1Y, it follows that f is continuous and isotone . Conversely, suppose $f:X \to Y$ is an isomorphism in <u>CrORR</u> , such that $C_1(f) = \psi$. Clearly f is bijective and since $(p_x \circ \psi)(h) = \psi(h)(x) = C_1(f)(h)(x) = (h \circ f)(x) = h(f(x)) = p_{f(x)}(h)$, we obtain 1) .

10.4 <u>REMARK</u>: If we define a partial order on C_1X by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$, then C_1X is a lattice .

10.5 <u>PROPOSITION</u>: The following statement is false :
" If $\psi: C_1Y \to C_1X$ is a lattice isomorphism, there exists an isomorphism in <u>COTS</u>, $f:X \to Y$ such that $\psi = C_1(f)$ ".

<u>PROOF</u>: Let $X := Y := 2$ and denote by (a,b) the function $(a,b):X \to R$ where $(a,b)(o) = a$ and $(a,b)(1) = b$. Define $\psi: C_1Y \to C_1X$ by $\psi(a,b) = (2a+1, 2b+1)$. Obviously ψ is a lattice isomorphism , but the above statement would imply the existence of a bijective $f:X \to Y$ such that $p_o \circ \psi = p_{f(o)}$ which means
$3 = p_o(3,3) = p_o \circ \psi(1,1) = p_{f(o)}(1,1) = 1$, a contradiction.

10.6 <u>REMARK</u>: The above example leaves open the question of whether there exists $f:X \cong Y$ in <u>COTS</u> , such that $C_1(f)$ is <u>another</u> isomorphism of C_1X and C_1Y , but we include the example here because the statements which we shall prove later are of the type just discussed .

11. Counterexamples

In the effort to generalize the Shirota Theorem mentioned in the introduction to this Chapter, we had to be content with more modest results, similar to those available for CX in Top . Since C_1X fails in general to have the algebraic structures considered for CX, we consider the subalgebras of CX generated by C_1X .

11.1 NOTATION: Let $F: \underline{CrORR} \to \underline{Cr}$ be the order-forgetful functor, $U: \underline{Cr} \to \underline{CrORR}$ the canonical inclusion, given by $X \mapsto (X,d)$. We set $CX := \underline{Cr}(FX, \mathbb{R})$, RX for the subring of CX generated by C_1X , LrX for the sub-ℓ-ring of CX generated by C_1X and LgX for the sub-ℓ-group of CX generated by C_1X . As an example, we remark that $R([a,b])$ is the set of continuous functions on $[a,b]$ which are of bounded variation . We shall introduce more functors when we need them .

11.2 PROPOSITION: The following statements for $X,Y \in \underline{COTS}$ are false :

1) $RY \cong RX$ in the category of rings, then $X \cong Y$ in COTS .
2) $LrY \cong LrX$ in the category of ℓ-rings, then $X \cong Y$ in COTS .
3) $LgY \cong LgX$ in the category of ℓ-groups, then $X \cong Y$ in COTS

PROOF: If we describe, as in 10.5, the functions in $C_1 2$ by $(a,b):2 \to R$ such that $(a,b)(o) = a$ and $(a,b)(1) = b$, $C_1 2$ is $\{(a,b) \in R^2 \mid a \leq b\}$ while $C2$ is the whole R^2. Since $a \not\leq b$ implies $b \leq a$ and therefore $-a \leq -b$, the group generated by $C_1 2$ is $C2$ and so is

$$Lg2 = Lr2 = R2 = C2 = C2 = R2 = Lr2 = Lg2 .$$

But it is clear that $2 \not\cong 2$.

12. Generalization of theorems on CX .

We introduce categories of pairs, with first component a certain algebraic system and second component a subset of the underlying set of the first. We show for RcOT and for some of these algebraic structures that the pair $(CX, C_1 X)$ characterizes the space X . A characterization which depends more strongly on $C_1 X$ is achieved for COTS where $(RX, C_1 X)$, $(LrX, C_1 X)$, $(LgX, C_1 X)$ characterize X .

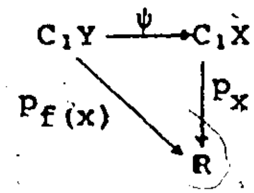
12.1 NOTATION: If AK denotes an algebraic category, we denote by p-AK the category whose objects are pairs (X,Y) such that $X \in$ AK and $Y \subset X$, and whose morphisms are $m:(X,Y) \to (Z,W)$ where $m:X \to Z$ is a AK-homomorphism and $m(Y) \subset W$.

12.2 DEFINITION: A translation lattice L with a nullary operation $z \in L$ will be called a pointed translation lattice.

12.3 REMARK: Clearly, CX and C_1X are pointed translation lattices, where we shall choose as its point the constant zero function. A PTL-homomorphism will be of course a function $f:L \to L'$ such that $f(z) = z'$, $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a+r) = f(a)+r$.

12.4 THEOREM: If $X, Y \in$ RcOT, and $\phi:(CY,C_1Y) \to (CX,C_1X)$ is a p-PTL-isomorphism, there exists $f:X \to Y$ a RcOT-isomorphism such that $\phi = C_1(f)$.

PROOF: For every $x \in X$, the map $p_x:CX \to R$ given by $p_x(f) = f(x)$ is a translation lattice-homomorphism (see 0.37). If we consider $CY \xrightarrow{\phi} CX \xrightarrow{p_x} R$ in the category TL (translation lattices), it follows by [21] Theorem 8 page 35, that there exists a unique point which we call $f(x)$ such that $p_x \circ \phi = p_{f(x)}$; the uniqueness arising from the fact that ρ_Y is injective. We show that the associating rule $x \mapsto f(x)$ defines a bijective function. By the uniqueness of $f(x)$, it is a function. Let $f(x) = f(y)$; then $p_x \circ \phi = p_{f(x)} = p_{f(y)} = p_y \circ \phi$ and since ϕ is an isomorphism and hence surjective, $p_x = p_y$. This shows for $X \in$ RcOT, that $x = y$. To show that f is surjective, let $y \in Y$ and consider $CX \xrightarrow{\phi^{-1}} CY \xrightarrow{p_y} R$. By the same argument as above, there exists a unique element of X, $g(y)$ such that $p_y \circ \psi^{-1} = p_{g(y)}$. Therefore: $p_y = p_y \circ \psi^{-1} \circ \psi = p_{g(y)} \circ \psi$ which means that $y = f(g(y))$. We have that $\psi(C_1 X) = C_1 Y$, that f is bijective and that, for every $x \in X$, the following diagram commutes:



It then follows from 10.3 that, $f: X \to Y$ is a RcOT-isomorphism.

12.5 COROLLARY: If X, Y are real compact spaces, and $\psi: CY \to CX$ a PTL-isomorphism, there exists a homeomorphism $f: X \to Y$ such that $\psi = C(f)$.

PROOF: We simply note that $C_1 X = CX$ and $C_1 Y = CY$.

12.6 LEMMA: For every $X \in Cr$, let SRX denote a subring of CX which contains all the constant functions. If $h: SRY \to SRX$ is a surjective ring homomorphism, then $h(\bar{r}) = \bar{r}$ for all $r \in R$.

PROOF: For $r = 1$, since $\bar{1} \in SRX$ and h is surjective, there exists $g \in SRY$ such that $\bar{1} = h(g) = h(g.\bar{1}) = h(g).h(\bar{1}) = \bar{1}.h(\bar{1}) = h(\bar{1})$. Suppose $h(\bar{n}) = \bar{n}$ for a positive integer n. Then $h(\overline{n+1}) = h(\bar{n}+\bar{1}) = h(\bar{n})+h(\bar{1}) = \bar{n}+\bar{1} = \overline{n+1}$. If n is a negative integer $\bar{o} = h(\overline{-n+n}) =$

$= h(\overline{-n}+\bar{n}) = h(\overline{-n})+h(\bar{n}) = \overline{-n}+h(\bar{n})$. Therefore $h(\bar{n}) = \bar{n}$. Moreover $\bar{1} = h(\bar{1}) = h(\bar{m}\cdot\overline{\frac{1}{m}}) = h(\bar{m})\cdot h(\overline{\frac{1}{m}}) = \bar{m}\cdot h(\overline{\frac{1}{m}})$ and we obtain $h(\overline{\frac{1}{m}}) = \overline{\frac{1}{m}}$. For rational numbers, $h(\overline{\frac{n}{m}}) = h(\bar{n}\cdot\overline{\frac{1}{m}}) = h(\bar{n})\cdot h(\overline{\frac{1}{m}}) = \bar{n}\cdot\overline{\frac{1}{m}} = \overline{\frac{n}{m}}$. If $r\varepsilon R$ and $r > o$, there exists $s\varepsilon R$, $r = s^2$. Then $h(r) = h(s)\cdot h(s) > o$. Let $r\varepsilon R$ and $r = \lim_{n\varepsilon N} r_n$ where r_n is rational for all $n\varepsilon N$. Let ε be an arbitrary positive rational number . Let $N\varepsilon N$, be such that whenever $n > N$, $r_n - r < \varepsilon$ or $r - r_n < \varepsilon$ and let $n > N$. If $r_n - r < \varepsilon$, $\varepsilon + r - r_n > o$, and $\bar{\varepsilon}+h(\bar{r})-h(\bar{r}_n) = h(\overline{\varepsilon+r-r_n}) > o$. Therefore $\bar{r}_n - h(\bar{r}) < \bar{\varepsilon}$ and, similarly, if $r - r_n < \varepsilon$, then $h(\bar{r})-\bar{r}_n < \bar{\varepsilon}$. For every $x\varepsilon X$, we then have : $r_n - h(\bar{r})(x) < \varepsilon$ or $h(\bar{r})(x) - r_n < \varepsilon$ which means that $h(\bar{r})(x) = \lim_{n\varepsilon N} r_n = r$ for all $x\varepsilon X$, and can be expressed as $h(\bar{r}) = \bar{r}$.

12.7 <u>PROPOSITION</u>: If $X,Y\varepsilon$<u>RcOT</u> , and $\phi:(CY,C_1Y)\rightarrow(CX,C_1X)$, is a <u>p-Ring</u>-isomorphism, there exists an <u>RcOT</u>-isomorphism $f:X\rightarrow Y$ such that $\phi = C_1(f)$.

<u>PROOF</u>: Since $\phi:CY\rightarrow CX$ is a Ring-isomorphism, it can be interpreted as a <u>PTL</u>-isomorphism because $\phi(o) = o$ and $\phi(f+\bar{r}) = \phi(f)+\phi(\bar{r}) = \phi(f)+\bar{r}$, see 12.6 . By 12.4 we obtain the desired result .

12.8 <u>COROLLARY</u>: If X,Y are realcompact spaces, and

$\psi: CY \to CX$ a ring isomorphism, there exists an homeomorphism $f: X \to Y$ such that $\psi = C(f)$.

PROOF: The proof is the same as in 12.5 .

12.9 DEFINITION: An ℓ-group G with an unary operation $1 \in G$ will be called a pointed ℓ-group , PLG . An PLG-homomorphism $h: G \to G'$ is an ℓ-group homomorphism such that $h(1) = 1'$.

12.10 PROPOSITION: If $X, Y \in$ RcOT, and $\phi: (CY, C_1Y) \to (CX, C_1X)$ is a p-PLG-isomorphism, there exists an RcOT-isomorphism $f: X \to Y$, such that $\phi = C_1(f)$.

PROOF: As in 12.4 , we consider for every $x \in X$, $CY \xrightarrow{\phi} CX \xrightarrow{p_x} R$ in PLG (for CZ we select the unary operation $1 \in CZ$, and for R , $1 \in R$) . By [21] Theorem 10 page 36, considering its proof, there exists a point $f(x)$ in Y , which is unique as $Y \in$ RcOT , such that $p_x \circ \phi = p_{f(x)}$. By following now all the steps in 12.4 , we finish this proof .

12.11 COROLLARY: If X, Y are real-compact spaces and $\psi: CY \to CX$ is a PLG-isomorphism, there exists an homeomorphism $f: X \to Y$ such that $\psi = C(f)$.

PROOF: The proof is the same as in 12.5 .

12.12 <u>NOTATION</u>: We define $Z:RX \to PX$ and $Z_1:LrX \to PX$ by $Zf := f^{-1}(o)$ and $Z_1 f := f^{-1}(o)$ respectively. We denote ZRX by ZX and $Z_1 LrX$ by $Z_1 X$. By a filter in ZX we mean a filter in the set ZX ordered by the inclusion.

12.13 <u>REMARK</u>:

i)	$Z\bar{o} = X$	iv)	$Zf \cdot g = Zf \cup Zg$
ii)	$Z\bar{1} = \emptyset$	v)	$Z(f^2+g^2) = Zf \cap Zg = Z_1(\lvert f\rvert + \lvert g\rvert)$
iii)	$Zf = Zf^n$	vi)	If $g = f \wedge \bar{1}$, then $Z_1 f = Z_1 g$.

12.14 <u>DEFINITION</u>: Let I be an ideal of the ring RX. We call I a <u>distinguished ideal</u> if $I \cap \{f \in RX \mid Zf = \emptyset\} = \emptyset$.

12.15 <u>LEMMA</u>: The intersection of any family of distinguished ideals is itself a distinguished ideal.

<u>PROOF</u>: trivial.

12.16 <u>PROPOSITION</u>: For every distinguished ideal I of RX, ZI is a filter of ZX.

<u>PROOF</u>: Since I is distinguished, $\emptyset \notin ZI$. Let Zf, Zg be two sets in ZI such that $f,g \in I$. Then $f^2+g^2 \in I$ and therefore $Zg \cap Zf = Z(f^2+g^2) \in ZI$. Let $f \in I$ and $Zf \subset Zg \in ZX$. Then $gf \in I$ and $Zg = Zf \cup Zg = Z(f \cdot g) \in ZI$.

12.17 <u>PROPOSITION</u>: If ϕ is a filter in ZX, $Z^{-1}\phi := \{f \in RX \mid Zf \in \phi\}$ is a distinguished ideal.

PROOF: Using the notation of 11.1 we remark that C_1UPX is CX and therefore RUPX = CX . Since ϕ has the finite intersection property and $\phi \subset ZUPX$, it generates a filter $\bar{\phi}$ in ZUPX . By [9] Theorem 2.3 b) page 25, $Z^{-1}\bar{\phi}$ is a proper ideal of CX , i.e. $Zf \neq \emptyset$ whenever $f \in Z^{-1}\phi$. Therefore $Z^{-1}\phi = Z^{-1}\bar{\phi} \cap RX$ is a distinguished ideal of RX .

12.18 REMARK: The ideal generated by f and g is distinguished exactly if $Zf \cap Zg \neq \emptyset$, i.e. if f^2+g^2 is non-invertible .

12.19 LEMMA: Let ϕ be a filter in ZX , and I a distinguished ideal of RX . Then $ZZ^{-1}\phi = \phi$ and $I \subset Z^{-1}ZI$.

PROOF: This is trivial, as we have defined $Z: RX \to PX$ as a map .

12.20 COROLLARY: If I is a maximal distinguished ideal of RX, then $I = Z^{-1}ZI$. If ϕ is a filter in ZX , there exists the distinguished ideal $I := Z^{-1}\phi$ such that $\phi = ZI$.

12.21 PROPOSITION: For every distinguished ideal M of RX , $Z^{-1}ZM$ is a maximal distinguished ideal if and only if ZM is an ultrafilter .

PROOF: Let M be a distinguished ideal . Suppose $Z^{-1}ZM$ is a maximal distinguished ideal and let ϕ be a filter in ZX such that $ZM \subset \phi$. We obtain that $Z^{-1}ZM \subset Z^{-1}\phi$

and since $Z^{-1}ZM$ is a maximal distinguished ideal, we have $Z^{-1}ZM = Z^{-1}\psi$. By 12.16 and 12.19, $ZM = ZZ^{-1}ZM =$ $= ZZ^{-1}\psi = \psi$. Conversely, suppose that ZM is an ultrafilter and I a distinguished ideal such that $Z^{-1}ZM \subset I$. Then $ZM = ZZ^{-1}ZM \subset ZI$ and since ZM is an ultrafilter, $ZM = ZI$. Therefore $Z^{-1}ZM = Z^{-1}ZI$. This, together with $Z^{-1}ZM \subset I \subset Z^{-1}ZI$, gives $Z^{-1}ZM = I$, which shows that $Z^{-1}ZM$ is a maximal distinguished ideal.

12.22 <u>LEMMA</u>: If u is an ultrafilter in ZX and $Zf \cap Zg \neq \varnothing$ for all $Zg \in u$, then $Zf \in u$. If M is a maximal distinguished ideal of RX and $Zf \cap Zg \neq \varnothing$ for all $g \in M$, then $f \in M$.

<u>PROOF</u>: This is immediate by the maximality of u and the fact that ZM is an ultrafilter. (See 12.21.)

12.23 <u>LEMMA</u>: For $x \in X$, if $p_x : RX \to R$ is the x-th projection, $Zp_x^{-1}(o)$ is an ultrafilter.

<u>PROOF</u>: $p_x^{-1}(o)$ is clearly a distinguished ideal since $x \in Zf$ for all $f \in p_x^{-1}(o)$. Since the map p_x is a surjective rings-homomorphism, $RX/p_x^{-1}(o) \cong R$ and $p_x^{-1}(o)$ is a maximal ideal. By 12.20 $p_x^{-1}(o) = Z^{-1}Zp^{-1}(o)$, and, by 12.21, $Zp_x^{-1}(o)$ is an ultrafilter.

12.24 <u>NOTATION</u>: We denote $Zp_x^{-1}(o)$ by A_x.

12.25 REMARK: In order to characterize all the maximal distinguished ideals of RX as sets of the form $p_x^{-1}(o)$, we need to discuss convergence in our filters .

12.26 REMARK: If we denote by N(x) the set of all (not necessarily open) neighbourhoods of x , a filter ϕ of ZX is said to converge to x if the set of all Zf in N(x) belongs to ϕ . This means that ϕ converges to x if and only if $A_x \cap N(x) \subset \phi$. A point x is said to be an adherence point of ϕ if $x \in \bigcap \phi$. If x is an adherence point of ϕ , $Zg \cap Zf \neq \emptyset$ whenever $Zg \in N(x)$ and $Zf \in \phi$. Therefore , in this last case, there exists an ultrafilter u such that $\phi \subset u$ and u converges to x .

12.27 LEMMA: If X is a locally compact space in NORC, every filter ϕ in ZX converges to at most one point.

PROOF: Suppose ϕ converges to two distinct points x and y . Without loss of generality $x \nleq y$. Since $\leq$ is continuous, there exist neighbourhoods U of x and V of y such that $LV \cap MU = \emptyset$. Without loss of generality , since X is locally compact, U and V are compact and therefore LV, MU are two disjoint closed sets which satisfy the hypothesis of [17] Prop.4 page 44. Let $f \in C_1X$ be such that $fLV = o$ and $fMU = 1$. Then $LV \subset f^{-1}(o) = Zf$ and therefore $Zf \in N(y)$. Similarly $MU \subset (f-1)^{-1}(o) = Z(f-1)$ which means that $Z(f-1) \in N(x)$. Having assumed that ϕ

converges to x and to y, $Zf, Z(f-1) \in \phi$ which is a contradiction as $Zf \cap Z(f-1) = \emptyset$ and ϕ is a filter.

12.28 LEMMA: Let $X \in$ COTS. If a filter ϕ of ZX converges to x, $\bigcap \phi = \{x\}$.

PROOF: Since X is compact and ϕ is a family of closed sets which satisfy the finite intersection property, $\bigcap \phi \neq \emptyset$. Let $y \neq x$. If $x \nleq y$ we reason as in 12.27 and find $f \in C_1 X$ such that $f(y) = o$ and $Z(f-1) \in N(x) \subset \phi$. Therefore $y \notin Z(f-1)$ and accordingly $y \notin \bigcap \phi$. If $y \nleq x$, we similarly find compact neighbourhoods V of y and U of x such that $LU \cap MV = \emptyset$, and a function g in $C_1 X$ such that $gLU = o$ and $gMV = 1$. It follows that $g^{-1}(o) = Zg \in N(x) \subset \phi$ and $y \notin Zg$. Therefore, in both cases, $y \notin \bigcap \phi$, a fact that together with $\bigcap \phi \neq \emptyset$ completes this proof.

12.29 LEMMA: If an ultrafilter ϕ of ZX has an adherence x, ϕ converges to x.

PROOF: We show that $A_x \cap N(x) \subset \phi$. Let $Zf \in A_x \cap N(x)$. For every $Zg \in \phi$, $x \in Zg \cap Zf$. Since ϕ is an ultrafilter, $Zf \in \phi$.

12.30 PROPOSITION: Let $X \in$ COTS. The ultrafilters in ZX are exactly $(A_x)_{x \in X}$.

PROOF: Since X is compact, by the finite intersection property every ultrafilter has an adherence point and therefore converges. We know from 12.23 that every A_x is an ultrafilter in ZX, and it is obvious that $A_x = Zp_x^{-1}(o)$ converges to x. Let ϕ be an arbitrary ultrafilter in ZX which converges to x. By 12.28, $\bigcap\phi = \{x\}$. Therefore, every set Zf of ϕ satisfies the following: $x \in Zf$, $f(x) = o$, $f \in p_x^{-1}(o)$, $Zf \in Zp_x^{-1}(o) = A_x$. This means that $\phi \subset A_x$ and, since ϕ is an ultrafilter, $\phi = A_x$.

12.31 REMARK: A_x is the unique ultrafilter of ZX which converges to x, and every filter ϕ which converges to x is a subset of A_x.

12.32 PROPOSITION: Let $X \in$ COTS. The maximal distinguished ideals of RX are exactly $(p_x^{-1}(o))_{x \in X}$.

PROOF: By 12.23 $p_x^{-1}(o)$ is a maximal distinguished ideal. Conversely, let M be an arbitrary maximal distinguished ideal of RX. By 12.20, $M = Z^{-1}ZM$ and, by 12.21, ZM is an ultrafilter. Since ZM satisfies the finite intersection property and X is compact, there exists an $x \in X$ such that ZM converges to x. By 12.31, since ZM is an ultrafilter, $ZM = A_x$. Therefore $M = Z^{-1}ZM = Z^{-1}A_x =$ $= Z^{-1}Zp_x^{-1}(o)$ and by 12.20 and 12.23, $Z^{-1}Zp_x^{-1}(o) = p_x^{-1}(o)$

12.33 NOTATION: We denote by MRX the set of all maximal distinguished ideals in RX, which by the above proposition is $\{p_x^{-1}(o) \mid x \in X\}$.

12.34 LEMMA: For every $X \in$ COTS, the natural map $b_X : X \to MRX$ given by $b_X(a) = p_a^{-1}(o)$ is a bijection .

PROOF: The surjectivity is obvious . Let $x \nleq y$ be given in X . Then $Mx \cap Ly = \phi$ and we can find a function $f \in C_1X \subset RX$ such that $f(y) = o \neq 1 = f(x)$. This means that $f \in p_y^{-1}(o)$ and $f \notin p_x^{-1}(o)$. Therefore $p_y^{-1}(o) \neq p_x^{-1}(o)$.

12.35 THEOREM : If $X, Y \in$ COTS, and $\phi : (RY, C_1Y) \to (RX, C_1X)$ is a p-Ring-isomorphism, there exists $\lambda : X \to Y$ a COTS-isomorphism such that $\phi = C_1(\lambda)$. Similarly, if $\sigma : (LrY, C_1Y) \to (LrX, C_1X)$ is a p-ℓ-Ring-isomorphism there exists $g : X \to Y$ a COTS-isomorphism such that $\sigma = C_1(g)$.

PROOF: Let X,Y be spaces in COTS such that $\phi : (RY, C_1Y) \to (RX, C_1X)$ is a p-Ring-isomorphism . Since $\phi : RY \to RX$ is a Ring-isomorphism, it induces a bijective function $\bar{\phi} : MRY \to MRX$. Define $\lambda : X \to Y$ as $\lambda := b_Y^{-1} \circ \bar{\phi}^{-1} \circ b_X : X \xrightarrow{b_X} MRX \xrightarrow{\bar{\phi}^{-1}} MRY \xrightarrow{b_Y^{-1}} Y$. Let $x \in X$; then $p_{\lambda(x)}^{-1}(o) = b_Y(\lambda(x)) = (b_Y \circ \lambda)(x) = (\bar{\phi}^{-1} \circ b_X)(x) = \phi^{-1} p_x^{-1}(o)$. Let $g \in p_{\lambda(x)}^{-1}(o)$; then $\phi(g)(x) = g(\lambda(x))$. We show next that $\phi(g) = g \circ \lambda$ for all $g \in RY$. Let $x \in X$ and $g \in RY$. Call $r := g(\lambda(x))$. Then $(g - \bar{r})(\lambda(x)) = o$ and therefore $g - \bar{r} \in p_{\lambda(x)}^{-1}(o)$. But, for this case, we have just shown that

$\psi(g-\bar{r})(x) = o = (g-\bar{r})(\lambda(x))$. By 12.6 $\psi(\bar{r}) = \bar{r}$ and we obtain $\psi(g)(x)-r = \psi(g)(x)-\psi(\bar{r})(x) = g(\lambda(x))-\bar{r}(\lambda(x))=$ $= (g\circ\lambda)(x)-r$. Since x was arbitrary, $\psi(g) = g\circ\lambda$ and we have shown that, for the bijection λ the following diagram commutes :

$$\begin{array}{ccc} RY & \xrightarrow{\psi} & RX \\ & \searrow^{p_{\lambda(x)}} & \downarrow p_x \\ & & \mathbb{R} \end{array}$$

Since $\psi(C_1Y) = C_1X$ we can interpret this diagram as

$$\begin{array}{ccc} C_1Y & \xrightarrow{\psi} & C_1X \\ & \searrow^{p_{\lambda(x)}} & \downarrow p_x \\ & & \mathbb{R} \end{array}$$

By 10.3, it follows that $\lambda:X\to Y$ is a CrORR-isomorphism; i.e. a COTS-isomorphism .

The statement about (LrY, C_1Y) is proved in an analogous way .

12.36 COROLLARY: If X,Y are compact spaces and $\psi:CY\to CX$ is a ring-isomorphism, there exists an homeomorphism $\lambda:X\to Y$ such that $\psi = C(\lambda)$.

PROOF: The proof is the same as that of 12.5 .

12.37 REMARK: We introduced in 12.9 the concept of pointed ℓ-group (ℓ-group with a unit), PLG . If X is a PLG we call a subset Y a sub-ℓ-group with unit of X if Y is an ℓ-group and has the same unit as X . The intersection of a family of sub-ℓ-groups with unit is clearly a sub-ℓ-group with unit .

12.38 NOTATION: Let $X\epsilon$COTS . We denote by Lqu(X) the sub-ℓ-group with $\bar{1}$ of CX which is the intersection of all L, sub-ℓ-groups with $\bar{1}$ of CX , which contain C_1X and such that, with $f\epsilon L$ and f invertible in CX , $f^{-1}\epsilon L$. We define Z_2:Lqu(x)→PX by $Z_2 f = f^{-1}(o)$ and, as in 12.12 by Z_2X we mean Z_2(Lqu(X)) .

12.39 LEMMA: Let $X\epsilon$COTS , and h:Lqu(X)→R be a surjective PLG-homomorphism . Then $h(\bar{r}) = r$ for all $r\epsilon R$.

PROOF: By the definition of h , $h(\bar{1}) = 1$, and since Lqu(X) is a lattice we can repeat the rest of the proof of 12.6, proceeding directly from $r_n - r < \varepsilon$ to $r_n - h(\bar{r}) = h(\overline{r_n - r}) \leq h(\bar{\varepsilon}) = \varepsilon$.

12.40 LEMMA: Let $X\epsilon$COTS and h:Lqu(X)→R a surjective LGO-homomorphism . If h(f) = o , then h(|f|) = o and $Z_2 f \neq \emptyset$.

PROOF: If h(f) = o , h(-f) = -h(f) = o . Therefore $h(|f|) = h(f \vee -f) = h(f) \vee h(-f) = o \vee o = o$. Suppose $Z_2 f = \emptyset$. Then o∉Im f and $f^{-1}\epsilon CX$. Therefore $f^{-1}\epsilon$Lqu(X) . We show that $h(|f^{-1}|) \neq o$. Suppose $h(|f^{-1}|) = o$. Since $h(|f|) = o$, $h(|f| \vee |f^{-1}|) = o$. But $\bar{1} \leq |f| \vee |f^{-1}|$ and

$1 = h(\bar{1}) \leq h(|f| \vee |f^{-1}|) = o$, a contradiction . Let $h(|f^{-1}|) = a > o$. We apply 12.39 and obtain : $h(|f^{-1}|-\bar{a}) = h(|f^{-1}|)-h(\bar{a}) = a-a = o$. From this it follows that $h((|f^{-1}|-a) \vee |f|) = o$. To obtain the contradiction needed to reject "$Z_2 f = \phi$" , let $e > 2a$ and $d := \min\{a, \frac{1}{e}\}$. We shall show that $(|f^{-1}|-\bar{a}) \vee |f| \geq d > o$. Since for each $x \in X$, $f(x) \neq o$, it follows that $f^{-1}(x) \neq o$ and $|f^{-1}(x)| \neq o$. If $o < |f^{-1}(x)| < a$, $\frac{1}{e} < \frac{1}{2a} < \frac{1}{a} < \frac{1}{|f^{-1}(x)|} = |f(x)|$ and we obtain $d \leq \frac{1}{e} < |f(x)|$. If $a \leq |f^{-1}(x)| < 2a$, then $\frac{1}{e} < \frac{1}{2a} < \frac{1}{|f^{-1}(x)|} = |f(x)|$. Finally, if $2a \leq |f^{-1}(x)|$, clearly $|f^{-1}(x)|-a \geq a \geq d$. This completes the proof .

12.41 <u>LEMMA</u>: Let $X \in$ <u>COTS</u>, and $h: Lgu(X) \to R$ be a surjective <u>PLG</u>-homomorphism. Then $Z_2 h^{-1}(o)$ is an ultrafilter in $Z_2 X$.

<u>PROOF</u>: Since h is a group-homomorphism, $h(\bar{o}) = o$, and we have $\bar{o} \in h^{-1}(o)$, $Z_2 \bar{o} = X \in Z_2 h^{-1}(o)$. Therefore $Z_2 h^{-1}(o) \neq \emptyset$. By 12.40, $\emptyset \notin Z_2 h^{-1}(o)$. Let $f, g \in h^{-1}(o)$; then $|f| \vee |g| \in h^{-1}(o)$ and $Z_2 f \cap Z_2 g = Z_2(|f| \vee |g|) \in Z_2 h^{-1}(o)$. To see that $Z_2 h^{-1}(o)$ is a filter in $Z_2 X$, we need now show that if $f \in h^{-1}(o)$ and $Z_2 f \subset Z_2 g$ for $g \in Lgu(X)$, then $g \in h^{-1}(o)$. Suppose $g \notin h^{-1}(o)$ and $h(g) = a \neq o$. Then $g-\bar{a} \in h^{-1}(o)$ and $Z_2(g-\bar{a}) \in Z_2 h^{-1}(o)$. From this we obtain a

contradiction, because it then follows that $Z_2 f \cap Z_2(g-\bar{a}) \subset Z_2 g \cap Z_2(g-\bar{a}) = \emptyset$. Having shown that $Z_2 h^{-1}(o)$ is a filter, suppose there exists a filter ϕ in $Z_2 X$ such that $Z_2 h^{-1}(o) \subsetneq \phi$. Let $Z_2 g \epsilon \phi \setminus Z_2 h^{-1}(o)$. Then $h(g) = a \neq o$, and we obtain again that $g-\bar{a} \epsilon h^{-1}(o)$ and $Z_2(g-\bar{a}) \epsilon Z_2 h^{-1}(o) \subset \phi$. Since, by assumption, $Z_2 g \epsilon \phi$ and $Z_2 g \cap Z_2(g-\bar{a}) = \emptyset$, this is a contradiction. Therefore $Z_2 h^{-1}(o)$ is an ultrafilter.

12.42 PROPOSITION: Let $X \epsilon$ COTS and $h: Lgu(X) \to R$ be a surjective PLG- homomorphism. There exists a unique $x \epsilon X$ such that $h(f) = f(x)$ for all $f \epsilon Lgu(X)$.

PROOF We have just shown that $Z_2 h^{-1}(o)$ is an ultrafilter in $Z_2 X$. Since X is in COTS, and is therefore locally compact and normally ordered with continuous order, we conclude as in 12.27 that $\bigcap Z_2 h^{-1}(o)$ has at most one element. Since X is compact, it follows by the finite intersection property of $Z_2 h^{-1}(o)$ that there exists $x \epsilon \bigcap Z_2 h^{-1}(o)$, and accordingly $\bigcap Z_2 h^{-1}(o) = \{x\}$.

Let $f \epsilon Lgu(X)$, and $h(f) = b$. Then $h(f-\bar{b}) = o$, and $f-\bar{b} \epsilon h^{-1}(o)$ follows. Therefore $x \epsilon \bigcap Z_2 h^{-1}(o) \subset Z_2(f-\bar{b})$, and we obtain $o = (f-\bar{b})(x) = f(x)-b$, which means $h(f) = b = f(x)$.

12.43 THEOREM: If $X, Y \in$ COTS , and $\phi: (Lgu(Y), C_1Y) \to (Lgu(X), C_1X)$ is a p-PLG-isomorphism there exists $f: X \to Y$, a COTS-isomorphism such that $\phi = C_1(f)$.

PROOF: If $\phi: Lgu(Y) \to Lgu(X)$ is the given PLG-isomorphism and $x \in X$, we consider $Lgu(Y) \xrightarrow{\phi} Lgu(X) \xrightarrow{p_x} R$, which is a surjective PLG-homomorphism, and, by 12.42, there exists a unique element of $Y, f(x)$ such that $p_x \circ \phi = p_{f(x)}$. As in 12.4, we see that the f thus defined is a bijective function and since $\phi C_1Y = C_1X$, the following diagram commutes :

$$\begin{array}{ccc} C_1Y & \xrightarrow{\phi} & C_1X \\ & {\scriptstyle p_{f(x)}} \searrow & \downarrow {\scriptstyle p_x} \\ & & R \end{array}$$

By 10.3 $f: X \to Y$ is a COTS-isomorphism .

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