PARTIALLY ORDERED, PRIMITIVE REGULAR SEMIGROUPS

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By

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SCOPE AND CONTENTS:

The theory of partially ordered, primitive regular semigroups is developed under the hypothesis that the partial order admits enough integral idempotents. This is done in analogy to the theory of partially ordered groups. In particular, results are given which - in a purely algebraic way - determine the existence of partial orders and characterize the semigroup of integral elements of a directed, primitive regular semigroup.

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Introduction

An element s in a partially ordered semigroup is called integral if $sx \leq x$ and $xs \leq x$ holds for all elements x in the semigroup. Not every partial order admits integral elements, but if they exist at all, then they form a subsemigroup. In the theory of partially ordered groups this subsemigroup, called the cone of the partial order, determines completely the order relation. Indeed the relation of two elements $a \leq b$ can be described in a purely algebraic way:

$$a \leq b \text{ if and only if } a \in CbC$$

where C denotes the cone of the partial order.
E. A. Behrens has shown in [2] that a similar powerful relationship between partial orders and the subsemigroup of integral elements can be found in completely simple semigroups if one introduces a fundamental hypothesis. This hypothesis postulates the existence of "enough" integral idempotents, i.e. to each element a there should exist integral idempotents e and f such that

$$a = eaf.$$

For partially ordered groups the existence of such idempotents is trivial, since the only idempotent, the identity of the group, is integral for every partial order.

Or, the other way around, every group can be partially ordered with enough integral idempotents, namely at least trivially. In general this situation is not true for primitive regular semigroups. However we can give necessary and sufficient conditions in order that a primitive regular semigroup can be partially ordered (Theorem 5.2). A similar characterization was given for (strictly) ordered inverse semigroups by T. Saito [11, Theorems 6.2 - 6.8].

The existence of enough integral idempotents not only allows us to describe the partial order algebraically, but also has strong effects on the algebraic structure of the semigroups itself. The most important feature is that it allows us to define an inverse for all (non-zero) elements (Proposition 3.5) and so partially ordered, primitive regular semigroups are close to inverse semigroups, a fact which is one of the keys to the whole theory. It has to be noted however that these inverses carry a more or less formal character: the usual rule for the inverse of products does not hold, except of course when the semigroup is inverse as well. Nevertheless we can show with the help of these inverses that many of the statements in partially ordered groups can be directly transferred to the theory of partially ordered, primitive regular semigroups. The most important one asserts that a completely 0-simple semigroup is directed if and only if it is the "quotient semigroup" of its integral elements (Theorem 4.3).

A theorem which goes back to John von Neumann and G. Birkhoff [3, Theorem 12] characterizes the cones of partially ordered groups by four properties. To prove this theorem one constructs the group of quotients over this semigroup and then shows that this group is partially ordered in a natural way. The partial order then turns out to be directed. We give a very similar theorem for completely 0-simple semigroups (Theorem 6.1), i.e. we give necessary and sufficient conditions for a semigroup in order that it constitutes the set of integral elements of a directed, completely 0-simple semigroup. The characterizing properties we found resemble very much the original ones for groups, besides having two among them which deal with the structure of the set of idempotents.

A similar system was found by E. A. Behrens [2] for the special case of the so-called quasiuniserial semigroups. The reason that these conditions only vaguely resemble ours may be found in the origin of quasiuniserial semigroups: they are isomorphic to the semigroup of ∪-irreducible ideals in certain arithmetical rings (cf. [1, Chapter IX]), a characterization which seems to be impossible to achieve for the semigroups under investigation in this thesis.

The problem of characterizing the semigroup of integral elements in a partially ordered, primitive regular semigroup is of course closely related with the problem of embedding a given semigroup into a primitive regular semigroup. In the context of this thesis we solved the embedding problem whenever the resulting primitive regular semigroup is a quotient semigroup of the subsemigroup of integral elements. The general problem, however, remains open, i.e. to the best of our knowledge no reasonable conditions have been found which characterize the semi-group of integral elements in an (unrestricted) partially ordered, primitive regular semigroup.

§1 Background and Fundamental Definitions

Following the standard reference book of A. H. Clifford and G. B. Preston [4] we call a semigroup H primitive regular, if H is regular in the sense of von Neumann and each non-zero idempotent is primitive. The structure of these semigroups is well known (cf. [4, Thm. 6.39]): each such semigroup H is the 0-disjoint union of completely 0-simple semigroups M_α and each of the M_α is an ideal in H. We shall make frequent use of the famous Rees-theorem [9] representing completely 0-simple semigroups as matrix semigroups M(G; I, A; P) and shall use this representation whenever appropriate.

All semigroups we consider are assumed to have a zero element denoted by 0, except when we talk about groups in section 2 and about completely simple semigroups in the last theorem of this thesis.

We want to investigate semigroups which are partially ordered. By a partial order $\leq$ on a semigroup H we mean a reflexive, transitive and antisymmetric relation which is compatible with the multiplication in H: whenever $a \leq b$ in H then it should follow that also $ac \leq bc$

and $ca \subseteq cb$ for all elements $c \in H$.

We shall restrict this general definition by requiring that a partial order admits enough integral idempotents: An element $s \in H$ is called <u>integral</u> (relative to a given partial order $\subseteq$) if and only if for every $a \in H$ the inequalities $sa \subseteq a$ and $as \subseteq a$ are satisfied. The identity in a semigroup has trivially this property. If integral elements exist at all then they form a semigroup, so it is plausible to require that the zero-element is integral, which then in turn makes it into the minimum of all elements, i.e. we have $0 \subseteq a$ for all elements a. By requiring the existence of "enough integral idempotents" we want to replace the single identity of a semigroup (which does not exist in most of the cases we are interested in) with a suitable family of idempotents. This is made more precise in the next definition.

<u>Definition 1.1.</u> A semigroup H has <u>enough</u> [<u>integral</u>] <u>idempotents</u> if and only if for each element $a \in H$ there exist [integral] idempotents e,f such that $a = eaf$.

In all we shall consider only those partial orders which satisfy our next definition.

<u>Definition 1.2.</u> A reflexive, transitive and antisymmetric relation $\subseteq$ on a semigroup H, compatible with

the multiplication, is called an <u>integral order</u> of H if and only if there exist enough integral idempotents and the zero-element is one of them.

In particular we shall be concerned only with those partial orders where the subsemigroup of integral elements is non-empty. This subsemigroup we shall call frequently <u>semicone</u> and we shall reserve the letter S for it. In the special case of a partially ordered group we follow the customary notation; we call the subsemigroup of integral elements the <u>cone</u> of the group and reserve the letter C for it.

The set $E = \{e_i | i \in I\}$ will always denote the set of non-zero integral idempotents or when there is no confusion the set of non-zero idempotents. We shall use E and I interchangeably as index sets, i.e. we shall index with $i \in I$ rather than with $e_i \in E$.

§2 Partially Ordered Groups

The results in this section are all long-established and well-known and can best be found in L. Fuchs' book [5]. Nevertheless we would like to state them here in order to put into perspective our further development, which will show a close connection between the results in this section and the theory of integrally ordered, primitive regular semigroups.

First of all it should be stated that in the case of partially ordered groups our definition 1.2 agrees with the usual one of a partial order: the identity of the group is clearly integral under any ordering and moreover it is "enough" in the sense of Definition 1.1.

Theorem 2.1. The cone C of a partially ordered group $(G, \leq)$ is characterised by

$$C = \{c \in G \mid c \leq 1\}.$$

Theorem 2.2. The cone C of a partially ordered group $(G, \leq)$ determines the partial order by

$$x \leq y \text{ if and only if } x \in CyC$$
$$\text{if and only if } xy^{-1} \in C.$$

<u>Theorem 2.3.</u> (L. Fuchs [5, Ch. II, Prop. 3]). The partial order $\leq$ in a group G is directed if and only if the cone C generates G, i.e.

$$G = \{xy^{-1} | x, y \in C\}.$$

<u>Theorem 2.4.</u> (L. Fuchs [5, Ch. II, Thm. 2]). A subset P of a group G is the cone of some partial order of G if and only if P satisfies:

(1) $P \cap P^{-1} = \{1\}$

(2) $PP \subseteq P$

(3) $xP = Px$ for all $x \in G$.

<u>Theorem 2.5.</u> (L. Fuchs [5, Ch.II, Thm. 4]). A semigroup P is the cone of some partially ordered group if and only if

(1) P has an identity 1

(2) P is cancellative

(3) $xy = 1$ implies $x = y = 1$

(4) $Px = xP$ for all $x \in P$.

§3 Partially Ordered Primitive Regular Semigroups

Many of the results in this section are modifications and extensions of the theory E. A. Behrens developed for completely simple semigroups in [2].

The key-observation for the whole theory is the following:

Theorem 3.1. (E. A. Behrens [2, Thm. 1.2]).
If in a partially ordered semigroup two integral idempotents generate the same one-sided ideal, then they are equal.

Proof. (loc. cit). Let $eH = fH$ with e and f integral idempotents in the semigroup H. Then $e = fe \subseteq f$ and $f = ef \subseteq e$. Hence $e = f$, q.e.d.

Keeping in mind that the fundamental Definition 1.2 requires the existence of enough integral idempotents we get the following:

Corollary 3.2. (E. A. Behrens [2, Thm. 1.4]).
Let H be an integrally ordered, primitive regular semigroup. Then for each $a \in H$ exist uniquely determined integral idempotents e and f such that

$$a = eaf.$$

This corollary allows us to describe the algebraic structure of integrally ordered, primitive regular semigroups more explicitly. The 0-minimal one-sided ideals are in one-to-one correspondence with the non-zero, integral idempotents, $E = \{e_i \mid i \in I\}$. We therefore can use the set I as index set in the representation of H as "matrix semigroup":

Theorem 3.3. (E. A. Behrens [2, Thm. 1.5]). Let H be an integrally ordered, primitive regular semigroup with $E = \{e_i \mid i \in I\}$ as set of non-zero integral idempotents. Then each completely 0-simple component of H is isomorphic to a Rees matrix semigroup of the particular form $M(G_\alpha, I_\alpha, I_\alpha, P_\alpha)$ where P_α satisfies

$$p_{ii}^{(\alpha)} = e_\alpha, \text{ the identity of } G_\alpha,$$

for all $i \in I_\alpha$, and the sets I_α form a partition of I.

Proof: Although E. A. Behrens proves the corresponding theorem for completely simple semigroups without using the matrix representation for that semigroup, the proof for Theorem 3.3 is implicit in his development.

Definition 3.4. A primitive regular semigroup H is called square normalized matrix semigroup if and only if H satisfies the conclusion of Theorem 3.3. The idempotents of the particular form $e_i = (e, i, i)$ are called *idempotents in the diagonal*.

Square normalized matrix semigroups behave in many respects like inverse semigroups. This is due to the fact that in these semigroups, the inverse of an element can be defined:

Proposition 3.5. Let H be a square normalized matrix semigroup. Let

$$0 \neq a = e_i a e_j$$

where e_i and e_j are idempotents in the diagonal. Then there exists one and only one element $b \in H$ such that

$$ab = e_i \quad \text{and} \quad ba = e_j.$$

Proof by an easy calculation with the matrix representation of H. If $a = (a_1; i, j)$, then $b = (a_1^{-1}; j, i)$ satisfies the conditions, since $p_{ii} = p_{jj} = e$ the identity of the structure group. If $x = (x_1; k, h)$ is another element with said properties, then

$$(x_1; k, h)(a_1; i, j) = (x_1 p_{hi} a_1; k, j) = (e; j, j)$$

implies $k = j$ and $x_1 = a_1^{-1} p_{hi}^{-1}$. Also

$$(a_1; i, j)(x_1; j, h) = (a_1 x_1; i, h) = (e; i, i)$$

and therefore $h = i$ and $x_1 = a_1^{-1}$, i.e. $x = b$, q.e.d. We make now the obvious

<u>Definition 3.6</u>. For $0 \neq a \in H$, H a square normalized matrix semigroup, define <u>the inverse of</u> a, denoted by a^{-1}, to be the unique element determined by Proposition 3.5.

It is clear that a^{-1} is a regular inverse, i.e. $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$, and so our definition agrees with the usual one for inverse semigroups in case that H is moreover an inverse semigroup, i.e. a 0-disjoint union of Brandt semigroups.

If $a \in e_iHe_j$, then $a^{-1} \in e_jHe_i$ and so when e_iHe_j is a subgroup in H, we see that the newly defined inverse a^{-1} agrees with the group inverse of a in e_iHe_j if and only if $e_i = e_j$. Therefore the usual rule for inverses of products does not hold, i.e. in general it is not true that $(ab)^{-1} = b^{-1}a^{-1}$. Actually this should not come as a surprise since the corresponding rule in inverse semigroups depends on the commutativity of the idempotents (cf. [4, Lemma 1.18]).

Each integrally ordered, primitive regular semigroup H with $E = \{e_i \mid i \in I\}$ as set of non-zero integral idempotents is the 0-disjoint union of its subsemigroups e_iHe_j:

$$H = \bigsqcup_{i,j \in I} e_iHe_j.$$

Each of the subsemigroups e_iHe_j is either a group, a zero semigroup or zero itself. In each case the partial order defined on H induces a partial order on the subsemigroup e_iHe_j. If e_iHe_j is a group, denote with C_{ij} the cone of the induced order. We shall see that $S \sqcap e_iHe_j$, i.e. the set of elements in e_iHe_j which are integral in the whole semigroup , forms only a subset of C_{ij} which is equal to C_{ij} if and only if $e_i = e_j$. If e_iHe_j is a zero semigruop, then all elements are integral in e_iHe_j, but only a subset is integral in the whole semigroup. The precise statement which is the equivalent of Theorem 2.1 is the following

Theorem 3.7. (E. A. Behrens [2, Thm. 1.4]).
Let H be an integrally ordered, primitive regular semigroup and denote with S its subsemigroup of integral elements. Then

$$S \sqcap e_iHe_j = \{a \in e_iHe_j \mid a \subseteq e_ie_j\}$$
$$= e_iSe_j.$$

Proof. (loc. cit.)

In the following corollaries let H be an integrally ordered, primitive regular semigroup, S its semicone and $E = \{e_i \mid i \in I\}$ the set of nonzero integral idempotents.

Corollary 3.8. Let e_iHe_j be a group. Then

$$C_{ij} = e_iSe_j \text{ if and only if } e_i = e_j.$$

Proof: Denote with e the identity of e_iHe_j. If $e_i = e_j$, then also $e = e_i = e_j$. If $C_{ij} = e_iSe_j$, then

$$C_{ij} = \{c \in e_iHe_j | c \subseteq e\}$$

$$= \{s \in e_iHe_j | s \subseteq e_ie_j\} = e_iSe_j.$$

Therefore $e_ie_j = e$. From

$$e = e^2 = e_ie_je_ie_j \subseteq e_ie_je_i \subseteq e_ie_j$$

we conclude $e_ie_je_i = e$ and analogously $e_je_ie_j = e$. But $e_iH \sqcap e_jH \neq 0$ forces (by Thm. 3.1) e_i and e_j to be equal, q.e.d.

Corollary 3.9. $e_ie_je_i = e_i$ if and only if $e_i = e_j$.

Proof. Since $e_je_i \neq 0$, e_iHe_j is a subgroup of H. Since $e_ie_j = e_ie_je_ie_j = (e_ie_j)^2$, the identity of it must be equal to e_ie_j. Now apply Corollary 3.8, q.e.d.

Corollary 3.10. If H is (strictly) ordered, then H is an ordered group with zero.

Proof. Let e_i and e_j two integral idempotents and assume $e_i \subseteq e_j$. Then

$$e_i \subseteq e_i e_j e_i \subseteq e_i e_j \subseteq e_i.$$

Hence $e_i e_j e_i = e_i$ and by Corollary 3.9 we conclude that H must be a group with zero, q.e.d.

Remark: T. Saito [8, Thm. 2] comes to similar moreover restricted results, even without assuming our fundamental hypothesis, the existence of enough integral idempotents.

Theorem 3.11. (E. A. Behrens [2, Thm. 1.4]). Let H, S, E be given as before. Let $a = e_i a e_j$ with $e_i, e_j \in E$. Then

$a \subseteq b$ if and only if $a \in Sbe_j$

if and only if $a \in e_i bS$

if and only if $a \in SbS$.

Proof: The proof has to be different from the one given in [2] since the latter one depends on the non-existence of a zero.

If $a \in SbS$, then by the very definition of integral elements we have $a \subseteq b$, so the last statement implies the first. Clearly the second and the third one both imply the last one.

Let then $a = e_i a e_j \subseteq b$. Since $\subseteq$ is compatible with the multiplication we have also

$$e_i a e_j \subseteq e_i b e_j .$$

If now $e_i b e_j = 0$, then the minimality of zero implies that $a = 0$ and then clearly $a = 0 \in Sbe_j$.

So we may assume $e_i b e_j \neq 0$, which is only possible if a and b lie in the same completely 0-simple component of H. So we can solve the two equations for x_1 and x_2:

$$e_i a e_j = e_i x_1 e_i \cdot e_i b e_j$$

$$e_i = e_i b e_j \cdot e_j x_2 e_i .$$

In order to prove the implication we have to show that $e_i x_1 e_i \in S$ or equivalently that $e_i x_1 e_i \subseteq e_i$ (cf. Theorem 3.7)

$$\begin{aligned} e_i x_1 e_i &= e_i x_1 e_i \cdot e_i b e_j \cdot e_j x_2 e_i \\ &= e_i a e_j \cdot e_j x_2 e_i \\ &\subseteq e_i b e_j \cdot e_j x_2 e_i \\ &= e_i . \end{aligned}$$

So $a \subseteq b$ implies the second statement. Symmetrically one shows that $a \subseteq b$ implies the third statement, q.e.d.

The last theorem, which is of course a partial equivalent to Theorem 2.2, reduces the partial order to the algebraic structure of H. We see that elements in different completely 0-simple components are not comparable with each other, or conversely given a primitive regular semigroup H, such that every completely 0-simple component is integrally ordered, then H itself is integrally ordered by the union of the various partial orders.

A similar consequence holds for Brandt semigroups: here two elements are comparable only if they lie in the same subsemigroup $e_i He_j$, since the product of different (integral) idempotents is always equal to zero.

Remark 3.12. At this point we would like to remark that the analogy between partially ordered groups and partially ordered primitive regular semigroups is not perfect. In groups we have the equivalence (Theorem 2.2)

$$x \leq y \text{ if and only if } xy^{-1} \in C.$$

The corresponding statement $a \subseteq b$ if and only if $ab^{-1} \in S$ is true only in one direction. If $a \subseteq b$ then $ab^{-1} \subseteq bb^{-1} \in E \subseteq S$, so $a \subseteq b$ implies $ab^{-1} \in S$.

But $ab^{-1} \in S$ does not imply that $a \subseteq b$. For example take $a = e_i a e_i$ and $b = e_i e_k \neq 0$ with $e_k e_i = 0$. Then a cannot be comparable with b, since $a \subseteq b$ implies

$$e_i a e_i = a \subseteq e_i b e_i = 0.$$

But $ab^{-1} = e_i a e_i \cdot (e_i e_k)^{-1} \in e_i He_i$ can be chosen to be smaller or equal to e_i, the identity of the group $e_i He_i$.

However the relation between inverses and the partial order is the same as in groups, i.e. $a \subseteq b$ if and only if $a^{-1} \supseteq b^{-1}$. One shows this best by looking at the matrix representation of H.

Theorem 3.13. Let H, S and E be as before. Let a and b both be different from zero. Then $a \subseteq b$ if and only if $b^{-1} \subseteq a^{-1}$.

Proof: Since $a = e_i a e_j$ and $b = e_k b e_h$ are different from zero the relation $a \subseteq b$ can only be valid if a and b lie in the same completely 0-simple component and if $e_i e_k \neq 0$ and $e_h e_j \neq 0$. So in the matrix representation we have $p_{ik} \neq 0$ and $p_{hj} \neq 0$.

Let $a = (a_1;\ i,\ j)$ and $b = (b_1;\ k,\ h)$

$$(a_1;\ i,\ j) \subseteq (b_1;\ k,\ h)$$

$$(a_1;\ i,\ j)(e\ ;\ j,\ i) \subseteq (e;\ i,\ i)(b_1;\ k,\ h)(e;\ j,\ i)$$

$$(a_1;\ i,\ i) \subseteq (p_{ik} b_1 p_{hj};\ i,\ i).$$

This inequality holds now in the group $e_i H e_i$ and so we may conclude

$$(a_1^{-1};\ i,\ i) \supseteq (p_{hj}^{-1} b_1^{-1} p_{ik}^{-1};\ i,\ i)$$

$$(e\ ;\ j,\ i)(p_{hj}^{-1} b_1^{-1} p_{ik}^{-1};\ i,\ i) \subseteq (e;\ j,\ i)(a_1^{-1};\ i,\ i)$$

$$(p_{hj}^{-1} b_1^{-1} p_{ik}^{-1};\ j,\ i) \subseteq (a_1^{-1};\ j,\ i) = a^{-1}.$$

$$b^{-1} = (b_1^{-1};\ h,\ k)$$

$$= (e;\ h,\ h)(p_{hj}^{-1}b_1^{-1}p_{ik}^{-1};\ j,\ i)(e;\ k,\ k)$$

$$\subseteq (p_{hj}^{-1}b_1^{-1}p_{ik}^{-1};\ j,\ i) \subseteq a^{-1},$$ q.e.d.

§4 Directed, Completely 0-simple Semigroups

In this section we shall investigate how Theorem 2.3 transfers to the theory of partially ordered, primitive regular semigroups, in other words we examine the connection between a partial order being directed and the semigroup M being generated as a "quotient semigroup" by elements of the semicone S.

Before going into details we would like to state a theorem of E. A. Behrens [2] which can easily be generalized to primitive regular semigroups.

Theorem 4.1. (E. A. Behrens [2, Thm. 1.6]).
Let M be an integrally ordered, completely 0-simple semigroup, $M = M(G; I, I; P)$. Then the partial order of moreover M induces a partial order on the structure group G by $x \leq y$ if and only if

$$(x; i, j) \subseteq (y; i, j)$$

independent of the indices $i, j \in I$. Each element $s \in S$ can be written as $s = (s_1 p_{ij}; i, j)$ for some $s_1 \in C$, the cone of G, and indices $i, j \in I$.

<u>Definition 4.2</u>. Let H, S and E be as usual. H is called a <u>left</u> [<u>right</u>] <u>quotient</u> <u>semigroup</u> of S if and only if each element $a \in H$ can be expressed as $a = s^{-1}t$ $[a = st^{-1}]$ with $s, t \in S$.

Since in a primitive regular semigroup the different, completely 0-simple components multiply to zero, we cannot obtain directedness for the partial order unless we restrict ourselves to completely 0-simple semigroups.

<u>Theorem 4.3</u>. Let $M = M(G; I, I; P)$ be an integrally ordered, completely 0-simple semigroup with S as semicone. The following are equivalent statements:

(1) $(M, \leq)$ is directed.

(2) The structure group G is directed under the induced partial order and for any two integral idempotents e_i, e_j there exist integral idempotents e_k, e_h such that

$$e_k e_i e_h \neq 0 \quad \text{and} \quad e_k e_j e_h \neq 0.$$

(3) The structure group G is directed under the induced partial order and M is a left as well as a right quotient semigroup of S.

<u>Proof</u>. (1) implies (2).

Let e_i and e_j be arbitrary. Since M is directed there exist an element $c = e_h c e_k$ such that

$$e_i \subseteq c \quad \text{and} \quad e_j \subseteq c.$$

From Theorem 3.11 we deduce that there exist $s, t \in S$ such that

$$e_i = e_i s e_h c e_k e_i$$

$$e_j = e_j t e_h c e_k e_j.$$

Clearly $e_k e_i \neq 0$ and $e_k e_j \neq 0$. Since $e_i s e_h$ and $e_j t e_h$ are different from zero it follows from Theorem 3.7 that also $e_i e_h \neq 0$ and $e_j e_h \neq 0$.

So in the matrix representation we have p_{ki}, p_{ih}, p_{kj} and p_{jh} all non-zero. Hence also $e_k e_i e_h \neq 0$ and $e_k e_j e_h \neq 0$.

Now we have to show that G under the induced order is directed.

Let $e_i a e_i$ and $e_i b e_i$ be given. There exists $c = e_k c e_h$ such that

$$e_i a e_i \subseteq e_k c e_h$$

$$e_i b e_i \subseteq e_k c e_h.$$

Multiplying with e_i on both sides we get

$$e_i a e_i \subseteq e_i e_k c e_h e_i$$

$$e_i b e_i \subseteq e_i e_k c e_h e_i$$

and the induced partial order on G is directed (Theorem 4.1).

(2) implies (1):

Let $(a_1; i, j)$ and $(b_1; k, h)$ be given. We want to construct an element c which is greater or equal to both of them. Choose integral idempotents e_m and e_n such that

$$e_i e_m \neq 0 \quad \text{and} \quad e_k e_m \neq 0$$

$$e_n e_j \neq 0 \quad \text{and} \quad e_n e_h \neq 0.$$

Since G is directed we can find $c_1 \in G$ such that

$$c_1 \geq p_{im}^{-1} a_1 p_{nj}^{-1}$$

$$c_1 \geq p_{km}^{-1} b_1 p_{nh}^{-1}.$$

(Note that all four entries of the sandwich matrix are unequal to zero). Now we can find elements s_1 and $t_1 \in C$ such that

$$a_1 = s_1 p_{im} c_1 p_{nj}$$

$$b_1 = t_1 p_{km} c_1 p_{nh}.$$

Hence

$$(a_1;\ i,\ j) = (s_1;\ i,\ i)(c_1;\ m,\ n)(e;\ j,\ j)$$

$$(b_1;\ k,\ h) = (t_1;\ k,\ k)(c_1;\ m,\ n)(e;\ h,\ h)$$

and M is directed.

(2) implies (3):

We show that M is a left quotient semigroup of S. Let $(a;\ i,\ j)$ be given. Choose e_k such that $e_k e_i \neq 0$ and $e_k e_j \neq 0$. Then also $p_{kj} \neq 0$ and $p_{ki} \neq 0$. Since G is directed we can find $s_1,\ t_1 \in C$ such that $p_{ki} a p_{kj}^{-1} = s_1^{-1} t_1$ (Theorem 2.3). Then we have

$$(s_1 p_{ki};\ k,\ i)^{-1}(t_1 p_{kj};\ k,\ j)$$

$$= (p_{ki}^{-1} s_1^{-1};\ i,\ k)(t_1 p_{kj};\ k,\ j)$$

$$= (a_1;\ i,\ j).$$

So $(a_1;\ i,\ j)$ is a left quotient of elements in S. $(a_1;\ i,\ j)$ can also be expressed as a right quotient: This one shows similarly using the existence of e_h such that $e_i e_h \neq 0$ and $e_j e_h \neq 0$.

(3) implies (2):

Let e_i and e_j be given. Pick $0 \neq a \in M$ such that $a = e_i a e_j$. Since a is a left quotient there exist s, t in S such that

$$a = s^{-1}t = e_i s^{-1} e_k t e_j.$$

By Theorem 3.7 we conclude that $e_k e_i \neq 0$ and $e_k e_j \neq 0$. Since a is also a right quotient we get the existence of e_h such that $e_i e_h \neq 0$ and that $e_j e_h \neq 0$. But then also $e_k e_i e_h \neq 0$ and $e_k e_j e_h \neq 0$, q.e.d.

§5 Existence of Partial Orders

Every group can be partially (integrally) ordered, at least by the identity relation. The situation for primitive regular semigroups is not as satisfactory, indeed the "identity relation" being an integral order restricts the semigroup quite seriously:

Proposition 5.1. Let H be a primitive regular semigroup. The identity relation together with all pairs of the form $(0, a)$, $a \in H$, is an integral order on H if and only if H is the 0-disjoint union of Brandt semigroups.

Proof: If the identity is a partial order then we have to show that the product of two different idempotents is always equal to zero. Since there are enough integral idempotents, it is sufficient to show this for integral idempotents. If $e_i e_j \neq 0$ then $0 \neq e_i e_j \subseteq e_i$ and the partial order is not trivial any more.

Conversely: If H is the 0-disjoint union of Brandt semigroups, then the set of all idempotents is a subsemigroup. It follows immediately from our next theorem (5.2) that this subsemigroup forms a semicone determining the identity relation on $H \setminus \{0\}$, q.e.d.

So in general the identity is not an integral order, which of course depends entirely on Definition 1.2, which requires the existence of enough integral idempotents. If we want to find partial orders in our sense, then in view of the special structure of integrally ordered, primitive regular semigroups (Theorem 3.3), we should not hope to obtain them in other than square normalized matrix semigroups. However in that case we can prove a theorem which is very similar to the corresponding one for groups (Theorem 2.4).

Theorem 5.2. Let H be a square normalized matrix semigroup with E as family of diagonal idempotents. A subset S of H is the subsemigroup of integral elements with respect to some integral order having E as set of integral idempotents if and only if S satisfies

(1) $S \cap S^{-1} = E$

(2) $SS \subseteq S$

(3) $eaSf = eSaf$ for all $a \in H$ and all idempotents $e, f \in E$.

Proof: Let S be the semicone with respect to the integral order $\leq$ of H.

(1) is satisfied: suppose $s = t^{-1}$ with $s, t \in S$ and both s and t unequal to zero. From the uniqueness of the inverse (Prop. 3.5) we conclude $t = s^{-1}$. Since always $s \subseteq ss^{-1}$ (Theorem 3.7) and since s and t are integral we have

$$s \subseteq ss^{-1} = st \subseteq t$$

$$t \subseteq tt^{-1} = ts \subseteq s.$$

Therefore $s = t$, i.e. s is its own inverse. This is only possible if s is a diagonal idempotent, i.e. $s \in E$.

(2) is satisfied, since S is a subsemigroup.

(3) is satisfied: Let $easf$ be given. Since s is integral we have

$$easf \subseteq eaf.$$

By Theorem 3.11 we conclude

$$easf \in S \cdot eaf \subseteq Saf.$$

Multiplying with e from the left we get

$$easf \in eSaf.$$

Similarily it follows that $eSaf \subseteq eaSf$.

Let now a subset S of H with properties (1), (2) and (3) be given. Define a relation $\subseteq$ on H by

$$a \subseteq b \text{ if and only if } a \in SbS.$$

This relation is reflexive since $E \subseteq S$ by (1). Since S is a subsemigroup (2) the relation is transitive.

$\leq$ is antisymmetric:

Let $e_i a e_j = a \leq b = e_k b e_h$ and also $b \leq a$. The definition of $\leq$ implies the existence of s_i and t_i in S; $i = 1, 2$; such that

$$a = e_i s_1 e_k b e_k t_1 e_j$$

$$b = e_k s_2 e_i a e_j t_2 e_k .$$

Then

$$a = e_i s_1 e_k s_2 e_i a e_j t_2 e_h t_1 e_j$$

$$aa^{-1} = e_i = e_i s_1 e_k s_2 e_i a e_j t_2 e_h t_1 e_j \cdot a^{-1}.$$

Therefore the element $e_i s_1 e_k s_2 e_i \in S$ has an inverse in the subgroup $e_i H e_i$, which is the same as the inverse in H. This inverse lies in S:

$$e_i a e_j t_2 e_h t_1 e_j a^{-1} \in e_i a S e_j a^{-1} = e_i S a e_j a^{-1}$$

$$= e_i S a a^{-1} \subseteq S$$

using property (3).

By (1) we conclude

$$e_i = e_i s_1 e_k \cdot e_k s_2 e_i .$$

The same argument for b instead of a yields

$$e_k = e_k s_2 e_i \cdot e_i s_1 e_k .$$

That means that the elements $e_i s_1 e_k$ and $e_k s_2 e_i$ are inverse of each other. By (1) this is only possible if

$$e_i = e_i s_1 e_k = e_k s_2 e_i = e_k .$$

The analogous argument for t_1 and t_2 shows finally that $a = b$.

The relation $\leq$ is compatible with the multiplication: Let $a \leq b = e_k b e_h$ and let $c = e_m c e_n$ be arbitrary. $a \leq b$ implies that

$$\begin{aligned} ac &\in SbSc \\ &= Se_k bSe_m c \\ &= Se_k Sbe_m c \\ &\subseteq Sbc \\ &\subsetneq SbcS \end{aligned}$$

making use of (3). The multiplication from the left is treated analogously.

Since $E \subseteq S$, $\leq$ has enough integral idempotents. It remains to show that S constitutes the semicone. That every element in S is integral is obvious from the very definition of the partial order. Take $a \in H$ integral. Then

$$a = aa^{-1}a \leq a^{-1}a$$

$$a \in Sa^{-1}aS \subseteq S, \text{ q.e.d.}$$

<u>Remark 5.3</u>. Condition (1) of Theorem 5.2 can be replaced by

(1') If $a, b \in S$ and $ab = e \neq 0$, $e \in E$, then $a = b = e$.

<u>Proof:</u> Let the conditions of Theorem 5.2 be satisfied. Assume $ab = e \neq 0$ for $a, b \in S$. Then we have in particular that $a \in eH$ and $b \in He$. Therefore

$$0 \neq ba = b \cdot e \cdot a = baba \leq e.$$

ba is an idempotent smaller or equal to the integral idempotent e. Since by the proof of Corollary 3.10 no two different idempotents are comparable, we conclude that $ba = e$, i.e. $a = b^{-1}$. By (1) of Theorem 5.2 we have $b = a = e$.

We now want to show that condition (1') implies condition (1) of Theorem 5.2. Let $s = t^{-1}$ with $s, t \in S$. Then $tt^{-1} = ts \in E$ and by (1') we have $s = t = e \in E$, i.e. we have (1), q.e.d.

§6 The Semigroup of Integral Elements

In this section we characterize those semigroups which can appear as the subsemigroups of integral elements in an integrally ordered, completely 0-simple semigroup. The corresponding theorem in the theory of partially ordered groups (Theorem 2.5) is proven by construction of the group of quotients. Then one shows that the so formed group can be partially ordered and is furthermore directed. We would like to state and prove a theorem which is the complete analogue to that situation. We give necessary and sufficient conditions in order that a semigroup constitutes the semicone of a directed, completely 0-simple semigroup.

<u>Theorem 6.1.</u> The semigroup S is the subsemigroup of integral elements in some directed, integrally ordered, completely 0-simple semigroup if and only if S satisfies

(A1) S has enough idempotents.

(A2) In S the following cancellation law is valid: Let e and f be idempotents such that $ef \neq 0$. Then

$$eae \cdot fbf = eae \cdot fcf \text{ implies } fbf = fcf$$

provided $eae \neq 0$ and

$$eae \cdot fbf = ece \cdot fbf \text{ implies } eae = ece$$

provided $fbf \neq 0$.

(A3) If $ab = e = e^2 \neq 0$ then $a = b = e$.

(A4) $eaSf = eSaf$ for all $a \in S$ and all idempotents e, f.

(A5) The idempotents are categorical at zero, i.e. $ef \neq 0$ and $fg \neq 0$ implies $efg \neq 0$.

(A6) For each pair of idempotents e and f there exist idempotents g and h such that $geh \neq 0$ and $gfh \neq 0$.

Remark: Conditions (A1)-(A4) reflect very closely the corresponding conditions for partially ordered groups (Theorem 2.5), which go back to J. von Neumann and G. Birkhoff (cf. [3, Theorem 12]) who gave similar conditions for lattice ordered groups. The change from the original system is made by replacing the single identity of a group by a family of idempotents, which then become the integral idempotents. In condition (A5) we followed the terminology of A. H. Clifford and G. B. Preston (cf. [4, Sec. 7.7]). The categorical behavior at zero of all elements plays an essential role in G. Lallement's and M. Petrich's development of matrix decompositions of semigroups [5] and generalizations of the Rees-theorem [6]. Condition (A6) originates of course from the directedness of the semigroup (Theorem 4.3).

The proof of the sufficiency of our conditions is based on the following ideas: The subsemigroups of the form eSe are all cones of partially ordered groups, as one can see from (A1)-(A4) letting $e = f$. (A2) and (A4) allow us to show that they are all isomorphic to each other. The connection between these special subsemigroups is given by (A4). (A4) implies in particular that the semigroups "off the diagonal" can be described by semigroups "on the diagonal", namely $eSf = eSe \cdot f$, thereby giving the idempotents the role of coordinates. In a special case the construction of the sandwich matrix is fairly direct: Under the assumption that there exists a "reference point" e_1 among the idempotents one composes the already mentioned isomorphisms in such a way that automorphisms of e_1Se_1 result, e.g.:

$$e_1Se_1 \to fSf \to gSg \to e_1Se_1.$$

These automorphisms of e_1Se_1 turn out to be inner automorphisms. The elements of e_1Se_1 determining them give rise to the entries in the sandwich matrix P, at which point one has to make use of (A5). (A6) makes it possible - by a direct limit construction - to overcome the restriction of having a fixed "reference point" and to prove the general result.

Proof. Let us first prove the necessity of (A1)-(A6): let M be a directed, integrally ordered, completely 0-simple semigroup with S as semicone and E the integral idempotents.

Since M has enough integral idempotents, i.e. idempotents in S, these idempotents are certainly enough for S, i.e. (A1) is satisfied.

(A2) can easily be deduced from the matrix representation of M: if $e = (e_1; i, i)$ and $f = (e_1; j, j)$ then $ef \neq 0$ implies $p_{ij} \neq 0$. Let now $eae \cdot fbf = eae \cdot fcf$. In the matrix representation this looks like

$$(a_1; i, i)(b_1; j, j) = (a_1; i, i)(c_1; j, j)$$

$$(a_1 p_{ij} b_1; i, j) = (a_1 p_{ij} c_1; i, j).$$

We can conclude that $b_1 = c_1$ in the structure group and hence $fbf = fcf$.

(A3) is condition (1') of Remark 5.3 .

(A4) is (3) of Theorem 5.2 restricted to elements of S. Since completely 0-simple semigroups are categorical at zero, (A5) is satisfied and we get (A6) by Theorem 4.3.

The proof of sufficiency as may be expected is broken up into several steps.

$E = \{e_i \mid i \in I\}$ will denote the set of non-zero idempotents in S. In order to save indices we shall index by I rather than by E.

<u>6.2 Step 1</u>. $S = \bigsqcup_{i,j \in I} e_i S e_j$.

<u>Proof</u>: Immediate from (A1).

<u>6.3 Step 2</u>. Each of the semigroups $e_i S e_j$ has the form

$$e_i S e_j = e_i S e_i \cdot e_j = e_i \cdot e_j S e_j.$$

<u>Proof</u> by (A4):

$$e_i S e_j = e_i \cdot e_i S \cdot e_j = e_i \cdot S e_i \cdot e_j$$

$$e_i S e_j = e_i \cdot S e_j \cdot e_j = e_i \cdot e_j S \cdot e_j.$$

<u>Corollary 6.4</u>. $e_i S e_j = 0$ if and only if $e_i e_j = 0$.

<u>6.5 Step 3</u>. S is the zero-disjoint union of the subsemigroups $e_i S e_j$.

<u>Proof</u>: Let $0 \neq a = e_i a e_j = e_k a e_h$. Then by Step 2 we can find an element a' such that

$$e_i a e_j = e_i \cdot e_j a' e_j.$$

Then

$$e_i \cdot e_j a' e_j = e_k e_i \cdot e_j a' e_j = e_i e_k e_i \cdot e_j a' e_j.$$

Since $e_j a' e_j$ is unequal to zero, (A2) allows us to conclude that

$$e_i = e_i e_k e_i = e_i e_k \cdot e_k e_i .$$

By (A3) we get

$$e_i = e_i e_k = e_k e_i$$

and again by (A3): $e_i = e_k$ similarly we show $e_j = e_h$, which then finishes the proof of Step 3.

Definition 6.6. For each pair of idempotents e_i, e_j such that $e_i e_j \neq 0$ define mappings

g_{ij}: $e_i S e_i \longrightarrow e_j S e_j$ and

f_{ji}: $e_j S e_j \longrightarrow e_i S e_i$ by

$e_i a e_i e_j = e_i e_j g_{ij}(a) e_j$ for $a \in e_i S e_i$

$e_i e_j b e_j = e_i f_{ji}(b) e_i e_j$ for $b \in e_j S e_j$.

Our definition makes sense because Step 2 (6.3) implies that g_{ij} and f_{ji} exist and the cancellation law (A2) implies that they both are well defined, i.e. they are mappings. Because of (A2) they are also one-to-one and it follows from Step 2 (6.3) that g_{ij} and f_{ji} are onto maps. Furthermore they are homomorphisms.

$$e_i e_j g_{ij}(a) g_{ij}(b) e_j = e_i e_j g_{ij}(a) e_j \; e_j g_{ij}(b) e_j$$

$$= e_i a e_i \; e_j g_{ij}(b) e_j$$

$$= e_i a b e_i \; e_j$$

$$= e_i e_j g_{ij}(ab) e_j.$$

Again (A2) implies that

$$g_{ij}(ab) = g_{ij}(a) g_{ij}(b).$$

Similarly we show f_{ji} is an homomorphism. So in all we have g_{ij} and f_{ji} to be isomorphisms. We note the following

Lemma 6.7. If $e_i e_j \neq 0$ then $f_{ji} g_{ij}$ is the identity on $e_i S e_i$, $g_{ij} f_{ji}$ is the identity on $e_j S e_j$.

6.8 Step 4. All subsemigroups of the form $e_i S e_i$ are isomorphic to each other.

Proof: Let e_i and e_j be given. By (A6) there exists an idempotent e_k such that $e_k e_i \neq 0$ and $e_k e_j \neq 0$. Then

$$g_{kj} f_{ik} : \; e_i S e_i \to e_j S e_j.$$

is an isomorphism from $e_i S e_i$ onto $e_j S e_j$, which proves Step 4.

We now would like to prove the theorem for a special case:

<u>6.9 Step 5</u>. If there exists a reference point among the idempotents, i.e. an idempotent $e_1 \in E$ such that $e_1 e_i \neq 0$ for all $e_i \in E$, then S can be embedded into $M(G_1; I, I; P)$ where the structure group G_1 is the quotient group over $e_1 S e_1 \setminus \{0\}$.

<u>Proof</u>: The proof is based on the fact that if one composes the g_{ij} and f_{ji} in such a way that an automorphism of $e_1 S e_1$ results then this automorphism is actually an inner automorphism giving rise to the sandwich matrix P.

<u>Definition 6.10</u>. Define elements $p_{ijk} \in S_1 = e_1 S e_1$ for each triple $(i, j, k) \in I^3$ as follows: if $e_i e_j e_k = 0$ then put $p_{ijk} = 0$, if $e_i e_j e_k \neq 0$ then p_{ijk} is determined by

$$e_i e_j e_k = e_i g_{1i}(p_{ijk}) e_i e_k.$$

<u>Remark</u>: the element p_{ijk} exists because of Step 2 (6.3) and it is unique because of (A2) and Cor. 6.4. We note that

$$e_1 p_{1ij} e_1 e_j = e_1 e_i e_j$$

and that

$$p_{1ii} = e_1 = p_{11i}.$$

Lemma 6.11. $p_{1ij} = 0$ if and only if $e_i e_j = 0$.

Proof: At this point (A5) becomes essential. If $e_i e_j = 0$, then $e_1 e_i e_j = 0$ and so is p_{1ij}. If $e_i e_j \neq 0$, then $e_1 e_i e_j \neq 0$ since $e_1 e_i \neq 0$. Hence $p_{1ij} \neq 0$, q.e.d.

Lemma 6.12. If $e_i e_k \neq 0$, then for all $b_1 \in S_1$, $f_{i1} f_{ki} g_{1k}(b_1)\, p_{1ik} = p_{1ik}\, b_1$.

Proof:

$$e_1 p_{1ik} b_1 e_1 e_k = e_1 p_{1ik}\, f_{k1} g_{1k}(b_1)\, e_1 e_k$$

$$= e_1 p_{1ik} e_1\, e_k\, g_{1k}(b_1)\, e_k$$

$$= e_1 e_i e_k\, g_{1k}(b_1)\, e_k$$

$$= e_1 e_i\, f_{ki} g_{1k}(b_1)\, e_i e_k$$

$$= e_1\, f_{i1} f_{ki} g_{1k}(b_1)\, e_1 e_i e_k$$

$$= e_1\, f_{i1} f_{ki} g_{1k}(b_1)\, p_{1ik} e_1 e_k.$$

Now (A2) implies the identity, q.e.d.

Lemma 6.13. If $e_i e_j e_k \neq 0$ then

$$p_{ijk} p_{1ik} = p_{1ij} p_{1jk}.$$

Proof:

$$e_1 p_{1ij} p_{1jk} e_1 e_k = e_1 p_{1ij} e_1 e_j e_k$$

$$= e_1 e_i e_j e_k$$

$$= e_1 e_i \, g_{1i}(p_{ijk}) \, e_i e_k$$

$$= e_1 \, f_{i1} g_{1i}(p_{ijk}) \, e_1 e_i e_k$$

$$= e_1 p_{ijk} e_1 e_i e_k$$

$$= e_1 p_{ijk} p_{1ik} e_1 e_k.$$

Again (A2) implies the formula, q.e.d.

Remark: The idea of resolving multiple products of idempotents in the indicated way goes back to E. A. Behrens [2], although he uses commutativity quite heavily.

Having thus removed the technicalities we proceed as follows:

Since $e_1 e_i \neq 0$ for all idempotents e_i, g_{1i} is defined for all indices $i \in I$. An arbitrary element $e_i a e_j$ in S can be written in the form $e_i a' e_i e_j$ by Step 2 (6.3). Since g_{1i} is an isomorphism we can express this element as

$$e_i g_{1i}(a_1) e_i e_j$$

for a uniquely determined $a_1 \in S_1$.

$S_1 \setminus \{0\}$ satisfies the conditions of Theorem 2.5. Therefore we can construct G_1 the group with zero of quotients of $S_1 \setminus \{0\}$. We claim that S can be embedded into $M = M(G_1; I, I; P)$ where the entries of the sandwich matrix are given by

$$p_{ij} = p_{1ij} \quad \text{for all } i, j \in I.$$

Define a map $\phi_1 : S \to M$ by

$$\phi_1(e_i q_{1i}(a_1) e_i e_j) = (a_1 p_{ij}; i, j) \quad \text{if } e_i e_j \neq 0$$

and

$$\phi_1(0) = 0.$$

It follows from the cancellation laws in S_1 and S that ϕ_1 is an one-to-one mapping. We have to show that it is a homomorphism.

Let $a, b \in S$ with

$$a = e_i q_{1i}(a_1) e_i e_j \quad \text{and} \quad b = e_k q_{1k}(b_1) e_k e_h .$$

We can exclude the trivial case where one of the elements is equal to zero, as well as the case where $e_j e_k = 0$, because then also $p_{jk} = 0 = p_{1jk}$ by Lemma 6.11. With regard to (A5) and Corollary 6.4 we may therefore assume that $e_i e_j e_k \neq 0$.

The following calculations will involve inverses of elements in S_1. This presents no problem, since we can always calculate in G_1 knowing that the result will end up in S_1.

$$\phi_1(a)\phi_1(b) = (a_1 p_{1ij};\ i,\ j)(b_1 p_{1kh};\ k,\ h)$$

$$= (a_1 p_{1ij} p_{1jk} b_1 p_{1kh};\ i,\ h)$$

$$ab = e_i q_{1i}(a_1) e_i e_j \cdot e_k q_{1k}(b_1) e_k e_h$$

$$= e_i q_{1i}(a_1 p_{ijk}) e_i e_k q_{1k}(b_1) e_k e_h$$

$$= e_i\, q_{1i}(a_1 p_{ijk})\ f_{ki} q_{1k}(b_1)\ e_i e_k e_h$$

$$= e_i q_{1i}\left\{a_1 p_{ijk}\ f_{i1} f_{ki} q_{1k}(b_1)\right\} e_i e_k e_h$$

$$= e_i q_{1i}\left\{a_1 p_{ijk} p_{1ik} b_1 p_{1ik}^{-1}\right\} e_i e_k e_h \quad \text{(Lemma 6.12)}$$

$$= e_i q_{1i}\left\{a_1 p_{ijk} p_{1ik} b_1 p_{1ik}^{-1} p_{ikh}\right\} e_i e_h$$

$$= e_i q_{1i}\left\{a_1 p_{1ij} p_{1jk} b_1 p_{1kh} p_{1ih}^{-1}\right\} e_i e_h \quad \text{(Lemma 6.13).}$$

Therefore

$$\phi_1(ab) = (a_1 p_{1ij} p_{1jk} b_1 p_{1kh} p_{1ih}^{-1} p_{1ih};\ i,\ h)$$

$$= \phi_1(a)\phi_1(b),$$

which finally proves Step 5.

6.14 Step 6. Under the hypothesis of Step 5

(6.9) $\phi_1(S)$ is the semicone for a directed, integral order of $M(G_1; I, I; P)$.

Proof: We show that the conditions of Theorem 5.2 are satisfied.

(1') of Remark 5.3 is satisfied because it is the exact equivalent of (A3) and because ϕ_1 is one-to-one. Equally trivially (2) is satisfied, since $\phi_1(S)$ is a subsemigroup.

re(3): We have to show that if

$$\phi_1(e_i),\ a,\ \phi_1(s),\ \phi_1(e_j)$$

are given then there exists an element $\phi(t)$ such that

$$\phi_1(e_i)a\phi_1(s)\phi_1(e_j) = \phi_1(e_i)\phi_1(t)a\phi_1(e_j).$$

Using the matrix representation we can argue

$$(e_1;\ i,\ i)(a_1;\ k,\ h)(s_1p_{mn};\ m,\ n)(e_1;\ j,\ j)$$

$$= (p_{ik}a_1p_{hm}s_1p_{mn}p_{nj};\ i,\ j)$$

$$= (t_1p_{ik}a_1p_{hj};\ i,\ j)$$

$$= (e_1;\ i,\ i)(t_1p_{ik};\ i,\ k)(a_1;\ k,\ h)(e_1,\ j,\ j).$$

Excluding the trivial case where the whole product collapses, we can always find an element $t_1 \in G_1$ like indicated, since G_1 is a group. We have to show that $t_1 \in S_1$:

In the next calculation we shall use the fact that S_1 is the cone of G_1, i.e. $S_1 x_1 = x_1 S_1$ for all $x_1 \in S_1$, and lemma 6.13:

$$t_1 = p_{ik} a_1 p_{hm} s_1 p_{mn} p_{nj} (p_{ik} a_1 p_{hj})^{-1}$$

$$= s_1' p_{ik} a_1 p_{hm} p_{mn} p_{nj} (p_{ik} a_1 p_{hj})^{-1}$$

$$= s_1' p_{1ik} a_1 p_{1hm} p_{1mn} p_{1nj} (p_{ik} a_1 p_{hj})^{-1}$$

$$= s_1' p_{1ik} a_1 p_{hmn} p_{hnj} p_{1hj} (p_{ik} a_1 p_{hj})^{-1}$$

$$\in S_1 p_{ik} a_1 S_1 p_{hj} (p_{ik} a_1 p_{hj})^{-1}$$

$$\in S_1 p_{ik} a_1 p_{hj} (p_{ik} a_1 p_{hj})^{-1} = S_1.$$

The partial order is directed because of the following reason (using Theorem 4.1):

$$(a_1;\ 1,\ 1) \subseteq (b_1;\ 1,\ 1)$$

$$(a_1;\ 1,\ 1) = (s_1;\ 1,\ 1)(b_1,\ 1,\ 1)$$

$$a_1 = s_1 b_1$$

$$a_1 \leq b_1$$

where $\leq$ denotes the partial order on G_1 defined by the cone S_1. $\leq$ is clearly directed since G_1 is the quotient group of S_1. The rest follows from (A6) and Theorem 4.3, and Step 6 has been proven.

6.15 Step 7. S without the restriction of Step 5 (6.9) can be embedded into a completely 0-simple semigroup.

Proof: For each idempotent $e_i \in E$ denote by $S(e_i)$ the following subsemigroup of S:

$$S(e_i) = \{e_k s e_h \mid e_i e_k \neq 0 \text{ and } e_i e_h \neq 0,\ s \in S\}.$$

Denote by

$$M(e_i) = M(G_i;\ I(i), I(i);\ P^{(i)})$$

the completely 0-simple semigroup constructed over $S(e_i)$ by Step 5 (6.9). We shall show that S can be embedded into the direct limit of the $M(e_i)$ in the following manner:

If $e_i e_j \neq 0$ then $e_j e_k \neq 0$ implies also $e_i e_k \neq 0$ i.e. in other words $S(e_j) \subseteq S(e_i)$. Therefore also the completely 0-simple semigroups should be somehow contained in each other. Indeed we can define certain embeddings of $M(e_j)$ into $M(e_i)$ which satisfy the usual universality property. The completely 0-simple semigroup to be constructed is then the direct limit of this system.

In the following we shall follow the terminology of G. Grätzer [6, § 21]. We now construct a direct family of completely 0-simple semigroups as follows:

The directed preordered set is given by $(E, \lesssim)$ where $\lesssim$ is defined by

$$e_j \lesssim e_i \text{ if and only if } e_i e_j \neq 0.$$

The relation is a preordering, because $e_i^2 \neq 0$ i.e. $e_i \lesssim e_i$ and if $e_k \lesssim e_j \lesssim e_i$ then $e_i e_j \neq 0$ and $e_j e_k \neq 0$. From (A5) and Corollary 6.4 we deduce that $e_i e_k \neq 0$, i.e. $e_k \lesssim e_i$. The preordering $\lesssim$ is directed because of (A6): let e_i, e_j be arbitrary: Then there exist $e_k \in E$ with $e_k e_i \neq 0$ and $e_k e_j \neq 0$, i.e. $e_i \lesssim e_k$ and $e_j \lesssim e_k$.

The semigroups in the direct family of semigroups are the $M(e_i) = M(G_i; I(i), I(i); P^{(i)})$ where the entries have the following meaning:

G_i is the group with zero consisting of the quotients of $e_i S e_i$ in the same way as G_1 was the group with zero of quotients of S_1 in Step 5.

$$I(i) = \{m \in I \mid e_i e_m \neq 0\}$$

$$p^{(i)}: I(i) \times I(i) \to G_i \quad \text{where} \quad p^{(i)}(m, n) = p_{mn}^{(i)}$$

is defined by

$$e_i e_m e_n = e_i p_{mn}^{(i)} e_i e_n.$$

It should be clear that this notation is nothing but a slight generalization of the notation used in Step 5 (6.9), confer in particular Def. 6.10. We have to keep in mind that in the general case a fixed reference point for all idempotents (like the idempotent e_1 in Step 5) is missing and that we construct $M(e_i)$ - according to Step 5 - with e_i as reference point for the idempotents in $I(i)$. It will be shown in the sequel that there exists a definite connection between the different sandwich matrices, i.e. between e.g. $p_{mn}^{(i)}$ and $p_{mn}^{(j)}$. This relationship enables us to show that $M(e_i)$ and $M(e_j)$ are isomorphic "on their intersection" and thus to construct the direct limit.

If $e_j \lesssim e_i$, i.e. $e_i e_j \neq 0$, we want to define an one-to-one homomorphism $\phi_{ji}: M(e_j) \to M(e_i)$. For this task [4, Cor. 3.12] gives the technical means: Since we do not want to permute the indices of the completely 0-simple semigroups, it is by the quoted theorem enough to give an isomorphism κ from G_j to G_i and elements

$v_m, u_n \in G_i$ such that

$$\kappa(p_{mn}^{(j)}) = v_m p_{mn}^{(i)} u_n \tag{6.16}$$

for all indices $m, n \in I(j) \subseteq I(i)$. We claim that 6.16 is satisfied if we take

$$\kappa = f_{ji}^*$$

the extension of f_{ji} (cf. Def. 6.6) to the quotient groups and

$$v_m = p_{jm}^{(i)} \quad \text{and} \quad u_m = v_m^{-1}.$$

With our choice we have

$$\phi_{ji}\{(a_j; m,n)\} = (p_{jm}^{(i)-1} f_{ji}^*(a_j) p_{jn}^{(i)}; m, n)$$

and ϕ_{ii} the identity. We have to show that (6.16) holds, i.e.

$$f_{ji}^*(p_{mn}^{(j)}) = p_{jm}^{(i)} p_{mn}^{(i)} p_{jn}^{(i)-1}$$

or

$$f_{ji}(p_{mn}^{(j)}) p_{jn}^{(i)} = p_{jm}^{(i)} p_{mn}^{(i)}.$$

This we show by the old technique:

$e_i\, f_{ji}(p_{mn}^{(j)})\, p_{jn}^{(i)} e_i e_n =$

$= e_i\, f_{ji}(p_{mn}^{(j)})\, e_i e_j e_n$ (def. of $p_{jn}^{(i)}$)

$= e_i e_j p_{mn}^{(j)} e_j e_n$ (def. of f_{ji})

$= e_i e_j e_m e_n$ (def. of $p_{mn}^{(j)}$)

$= e_i p_{jm}^{(i)} e_i e_m e_n$ (def. of $p_{jm}^{(i)}$)

$= e_i p_{jm}^{(i)} p_{mn}^{(i)} e_i e_n$ (def. of $p_{mn}^{(i)}$).

As usual (A2) gives the desired equality.

We furthermore have to show that in case $e_k \lesssim e_j \lesssim e_i$, i.e. $e_i e_j e_k \neq 0$, we have

$$\phi_{ki} = \phi_{ji}\phi_{kj}.$$

In order to prove this we have to show two lemmata:

Lemma 6.17. $f_{ji}f_{kj}(a) = p_{jk}^{(i)} f_{ki}(a)\, p_{jk}^{(i)-1}$, $a \in e_k S e_k$.

Proof: The equation is equivalent to

$$f_{ji}f_{kj}q_{ik}(a_i) = p_{jk}^{(i)} a_i p_{jk}^{(i)-1}$$

for the element $a_i \in e_i S e_i$ such that $q_{ik}(a_i) = a$. The latter equation follows immediately from Lemma 6.12 keeping in mind that e_i rather than e_1 is the reference point, i.e. we have $p_{jk}^{(i)}$ instead of p_{1jk}, q.e.d.

<u>Lemma 6.18</u>. $p_{jk}^{(i)} p_{km}^{(i)} = f_{ji}(p_{km}^{(j)})\ p_{jm}^{(i)}$.

<u>Proof:</u>

$$e_i\ f_{ji}(p_{km}^{(j)})\ p_{jm}^{(i)} e_i e_m$$

$$= e_i\ f_{ji}(p_{km}^{(j)})\ e_i e_j e_m$$

$$= e_i e_j p_{km}^{(j)} e_j e_m$$

$$= e_i e_j e_k e_m$$

$$= e_i p_{jk}^{(i)} p_{km}^{(i)} e_i e_m, \text{ q.e.d.}$$

Now for the proof that $\phi_{ki} = \phi_{ji}\phi_{kj}$:

$$\phi_{ji}\phi_{kj}(a_k, m, n)$$

$$= \phi_{ji}\ (p_{km}^{(j)-1}\ f_{kj}^*(a_k)\ p_{kn}^{(j)};\ m, n)$$

$$= (p_{jm}^{(i)-1}\ f_{ji}^*\left\{p_{km}^{(j)-1}\ f_{kj}^*(a_k)\ p_{kn}^{(j)}\right\} p_{jn}^{(i)};\ m, n)$$

$$= (p_{jm}^{(i)-1}\ f_{ji}(p_{km}^{(j)})^{-1}\ f_{ji}^* f_{kj}^*(a_k)\ f_{ji}(p_{kn}^{(j)})\ p_{jn}^{(i)};\ m, n)$$

$$= (p_{jm}^{(i)-1}\ f_{ji}(p_{km}^{(j)})^{-1}\ p_{jk}^{(i)} f_{ki}^*(a_k)\ p_{jk}^{(i)-1}\ f_{ji}(p_{kn}^{(j)})\ p_{jn}^{(i)};\ m,n)$$

by Lemma 6.17

$$= (p_{km}^{(i)-1}\ f_{ki}^*(a_h)\ p_{kn}^{(i)};\ m, n) \quad \text{by Lemma 6.18}$$

$$= \phi_{ki}\ (a_k;\ m, n)\ .$$

We have therefore shown that the semigroups $M(e_i)$ together with the maps ϕ_{ij} if $e_i e_j \neq 0$ form a directed family of semigroups. Denote with M the direct limit of this system and with

$$\phi_{i\infty}: M(e_i) \to M$$

the resulting limit maps.

It is advantageous to describe the underlying set of M. It is the disjoint union of the underlying sets of the $M(e_i)$ modulo an equivalence relation ρ which is defined as follows:

$$(a_i; m, n)\rho(b_j; q, r)$$

if and only if for some e_k with $e_k e_i \neq 0$ and $e_k e_j \neq 0$ we have

$$\phi_{ik}\{(a_i; m, n)\} = \phi_{jk}\{(b_j; q, r)\},$$

where the index i in the first component of a triple means that the whole element is in $M(e_i)$.

Since the ϕ_{ik} do not change the indices we get $m = q$ and $n = r$. Furthermore we have

$$p_{im}^{(k)-1} f_{ik}^{*}(a_i)\, p_{in}^{(k)} = p_{km}^{(k)-1}\, f_{jk}^{*}(b_j)\, p_{jn}^{(k)}$$

or in particular as

Formula 6.19:

$$(p_{im}^{(k)-1} f_{ik}^{*}(a_i)\, p_{in}^{(k)};\, m,\, n)\ \rho\ (a_i; m, n).$$

Denote the equivalence class of the element $(a_i: m, n)$ by $[a_i; m, n]$. The multiplication between two equivalence classes $[a_i: m, n]$ and $[b_j; q, r]$ is defined as follows: Take e_k with $e_k e_i \neq 0$ and $e_k e_j \neq 0$ and elements a_k and b_k such that

$$[a_k; m, n] = [a_i; m, n] \quad \text{and}$$

$$[b_k; q, r] = [b_j; q, r].$$

Then the product is defined as $[a_k p^{(k)}_{nq} b_k; m, r]$.

Now we can show that M is completely 0-simple: let a, b be arbitrary non-zero elements in M. By the previous remark we may assume that

$$a = [a_k; m, n] \quad \text{and} \quad b = [b_k; q, r].$$

Since $M(e_k)$ is completely 0-simple there exist elements x, y $M(e_k)$ such that

$$x(a_k; m, n)y = (b_k; q, r) \in M(e_k).$$

Hence

$$\phi_{k\infty}(x)\,[a_k; m, n]\,\phi_{k\infty}(y) = [b_k; q, r] \in M$$

and M is 0-simple.

Let f, g be idempotents in M such that $fg = gf = g$. Again this gives rise to an equivalent statement in one of the $M(e_k)$. Keeping in mind that

$\phi_{k\infty}$ is a one-to-one mapping we conclude that $q = 0$ or $f = q$. So M is completely 0-simple.

Denote by $\phi_i: S(e_i) \to M(e_i)$ the embedding constructed in step 5 (6.9). Define a map $\psi: S \to M$ as follows

$$\psi(e_i a e_j) = \phi_{i\infty}\phi_i(e_i a e_j) \quad \text{for} \quad e_i a e_j \neq 0 \quad \text{and}$$

$$\psi(0) = 0 .$$

We claim that ψ is a homomorphism:

Let $e_i a_i e_i e_j$ and $e_k b_k e_k \cdot e_h$ represent two different, arbitrary elements in S with $a_i \in e_i S e_i$ and $b_k \in e_k S e_k$. Then

$$\psi(e_i a_i e_i e_j \cdot e_k b_k e_k e_h)$$

$$= \phi_{i\infty}\{\phi_i(e_i a_i e_i e_j)\cdot\phi_i(e_k b_k e_k e_h)\}$$

$$= \phi_{i\infty}\{(a_i p_{ij}^{(i)};\ i,\ j)\cdot\phi_i(e_k\ q_{ik} f_{ki}(b_k)\ e_k e_h)\}$$

$$= \phi_{i\infty}\{(a_i;\ i,\ j)(f_{ki}(b_k)\ p_{kh}^{(i)};\ k,\ h)\} \quad (\text{note: } p_{ij}^{(i)} = e_i)$$

$$= \phi_{i\infty}\{(a_i p_{jk}^{(i)}\ f_{ki}(b_k)\ p_{kh}^{(i)};\ i,\ h)\}$$

$$= \left[a_i p_{jk}^{(i)}\ f_{ki}(b_k)\ p_{kh}^{(i)};\ i,\ h\right]$$

$$\psi(e_i a_i e_i e_j)\ \psi(e_k b_k e_k e_h)$$

$$= \left[a_i;\ i,\ j\right]\left[b_k;\ k,\ h\right]$$

$$= \left[a_i;\ i,\ j\right]\left[p_{kk}^{(i)-1}\ f_{ki}(b_k)\ p_{kh}^{(i)};\ k,\ h\right] \text{ by (6.19)}$$

$$= \left[a_i p_{jk}^{(i)} f_{ki}(b_k)\ p_{kh}^{(i)};\ i,\ h\ \right]$$

ψ is furthermore one-to-one: if

$$\psi(e_i a_i e_i e_j) = \psi(e_k b_k e_k e_h)$$

then $\left[a_i;\ i,\ j\right] = \left[b_k;\ k,\ h\right]$. This implies first of all that $i = k$ and $j = h$ and since ϕ_i and $\phi_{i\infty}$ are both one-to-one we get $a_i = b_k$.

So S is actually embedded into M. We proceed to show that $\psi(S)$ is the semicone of a directed integral order of M having $\psi(E)$ as integral idempotents. As usual we verify conditions (1'), (2) and (3) of theorem 5.2. For (1') and (2) there is nothing to show.

Let then $\psi(e)a\psi(s)\psi(f)$ be given. This has the form

$$\left[e_i;\ i,\ i\right]\left[a_k;\ m,\ n\right]\left[s_h;\ h,\ \ell\right]\left[e_j;\ j,\ j\right].$$

If e_l is such that $e_l e_i \neq 0,\ e_l e_k \neq 0,\ e_l e_h \neq 0$ and $e_l e_j \neq 0$, we have in $M(e_l)$:

$$(e_1;\ i,\ i)(p_{km}^{(1)^{-1}} f^*_{k1}(a_k)p_{kn}^{(1)};\ m,\ n)\cdot$$

$$(p_{hh}^{(1)^{-1}} f^*_{h1}(s_h)p_{h\ell}^{(1)};\ h,\ \ell)(e_1;\ j,\ j).$$

Since $(p_{hh}^{(1)^{-1}} f^*_{h1}(s_h)p_{h\ell}^{(1)};\ h,\ \ell) = (f_{h1}(s_h)p_{h\ell}^{(1)};\ h,\ \ell)$ $\in \phi_1(S(e_1))$ and since in $M(e_1)$ we have condition (3) satisfied we can find an element $t = e_i g_{1i}(t_1)e_i e_m \in S(e_1) \subseteq S$ such that

$$(e_1;\ i,i)(p_{km}^{(1)^{-1}} f^*_{k1}(a_k)p_{kn}^{(1)};\ m,\ n)\cdot$$

$$\cdot(f_{h1}(s_h)p_{h\ell}^{(1)};\ h,\ \ell)(e_1;\ j,\ j) =$$

$$= (e_1;\ i,\ i)(t_1p_{im}^{(1)};\ i,\ m)(p_{km}^{(1)^{-1}} f^*_{k1}(a_k)p_{kn}^{(1)};\ m,\ n)\cdot$$

$$\cdot(e_1;\ j,\ j)$$

Applying the homomorphism $\phi_{1\infty}$, the limit map, on this equation we get:

$$\phi(e)a\phi(s)\psi(f) =$$

$$= \phi(e)\left[t_1p_{im}^{(1)};\ i,\ m\right] a\phi(f)$$

we have to show that $\phi(t) = \left[t_1\ p_{im}^{(1)};\ i,\ m\right]$ or, what is the same, that $(g_{1i}(t_1);\ i,m)\rho(t_1p_{im}^{(1)};\ i,m)$. This follows from Formula 6.19 as follows:

$$(t_1 p_{im}^{(1)}; i,m) = (p_{1i}^{(1)^{-1}} f_{i1}^{*} g_{1i}(t_1) p_{im}^{(1)}; i, m)\rho$$

$$\rho(g_{1i}(t_1); i, m)$$

so we have indeed that

$$\psi(e)a\psi(s)\psi(f) = \psi(e)\psi(t)a\psi(f).$$

It remains to show that the integral order on M (now defined) is directed. The usual pattern again applies: given two elements, find a semigroup $M(e_k)$ in which both of them are contained. In this semigroup $M(e_k)$ there exists an element greater or equal than both elements, so the image under $\phi_{k\infty}$ of this element will be greater or equal than both the original elements, q.e.d.

§7 The Semiprime Case and the Case without Zero

The whole theory so far developed becomes particularly satisfying for semiprime semigroups and semigroups without a zero.

If one defines a semiprime semigroup as a semigroup having no nilpotent elements, then a semiprime primitive regular semigroup can be represented as a 0-disjoint union of completely 0-simple semigroups which have no zero-divisors, i.e. each component is a completely simple semigroup with a zero adjoint to it. This property reflects itself in the sandwich matrix by the fact that then all entries are different from zero. Let $p_{\lambda i}$ be arbitrary. Since $(e, i, \lambda) \neq 0$ we must have $(p_{\lambda i}, i, \lambda) = (e, i, \lambda)^2 \neq 0$, so $p_{\lambda i} \neq 0$. This strong restriction on the sandwichmatrix, which governs the behaviour of the idempotents, allows us to prove

<u>Theorem 7.1.</u> The semiprime semigroup S is the subsemigroup of the integral elements of an integrally ordered, semiprime, primitive regular semigroup if and only if S satisfies the following conditions:

(B1) S has enough idempotents.

(B2) For each idempotent e the set $eSe\setminus\{0\}$ is a cancellative subsemigroup.

(B3) If $ab = e = e^2 \neq 0$ then $a = b = e$

(B4) $eaSf = eSaf$ for all $a \in S$ and all idempotents e and f.

(B5) The idempotents are categorical at zero.

<u>Proof</u>: Since (B1) - (B5) is a weaker system than (A1) - (A5) of Theorem 6.1, necessity is clear. In order to show the sufficiency of the conditions we make two short remarks beforehand:

(1) $ef \neq 0$ if and only if $fe \neq 0$: if $ef \neq 0$ then, since S is semiprime also $efef = (ef)^2 \neq 0$. Hence $fe \neq 0$. In particular we have $efe \neq 0$ whenever $ef \neq 0$.

(2) (A2) of Theorem 6.1 is satisfied: let $ef \neq 0$ be given and let $eae \cdot fbf = eae \cdot fcf$. By multiplication from the right with e we get

$$eae \cdot efbfe = eae \cdot efcfe.$$

Since eSe is cancellative (B2) we get

$$efbfe = efcfe$$

$$fef \cdot fbf \cdot fef = fef \cdot fcf \cdot fef$$

$$fbf = fcf.$$

Similarly we show that if $eae \cdot fbf = ece \cdot fbf$ then $eae = ece$.

Define a relation Q on the set of idempotents E by

$$e_i Q e_j \text{ if and only if } e_i e_j \neq 0.$$

In the semiprime case this is an equivalence relation since it is clearly reflexive and by remark (1) also symmetric.

Q is transitive: if $e_i Q e_j Q e_k$ then by remark (1) we have $e_i e_j \neq 0$ and $e_j e_k \neq 0$. Now (R5) implies that $e_i e_j e_k \neq 0$ and by Corollary 6.4, which clearly can be deduced in this context, we have $e_i e_k \neq 0$.

In each of the equivalence classes of Q we have the following situation: If $\overline{e_1}$ is such a class and $e_i \in \overline{e_1}$ an arbitrary element in there, then $e_1 e_i \neq 0$, i.e. the hypothesis of Step 5 (6.9) is satisfied and we can embed

$$S(e_1) = \{e_i S e_j \mid e_1 e_i \neq 0 \text{ and } e_1 e_j \neq 0;\ s \in S\}$$

$$= \{e_i S e_j \mid e_i,\ e_j \in \overline{e_1};\ s \in S\}$$

into $M(e_1)$. Defining then

$$M(e_i) M(e_j) = 0 \text{ if } \overline{e_i} \neq \overline{e_j}$$

we get the theorem, q.e.d.

We achieve the best analogy to the theory of partially ordered groups in the case of completely simple semigroups:

Theorem 7.2. The semigroup S is the subsemigroup of the integral elements of some integrally ordered completely simple semigroup if and only if S satisfies the following conditions:

(C1) There exist enough idempotents

(C2) For each idempotent e the subsemigroup eSe is cancellative

(C3) if $ab = e = e^2$ then $a = b = e$

(C4) $eaSf = eSaf$ for all $a \in S$ and all idempotents e and f.

Proof: The proof follows immediately from the semiprime case.

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