STUDIES IN NONLOCAL POTENTIA

· · ·

STUDIES IN NONLOCAL POTENTIALS

Bv

YOGESHWAR SINGH, M.Sc.

🗇 A Thesi's

Submitted to the School of Graduate Studies in Partial Fulfilment of the Requirements

> for the Degree Doctor of Philosophy

McMaster University October 1972 DOCTOR OF PHILOSOPHY (1972) (Physics)

McMASTER UNIVERSITY Hamilton, Ontario.

TITLE: Studies in Nonlocal Potentials AUTHOR: Yogeshwar Singh, B.Sc. (Lucknow University)

M.Sc. (Lucknow University)

SUPERVISOR: Professor R. K. Bhaduri

NUMBER OF PAGES: viii, 142

SCOPE AND CONTENTS:

The two-body elastic scattering data and the binding energy of the deuteron do not uniquely determine the nucleonnucleon interaction. Many potentials, which fit these data, may give different results in many-body problems due to their different off-energy-shell behaviour. It is also expected that the nucleon-nucleon interaction would be nonlocal at short-relative distances and local at comparatively larger $(r \ge 2 \text{ Fm})$ distances. Thus it is of interest to construct and study partly nonlocal potentials.

We have generalised the Jost-Pais theorem for nonlocal central and noncentral potentials. A method, based on the formalism of Fuda, has been developed to construct partly nonlocal central phase equivalent potentials. These potentials, constructed in the S-state, have an attractive local part, superimposed with a short range repulsive rank-one separable potential. We study the behaviour of such partly nonlocal potentials in nuclear matter, with a view to examine if these differ significantly from corresponding phase equivalent separable potentials.

We have also studied the dependence of the threshold cross section for the reaction $p+p + \pi^++d$ on the nature of the deuteron wave function. Phase equivalent potentials with varying D-state probability of the deuteron have been generated using unitary transformations on the two-body Hamiltonian which includes a local or separable potential. It has been shown that the cross section depends sensitively not only on the D-state probability of the deuteron but also on the form of the wave function at short relative distances, and hence to the off-shell behaviour and the nonlocality of the interaction.

(iii)

ACKNOWLEDGMENTS

The guidance and encouragement of Professor R. K. Bhaduri during-the course of this work have been invaluable. I am grateful to him.

I am deeply indebted to Dr. C. S. Warke for his continued interest, advice and assistance.

I am grateful to Dr. H. C. Pradhan and Professor D. W. L. Sprung for their help and criticism.

I thank Dr. Y. Nogami and Dr. M. L. Tiku, members of my supervisory committee for their constructive suggestions.

It is a pleasure to thank my colleagues and friends who made my stay at McMaster a memorable one.

The financial assistance given by the McMaster University and the study leave granted by the Roorkee University, are gratefully acknowledged.

Finally, I thank Miss Erie Long for her swift and competent typing of the thesis.

(iv)

TABLE OF CONTENTS

X

REFERENCES

CHAPTER		PAGE
I	INTRODUCTION	1
II	JOST FUNCTION IN SCATTERING THEORY	15
III	CONSTRUCTION OF PARTLY NONLOCAL POTENTIALS	40
IV	NUCLEAR MATTER AND DEFECT WAVE FUNCTIONS	70
v	OFF-SHELL EFFECTS IN THE REACTION	•
I	$p+p \rightarrow \pi^+ + d$ AT THRESHOLD	90 ⁻
VI	CONCLUSIONS	115
APPENDI	X	
• A	GENERALISATION OF JOST-PAIS THEOREM TO	
• • •	NONLOCAL NONCENTRAL POTENTIALS	118
. B	CALCULATION OF n(k) APPEARING IN III-2.18	124

127

(v)

LIST OF FIGURES

FIGURE 1: Half-off-shell R-matrix elements R(p,k;k²) given by III-3.1 are plotted. These plots are for A0, A3, B0, B3 and Standard Tabakin (ST) potentials. k is taken to be 0.7 Fm⁻¹.

FIGURE 2: Half-off-shell R-matrix elements R(p,k;k²) given by III-3.1 are plotted. These plots are for A0, A3, B0, B3 and Standard Tabakin (ST) potentials. k is taken to be 1.0 Fm⁻¹.

FIGURE 3: Diagonal matrix elements <k |V|k> of class A potentials given by III-2.15 and the phaseequivalent class B potentials given by III-2.19 are plotted. Figure 3(a) displays the pair A0,B0 for which the attractive parts are also phaseequivalent. Figures 3(b) - 3(d) are for the pairs A1,B1; A2,B2 and A3,B3 respectively.

FIGURE 4:

The bound state wave function $\phi_B(k)$ generated by the attractive Bargmann potential III-2.15 for the limiting case A3 and the corresponding $\eta(k)$ which is orthogonal to $\phi_B(k)$ are displayed. Also shown, for comparison, are the bound state wave function $\phi_B^*(k)$ generated by the attractive component of B3 and the corresponding repulsive form factor $\gamma(k)$.

(vi)

FIGURE 5: The defect wave function $\chi_{k_0}(r)$, defined by IV-2.5, is plotted against r for the potentials A0,A2 and B0,B2. The wave number k_0 in each case is taken to be at 0.55 k_F^S , where k_F^S is the saturation value shown in Table 4.1. FIGURE 6: Off-diagonal matrix elements, $G(k,k_0;\overline{K})$ and $V(k,k_0)$ given in IV-1.13, are plotted for the potentials A0,A3 and B0,B3. k_0 is taken as 0.55 k_F^S . Fermi momentum at saturation (k_F^S) is given in Table 4.1.

IST OF TABLE

t

TABLE	PAGE
3.I.	Parameters for class A and class B potentials 61
4.1	Nuclear matter results for class A, class B
a .	and Standard Tabakin (ST) potentials 86
5.1	Parameters of unitary transformations in
	$^{3}S_{1}^{-3}D_{1}$ channel
5.2	Cross section for threshold s-wave pion •
· .	production for various separable and their
•	phase equivalent potentials 107
~_ 5.3	Cross section for threshold s-wave pion
	production for Reid-soft-core and its phase
	equivalent potentials 110
5.4	Radial integrals $(I_1 \dots I_6)$, defined in
	V-1.9, for various potentials

(viii)

CHAPTER I

INTRODUCTION

The study of the two-body nucleon-nucleon (N-N) interaction is of central importance for the understanding of nuclear properties on a microscopic viewpoint. Although in a system of many nucleons there may exist three- and many-body forces, the role of the NN interaction remains predominant. Generally, in nuclear physics nucleons are considered as nonrelativistic particles interacting through the N-N interaction which is represented by a potential. Experimental information about the N-N interaction is obtained from nucleon-nucleon scattering experiments. This is supplemented by the properties of the deuteron which is the only bound state of the two nucleon system. Analyses of these data yield phase-shifts 1,2) for the different partial waves contributing to the N-N scattering. These scattering experiments and deuteron properties strongly indicate that N-N interaction has the following broad features. the

 The N-N interaction is of short range (~2.5 Fm) and is charge independent.

It is spin dependent and has a noncentral component.
 It is repulsive in nature at small relative distances.

ø

If we incorporate these features of the N-N interaction in a potential we can explain the scattering data and the properties of the deuteron. However, for our study, we made the simplification that the potential acts only in the S-state and is spin independent. Since the deuteron is a very loosely bound system, and the interaction in the ${}^{1}S_{0}$ state falls just short of binding the n-p system, the above assumption is not a drastically unrealistic one.⁴

We shall often use the term "low energy" for energies below 10 MeV lab. In this range the low energy parameters like scattering length, effective range and the shape parameter are sufficient to describe the experimental phase shifts. We shall use the term "high energy" to mean energies in the vicinity of the threshold of pion production, which is ~280 MeV lab. Although at higher energies the phase shifts are not real because of pion production, the experimental elastic cross sections are fitted with the real phase shifts to obtain a phenomenological potential. In the study of the phase equivalent potentials (defined later), we assume the existence of real phase shifts at all energies and obtain a real potential to be used in the region where phase shifts are real.

A number of phenomenological potentials $^{3-6)}$ have been constructed to fit the phase shift data and to reproduce the properties of the deuteron. A theoretical





basis is sought for such potentials $^{7-13}$ by using field théoretical considerations in their derivations. It is supposed that the long range (>2.5 fm) part of the N-N interaction is given by the one-pion-exchange potential (OPEP). This is confirmed by the fact that the phase shifts for higher partial waves (L \gtrsim 4) in the entire energy range 0-300 MeV lab are very well reproduced by the one-pionexchange potential alone.

The situation for the short range part (r < 1 fm) of the interaction is not so clear. To probe the short range part of the interaction one requires scattering data at higher energies, which are limited due to the complications arising from 1) meson production and 2) the contribution due to an increasing number of partial waves taking part in the scattering. At high energies the validity of the potential theory is itself questionable, because the multiparticle production and relativistic effects cannot be accounted for in a static potential approach. It appears that the short range part of the interaction mainly arises due to multipion exchange processes when the recoil of the nucleons can hardly be ignored. Under the circumstances the N-N interaction cannot be represented by one radial variable r, i.e., the relative separation of two interacting nucleons. It will also depend on the immediate vicinity of the two interacting nucleons to account for the effects of

the recoil. Thus the N-N interaction should be represented by V(rr') which depends on two variables r and r'. Note that V(rr') has the dimensions of energy density rather than that of energy. The wave function $\psi(r)$ at any point r will now depend on its value at other neighbouring points r' through the interaction V(rr'). The Schrödinger equation for such an interaction is

$$\frac{\hbar^2}{M} \nabla^2 \psi(\mathbf{r}) + \int V(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}' = E\psi(\mathbf{r}) , \quad I-1.1$$

where M/2 is the reduced mass of the two nucleons. The kernel of this integral equation is V(rr') which is a nonlocal potential. Since V(rr') is no longer a diagonal matrix, it is not possible to obtain a solution analytically, in general, for any arbitrary form of V(r,r'). However, if we impose the condition that the potential is local, i.e., $V(rr') = V(r)\delta(r-r')$, we obtain the familiar form of the Schrödinger equation from I-1.1, which is

Thus the local potential $V(r) \delta(r-r^*)$ is a limiting form of a general nonlocal potential. The other convenient representation of the nonlocal potential is the separable potential. This form in configuration space is defined as



 $V(\mathbf{r},\mathbf{r}') = \sum_{i} g_{i}(\mathbf{r}) g_{i}(\mathbf{r}')$

بعسح

I-1.3

In practice one terminates this series after two or three The form of g(r) is rather arbitrary, and different terms. terms may have different forms. These potentials are also separable in momentum space. Separable potentials are very convenient from the mathematical point of view because these reduce I-1.1 into a set of algebraic equations which are easy to solve. However, physically such potentials are not a good representation because the nonlocality is diffused over the whole range of the interaction. Many separable potentials 15-18) have been constructed to reproduce the two-body phase shift data and the deuteron binding energy, but these do not reproduce other deuteron properties like form factors, D-state probability etc. realistically. Some separable potentials have been constructed by Serduke 19) which fit the two body elastoc scattering data and also have fairly realistic deuteron properties. Separable potentials have been used in different many-body calculations like nuclear matter ²⁰⁾, triton ²¹⁾ and finite nuclei ²²⁾. These potentials, because of their simplicity, are almost exclusively used for the study of three-body 23-24) problems in nuclear physics.

I-2

The two body scattering data (even if known at all energies) are, in general, not sufficient to determine a unique potential. However, if certain restrictions are imposed on the potential, i.e., it is local and does not support any bound state, a unique potential for each partial wave can be obtained. This procedure of obtaining the potential starting from the phase shifts is called the inverse problem in scattering. For local potentials one can use the method of Gelfand and Levitan ²⁵⁾ or of Marchanko ²⁶⁾ In the presence of a bound state, a family of potentials can be generated reproducing the given phase shifts and the bound state energy. The inverse scattering problem is also studied for separable 27-29 and Yukawa type 30 potentials. When, as in the realistic situation, the phase shift changes sign at high energy, the inverse scattering problem cannot be solved in general for a one-term separable potential. At least two terms are required to incorporate the sign change for the phase shift*. Since the phase shifts at high energies

*Tabakin ¹⁶⁾ has constructed a one-term separable potential fitting the experimental phase shifts. This potential has a positive energy bound state in ¹S₀. are not known experimentally, many potentials with different high energy extrapolations can be constructed. Some separable potentials with very different high energy phase shifts have been studied by us ³¹⁾ in nuclear matter. We observe that the nuclear matter binding energy is sensitive to the changes in the high energy phase shifts provided the form factor of the separable potential is of very short range. Fiedeldey ³²⁾ has done a similar study for triton and found that the binding energy phase shifts.

In two-body elastic scattering, the two-particle energy is conserved, therefore, such data can only provide the on-energy-shell (on-shell) transition-matrix (T-matrix) elements defined as

$$\langle \underline{p} | T(w) | \underline{k} \rangle = \langle \underline{p} | V | \underline{k} \rangle + \langle \underline{p} | V \frac{1}{w - H_0 + i\varepsilon} V | \psi_{\underline{k}}^{+} \rangle$$
, I-2

with $|\mathbf{p}|^2 = |\mathbf{k}|^2 = |\mathbf{w}|$. Using the partial wave expansion (discussed in Chapter II) the on-shell T-matrix element for the ℓ^{th} partial wave can be directly related to the differential cross section and obtained from the experimental data. However, in a many-body system, the scattering of two particles does not conserve energy, because other particles can share some of the energy. Thus one requires the knowledge of the off-energy-shell (off-shell) T-matrix elements, defined by $|\mathbf{p}|^2 \neq |\mathbf{k}|^2 \neq |\mathbf{w}|$, for such a study.

.1

These can be determined uniquely only if some form of the two-nucleon potential is assumed. This is equivalent to assuming some specific extension of the on-shell elements of the T-matrix to the off-shell region. There can be infinitely many such potentials as pointed out by Ekstein 33, all of them would be phase equivalent, i.e., generating the same phase shifts at all energies but having different off-shell behaviour. All such potentials are called phase shift equivalent' (phase equivalent) potentials. This arbitrariness of the off-shell behaviour automatically introduces some model dependence in the many-body calculations. In principle, such a model dependence can be removed if we had complete knowledge of the off-shell behaviour. Experiments like electron-deuteron scattering, neutrondeuteron scattering, proton-proton or neutron-proton bremsstrahlung (ppy, and npy) etc. which depend on the offshell behaviour can provide such information. Some of these experiments are difficult to perform and equally difficult to interpret. The cross section is dominated by elastic processes. Proton-proton bremsstrahlung (ppy) experiments are comparatively easy to interpret because the final state of this reaction consists of only two strongly interacting particles and a photon which interacts only through the electromagnetic interaction. Typical p-p bremsstrahlung cross section is smaller by a factor of 10⁴ than that of p-p elastic scattering. Therefore, a good statistical

accuracy for such experiments is hard to achieve. Experimental data 34-36) for incident protons of (10 - 150 MeV lab) are available and a number of calculations with different potential models 37-39) have been done to estimate the offshell effects. The present feeling is that these experiments at these limited energies are not reliable indicators of the off-shell effects. For these experiments to be useful in discriminating between various potential models, they should be performed at high enough energies such that the faroff-shell region is explored. Other experiments like neutrondeuteron scattering at high energies are more difficult to interpret because the final state contains more than two strongly interacting particles. Apart from these experimental difficulties, there is another problem; namely the analytic properties of the off-shell T-matrix are not fully known. Thus an explicit parameterization of the off-shell elements is not possible. The other alternative is to study phase equivalent potentials which may give some suitable criterion to restrict the off-shell behaviour. The earliest studies 40-42 of this problem were made with potentials which were not exactly phase equivalent, their phase shifts roughly coinciding up to 300 MeV lab, and these were used to find out the differences in the binding energy and the saturation property of nuclear matter, or to investigate the properties of the triton. Such differences may not be rigourously

ascribed to the variations in the off-shell properties because these potentials have different on-shell elements at high energy.

There are some practical methods to construct phase equivalent potentials. In one method used by Fiedeldey 43), a family of phase equivalent rank-two separable potentials is obtained by choosing, arbitrarily; one of the potential. form factors and calculating the other using the given phase shift data. This is done by solving the inverse scattering problem. He constructs these potentials to reproduce the phase shift and bound state energy given by the Tabakin 42) potential. These potentials are then used to study the changes in the binding energy of triton ²¹⁾. Some of these potentials were studied by us 44) to calculate nuclear matter binding energy. Another method has been followed by Coester et al. 45, and Haftel and Tabakin 46. In this method a family of phase equivalent potentials are generated using unitary transformations on a two-body Hamiltonian. The twobody Hamiltonian may be chosen with any local or nonlocal potential. The rank-two separable type of unitary transformations are used in these calculations; some details of these transformations are given in Chapter V. Coester et al. 45) used these transformed potentials to calculate the binding energy in nuclear matter and found that it increased for those potentials which introduce less distortion in the relative two-body wave function. Haftel and Tabakin 47) used

these transformations on the Reid-soft-core potential to generate a set of potentials, which are used in nuclear matter calculations. They found that the binding energy and saturation density of nuclear matter change significantly for the transformed potentials. Some of these potentials are recently used to calculate the ground state energy of the oxygen nucleus $^{48)}$ and binding energy of triton $^{49)}$. A third method to generate a family of momentum-dependent $(p^2$ dependent) phase equivalent potentials 50-52 is to use the short range radial scale distortions on a two-body Hamilton-Such potentials have been used in nuclear matter 53,45) ian. proton-proton bremsstrahlung $\frac{54}{100}$ and deuteron photon disintegration ⁵⁵⁾. Srivastava ⁵⁶⁾ has studied the off-shell behaviour of some p²-dependent potentials in comparison with local and separable phase equivalent potentials. He also constructed ⁵⁷⁾ some local potentials phase equivalent to separable potentials of Tabakin 15 and of Mongan 17 using the Marchanko method.

We are interested in constructing such phase equivalent potentials which are attractive and local at long range, and are repulsive and nonlocal at short range. This form of the potentials is predicted from meson theoretical considerations. We call such potentials partly nonlocal. The method used by us is based on the formalism of Fuda ⁵⁸⁾ and is very ensemblent for our purpose. In the next section we describe the plan that is followed in this thesis.

I-3

We have studied nuclear matter and the threshold cross section of the reaction $p+p \rightarrow \pi^+ + d$ with phase equivalent potentials, to determine the changes produced due to the different off-shell behaviour. We also investigate the variations in the results of nuclear matter when partly nonlocal potentials are replaced by separable rank-two phase equivalent potentials. Since we use a method to construct the phase equivalent potentials which requires information about Jost; functions for nonlocal potentials, we have given some simple properties of the Jost function for the local potentials in Chapter II. Unlike the local situation, the Jost function for a nonlocal potential is not equivalent to the Fredholm determinant of the kernel of the Lippmann-Schwinger equation. The expression for the S-state Jost function for a nonlocal central potential was derived by Warke and Bhaduri ⁵⁹⁾, we have generalised this result for higher partial waves in Chapter II. Furthermore, in Appendix A, we derive the expressions for Jost functions for nonlocal, noncentral potentials' as well. The basic mathematical framework for the two-body scattering problem is also developed /briefly in Chapter II.

Our method of generating phase equivalent potentials is given in Chapter III. This chapter also includes the construction of separable rank-two phase equivalent potentials using our method, and a discussion of some features of these potentials. We have not used a realistic form of the local potential because in that case analytical solutions are not possible, instead we have chosen a Bargmann⁶⁰⁾ potential as the local part. This potential has analytical solutions for the scattering and bound state problem. The short range repulsive part is always taken to be a rank-one separable potential. The partly nonlocal and rank-two separable potentials are used in nuclear matter to calculate the binding energy and the saturation density. The wound integrals for all these potentials are also calculated. The details of this calculation and the results are reported in Chapter IV. These calculations are confined only to the S-state, because the S-state contributes most to the binding energy of nuclear matter⁶¹⁾

In Chapter V we studied the effect of the off-shell behaviour in the production cross section of the reaction $p+p + \pi^+ + d$, near threshold. It was suggested by Thomas and Afnan ⁶²⁾ that the study of the cross section for the reaction $p+p + \pi^+ + d$ near threshold may give considerable information about the tensor component of the N-N interaction, and thus restrict the D-state probability in the deuteron to a more precise



value than the present one $(4 \sim 8\%)$. We investigate this problem with different phase equivalent potentials. Some of our potentials are phase equivalent to RSC and others are separable potentials ⁶³⁾ with varying D-state probability. This study is undertaken with the intention to see how far the off-shell behaviour alters the cross section and whether it is really possible to pin down the D-state probability through such a study.

Finally we summarize our conclusions in the last chapter. Two appendices are given to supplement Chapters II and III.

CHAPTER II

JOST FUNCTIONS IN SCATTERING THEORY

The Jost functions are important for the study of the analytical properties of the scattering matrix (S-matrix) and in potential theory. Recently 58) these have also been used to generate phase-equivalent potentials. Jost and Pais $^{64)}$, as well as Newton ⁶⁵⁾, have studied the analytical properties of the Jost functions for local potentials and their relationship with the S-matrix. In this chapter, we shall examine the Jost function for a nonlocal potential and derive some new relations. A brief review of the mathematical framework for the scattering theory is given in Section II-1. Section II-2 deals with the definition of the Jost functions, their analytical properties and their relation with the S-matrix. Finally in Section II-3, we derive the expressions for the Jost function of a nonlocal central potential for any partial wave. The more general Jost function of a nonlocal, noncentral potential is given in Appendix A.



II-1 Mathematical Framework

We start with the time independent Schrödinger equation for two particles in the center of mass system:

$$(\underline{v}^2 + \underline{k}^2)\psi_{\underline{k}}(\underline{r}) = \frac{2u}{h^2} \int \langle \underline{r} | u | \underline{r}' \rangle \psi_{\underline{k}}(\underline{r}') d\underline{r}'$$
; II-1.1

where r and r' are the relative coordinates, u the two body potential, and μ the reduced mass. The above equation can be rewritten by defining

$$\langle \underline{r} | \underline{v} | \underline{r}' \rangle = \frac{2u}{\hbar^2} \langle \underline{r} | u | \underline{r}' \rangle$$
, II-1.2

as

$$(\nabla^2 + k^2)\psi_k(r) = \int \langle r|V|r' \rangle \psi_k(r')dr'$$
 II-1.3

In momentum space, II-1.3 is equivalent to the expression

$$(k^{2} - p^{2})\psi_{k}(p) = \int \langle p | v | p' \rangle \psi_{k}(p') dp' \qquad \text{II-1.4}$$

The transformation from r-space to momentum space is such

that

Ţ.

$$\langle \mathbf{r} | \mathbf{k} \rangle \langle \mathbf{k} | \mathbf{r}' \rangle d\mathbf{k} = \delta (\mathbf{r} - \mathbf{r}')$$

It will be convenient to replace II-1.3 by the Lippmann-Schwinger ⁶⁶⁾ equation, which incorporates the boundary condition that at large distances the scattering wave function consists of a plane wave superimposed with the outgoing spherical wave. In this form the outgoing scattering wave $|\psi_k^+\rangle$ is given by

$$|\psi_{k}^{+}\rangle = |k\rangle + (k^{2} - H_{0} + i\epsilon)^{-1} V |\psi_{k}^{+}\rangle$$
 II-1.5

where H_0 is the kinetic energy operator in the units of $2\mu/\hbar^2$. The S-matrix and the transition matrix (T-matrix) are defined by the following equations ⁶⁷⁾.

= $\langle p|k \rangle - 2\pi i \delta (p^2 - k^2) \langle p|T(k^2)|k \rangle$

$$\langle \underline{p} | \mathbf{s} | \underline{k} \rangle = \langle \psi \underline{p} | \psi \underline{k} \rangle$$

with

$$\langle \underline{p} | T(k^2) | \underline{k} \rangle = \langle \underline{p} | V | \psi_{\underline{k}}^+ \rangle$$

$$= \langle \underline{\mathbf{p}} | \mathbf{V} | \underline{\mathbf{k}} \rangle + \langle \underline{\mathbf{p}} | \mathbf{V} \frac{1}{\mathbf{k}^2 - \vartheta \mathbf{H}_0} + \mathbf{i} \varepsilon = \mathbf{V} | \psi_{\mathbf{k}}^+ \rangle$$
III-1.7

Note that the S-matrix elements $\langle p|S|k \rangle$ are defined only on the energy-shell, i.e., $p^2 = k^2$, while the definition II-1.7 for the T-matrix elements holds even for cases when $p^2 \neq k^2$.

N

II-1.6

Using II-1.5 in r-space, we get, for V = 0, the wave function

$$\langle \mathbf{r} | \mathbf{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

we expand the plane wave $e^{i \underbrace{k} \cdot \underbrace{r}{2}}$ into its partial wave components as follows

$$\tilde{e}^{ik \cdot r} = \sum_{\ell=0}^{\infty} i^{\ell} (2\ell+1) \frac{u_{\ell}(kr)}{kr} p_{\ell}(\cos\theta) , \qquad \text{II-1.9}$$

where θ is the angle between the vectors \hat{k} and \hat{r} . The coefficients $u_{\ell}(kr)$ are the Riccati-Bessel functions defined in terms of the cylindrical Bessel functions $^{68)} J_{11}(2)$, as

$$u_{\ell}(Z) = Zj_{\ell}(Z) = \sqrt{\frac{\pi Z}{2}} J_{\ell+\frac{1}{2}}(Z)$$
. II-1.

Z)

These can be decomposed into the outgoing and the incoming wave; using the Riccati-Hankel functions, as follows:

$$w_{\ell}^{+}(Z) = ie^{i\pi\ell} Zh_{\ell}^{(1)}(Z)$$
$$= ie^{i\pi\ell} \sqrt{\frac{\pi Z}{2}} H_{\ell+\frac{1}{2}}^{(1)}(Z)$$
$$w_{\ell}^{-}(Z) = w_{\ell}^{+}(-Z) = e^{i\pi\ell} w_{\ell}^{+}(Z)$$

 $= -izh_{l}^{2}(2)$ (2)

II-1.11

חר

II-1.8

The function $u_{q}(Z)$ is a regular solution of the differential equation

$$\frac{d^2 u_{\ell}}{dz^2} - \frac{\ell(\ell+1)}{z^2} u_{\ell}(z) + u_{\ell}(z) = 0 , \qquad \text{II-1.12}$$

while $v_{\ell}(Z)$ is the irregular solution of II-1.12. The asymptotic behaviour of the $u_{\ell}(Z)$ and $v_{\ell'}(Z)$ for large and small Z is,

For $\mathbf{Z} \xrightarrow{\sim} \infty$

$$u_{\ell}(Z) = \sin(Z - \frac{\ell\pi}{2})$$

$$v_{\ell}(Z) = -\cos(Z - \frac{i\pi\ell}{2})$$

$$w_{\ell}^{\pm}(Z) = e^{\pm iZ \pm \frac{i\pi\ell}{2}}$$

and for $Z \neq 0$ $u_{\ell}(Z) = \frac{Z^{\ell+1}}{(2\ell+1)!!} - \frac{1}{2} \frac{Z^{\ell+3}}{(2\ell+3)!!} + O(Z^{\ell+5})$ $v_{\ell}(Z) = -Z^{-\ell}(2\ell-1)!! - \frac{1}{2} Z^{-\ell+2}(2\ell-3)!!$

+ 0(2⁻²⁺⁴) .

II-1.14

II-1.13

It is convenient to write down the radial part of the Schrödinger equation II-1.3 for 1th partial wave using



II-1.9 and the expansion for potential $\langle r | V | r' \rangle$:

$$\begin{aligned} \tilde{\mathbf{r}} |\mathbf{V}| \mathbf{r}' > &= \sum_{lm} \mathbf{Y}_{lm}^{\star}(\mathbf{\hat{r}}) \mathbf{Y}_{lm}^{\star}(\mathbf{\hat{r}}') \frac{1}{\mathbf{r}} \langle \mathbf{r} | \mathbf{V}_{l} | \mathbf{r}' > \frac{1}{\mathbf{r}'} \\ &= \sum_{l} (2l+1) \mathbf{P}_{l}^{\star}(\cos\theta) \frac{\langle \mathbf{r} | \mathbf{V}_{l} | \mathbf{r}' >}{\frac{4\pi \mathbf{r} \mathbf{r}'}{\mathbf{r}'}} \quad \text{II-1.15} \end{aligned}$$

Substituting II-1.15 and II-1.9 into II-1.3 we obtain:

$$\frac{-d^2}{dr^2} \psi_{\ell}(kr) + \frac{\ell(\ell+1)}{r^2} \psi_{\ell}(kr) + \int \langle r | V_{\ell} | r' \rangle \psi_{\ell}(kr') dr'$$

= $k^2 \psi_{\ell}(kr)$,

where $\psi_{g}(\mathbf{kr})$ are defined from

as

$$\langle r | \psi_k \rangle = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} i^{\ell} (2\ell+1) \frac{\psi_{\ell}(kr)}{kr} P_{\ell}(\cos\theta)$$

For the spherically symmetric potential, $\psi_{\underline{k}}^{\dagger}$ and T have the following partial wave expansions:

$$r_{k}^{+} = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} i^{\ell} (2\ell+1) \frac{\psi_{\ell}^{+}(kr)}{kr} P_{\ell}(\cos\theta)$$
 II-1.17

$$\langle \mathbf{p} | \mathbf{T} | \mathbf{k} \rangle = \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1) t_{\ell}(\mathbf{p},\mathbf{k};\mathbf{k}^2) \mathbf{p}_{\ell}(\cos\theta)$$
. II-1.18

and the $t_{l}(p,k;k^{2})$ are defined for a general nonlocal potential



$$t_{g}(p,k;k^{2}) = \frac{1}{pk} \int dr dr' u_{g}(pr) \langle r | V_{g} | r' \rangle \psi_{g}^{+}(kr')$$
.

21

The Lippmann-Schwinger equation for the Lth partial wave can be written as

$$\psi_{\ell}^{+}(kr) = u_{\ell}(kr) + \frac{2}{\pi} \int_{0}^{\infty} \frac{u_{\ell}(qr)t_{\ell}(q,k;k^{2})kqdq}{k^{2} - q^{2} + i\epsilon} \quad \text{II-1.20}$$

It is often convenient to regard the T-matrix as a function of the complex variable $w = k^2 + i\epsilon$, and rewrite II-1.7 as

$$\langle \underline{p} | T(w) | \underline{k} \rangle = \langle \underline{p} | V | \underline{k} \rangle + \langle \underline{p} | V \frac{1}{w - H_0} T(w) | \underline{k} \rangle$$
 II-1.21

As the energy is conserved in the scattering process, it is not possible to construct the complete T-matrix from scattering data alone. However, if the potential matrix $\langle p|V|k \rangle$ is known, we can calculate the complete T-matrix using II-1.21. Elastic scattering data can provide the information for those T-matrix elements which satisfy energy conservation. These are called "onthe-energy-shell" (on-shell) T-matrix elements. Those which do not satisfy energy conservation are called "off-theenergy-shell" (off-shell) elements. The completely off-shell elements are functions of three variables; that is why we used three parameters to define the partial wave T-matrix" elements in II-1.18. For completeness we write down the



T-matrix in the 1th partial wave for all the three possible cases as follows:

 $t_{\ell}(p,k;w)$, $p^2 \neq k^2 \neq |w|$: off-shell;

$$t_{g}(p,k;k^{2})$$
, $p^{2} \neq k^{2}$ $|w| = k^{2}$: half-off-shell;

 $t_{\ell}(k,k;k^2) \equiv t_{\ell}(k)$, $p^2 = k^2 = |w|$: on-shell. II-1.22

The on-shell elements $t_{l}(k)$ are related to the l^{th} partial wave phase shifts $\delta_{l}(k)$ through the following relation

$$t_{g}(k) = -\frac{1}{k} e^{i\delta_{g}(k)} \sin \delta_{g}(k)$$
. II-1.23

Equation II-1.21 may be rewritten in the partial wave expansion as

$$t_{g}(p,k;w) = V_{g}(p,k) + \frac{2}{\pi} \int_{0}^{\infty} \frac{V_{g}(p,q) t_{g}(q,k;w) q^{2} dq}{w-q^{2}}$$

In relation II-1.24, $V_{g}(p,k)$ is related to $\langle r | V_{g} | r' \rangle$ as follows

$$V_{g}(p,k) \equiv \langle p | V_{g} | k \rangle = \int \frac{u_{g}(pr)}{p} \langle r | V_{g} | r' \rangle \frac{u_{g}(kr')}{k} dr dr'$$
.
II-1.25

In actual calculations it is convenient to use the reaction matrix (R-matrix) defined for real w with a principal value integration of II+1.24:

$$R_{\ell}(p,k;w) = V_{\ell}(p,k) + \frac{2}{\pi} P \int_{0}^{\infty} \frac{V_{\ell}(p,q)R_{\ell}(q,k;w)q^{2}dq}{w-q^{2}}$$
III-1.26

The $t_l(p,k;k^2)$ can be obtained from $R_l(p,k;k^2)$ using the Heitler damping equations:

$$t_{\ell}(p,q;k^{2}) = R_{\ell}(p,q;k^{2}) - \frac{ik\theta(k^{2})R_{\ell}(p,k;k^{2})R_{\ell}(q,k;k^{2})}{1 + ikR_{\ell}(k,k;k^{2})}$$
II-1.27

or conversely

$$R_{\ell}(p,q;k^{2}) = t_{\ell}(p,q;k^{2}) + \frac{ik\theta(k^{2})t_{\ell}(p,k;k^{2})t_{\ell}(q,k;k^{2})}{1 + ikt_{\ell}(k,k;k^{2})}$$
II-1.2

where $\theta(x)$ is the step function defined as

$$\theta(x) = \frac{1}{2}(1 + \frac{x}{|x|})$$

The above equations are the generalisation of the on-shell relation between $t_{g}(k)$ and $R_{g}(k)$

$$t_{g}(k) = \frac{R_{g}(k)}{1 + ikR_{g}(k)}$$

II-1.29

The relations II-1.28 and II-1.23 immediately give the relation of $R_{l}(k)$ to the phase shift of the l^{th} partial wave.

$$R_{\ell}(k) \equiv R_{\ell}(k,k;k^2) = -\frac{1}{k} \tan \delta_{\ell}(k)$$
 . II-1.30

The R-matrix is real and hermitian and has no cut on the real energy axis from 0 to ∞ . The poles of the R-matrix will appear only at the actual positions of the resonances on the $w = k^2$ plane, and not on the unphysical sheet, as is the case with the T-matrix ⁶⁹⁾.

II-2 Jost Function for a Local Potential

The radial Schrödinger equation for a local potential $\langle \mathbf{r} | \mathbf{V} | \mathbf{r}' \rangle = \mathbf{V}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}')$ acting in the relative S-state ($\ell = 0$) is given by

II-2.1

$$-\psi^{\mathbf{m}}(\mathbf{kr}) + \mathbf{V}(\mathbf{r})\psi(\mathbf{kr}) = \mathbf{k}^{2}\psi(\mathbf{kr})$$

The regular solution $\phi(kr)$ is defined by the boundary conditions at r = 0 such that

 $\phi(\mathbf{kr}) = 0$

 $\phi'(kr) = 1$

However, in the scattering problem one is interested in the solutions of the Schrödinger equation for asymptotic r, therefore, one should define the solutions with the boundary conditions at $r = \infty$. Since $r = \infty$ is an irregular singular point of the differential equation II-2.1 and k^2 multiplies the term of highest singularity at $r = \infty$, the boundary conditions at $r = \infty$ cannot be independent of k. We can define the solutions $f^{\pm}(kr)$ of II-2.1 with the following boundary condition

for r = 0

$$\lim_{r \to \infty} e^{\frac{1}{r}ikr} f^{\pm}(kr) = 1$$

II-2.2

II-2.5

and these are called the Jost solutions. They are linearly independent, except at k = 0, and satisfy the following integral equation:

$$f^{\pm}(kr) = e^{\pm ikr} - \frac{1}{k} \int_{r}^{\infty} dr' \sin k(r-r') V(r') f^{\pm}(kr)$$

Jost and Pais have shown that $f^+(kr)$ for real r is an analytic function of k, with a continuous k derivative, in the region Im k \geq 0. If the potential satisfies the condition

 $\int_{\infty}^{\infty} dr V(r) re^{2ar} < \infty , \qquad II-2.4$

then $f^+(kr)$ is analytic for Im $k \ge -a$. If the potential is of finite range, such that after some finite distance it vanishes identically, then $f^+(kr)$ is an entire analytic function of k. Analogous statements hold for $f^-(kr)$. Since $f^{\pm}(kr)$ are linearly independent solutions of II-2.1, these may be used to construct other solutions of II-2.1. Let us define the Wronskian W for two arbitrary functions, g and h,

 $W(g,h) = g \cdot \frac{dh}{dr} - \frac{dg}{dr} \cdot h$

as:

then $W(f^{\pm},\phi) = f^{\pm}(kr) \cdot \frac{d}{dr} \phi(kr) - \frac{d}{dr} f^{\pm}(kr) \phi(kr)$

 $\equiv \int_{\pm}^{1} (\mathbf{k})$.

II-2.6

Evaluating J(k) at r = 0 from II-2.6 and using $W(f^{+}(kr), f^{-}(kr)) = -2ik$, we can write $\phi(kr)$ as:

$$\phi(kr) = \frac{1}{2ik} \left[\int_{-}^{+} f'(kr) - \int_{+}^{0} f'(kr) \right] .$$
 II-2.7

Using II-2.6 we can define the Jost function:

$$f_{+}(k,0) \equiv J_{+}(k)$$
 and $J_{-}(k) = J_{+}(ke^{i\pi})$. II-2.8

The integral representation of the Jost function can be obtained from II-2.3:

$$f_{+}(k) = 1 + \frac{1}{k} \int_{0}^{\infty} dr \sin kr V(r) f^{+}(kr)$$
. II-2.9

The Jost function satisfies the following dispersion relation:

Re
$$\log f_{+}(k) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} \log f_{+}(k') dk'}{k'-k}$$
. II-2.10

Writing $f_{+}(k) = |f(k)|e^{-i\delta(k)}$, we have the relation:

$$|f(k)| = \exp[\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{dk^* \delta(k^*)}{k-k^*}]$$
 II-2.11
If we assume that $\delta(-k) = -\delta(k)$, II-2.11 reduces to

$$|f(k)| = \exp[-\frac{2}{\pi}P \int_{0}^{\infty} \frac{dk'k'\delta(k')}{k'^2 - k^2}]$$
, II-2.12

28

for Im $k \ge 0$. Equation II-2.11 is true only if the potential V(r) does not support any bound state. If, on the other hand, V(r) has a finite number of bound states at energies $E_n = -k_n^2$, then II-2.12 is modified, and is given by

$$|f(k)| = \prod_{n} (1^{\circ} + \frac{k_{n}^{2}}{k^{2}}) \exp(-\frac{2}{\pi} P \int_{0}^{\infty} \frac{dk^{\circ} k^{\circ} \delta(k^{\circ})}{k^{\circ} - k^{2}}) . \quad \text{II-2.1}$$

The relation II-2.13 is very useful because it gives a direct method of constructing the Jost function in terms of the phase shifts $\delta(\mathbf{k})$ on the real k axis and the bound state energies. These quantities can be experimentally obtained. Jost and Pais have shown that the Jost function, $f_+(\mathbf{k})$, for the local central potential is identical to the Fredholm determinant of the kernel of the Lippmann-Schwinger equation for the outgoing wave. This is referred to as the Jost-Pais theorem. Newton has extended the definition of Jost and Pais to include the nonzero angular momenta as well as noncentral local potentials. For the case $\ell \neq 0$, the Schrödinger equation II-2.1 includes the centrifugal term, and is given by

$$-\psi_{\ell}^{n}(kr) + \frac{\ell(\ell+1)}{r^{2}}\psi_{\ell}(kr) + V(r)\psi_{\ell}(kr) = k^{2}\psi_{\ell}(kr)$$

In the case of a noncentral force, the orbital angular momentum ℓ does not remain a conserved quantum number. Such cases are dealt with in Appendix A along with nonlocal forces. The regular solution of II-2.14 is again defined by the boundary condition at r = 0, which now stands as

$$\lim_{r \to \infty} r^{-(\ell+1)} \phi_{\ell}(kr) = 1 .$$
 II-2.15

This dependence of $\phi_{\ell}(kr)$ on ℓ is due to the fact that r = 0is a regular singular point for $\ell \neq 0$. The irregular solutions $f_{\ell}^{\pm}(kr)$ are still defined by the boundary conditions of II-2.2. The Jost solutions now satisfy the following integral equation, which is a generalisation of II-2.3:

$$f_{\ell}^{\pm}(kr) = f_{\ell\pm}^{0}(kr) - \int_{r}^{\infty} dr' g_{\ell}(r,r';k^{2}) V(r') f_{\ell}^{\pm}(kr') ;$$
II-2.16

where $g_{\ell}(r,r';k^2)$ is the ℓ^{th} partial wave component of the Green's function for the free particle. Functions $f_{\ell \pm}^0(kr)$ and $g_{\ell}(r,r';k^2)$ are given in terms of the Riccati-Bessel functions of II-1.9 and II-1.10 as:

$$f_{l\pm}^{0}(kr) = w_{l}^{\pm}(kr)e$$

29

IT-2.14

II-2.17

- -

and

$$g_{g}(r,r';k^{2}) = \frac{1}{k} [u_{g}(kr')v_{g}(kr) - u_{g}(kr)v_{g}(kr')]$$

$$r' > r$$

$$= 0 \qquad r' < r$$

$$II=2 18$$

The Jost function $f_{l+}(k)$ is defined in analogy with II-2.8 as

$$f_{l+}(k) = \frac{k^{l} e^{w(f_{l+},\phi_{l})}r=0}{(2l+1)!!}$$
, II-2.19

and has the following integral representation:

$$f_{l+}(k) = 1 + \frac{1}{k} e^{-\frac{i\pi l}{2}} \int_{0}^{\infty} dr \ u_{l}(kr) V(r) f_{l+}(kr) .$$

Note that II-2.20 reduces to II-2.9 for l = 0. The analytic properties of the Jost function are derived through the analytic properties of $f_{l\pm}(kr)$, and are given in Ref. 70) we summarize some of them here:

(i) For each real L and k

$$f_{l+}^{*}(k) = f_{l-}^{\infty}(k)$$

(ii) For each real L and complex k

$$f_{\ell+}^{\pm}(k^{\alpha}) = f_{\ell-}(k)$$

(iii) f_l(k) is a meromorphic function in the lower half
 k-plane.

The S-matrix $S_{l}(k)$ is defined in terms of $f_{l\pm}(k)$ as:

$$S_{\ell}(k) = \frac{f_{\ell-}(k)}{f_{\ell+}(k)} = e^{2i\delta_{\ell}(k)}$$
 II-2.21

It is unitary for real k and satisfies the relation;

$$S_{\ell}^{\star}(k) = \frac{1}{S_{\ell}(k)} = S_{\ell}(-k)$$
 . II-2.22

The phase shifts $\delta_{\underline{\ell}}(k)$ are real and are odd functions of k due to the unitarity of the S-matrix. Normally we define $S_{\underline{\ell}}(k)$ for the real positive values of k. This implies that E lies on the upper rim of the right hand cut from E = 0 to $E = \infty$. This requires that $S_{\underline{\ell}}(k)$ should be calculated using the outgoing wave Green's function in the Lippmann-Schwinger equation II-1.5. For well-behaved potentials the following properties of the S-matrix are important:

i. Poles of the S-matrix on the physical sheet, (upper half k-plane), which are due to the zeros of $f_{l+}(k)$ correspond to the bound states.

ii. Poles of the S-matrix on the unphysical sheet, (lower half k-plane), give rise to resonances provided these poles are close to the positive real axis. If such poles appear on the negative real axis, they denote the virtual or antibound states.

32

II-3 Jost Function for a Nonlocal Central Potential

In this section we will derive the expression for the Jost function for any angular momentum state due to a nonlocal central potential. Our expressions will reduce to those of Warke and Bhaduri ⁵⁹⁾ for the l = 0 case. We show that the Jost function for a nonlocal potential is equal to the ratio of the Fredholm determinants for the scattering solution to that of the regular solution. The case of the nonlocal noncentral potentials is given in Appendix A.

We start with the radial part of the Schrödinger equation II-1.16, and introduce a strength parameter λ for convenience. Throughout this section we will suppress the k-dependence of the functions in order to simplify the ontation. The corresponding Lippmann-Schwinger equation for the outgoing scattering solution is

$$\langle \mathbf{r} | \psi_{g}^{\dagger} \rangle = \langle \mathbf{r} | \psi_{g}^{0} \rangle + \lambda \langle \mathbf{r} | G_{g}^{\dagger} \nabla | \psi_{g}^{\dagger} \rangle$$

with

$$\langle \mathbf{r} | \psi_{\ell}^{\dagger 0} \rangle = \mathbf{u}_{\ell} (\mathbf{r})$$

 $<\mathbf{r}|G_{\ell}^{\dagger}|\mathbf{r}'> = -\frac{1}{k}e^{-i\pi\ell}u_{\ell}(kr_{<})w_{\ell}^{\dagger}(kr_{>})$

II-3.1

II-3.

II-3.3

33

The functions $u_{g}(Z)$, $v_{g}(Z)$, $w_{g}^{\pm}(Z)$ are defined in Section II-1 of this chapter and their properties are given in II-1.10 to II-1.14. Similarly the integral equation for the regular solution is

$$|\phi_{\ell}\rangle = \langle r |\phi_{\ell}^{0}\rangle + \lambda \langle r |G_{\ell}V|\phi_{\ell}\rangle$$
,

with $\langle r | \phi_{\ell}^{0} \rangle = (2\ell+1) \prod_{k} k^{-(\ell+1)} u_{\ell}(kr)$

 $<\mathbf{r}|G_{g}|\mathbf{r}'> = \frac{1}{k} \{u_{g}(k\mathbf{r}')v_{g}(k\mathbf{r}) - u_{g}(k\mathbf{r})v_{g}(k\mathbf{r}')\} \mathbf{r}' < \mathbf{r}$

= 0 r! \geq r , II-3.5

and the boundary condition

$$\lim_{r \to 0} r^{-(\ell+1)} < r |\phi_{\ell}^{0} > = 1 .$$
 II-3.6

The Jost solution is given by

$$\langle \mathbf{r} | \mathbf{f}_{\ell}^{\pm} \rangle = \langle \mathbf{r} | \mathbf{f}_{\ell\pm}^{0} \rangle + \lambda \langle \mathbf{r} | \tilde{\mathbf{G}}_{\ell} \nabla \mathbf{f}_{\ell}^{\pm} \rangle$$
, II-3.7

, where \tilde{G}_{g} is the transpose of G_{g} defined in II-3.5, with

$$<\mathbf{r}|\mathbf{f}_{2\pm}^{0}>=\mathbf{e}^{-\frac{i\pi 2}{2}}\mathbf{w}_{2}^{\pm}(\mathbf{kr})$$
 II-3.8

11-3.4

The boundary condition for $\langle r | f_{g}^{\pm} \rangle$ is

$$\lim_{r \to \infty} e^{\mp i k r} \langle r | f_{g}^{\pm} \rangle = 1 . \qquad \text{II-3.9}$$

The Jost functions are defined in terms of the Jost solutions:

$$f_{\ell}^{\pm}(k) = \frac{k^{\ell} e^{\pm i\pi \ell/2}}{(2\ell-1)!!} \lim_{r \to 0} r^{\ell} < r | f_{\ell}^{\pm} > .$$
 II-3.10

Using the definition of the Wronskian (II-2.5) and the relations II-3.1 and II-3.9, it can be shown in general that

$$W[\langle r | f_{l}^{+} \rangle, \langle r | f_{l}^{-} \rangle]_{r=\infty} = W[\langle r | f_{l}^{+} \rangle, \langle r | f_{l}^{-} \rangle]_{r=0}$$

The equations II-3.4, II-3.7 and II-3.11 lead to the following relation for the regular solution

$$\langle \mathbf{r} | \phi_{\underline{k}} \rangle = \frac{(2\underline{k}+1)\underline{11}}{2i k^{2}} \left[e^{-\frac{i\pi\underline{k}}{2}} f_{\underline{k}}^{-}(\mathbf{k}) \langle \mathbf{r} | f_{\underline{k}}^{+} \rangle - e^{-\frac{i\pi\underline{k}}{2}} f_{\underline{k}}^{+}(\mathbf{k}) \langle \mathbf{r} | f_{\underline{k}}^{-} \rangle \right] .$$
 II-3.12

Using the reflection property of the Jost solutions

$$f_{\ell}^{+}(k) = f_{\ell}^{-*}(k) = 1 + \frac{\lambda k^{\ell} e^{-i\pi \ell}}{(2\ell+1)!!} < w_{\ell}^{+*}(V) |\phi_{\ell}>$$
 II-3.13

. Weren

II-3.11

35

and II-3.4, II-3.7, one can show that

$$\frac{d}{dr} f_{g}^{+}(k) = \frac{k^{2} e}{(2l+1)!!} < f_{g}^{-} |V| \phi_{g} > . \qquad II-3.14$$

Following an approach similar to that used by Warke and Bhaduri 59, we define a function F_{g} :

$$|\mathbf{r}|_{\mathcal{L}}|\mathbf{r}'\rangle = \langle \mathbf{r}|_{\mathcal{L}}^{+}|\mathbf{r}'\rangle - \langle \mathbf{r}|_{\mathcal{L}}^{0}|\mathbf{r}'\rangle$$
, II-3.15

where the interacting Green's functions l_{l}^{+} and l_{l}^{+} are defined in terms of free Green's functions G_{l}^{+} and G_{l}^{-} :

11-3.16

36

The function $\langle r|F_{g}|r' \rangle$ will satisfy the differential equations

 $\frac{(-d^2)}{dr^2} + \frac{\ell(\ell+1)}{r^2} - k^2 < r|F_{\ell}|r'> + \lambda < r|VF_{\ell}|r'> = 0$

$$\left(\frac{-d^{2}}{dr'^{2}} + \frac{\ell(\ell+1)}{r'^{2}} - k^{2}\right) < r|F_{\ell}|r' > + \lambda < r|F_{\ell}V|r' > = 0 ,$$

with the following boundary conditions

$$\langle \mathbf{r} | \mathbf{F}_{g} | \mathbf{r}^{*} \rangle = 0 \qquad \mathbf{r} + 0 ,$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{r}} < \mathbf{r} |\mathbf{F}_{\ell}| \mathbf{r} > \Big|_{\tau=\tau' + 0} = -\frac{\ell+1}{2\ell+1}$$

 $<\mathbf{r}|\mathbf{F}_{\ell}|\mathbf{r}'> = e^{\mathbf{i}\mathbf{k}\mathbf{r}'} \times (\text{Function of }\ell, \, \mathbf{k}, \, \mathbf{r})$. II-3.18

The solution of II-3.17 which satisfies the boundary conditions (II-3.18) is

$$\langle \mathbf{r} | \mathbf{F}_{\ell} | \mathbf{r}' \rangle = - \frac{k^{\ell} e}{(2\ell+1)!! f_{\ell}^{+}(k)} \langle \mathbf{r} | \phi_{\ell} \rangle \langle \mathbf{r}' | f_{\ell}^{+} \rangle$$
 II-3.19

Using the property that $\langle r | f_{\ell}^{-} \rangle$ is the complex conjugate of $\langle r | f_{\ell}^{+} \rangle$ and II-3.19, it can be shown that

Note that $f_{l+}(k) + 1$ as $\lambda + 0$. It is now straightforward to prove from II-3.15, II-3.16 and II-3.20 that

$$f_{\ell+}(k) = \frac{\exp[\operatorname{Tr} \log(1-\lambda G_{\ell}^{\dagger} V)]}{\exp[\operatorname{Tr} \log(1-\lambda G_{\ell} V)]} = \frac{\operatorname{Det}(1-\lambda G_{\ell}^{\dagger} V)}{\operatorname{Det}(1-\lambda G_{\ell} V)} = \frac{D_{\ell}^{\dagger}(k)}{E_{\ell}(k)}$$

II-3.21

In the derivation of II-3.21 we have used the relation

Det
$$M = \exp[Tr \log M]$$
. II-3.22

If the nonlocal central potential is separable, the relation II-3.21 reduces to

$$f_{l+}(k) = \frac{(1-\lambda Tr \ G_{l}^{+1})}{(1-\lambda Tr \ G_{l}^{-1})}$$
 II-3.23

A detailed study of the Fredholm determinants for nonlocal potentials have been made by Bertero et al. ⁷¹⁾. For $\pounds = 0$, II-3.21 reduces to the relation given by Warke and Bhaduri ⁻⁵⁹⁾. For the local potentials,

$$< r | V | r' > = V(r) \delta(r-r')$$
 II-3.24

$$Det(1-\lambda G_{\chi}V) = exp[Tr log(1-\lambda G_{\chi}V)]$$

$$= \exp\left[-\mathrm{Tr}\left\{\lambda G_{\ell}^{2}V + \frac{\lambda^{2}G_{\ell}VG_{\ell}V}{2} + \ldots\right\}\right] ,$$
[1]

$$\operatorname{Tr}\lambda G_{\ell} V = \lambda \int_{0}^{\infty} G_{\ell}(\operatorname{rr}) V(r) dr$$

$$0$$
 , since G, (rr) = 0

$$\mathrm{Tr}\lambda^{2}\mathrm{G}_{g}\mathrm{V}\mathrm{G}_{g}\mathrm{V} = \lambda^{2}\int_{0}^{\infty}\mathrm{d}r\int_{0}^{r}\mathrm{d}r^{*}\mathrm{G}_{g}(\mathrm{rr}^{*})\mathrm{V}(\mathrm{r}^{*})\mathrm{G}_{g}(\mathrm{r}^{*}\mathrm{r})\mathrm{V}(\mathrm{r})$$

= 0, since $G_{g}(r'r) = 0$ for r' < r. II-3.26

-3.25

Similarly it can be shown that the trace of the higher order terms is zero. Thus

 $Det(1-\lambda G_{g}V) = exp(0) = 1$

and

$$f_{\ell+}(k) = \text{Det}(1-\lambda G_{\ell}^{\mathsf{T}} V)$$

This relation is the statement of Jost-Pais theorem for the local potential and the relation II-3.21 is the generalised Jost-Pais theorem for the nonlocal potential. The denominator of II-3.21, which is k dependent, can be removed by modifying the boundary condition for the Jost solution as

$$\lim_{r \to \infty} \langle r | f_{\ell}^{\pm} \rangle = \frac{e^{\pm ikr}}{\text{Det}(1-G_{\ell}V)} \quad \text{II-3.28}$$

The derivations given in this section have already been published by us, and there (section 3) we have generalised \Rightarrow these relations for the case of nonlocal noncentral potentials. This paper is attached as Appendix A.

CHAPTER III

CONSTRUCTION OF PARTLY NONLOCAL PHASE EQUIVALENT POTENTIALS

In this chapter we describe our method of generating phase equivalent potentials. There are many different ways of generating phase equivalent potentials, some of which are outlined briefly in Chapter I. The long range part of the N-N interaction (\geq 2.5 Fm) is well established. Both meson theory and experimental data suggest that it should be given by the one-pion-exchange-potential. However, the form of the short range part of the N-N interaction is uncertain. As discussed earlier we expect the short range part of the potential to be nonlocal, but it is difficult to decide on a specific form of nonlocality to represent the short range part. Thus we have a lot of freedom to vary the short range part of the potential.

The class of phase equivalent potentials considered by us has a long range attractive local part superimposed with a repulsive one-term separable potential. These are partly nonlocal potentials and referred to as class A potentials in the text. We also construct a set of rank-two separable phase equivalent potentials referred to as class B potentials. The separable potentials are such that the attractive part is independently phase equivalent to the

40

attractive local part of the class A type. These potentials are constructed to compare the differences produced by introducing the separability in the interaction.

For our class A potentials, we have chosen the local part to be given by a Bargmann potential rather than the more realistic Yukawa form. This is done because for the Bargmann potential all the required expressions can be derived analytically. The Bargmann potential has an exponential tail and decays slowly in comparison with the Yukawa potential.

Sections 2 and 3 of this chapter deal with the details of the method and the numerical computations of these model potentials. In Section 4 we discuss and compare some features of partly nonlocal potentials to those of purely separable potentials.

III-1 Phase Equivalent Potentials

We adapt a method suggested by Fuda $^{58)}$ to construct phase equivalent potentials. This formalism is similar to the work of Chadan $^{72)}$, but is more concise. We will consider, for simplicity, only the relative S-state interaction. The basis of orthonormal wave functions in the S-state is taken to be

$$\langle \mathbf{r} | \mathbf{p} \rangle = \sqrt{\frac{2}{\pi}} \sin \mathbf{p} \mathbf{r}$$

with

$$\int_{a}^{b} \langle \mathbf{r} | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{r}' \rangle d\mathbf{p} = \delta(\mathbf{r} - \mathbf{r}')$$

In the literature, it is usual to take a three dimensional basis:

$$\langle r | p \rangle = \frac{1}{(2\pi)^{3/2}} e^{ik \cdot r}$$
,

and define the S-state separable potential in coordinate space as

$$\langle r | V | r' \rangle = \frac{h^2}{M} \frac{1}{2\pi^2} g(r) g(r')$$

III-1.3

III-1.2

III-1.1

or in k-space as,

$$\langle k | V | k' \rangle = \frac{\frac{1}{N}^2}{M} \frac{1}{2\pi^2} g(k) g(k')$$
, III-1.

with $g(k) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(r) j_{0}(kr) r^{2} dr$.

In our basis the same potential becomes

 $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$

$$<\mathbf{r}|\mathbf{V}|\mathbf{r}'> = \frac{\hbar^2}{M}\frac{2}{\pi} rg(\mathbf{r})r'g(\mathbf{r}')$$

or

$$k|V|k' > = \frac{h^2}{M} \frac{2}{\pi} kg(k)k'g(k')$$
.

Let us consider the S-wave phase shifts $\delta(k)$ given for all k, such that $\delta(k) \neq 0$ as $k \neq \infty$. These could be, for example, the known experimental phase shifts with smooth high energy extrapolation or could be generated by a given potential without a hard core. Our problem is to generate a set of phase equivalent potentials of the type

III-1.7

III-1.5

III-1.6

where V_1 is local and attractive while V_2 is nonlocal and repulsive. These must reproduce the input phases, for all values of k. The input data may specify a finite number of bound state eigenvalues which should also be reproduced by the potential in III-1.7. We may also consider the case where the input data do not have any bound states at all. This is the situation with ${}^{1}S_{0}$ potentials, but V_{1} may still have a bound state, which is eliminated by adding a repulsive potential V_{2} . Thus a set of phase equivalent potentials for the ${}^{1}S_{0}$ state are generated, such that the attractive part may or may not have a bound state.

The Fredholm determinant $D^+(k)$ for the scattering solution can be calculated with the given input data from the equation, which is similar to II-2.13,

 $D^{+}(k) = \prod_{n} (1 + \frac{k_{n}^{2}}{k^{2}}) \exp\left[-\frac{2}{\pi}P \int_{0}^{\infty} \frac{\delta(p)pdp}{p^{2} - k^{2}}\right] \exp\left(-i\delta(k)\right) .$ III-1.8

Note that we have used the Fredholm determinant $D^+(k)$ rather than the Jost function $f_+(k)$ because for the nonlocal potential the two are not the same as discussed in Chapter II. The bound state energies for the n bound states are given by $-h^2k_n^2/M$. In practice the triplet S-state has only one bound state. To ensure phase equivalence it is sufficient that the potentials III-1.7 generate an identical Fredholm determinant. The Fredholm determinant for the potential in III-1.7 should thus be equated to that given by $D^+(k)$,

$$D^{+}(k) = Det(1 - G_{0}^{+}V)$$
,

III-1.9

where the free Green's function $G_0^+(k^2)$ is

$$G_0^+(k^2) = (k^2 - H_0 + i\epsilon)^{-1}$$
. III-1.10

We have put $h^2/M = 1$ for convenience and H_0 is the kinetic energy operator. The right hand side of III-1.9 can be written as a product of two determinants using the identities of the Green's function. Let us define $G_1^+(k^2)$ as the Green's function in the presence of potential V_1 only; it satisfies the following relations:

$$G_{1}^{+}(k^{2}) = G_{0}^{+}(k^{2}) + G_{0}^{+}(k^{2}) V_{1}G_{1}^{+}(k^{2})$$
$$= G_{0}^{+}(k^{2}) + G_{1}^{+}(k^{2}) V_{1}G_{0}^{+}(k^{2}) .$$

Using III-1.11

$$(1 - G_0^+ V) = 1 - G_1^+ V_1 - G_1^+ V_2 + G_1^+ V_1 - G_0^+ V_1 + G_1^+ V_2 - G_0^+ V_2$$
$$= 1 - G_0^+ V_1 - (G_1^+ V_2 - G_0^+ V_1 G_1^+ V_2)$$
$$= (1 - G_0^+ V_1) (1 - G_1^+ V_2) , \qquad \text{III-1.12}$$

therefore,

$$Det(1 - G_0^{\dagger}V) = Det(1 - G_0^{\dagger}V_1)Det(1 - G_1^{\dagger}V_2) ,$$



III-1.13

III-1.11

and

$$D^{+}(k) = D_{1}^{+}(k)\tilde{D}_{2}^{+}(k)$$

III-1.14

where $D_1^+(k)$ is the Fredholm determinant for the local potential V_1 and $\tilde{D}_2^+(k)$ is the Fredholm determinant for the potential V_2 but given in the basis of the solutions generated by V_1 :

$$\tilde{D}_{2}^{+}(k) = \text{Det}(1 - G_{1}^{+}V_{2})$$

= exp(Tr $ln(1 - G_1^+V_2))$

III-1.15

If V_2 is a separable potential of rank-one, then

$$\exp(\operatorname{Tr} \ln(1 - G_1^{\dagger}V_2)) = (1 - \operatorname{Tr} G_1^{\dagger}V_2)$$
. III-1.16

Since, the operator

$$G_1^{+}(k^2) = (k^2 - (H_0 + V_1) + i\epsilon)^{-1}$$
 III-1.17

is diagonal in the basis of the eigenfunctions generated by the potential V_1 , it is simple to find its trace in this basis. Let the regular solution of the operator $H_1 = (H_0 + V_1)$ be denoted by $|\phi_p\rangle$, and the bound state solution by $|\phi_B\rangle$ with eigenvalue $-k_B^2$:

$$H_{1}|\phi_{p}\rangle = p^{2}|\phi_{p}\rangle \qquad p^{2} > 0$$
$$H_{1}|\phi_{B}\rangle = -k_{B}^{2}|\phi_{B}\rangle \qquad k_{B}^{2} > 0$$

then, with properly normalised $|\phi_p>$ and $|\phi_B>$, the trace in the III-1.16 is,

$$\operatorname{Tr} \ \mathbf{G}_{1}^{+}\mathbf{V}_{2} = \sum_{\mathbf{p}} \langle \phi_{\mathbf{p}} | \mathbf{G}_{1}^{+}\mathbf{V}_{2} | \phi_{\mathbf{p}} \rangle + \langle \phi_{\mathbf{B}} | \mathbf{G}_{1}^{+}\mathbf{V}_{2} | \phi_{\mathbf{B}} \rangle$$
$$= \sum_{\mathbf{p}} \frac{1}{k^{2} - p^{2} + i\varepsilon} \langle \phi_{\mathbf{p}} | \mathbf{V}_{2} | \phi_{\mathbf{p}} \rangle + \frac{\langle \phi_{\mathbf{B}} | \mathbf{V}_{2} | \phi_{\mathbf{B}} \rangle}{k^{2} + k_{\mathbf{B}}^{2}} \quad \text{. III-1.19}$$

We have assumed in the above analysis that the potential V_2 is separable and of rank-one; let its form in our basis be given by

$$\langle r | V_2 | r' \rangle = \eta(r) \eta(r')$$
, III-1.20

then it follows that,

$$<\phi_{\rm p} |V_2|\phi_{\rm p}> = (\tilde{n}({\rm p}))^2$$
,

III-1.21

47

III-1.18

with

$$\tilde{n}(p) = \int_{0}^{\infty} \phi_{p}(r) \eta(r) dr$$

111-1.22



and

$$\langle \phi_{\rm B} | V_2 | \phi_{\rm B} \rangle = \widetilde{n}_{\rm B}^2$$
,

where

$$\tilde{n}_{B} = \int \phi_{B}(r) \eta(r) dr$$

With these simplifications III-1.19 reduces to

In
$$G_1^+ V_2 = \int \frac{(\tilde{n}(p))^2 dp}{k^2 - p^2 + i\epsilon} + \frac{\tilde{n}_B^2}{k^2 + k_B^2}$$
. III-1.25

Using III-1.14 and III-1.16 we get

$$\frac{D^{+}(k)}{D_{1}^{+}(k)} = 1 - \int \frac{(\tilde{\eta}(p))^{2} dp}{k^{2} - p^{2} + i\epsilon} - \frac{\tilde{\eta}_{B}^{2}}{k^{2} + k_{B}^{2}} . \qquad \text{III-1.26}$$

Equating the real and imaginary part of III-1.26 we obtain,

$$\operatorname{Re}\left[\frac{D^{+}(k)}{D_{1}^{+}(k)}\right] = 1 - P \int_{0}^{\infty} \frac{\left(\tilde{n}(p)\right)^{2} dp}{k^{2} - p^{2}} - \frac{\tilde{n}_{B}^{2}}{k^{2} + k_{B}^{2}} \qquad \text{III-1.27}$$

and

$$Im[\frac{D^{+}(k)}{D_{1}^{+}(k)}] = \frac{\pi}{2k} (\tilde{\eta}(k))^{2} .$$
 III-1.28

Since $D^+(k)$ is given, by choosing a suitable V_1 with known $D_1^+(k)$, we can determine the form factor $\tilde{\eta}(k)$ of the potential V_2 in the basis generated by V_1 .

Some useful relations can at once be derived using III-1.27 and III-1.28. If we assume that the total potential

III-1.23

III-1.24

V has only one bound state of energy $-k_D^2$, then $D^+(ik_D) = 0$. Using III-1.26 at $k = ik_D$ we obtain

$$\tilde{n}_{B}^{2} = (k_{B}^{2} - k_{D}^{2}) \left[1 + \int_{0}^{\infty} \frac{(\tilde{n}(p))^{2} dp}{p^{2} + k_{D}^{2}}\right] \quad \text{III-1.29}$$

Since V_1 is more attractive than V, $k_B^2 > k_D^2$ and in the limiting case when $k_B^2 + k_D^2$, III-1.29 implies that

$$\tilde{n}_{B} = \int_{0}^{\infty} \phi_{B}(r) \eta(r) dr + 0$$
 as $k_{B}^{2} + k_{D}^{2}$. III-1.30

This could be transformed to the momentum space by defining

$$n(p) = \int_{\pi}^{2} \int_{0}^{\infty} n(r) \sin pr dr , \qquad \text{III-1.31}$$

and

$$\phi_{\rm B}({\rm p}) = \int_{\pi}^{2} \int_{0}^{\infty} \phi_{\rm B}({\rm r}) \sin {\rm pr} \, {\rm dr}$$

We can then rewrite III-1.30 as

$$\tilde{n}_{B} = \int_{0}^{\infty} \phi_{B}(p) \eta(p) dp = 0$$
 $k_{B}^{2} = k_{D}^{2}$. III-1.33

Equations III-1.30 and III-1.33 imply that for the limiting situation $k_B^2 = k_D^2$, the form factor of V_2 is orthogonal to the two body bound state wave function ϕ_B generated by V_1 .



III-1.32

Such potentials have quite peculiar effects on the binding energy of triton and in nuclear matter, as shown by Fiedeldey ²¹⁾ and by us ⁴⁴⁾. This effect will be discussed in detail at a later stage. If the phase shifts generated by potentials V and V₁ are denoted by δ and δ_1 respectively, and we define

$$\delta^* = \delta - \delta_1$$

then using III-1.8 and III-1.28 we get

$$Im[\frac{D^{+}(k)}{D_{1}^{+}(k)}] = -(\frac{k^{2}+k_{D}^{2}}{k^{2}+k_{B}^{2}})exp\{-\frac{2}{\pi}P\int_{0}^{\infty}\frac{\delta'(p)pdp}{p^{2}-k^{2}}\}sin \delta'(k)$$
$$= \frac{\pi}{2k}[\tilde{\eta}(k)]^{2}.$$
 III-1.

The above equation demands that for $\tilde{\eta}(k)$ to be real, we must, have

 $\delta^* = \delta - \delta_1 \leq 0, \quad .$

/ III-1.36

III-1.34

35

Furthermore by multiplying III-1.35 by $(k^2+k_B^2)$, and putting $k = ik_B$, we find that

$$\sin \delta^{\prime}(ik_{n}) = 0$$



Since $\tilde{\eta}(k)$ is finite, we can rewrite III-1.27, using III-1.8, as

$$\operatorname{Re}\left[\frac{D^{+}(k)}{D_{1}^{+}(k)}\right] = \left(\frac{k^{2}+k_{D}^{2}}{d^{2}+k_{B}^{2}}\right) \exp\left[-\frac{2}{\pi}P\int_{0}^{\infty}\frac{\delta'(p)pdp}{p^{2}-k^{2}}\right] \cos \delta'(k)^{2}.$$
III-1.38

Evaluating III-1.36 at $k = ik_B$, and using III-1.35, we get

$$[\tilde{n}_{B}]^{2} = (k_{B}^{2} - k_{D}^{2}) \exp\{-\frac{2}{\pi} P \int_{0}^{\infty} \frac{\delta'(p) p dp}{p^{2} + k_{B}^{2}} \} . \qquad \text{III-1.39}$$

We can also consider the case when $D^+(k)$ corresponds to some given input data without any bound state, e.g., in the ${}^{1}S_{0}$ state. The equations III-1.27 to III-1.39 are modified because $D^+(ik_{D})$ is no more defined. For such cases we have the freedom to choose V_{1} with or without a bound state. Let us first consider the case when V_{1} does not have a bound state. Equations III-1.35 and III-1.38 will be modified in the following way

$$Im[\frac{D^{+}(k)}{D_{1}^{+}(k)}] = -\exp[-\frac{2}{\pi}P\int_{0}^{\infty}\frac{\delta^{*}(p)pdp}{p^{2}-k^{2}}]\sin \delta^{*}(k)$$
$$= \frac{\pi}{2k}[\tilde{n}(k)]^{2}, \qquad III-1.40$$
$$Re[\frac{D^{+}(k)}{D_{1}^{+}(k)}] = \exp\{-\frac{2}{\pi}P\int_{0}^{\infty}\frac{\delta^{*}(p)pdp}{p^{2}-k^{2}}\}\cos \delta^{*}(k) . III-1.41$$

and

We do not have any $\tilde{\eta}_B$ because $\phi_B(k)$ is not defined as there is no bound state in the attractive potential V_1 .

For the case when V_1 has a bound state of energy $-k_B^2$, we cannot calculate \tilde{n}_B using III-1.28, because this equation was the outcome of the fact that $D(ik_D) = 0$ which is no more valid. Therefore, we have to find out some other way to calculate \tilde{n}_B . We can rewrite III-1.27 as

$$\frac{\tilde{\eta}_{B}^{2}}{k^{2}+k_{B}^{2}} = \left[1 - P \int_{0}^{\infty} \frac{\tilde{\eta}^{2}(p)dp}{k^{2}-p^{2}} - \operatorname{Re}\left(\frac{D^{+}(k)}{D_{1}^{+}(k)}\right)\right] \quad \text{III-1.42}$$

This relation can be used for any real value of k, as \tilde{n}_B is a constant and independent of k. Since V_1 has a bound state, the phase shift δ_1 will start from π according to the Levinsion's theorem. If we superimpose a one-term separable repulsive potential, only one bound state can be eliminated, because the S-matrix for the one term separable potential can have only one simple pole. Thus, for the present case, we can have only one bound state in the local potential V_1 . For a general case one can start with N bound states in the local potential and eliminate these one at a time by N rank-one separable potentials. This can be understood with the help of the generalised Levinson's theorem for the nonlocal potentials given by Martin ⁷³⁾, which states that

$$\delta(0) - \delta(\infty) = (\nu + \sigma)\pi$$

$$\delta_1(0) - \delta_1(\infty) = n\pi ,$$

 $\delta^{\dagger}(0) - \delta^{\dagger}(\infty) = (\gamma - n)\pi + \sigma\pi .$

The phase shifts are defined in the notations of III-1.34, and ν is the number of bound states in the total potential, while n is the number of the bound states in the local potential. The number of degenerate states of positive energies is denoted by σ . Martin ⁷³⁾ has shown that the only negative value permitted in III-1.44 is $-\pi$. This value occurs when we take $\nu = n-1$, $n \neq 0$ and $\sigma = 0$. Our case of ¹S₀ corresponds to the choice of n = 1 and $\nu = 0$.

III-1.44

III-1.43 ·

53

III-2 Numerical Construction of Phase-Equivalent Potentials

In order to construct a set of phase equivalent partly nonlocal potentials, we first have to decide what input phases $\delta(k)$, or equivalently, the Fredholm determinant $D^+(k)$ to choose. We choose, for simplicity, the input $\delta(k)$ generated by the rank-two separable potential of Tabakin 42 *. These phase shifts are also used by Fiedeldey $^{43)}$ to construct rank-two phase equivalent separable potentials. In our basis the potential is of the form

$$\langle k | V | k' \rangle = \frac{h^2}{M} \frac{2}{\pi} kk' [-g(k)g(k') + h(k)h(k')]$$
, III-2.1

with

$$g(k) = \frac{\rho_1}{k^2 + a_0^2},$$

$$h(k) = \frac{\rho_2 k^2}{((k-d_0)^2 + b_0^2)((k+d_0)^2 + b_0^2)}.$$
III-2.2

This spin independent potential is an average of the potentials acting in the ${}^{1}S_{0}$ and ${}^{3}S_{1}$ states. It has a small bound state energy equal to -0.428 MeV. The parameters for this potential are:

The potential we actually used was potential 1 of Table 1 of Ref. 42) and is referred to as (ST) in this text.

54

$$\rho_1 = 2.074 \text{ Fm}^{-3/2}$$
, $\rho_2 = 2.664 \text{ Fm}^{-3/2}$

$$a_0 = 1.199 \text{ Fm}^{-1}$$
, $b_0 = 1.248 \text{ Fm}^{-1}$, $d_0 = 1.441 \text{ Fm}^{-1}$

$$E_{\rm D} = 0.428 \text{ MeV or } k_{\rm D} = 0.101603 \text{ Fm}^{-1}$$
. III-2.3

The Fredholm determinant for a potential of the above form is given by the following expression

$$D^{+}(k) = (1 - G_{11}^{+}(k)) (1 + G_{22}^{+}(k)) - G_{12}^{+}(k)^{2}$$
, III-2.4

with

$$G_{11}^{+} = \frac{2}{\pi} \int_{0}^{\infty} \frac{q^{2}(p) p^{2} dp}{k^{2} - p^{2} + ig},$$

$$G_{22}^{+} = \frac{2}{\pi} \int_{0}^{\infty} \frac{h^{2}(p) p^{2} dp}{k^{2} - p^{2} + i\varepsilon},$$

$$G_{12}^{+} = \frac{2}{\pi} \int_{0}^{\infty} \frac{q(p)h(p) p^{2} dp}{k^{2} - p^{2} + i\varepsilon}.$$
III-2.5

The above quantities are obtained analytically using III-2.2 and III-2.5, and are found to be 56:

$$G_{11}^{+}(k) = (\frac{k^2 - a_0^2}{2a_0} - ik)g^2(k)$$

$$G_{22}^{+}(\bar{k}) = \left[\left\{ \frac{(5b_{0}^{6} - 3d_{0}^{6} + 7b_{0}^{4}d_{0}^{2} - d_{0}^{4}b_{0}^{2})}{k^{2}} + \frac{(b_{0}^{2} + d_{0}^{2})^{4}}{k^{4}} - k^{2}(5b_{0}^{2} + d_{0}^{2}) + 3d_{0}^{4} + 15b_{0}^{4} + 2d_{0}^{2}b_{0}^{2} \right] / 16b_{0}^{3} + ik \right]h^{2}(k) ,$$

$$G_{12}^{+}(k) = \left[\left\{ \left(\frac{a_{0}^{2}(d_{0}^{2} + b_{0}^{2})^{2}}{k^{2}} + (d_{0}^{2} + b_{0}^{2})^{2} + a_{0}(2b_{0} - a_{0}) \right. \right. \right] \\ \times \left(d_{0}^{2} + b_{0}^{2} \right) + 4a_{0}^{2}b_{0}^{2} - k^{2}(d_{0}^{2} + b_{0}^{2} + 2a_{0}b_{0}) \right] \\ \times \left(2b_{0}(d_{0}^{2} + (a_{0} + b_{0})^{2}) \right] i - k h(k)g(k) .$$

$$III-2.6$$

The Fredholm determinant for the attractive local Bargmann⁶⁰⁾ potential V_1 is simple, and is given as

$$D_1^+(k) = \frac{k - ia}{k + ib}$$
, a, b > 0, III-2.7

and $D_1^+(k = ia) = 0$, giving only one bound state. The binding energy of this bound state, in units of \hbar^2/M , is a^2 . If there is no bound state, $D_1^+(k)$ is given as

$$D_1^+(k) = \frac{k + ia}{k + ib}$$
, a,b > 0. III-2.8

The constants a and b are real. The Bargmann potential in terms of these constants is given as:



$$V_1(r) = -8b^2\beta \frac{e^{-2br}}{(1 + \beta e^{-2br})^2}$$

with $\beta = \frac{b+a}{b-a}$ for the case when there is a bound state and $\beta = \frac{b-a}{b+a}$ otherwise.

The S-state phase shifts δ_1 generated by the potential V_1 are,

$$\tan \delta_{1}(k) = -\frac{\operatorname{Im} D_{1}^{+}(k)}{\operatorname{Re} D_{1}^{+}(k)}$$
.

The Bargmann potential III-2.9 generates an orthonormal set of regular wave functions given by Newton and Fulton $^{74)}$:

$$r|\phi_{p}\rangle \equiv \phi_{p}(r) = \left(\frac{p}{|b_{1}^{+}(p)|}\right) \sqrt{\frac{2}{\pi}} \left[\frac{\sin pr}{p} \left(1 - \frac{(b-a)b}{p^{2}+b^{2}}\right) + \frac{(1 + e^{-2br})}{(1 + \frac{b-a}{b+a}e^{-2br})} + \cos pr \left\{\frac{(b-a)(1 - e^{-2br})}{(p^{2}+b^{2})(1 + \frac{b-a}{b+a}e^{-2br})}\right\}.$$

If we change a to -a in III-2.11 we get the expression for the case when there is a bound state of energy $-a^2$. The normalisation of $\phi_p(r)$ is such that,

$$\iint_{0}^{\infty} \phi_{p}(r) \phi_{p}(r) dr = \phi(p-p') , \qquad III-2.12$$

. and it has the following asymptotic behaviour



III-2.9

$$\phi_{p}(r) + 0 \quad as \quad r + 0 ,$$

$$III-2.$$

$$+ \sqrt{\frac{2}{\pi}} \sin(pr + \delta_{1}) \quad as \quad r + \infty .$$

When there is a bound state, the bound state's normalised wave function generated by this potential is,

$$\langle r | \phi_{B} \rangle \equiv \phi_{B}(r) = \left(\frac{2a}{b^{2}-a^{2}}\right)^{1/2} (a+b) \frac{e^{-ar}(1-e^{-2br})}{(1+\beta e^{-2br})}$$
 III-2.14

The set of the phase equivalent class A potentials that we generate are all of the form

$$\langle k | V_{A} | k' \rangle = \frac{h^{2}}{M} [\langle k | V_{1} | k' \rangle + \eta(k) \eta(k')]$$
 . III-2.15

These potentials reproduce the input Fredholm determinant $D^+(k)$, i.e., the input phase shifts as well as the input bound state energy. Once the parameters a and b are fixed for the potential V_1 , the form factor $\tilde{\eta}(k)$ can be immediately obtained from III-1.27. For calculations we require $\eta(k)$ in the basis defined in III-1.1:

$$n(k) \equiv \langle k | n \rangle = \sum \langle k | \phi_p \rangle \langle \phi_p | n \rangle + \langle k | \phi_B \rangle \langle \phi_B | n \rangle$$

$$p$$

$$= \int_0^\infty \phi_p(k) \tilde{\eta}(p) dp + \phi_B(k) \tilde{\eta}_B \quad \text{III-2.1}$$



The regular solution $|\phi_p\rangle$ in III-2.16 obeys the following integral equation

$$|\phi_{p}\rangle = \cos \delta_{1}(p) |p\rangle + P \frac{1}{p^{2} - H_{0}} V_{1} |\phi_{p}\rangle$$
, III-2.17

 $\cos \delta_1(p)$ in the first term of the right hand side is for the proper asymptotic form of $|\phi_p\rangle$; without it, the asymptotic form would be $\phi_p(r) \sim \sin(pr + \delta_1)/\cos \delta_1$. Substituting III-2.17 into III-2.16 we get

$$n(k) = \cos \delta_{1}(k) \tilde{\eta}(k) + P \int_{0}^{\infty} \frac{\tilde{\eta}(p) \langle k | V_{1} | \phi_{p} \rangle}{\tilde{p}^{2} - k^{2}} dp$$

$$+ \phi_{\rm R}(k) \tilde{n}_{\rm R}$$

The above relation is used for numerical computation of $\eta(k)$ from $\tilde{\eta}(k)$. The details of numerical calculations are given in Appendix B.

We also construct the class B type of phase equivalent rank-two separable potentials:

$$k|V_{B}|k^{*}\rangle = \frac{\hbar^{2}}{M} \left[-\xi(k)\xi(k^{*}) + \gamma(k)\gamma(k^{*})\right]$$
, III-2.19

which reproduce the same input data as the partly nonlocal potentials of III-2.15. Additionally, the separable attractive part $-\xi(k)\xi(k')$ is chosen to be phase equivalent to the Bargmann potential $\langle k | V_1 | k' \rangle$, which yields, using

III-2.18



III-1.28,

Im
$$D_1^+(k) = -\frac{\pi}{2k} \xi^2(k) = -\frac{k(b+a)}{k^2 + b^2}$$
,
 $\xi(k) = \sqrt{\frac{2}{\pi}} \frac{k(b+a)^{1/2}}{(k^2 + b^2)^{1/2}}$. III-2.20

60

Inversity Linner

The bound state wave function, normalised to unity, generated by the separable potential $-\xi(k)\xi(k')$ is

$$\phi_{B}^{*}(k) = \sqrt{2a(b+a)} \xi(k)/(k^{2} + a^{2})$$
 . III-2.21

The bound state energy for the attractive separable part is again $-a^2$. Once the parameters of $\xi(k)$ are fixed, $\gamma(k)$ can bootained as before.

We construct the following potentials of class A and class B type. The parameters a and b are the same for both of them and are given in Table 3.1 ABLE 3.1 The partly nonlocal potentials of class A are denoted by A0 to A3, while those of class B are denoted by B0 to B3.

Ś

Potentials a	in Fm ⁻¹ b	o in Fm ⁻¹
A0 , B0	0.5000	1.0
Al, Bl 🕫	0.2500	1.0
A2, B2	0.1100	1.0
A3 , B3	0.1016	1.0

TABLE 3.1

TH UNIVERSITY LIGEARY

III-3 Half-Off-Shell T-Matrix Elements

The half-off-shell T-matrix has been defined earlier in II-1.7. In practice it is convenient to calculate the R-matrix defined in II-1.26 for real w. The half-off-shell R-matrix elements $R(p,k;k^2)$ in our basis III-1.1 for the relative S-state are given as

$$R(p,k;k^{2}) = V(p,k) - P \int_{0}^{\infty} \frac{V(p,q) \hat{R}(q,k;k^{2}) dq}{q^{2} - k^{2}} . III-3.1$$

We have used the method of matrix inversion for solving the integral equation III-3.1. This method is similar to one used by Haftel and Tabakin $^{46)}$. To replace the principal value condition by a smooth integral we add a zero term in III-3.1 and rewrite this as,

$$R(p,k;k^{2}) = V(p,k) - \int_{0}^{\infty} \frac{[V(p,q)R(q,k;k^{2})-V(p,k)R(k,k;k^{2})]dq}{q^{2}-k^{2}}$$
III-3.2

The infinite integral has been mapped to a finite sum using the transformation

$$q = C Tan \frac{\pi}{4} (1 - y) - 1 \le y \le 1$$
. III-3.3

62

TIVE BOILY
The constant C is adjusted to distribute the integration points into the most important region of integration. We choose the mesh for q such that no point of it is equal to k. We require to invert a $(N+1) \times (N+1)$ matrix, where N is the number of points in the Gauss quadrature used for integration. The starting energy corresponds to k which is always denoted by the (N+1)th point in the integration mesh. Using III-3.3 we can rewrite III-3.2 as

$$V(k_{i},k_{N+1}) = \sum_{j=1}^{N+1} F(k_{i},k_{j})R(k_{j},k_{N+1};k_{N+1}^{2}) . III$$

The F-matrix in III-3.4 is defined as:

$$F(k_{i},k_{j}) = \delta_{ij} + \omega_{j}' V(k_{i},k_{j}) ,$$

with

$$\omega_j = \omega_j / (k_j^2 - k^2) \qquad j \le N$$

 $= -\frac{N}{m} \frac{\omega_{m}}{(k_{m}^{2} - k^{2})} \quad j = N+1 \quad . \qquad III-3.6$

where the ω_j^* 's are the corresponding Gauss weights. The nonsingular F-matrix is inverted to obtain on- and off-energyshell R-matrix elements:

63

-3.4

III-3.5

DUNALUTA TUDDAD

$$R(k_{i},k_{N+1};k_{N+1}^{2}) = \sum_{j=1}^{N+1} F^{-1}(k_{i},k_{j})V(k_{j},k_{N+1}) \quad . \quad III-3.7$$

The on-shell R-matrix element denoted by R(k) is such that $k_i = k_{N+1}$ and related to the phase shift, $\delta(k)$, through the relation:

$$R(k) = -\frac{\otimes k}{\pi} \tan \delta(k) \quad . \qquad \text{III-3.8}$$

We have calculated $R(p,k;k^2)$ for the potentials of Table 3.1 for different k. Some of these are plotted in Figs. 1 and 2. The half-off-shell R-matrix elements are plotted in Fm⁻¹ against p in Fm⁻¹. The values of k chosen are 0.7 Fm⁻¹ and 1.0 Fm⁻¹ corresponding to 40.64 and 82.94 MeV lab. These values of k are chosen because the behaviour of the R-matrix corresponding to these energies is important for nuclear matter. We observe the following features from these plots.

- 1. When p > k, different potentials give different halfoff-shell behaviour. At p = k all potentials have the same value of R(p,k;k²) due to their phase equivalence.
- For p < k, the differences are not large. These
 differences increase for large k as shown in Fig. 2.

- 3. The half-off-shell elements for our potentials become a constant for large values of p. This is due to the fact that $R(p,k;k^2)$ in our basis has an extra factor of $\frac{2pk}{p}$ compared to the three-dimensional basis.
- The half-off-shell elements for the standard Tabakin 42) potential in our basis goes to zero indicating that the decay of the $R(p,k;k^2)$ for Tabakin potential is much faster*than our potentials.
- The half-off-shell variations given by the partly nonlocal 5. potentials (A0 , A3) are more than given by the correspondingly separable phase equivalent potentials (B0 , B3).

A simple qualitative explanation for the above observations is obtained through the following picture. The R-matrix for S-state may be written as 74 :

$$R(p, k^{2}) = \frac{p}{k} R(k, k; k^{2}) + \sqrt{\frac{2}{\pi}} (p^{2} - k^{2}) \int_{0}^{\infty} dr \frac{\sin pr}{r}$$

$$(\overline{u}(kr) - u(kr)) = U(kr)$$
III-3.9

where u(kr) is the wave function produced by the potential and $\overline{u}(kr)$ is the asymptotic form of this wave function interpolated up to r = 0. For a fixed k and p the first term in RHS of III-3.9 will remain the same for all the phase equivalent potentials. Thus the differences in

 $(\overline{u}(kr) - u(kr))$

 $R(p,k;k^2)$ will be coming due to the differences in the integral of III-3.9. When p > k, and large, the significant contribution to the integral will come from the term $(\overline{u}(kr) - u(kr))$ for small values of r. Since $\overline{u}(kr)$ is the same for all the phase equivalent potentials, it is u(kr) which will decide the magnitude of $R(p,k;k^2)$. We are looking at small values of r, where u(kr) will be suppressed if the potential has a large short range repulsion. Our potentials are such that the strength of the repulsive term falls as we approach towards the limiting case, i.e., A0 to A3 or B0 to B3. Thus u(kr), for small r, will be enhanced for the limiting cases, reducing the overall value of $(\overline{u}(kr) - u(kr))$. This reduction is seen as a fall in . $R(p,k;k^2)$ for the limiting cases. It is also to be noted from Figs. 1 and 2 that as a class, the partly nonlocal potentials have more effective short range repulsion than ST, whereas the separable potentials have less short range repulsion. Thus for large p the class A potentials have larger values of R(p,k;k²) than Standard Tabakin and the opposité is the case for class B potentials. This behaviour of $R(p,k;k^2)$ also suggests that the wave function u(kr) for the separable potentials is larger for small r than the corresponding wave function for the partly nonlocal potential. A similar trend is observed for the bound state wave functions of these phase equivalent potentials. It is known from earlier



studies ⁷⁶⁾ that the wave function for the separable potentials is larger at small r than those of the local potentials.

In the region p < k and small p, we have to examine the behaviour of $(\overline{u}(kr) - u(kr))$ for large values of r. In this region u(kr) is almost equal to $\overline{u}(kr)$ and $(\overline{u}(kr) - u(kr))$ is very much reduced. Therefore, we do not observe a large variation in the half-off-shell R-matrix elements. However, if we increase k some differences are observed due to changes in u(kr).

Although all these potentials are phase equivalent, they have considerably different matrix elements. The diagonal matrix elements in k-space are plotted in Fig. 3. For convenience in comparison, the pair (A0, B0) is plotted in Fig. 3a; likewise the other three pairs are plotted in Figs. 3b-3d. The following characteristics should be noted:

- (i) All the potentials have matrix elements which are attractive for low values of k (less than about 1:5 Fm⁻¹) but turn repulsive for higher momenta, reflecting the change in sign of the s-wave phase shift.
- (ii) Potentials of class A are more attractive for lower values of k, but become much more repulsive for higher momenta than the potentials of class B.

(iii) Figure 3c shows the near-limiting case with k_B^2 slightly greater than k_D^2 , while Fig. 3d shows the limiting situation when $k_B^2 = k_D^2$. It will be seen from these figures that in potentials of class B, a large repulsive hump develops for low values of k. This hump is particularly prominent in Fig. 3d for the potential B3, and results in its being much less attractive than the corresponding potential A3. The repulsive hump is absent in cases where k_B^2 is quite different from k_D^2 , as in Figs. 3a and 3b. This characteristic of the potentials has an important bearing in the nuclear matter calculation, and will be discussed later.

The origin of the repulsive hump that is produced for the limiting potentials can be simply understood in terms of the behaviour of the two-body bound state wave functions ϕ_B and ϕ_B^* that are generated by the attractive parts V_1 and $-\xi(k)\xi(k')$ respectively. Note, from III-1.33, that in the limiting case the overlap integral $\tilde{\eta}_B = \int_0^0 \phi_B(k)\eta(k)dk = 0$, and likewise for class B potentials, $\tilde{\gamma}_B = \int_0^0 \phi_B^*(k)\gamma(k)dk = 0$. In Fig. 4, we have plotted $\phi_B(k)$ generated by the Bargmann potential III-2.9 and $\phi_B^*(k)$ III-2.21 generated by $\xi(k)$ for the limiting situation. It will be noted that while both ϕ_B and ϕ_B^* are positive throughout, ϕ_B^* of the separable potential has substantially more high-k components, and, therefore, decays more slowly in k-space. Since the overlap integrals $\tilde{\eta}_{\rm B}$ and $\tilde{\gamma}_{\rm B}$ are identically zero in this case, both $\eta(k)$ and $\gamma(k)$ have to develop negative humps at lower values of k, as shown in Fig. 4; but the negative hump in $\gamma(k)$ is more pronounced to counterbalance the slower decay of $\phi_{R}^{\dagger}(k)$ with increasing k. This effect is enhanced in the diagonal matrix elements of the potentials where the repulsive parts appear as $\eta^2(k)$ and $\gamma^2(k)$. Fiedeldey 75), working with phase equivalent rank-two separable potentials of Yamaguchi type form factors, also noted similar effects in the limiting cases. According to him, this effect is responsible for the enhancement of the triton binding energy for the limiting potentials.





CHAPTER IV

NUCLEAR MATTER AND DEFECT WAVE FUNCTIONS

Nuclear matter is essentially an infinite medium with an equal number of neutrons and protons, with the Coulomb force between the protons switched off. Such an idealisation of the realistic nucleus is necessary to simplify the many-body problem, and nuclear matter may resemble the interior of a heavy nucleus where surface effects are small. The physical quantities associated with nuclear matter are the binding energy per nucleon (BE/A, A being the number of nucleons) and the saturation density ρ_s . Saturation in nuclear matter is necessary because we find from experiments that the average central density for the finite nuclei is practically constant. These quantities for nuclear matter can be calculated for a given two-body potential, and compared with those obtained from the mass formula 77 and high energy electron scattering 78experiments respectively. In nuclear matter, it is important to examine the variation of the BE/A and ρ_s with various phase equivalent potentials. These calculations are carried out with the hope that the results would lead us to discriminate between the various phase equivalent potentials.

These calculations are carried out in the framework of the Brueckner-Bethe-Goldstone many-body theory valid for strongly interacting fermions, particularly at low density. Good reviews $^{79-81}$ of nuclear matter theory and details of calculations are available in the literature.

In Section 1 we mention some salient steps and approximations which we use to calculate BE/A and ρ_s for our phase equivalent potentials. Section 2 is devoted to a description of the defect wave function and the calculation of the wound integral for our potentials. Finally in the last section we report and discuss our results.

IV-1 Nuclear Matter Calculation

Our method of calculating BE/A and ρ_s for nuclear matter is the same as that of Haftel and Tabakin ⁴⁶⁾, but our expressions are in a different basis. We have not tried to rederive the general expressions available in the literature. Only those expressions are given which are necessary for our calculation. In nuclear matter, the single particle wave function^{*} ϕ_k and the corresponding energy E(k) are

$$k_{i}(\mathbf{r}_{i}) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k}\cdot\mathbf{r}_{i}}$$
IV-1.1

 $E(k_{i}) = T(k_{i}) + U(k_{i})$ IV-1.2

where $T(k_i)$ is the kinetic energy, and $U(k_i)$ is the potential energy of a nucleon in state k_i . The latter is obtained from the interaction in a self consistent way. In nuclear matter each single particle state k_i is four-fold degenerate, and all states up to the Fermi momentum k_F are occupied. The Fermi momentum k_F is related to the nuclear matter density by the relation $\rho = (2/3\pi^2)k_F^3$. In Brueckner theory the effective N-N interaction is given by a G-matrix as

* The wave function ϕ_k is normalised in the volume $\widehat{\alpha}_i$ such that as A and $\Omega + \infty$, $\rho = A/\Omega$ remains constant.

 $\langle k_1 k_2 | G^{S,T} | k_{10} k_{20}; w \rangle = \langle k_1 k_2 | v^{S,T} | k_{10} k_{20} \rangle$

 $-\sum_{\substack{k_{1} \\ k_{2} \\ k_{1} \\ k_{2} \\ k_{2} \\ k_{1} \\ k_{2} \\ k_{2}$

In IV-1.3 V (the free N-N interaction) and G depend on the spin (S) and iso-spin (T) states of the nucleons. However, for simplicity in writing we have dropped the superscript S and T on V and G. The Pauli operator Q is such that it is 1 if $|k_1'|$, $|k_2'| > k_F$ and 0 otherwise. The starting energy w depends on the initial states of the interacting pair; in the special case when both k_{10} and k_{20} are occupied, it is

 $w = E(k_{10}) + E(k_{20})^{2}$

IV-1.4

IV-1.3

The relative and center of mass momenta are defined as

 $k_1 + k_2 = k_1 + k_2 = k_{10} + k_{20} = 2K$

 $k_1 - k_2 = 2k$. IV-1.5

From IV-1.3 and IV-1.5 the G-matrix becomes a function of $k \not k_0 \not K$ and w, which we denote by $G(k, k_0; K \ w)$. The initial and final state relative momenta are denoted as k_0 and k

respectively. The total nuclear matter energy E is:

$$E = 4 \sum_{\substack{k_1 < k_F}} \frac{\langle k_{10} | T | k_{10} \rangle + \frac{1}{2}}{k_1 k_2 \langle k_F | S, T} \sum_{\substack{(2S+1) (2T+1)}} \frac{\langle 2T+1 \rangle}{\langle k_1 | k_2 | G^{S, T} | W \rangle} + \frac{1}{2} \sum_{\substack{k_1 \\ k_2 < k_F | S, T}} \sum_{\substack{k_1 \\ k_2 > k_1 \rangle} \sum_{\substack{k_2 \\ k_1 > k_2 \rangle} \sum_{\substack{k_1 \\ k_2 > k_1 \rangle} \sum_{\substack{k_1 \\$$

where T is the single particle kinetic energy operator. Using IV-1.4 the G-matrix expression IV-1.3 becomes:

$$G(k, k_0; K w) = V(k, k_0) - \int dk' \frac{V(kk')Q(k', K)G(k', k_0; K w)}{E(k', K) - E(k_0, K)}$$

$$IV-1.7$$

with

$$E(k',K) - E(k_0,K) = \frac{h^2}{M} [(k'^2 - k_0^2) + U(|k+k'|) + U(|K-k'|)]$$

 $- U(|K+k_0|) - U(|K-k_0|)$. IV-1.8

The above G-matrix integral equation is difficult to solve because of the dependence on the center of mass momentum vector K. In order to simplify we follow the usual approach of using the angle-averaged Pauli operator and the effective mass approximation. The angle averaged Pauli operator \overline{Q} is

$$\overline{Q}(k',K) = 0 \qquad k' \leq \sqrt{k_F^2 - K^2}$$

$$= 1 \qquad k' \geq k_F + K$$

$$= \frac{K^2 + k'^2 - k_F^2}{2Kk'} \qquad \sqrt{k_F^2 - K^2} < k' \leq k_F + K , IV-1.9$$

and the single particle energy in the effective mass approximation is

$$E(k_{\alpha}) = \frac{h^{2} k_{\alpha}^{2}}{2M^{\star}} - U_{0} \qquad k_{\alpha} \leq k_{F}$$

$$= \frac{h^{2} k_{\alpha}^{2}}{2M} \qquad k_{\alpha} > k_{F} , \qquad IV-1.10$$

where M^* is defined as the effective mass of the nucleon. The G-matrix expression IV-1.7 along with the single particle energy of IV-1.10 is given by

$$G(k,k_{0};K) = V(k,k_{0}) - \int_{0}^{\infty} dk \frac{V(kk')\overline{Q}(k',K)G(k',k_{0};K)}{E_{>}(k',K) - E_{<}(k_{0},K)}$$

where

$$\frac{h^2}{M} E_{>} = \frac{h^2}{M} (K^2 + k'^2).$$

$$\frac{\hbar^2}{M} E_{<} = \frac{\hbar^2}{M^*} ((K^2 + k_0^2) - 2U_0)$$

IV-1.12

IV-1.11



בה וואיתבמכובר השבעמת

Using the method of partial wave expansion, we obtain the S-state G-matrix equation in our basis, defined in III-1.1, as:

$$G(k,k_{0};K) = V(k,k_{0}) - \int_{0}^{\infty} dk' \frac{V(kk')\overline{Q}(k',K)G(k',k_{0};K)}{E_{>}(k'K) - E_{<}(k_{0}K)}$$

$$IV-1.13$$

The above integral equation for G is solved by the matrix inversion method similar to the one used in solving III-3.1. The single particle spectrum $U(k_1)$ is calculated using the diagonal elements of the G-matrix. Following Haftel and Tabakin ⁴⁶⁾, we replace K by its average (K_{av}) in the calculation of G, where k_{av}^2 is:

$$\kappa_{av}^2 = k_1^2 + k_0^2 \qquad 2k_0 \leq k_F \leq k_1$$

$$= k_{1}^{2} + k_{0}^{2} - \frac{1}{4} (2k_{0} + k_{1} - k_{F}^{2}) (2k_{0} + k_{1} + k_{F})$$

for $k_F - k_i \leq 2k_0 \leq k_F + k_i$

IV-1.14

The single particle potential energy is given by

$$J(k_{1}) = \frac{\hbar^{2}}{M} 24 \left[\int_{0}^{\frac{1}{2}(k_{F}-k_{1})} dk_{0} G(k_{0},k_{0};K_{av}) + \frac{1}{2k} \int_{\frac{1}{2}(k_{F}+k_{1})}^{\frac{1}{2}(k_{F}+k_{1})} dk_{0} \frac{1}{2}(k_{F}-k_{1}) \frac$$

Since our potentials (A0-A3, B0-B3) are spin independent and are acting only in the S-state, we get the factor of 24 by summing over the spin and iso-spin states. The G-matrix equation IV-1.13 is solved with Brueckner's self consistency condition. Since the G-matrix depends on the U_0 and M* (IV-1.10), one starts with some suitable choice of U_0 and M* and calculates U(k). The new U(k) is then used to obtain another set of values for U_0 and M*. This process is repeated until self consistency is achieved. For obtaining the values of U_0 and M* from U(k) we use the following relations:

$$-U_0 = \frac{1}{4} (25\overline{U}_2 - 21\overline{U}_4)$$

IV-1.16

$$\frac{1}{m^{\star}} \equiv \frac{M}{M^{\star}} = (1 + \frac{35}{2}(\overline{U}_4 - \overline{U}_2)/k_F^2)$$

where the nth moment of U(k) is

$$\overline{U}_{n} = \frac{n+1}{k_{E}^{n}} \int_{0}^{k_{F}} U(k_{i}) k_{i}^{n} dk_{i} . \qquad IV-1.17$$

The values of U_0 and M* obtained in this way are dependent on the whole single particle spectrum $U(k_0)$, rather than just two values of $U(k_1)$ normally used. Once the selfconsistent values of U_0 and M* are obtained, the evaluation of the potential energy per particle in nuclear matter (PE/A) (IV-1.6) is straightforward, and is given by

$$\frac{PE}{A} = \frac{h^2}{M} \frac{12}{12} \int_{0}^{k_{\rm F}} dk_0 \left(1 - \frac{3}{2} \frac{k_0}{k_{\rm F}} + \frac{1}{2} \frac{k_0^3}{k_{\rm F}^3} \right) G(k_0, k_0; \vec{K}) .$$
IV-1

In the derivation of IV-1.18 we replaced the center of mass momentum K by its average value \overline{K} which is

$$\overline{K}^{2} = \frac{3}{5} k_{F}^{2} (1 - \frac{k_{0}}{k_{F}}) (1 + \frac{k_{0}/k_{F}}{3(2 + k_{0}/k_{F})}) \qquad \text{IV-1.19}$$

Note that $\overline{K} \neq K_{av}$ of IV-1.14. The average center of mass momentum (K_{av}) is defined for a pair of nucleons with fixed relative momentum, and the momentum of one of the particles is also fixed; whereas \overline{K} is the average when only the relative momentum of the pair is fixed. The BE/A is the sum of the average kinetic energy and the potential energy per particle. This is written as:

$$-\frac{BE}{A} = \frac{3}{10}\frac{k^2}{M}k_F^2 + PE/A$$

IV-1.20

78 ·

.18

IV-2 Calculation of the Defect Wave Function and the Wound Integral

In nuclear matter the unperturbed wave function of the relative motion of a pair of nucleons, $|\phi_{k}\rangle$, is a plane wave, but the perturbed wave function $|\psi_k\rangle$ is very different from the unperturbed wave function. The latter is also different from the scattered wave function, because it rapidly approaches the plane wave without any phase shift. The distance at which these two become the same is called the healing distance. The difference between the unperturbed and the perturbed wave function is defined as the defect wave function $|\chi_k\rangle$. The absence of the phase shift in the perturbed wave function of nuclear matter is due to the fact that all the scattering states are virtual. These virtual scattering states are considered to incorporate the high k components in the nuclear matter wave function. The defect wave function is given by

$$|\chi_{\mathbf{k}}\rangle = |\phi_{\mathbf{k}}\rangle - |\psi_{\mathbf{k}}\rangle$$

Using the following definition of ψ_k >,

$$\mathbf{V}|\psi_{\mathbf{k}}\rangle = \mathbf{G}|\phi_{\mathbf{k}}\rangle$$

IV-2.2

IV-2.1

and the equation IV-1.3, one obtains the relation of $|\chi_{\rm k}\rangle$ with G:

$$|\chi_k\rangle = \frac{Q}{e} G |\phi_k\rangle$$
, IV-2.3

where e is the energy denominator of the second term in IV-1.3. The healing property demands that

$$\langle r | \chi_{L} \rangle \rightarrow 0$$
 as $r \rightarrow \infty$. IV-2.4

The coordinate and the momentum space representations of the defect wave function are as follows:

$$\langle \mathbf{r} | \chi_{\mathbf{k}_{0}} \rangle \equiv \chi_{\mathbf{k}_{0}}(\mathbf{r}) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \langle \mathbf{k} | \frac{\overline{Q}}{\overline{e}} G(\mathbf{k}, \mathbf{k}_{0}; \overline{K}) | \mathbf{k}_{0} \rangle \sin \mathbf{k} r d\mathbf{k},$$

IV-2.5

and

$$\langle k | \chi_{k_0} \rangle \equiv \chi_{k_0}(k) = \frac{\overline{Q}(k, \overline{K}) G(k, k_0; \overline{K})}{E_{>}(k, \overline{K}) - E_{<}(k_0, \overline{K})}$$
. IV-2.6

one can define the wound integral κ as the integral of the square of $\langle k | \chi_{k_0} \rangle$ or $\langle r | \chi_{k_0} \rangle$ over the whole space, multiplied by the density ρ .

In order to make κ a dimensionless number and equivalent to the definition normally used in the literature, e.g., Haftel and Tabakin ⁴⁶⁾, we define it as:

 $\kappa = 60 \frac{\pi^2}{4k_0^2} \int_0^\infty |\langle k|\chi_{k_0}\rangle|^2 dk$

IV-2.7



where, the factor of 6 is due to the spin and iso-spin weighting, and $\pi^2/4k_0^2$ is due to the relation of G in our basis, to that of the three-dimensional basis generally used. This relation for the S-state is

$$G(k,k_0) = \frac{1}{4\pi k k_0} G(k,k_0)$$
 . IV-2.8

We have used IV-2.5 for the calculation of $\chi_{k_{a}}(r)$. For this calculation the Fermi momentum $k_{\rm F}$ is taken at the saturation point k_F^S and k_0 is taken to be 0.55 k_F^S . We have plotted the defect wave function for our potentials A0,A2 and B0,B2 (Table 3.1) in Fig. 5. We notice that for the partly nonlocal potentials (A0,A2) $\chi_{k_0}(r)$ is positive at small r indicating a short range repulsion, whereas for the purely separable potentials (B0,B2) the defect wave function is negative showing that the interaction is attractive. This behaviour can be understood from IV-2.5. For small distances, the main contribution to $\chi_{k_n}(r)$ comes from large values of k. The factor \overline{Q}/e would be positive for $k > k_{p}$ and zero for $k \leq \sqrt{k_F^2 - \overline{K}^2}$. Therefore, the sign of $\chi_{k_0}(r)$ will be decided by whether $G(k,k_0;\overline{K})$ is attractive or repulsive for large k. For our potentials the contribution to G from the second term in IV-1.13 is small compared to that of the first term, i.e., V(k,k₀) except for A0. This could be seen from the plots of $V(k,k_0)$ and $G(k,k_0;\overline{K})$ given in Fig. 6. Thus the sign of the G for large

k is determined by that of $V(k,k_0)$. In the case of partly nonlocal potentials, (A0,A3), $V(k,k_0) = V_B(k,k_0) + \eta(k)\eta(k_0)$. The first term $V_B(k,k_0)$ is attractive and goes to zero as $k \rightarrow \infty$; therefore, for a fixed k_0 and large k, $V(k,k_0) \rightarrow$ positive constant $\times \eta(k_0)$. The form factor $\eta(k_0)$ being positive, the sign of G will be positive for large k. Therefore, $\chi_{k_{\Lambda}}(r)$ will be positive and the magnitude of $\chi_{k_0}(r)$ will depend on the strength $n(k_0)$. In case of separable potentials $V(k,k_0) = -\xi(k)\xi(k_0) + \gamma(k)\gamma(k_0)$ (III-2.19) and the form factors are such that $\xi(k)$ or $\gamma(k)$ goes to a positive constant (C) depending on the parameters of the attractive part as k goes to ., hence for large k, $V(k,k_0) \neq C(-\xi(k_0) + \gamma(k_0))$. Thus the sign of $V(k,k_0)$ for large k will depend on the relative strengths of the two terms and not only on the repulsive form factor $\gamma(k_0)$. $k_0 = 0.55 k_F^S$ for the potentials (B0-B3), $\xi(k_0)$ is stronger than $\gamma(k_0)$, hence both $V(k,k_0)$ and G will be attractive. The defect wave function is negative for such potentials. It is to be noted that for some other choice of k₀, the signs may change. This reflects the peculiar characteristic of the separable potentials that attraction and repulsion are diffused over the whole range of the interaction in the momentum space. It is their relative strengths which decide whether the total interaction is attractive or repulsive. The same characteristic of separable potentials is also



responsible for the variation of $R(p,k;k^2)$ as noted in Chapter III-3.

IV-3 Nuclear Matter Calculations and Results

The partly nonlocal potentials (A0-A3) as well as the two term separable phase equivalent potentials (B0-B3) are used to obtain BE/A and ρ_s for the nuclear matter. Results for the standard Tabakin potential are included for comparison. The saturation density ρ_s is obtained by plotting the BE/A as a function of k_p and then noting the minimum of the curve. Self consistent calculations for a few k_p around the minimum are sufficient to provide an accurate saturation density. The direct matrix inversion method given in Chapter III has been used to solve the matrix equation, IV-1.13. The aim of this model study is twofold:

- (i) To compare the nuclear matter results for partly nonlocal potentials with those of separable potentials, and
- (ii) To study the dependence of these results on the limiting parameters for the two sets of phase equivalent potentials. These limiting parameters are defined in Chapter III.

Many nuclear matter studies have been made earlier with phase equivalent potentials. However, all such studies are made either with two term separable potentials 14-20) with Yamaguchi type form factors or with the potentials obtained by applying unitary transformations 45-47 to the two body Hamiltonian with soft core local interactions. As far as we are aware no systematic study has been made with partly nonlocal potentials. We confine ourselves to only S-state potentials as their contribution to the BE/A is by far the largest ⁶¹⁾. The results are presented in Table 4.1. The numbers in the first row refer to the saturation k_F^S in Fm⁻¹. The next row lists the values of κ evaluated by using IV-2.7° at 0.55 k_F^S . The average kinetic energy per nucleon at k_F^S is given in the next row. The potential energy per particle (PE/A), given by IV-1.18, can be split into two parts by using the relation:

$$G(k,k) = V(k,k) - \int_{0}^{\infty} dk' V(k,k') \chi_{k}(k')$$
, IV-3.1

where we have suppressed the \overline{K} dependence in the notation. We have accordingly shown the first-order contribution to PE/A, coming from V(k,k), and the higher order contributions arising from the last term in IV-3.1, separately in Table 4.1. It will be seen that all the potentials that we consider here are quite soft except for A0 and B3, with most





TABLE 4.1 Nuclear matter results for partly nonlocal phase equivalent (class A), rank-two separable phase equivalent (class B) and Standard Tabakin (ST) potentials. All energies are in MeV.

Q.		· 57	· ·	· · .				and the second sec	
	Partly nonlocal phase equivalent				Separable rank-two phase equivalent				
	potenti	als			potentials .				
	AO	A1	A2	А3	BO	o Bl	B 2	вź	ST
k ^S _F in Fm ⁻¹	1.85	2.05	2.08	2.1	2.05	2.0	1.89	1.83	2.1
ĸ	0.0296	0.0099	0.0040	0,0036	0.0050	0.0071	0.0129	0.0165	0.001
Kinetic energy per nucleon at k_F^S	42.58	52.28	53.82	54.86	52.28	49.76	44.44	5 41.66	54.86
lst order contribution to PE/A	-44.69	-76.64	-77.24	-73.94	-67.92	-58.29	-35.55	-24.29	-82.61
Higher order contribution to PE/A	·	- 9.11	- 4.90	- 5.91	-10.09	-13.60	-22.49	-26.93	- 2.06
Total PE/A	-71.58	-85.75	-82.14	-79.85	-78.01	-7.1.89	-58.04	-51.22	-84.67
BE/A	29.00	33.47	28.32	24.99	25.73	22.13	13.60	9.56	29.81

TABLE 4.1

of the binding coming from the first-order term. Accordingly, our calculated values of $\langle k | G | k \rangle$ look very similar to $\langle k | V | k \rangle$ as a function of k, and we do not plot these separately. We now go on to point out some interesting aspects of the results given in Table 4.1.

The first feature which we observe from the results is that the wound integral K decreases with the increase in the saturation density, in spite of a multiplicative factor o in the definition of κ given in IV-2.7. This is found to be true for potentials of both types. As was pointed out by Bethe 82), the contribution of sthe second term in the equation IV-3.1 to the PE/A, which is calculated using the equation IV-1.18, will increase less fast with k, as compared to that of the first term due to V(k,k). In the context of perturbation theory, this comes about due to the Pauli operator that effectively reduces the available phase space for the intermediate states with increasing kp. Additionally, with increasing k_p, the intermediate states k' has to increase, thereby damping the matrix elements $V(k,k^*)$, and also increasing the energy denominator. It is also to be noted that when higher order terms are appreciable, the wound integral k is large. Since these higher order terms help in saturation; larger values of k also imply, in general, smaller saturation densities, despite the multiplicative factor p in the definition of K.

A very interesting feature emerges from a comparison of the wound integral of class A and class B type potentials. For class A potentials as these approach the limiting situation, the wound integral k decreases monotonically,while just the opposite is the case for class B potentials. This behaviour can be understood (in the following way) Consider first the potential A0, where $k_{\rm B} = 0.5 >> k_{\rm D} = 0.1016 \, {\rm Fm}^{-1}$. In this potential, both the attractive and repulsive components are strong and this yields a large κ as well as a large higher order contribution to the potential energy per nucleon (Table 4.1). As the limiting situation is approached, both the attractive and repulsive components become weaker, though not the net $\langle k | V | k \rangle$. This makes the defect wave function $| \chi \rangle$ and hence κ smaller as observed in Table 4.1. This can also be seen 5, as $\chi_{k_{x}}(r)$ for A2 (nearly limiting) is smaller from Fig. in magnitude than that of A0. In the previous section of this chapter we have shown that for the class A type potentials $\chi_{k_{n}}(r)$ will depend on the strength of the repulsive term. An anamolous situation exists in this respect for the class β potentials. As we have discussed before, χ_k these potentials depend on the relative strengths ofattractive and repulsive part of the potentials. This relative strength corresponding to k_0 (= 0.55 k_p^S) increases as the limiting condition is approached, hence the wound

integral κ is increased. For the separable potentials of class B a large hump develops in the repulsive form factors (Fig. 4) mainly because of the slow decay of the bound state wave function $\phi_B^i(k)$ in k-space. This repulsive hump for potentials B2 and B3 not only reduces the attractive firstorder contribution from $\langle k | V | k \rangle$, but also increases the defect wave function and hence the wound integral. Note that this has the effect of increasing the higher order contributions in the potential energy as shown in Table 4.1. This increase, however, is not enough to compensate for the drastic fall in the first order contribution, with the net result that the binding energy per nucleon drops sharply as the limiting condition is approached.

Finally, we note that the partly nonlocal potentials A0-A3 give more binding in nuclear matter than the potentials B0-B3 when compared pair-wise. The maximum fall in BE/A for class A potentials is 8.47 MeV, while for class B potentials it is 16.17 MeV. Similar sharp fall in the BE/A for the limiting potentials have been observed by us ⁴⁴⁾ while working with rank-two separable phase equivalent potentials with Yamaguchi type form factors. We suggest that such a sharp drop is associated with the behaviour of the two-body bound state wave function of separable potentials in general and is absent for local or partly nonlocal potentials.

CHAPTER V.

OFF-SHELL EFFECTS IN THE REACTION p+p $\rightarrow \pi^{T}$ +d at threshold

As seen before the two-body experimental data, (i.e., the phase shifts and the energy of the bound state) do not, uniquely determine the N-N interaction. To investigate the interaction further we have constructed phase equivalent potentials which are all on the same footing as far as the above data is concerned, and examined nuclear matter. Another alternative would be to investigate the nature of the bound state wave functions that these phase equivalent potentials generate.

To be specific, we wish to analyse an experimental situation where the observable result is very sensitive not only to the tail of the deuteron wave function but also dependent on its short range characteristics. Such is the case in the behaviour of the cross section reaction $p+p + \pi^++d$ near the threshold of pion production. Woodruff ⁸³⁾ first calculated this cross section; his calculation was repeated by Koltun and Reitan ⁸⁴⁾ who obtained good agreement with the experimental results available at that time. They emphasized the sensitivity of the cross section to the shape of the deuteron wave function, a sensitivity that comes about due to the cancellation of the contributions of the S- and

D-wave components. It is expected that such a calculation will be sensitive to the D-state probability, P_D , as well as the shape of the wave function. Even if the deuteron binding energy as well as P_D are fixed, the wave function will change with the form of the interaction, i.e., it will be dependent on the off-shell behaviour of the T-matrix. The calculation of the threshold cross section may be of importance in determining the D-state probability P_D , and thus give valuable information about the tensor component of the N-N interaction in the ${}^3S_1 - {}^3D_1$ state. Recently Thomas and Afnan ${}^{62)}$ investigated this problem using various local and nonlocal separable potentials and found that the cross section is sensitive to both P_D and the nonlocality of the interaction.

We have undertaken the same calculation with the aim to sort out the various factors contributing to the sensitivity, in particular P_D of the deuteron and the off-shell behaviour of the interaction. We feel that only after such a study can any conclusions about fixing P_D from such an experiment be drawn. Keeping the interaction in the p-p channel fixed, which is taken to be either a Reid-Soft-Core ⁵ (RSC) local potential or a Tabakin ¹⁵) separable potential in ${}^{3}P_1$ -state. We study the variation of the threshold cross section with different deuteron wave functions generated by a set of phase equivalent potentials. These are generated by subjecting the deuteron wave functions for RSC and for our rank-two separable potentials ⁶³⁾ with varying P_D to the unitary transformation of Ref. 47). The deuteron wave function for RSC is taken from Reid's paper whereas analytical expressions for the wave functions of our potentials are used.

In Section V-1 we briefly review the pion-nucleon interaction and the method of calculating the cross section. In Section V-2 we report our results and conclusions.

V-1 Theory

The pion-nucleon interaction density has been taken to be of the form used by Woodruff $^{83)}$, and Koltun and Reitan $^{84)}$,

$$V(j) = H_0(j) + H_1(j) + H_2(j)$$

where j denotes the jth nucleon at position x_i ,

$$H_0(x) = (4\pi)^{1/2} (f/\mu) i \sigma \cdot \{ \nabla_{\pi} [\tau \cdot \phi(x)] \}$$

+
$$(2M)^{-1}[p\tau \cdot \pi(x) + \tau \cdot \pi(x)p]$$

$$H_1(x) = 4\pi \lambda_1 \mu^{-1} \phi^2(x)$$

$$H_2(x) = 4\pi\lambda_2 \mu^{-1}\tau \cdot \phi(x) \times \pi(x)$$
 . V-1.3

1.1

Here, σ and τ are nucleon spin and iso-spin operators, p is the nucleon momentum operator, and M and μ are the masses of the nucleon and pion respectively. The gradient ∇_{π} operates only on the pion field $\phi(x)$ and its conjugate $\pi(x)$. The Hamiltonian H₀ is the pion-nucleon interaction density. The

pseudoscalar coupling constant f^2 is related to g^2 by $f^2 = \frac{g^2}{4\pi} \left(\frac{\mu}{2M}\right)^2$. The first term in H₀ is the usual static A P-wave interaction density. The second term represents the S-wave part included to make H₀ Galilean-invariant. The terms H_1 and H_2 are direct and charge exchange interactions introduced 85,86) to explain the low-energy S-wave pion-nucleon scattering. Since at the threshold the pion emerges in the S-wave, the lowest order process is given by the second term in H_0 . It is customary to include the second order process which represents the pion produced at one nucleon, through either of the terms in H_0 , and rescattered into the S-wave by the second nucleon through H_1 or H_2 . These second order processes will compete with the lowest order process partly because of the factor μ/M in the second term of H₀. The coefficients λ_1 and λ_2 are fixed phenomenologically, although in principle these can be obtained from meson theory through a Foldy-Wouthuysen transformation 87). The transition amplitude up to the second order is given by

> $T_{dp} = \langle \chi_{d} | T | \chi_{p}^{+} \rangle$ $T = T_{0} + T_{1}^{+} + T_{2}^{-}$

 $T_0 = \Sigma H_0(i)$ i=1,2

 $T_{1,2} = \sum_{i \neq j}^{E} H_{1,2}(i) (E - H + ic)^{-1} H_{0}(j)$

V-1.4

The total Hamiltonian for a system of two interacting nucleons and a free pion is denoted by H in V-1.4 and E is the initial channel energy in the center of mass frame. The initial and final channel wave functions χ_p^+ and χ_d are the scattering wave function for the two protons and the wave function for the deuteron (ψ_d) plus a free pion, respectively. These are given as:

$$\chi_{p}^{+}(r) = \frac{1}{4}(4\pi \cdot 3)^{1/2} i \exp^{i\delta_{1,1}} \frac{u_{1,1}(r)}{r} y_{\pm 11}^{111}$$
 v-1.5

$$\psi_{d}(r) = \frac{u(r)}{r} y_{M0}^{0110} + \frac{w(r)}{r} y_{M0}^{2110}$$
. V-1.6

Here $y_{MT_z}^{LSJT}$ is the generalized spherical harmonic, \vec{r} is the relative coordinate of the two nucleons and the normalizations are given by

$$u_{LJ}(r)/r + \sin(pr) - \frac{1}{2} \pi L + \delta_{LJ})/pr$$
,

for the scattering wave and

$$\int_{0}^{\infty} (u^2 + w^2) dr = 1$$

v-1.7

for the deuteron. All coulomb interactions are ignored. Following Koltun and Reitan we first evaluate the matrix elements of T_{dp} between the initial (no pion) and final





 $(\pi^+ \text{ with momentum } q)$ pion states, and then using the nucleon states, V-1.5 and V-1.6, obtain the expression for the absolute square of the transition amplitude summed over initial and final spin states,

$$\sum_{\text{spin}} |T_{dp}|^2 = 8 (4\pi)^2 f^2 M^{-2} \mu^{-3} |\sum_{i=1}^{6} |I_i|^2 , \quad v-1.8$$

where

$$I_{1} = -\mu \int_{0}^{\infty} dr \ r^{2} \ \frac{u(r)}{r} \ (\frac{d}{dr} + \frac{2}{r}) \ \frac{u_{1,1}}{r} ,$$

$$I_{2} = -\frac{1}{\sqrt{2}} \int_{0}^{\infty} dr \ r^{2} \ \frac{w(r)}{r} \ (\frac{d}{dr} - \frac{1}{r}) \ \frac{u_{1,1}}{r} ,$$

$$I_{3} = C_{1} \int_{0}^{\infty} dr \ r^{2} \ \frac{u(r)}{r} \ F(r) \ (\frac{d}{dr} + \frac{2}{r}) \ \frac{u_{1,1}}{r} ,$$

$$I_{4} = C_{1} \ \frac{1}{\sqrt{2}} \int_{0}^{\infty} dr \ r^{2} \ \frac{w(r)}{r} \ F(r) \ (\frac{d}{dr} - \frac{1}{r}) \ \frac{u_{1,1}}{r} ,$$

$$I_{5} = C_{27} \int_{0}^{\infty} dr \ r^{2} \ \frac{u(r)}{r} \ F'(r) \ \frac{u_{1,1}}{r} ,$$

$$I_{5} = C_{1} \ \frac{1}{\sqrt{2}} \int_{0}^{\infty} dr \ r^{2} \ \frac{u(r)}{r} \ F'(r) \ \frac{u_{1,1}}{r} ,$$

and

$$C_1 = \lambda_1^{\mu} + 1.5\lambda_2$$
, $C_2 = \frac{M}{\mu} (2 + \frac{\mu}{2M})C_1$,

² 1/2

$$F(r) = \frac{1}{r} \exp(-\sqrt{\frac{3}{4}} \mu r)$$
, $F'(r) = \frac{d}{dr}F(r)$. V-1.10

The lowest order term of the T-matrix, T_0 , gives rise to I_1 and I_2 . The next and only higher order terms included, T_1 and T_2 , yield the other integrals $I_3 - I_6$. The cross section for the reaction $p+p + \pi^++d$ at the threshold is given by

$$\sigma = 16\pi f^{2} (\mu M)^{-3/2} \eta | \sum_{i=1}^{6} I_{i} |^{2} , \qquad v-1.11$$

where $n = |q|/\mu$. We have used two sets of the potentials for the initial p-p channel: i) RSC and ii) Tabakin. In the case of RSC, u_{1,1}(r) is obtained by solving the Schrödinger equation numerically; whereas for the Tabakin interaction it is obtained by analytically solving the Schrödinger equation in the momentum space and then fourier-transforming the momentum space wave function. Our deuteron wave functions are generated by i) RSC and a set of phase equivalent potentials to it and ii) a number of nonlocal rank-two separable potentials with varying P_D, and a set of phase equivalent potentials to each of them. It is not necessary to solve for the bound state of the Schrödinger equation in the case of every phase equivalent potential, rather one can directly obtain the corresponding deuteron wave function $\tilde{\psi}_{\mathbf{d}}$ by noting that $\tilde{\psi}_d = U\psi_d$ where U is the unitary transformation such that it generates a phase equivalent potential as follows:

Ð
$$H = T + V$$

$$\tilde{H} = UHU^{\dagger} = T + [UTU^{\dagger} - T + UVU^{\dagger}]$$

$$= \mathbf{T} + \tilde{\mathbf{V}}$$
,

V-1.12

where \tilde{v} is the transformed or phase equivalent potential. Following Haftel and Tabakin ⁴⁷⁾ we have considered the unitary transformation U such that,

$$U = 1 - 2\Lambda$$
 with $\Lambda^2 = \Lambda$, V-1.13

and

$$\langle \mathbf{r} | \Lambda | \mathbf{r}' \rangle = \sum_{\substack{JSMTT\\LL'}} g_{L}^{JST}(\mathbf{r}) g_{L'}^{JST}(\mathbf{r}') Y_{MT_{z}}^{LSJT}(\hat{\mathbf{r}}) Y_{MT_{z}}^{*L'SJT}(\hat{\mathbf{r}}) A_{LL'}^{JST}$$

The matrix A is such that $A_{LL}^{JST} = 1$ for uncoupled channels and

 $A_{LL}^{JST} = \delta_{LL}, \text{ or } \begin{pmatrix} \sin^2 \theta & \sin\theta \cos\theta \\ \sin\theta \cos\theta & \cos^2 \theta \end{pmatrix}, V-1.15$

for coupled channels. The radial functions $g_L(r)$ are real and square integrable. If we demand that $\langle r|\Lambda|r' \rangle + 0$, as $r + \infty$, $r' + \infty$, sufficiently rapidly, not only does \tilde{V} give the same phase shifts as V, but it also retains the long range behaviour of V. For the ${}^{3}S_{1} - {}^{3}D_{1}$ states we need only g_{0}^{110} and g_{2}^{110} , which we take of the form,

$$g_0(r) = C_0 \exp(-\alpha_0 r) (1 - \beta_0 r)$$

$$g_2(r) = C_3 \exp(-\alpha_2 r)r(1 - \beta_2 r)$$

or
$$C_3^{i} \exp(-\alpha_2^{i}r)r^2(1-\beta_2^{i}r)$$
, V-1.16

where we have omitted the superscript JST. The constants C_0 and $C_3(C_3^i)$ are determined by requiring $\int_0^{\infty} r^2 dr g_L^2(r) = 1$. We have five free parameters α_0 , β_0 , α_2 , $\beta_2(\alpha_2^i, \beta_2^i)$ and θ . The transformed deuteron wave function is:

$$\tilde{\Psi}_{d}(\mathbf{r}) = \Psi_{d}(\mathbf{r}) + g_{0}(\mathbf{r}) \gamma_{M0}^{0110} (B_{0} \sin^{2}\theta + B_{2} \sin\theta \cos\theta)$$

$$g_2(r) y_{M0}^{2110} (B_0 \sin\theta \cos\theta + B_2 \cos^2\theta)$$
 V-1.17

with B_0 and B_2 given by,

$$B_{0} = \int_{0}^{\infty} g_{0}(r) u(r) r dr ,$$

$$B_{2} = \int_{0}^{\infty} g_{2}(r) w(r) r dr .$$

V-1.18

The choice of $\theta = \pi/2$ (0) affects only the S (D) components of the deuteron wave function. Intermediate values of sin θ give mixed transformations, which change the D-state probability. The separable potentials with varying P_D are constructed using the formalism of Mehrotra and Sen Gupta ⁸⁸⁾ and modified by us $^{63)}$. The form of the potential acting in ${}^{3}S_{1} - {}^{3}D_{1}$ eigenchannel is:

$$\langle k | V | k' \rangle = - \frac{\lambda}{M} \{ g(k)g(k') - h(k)h(k') \}$$
. V-1.19

with $g(\underline{k}) = C(\underline{k}) + T(\underline{k})S(\underline{k})/\sqrt{8}$, and the forms of various factors are:

$$C(k) = 1/(\beta^2 + k^2)$$

$$f(k) = -tk^2/(\gamma^2 + k^2)^2$$

$$h(k) = nk^2/(\rho^2 + k^2)^2$$

and the momentum space tensor operator

S

$$(k) = \frac{3}{k^2} \sigma_1 \cdot k \sigma_2 \cdot k - \sigma_1 \cdot \sigma_2$$
. V-1.20

The parameter λ gives the strength of the potential, whereas t and n represent respectively the strength of the tensor component and the repulsion relative to the central attraction. The remaining three parameters β , γ and ρ are the so-called range parameters of the central attraction, tensor and repulsive form factors, respectively. It is possible to obtain, using this potential, analytical expressions for the binding energy α and quadrupole moment Q_d of the deuteron, and the triplet scattering length a_t . Given experimental values of these quantities, analytical expressions determine the strength parameters (λ , t and n) in terms of the range parameters (β , γ and ρ). We adjust these parameters until a reasonable phase shift is obtained. These potentials give different P_D . We take $Q_d = 0.282 \text{ Fm}^2$, $a_t = 5.4 \text{ Fm}$ and $\alpha = 0.2317 \text{ Fm}^{-1}$. The potentials we list yield effective range $r_{0t} = 1.78 \text{ Fm}$, the asymptotic ratio of the D- and S-wave component is obtained to be 0.025 to 0.033, and P_D is varied between 2.69 and 7.94%. The deuteron wave function may be easily obtained in analytic form as,

$$(\mathbf{r}) = \frac{\sqrt{2} \pi N_{1}}{\beta^{2} - \alpha^{2}} (e^{-\alpha \mathbf{r}} - e^{-\beta \mathbf{r}}) + \frac{\sqrt{2} \pi N_{2} \alpha^{2}}{(\rho^{2} - \alpha^{2})^{2}} \times [e^{-\alpha \mathbf{r}} - e^{-\rho \mathbf{r}} (1 + \frac{\rho^{2} - \alpha^{2}}{2\alpha^{2}} \rho \mathbf{r})]^{2}, \quad \nabla$$

and

$$w(\mathbf{r}) = \frac{\sqrt{2} \pi N_2 t}{(\gamma^2 - \alpha^2)^2} \left[(\alpha^2 + \frac{3\alpha}{r} + \frac{3}{r^2}) e^{-\alpha r} - (\gamma^2 + \frac{3\gamma}{r} + \frac{3}{r^2}) e^{-\gamma r} + \frac{(\alpha^2 - \gamma^2)^2}{2} (1 + \gamma r) e^{-\gamma r} \right],$$

$$v-1.22$$

where $N_2 = ZN_1$, 2 is a function of the parameters of the potential and N_1 is determined through the normalization condition $\int_0^\infty (u_1^2 + w_1^2) dr = 1$.

1.21

V-2 Results and Discussion

In this section we present the details and the results of our calculations. First we list in a schematic way the various interactions we studied.

Initial p+p

Final T+d

channel

channel

i. RSC

ii. RSC

iii. Tabakin

iv. Tabakin

RSC wave function and its unitary transforms wave function obtained from our

separable potentials and their unitary transforms

RSC wave function and its unitary transforms

wave function obtained from our separable potentials and their unitary transforms.

We use the same values of pion-nucleon coupling constants as those used by Koltun and Reitan ⁸⁴, $f^2 = 0.088$, $\lambda_1 = 0.005$ and $\lambda_2 = 0.045$. More recent values ⁸⁹, $f^2 = 0.0822$, $\lambda_1 = 0.0058$ and $\lambda_2 = 0.0487$, slightly change the



results but in no way affect our conclusions. We use $\mu = 139.4$ MeV and M/ $\mu = 6.73$. Table 5.1 lists the parameters of the unitary transformations considered, some of which are taken from Haftel and Tabakin ⁴⁷⁾. Transformations 1-5, 8-9 and 11-13, when applied to the wave function obtained for RSC, give the same electric form factor of the deuteron as obtained for RSC, within experimental error. Transformation 6 borders on an unacceptable fit to the form factor. We also study the transformations 7-10 and 14-16 which alter the form factor beyond the limits specified by the experimental error. Transformation 1-7 affect only ${}^{3}S_{1}$ component of the deuteron wave function, 8-10 affect only ${}^{3}D_{1}$ component, and the rest are mixed (0° < 0 < 90°). The mixed transformations change the deuteron D-state probability P_D.

Our two term separable potentials do not yield a good fit to the form factor data. This is in accordance with the fact that the two term separable potentials with simple analytical forms yield a Hulthen-type ⁹⁰⁾ of deuteron wave function $\psi_d(\mathbf{r})$ (modulated at short distances for the Tabakin potential), which, unlike the wave function of a local potential with repulsive core, does not fall rapidly enough for small r, as demanded by the deuteron form factor data. No fixed criterion exists, therefore, to classify, as in the case of RSC, the unitary transformations for the separable potentials, we note that generally all our unitary

TABLE 5.1 Parameters of unitary transformations in the ${}^{3}S_{1} - {}^{3}D_{1}$ channel. The asterisk in the first column indicates a transformation which on application to the RSC deuteron wave function does not yield a satisfactory fit to the deuteron < electric form factor as explained in the text.

ABLE 5.1

Transfor- mation	α ₀ (F ⁻¹)	^β 0 (F ⁻¹)	α ₂ (F ⁻¹)	β ₂ (F ⁻¹)	α ₂ (F ⁻¹)	β <mark>2</mark> (F ⁻¹)	θ (degrees)
ì	2.4	0.80	ý -	. –	· · · · ·	• ` <u>-</u>	90
2	2.4	0.83	_	-	-	-	9 <u>0</u>
3	2.7	0.88	-	-		_	90
4	2.7	0.91	-			, ∖ −	90
5	4.0	1.30	. .	, -		-	90
×6*	2.2	0.80	-	-	 _	· •	90`
7*	2.2	0.83			€ , ➡ -	-	90
8	· -	s –	. —		3.50	0.80	0
9	_	-	2.4	0.72	-		0
10*	.	-		· . –	3.50	0.50	0
- 11	2.4	0.83	2.4	0.72	-	, .	60
12	2.4	0.83	2.4	0.72			45
13	2.4	0.83	2.4	0.72	· .		30
14*	2.7	0.88	-	- -	3.50	0.50	60
15*	2.7	0.88	• •	-	3.50	0.50	45
16*	2.7	0.88		• •	3.50	0.50	. 30 .

Ł

transformations change the form factor in the direction of agreement with the experiments.

In Table 5.2 we list our results for separable potentials with varying P_D in the π^+d channel. For each separable potential we have generated phase equivalent potentials as shown in column 2. For each such potential in the π^+d channel we calculate the cross section σ_R/η with RSC in the p-p channel and σ_S/η with the Tabakin potential in the p-p channel. Table 5.3 gives similar results for RSO in the π^+d channel where we also show the deuteron quadrupole moment Q_d for the various phase equivalent potentials. Results for the mixed transformation are listed only for the RSC and separable potentials V_2 and V_5 ; out of $V_1 - V_6$, V_2 yields the best phase shift fit and V_5 has $P_D \gtrsim 7$ %. For the mixed transformation the changed values of P_D are also indicated.

Tables 5.2 and 5.3 reveal the following trends:

1) For every case $\sigma_R < \sigma_s$. Thus the cross section depends on the non-locality of the nucleon-nucleon interaction. The scattering wave function $\chi_p^+(r)$ for small r is smaller for RSC than for Tabakin potentials, hence $\sigma_R < \sigma_s$. A comparison of the cross sections with RSC and V_5 ($P_D \gtrsim 7$ %) in the π^+d channel leads to the same conclusion.

- 2) The cross section, either σ_R or σ_s , generally decreases with increasing P_D , i.e., with increasing strength of the tensor force. However, this is not the case for the mixed transformations which change P_D by distorting the short range tensor-component.
- 3) The cross section is sensitive to the off-shell behaviour of the nucleon-nucleon interaction. This sensitivity is very much reduced in the case of RSC in π^+d , especially if we omit transformations 6, 7, 10, 14, 15 and 16 that effect the form factor considerably. We find that offshell variations may produce very different results for separable potentials than for a local interaction, e.g., transformation 1 which changes σ_R for RSC by less than 7% changes it for the separable potentials by 200-300%.

It is thus clear that the cross section for the threshold s-wave pion production is sensitive to the shape of the entire deuteron wave function, not just the P_D which is an integral over it. This point is further illustrated by the structure of the integrals $I_1 - I_6$ defined in V-1.9, that enter the expression V-1.8 for the cross section. The presence of the damping factor F(r) means that relatively short range (r < μ^{-1}) part of the deuteron wave function is important in $I_3 - I_5$. The integrals I_4 and I_6 involving w(r) are expected to be smaller than I_3 and I_5 , which depend on u(r). Further I_5 is enhanced over I_3 by the factor

TABLE 5.2 Cross section for threshold s-wave pion production for a number of non-local separable potentials with varying deuteron D-state probability P_D and the phase-equivalent potentials for each of these. The last two columns give σ_R/n and σ_S/n which are the cross sections obtained with the 3P_1 Reid softcore and Tabakin potentials, respectively in the p-p channel. The index 0 in the second column refers to the untransformed potential.

				•
•			· · · · · ·	
•	TABLE	5.2		
Potential in the	Unitary	•		
T [†] d channel	Transfor-	PD	σ _R /η	σ _s /η
$(3_{\rm S} - 3_{\rm D})$	mation	(\$)	(µb)	(μb)
(3, 21			•	
v ₁	0	2.69	312.7	349.6
	. 1	ن	94.0	116.0
•	2	· · ·	128.9	155.2
	5		288.4	349.7
•	7		174.9	201.8
а А	9		345.0	385.8
•	10	a	262.2	293.1
v _a	0	3.76	* 264.3	295.3
4	1		88.2	107.5
	`2		120.4	143.4
	3	•	135.0	165.3
	· · · · · · · · · · · · · · · · · · ·	· · · · · · · · · · · · · · · · · · ·	160.7	193.1
x	5		248.2	295.3
•	6		120.6	* 140.2
~	7		166.7	190.7
```	8		- 286.6	321.4-
<u>_</u>	9	•	280.6	313.4
	10		196-4	219.3
	11	4.48 ~	109.5	128.3
		<b>6</b> 2	134.7	154.7
ب ج		e.		

• •	•	<b>v</b> .	
	7	,	
TABLE	5.2 - continued		`108
13	4.34	178.2	201.4
. 14	3.43	88.5	111.7
15	4.16	91.7	112.9
16	4.70	112.5	133.3
v. 0	4.38	226.9	255.1
3	L	-103.6	123.7
2	2	135.8	158.9
(	5	223.8	255.1
	1.	187.7	213.1
ç	• • • • • • • • • • • • • • • • • • •	232.3	261.1
· 16	)	153.6	172.8
V. (	5.91	193.6	216.1
4		90.2	106.4
	2	119.3	138.1
	5	192.1	216.1
	7	168.8	189.7
·	9	174.8	195.1
10	0	105.0	117.1 。
V_ \ (	6.87	172.2	191.1
5	1	84.2	98.2
	2	111.5	127.7
· · · · · · · · · · · · · · · · · · ·	3	114.5	133.8
	4	134.1	154.0
5 4	5	172.2	191.1
	6	120.1	135.3
	7	160.3	178.4
•		s f succession	

TABLE 5.2 - continued

v₆

157.2 173.8 8 156.4 140.9 9 87.3 10 78.9 98.5 6.89 85.8 11 7.02 87.3 99.3 12 110.8 7.07 13 98.4 57.9 4.95 44.7 14 32.3 42.6 5.15 15 42.3 33.4 6.20 16 7.94 178.5 196.2 0 76.5 . 64.9 1 90.3 104.1 2 196.2 170.7 5 132.3 147.3 7 144.2 . 131.5 9 78.8 72.4 10

TABLE 5.3 Cross section for threshold s-wave pion production for Reid soft-core and its phase-equivalent potentials. The notation is the same as in Table 5.2.

TABLE 5.3

Potential in the Unitary P D  $\pi^+$ d channel σ_R/η σ_s/η Q_d (F²) Transfor-(µb)  $({}^{3}S_{1} - {}^{3}D_{1})$ (**8**) (µb) mation

未

RSC

3		· +	<i>.</i> *		•
0		6.47	.280	152.6	162.0
1	2	<b>.</b> .	.280	142.6	151.7
2	•	<b>,</b>	.280	172.9	182.7
3			.279	161.1	170.2
° <b>4</b>		•	.280	179.4	187.3
5		• .,	.279	167.9	162.1
6	7	0 · ·	.279	195.5	206,8
7	· · ·		.277	239.9	252.6
8			.278	162.6	173.5
9	· · · ·	50 1	.281	147.2	156.2
10 .	н Т. 		.272	73.5	76.2
.11		6.49	.277	172.2	182.5
12	3	6.49	.278	166.6	176.7
13 .		6.48	.278	159.2	169.0
14		5.37	.279	84.2	93.2
15	ی • 	4.90	.276	60.2	67.2
16		5.21	.276	50.6	56.1

 $C_2/C_1 \gtrsim 12$ . The asymptotic part of the deuteron wave function becomes important through I, and I,. To understand the nature, of the contribution of these integrals we replaced u(r) and w(r) by Hulthen wave functions. This, in fact, turns out to be a very good approximation for  $I_1$  and I₂. We used the fourier expansion  $g_{11}(k)j_1(kr)k^2dk$  for  $u_{1,1}/r$ . The operators (d/dr + 2/r) and (d/dr - 1/r)operating on  $j_1(kr)$  yield  $kj_0(kr)$  and  $(-kj_2(kr))$  respectively. One immediately expects from this that  $I_1$  and  $I_2$  (and similarly  $I_3$  and  $I_4$ ) would come out preferably with opposite signs. Moreover, the integrals  $dr r^2(u/r)j_0(kr)$  and dr  $r^2(w/r)j_2(kr)$  which determine the relative magnitudes of I, and I, can be easily worked out analytically in the Hulthen approximation, and this reveals the curious fact that the contribution from w(r) is so enhanced that  $I_1 \ 2 \ I_2$  for  $P_D \stackrel{\sim}{\sim} 6$ %. We note that such enhancement is suppressed in  $I_4$ by the damping factor F(r). We list the values of  $I_1 - I_6$ for the untransformed potentials  $V_1$ ,  $V_4$ ,  $V_5$  and RSC in the  $\pi^{T}d$  channel and both RSC and Tabakin in the p-p channel in Table 5.4. This table confirms our observations about the structure of the integrals given in V-1.9. The largest contribution to the cross section comes from I₅ and off-shell variations affect it considerably. At the same time, it is pointed out that extremely short range transformations, e.g., transformation 5 of Table 5.1, distort  $\psi_d(r)$  only for very small r (r <<  $\mu^{-1}$ ) and hence do not alter the cross section.



	Potential in	Potential in	• .	<b>7</b> 1.	с. С	, , , , , , , , , , , , , , , , , , ,		1 <b></b>
•	the $\pi^+$ d channel	the p-p channel	I.	1 ₂ ,	1 ₃	1 ₄	I I	I ₆
	( ³ s ₁ - ³ D ₁ )	( ³ P ₁ )			ر •			<b>*</b> * *
	v ₁	Tabakin	1043	.0462	.0157	0008	1613	0161
	(P _D ² = 2.69%)	RSC	1008	.0434	.0146	0007	1502	0148
•	v ₄	Tabakin	0809	.0807	.0134	0012	1504	-`.0351
	(P _D = 5.91%)	RSC	0799	.0745	.0123	0010	1378	0321
	V	Tabakin ,	0785	.0928	.0134	0012	1472	0446
	(P _D = 7.94%)	RSC	0771	.0850	.0124	0011	1361	0407
	RSC	Tabakin	0665	.0858	.0111	0014	1417	0373
	$(P_{D} = 6.478)$	RSC	0695	.0779	.0103	0012	<b>1294</b>	0339
	· · · · · · · · · · · · · · · · · · ·	,	,	•				

TABLE 5.4

significantly. If  $P_D$  is increased the cross section is found to reduce, since the change in  $(I_1 + I_2)$  tends to cancel the change in  $(I_5 + I_6)$ .

Are we in a position to determine P, and, therefore, the strength of the tensor component of the N-N interaction, as conjectured by Thomas and Afnan, ⁶²⁾, from the threshold s-wave pion production? The answer can hardly be affirmative, at least, in the context of one or two term separable potentials of simple analytical form. It might be possible that this reaction enables one to fix  $P_n$  and, therefore, the strength of the tensor force in the context of a local interaction, such as RSC, which reproduces deuteron form factor data in addition to the two body data. We note that the variation in the cross section is severely constrained (to about 10%) in the case of potentials, phase equivalent to a local potential (RSC), which reproduce the deuteron form factor within experimental error. Similar constraints on the variation of the binding energy of the triton have' been observed recently by Haftel 49). However, we note that the value of the threshold cross section ( $\sigma = 170.2 \ \mu b$ ) we get using RSC in both p-p and  $\pi^+d$  channels and the more recent values of the parameters  $f^2$ ,  $\lambda_1$  and  $\lambda_2$  (Samaranayake and Woolcock (89) fall considerably short of the experimental result ( $\sigma = 240 \pm 20$  µb) of Rose ⁹¹. The experimental result of Rose itself is in considerable disagreement with

earlier work by Crawford and Stevenson ⁹²⁾ who found  $\sigma = 138 \pm 15 \ \mu$ b. Moreover the contribution to the cross section from terms of higher-order in pion-nucleon interaction remains to be estimated. We have seen that the contribution of the second-order terms (sum of the integrals  $I_3 - I_6$ ) dominates over the contribution of the first-order terms (the sum of  $I_1$  and  $I_2$ ). This suggests that the higherorder effects may be important. Until the theoretical and experimental situation becomes clearer, we feel that, the cross section for s-wave pion production near the threshold in p+p +  $\pi^+$ +d can at best only support other more conclusive sources, e.g.,  $\pi$ -d elastic scattering data (Michael and Wilkin ⁹³⁾), in determining the D-state probability of the deuteron and hence the relative strength of the tensor to the central component of the nucleon-nucleon interaction.

# CHAPTER VI

## CONCLUSIONS

In this chapter we summarize our main conclusions of this study.

We have generalised the Jost-Pais theorem for nonlocal central and noncentral potentials. We have shown that, unlike the local case, the Jost function for the nonlocal potential is given by the ratio of the Fredholm determinants of the kernel of the Lippmann-Schwinger equation for the scattering solution to that of the regular solution.

Adapting the formalism of Fuda, we have developed a method of constructing partly nonlocal phase equivalent potentials. These potentials have a local attractive part superimposed with a repulsive rank-one separable potential. The attractive part has a comparatively larger range than the repulsive part. Although we constructed such potentials only in the S-state, the method can be used in any uncoupled channel. Rank-two separable phase equivalent potentials can also be generated using the same method.

The study of the partly nonlocal potentials along with rank-two separable potentials reveals the importance of the role of the two-body bound state wave function in nuclear matter results. The separable potentials, for this study, have been constructed such that their attractive part is independently phase equivalent to the attractive part of the partly nonlocal potentials. The special case when the repulsive form factor of the potential is orthogonal to the two-body bound state wave function is referred to as the limiting potential, which is unique for a given attractive potential. For such a situation we have found a large drop in nuclear matter binding energy for the separable case. No such drastic drop has been observed in the corresponding situation of the partly nonlocal potentials. This is attributed to the high momentum components of the bound state  $\heartsuit$ wave function of the attractive separable potential, which is very different from that generated by the local Bargmann potential. These potentials have been constructed and studied for the S-state only, because its contribution to nuclear matter binding energy is the largest.

Different phase equivalent potentials have been generated by using short range unitary transformations on the two-body Hamiltonian with a Reid soft-core or separable potential. These potentials have a tensor component and generate different deuteron wave functions. The deuteron wave functions generated by different phase equivalent potentials have been used in the study of the threshold cross section for the reaction  $p+p + \pi^++d$ . It has been shown that the cross section is sensitive to the form of the entire deuteron wave function as well as the D-state probability of the deuteron. Thus it would not be reasonable to expect that this reaction would enable us to fix the D-state probability, which is only an integrated effect over the square of the D-state deuteron wave function. It has been shown that the variation in the cross section is reduced when the phase equivalent potentials are constrained to reproduce good deuteron form factors.

The results of nuclear matter and the threshold cross section for  $p \pm p \neq \pi^+ \pm d$  strongly indicate that for a meaningful comparison of the off-shell effects in many-body problems, the phase equivalent potentials should be constrained to generate deuteron wave functions which can fit the available experimental data on the form factor.

#### APPENDIX A

### Generalizations of the Jost-Pais Theorem for Nonlocal Potentials¹

## YOGESHWAR SINGH² AND C. S. WARKE³

Physics Department, McMaster University, Hamilton, Ontario

Received December 4, 1970

We derive general expressions for the Jost function of a nonlocal central potential in any angularmomentum state, and generalize these results when there is a tensor component in the nonlocal potential. These results relate the Jost function to the Fredholm determinants and are generalizations of the Jost-Pais theorem for the corresponding local case.

Nous déduisons des expressions générales pour la fonction de Jost d'un potentiel central non local dans n'importe quel état de moment cinétique, et généralisons ces résultats au cas où une composante tensorielle est présente dans le potentiel non local. Ces résultats relient la fonction de Jost aux déterminants de Fredholm et constituent des généralisations du théorème de Jost-Pais pour le cas local correspondant.

Canadian Journal of Physics, 49, 1029 (1971)

#### 1. Introduction

The theorem that the s-state Jost function of a central local potential is equal to the /Fredholm determinant of the integral equation of a scattering solution (Jost and Pais 1951) has been generalized by Newton to higher partial waves (Newton 1961). It is also, known (Newton 1966) that the determinant of the Jost matrix, for the local noncentral potential, is equal to the Fredholm determinant of the set of the integral equations of the scattering solution. Recently, it has been of the spectral function to the new Jost function. shown (Warke and Bhaduri 1971) that the s-state Jost function for a nonlocal potential is equal to the ratio of the Fredholm determinants of the integral equations of the scattering solution to that of the integral equation of the regular solution. In the first section of this paper we generalize this result to the nonzero angular momenta. The derivation of a similar relation between the determinant of a Jost matrix of a noncentral, nonlocal potential and the Fredholm determinants of the corresponding integral

equations is given in the second section. This change in the relation of the Jost function  $f_i(k)$ to the Fredholm determinant, in going from local to the nonlocal potential, gives rise to additional zeros of  $f_i(k)$  on the positive imaginary k axis. These poles do not correspond to bound states. These redundant zeros of  $f_i(k)$  could easily be removed with the change of asymptotic boundary, condition for the Jost solution of a nonlocal potential. This would trivially modify the relation

#### 2. Jost Function of Higher Angular Momenta with a Central Nonlocal Potential

In this section we assume a central, nonlocal interaction between two scattering particles. Inorder to clarify the notations and the approach of the derivation of the Jost function, we will treat this section in more detail. The usual expansion of the wave function in partial waves gives the radial Schrodinger equation

[1] 
$$-\frac{d^2}{dr^2}\langle r|\psi_i\rangle + \frac{l(l+1)}{r^2}\langle r|\psi_i\rangle + \gamma\langle r|V\psi_i\rangle = k^2\langle r|\psi_i\rangle$$

where the nonlocal two-body interaction  $\gamma U$  in the diagonal  $l_j$  channel is related to V as follows:

$$\frac{2\mu}{\hbar^2} \gamma r \langle r | U | r' \rangle r' = \gamma \langle r | V' | r' \rangle$$

Here y is the strength of the potential and µ is the reduced mass. Throughout this paper we will suppress the k dependence of the functions in order to simplify the notation. The corresponding Lippmann-

On leave of absence from the Tata Institute of Fundamental Research, Bombay, India.

Work supported by the National Research Council of Canada.

^{&#}x27;On leave of absence from Roorkee University, Roorkes, India.

## CANADIAN JOURNAL OF PHYSICS. VOL 49, 1971

Schwinger equation of the outgoing scattering solution is

[3] 
$$\langle r|\psi_{i}^{+0}\rangle = \langle r|\psi_{i}^{+0}\rangle + \gamma \langle r|G_{i}^{+}V\psi_{i}^{+}\rangle$$
  
where  $\langle r|\psi_{i}^{+0}\rangle = \mu fr$ 

and

[4] 
$$\langle r|G_{l}^{+}|r'\rangle = -\frac{1}{k}e^{-i\epsilon l}u_{l}(kr_{l})w_{l}^{+}(kr_{l})$$

The functions  $u_1(z)$ ,  $v_1(z)$ , and  $w_1^{\pm}(z)$  are defined in terms of spherical Bessel functions  $j_1(z)$  and Neuman functions  $n_1(z)$ 

$$[5] u_l(z) = zj_l(z), v_l(z) = zn_l(z), w_l^-(z) = -[v_l(z) + iu_l(z)], w_l^+(z) = e^{iu_l}w_l^-(z^*)$$

Similarly the integral equation of the regular solution is.

$$\langle r|\phi_i\rangle = \langle r|\phi_i^0\rangle + \gamma \langle r|G_iV\phi_i\rangle$$

where

$$\langle r | \phi_l^{u} \rangle = (2l+1)!! k^{-(l+1)} u_l(kr)$$

[7] 
$$\langle r[G_i|r' \rangle = [u_i(kr')v_i(kr) - u_i(kr)v_i(kr')]/k$$
 for  $r' < i$ 

And it obeys the boundary condition

$$\lim_{t \to 0} r^{-(l+1)} \langle r | \phi_l \rangle = 1$$

Finally the Jost solutions are given by

[9] 
$$\langle r U_i^{\dagger} \rangle = \langle r U_i \cdot \partial \chi + \gamma \langle r | G_i V f_i^{\dagger} \rangle$$

In [9],  $G_1$  is the transpose of  $G_1$  defined in [7], and

$$[10] \qquad \langle r|f_{1}, 0 \rangle = e^{-i\pi t/2} w_{1}^{2} (k$$

The boundary conditions for  $\langle r|f_1^{\pm} \rangle$  are

$$\lim_{r \to \infty} e^{\frac{1}{r} \Delta r} \langle r | f_i^{\pm} \rangle = 1$$

In terms of these solutions the Jost functions are defined as follows

[12] 
$$f_{l}^{\pm}(k) = \frac{k^{l} e^{\pm i t / 2}}{(2l-1)!!} \lim_{r \to 0} r^{l} \langle r | f_{l}^{\pm} \rangle$$

From [L] and [11] one can prove in general that the Wronskian

$$[13] \qquad \qquad \mathcal{W}[\langle r|f_i^+\rangle, \langle r|f_i^-\rangle]_{r=0} = \mathcal{W}[\langle r|f_i^+\rangle, \langle r|f_i^-\rangle]_{r=0} = -2ik$$

Using [6], [9], and [13] one obtains ---

$$[14] \quad \langle r|\phi_i \rangle = \frac{(2l+1)!!k^{-1}}{2ik} [e^{-\frac{kk}{2}}f_i^{-}(k)\langle r|f_i^{+}\rangle - e^{\frac{kk}{2}}f_i^{+}(k)\langle r|f_i^{-}\rangle]$$

and . [15]

$$f_i^{+}(k) = f_i^{-*}(k) = 1 + \gamma k^2 e^{-kk} \langle w_i^{+*} | \mathcal{V} | \phi_i \rangle / (2l+1)$$

From [6], [9], and [15] it can further be shown that

[16] 
$$\frac{d}{dr}f_i^+(k) = k^1 e^{-int/2} \langle f_i^- | V | \phi_i \rangle / (2l+1)!!$$

## ENCH AND WARKE: GENERALIZATIONS OF THE ROST-PAIS THEOREM

Following an approach similar to that used in Warke and Bhaduri (1971) we define a function

$$\langle r|\mathcal{F}_i|r\rangle = \langle r|\mathcal{G}_i^+|r'\rangle - \langle r|\mathcal{G}_i|r'\rangle$$

Where the interacting Green's functions are given by

[18] 
$$\mathcal{I}_{l}^{+} = G_{l}^{+} + \gamma G_{l}^{+} \mathcal{V} \mathcal{I}_{l}^{+} = G_{l}^{+} + \gamma \mathcal{I}_{l}^{+} \mathcal{V} G_{l}$$
$$\mathcal{I}_{l} = G_{l} + \gamma G_{l} \mathcal{V} \mathcal{I}_{l} = G_{l} + \gamma \mathcal{I}_{l} \mathcal{V} \mathcal{G}_{l}$$

It can be derived from [4], [7], and [18] that

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}-k^2\right)\langle r|F_l|r'\rangle+\gamma\langle r|VF_l|r'\rangle=0$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r'^2}+\frac{l(l+1)}{r'^2}-k^2\right)\langle r|F_i|r'\rangle+\gamma\langle r|F_iV|r'\rangle=0$$

and that  $F_i$  obeys the boundary conditions

$$\langle r|F_{i}|r' \rangle = 0, \frac{d}{dr} \langle r|F_{i}|r' \rangle = -\frac{(l+1)}{(2l+1)}$$

[20]

$$\langle r|F_l|r \rangle = e^{\alpha r} x$$
 (a function of l, k, and r)

The solution  $F_1$  of [19] that satisfies the boundary conditions [20] is

$$[21] \qquad \langle r|F_i|r'\rangle = -\frac{k^i e^{-ikl/2}}{(2l+1)!!f_i^+(k)} \langle r|\phi_i\rangle \langle r'|f_i^+\rangle$$

Using [21] and the property that  $\langle r|f_1^- \rangle$  is the complex conjugate of  $\langle r|f_1^+ \rangle$ , eq. [16] further reduces to

$$\log f_1^*(k) = -\int_0^{\pi} \operatorname{Tr} F V \, \mathrm{d} \gamma'$$

In the derivation of [22] we also used the fact that  $f_t^*(k) \to 1$  as  $\gamma \to 0$ . It is now straightforward to prove from [17], [18], and [22] that

[23] 
$$f_i^+(k) = \exp\left[\operatorname{Tr}\log\left(1 - \gamma G_i^+ V\right)\right]/\exp\left[\operatorname{Tr}\log\left(1 - \gamma G_i V\right)\right]$$
$$= \operatorname{Det}\left(1 - \gamma G_i^+ V\right)/\operatorname{Det}\left(1 - \gamma G_i V\right)$$

In the case of a local potential the determinant in the denominator of [23] becomes

[24] 
$$Det(1 - \gamma G_i V) = 1$$

Thus the  $f_1^+(k)$  in [23] reduces to the Fredholm determinant of the integral equation for the scattering solution. In the case of a nonlocal potential the denominator in [23] introduces redundant zeros of  $f_1^+(k)$  which do not correspond to bound states. This k-dependent constant factor can be removed by modifying the asymptotic boundary condition as follows:

25] 
$$\lim_{t\to\infty} \langle r|f_1^{\pm} \rangle = e^{\pm \alpha r} / \text{Det} (1 - \gamma G_i V)$$

It can also be shown that the spectral function corresponding to a nonlocal potential has the same relation to its Jost function as that in the case of a local potential. Thus any multiplicative factor, as in [25], would introduce a trivial modification in this relation. For a special case of a separable potential, eq. [23] reduces to

[26] 
$$f_i^+(k) = (1 - \gamma \operatorname{Tr} G_i^+ V)/(1 - \gamma \operatorname{Tr} G_i V)$$

120

<

[19]

#### CANADIAN JOURNAL OF PHYSICS, VOL. 49, 1971

#### 3. Determinant of the Jost Matrix for a Nonlocal, Noncentral Potential

In this section we assume that besides the central and spin-orbit forces there is also a noncentral tensor force component present in the nonlocal interaction  $\langle r|U|r' \rangle$ . This interaction conserves total spin s and the total angular momentum *j*, but not the relative angular momentum *l*. The partial wave, analysis now yields a set of coupled integral equations analogous to eqs. [3], [6], and [9].

 $\langle r | \phi_i \rangle = \langle r | \phi_i^0 \rangle + \gamma \langle r | G_i V \phi_i \rangle$ 

$$[27] \qquad \langle r|\psi_j^*\rangle = \langle r|\psi_j^\circ\rangle + \gamma \langle r|G_j^*V\psi_j^*\rangle$$

[28]

[29]

$$\langle r|f_{j}^{\pm}\rangle = \langle r|f_{j}^{\bullet}\rangle + \gamma \langle r|\bar{G}_{j}Vf_{j}^{\pm}\rangle$$

Because the angular momentum component  $l_1 = l$  is coupled by V to the  $l_2 = l + 2$  component, the solutions of [27] to [29] now will be two component vectors. There would be two such independent solutions of each equation depending on the incident wave component ( $\gamma = 0$  solutions) present in either the  $l_1$  or  $l_2$  channel. It is convenient to carry both these solutions together in a matrix form where the two columns of this matrix are the two vector solutions of the integral equation. Thus all the symbols used in [27] to [29] are  $2 \times 2$  matrices in the *l* space. In this section the Green's functions and the interaction will be considered 's matrices in the *r* as well as in the *l* space together. For the sake of completeness we quote the det. tions of various quantities used in [27] to [29]:

$$\frac{2\mu}{\hbar^2} r(Y_{l_1xj}|U(r,r')|Y_{l_2xj})r' = \langle r_{l_1}|V|r'l_2 \rangle$$

where  $Y_{ij}(P, \sigma)$  denotes the usual normalized wave function with spin s and orbital angular momentum *j* coupled to total angular momentum *j*. Because of the rotational invariance of the interaction, *V* will not depend on the projection of *j* on the *z* axis. Therefore its dependence is not indicated explicitly in [30]. On the left-hand side (LHS) of [30] the angular integrations and the spin summations are carried out. The  $\gamma = 0$  functions in [27] to [29] are

$$[31], \qquad \langle r|\chi_{J}^{\circ}\rangle = \begin{pmatrix} \langle r|\chi_{I_{1}}^{\circ}\rangle & 0\\ 0 & \langle r|\chi_{I_{2}}^{\circ}\rangle \end{pmatrix}$$

where  $\langle r|\chi_i^0 \rangle$  can be  $\langle r|\psi_i^0 \rangle$ ,  $\langle r|\phi_i^0 \rangle$ , or  $\langle r|f_i^0 \rangle$ , which are defined in the previous section. Similarly, the unperturbed Green's functions are also defined in terms of their earlier definitions.

$$[32] \qquad \langle r|G_j|r'\rangle = \begin{pmatrix} \langle r|G_{l_1}|r'\rangle & 0\\ 0 & \langle r|G_{l_2}|r'\rangle \end{pmatrix}$$

Lastly the boundary conditions on the various matrix solutions are

$$\lim_{r \to 0} \langle r | \phi_j \rangle = r^{L+1}$$

$$\lim_{r \to 0} \langle r | f_i^{\pm} \rangle = 1 e^{\pm i t t}$$

[34]

[33]

We will assume here that the off-diagonal elements of V in the matrix eqs. [27] to [29] arising in the matrix product are not divergent at r = 0. This condition on the potential can be avoided by modifying the inhomogeneous term in these equations (Newton 1966). In [33] and [34], L is a diagonal  $2 \times 2$  matrix with elements  $I_1$  and  $I_2$  and 1 is a unit matrix. The Wronskian matrix is defined as

$$[35] \qquad \qquad W[f(r),g(r)] = \overline{f(r)}g'(r) - \overline{f'(r)}g(r)$$

The bar at the top of a matrix f denotes its transpose in the  $2 \times 2$  (space. From the matrix differential equations of  $\langle r|f_j^* \rangle$  and  $\langle r|f_j^- \rangle$  it can easily be proved that

[36] 
$$W[\langle r|f_{j}^{+}\rangle, \langle r|_{j}^{-}\rangle]_{r=0} = W[\langle r|f_{j}^{+}\rangle, \langle r|f_{j}^{-}\rangle]_{r=0} = -22k$$

#### ENGH AND WARKE: GENERALIZATIONS OF THE JOST-PAIS THEOREM

The right-hand side (RHS) of [36] is a  $2 \times 2$  unit matrix. Proceeding in exactly the same way as we did in deriving [14] to [16], one obtains the corresponding matrix equations from [28], [29], and [36].

$$[37] \qquad \langle r|\phi_j \rangle = \frac{1}{2ik} \left[ \langle r|f_j^+ \rangle f_j^-(k) - \langle r|f^- \rangle f_j^+(k) \right] e^{ikL/2} k^{-L} / (2L+1)!!$$

. The Jost matrices  $f_1^{\pm}(k)$  are given by

[38] 
$$f_{j}^{+}(k) = [f_{j}^{-}(k)]^{*} = 1 + \gamma \int_{0}^{\infty} \langle r|f_{j+}^{0} \rangle \langle r|V_{j} \rangle dr e^{-ikL/2} k^{L} / (2L + 1)!!$$

Differentiating [38] with respect to  $\gamma$  and using [28] and [29], we get the following relation

[39] 
$$\frac{\mathrm{d}}{\mathrm{d}\gamma}f_{j}^{+}(k) = \int_{0}^{\infty} \mathrm{d}r\langle r|f_{j}^{-}\rangle^{+}\langle r|V\phi_{j}\rangle \,\mathrm{d}r \,\mathrm{e}^{-\ln L/2}k^{L}/(2L+1)!!$$

The matrix generalization of [17] is straightforward. The interacting Green's matrices are

[41]

[42]

[43]

$$\mathcal{G}_{j}^{+} = G_{j}^{+} + \gamma G_{j}^{+} V \mathcal{G}_{j}^{+} = G_{j}^{+} + \gamma \mathcal{G}_{j}^{+} V G_{j}^{+}$$
$$\mathcal{G}_{j} = G_{j} + \gamma G_{j} V \mathcal{G}_{j} = G_{j} + \gamma \mathcal{G}_{j} V G_{j}$$

and the F matrix we take as

$$\langle r|F|r'\rangle = \langle r|\mathcal{G}_{j}^{*}|r'\rangle - \langle r|\mathcal{G}_{j}$$

From this definition and [40] one obtains

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{L(L+1)}{r^2} - k^2\right) \langle r|F|r' \rangle + \gamma \langle r|VF|r' \rangle = 0$$
$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r'^2} + \frac{L(L+1)}{r'^2} - k^2\right) \langle r|\overline{F}|r' \rangle + \gamma \langle r|\overline{FV}|r' \rangle = 0$$

With our assumption of the behavior of  $\langle r | V | r' \rangle$  for  $r, r' \rightarrow 0$ , it is not difficult to find out the boundary conditions of F from [40] and [41].

$$\langle r|F|r' \rangle|_{r=0} = 0, \frac{d}{dr} \langle r|F|r' \rangle|_{r=r'=0} = -(L+1)/(2L+1)$$

 $\langle r|F|r' \rangle|_{r' \to \infty} = e^{\beta r'} x$  (a matrix function of *l*, *k*, and *r*)

The solution of the coupled differential eqs. [42] which satisfy the matrix boundary conditions [43] is

[44] 
$$\langle r|F|r' \rangle = - \langle r|\phi_j \rangle \frac{k^L e^{-kL/2}}{(2L+1)!!} f^+_{j}^{-1}(k) \langle \overline{r'|f_j}^+ \rangle$$

Multiplying [39] on both sides with a matrix  $[f_j^+(k)]^{-1}$  and taking the trace over the *l* space, one obtains

[45] 
$$\frac{\mathrm{d}}{\mathrm{d}r}\log\left[\operatorname{Det} f_{j}^{+}(k)\right] = -\operatorname{Tr} FV$$

The trace in [45] is now over the r as well as over the l space. In deriving [45] we used the invariance of a trace with respect to the cyclic permutations of a matrix product along with [44] with the property that

$$\langle \mathbf{r} | \mathbf{J}_{1}^{-} \rangle^{+} = \langle \mathbf{r} | \mathbf{J}_{1}^{+} \rangle$$

From [40] and [45] it can now be shown that

[47] 
$$Det f_j^+(k) = Det (1 - \gamma G_j^+ V)/Det (1 - \gamma G_j^V)$$

The determinant on the LHS is over the  $2 \times 21$ space while that on the RHS is over both the rand I space. Equation [47] proves the desired result that the determinant of a Jost matrix is equal to the ratio of the Fredholm determinants of the coupled integral equations of  $\langle r | \psi_j^+ \rangle$  and  $\langle r | \phi_i \rangle$  respectively.

#### Acknowledgments

The authors would like to thank Professor R. K. Bhaduri for many useful discussions and for the critical reading of the manuscript. C.S.W.

would like to thank the National Research Council of Canada for the award of a Research Associateship and Professor M. W. Johns for the hospitality of the Physics Department at McMaster. Y.S. would like to thank McMaster University for the award of a Research Assistantship.

To be published.

## APPENDIX B

In order to calculate  $\langle k | V_A | k' \rangle$  given by III-2.15, we have to evaluate  $\eta(k)$  by III-2.18 as well as the matrix elements of the Bargmann potential  $\langle k | V_1 | k' \rangle$  numerically. The calculation of  $\eta(k)$  involves the evaluation of  $\phi_p(k)$ and  $\phi'_B(k)$  which are defined in the coordinate space by III-2.11 and III-2.14. For example,

$$\phi_{\rm B}(k) = \sqrt{\frac{2}{\pi}} (2a\beta)^{1/2} \int_{0}^{\infty} dr \, \frac{e^{-ar} (1 - e^{-2br})}{(1 + \beta e^{-2br})} \sin kr$$
$$= \sqrt{\frac{2}{\pi}} (2a\beta)^{1/2} \left[ -\frac{k}{\beta (k^2 + a^2)} + \frac{1 + \beta}{\beta} I(k) \right] \quad B-1$$

with  $I(k) = Im \int_{0}^{\infty} \frac{e^{-\alpha r} e^{ikr}}{(1+\beta e^{-\alpha r})} dr$ , where  $\alpha = 2b$ . Making the transformation

$$\beta e^{-\alpha r} = y$$

we obtain

$$I(k) = Im \left[\frac{1}{\alpha\beta} \beta^{1-\frac{a}{\alpha}+\frac{ik}{\alpha}}\right]^{\beta} dy(1+y)^{-1} y^{-1+\frac{a}{\alpha}-\frac{ik}{\alpha}}].$$

Note that  $\beta > 1$ ; hence the above integral can be decomposed into two parts: from  $0 \le y \le 1$  and from  $1 \le y \le \beta$ . The integrand in the two parts can be expanded in power series of y and  $y^{-1}$  and integrated term by term. The final result, although complicated, can be evaluated very fast in the computer:

$$I(k) = \frac{k}{\alpha^2 \beta} S_1 + \beta^{(1-a/\alpha)} [\sin(\frac{k \log \beta}{\alpha}) - \frac{a\alpha}{a^2 + k^2}]$$
$$+ \frac{2a}{\alpha} S_2 - \frac{2a}{\alpha^3} (a^2 + k^2) S_3 + \cos(\frac{k \log \beta}{\alpha})$$
$$\times \frac{k\alpha}{k^2 + a^2} - \frac{2k}{\alpha} S_2 - \frac{2k}{\alpha^3} (a^2 + k^2) S_3 ]$$

where  $S_1$ ,  $S_2$  and  $S_3$  are defined as follows:

$$S_{1} = \sum_{n=0}^{\infty} \frac{(-)^{n} \beta^{-n}}{(n+1)^{2} + \frac{a^{2}}{\alpha^{2}} - \frac{2(n+1)a}{\alpha} + \frac{k^{2}}{\alpha^{2}}}$$

$$2 = \sum_{n=0}^{\infty} \frac{(-)^{n} (n+1)^{2}}{[(n+1)^{2} + \frac{k^{2}}{\alpha^{2}} - \frac{a^{2}}{\alpha^{2}}]^{2} + \frac{4k^{2}a^{2}}{\alpha^{4}}}$$

nd 
$$S_3 = \sum_{n=0}^{\infty} \frac{(-)^n}{[(n+1)^2 + \frac{k^2}{\alpha^2} - \frac{a^2}{\alpha^2}]^2 + \frac{4k^2a^2}{\alpha^4}}$$

aı

Combining B-3 and B-1 we obtain  $\phi_{\rm B}(k)$  accurately even for very large values of k. A similar method is used in evaluating integrals coming in  $\phi_{\rm p}(k)$  and  $\langle k | V_1 | k^* \rangle$ , and this turns out to be much faster than using Simpson's rule

B-3



directly. We do not give the detailed expressions for these, which are obtained in a straightforward manner.

## REFERENCES

1. NIGAM, B. P., Rev. Mod. Phys. 35 (1963) 117

It contains many references of original papers in phase shift analysis.

- 2. SEAMON, R. E., FRIEDMAN, K. A., BREIT, G., HARACZ, R. D., HOLT, J. M. and PRAKASH, A., Phys. Rev. <u>165</u> (1968) 1579
- 3. GAMMEL, J. L. and THALER, R. M., Phys. Rev. <u>107</u> (1957) 291, 1337
- 4. HAMADA, T. and JOHNSTON, I. D., Nucl. Phys. 34 (1962)

5. REID, R. V., Jr., Ann. Phys. (N.Y.) 50 (1968) 411

6. BRESSEL, C. N., KERMAN, A. K. and ROUBEN, B., Nucl.

Phys. A124 (1968) 624

7. BRYAN, R. A. and SCOTT, B. L., Phys. Rev. <u>135B</u> (1964) 434

SCOTTI, A. and WONG, D. Y., Phys. Rev. <u>138B</u> (1965) 145
 LOMON, E. L. and FESHBACH, H., Rev. Mod. Phys. <u>39</u>

(1967) 611

382

10. KIANG, D., PRESTON, M. A. and YIP, P. C., Phys. Rev.

<u>170</u> (1968) 907

11. SUGWARA, H. and VON HIPPEL, F., Phys. Rev. 172 (1969)

127

	•
12.	INGBER, L., Phys. Rev. <u>174</u> (1968) 1250
13.	UEDA, T. and GREEN, A. E. S., Phys. Rev. <u>174</u> (1968) 1350-
14.	TABAKIN, F. and DAVIS, K. T. R., Phys. Rev. 150 (1966)
	79
15.	TABAKIN, F., Ann. Phys. (N.Y.) 30 (1964) 51
16.	TABAKIN, F., Phys. Rev. <u>174</u> (1968) 1208
17.,	MONGAN, T. R., Phys. Rev. 175 (1968) 1260
18.	MONGAN, T. R., Phys. Rev. <u>178</u> (1969) 1597
19.	SERDUKE, F. J. D., Ph.D. Thesis (University of Cali-
τ	fornia, Davis, U.S.A.) 1970
20.	CLEMENT, D. M., SERDUKE, F. J. D. and AFNAN, I. R.,
	Nucl. Phys. <u>A139</u> (1969) 407
21.	FIEDELDEY, H., Phys. Lett. 30B (1969) 603
22.	CLEMENT, D. M. and BARANGER, E. U., Nucl. Phys. A108
đ	(1968) 27
23.	McKEE, J. S. C., Reports on progress in Physics 33
	(1970) 691; MITRA, A. N., Nucl. Phys. <u>32</u> (1962) 529
24.	AMADO, R. D., Ann. Rev. Nucl. Sci. 19 (1969) 61
	The above two articles contain other relevant
*	references both pertaining to theory and experiments
<u>.</u>	for the study of three-body problem
25.	GELFAND, I. M. and LEVITAN, B. M., Izvestia Akad. Nauk.
	S.S.S.R. Ser. Math. 15 (1951) 309
26.	MARCHENKO, V. A., Dokl. Akad. Nauk. S.S.S.R. 72 (1950)

47 (See also V. de Alfaro and T. Regge, "Potential Scattering", North Holland Publ. Co., Amsterdam 1965)

128

27. OMENS, R., Nuovo. Cimento. 21 (1961) 524 28. BOLSTERLI, M. and MACKENZIE, J., Physics (U.S.A.) 2 (1965) 141 TABAKIN, F., Phys. Rev. 177 (1969) 1443 29. MARTIN, A., Nuovo. Cimento. 19 (1961) 1257 30. CHONG, K. F., SINGH, Y., SPRUNG, D. W. L. and 31. SRIVASTAVA, M. K., Phys. Lett. 38B (1972) 132 FIEDELDEY, H., Nucl. Phys. A156 (1970) 242 32. 33. EKSTEIN, H. Phys. Rev. 117 (1960) 1590 ROTHE, K. W., KOEHLER, P. F. M. and THORNDIKE, E. H., 34. Phys. Rev. 157 (1967) 1247 35. MASON, D. L., HALBERT, M. L. and van der WOUDE, A., Phys. Rev. 179 (1969) 940 WILLIS, A., COMPARAT, V., FRASCARIA, R., MARTY, N. 36. MORLET, M. and WILLIS, N., Phys, Rev. Lett. 28 (1972) 1063 37. BROWN, V. R., Bull. Am. Phys. Soc. 12 (1967) 471 38. DRESCHSEL, D., MAXIMON, L. C. and WARNER, R. E., Phys. Rev. 181 (1969) 1720 BROWN, V. R., Phys. Rev. C 6 (1972) 1110 39. MOSZKOWSKI, S. A., Phys. Rev. 129 (1963) 1901 40. PRESTON, M. A. and BRADURI, R. K., Phys. Lett. 6 (1963) 41. 193 TABARIN, F., Phys. Rev. 137 (1965) 875 42. FIEDELDEY, H., Nucl. Phys. A135 (1969) 353 43.
44. SRIVASTAVA, M. K., SINGH, Y. and BHADURI, R. K., Phys. Lett. <u>32B</u> (1970) 333
45. COESTER, F., COHEN, S., DAY, B. and VINCENT, C. M., Phys. Rev. C <u>1</u> (1970) 769

46. HAFTEL, M. I. and TABAKIN, F., Nucl. Phys. A<u>158</u> (1970) 1
47. HAFTEL, M. I. and TABAKIN, F., Phys. Rev. C <u>3</u> (1971) 921
48. HAFTEL, M. I., LAMBERT, R. and SAUER, P. U., Nucl. Phys. Al92 (1972) 225

49. HAFTEL, M. I., Phys. Rev. (to be published)

MITTLSTAEDT, P. and RISTIG, M., Z. Physik <u>193</u> (1966) 349
 RISTIG, M., Z. Physik <u>199</u> (1967) 325

52. BAKER, G. A., Jr., Phys. Rev. 128 (1962) 1485

53. RISTIG, M. and KISTLER, S., Z. Physik 215 (1968) 419

54. MILLER, M. D., SHER, M. S., SIGNELL, P., YODER, N. R.

and MARKER, D., Phys. Lett. 30B (1969) 157

55. KISTLER, S., Z. Physik 223 (1969) 447

56. SRIVASTAVA, M. K., Ph.D. Thesis (McMaster University, Hamilton, Canada) 1970

57. SPRUNG, D. W. L. and SRIVASTAVA, M. K., Nucl. Phys.

<u>A139</u> (1969) 605

58. FUDA, M. G., Phys. Rev. C 1 (1970) 1910

59. WARKE, C. S. and BHADURI, R. K., Nucl. Phys. <u>A162</u> (1971) 289

60. BARGMANN, V., Rev. Mod. Phys. 21 (1949) 488

61. BHARGAVA, P. C. and SPRUNG, D. W. L., Ann. Phys. (N.Y.)

42 (1967) 222

62. THOMAS, A. W. and AFNAN, I. R., Phys. Rev. Lett. 26

» (1971) 906

63. PRADHAN, H. C. and SINGH, Y., submitted for publication in the Phys. Rev. C)

- 64. JOST, R. and PAIS, A., Phys. Rev. 82 (1951) 840
- 65. NEWTON, R. G., J. Math. Phys. 1 (1960) 319;

Phys. Rev. 114 (1959) 1611

- 66. LIPPMANN, B. A. and SCHWINGER, J., Phys. Rev. <u>79</u> (1950) 469
- 67. GOLDBERGER, M. L. and WATSON, K. M., "Collision Theory" (Wiley, New York, 1964)

68. WATSON, G. N., "Bessel Functions" (Cambridge, New York, 1958)

69. KOWALSKI, K. L. and FELDMAN, D., J. Math. Phys. 2 (1961) 499; 4 (1963) 507

70. NEWTON, R. G., "Scattering Theory of waves and particles" (McGraw Hill, New York, 1966)

71. BERTERO, M., TALENTI, G. and VIANO, G. A., Nucl. Phys.

A113 (1968) 625; A115 (1968) 395

72. CHADAN, K., Nuovo. Cimento. <u>10</u> (1958) 892; <u>47</u> (1967) 510 73. MARTIN, A., Nuovo. Cimento. <u>7</u> (1958) 607

74. NEWTON, R. G. and PULTON, T., Phys. Rev. <u>107</u> (1957) 1103 75. FIEDELDEY, H. and McGURK, N. J., Nucl. Phys. <u>A189</u> (1972)

83 -

76. ROUBEN, B., Ph.D. Thesis (Massachusetts Institute of Technology, U.S.A.) 1969 MYERS, W. D. and SWIATECKI, W. J., Nucl. Phys. 81 (1966) 77. 1 HOFSTADTER, R., BUMILLER, F. and YEARIAN, M. K., Rev. 78. Mod. Phys. 30 (1958) 482 (Also see Nuclear and Nucleon Structure, Benjamin, New York, 1963) 79. DAY, B., Rev. Mod. Phys. 39 (1967) 719 80. BETHE, H. A., Ann. Rev. Nucl. Sci. 21 (1971) 93 SPRUNG, D. W. L., Advances in Nuclear Phys. 5 (1972) 81. BETHE, H. A., Phys. Rev. 103 (1956) 1353 82. 83. WOODRUFF, A. E., Phys. Rev. 117 (1960) 1113 84. KOLTUN, D. S. and REITAN, A., Phys. Rev. 141 (1966) 1413 85. KLEIN, A., Phys. Rev. 99 (1955) 998 86. DRELL, S. D., FRIEDMAN, M. H. and ZACHARIASEN, F., Phys. Rev. 104 (1956) 236 BERGER, J. M., FOLDY, L. L. and OSBORN, R. K., Phys. Rev. 87. 87 (1952) 1061 MEHROTRA, R. and SEN GUPTA, K., Phys. Rev. D 1 (1966) 88. 1413 SAMARANAYAKE, V. K. and WOOLCOCK, W. S., Phys. Rev. Lett. 89. 15 (1965) 936 90. HULTHEN, L. and SUGAWARA, M., Handbuch der Physik, pp. 32 (Springer-Verlag, Berlin, 1957) ROSE', C. M., Jr., Phys. Rev. 154 (1967) 1305 91.

.132 .

92. CRAWFORD, F. S., Jr. and STEVENSON, M. L., Phys. Rev.

## 97 (1955) 1305

93 MICHAEL, C. and WILKIN, C., Nucl. Phys. Bll (1969) 99

















