COMMUTATIVE COHERENT RINGS

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ABSTRACT

Those modules over a commutative Noetherian ring which are finitely generated (and therefore automatically finitely presented) have especially pleasant properties. For example, any such module has a finitely generated projective resolution. Furthermore, any ideal contained in the set of zero-divisors of a non-zero finitely generated module M is actually annihilated by some non-zero element of M. Now the property that any finitely presented module has a finitely generated projective resolution actually characterizes coherent rings. Those commutative coherent rings whose non-zero finitely presented modules possess the second property mentioned above with respect to finitely generated ideals are herein entitled "pseudo-Noetherian" rings. This thesis is devoted to the study of these rings.

It is demonstrated that a faithfully flat directed colimit of such rings is again pseudo-Noetherian and this observation leads to non-trivial examples of pseudo-Noetherian rings. Equipped with a suitable definition for the "depth" of a non-zero finitely presented module M over a local pseudo-Noetherian ring R one may establish the following extensions of results known in the Noetherian situation: Depth M equals the length of any maximal R-sequence on M. Moreover, if M = R, this number equals the supremum of the projective dimensions of those finitely presented modules which have finite projective dimension. Furthermore, if M has finite projective dimension, (p. dim M) + (depth M) = depth R. These last two statements may be sharpened

by substituting "Gorenstein dimension" for "projective dimension" wherever the latter occurs.

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INTRODUCTION

In recent years many algebraists have directed their attention toward the study of coherent rings. Although these rings are introduced and examined by Bourbaki in ([6], pp. 62-63, Exercises 11 and 12), parts (a) and (g) of Exercise 12 first appeared in [8] along with further properties and examples of coherent rings. It is often useful to regard these rings as generalizations of Noetherian rings. In fact, many homological results which hold for Noetherian rings remain valid for coherent rings providing that finitely generated modules are replaced by finitely presented modules. This replacement technique is, however, not adequate to generalize those theorems in the theory of local Noetherian rings which unite the algebraic concept of R-sequence with the homological concepts of depth and finitistic dimension (cf. Theorem 1.7 of [3], Theorem 173 of [17] and Theorem 4.13 of [2]).

The crucial concept introduced here which allows extension of these theorems to non-Noetherian rings is that of a "pseudo-Noetherian" ring (or module). A non-zero module M over a coherent ring R is called a pseudo-Noetherian module if any finitely generated ideal whose elements are zero-divisors of M is actually annihilated by a single non-zero element of M. For example, it is well known that any finitely generated module over a Noetherian ring is pseudo-Noetherian. However, a finitely presented module over a coherent ring need not be pseudo-Noetherian (cf. Chapter II, Example 9). A coherent ring with the property that all non-zero finitely presented modules are pseudo-Noetherian will be called a pseudo-Noetherian ring. The greater part of this work is devoted to the study of such rings.

This pseudo-Noetherian property may be described in terms of prime ideals. If M is a non-zero module over a coherent ring and P is an ideal maximal in the set of all ideals which have the property that finite subsets are annihilated by non-zero elements of M, then P is prime. The set of all such maximal ideals is denoted $\underline{a}(M)$ and M is pseudo-Noetherian if and only if this set coincides with the set of (usual) maximal primes of M. The set $\underline{a}(M)$ is examined and some insight obtained with respect to the manner in which it is related to other types of associated primes. In the case of pseudo-Noetherian rings and finitely presented modules, primes of $\underline{a}(M)$ are directed unions of annihilators of elements of M and may occasionally be employed to generalize results true in the Noetherian situation. (e.g. Theorem 2.11)

The concept of faithful flatness for modules and ring extensions has been employed by other authors (e.g. M.E. Harris, [12]) in their investigations of coherent rings. Here, it is found useful for generating examples of pseudo-Noetherian rings. It is demonstrated that a faithfully flat directed union of pseudo-Noetherian rings is again pseudo-Noetherian. Those coherent rings which are Z.D. rings in the terminology of E.G. Evans ([9]), provide further examples of pseudo-Noetherian rings.

Our interest in pseudo-Noetherian rings first arose in connection with the examination of certain results concerning Noetherian local rings.

Chapters III and IV present the generalizations of these results. A concept of depth is introduced in a purely homological manner for a non-zero finitely presented module M over a local pseudo-Noetherian ring R. This depth is then shown to be equal to the length of any maximal R-sequence on M. Moreover, it is demonstrated that if the projective dimension of M is finite, then (p. dim M) + (depth M) = depth R (1). A version of the Third Change of Rings Theorem for coherent rings and finitely presented modules is an important tool in these considerations. Of course, an essential difference between such results and their Noetherian analogoues is the fact that both R-sequences and depth may be infinite in the coherent case.

In [2], M. Auslander and M. Bridger introduce the concept of Gorenstein dimension (G-dim) of a finitely generated module over a Noetherian ring. This dimension may be defined in the same manner for a finitely presented module M over a coherent ring R. It is shown that in the case where R is a pseudo-Noetherian local ring and $0 \neq M$ has finite Gorenstein dimension, (G-dim M) + (depth M) = depth R. This further generalization of (1) is followed by a few concluding remarks on the FP-injective dimension of a coherent ring.

CHAPTER I

COHERENT RINGS AND FINITELY PRESENTED MODULES

Before presenting some fundamental results, a few remarks on notation are necessary.

All rings will be associative and commutative with unit, all modules unitary, and R-Mod will represent the category of such modules. $\mathbb{N}$ will denote the natural numbers, $\mathbb{Z}$, the integers and $\mathbb{Q}$, the rationals. If S is a set, $|S|$ will represent the cardinality of S. Moreover, if S is a subset of a module (resp. ring), (S) will denote the submodule (resp. ideal) generated by S. If M is an R-module, S_R a subset of R, and S_M a subset of M, then $\mathrm{ann}_{S_R} S_M$ will denote the set of all r in S_R with $rS_M = 0$ and $\mathrm{ann}_{S_M} S_R$ will denote the set of all m in S_M with $S_r m = 0$. The elements of $\bigcup_{0 \neq m \in M} \mathrm{ann}_R(m)$ are called the zero-divisors of M and this set is denoted by Z(M). The Jacobson radical of a ring will be henceforth simply referred to as "the radical". The term "local ring" will mean a (not necessarily Noetherian) ring R with unique maximal ideal $\underline{m}$ and such a ring will often be represented by the pair $(R,\underline{m})$. It will be assumed that the reader is familiar with the fundamental aspects of the theory of commutative rings and modules as well as with those of homological algebra.

The first chapter lays the ground work for the material presented in later chapters. It was not deemed necessary to reproduce the proofs of many of the known results recorded in Chapter I. The interested reader may, however, consult the indicated literature for verification.

1. FINITELY PRESENTED MODULES

DEFINITION 1.1 Let R be a ring. An R-module M is called *finitely presented* if there exists a short exact sequence

$$0 \to K \to P \to M \to 0$$

in which P is a finitely generated projective module and K is a finitely generated module. Such a sequence will be called a finite presentation of M .

REMARK 1.2

(a) If $0 \to K' \to P' \to M \to 0$ is another short exact sequence for M in which P' is finitely generated and projective, $K \oplus P' \cong K' \oplus P$ by Schanuel's lemma and hence, K' is also finitely generated.

(b) An R-module is finitely presented if and only if the functor $\mathrm{Hom}_R(M, \ ): R\text{-Mod} \to R\text{-Mod}$ commutes with directed colimits ([11]).

Bourbaki records the following basic lemma concerning finitely presented modules ([6], Chapter 1, §2, no. 8, Lemma 9).

LEMMA 1.3 Let R be a ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of R-modules. Suppose M" is finitely presented and M is finitely generated. Then M' is finitely generated.

The next lemma is needed in the proof of Theorem 2.11 of Chapter II.

<u>LEMMA 1.4</u> Let R be a ring and M a finitely presented R-module. Suppose I is an ideal of R. Then M/IM is a finitely presented R/I-module.

<u>PROOF</u> Let $0 \to K \to F \xrightarrow{\alpha} M \to 0$ be a finite presentation of M with F a finitely generated free R-module and K a finitely generated submodule of F. Denote by ν_F (resp. ν_M) the natural map $F \to F/IF$ (resp. $M \to M/IM$) and let $\bar{\alpha} : F/IF \to M/IM$ be that map defined by the equation $\bar{\alpha}(\nu_F(f)) = \nu_M\alpha(f)$ for any f in F. It is easy to see that $\bar{\alpha}$ is a surjective R/I-homomorphism and that the square in the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & F \arrow[r, "\alpha"] \arrow[d, "\nu_F"] & M \arrow[r] \arrow[d, "\nu_M"] & 0 \\
 & & F/IF \arrow[r, "\bar{\alpha}"] & M/IM \arrow[r] & 0
\end{tikzcd}$$

Notice now that $\operatorname{Ker} \bar{\alpha} = \nu_F(K) = \frac{K + IF}{IF}$. In fact, it is clear that if $k \in K$, $\bar{\alpha}(\nu_F(k)) = \nu_M(\alpha(k)) = 0$. On the other hand, if $\nu_F(f) \in \operatorname{Ker} \bar{\alpha}$ for $f \in F$, then $\nu_M(\alpha(f)) = 0$. This means $\alpha(f) \in IM = \alpha(IF)$. Thus, there exists an element g in IF such that $\alpha(f) = \alpha(g)$. Since $f - g \in K$ and $f = (f-g) + g$, $\nu_F(f) = \nu_F(f-g) \in \nu_F(K)$.

Now $0 \to \nu_F(K) \to F/IF \to M/IM \to 0$ is a short exact sequence of R/I-modules with F/IF finitely generated free and $\nu_F(K)$ finitely generated. Hence M/IM is a finitely presented R/I-module.

A proof of the next lemma may be found in [13].

<u>LEMMA 1.5</u> Let R be a ring and I a finitely generated ideal of R. If M is a finitely presented R/I-module, it is also a finitely presented R-module.

It is well known that finitely generated modules behave especially well for Noetherian rings and it will be seen that those rings for which finitely presented modules have particularly pleasant properties are the "coherent" rings.

A few fundamental results concerning coherent rings will now be presented.

2. COHERENT RINGS

<u>THEOREM 1.6</u> For an arbitrary commutative ring R, the following statements are equivalent:

(a) Every finitely generated ideal of R is finitely presented.

(b) Every finitely generated submodule of a finitely generated free module is finitely presented.

(c) Every finitely generated submodule of a finitely presented module is finitely presented.

(d) The direct product of any family of flat modules is flat.

(e) (i) If I and J are any two finitely generated ideals of R, $I \cap J$ is also finitely generated.

(ii) If $r \in R$, $\mathrm{ann}_R(r)$ is finitely generated.

(f) For any $n \geq 0$ and any finitely presented module M, the functor $\mathrm{Ext}^n_R(M, \ )$ commutes with directed colimits.

A ring for which any one of these equivalent conditions holds is called a "coherent" ring.

REMARK The equivalence of (a), (b), (d) and (e) is established in [8]. The equivalence of (c) and (d) is an exercise in Bourbaki ([6], p. 63, Exercise 12 (a)). (The French Mathematicians have spent a great deal of time examining the concept of coherence.) Finally the equivalence of (c) and (f) is proven in Theorem 3.2 of [32]. Notice that coherence is exactly what is needed to generalize the necessity implication of Remark 1.2 (b).

It is easy to see that Noetherian rings and semi-hereditary rings are coherent. Hence Bézout domains (in particular, valuation domains) and Von Neumann regular rings are coherent. Further examples will occur throughout the text.

The following results indicate how new coherent rings may be generated from those already known.

LEMMA 1.7 . If R is a coherent ring and I is a finitely generated ideal, R/I is a coherent ring.

LEMMA 1.8 The ring direct product of finitely many coherent rings is coherent.

PROOF ([13], Corollary 2.1).

LEMMA 1.9 If R is a coherent ring and S is a multiplicatively closed set in R, R_S is a coherent ring.

PROOF ([12], Corollary 3.1).

Let D be a up-directed set and $S = (R_d, f_{dd'})_{d \leq d'}$ a corresponding directed system of rings and ring homomorphisms. S is called a flat directed system if for each $(d,d') \in D \times D$ with $d \leq d'$, $R_{d'}$ is a flat R_d-module via the map $f_{dd'} : R_d \to R_{d'}$. In this case the colimit of S is called a flat directed colimit of the family of rings $(R_d)_{d \in D}$. The following proposition is important in Chapter II.

PROPOSITION 1.10 A flat directed colimit of a directed system of coherent rings is coherent.

PROOF ([31], Proposition 20).

By employing finitely presented modules, results analogous to those for finitely generated modules over Noetherian rings may now be established.

LEMMA 1.11 Let R be a coherent ring and M a finitely presented module. Then the functor $M \otimes_R - : R\text{-Mod} \to R\text{-Mod}$ preserves products.

PROOF This generalization is established within the proof of Theorem 2.1 of [8].

LEMMA 1.12 Let R be a coherent ring, I a finitely generated ideal of R and M a finitely presented R-module. Then $\mathrm{ann}_M I$ is a finitely generated submodule of M.

PROOF Suppose I is cyclic and $I = (r)$ for some r in R. Then

$$0 \to \mathrm{ann}_M(r) \to M \to rM \to 0 \quad \text{is exact}$$

and since rM is a finitely generated, and hence finitely presented, submodule of M, $\mathrm{ann}_M(r)$ is finitely generated by Lemma 1.3. If I is generated by more than one element - i.e. $I = (x_1, \ldots, x_n)$ - then $\mathrm{ann}_M I = \bigcap_{i=1}^{n} \mathrm{ann}_M (x_i)$. The reader may check that since R is coherent, a finite intersection of finitely generated submodules of M must again be finitely generated. Hence $\mathrm{ann}_M I$ is finitely generated.

The proofs of the following two lemmas are straightforward.

<u>LEMMA 1.13</u> Let R be a coherent ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of R-modules. Then if any two of these modules are finitely presented, so is the third.

<u>LEMMA 1.14</u> If R is a coherent ring and M is a finitely presented R-module, $\mathrm{Hom}_R(M,R)$ is also a finitely presented R-module.

3. THE CHANGE OF RINGS THEOREM AND OTHER HOMOLOGICAL RESULTS

At this point a few fundamental homological results concerning coherent rings and finitely presented modules will be introduced.

It is now well known that for an arbitrary ring every R-module is a directed colimit of finitely presented modules and every flat module is a directed colimit of finitely generated free modules (cf. [22]). Also, the remarks on p. 120 of [7] indicate that if R is a coherent ring, M is a finitely presented module, and X is any arbitrary module, then

$$\mathrm{Ext}^n_R(M, X^*) \cong (\mathrm{Tor}^R_n(M, X))^* \quad \text{and}$$

$$\mathrm{Tor}^R_n(M, X^*) \cong (\mathrm{Ext}^n_R(M, X))^* \quad \text{for any non-negative}$$

integer n where $Y^* = \mathrm{Hom}_{\mathbf{Z}}(Y, \mathbf{Q}/\mathbf{Z})$ for any R-module Y.

LEMMA 1.15 Let R be a coherent ring and M a finitely presented module. Then p.dim M = w.dim M.

PROOF It is always true that w.dim M $\leq$ p.dim M. If w.dim M = ∞ the equality is obvious. Assume w.dim M = n < ∞. Then $(\mathrm{Ext}_R^{n+1}(M, X))^* \cong \mathrm{Tor}_{n+1}^R(M, X^*) = 0$ for any R-module X. Since * is a zero reflecting functor on R-Mod, $\mathrm{Ext}_R^{n+1}(M, X) = 0$ for all R-modules X. This shows p.dim M $\leq$ n and hence p.dim = n.

LEMMA 1.16 Let R be a coherent ring. Then w. gl. dim R = sup.{p. dim M|M a finitely presented module}.

PROOF By Lemma 1.15, it is clear that sup.{p. dim M|M a finitely presented module} $\leq$ w. gl. dim R. Hence, equality is obvious if the supremum is infinity. Suppose the supremum is n for some non-negative integer n. Then for any module X, $\mathrm{Tor}_{n+1}^R(M, X) = 0$ for all finitely presented modules M. Therefore if Y is any other module, Y is a directed colimit of finitely presented modules and since $\mathrm{Tor}_{n+1}^R(\ , X)$ commutes with directed colimits, $\mathrm{Tor}_{n+1}^R(Y, X) = 0$. This shows w. gl. dim R $\leq$ n and hence, w. gl. dim R = n.

Using Lemma 1.13 and induction on the number of generators of finitely presented modules it is possible to sharpen this result by showing that under the same conditions,

w. gl. dim R = sup.{p. dim M|M a cyclic finitely presented module}

The Change of Rings Theorems for Noetherian rings may be found in [16]. The Third Change of Rings Theorem is very useful in the theory of depth and R-sequences and the coherent analogue of this theorem given below will be used repeatedly in Chapter III. Because of it's importance a proof will be provided. (Although this result is certainly known, the author could not find it recorded in the literature.) A few preliminary results are now needed.

THEOREM 1.17 (First Change of Rings Theorem) Let R be a commutative ring and x a non-unit, non-zero-divisor in R. If M is a non-zero $R/(x)$ -module for which p. dim $_{R/(x)}$ M = n, then p. $\dim_R$ M = n + 1.

THEOREM 1.18 (Second Change of Rings Theorem) Let R be a commutative ring and x a non-unit, non-zero-divisor in R. If M is an R-module upon which x acts faithfully, then

$$\text{p. dim}_{R/(x)}\, M/xM \leq \text{p. dim}_R\, M.$$

DEFINITION 1.19 Let X and Y be two R-modules. An epimorphism f : X → Y is called "cover" if any morphism g from a module Z into X with the property that fg is an epimorphism must be an epimorphism. Usually X is simply called a cover of Y.

LEMMA 1.20 (Strooker, [33]) Let R be a commutative ring and I an ideal of R in rad R. If N is a finitely generated projective non-zero R/I-module, M is a finitely presented R-cover of N, and $\text{Tor}_1^R(R/I, M) = 0$ then M is uniquely determined up to isomorphism and is projective. Moreover, $M/IM \cong N$.

THEOREM 1.21 (Third Change of Rings Theorem for coherent rings) Let R be a coherent ring, M a finitely presented module and x an element of the radical of R which acts faithfully on both R and M. Then

$$\text{p. dim}_R\, M = \text{p. dim}_{R/(x)}\, M/xM .$$

PROOF If $\text{p. dim}_{R/(x)}\, M/xM = \infty$ then $\text{p. dim}_R\, M = \infty$ by Theorem 1.18. The remainder of the proof follows by induction on $\text{p. dim}_{R/(x)}\, M/xM$.

Suppose $\text{p. dim}_{R/(x)}\, M/xM = 0$. Considering the short exact sequence $0 \to (x) \to R \to R/(x) \to 0$ one obtains that the following initial portion of the long exact sequence for Tor is exact:

$$0 \to \text{Tor}_1^R (R/(x), M) \to (x) \otimes_R M \to R \otimes_R M \to R/(x) \otimes_R M \to 0$$

But $(x) \otimes_R M \cong xM$ because x acts faithfully on M and since $0 \to xM \to M \to M/xM \to 0$ is exact, $\text{Tor}_1^R (R/(x), M) = 0$. Furthermore, Nakayama's Lemma ensures that M is a cover of M/xM via the natural epimorphism. Therefore, by Theorem 1.20, M is projective over R.

Now assume that $\text{p. dim}_{R/(x)}\, M/xM = n > 0$ and that if M' is any other finitely presented module upon which x acts faithfully and for which $\text{p. dim}_{R/(x)}\, M'/xM' < n$, then $\text{p. dim}_R\, M' = \text{p. dim}_{R/(x)}\, M'/xM'$. Let $0 \to K \to F \to M \to 0$ be a finite presentation of M with F finitely generated free and K finitely generated. Then constructing the long exact sequence for Tor, one obtains that

$$0 \to \operatorname{Tor}_1^R (R/(x), M) \to R/(x) \otimes_R K \to R/(x) \otimes_R F \to R/(x) \otimes_R M \to 0$$

is exact. Since $\operatorname{Tor}_1^R (R/(x), M) = 0$ for the same reasons as those given above and since $R/(x) \otimes_R Y \simeq Y/xY$ for any R-module Y,

$$0 \to K/xK \to F/xF \to M/xM \to 0 \qquad \text{is exact.}$$

Now p. $\dim_{R/(x)} K/xK = n - 1$ and x acts faithfully on K since K is a submodule of a free R-module and x is a non-zero-divisor. Furthermore K is a finitely presented R - module by Theorem 1.6. Hence, by the induction hypothesis, p. $\dim_R K = n - 1$ and therefore p. $\dim_R M = n$ as required.

Notice that the conclusion of this theorem could have been formulated

$$\text{w. } \dim_R M = \text{w. } \dim_{R/(x)} M/xM$$

by Lemma 1.15, since both R and $R/(x)$ are coherent rings (Lemma 1.7) and M/xM is finitely presented $R/(x)$ -module (Lemma 1.4).

It could now be shown that if R is a coherent ring, x is a non-zero-divisor in the radical of R and w. gl. $\dim R/(x) < \infty$, then w. gl. dim R = w. gl. $\dim R/(x) + 1$. The condition " w. gl. $\dim R/(x) < \infty$" is really needed here. (Let k be a field and R = k[[x,y]] for two indeterminates x and y. Then w. gl. dim R = 2 but w. gl. $\dim R/(x^2 + y^3) = \infty$.) However, making this assumption is very inconvenient and accordingly a new type of dimension will be defined for coherent rings which will simplify these matters.

<u>DEFINITION 1.22</u> For a coherent ring R, the supremum of the projective dimensions of finitely presented modules which have finite projective dimension will be called the finitistic dimension of R and be denoted "f. dim R".

From Lemma 1.16 which motivates this definition, it may be seen that f. dim R $\leq$ w. gl. dim R. Unlike the case for weak global dimension, no finiteness condition is required on f. dim $R/_{(x)}$ in the following useful proposition.

<u>PROPOSITION 1.23</u> Let R be a coherent ring and x a non-zero-divisor in the radical of R. Then,

$$\text{f. dim } R = \text{f. dim } R/_{(x)} + 1$$

<u>PROOF</u> Suppose f. dim $R/_{(x)} = n < \infty$. Then there exists a finitely presented non-zero $R/_{(x)}$-module N such that p. $\dim_{R/_{(x)}} N = n$ and by Theorem 1.17, p. $\dim_R N = n + 1$. However, N is finitely presented over R by Lemma 1.5 and hence f. dim R $\geq$ n + 1.

It will now be shown that f. dim R $\leq$ n + 1. For this purpose let M be an arbitrary finitely presented R-module with finite projective dimension. It is only necessary to demonstrate that p. dim M $\leq$ n + 1. If p. dim M = 0 this is obvious. Suppose p. dim M = k ($0 < k < \infty$) and let $0 \to K \to F \to M \to 0$ be a finite presentation of M with F finitely generated free. Now since K is a finitely presented R-module, x acts faithfully on K, and p. $\dim_R K = k - 1$, then p. $\dim_{R/_{(x)}} K/_{xK} = k - 1$

by Theorem 1.21. Moreover $K/_{xK}$ is a finitely presented $R/_{(x)}$-module (Lemma 1.4), and hence $k-1 \leq n$ since f. dim $R/_{(x)} = n$. This shows $k \leq n + 1$ as required. Therefore f. dim $R \leq n + 1$ and since $n + 1 \leq$ f. dim R, $n + 1 =$ f. dim R.

If f. dim $R/_{(x)} = \infty$ there exists a finitely presented $R/_{(x)}$-module N with p. $\dim_{R/_{(x)}} N > n$ for any previously given natural number n. Again, N is finitely presented over R and Theorem 1.17 shows that p. $\dim_R N > n + 1$. Hence f. dim $R = \infty$.

The generalization of ([4],Corollary 5.6) appearing below may be proven using the techniques of [4] with only minor alterations.

PROPOSITION 1.24 The following statements are equivalent for a coherent ring R:

(a) f. dim $R = 0$

(b) Every finitely generated proper ideal of R has a non-zero annihilator.

4. M-SEQUENCES AND DEPTH

In [3] Auslander and Buchsbaum introduce the concept of an M-sequence for a finitely generated module M over a Noetherian ring. Because of the ascending chain condition such sequences are always finite. The following more general definition allows the possibility of infinite M-sequences for non-Noetherian rings and their modules.

DEFINITION 1.25 Let R be a commutative ring, M a non-zero R-module and suppose $(x_\alpha)_{\alpha < \gamma}$ is a family of elements of R indexed by all ordinals α less than a particular ordinal γ . Then $(x_\alpha)_{\alpha < \gamma}$ is called an R-sequence on M or simply an M-sequence if the following two conditions hold:

(a) $(\sum_{\alpha<\gamma} x_\alpha R)M \neq M$

(b) $x_0 \notin Z(M)$ and if $0 < \beta < \gamma$, $x_\beta \notin Z\left(\frac{M}{(\sum_{\alpha<\beta} x_\alpha R)M}\right)$

If γ is an infinite ordinal, the M-sequence is said to have length infinity. If γ is finite, the length of the M-sequence is $\gamma - 1$. An M-sequence $(x_\alpha)_{\alpha < \gamma}$ is called maximal if there is no element (denoted x_γ for convenience) such that $(x_\alpha)_{\alpha < \gamma + 1}$ is still an M-sequence.

It will only be necessary to distinguish between finite and infinite R-sequences in the sequel and the definition above may seem more elaborate than necessary. The reference to infinite ordinals was included only for the sake of clarity and convenience.

In Theorem 118 of [17], Kaplansky shows that if $x_1, \ldots, x_n$ is a finite M-sequence, the sequence obtained by interchanging x_i and x_{i+1} is an M-sequence if and only if $x_{i+1} \notin Z\left(\frac{M}{(x_1, \ldots, x_{i-1})M}\right)$

<u>PROPOSITION 1.26</u> Let R be a coherent ring, M a non-zero finitely presented module, and $x_1, \ldots, x_n$ an M-sequence with $\{x_1, \ldots, x_n\}$ contained in the radical of R. Then any permutation is again an M-sequence. (This result generalizes Theorem 119 of [17]).

<u>PROOF</u> It is enough to consider an M-sequence x,y of length two (see [17]) and by the above remark it suffices to prove $y \notin Z(M)$. Let $N = \mathrm{ann}_M(y)$.

If $n \in N$, $n \in xM$ because $y \notin Z(M/xM)$. Hence $n = xm$. Now since $x \notin Z(M)$ and $yxm = 0$, $m \in N$. This shows $N = xN$. By Lemma 1.12, N is finitely generated and since x is in the radical of R, Nakayama's Lemma shows that $N = 0$ which means $y \notin Z(M)$.

Notice that if $(R,\underline{m})$ is a local ring, M is a finitely generated module and $\{x_\alpha | \alpha<\gamma\} \subseteq \underline{m}$, then condition (a) of Definition 1.25 is automatic by Nakayama's Lemma. Furthermore the reader may easily check that a generalized valuation ring (local Bézout ring) cannot have any R-sequence of length greater than one. Examples of local coherent rings with infinite R-sequences will be presented in Chapter II.

For the purposes of the following proposition, let $\ell(s)$ denote the length of an R-sequence s.

<u>PROPOSITION 1.27</u> Let $(R,\underline{m})$ be a local coherent ring. Then,

$$\sup.\ \{\ell(s) | s \text{ an R-sequence}\} \leq \text{f. dim } R.$$

<u>PROOF</u> Let s be any R-sequence. If f. dim R = 0, Proposition 1.24 insures that every non-unit is a zero-divisor and hence $\ell(s) = 0$. Now assume f. dim R = n > 0 and the result holds for any local coherent ring with finitistic dimension less than n. Suppose x is the first element of s and s' represents the sequence with x deleted. Denoting the natural map $R \to R/(x)$ by ν, $\nu(s')$ will denote the sequence of elements in $R/(x)$ obtained by acting ν on each element of s'. $\nu(s')$ is easily seen to be an $R/(x)$-sequence. By Proposition 1.23, f. dim $R/(x) = n - 1$ and hence by the induction hypothesis, $\ell(\nu(s')) \leq n - 1$. Then $\ell(s) \leq n$ as required.

In [30], D. Rees defines the grade of a finitely generated module M over a Noetherian ring R to be the least integer k such that $\operatorname{Ext}_R^k(M,R) \neq 0$. Using different terminology M. Auslander and M. Bridger extend this definition in [2] as follows.

<u>DEFINITION 1.28</u> Let R be a commutative ring and M and N two R-modules. Then the least integer k such that $\operatorname{Ext}_R^k(M,N) \neq 0$ is called M-depth $_R$ N or simply M-depth N when there is no confusion with respect to the ring. If $\operatorname{Ext}_R^k(M,N) = 0$ for all non-negative integers k, M depth N $= \infty$.

(The "depth" terminology is used here rather than Kaplansky's concept of "grade" to insure that the notation of Chapter IV is consistent with that of [2]).

For a Noetherian local ring (R, $\underline{m}$) and finitely generated module M , $R/\underline{m}$-depth $_R$ M is abbreviated depth $_R$M (depth M). In Chapter III, a concept of depth is introduced (for finitely presented modules over a certain type of coherent local ring) which generalizes this notion.

In that chapter the close relationships between the concepts of M-sequence and depth will be examined.

<u>5. LOCAL COHERENT RINGS AND FAITHFUL FLATNESS</u>

The following proposition is needed in Chapter III.

<u>PROPOSITION 1.29</u> Let (R, $\underline{m}$) be a coherent local ring, M a finitely presented R-module and $n (n \geq 0)$ an integer. Then p. dim M $\leq$ n

if and only if $\operatorname{Tor}_{n+1}^{R}(M, R/\underline{m}) = 0$.

The proof of this proposition follows that of Theorems 13 and 14 in Chapter 9 of [25] with very little alteration necessary. Notice now that for such a ring, w. gl. dim R = w. $\dim_R R/\underline{m}$. Indeed, if w. $\dim_R R/(\underline{m}) = \infty$, then obviously w. gl. dim $R = \infty$. If w. $\dim_R R/\underline{m} = n < \infty$ and M is any finitely presented R-module $\operatorname{Tor}_{n+1}^{R}(M, R/\underline{m}) = 0$. By Proposition 1.29, p. dim $M \leq n$. This shows by Lemma 1.16 that w. gl. dim $R \leq$ w. $\dim_R R/\underline{m}$ and since w. $\dim_R R/\underline{m} \leq$ w. gl. dim R in any case, the equality is established.

The concept of faithful flatness becomes very important in the sequel and a few remarks will now be made in this direction.

<u>DEFINITION 1.30</u> Let R be a commutative ring. An R-module X is called faithfully flat if and only if X satisfies one of the following equivalent conditions ([6], Chapter 1, §3).

(a) For any modules Y, Y' and Y''

$0 \to Y' \to Y \to Y'' \to 0$ is exact if and only if

$0 \to X \otimes_R Y' \to X \otimes_R Y \to X \otimes_R Y'' \to 0$ is exact.

(b) X is a flat R-module and for any module Y, $Y \otimes_R X = 0$ implies $Y = 0$.

(c) X is a flat R-module and for any maximal ideal $\underline{m}$ of R, $\underline{m} X \neq X$.

Recall that for a commutative ring R and modules X and Y with $X \overset{\nu}{\hookrightarrow} Y$, X is called a pure submodule of Y if for any finitely presented module M, $M \otimes_R X \xrightarrow{M \otimes \nu} M \otimes_R Y$ is still injective, or equivalently, if any finite set of linear equations with coefficients in R and constants in X has a solution in X if it has a solution in Y.

<u>PROPOSITION 1.31</u> (a) If $\phi : A \to R$ is a ring homomorphism between two rings A and R such that R is a faithfully flat A-module via ϕ, then ϕ is injective.

(b) If A is a subring of a ring R, R is a faithfully flat A-module if and only if R is a flat A-module and A is a pure A-submodule of R.

(c) If $\phi : (A, \underline{m}_A) \to (R, \underline{m}_R)$ is a local homomorphism between two local rings $(A, \underline{m}_A)$ and $(R, \underline{m}_R)$ (i.e. $\phi(\underline{m}_A) \subseteq (\underline{m}_R)$), then as an A-module via ϕ, R is flat if and only if R is faithfully flat.

(d) If A is a subring of a ring R, R is (faithfully) flat as an A-module if and only if R[x] is (faithfully) flat as an A[x]-module.

<u>PROOF</u>. Proofs of parts (a) and (b) may be found in §3 of Chapter 1 of [6] along with much more information on faithful flatness. For part (c) notice that since $\underline{m}_A R \subseteq \underline{m}_R$, $\underline{m}_A R \neq R$ and this shows that if R is flat over A, R is faithfully flat over A by Definition 1.30 (c).

Now consider part (d) and observe first that the polynomial extension of any ring is always faithfully flat over the base ring. Hence, if R[x] is a (faithfully) flat A[x]-module, it is a (faithfully) flat A-module because (faithful) flatness is transitive. Since R[x] is (faithfully) flat over A and faithfully flat over R, R must be a (faithfully) flat A-module. (This is an example of the use of the property "descent". The reader may easily verify for himself that the concept of faithful flatness is transitive and obeys the descent law).

Now assume that R is a flat A-module and notice that $A[x] \otimes_A R \cong R[x]$. If M is an arbitrary $A[x]$-module

$M \otimes_{A[x]} R[x] \cong M \otimes_{A[x]} (A[x] \otimes_A R) \cong (M \otimes_{A[x]} A[x]) \otimes_A R \cong M \otimes_A R$ and call the composition of all of these isomorphisms i_M. Then the collection of all such isomorphisms (for each M in $A[x]$-Mod) defines a natural equivalence between the two $A[x]$-Mod endofunctors $- \otimes_A R$ and $- \otimes_{A[x]} R[x]$.

Hence if a map ν embeds a finitely generated ideal I of $A[x]$ into $A[x]$, the following diagram commutes:

$$\begin{tikzcd}
I \otimes_A R \arrow[r, "\nu \otimes_A R"] & A[x] \otimes_A R \\
I \otimes_{A[x]} R[x] \arrow[u, "i_I"] \arrow[r, "\nu \otimes_{A[x]} R[x]"'] & A[x] \otimes_{A[x]} R[x] \arrow[u, "i_{A[x]}"']
\end{tikzcd}$$

Since $\nu \otimes_A R$ is a monomorphism because R is flat over A, $\nu \otimes_{A[x]} R[x]$ is also a monomorphism. This shows $R[x]$ is a flat $A[x]$-module. Furthermore, if R is faithfully flat over A, and M is an $A[x]$-module, $M \otimes_{A[x]} R[x] = 0$ means $M \otimes_A R = 0$ which in turn shows $M = 0$. Thus $R[x]$ is faithfully flat over $A[x]$. (Definition 1.30 (b)).

<u>LEMMA 1.32</u> Let A be a subring of a ring R and suppose R is a flat A-module. If A is coherent, M is a finitely presented A-module and N is an arbitrary A-module, then

$$R \otimes_A \operatorname{Ext}_A^n (M, N) \cong \operatorname{Ext}_R^n (R \otimes_A M, R \otimes_A N)$$

for all non-negative integers n.

PROOF The Noetherian version of this lemma is treated in Exercise 11, p. 123 of [7]. The proof depends heavily on the fact that M has a finitely generated projective resolution. This of course is also true in the case treated here and the proof follows analogously.

LEMMA 1.33 Let A be a subring of R and suppose R is a faithfully flat A-module. Then if R is coherent, A is coherent.

PROOF Because this result has also been observed by M.E. Harris, the reader is referred to ([12], p. 476, Corollary 2.1) for the proof.

The two lemmas above assist in establishing the following result.

PROPOSITION 1.34 Let A be a subring of a coherent ring R and suppose R is a faithfully flat A-module. Then:

(a) w. gl. dim A $\leq$ w. gl. dim R.

(b) f. dim A $\leq$ f. dim R.

PROOF (a) If w. gl. dim R $= \infty$ the result is obvious. Suppose, then, that w. gl. dim R $= n < \infty$ and let M be an arbitrary finitely presented A-module. Then $R \otimes_A M$ is a finitely presented R-module ([6], Chapter 1, §3, no. 6, Prop. 11) and hence, p. $\dim_R R \otimes_A M =$ w. $\dim_R R \otimes_A M \leq n$ (Lemma 1.15). Now if N is an arbitrary A-module, this means $\mathrm{Ext}_R^{n+1}(R \otimes_A M, R \otimes_A N) = 0$. Since Lemma 1.33 insures that A is coherent, $R \otimes_A \mathrm{Ext}_A^{n+1}(M, N) = 0$ by Lemma 1.32. Thus $\mathrm{Ext}_A^{n+1}(M,N) = 0$ (Definition 1.30 (b)) and therefore p. $\dim_A M \leq n$. Since M was an arbitrary finitely presented A-module w. gl. dim A $\leq n$ by Lemma 1.16.

(b) Suppose f. dim R = n < ∞ and M is a finitely presented A-module with finite projective dimension. Then since R is a flat A-module p. $\dim_R (R \otimes_A M) \leq$ p. $\dim_A M < \infty$. Therefore because $R \otimes_A M$ is ~~still~~ finitely presented over R, p. $\dim_R R \otimes_A M \leq n$ by the definition of f. dim R. Now the proof continues exactly as in (a) establishing that p. $\dim_A M \leq n$. Since M was an arbitrary finitely presented A-module with finite projection dimension, f. dim A $\leq$ n.

A local Noetherian ring is called "regular" if gl. dim R < ∞. ~~J. Bertin~~ has extended the concept of regularity to coherent local rings as follows:

<u>DEFINITION 1.35</u> (cf. [5]) A coherent local ring is called regular if every finitely presented module has finite projective dimension.

It is a well known theorem of Auslander - Buchsbaum that every Noetherian local regular ring is a unique factorization domain. A coherent local regular ring R is always a domain ([5]) and is a unique factorization domain if and only if R satisfies the ascending chain condition for principal ideals ([29]). Notice that for a coherent local regular ring the finitistic dimension and the weak global dimension coincide. Some of the rings used for examples in later chapters are regular.

The concept of a flat directed colimit of a directed system of rings and ring homomorphisms has been previously explained. A faithfully flat directed colimit is described analogously and Proposition 1.31 (a) illustrates that one may as well consider directed unions in this case. The following proposition may now be proven.

PROPOSITION 1.36 A faithfully flat directed colimit of regular coherent local rings is a regular coherent local ring.

PROOF Suppose R is a faithfully flat directed union of a family of coherent local regular rings $(R_s)_{s\in S}$ for some infinite set S. Let I be a finitely generated ideal of R generated by the set of elements $\{x_1,\ldots,x_n\}$. Now since the union is directed, there exists some $s_o \in S$ such that $\{x_1,\ldots,x_n\} \subseteq R_{s_o}$. Therefore define $I_o = \sum_{i=1}^{n} R_{s_o} x_i$ to be the corresponding ideal of R_{s_o}. Since R is a flat R_{s_o}-module (R is a directed union of flat R_{s_o}-modules) $\text{p. dim}_R\,(R \otimes_{R_{s_o}} I_o) \leq \text{p. dim}_{R_{s_o}} I_o$ and because $R \otimes_{R_{s_o}} I_o \cong I$ and R_{s_o} is regular this shows that $\text{p. dim}_R\, I < \infty$. Hence, every finitely generated ideal of R has finite projective dimension. The reader may easily check that since R is coherent (Proposition 1.10) and local, this is equivalent to regularity.

CHAPTER II

A CLASS OF COHERENT RINGS

In his book Commutative Rings, I. Kaplansky states that the following result is "among the most useful in the theory of commutative rings".

THEOREM ([17], Theorem 82) Let R be a commutative Noetherian ring, M a finitely generated non-zero R-module, and S a subring contained in Z(M). Then there exists a single non-zero element m in M with Sm = 0.

For applications, S is usually an ideal of the ring. In view of the importance of this theorem, the crucial property involved is isolated in the following definition.

DEFINITION 2.1 Let R be a commutative ring and M a non-zero R-module. M is called a psuedo-Noetherian module if for any finitely generated ideal I contained in Z(M), there exists a non-zero m in M with Im = 0.

A commutative ring R is called pseudo-Noetherian if it is coherent and has the property that every non-zero finitely presented module is pseudo-Noetherian.

Chapter III will explore the important role played by pseudo-Noetherian rings in the theory of R-sequences, depth and codimension. The present chapter is devoted to examining pseudo-Noetherian modules and to investigating the incidence of pseudo-Noetherian rings in the class of all commutative rings.

The theory of associated prime ideals is a very well developed and useful tool when employed in the study of Noetherian rings. Indeed, the theorem presented above is proven using this theory. Taking advantage of the concept of the pseudo-Noetherian module, and avoiding any discussion of prime ideals, some later results are established here, whose analogues in the Noetherian situation are proven by the use of associated primes. However, the pseudo-Noetherian condition may be described in terms of prime ideals and the related theory will now be presented.

1. PRIME IDEALS ASSOCIATED TO PSEUDO-NOETHERIAN MODULES

Before embarking on this discussion a few definitions will be recorded for later reference.

DEFINITION 2.2 Let R be a commutative ring and M a non-zero R-module.

(a) A prime ideal P is called a maximal prime of M if P is maximal within Z(M). (Zorn's Lemma insures the existence of maximal primes of M.)

(b) Ass M is the set of prime ideals P for which there exists a non-zero m in M with $P = \mathrm{ann}_R(m)$.

(c) $\mathrm{Ass}_f M$ is the set of prime ideals P for which there exists a non-zero m in M with the property that P is minimal among those prime ideals containing $\mathrm{ann}_R(m)$.

(d) $\widetilde{\mathrm{Ass}}_f M$ is the set of prime ideals P for which there exists a flat module F with the property that P is a member of $\mathrm{Ass}_f (M \otimes_R F)$. (i.e. $\widetilde{\mathrm{Ass}}_f M = \bigcup_{F \text{ flat}} \mathrm{Ass}_f (M \otimes_R F)$.)

Further remarks on this material may be found in ([6], Chapter 4). Notice that $Z(M) = \bigcup \mathrm{Ass}_f M = \bigcup \widetilde{\mathrm{Ass}}_f M$ and that if R is Noetherian and M is finitely generated $\mathrm{Ass}\ M = \mathrm{Ass}_f M = \widetilde{\mathrm{Ass}}_f M$ where the primes maximal in this set are just the maximal primes of M.

DEFINITION 2.3 Let R be a commutative ring and M a non-zero R-module. Denote by $\underline{c}(M)$ (or $\underline{c}_R M$) the collection of all ideals I in R having the property that for any finite set F of I there exists a non-zero m in M (depending on F) with $F \subseteq \mathrm{ann}_R(m)$. By Zorn's Lemma, it is clear that maximal elements exist in $\underline{c}(M)$ and the collection of these will be denoted by $\underline{a}(M)$ (or $\underline{a}_R(M)$).

It is obvious that every member of $\underline{c}(M)$ is contained in some member of $\underline{a}(M)$ and that $Z(M) = \bigcup \underline{a}(M)$. Notice that a non-zero module M is pseudo-Noetherian if and only if every finitely generated ideal contained in $Z(M)$ lies within some member of $\underline{a}(M)$.

PROPOSITION 2.4 Let R be a commutative ring and M a non-zero R-module. Then $\underline{a}(M) \subseteq \mathrm{Spec}\ R$.

PROOF Suppose $I \in \underline{a}(M)$ and I is not prime. Then there exist elements x and y in R with the property that $xy \in I$ but $x \notin I$ and $y \notin I$. Since $x \notin I$ and I is maximal in $\underline{c}(M)$ the ideal $I + (x)$ is not a member of $\underline{c}(M)$. Hence, there exists a finite subset $F_x = \{f_1,\ldots,f_n\}$ of I and a finite subset $\{r_1,\ldots,r_n\}$ of R such that $\bigcap_{i=1}^{n} \mathrm{ann}_M (f_i + r_i x) = 0$. Thus $\mathrm{ann}_M F_x \cap \mathrm{ann}_M(x) = 0$. For the same reasons, there exists a finite subset F_y of I with $\mathrm{ann}_M F_y \cap \mathrm{ann}_M(y) = 0$. Since I ε $\underline{c}(M)$, there exists a non-zero element m in M with $F_x \cup F_y \cup \{xy\} \subseteq \mathrm{ann}_R(m)$. Now on the one hand $ym \neq 0$ because otherwise $m \varepsilon \mathrm{ann}_M F_y \cap \mathrm{ann}_M (y) = 0$. On the other hand $ym \varepsilon \mathrm{ann}_M F_x \cap \mathrm{ann}_M (x) = 0$. This contradiction shows that the original assumption was incorrect and hence, that I is prime.

The following proposition shows that in the coherent situation the elements of $\underline{a}(M)$ have, by virtue of their maximality, a stronger property than that defining ideals of $\underline{c}(M)$.

<u>PROPOSITION 2.5</u> Let R be a coherent ring and M a non-zero finitely presented R-module. Suppose P is an ideal in $\underline{a}(M)$. Then for any finite subset F of P, there exists a non-zero m in M with $F \subseteq \mathrm{ann}_R(m) \subseteq P$. (i.e. P is a directed union of annihilators of elements of M.)

<u>PROOF</u> Since R is coherent and M is finitely presented $\mathrm{ann}_R(m)$ is a finitely generated ideal of R for any $m \varepsilon M$. Thus the last statement is clear if the first part of the proposition is proven.

Let F be a finite subset of P and let $N = \mathrm{ann}_M(F)$. By Lemma 1.12 N is a finitely presented submodule of M. Suppose N is generated by finitely many elements $n_1,\ldots,n_s$. ($s \varepsilon \mathbb{N}$) and assume by way of contradiction that $\mathrm{ann}_R(n_i) \nsubseteq P$ for all i, $1 \leq i \leq s$. Then for each i, there exists x_i in $\mathrm{ann}_R(n_i)$ with $x_i \notin P$. Since P is prime, $x = \prod_{i=1}^{s} x_i \notin P$ but $x \varepsilon \mathrm{ann}_R N$. As in the proof of Proposition 2.4 above, the fact that $P + (x) \notin \underline{c}(M)$ implies there exists a finite subset F' of P with the property that $\mathrm{ann}_M F' \cap \mathrm{ann}_M(x) = 0$. Thus $\mathrm{ann}_M(F' \cup F) \cap \mathrm{ann}_M(x) = 0$. Since P ε $\underline{a}(M)$, there exists a non-zero m in M with $m \varepsilon \mathrm{ann}_M(F' \cup F)$. However, because $m \varepsilon N$, $xm = 0$ and hence $m \varepsilon \mathrm{ann}_M(F' \cup F) \cap \mathrm{ann}_M(x) = 0$. This contradiction establishes that there exists a generator n of N such that $P \subseteq \operatorname{ann}_R(n) \subseteq P$.

<u>REMARK 2.6</u>

(a) If R is a Noetherian ring and M is a non-zero finitely generated module the primes of $\underline{a}(M)$ are exactly the maximal primes of M. This is clear upon noticing that the maximal primes of M are all in Ass M and hence in $\underline{c}(M)$.

(b) If R is a commutative ring and M is a non-zero R-module, the following statements are equivalent:

(1) M is pseudo-Noetherian.

(2) $\underline{a}(M)$ consists exactly of the maximal primes of M.

(3) Every maximal prime of M is in $\underline{a}(M)$.

Indeed, if M is pseudo-Noetherian and P is a maximal prime of M, P is obviously in $\underline{c}(M)$ and hence in $\underline{a}(M)$ (by maximality). Furthermore, every prime in $\underline{a}(M)$ is contained in, and hence equals, some such maximal prime. If every maximal prime of M is in $\underline{a}(M)$ and I is a finitely generated ideal contained in $Z(M)$, then I is contained in some maximal prime P. Since $P \in \underline{a}(M)$, $I \subseteq \operatorname{ann}_R(m)$ for some non-zero m in M.

The primes in $\underline{a}(M)$ do not seem to behave well under the formation of arbitrary rings of fractions. However the following is true.

<u>PROPOSITION 2.7</u> Let R be a commutative ring and M a non-zero R-module. Suppose S is a multiplicatively closed set in R with the property that $S \subseteq R - Z(M)$ and assume P is a prime ideal. Then, $P \in \underline{a}_R(M)$ if and only if $P_S \in \underline{a}_{R_S}(M_S)$.

PROOF Claim 1 If $P \in \underline{a}_R(M)$ then $P_S \in \underline{c}_{R_S}(M_S)$.

Proof Let $\{\frac{p_1}{s_1}, \ldots, \frac{p_n}{s_n}\}$ be a finite subset of P_S $(1 \leq n < \infty)$.

Since $P \in \underline{a}_R(M)$, there exists a non-zero m in M with $\{p_1, \ldots, p_n\} \subseteq \mathrm{ann}_R(m) \subseteq P$ by Proposition 2.5. Because $P \subseteq Z(M), P \cap S = \phi$ and therefore $\frac{m}{1} \neq 0$ in M_S. Hence $\{\frac{p_1}{s_1}, \ldots, \frac{p_n}{s_n}\} \subseteq \mathrm{ann}_{R_S}(\frac{m}{1})$ and $P_S \in \underline{c}_{R_S}(M_S)$.

Claim 2 Let $\mathcal{P} \in \underline{a}_{R_S}(M_S)$ and let P be the unique prime in R with $P \cap S = \phi$ and $P_S = \mathcal{P}$. Then $P \in \underline{c}_R(M)$.

Proof Suppose $\{p_1, \ldots, p_n\}$ is a finite subset of P $(1 \leq n < \infty)$. Since $P_S \in \underline{c}_{R_S}(M_S)$ there exists $\frac{m}{1} \neq 0$ in M_S with $\frac{p_i m}{1} = 0$ in M_S for all i. This means that for each i, there exists $s_i \in S$ with $p_i s_i m = 0$ in M. Hence $\{p_1, \ldots, p_n\} \subseteq \mathrm{ann}_R((\prod_{i=1}^{n} s_i)m)$ and $(\prod_{i=1}^{n} s_i)m \neq 0$ because $\frac{m}{1} \neq 0$ in M_S. This shows $P \in \underline{c}_R(M)$.

Now assume $P \in \underline{a}_R(M)$. From Claim 1 $P_S \in \underline{c}_{R_S}(M_S)$. Suppose $\mathcal{P}' \in \underline{a}_{R_S}(M_S)$ and $P_S \subseteq \mathcal{P}'$. Then if P' is the unique prime ideal in R with $P' \cap S = \phi$ and $P'_S = \mathcal{P}'$, Claim 2 shows that $P' \in \underline{c}_R(M)$. Since $P \in \underline{a}_R(M)$ and $P \subseteq P'$, $P = P'$ and hence $\mathcal{P}' = P_S$. Thus P_S is a maximal element in $\underline{c}_{R_S}(M_S)$ (i.e. a member of $\underline{a}_{R_S}(M_S)$.) The implication from $P_S \in \underline{a}_{R_S}(M_S)$ to $P \in \underline{a}_R(M)$ is proven similarily.

COROLLARY 2.8 Let R be a commutative ring, M a non-zero R-module, and S a multiplicatively closed set in R such that S consists entirely of non-zero-divisors of M. Then M is a pseudo-Noetherian R-module if and only if M_S is a pseudo-Noetherian R_S-module.

PROOF If $P \in \operatorname{Spec} R$ under the above conditions P is a maximal prime of M if and only if P_S is a maximal prime of M_S (in R_S). Remark 2.6 together with Proposition 2.7 now gives the result.

The above corollary demonstrates that if P is a prime ideal containing $Z_R(M)$ and M is a non-zero finitely presented R-module over a pseudo-Noetherian ring R, then M_P is a pseudo-Noetherian R_P-module. There are other prime ideals for which such an implication holds. For example suppose P is a maximal prime of M. If $\mathcal{J}$ is a finitely generated ideal contained in $Z_{R_P}(M_P)$, then $\mathcal{J} = I_P$ where I is a finitely generated ideal contained in P. Since P is in $\underline{a}(M)$ because M is pseudo-Noetherian (Remark 2.6(b)) there exists a non-zero m in M with $I \subseteq \operatorname{ann}_R(m) \subseteq P$. Now $\frac{m}{1}$ is a non-zero element of M_P which annihilates $\mathcal{J}$.

Let R be a valuation domain. A prime ideal P of R is called a "branched prime" if P is the radical of a principal ideal ([21], p. 119, Exercise 6 (b)). Now consider a cyclic module $M = R/(x)$, $x \neq 0$. It is an easy matter to check that any prime in $\operatorname{Ass}_f(M)$ is a branched prime and that $\underline{a}(M)$ possesses only one member, the maximal ideal of the ring. Thus if the valuation domain R is chosen so that its maximal ideal is unbranched (e.g. the example presented at the end of Chapter III), R is a pseudo-Noetherian ring possessing a non-zero finitely presented module M for which $\underline{a}(M) \nsubseteq \operatorname{Ass}_f(M)$. However, $\underline{a}(M) \subseteq \widetilde{\operatorname{Ass}}_f(M)$ as is shown by the following theorem.

THEOREM 2.9 Let R be a coherent ring and M a non-zero finitely presented R-module. Then, $\underline{a}(M) \subseteq \tilde{\mathrm{Ass}}_f(M)$.

PROOF Let P be a prime ideal in $\underline{a}(M)$. By Definition 2.2 (d), it is necessary to find a flat module N and element e in $M \otimes_R N$ with the property that P is minimal among those prime ideals containing $\mathrm{ann}_R(e)$. Denote the set of finite subsets of P by $\mathcal{P}_f(P)$. For each $F \in \mathcal{P}_f(P)$ let $\langle F \rangle = \{H \in \mathcal{P}_f(P) \mid H \supseteq F\}$ and for elements F and G of $\mathcal{P}_f(P)$ with $F \subseteq G$ define $\phi_{F,G} : R^{\langle F \rangle} \to R^{\langle G \rangle}$ by stipulating $\pi_H(\phi_{F,G}(x)) = \pi_H(x)$ for all $H \in \langle G \rangle$ where π_H represents the H'th projection. The pair

$$((R^{\langle F \rangle})_{F \in \mathcal{P}_f(P)}, (\phi_{F,G})_{\substack{F,G \in \mathcal{P}_f(P) \\ F \subseteq G}})$$

clearly represents a compatible system of modules and module homomorphisms defined over the directed set $\mathcal{P}_f(P)$. Denote the directed colimit of this system by N and represent the colimit maps $R^{\langle F \rangle} \to N$ by ϕ_F for each $F \in \mathcal{P}_f(P)$. Notice that N is flat because it is the directed colimit of flat modules (Proposition 1.6(d)).

Now "tensor the system" with M. Since R is coherent and M is finitely presented, $M \otimes_R R^{\langle F \rangle} \cong M^{\langle F \rangle}$ for each $F \in \mathcal{P}_f(P)$ (Lemma 1.11) via an isomorphism i_F for which $\pi_H(i_F(m \otimes_R x)) = \pi_H(x)\, m$ for all $H \in \langle F \rangle$ where $x \in R^{\langle F \rangle}$ and $m \in M$. Now $M \otimes_R N$ is the directed colimit of the directed system

$$((M^{\langle F \rangle})_{F \in \mathcal{P}_f(P)}, (\psi_{F,G})_{\substack{F,G \in \mathcal{P}_f(P) \\ F \subseteq G}})$$

for which

$\psi_{F,G} = i_G \circ (M \otimes_R \phi_{F,G}) \circ i_F^{-1}$ for each pair $(F,G) \in (\mathcal{P}_f(P))^2$ with $F \subseteq G$.

Denote the colimit maps of this system by ψ_F for each $F \in \mathcal{P}_f(P)$ where

$\psi_F = (M \otimes_R \phi_F) \circ i_F^{-1}$.

Since $P \in \underline{h}(M)$, pick, for each $F \in \mathcal{P}_f(P)$, one non-zero member m_F of M with the property that $F \subsetneq \mathrm{ann}_R(m_F) \subseteq P$ (Proposition 2.5) and keep those elements of M fixed throughout the remainder of the proof. For each F in $\mathcal{P}_f(P)$, denote by "e_F" that member of $M^{\langle F\rangle}$ described by the equations $\pi_H(e_F) = m_H$ $(H \in \langle F\rangle)$. Notice that if F and G are elements of $\mathcal{P}_f(P)$ with $F \subseteq G$, then $\Psi_{F,G}(e_F) = e_G$. Let e represent the colimit of the family $(e_F)_{F \in \mathcal{P}_f(P)}$. (i.e. $e = \Psi_F(e_F)$ for any $F \in \mathcal{P}_f(P)$.)

It will now be shown that $\mathrm{ann}_R(e) = P$. If $p \in P$, $pe_{\{p\}} = 0$ and hence $pe = 0$. Thus $P \subsetneq \mathrm{ann}_R(e)$. On the other hand if $r \in \mathrm{ann}_R(e)$, there exists some $F \in \mathcal{P}_f(P)$ such that $re_F = 0$. Hence $rm_F = 0$. But since $\mathrm{ann}_R(m_F) \subseteq P$ by construction, $r \in P$. Hence $\mathrm{ann}_R(e) \subseteq P$. (Observe that P has been shown to be a member of $\mathrm{Ass}(M \underset{R}{\oplus} N)$ which is an even stronger result than was needed.)

The following important corollary is obtained by restricting to pseudo-Noetherian rings.

<u>COROLLARY 2.10</u> Let R be a pseudo-Noetherian ring and M a non-zero finitely presented module. Then $\underline{a}(M)$ consists exactly of the maximal members of $\widetilde{\mathrm{Ass}}_f(M)$.

<u>PROOF</u> First suppose $P \in \underline{a}(M)$. By the above theorem $P \in \widetilde{\mathrm{Ass}}_f(M)$. Now if $Q \in \widetilde{\mathrm{Ass}}_f(M)$ with $P \subseteq Q$; $Q \subseteq Z(M)$ and hence $Q \in \underline{c}(M)$ since R is pseudo-Noetherian. Thus $P = Q$ since P is maximal in $\underline{c}(M)$. This shows P is also maximal in $\widetilde{\mathrm{Ass}}_f(M)$. Now assume that P is a prime maximal in $\widetilde{\mathrm{Ass}}_f(M)$. As above, since $P \subseteq Z(M)$, $P \in \underline{c}(M)$. Hence there exists a Q in $\underline{a}(M)$

with $P \subseteq Q$. But by the theorem Q is then in $\widetilde{\mathrm{Ass}}_f(M)$ and this means $P = Q$ showing $P \in \underline{a}(M)$.

The next theorem, which will be useful in Chapter IV, presents the relationship between $\mathrm{ann}_R M$ and $Z(R)$ for a non-zero finitely presented module M over a pseudo-Noetherian ring R. The reader should be familiar with the concept of the Euler Characteristic (denoted "$\chi(M)$" and described in ([17], Section 4-3)) of a module M with a finite free resolution (FFR). The following results will be needed for Theorem 2.11.

(i) ([17], Theorem 190) Let M be an FFR module over a ring R, and let S be a multiplicatively closed set in R. Then M_S is an FFR module over R_S and $\chi_{R_S}(M_S) = \chi_R(M)$.

(ii) ([17], Theorem 191) Let $(R,\underline{m})$ be a local ring. If every finite subset of $\underline{m}$ possesses a non-zero annihilator, every FFR module is free.

(iii) If R is a coherent ring and $P \in \underline{a}(R)$, then f. dim $R_P = 0$

PROOF Since R_P is again coherent (Lemma 1.9) it is enough to show that any proper finitely generated ideal of R_P has non-zero annihilator (Proposition 1.24). Let $\mathcal{J}$ be a finitely generated ideal contained in P_P generated by $i_1/s_1, \ldots, i_n/s_n$ where $i_k \in P (1 \le k \le n)$ and $s_k \notin P (1 \le k \le n)$. By Proposition 2.5, there exists $x \neq 0$ in R with $\{i_1, \ldots, i_n\} \subseteq \mathrm{ann}_R(x) \subseteq P$. Thus $\mathcal{J} \subseteq \mathrm{ann}_{R_P}(x/1)$ and $x/1$ is a non-zero element of R_P.

(iv) Let R be a domain and M a non-zero finitely presented faithful R-module. Then $\mathrm{Hom}_R(M,R) \neq 0$.

PROOF cf.([30], Lemma 1.31).

(v) Let R be a commutative ring, I an ideal in R and M a finitely generated R-module. Then, $(IM:M)$ has the same radical as the ideal $I + \mathrm{ann}_R M$.

PROOF ([30], Lemma 1.32)

D. Rees proved part (a) of the following result in the Noetherian case ([30], Lemma 1.33). The Noetherian analogue of part (b) may be found in ([17], Theorem 196). Both proofs employed the primes maximal in $Z(R)$. In like manner, primes of $\underline{a}(R)$ are used in the coherent situation.

THEOREM 2.11 Let R be a pseudo-Noetherian ring and M a non-zero finitely presented R-module. Then:

(a) $\mathrm{ann}_R M \nsubseteq Z(R)$ if and only if $\mathrm{Hom}_R(M,R) = 0$

(b) $\mathrm{ann}_R M \nsubseteq Z(R)$ if and only if $\mathrm{ann}_R M \neq 0$ if M is a module with a finite free resolution.

PROOF (a) First suppose $\mathrm{ann}_R M \nsubseteq Z(R)$ and let x be a non-zero-divisor in R contained in $\mathrm{ann}_R M$. Then for any $f \in \mathrm{Hom}_R(M,R)$, $x(\mathrm{im}\ f) = 0$ showing that $f = 0$. Hence $\mathrm{Hom}_R(M,R) = 0$.

Now assume $\mathrm{ann}_R M \subseteq Z(R)$. Since R is coherent and M is finitely presented $\mathrm{ann}_R M$ is finitely generated (Theorem 1.6). Because R is pseudo-Noetherian there exists a prime ideal P in $\underline{a}(R)$ with $\mathrm{ann}_R M \subseteq P$. Now R/P is a domain and M/PM is a finitely presented non-zero R/P-module by by Lemma 1.4. Furthermore $M/_{PM}$ is a faithful $R/_P$-module because $\mathrm{rad}(PM:M) = \mathrm{rad}(P + \mathrm{ann}_R M) = \mathrm{rad}P = P$ (by (v)) and therefore $(PM:M) \subseteq P$. Thus all conditions for (iv) are fulfilled and $\mathrm{Hom}_{R/_P}(M/_{PM}, R/_P) \neq 0$.

As in Lemma 1.4 the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu_F|K} & & \downarrow{\scriptstyle \nu_F} & & \downarrow{\scriptstyle \nu_M} & & \\
0 & \longrightarrow & \frac{K+PF}{PF} & \longrightarrow & F/_{PF} & \longrightarrow & M/_{PM} & \longrightarrow & 0
\end{array}$$

(F a finitely generated free module, K a finitely generated R-module.)

Let $\{\mu_1,\ldots,\mu_n\}$ be a basis of F and let $\{\omega_j \mid j=1,\ldots,m,\ \omega_j = \sum_{i=1}^{n} a_{ji}\mu_i,\ a_{ji} \in R\}$ be a generating set for K. For $a \in R$, let $\bar{a}$ represent the image of a under the natural map $R \to R/_P$. $\mathrm{Hom}_{R/_P}(M/_{PM}, R/_P) \neq 0$ means the following system of m equations has a non trivial solution in $R/_P$.

$$\left(\sum_{i=1}^{n} \bar{a}_{ji} X_i = \bar{0}\right)_{1 \leq j \leq m}$$

That is, there exist $x_1,\ldots,x_n$ in R and $b_1,\ldots,b_m$ in R with $\{b_j \mid 1 \leq j \leq m\} \subseteq P$ and $\{x_i \mid 1 \leq i \leq n\} \nsubseteq P$ such that

$$\left(\sum_{i=1}^{n} a_{ji} x_i = b_j\right)_{1 \leq j \leq m}$$

<u>Claim</u> If x is a non-zero element of R and $xb_j = 0$ $(1 \leq j \leq m)$, then there exists a number i_o $(1 \leq i_o \leq n)$ such that $xx_{i_o} \neq 0$.

<u>Proof</u> Suppose to the contrary that if $xb_j = 0$ $(1 \leq j \leq m)$ then $xx_i = 0$ $(1 \leq i \leq n)$. Let $Q = P + (x_1,\ldots,x_n)$ where $(x_1,\ldots,x_n)$ denotes the ideal

generated by $x_1,\ldots,x_n$. Let $\{p_s + y_s | 1 \le s \le t,\ p_s \in P,\ y_s \in (x_1,\ldots,x_n)\}$ be an arbitrary finite set in Q. Now $\{p_s | 1 \le s \le t\} \cup \{b_j | 1 \le j \le m\} \subseteq P$ and since $P \in \underline{a}(R)$, there exists a non-zero x in R such that $xp_s = 0$ $(1 \le s \le t)$ and $xb_j = 0$ $(1 \le j \le m)$. By hypothesis $xx_i = 0$ $(1 \le i \le n)$. Therefore $x(p_s + y_s) = 0$ $(1 \le s \le t)$ and this shows $Q \in \underline{c}(R)$. Hence $Q = P$ and $\{x_i | 1 \le i \le n\} \subseteq P$ giving a contradiction.

Now $\{b_j | 1 \le j \le m\} \subseteq P$ so there exists a non-zero x in R with $xb_j = 0$ $(1 \le j \le m)$. However, there exists $i_o (1 \le i_o \le n)$ such that $xx_{i_o} \neq 0$ by the above claim. Therefore

$$\left(\sum_{i=1}^{n} a_{ji} x x_i = xb_j = 0\right)_{1 \le j \le m}$$

and this means there is a non-zero homomorphism in $\mathrm{Hom}_R(M,R)$ completing the proof of (a).

(b) Suppose M is a module with a finite free resolution and $\mathrm{ann}_R M \subseteq Z(R)$. If $\chi_R(M) = 0$, $\chi_{R_P}(M_P) = 0$ for each P in $\underline{a}(R)$ by (i). Furthermore by (ii) and (iii) M_P is a free R_P module and hence $M_P = 0$ for all P in $\underline{a}(R)$. Now since $\mathrm{ann}_R M$ is finitely generated, $\mathrm{ann}_R M \subseteq Z(R)$, and R is pseudo-Noetherian, there exists a certain P_o in $\underline{a}(R)$ with $\mathrm{ann}_R M \subseteq P_o$. However, because M is finitely generated and $M_{P_o} = 0$, there exists an $s \notin P_o$ with $sM = 0$. This contradiction shows $\chi_R(M) \neq 0$.

Since $\chi_R(M) = \chi_{R_P}(M_P) \neq 0$, M_P is a non-zero free R_P-module for each P in $\underline{a}(R)$. Therefore, $\mathrm{ann}_{R_P}(M_P) = 0$ and hence $(\mathrm{ann}_R M)_P = 0$ for each P in $\underline{a}(R)$. Therefore, $\mathrm{ann}_R(\mathrm{ann}_R M) \nsubseteq Z(R)$ by arguments analogous to those in the paragraph above. This means $\mathrm{ann}_R M = 0$.

Since the reverse implication in (b) is obvious the proof is complete.

The assumption that R is pseudo-Noetherian is actually a stronger condition than is needed. It suffices to assume that R is a pseudo-Noetherian R-module or equivalently f. dim Q = 0 when Q is the total quotient ring of R.

The following discussion serves to illustrate further the significance of $\underline{a}(M)$ *in the theory of prime ideals for a coherent ring. The material presented here has no direct bearing on the sequel. R will denote a coherent commutative ring throughout.*

Define $M(P) = \{m \in M \mid \mathrm{ann}_R(m) \nsubseteq P\}$ *for a non-zero module M and a prime ideal P. P is called a Krull prime of M or (K)-prime of M (cf.([19], Definition 1)) if and only if P is a maximal prime of* $M/M(P)$ *(k(M) denotes the collection of such primes). If* $P \in \mathrm{Supp}\, M$ *this condition may be shown to be equivalent to the following property of P: for every element p of P, there exists an m in M such that* $p \in \mathrm{ann}_R(m)$ *and* $\mathrm{ann}_R(m) \subseteq P$. *In accordance with this, those prime ideals P with the property that to each finite subset F of P then exists at least one m in M with* $F \subseteq \mathrm{ann}_R(m) \subseteq P$, *will be called "strong" (K)-primes of M. For example, if M is a non-zero finitely generated R-module, any prime ideal minimal over* $\mathrm{ann}_R M$ *is a strong (K)-prime of M. Notice that for a finitely presented non-zero module M, the strong (K)-primes of M are directed unions* *of annihilators of elements of M.*

Denote the strong (K)-primes of a module M by $sk_R(M)$ *or simply* $sk(M)$. *The following facts may be verified:*

(0) *If M is isomorphic to M'*, $sk(M) = sk(M')$

(1) $M \neq 0$ *if and only if* $sk(M) \neq \phi$

(2) *Let N be a submodule of M. Then*

$$sk(M) \subseteq sk(N) \cup sk(M/N)$$

(3) *For any multiplicatively closed set S,* $P \to P_S$ *induces a bijection between* $sk(M) \cap \{P \in \mathrm{Spec}\, R \mid P \cap S = \phi\}$ *and* $sk_{R_S}(M_S)$.

(J. Merker has termed a "quasi-Ass" any function which associates to each R-module a set of primes satisfying (0) - (3) above ([24]))

Proposition 2.5 shows that if M is any non-zero finitely presented module each member of $\underline{a}(M)$ *is a strong (K)-prime of M. In fact, in this case* $\underline{a}(M)$ *consists exactly of the maximal elements of* $sk(M)$. *By the same proof as is found in Theorem 2.9* $sk(M) \subseteq \widetilde{\mathrm{Ass}}_f(M)$. *Furthermore* $\mathrm{Ass}_f(M) \subseteq sk(M) \subseteq k(M)$ *for any non-zero finitely presented module M. Finally, the following statements are equivalent for any such module:*

(1) *M is a pseudo-Noetherian module*

(2) *Every maximal prime of M is in* $sk(M)$

(3) *Every maximal prime of M is in* $\underline{a}(M)$.

In the following section, the problem of manufacturing new pseudo-Noetherian rings from those already known is discussed.

2. NEW PSEUDO-NOETHERIAN RINGS FROM OLD

LEMMA 2.12 Let R be a pseudo-Noetherian ring and I a finitely generated ideal of R. Then R/I is also a pseudo-Noetherian ring.

PROOF That R/I is coherent is clear from Lemma 1.7. Let M be a non-zero finitely presented R/I-module and let $\mathcal{J}$ be a finitely generated ideal of R/I with $\mathcal{J} \subseteq Z_{R/I}(M)$. Denote by J the inverse image of $\mathcal{J}$ under the canonical map $R \to R/I$ and notice that J is a finitely generated ideal of R. Furthermore, M is a finitely presented R-module by Lemma 1.5. $\mathcal{J} \subseteq Z_{R/I}(M)$ implies $J \subseteq Z_R(M)$ and since R is pseudo-Noetherian, there exists $m \neq 0$ in M with $Jm = 0$. Therefore $\mathcal{J}m = 0$, completing the proof.

The above result allows the use of induction in proofs involving pseudo-Noetherian rings in the case where I is generated by an R-sequence.

LEMMA 2.13 Let $\{R_i \mid 1 \leq i \leq n\}$ be a set of finitely many pseudo-Noetherian rings. Then the product ring $R = \prod_{i=1}^{n} R_i$ is also pseudo-Noetherian.

PROOF Straightforward.

The theorem appearing below is a useful tool in providing more examples of pseudo-Noetherian rings. The following remark concerning flatness with respect to an ideal will be needed.

REMARK 2.14 ([7] ; p. 122, Exercise 5) Let R be a commutative ring, M an R-module, and I an ideal of R. The following statements are equivalent.

(a) The tensor map $M \otimes_R I \to M \otimes_R R$ is injective.

(b) For any exact sequence R-modules, $0 \to N \to P \to M \to 0$ with P projective, $N \cap PI = NI$.

(c) There exists an exact sequence of R-modules as in (b) with $N \cap PI = NI$.

LEMMA 2.15 Let A be a subring of a ring R. Let M be a finitely presented R-module with finite presentation

$$0 \to K \to R^n \xrightarrow{\beta} M \to 0$$

such that the finitely many generators $k_1,\ldots,k_m$ of K lie in A^n.

Let $K' = \sum_{i=1}^{m} Ak_i$ and consider

$$0 \to K' \to A^n \xrightarrow{\beta'} M' \to 0 \quad (M' = \text{coker } (K' \to A^n)).$$

Then $M' \otimes_A R \simeq M$.

PROOF $R^m \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$ is exact. $\alpha(r_1,\ldots,r_m) = \sum_{i=1}^{m} r_i k_i$

$A^m \xrightarrow{\alpha'} A^n \xrightarrow{\beta'} M' \to 0$ is exact. $\alpha'(a_1,\ldots,a_m) = \sum_{i=1}^{m} a_i k_i$

Define $A^m \otimes_A R \xrightarrow{h} R^m$ by $h((a_1,\ldots,a_m) \otimes_A r) = (a_1 r,\ldots,a_m r)$.

Define $A^n \otimes_A R \xrightarrow{g} R^n$ by $g((a_1,\ldots,a_n) \otimes_A r) = (a_1 r,\ldots,a_n r)$.

h and g are isomorphisms and the following diagram commutes with exact rows.

$$\begin{array}{ccccccc}
A^m \otimes_A R & \xrightarrow{\alpha' \otimes_A R} & A^n \otimes_A R & \xrightarrow{\beta' \otimes_A R} & M' \otimes_A R & \to & 0 \\
\downarrow h & & \downarrow g & & & & \\
R^m & \xrightarrow{\alpha} & R^n & \xrightarrow{\beta} & M & \longrightarrow & 0
\end{array}$$

The induced map between the cokernels which makes the second square commute is clearly onto and straightforward diagram chasing shows it also to be injective. This map is the required isomorphism.

<u>THEOREM 2.16</u> If a ring is R a faithfully flat directed colimit of pseudo-Noetherian rings, then R is pseudo-Noetherian.

<u>PROOF</u> First observe that R is coherent by Proposition 1.10 and that R is actually a directed union of pseudo-Noetherian rings by Proposition 1.31 (a). Therefore let $R = \underset{\beta \in B}{\overrightarrow{\bigcup}} R_\beta$ where B is some infinite set, R_β is pseudo-Noetherian for each $\beta \in B$, and the union is directed. Now let I be a finitely generated ideal of R and M a non-zero finitely presented R-module with $I \subseteq Z(M)$. The goal of this proof is to exhibit a non-zero m in M with $Im = 0$.

Suppose M has the following finite presentation:

$$0 \to K \to R^s \to M \to 0 .$$

Without loss of generality assume that K is a submodule of R^s generated by $\{k_i \mid 1 \leq i \leq t\}$ where $k_i = (k_{i1},\ldots,k_{is})$. Moreover, suppose I is generated by $\{a_\ell \mid 1 \leq \ell \leq u\}$. (s,t and u are natural numbers). Now choose $\beta_o \in B$ such that $\{k_{ij} \mid 1 \leq i \leq t,\ 1 \leq j \leq s\} \cup \{a_\ell \mid 1 \leq \ell \leq u\} \subsetneq R_{\beta_o}$.

Let $I' = \sum_{\ell=1}^{u} R_{\beta_o} a_\ell$, $K' = \sum_{i=1}^{t} R_{\beta_o} k_i$ and suppose M' is given by the short exact sequence,

$$0 \to K' \to R_{\beta_o}^s \to M' \to 0.$$

Then I' is a finitely generated ideal of R_{β_o} and M' is a non-zero finitely presented R_{β_o}-module with $M' \otimes_{R_{\beta_o}} R \cong M$. (Lemma 2.15).

<u>Claim</u> $I' \subseteq Z(M')$.

<u>Proof</u> Let $x \in I' \subseteq I$. Since $I \subseteq Z(M)$, there exists a non-zero m in M with $xm = 0$. Suppose that for all $m' \in M'$, $xm' = 0$ implies $m' = 0$. This means $xR_{\beta_o}^s \cap K' = xK'$. By Remark 2.14

$$M' \otimes_{R_{\beta_o}} xR_{\beta_o} \to M' \quad \text{is 1-1.}$$

Tensoring this map with R over R_{β_o}

$$M' \otimes_{R_{\beta_o}} xR_{\beta_o} \otimes_{R_{\beta_o}} R \to M' \otimes_{R_{\beta_o}} R \quad \text{is still 1-1.}$$

Now $M' \otimes_{R_{\beta_o}} R \cong M$ and $xR_{\beta_o} \otimes_{R_{\beta_o}} R \cong xR$.

Thus, $M' \otimes_{R_{\beta_o}} xR \to M$ is 1-1

giving $M \otimes_R xR \to M$ is 1-1, again because $M' \otimes_{R_{\beta_o}} R \cong M$. However, this means $xR^s \cap K = xK$ by Remark 2.14 which in turn signifies that $m = 0$ since $xm = 0$. This is a contradiction. Therefore the supposition made at the beginning is false and $I' \subseteq Z(M')$ as required.

Now since R_{β_o} is pseudo-Noetherian, there exists a non-zero m' in M' with I'm' = 0. Since R is a directed union of rings faithfully flat over R_{β_o}, R is faithfully flat over R_{β_o}. Hence, R_{β_o} is pure submodule of R and therefore the middle map in the following sequence is injective.

$$M' \cong M' \underset{R_{\beta_o}}{\otimes} R_{\beta_o} \to M' \underset{R_{\beta_o}}{\otimes} R \cong M$$

Since m' can now be thought of as lying in M and since I' and I are generated by the same elements (albeit over different rings), I'm' = 0 implies Im' = 0 completing the proof.

Theorem 2.16 is used repeatedly in the following final section of this chapter which presents a list of examples and counterexamples serving to further clarify the pseudo-Noetherian concept.

3. EXAMPLES

Example 1 Every coherent Bézout ring is pseudo-Noetherian.

Direct reference to Definition 2.1 establishes that such rings are pseudo-Noetherian because all of their finitely generated ideals are principal. Hence, in particular Von Neumann regular rings and coherent generalized valuation rings ([35]) are pseudo-Noetherian. By Theorem 1.6 Bézout domains are necessarily coherent because they are GCD-domains and such rings have the property that the intersection of two principal ideals is principal (([17], p. 32) and ([15], p. 81, Theorem 4)).

The following four examples all make use of Theorem 2.16.

EXAMPLE 2 A pseudo-Noetherian ring of infinite weak global dimension.

Let A be a Noetherian ring and $\{x_n \mid n \in N\}$ an infinite set of indeterminates. Denote by R the subring of $A[[(x_n)_{n \in N}]]$ consisting of all those elements whose expansions contain only finitely many indeterminates. Represent the set of finite subsets of N by $\mathcal{P}_f(N)$ and let $R_F = A[[(x_n)_{n \in F}]]$ for $F \in \mathcal{P}_f(N)$. R_F is Noetherian (hence pseudo-Noetherian) and R is the directed union of $\{R_F \mid F \in \mathcal{P}_f(N)\}$. Furthermore, this is a faithfully flat directed union. In fact, by Proposition 1.31 (b), it is enough to show that if F and F' are two elements of $\mathcal{P}_f(N)$ with $F \subseteq F'$, then $R_{F'}$ is a flat R_F-module and R_F is pure in $R_{F'}$.
Since $R_{F'} \simeq R_F[[(x_n)_{n \in F'-F}]]$ as R_F-modules, $R_{F'}$ is isomorphic to a direct product of copies of R_F and because R_F is coherent, $R_{F'}$ is a flat R_F-module by Theorem 1.6. To show R_F is pure in $R_{F'}$ assume $\{\sum_{i=1}^{n} Y_i f_{ij} = f_j \mid 1 \leq j \leq m\}$ is a system of linear equations with $n,m \in N$ and $f_{ij}, f_j \in R_F$ for all i and j, which has a solution $g_1,\ldots,g_n$ in $R_{F'}$. For fixed i, $1 \leq i \leq n$ let g_i' denote the power series which arises by deleting all terms of g_i which contain indeterminates indexed by members of $F' - F$. Then $g_1',\ldots,g_n'$ is a solution for the set of equations which lies in R_F. Thus R_F is pure in $R_{F'}$. Hence R is a faithfully flat directed union of pseudo-Noetherian rings and, therefore, is itself pseudo-Noetherian by Theorem 2.16. Notice that if A were a local Noetherian regular ring - in particular a field - R would be a local ring, regular in the sense of Bertin. (Definition 1.35) In any case R has infinite weak global dimension. (Proposition 1.34).

EXAMPLE 3 A modification of Example 2 yields a pseudo-Noetherian local non-domain.

Let K be a field and $\{x_n \mid n \in \mathbb{N}\}$ an infinite set of indeterminates. For each $n \in \mathbb{N}$ denote by R_n the Noetherian local ring $\dfrac{K[x_1,\ldots,x_n]}{(x_1, x_2^2,\ldots,x_n^n)}$.

R_n may be embedded naturally in R_{n+1} and these indexed rings therefore form a chain. Let R be the union of this chain. Now it is easy to see that this is a faithfully flat directed union. Indeed, if m and n are members of $\mathbb{N}$ with $m \leq n$, R_n is isomorphic (as an R_m-module) to a finite direct sum of copies of R_m and this insures that R_n is a flat R_m-module and R_m is pure in R_n. Thus R is pseudo-Noetherian by Theorem 2.16. Note that w. gl. dim R = ∞.

EXAMPLE 4 Let n be a positive integer. The following is an example of a local pseudo-Noetherian domain of weak global dimension n.

Let K be a field and $\{x_{ij} \mid 1 \leq i \leq n,\ j \in \mathbb{N}\}$ an infinite set of indeterminates. For each $m \in \mathbb{N}$ define $R_m = K[[x_{1m},\ldots,x_{nm}]]$ and a map $f_m^{m+1} : R_m \to R_{m+1}$ given by $f_m^{m+1}(x_{im}) = x_{i\,m+1}^2$ $(1 \leq i \leq n)$. R_m is of course Noetherian and f_m^{m+1} is an injective ring homomorphism. Let R be the colimit of the corresponding directed system of rings and ring homomorphisms.

Now observe that if m and m' are natural numbers and $m \leq m'$, $R_{m'}$ is a finitely generated free R_m-module. Indeed, $\{x_{1m'}^{s_1}, x_{2m'}^{s_2},\ldots, x_{nm'}^{s_n} \mid 0 \leq s_i < 2^{m'-m}\ (1 \leq i \leq n)\}$ is a free R_m-basis for $R_{m'}$.

Thus, as in the analogous situation in Example 3, $R_{m'}$ is a flat R_m-module and R_m is pure in $R_{m'}$, showing by Proposition 1.31 (b) and Theorem 2.16 that R is pseudo-Noetherian.

R may be described as that subring of the power series ring $K[[x_1, x_1^{1/2}, x_1^{(1/2)^2}, \ldots x_2, x_2^{1/2}, \ldots x_n, x_n^{1/2}, \ldots]]$ consisting of all elements for which there exists an $m \in \mathbb{N}$ (depending on the element) such that there are no terms in the expansion of the element which contain roots $x_i^{(1/2)^k}$ for $k \geq m$ and $1 \leq i \leq n$. With this in mind, it is immediately clear that R is non-Noetherian for $(x_1) \subset (x_1^{1/2}) \subset (x_1^{(1/2)^2}) \ldots$ is an infinite properly ascending chain of ideals. Since R is a faithfully flat directed union of local domains, R itself is a local domain faithfully flat over each R_m ($m \in \mathbb{N}$). (Proposition 1.31 (b)). Now w. gl. dim $R_m = n$ ($m \in \mathbb{N}$) and since R is a directed colimit of rings with weak global dimension n, w. gl. dim $R \leq n$ ([7], p. 125, Exercise 17). On the other hand, by Proposition 1.34 (a), $n =$ w. gl. dim $R_m \leq$ w. gl. dim R. Hence w. gl. dim R = n. (Note: By a result of B. Osofsky, the global dimension of R is less than or equal to n+1 since R is a countable directed colimit of rings with global dimension equal to n ([28], p. 320, Proposition 2.1)).

<u>EXAMPLE 5</u> Let R be a Noetherian ring and G an arbitrary abelian group. Then the group ring RG is pseudo-Noetherian.

Notice first that if H and H' are two subgroups of G and $H \subseteq H'$, then RH' is a free RH-module. To see this, consider the factor group H'/H and let $\{x_\alpha \mid \alpha \in A\}$ be a representative set for the cosets associated with this quotient. Thus $\{x_\alpha \mid \alpha \in A\} \subseteq H'$. Let $\sum_{\alpha \in A} \oplus RH$ be the free

RH-module for which the cardinality of the basis is the same as that of A. Now for F' a finite subset of H' let $x = \sum_{h' \in F'} r_{h'} h'$ be a member of RH' ($r_{h'} \in R$ for $h' \in F'$). Each h' is in exactly one coset $x_{\alpha_{h'}} H$. i.e. $h' = x_{\alpha_{h'}} h_{h'}$ where $h_{h'}$ is uniquely determined by h'. Let A_x be the set of α's determined in this manner by the members of F'. Then $x = \sum_{\alpha \in A_x} \Big(\sum_{\substack{\alpha = \alpha_{h'} \\ h' \in F'}} r_{h'} h_{h'} \Big) x_\alpha$. Define a map $\phi\colon RH' \to \sum_{\alpha \in A} \oplus RH$ as follows. The image of x is that member $\phi(x) \in \sum_{\alpha \in A} \oplus RH$ with $\phi(x)(\alpha) = \begin{cases} \sum_{\substack{\alpha = \alpha_{h'} \\ h' \in F'}} r_{h'} h_{h'} & \alpha \in A_x \\ 0 & \alpha \notin A_x \end{cases}$

This map may be verified to be an isomorphism.

Let $\mathcal{P}_f(G)$ be the set of all finite subsets of G and for $F \in \mathcal{P}_f(G)$ let (F) denote the subgroup of G generated by F. Clearly RG is the directed union of all R(F) for $F \in \mathcal{P}_f(G)$. It is well known that each R(F) ($F \in \mathcal{P}_f(G)$) is Noetherian ([20], p. 153, Proposition 1(c)). Moreover, by the above remarks, if $F, F' \in \mathcal{P}_f(G)$ and $F \subseteq F'$, R(F') is a free R(F)-module and hence this directed union is faithfully flat (as in the preceeding two examples). Thus by Theorem 2.16 R is pseudo-Noetherian.

EXAMPLE 6 The Noetherian-like rings of E.G. Evans.

In [9] a ring R is called a zero-divisor ring (Z.D. ring) if $Z_R(R/I)$ is a finite union of prime ideals for all proper ideals I of R. A non-zero module M is called a zero-divisor module (Z.D. module) if $Z_R(M/M')$ is a finite union of primes for all proper submodules M' of M. Evans shows that if R is a Z.D. ring, any finitely generated non-zero module is a Z.D. module ([9], Corollary 5). Furthermore every non-zero Z.D. module is pseudo-Noetherian ([9], Lemma 14) and hence, every coherent Z.D. ring is pseudo-Noetherian.

The following proposition gives a bound for the number of primes in $\underline{a}(M)$ ($|\underline{a}(M)|$).

PROPOSITION 2.17 Let R be a commutative ring and M a non-zero R-module. Then $|\underline{a}(M)| \leq$ Goldie dimension M.

PROOF Suppose the Goldie dimension of M is $n < \infty$ and $|\underline{a}(M)| > n$. Accordingly, let $P_1, \ldots, P_{n+1}$ be n+1 different primes in $\underline{a}(M)$ (cf. Proposition 2.4). Now for any fixed i $(1 \leq i \leq n+1)$, $P_i \nsubseteq \bigcup_{k \neq i} P_k$ and therefore, there exists an x_i in P_i such that $x_i \notin P_k$ for all $k \neq i$. Choose such an x_i for each i $(1 \leq i \leq n+1)$ and define $y_i = x_1 x_2 \ldots x_{i-1} \hat{x}_i x_{i+1} \ldots x_{n+1}$ (i.e. x_i is omitted in the product). Hence $y_i \in \bigcap_{k \neq i} P_k$ and $y_i \notin P_i$ $(1 \leq i \leq n+1)$. Now since $P_i \in \underline{a}(M)$ and $y_i \notin P_i$, $P_i + (y_i) \notin \underline{c}(M)$. Thus there exists a finite subset F_i of P_i such that $\operatorname{ann}_M F_i \cap \operatorname{ann}_M(y_i) = 0$ for each i (see proof of Proposition 2.4). Without loss of generality, it may be assumed that $x_i \in F_i$ $(1 \leq i \leq n+1)$. Since $P_i \in \underline{a}(M)$ there exists $0 \neq m_i \in M$ for each i such that $F_i m_i = 0$ and hence $m_i \in \bigcap_{k \neq i} \operatorname{ann}_M(y_k)$, $(1 \leq i \leq n+1)$.

Now it may be seen that the sum $(m_1) + (m_2) + \ldots + (m_{n+1})$ of the principal non-zero submodules (m_i) of M $(1 \leq i \leq n+1)$ is direct. Indeed, suppose $\sum_{i=1}^{n+1} r_i m_i = 0$ $(r_i \in R)$. Then for any k, $1 \leq k \leq n+1$, multiplication of this sum by y_k yields $y_k r_k m_k = 0$. Hence $r_k m_k$ is a member of $\operatorname{ann}_M F_k \cap \operatorname{ann}_M(y_k)$ and is therefore zero. Since this is true for all $k (1 \leq k \leq n+1)$ the sum is direct and the existence of this sum contradicts the assumption that Goldie dimension M = n. Hence $|\underline{a}(M)| \leq$ Goldie dimension M.

COROLLARY 2.18 (a) A ring with the property that all cyclic modules have finite Goldie dimension is a Z.D. ring.

(b) A coherent ring with the property that all cyclic finitely presented modules have finite Goldie dimension is a pseudo-Noetherian ring.

PROOF (a) Since $\cup\, \underline{a}(M) = Z(M)$ for any non-zero module, the result follows from Proposition 2.17.

(b) A simple induction argument on the number of generators shows that any finitely presented module M has finite Goldie dimension. Hence, $|\underline{a}(M)| < \infty$ by Proposition 2.17. If I is a finitely generated ideal contained in Z(M), I must be contained in one of the primes of $\underline{a}(M)$ since $\cup\, \underline{a}(M) = Z(M)$ and $|\underline{a}(M)| < \infty$. Therefore there exists a non-zero m in M which annihilates I. This shows that any finitely presented non-zero module is pseudo-Noetherian, and hence, that R is pseudo-Noetherian.

It is known that a ring with (Gabriel-Rentschler) Krull dimension has the property that all cyclic modules have finite Goldie dimension ([18]). Hence, Proposition 2.17 provides one way of seeing that rings with Krull dimension are Z.D. rings. Of course, Noetherian rings and valuation domains are trivially also (coherent) Z.D. rings.

The imposition of coherence does not seem to place great restrictions on Z.D. rings. For example, a non-discrete valuation domain is coherent and Z.D. but still not Noetherian. A coherent ring is of course not necessarily a Z.D. ring. A Von Neumann regular ring R with infinitely many maximal ideals is an example of a pseudo-Noetherian ring which is not Z.D. Examples of local pseudo-Noetherian rings which are not Z.D. are not so easy to find. The following example is due to B. J. Mueller.

EXAMPLE 7. A local pseudo-Noetherian ring which is not a Z.D. ring. Let $\{y_n \mid n \in \mathbb{N}\} \cup \{y,x\}$ be a set of indeterminates and let K be a field. Denote by R the subring of the power series ring $K[[y, (y_n)_{n \in \mathbb{N}}]]$ consisting of all elements whose expansions contain at most finitely many variables. Representing the unique maximal ideal of R by $\underline{m}$, let S signify the local ring $R[x]_{(x,\underline{m})}$. It will be shown that S is the required example.

First, it is necessary to observe that S is pseudo-Noetherian. For each $n \in \mathbb{N}$, let $R_n = K[[y,y_1,\ldots,y_n]]$ and denote it's maximal ideal by $\underline{m}_n$. Now $S_n = R_n[x]_{(x,\underline{m}_n)}$ is a Noetherian local ring for each $n \in \mathbb{N}$. Moreover, since for $n' \geq n, (x, \underline{m}_n) = (x, \underline{m}_{n'}) \cap R_n[x]$, S_n may be locally embedded into $S_{n'}$, and S can now be seen to be the union of this chain of rings $(S = \bigcup_{n \in \mathbb{N}} S_n)$. By Theorem 2.16 and Proposition 1.31 (c) it is only necessary to prove that if n and n' are two natural numbers with $n' \geq n$, then $S_{n'}$ is a flat extension of S_n. Now $R_{n'}$ is a flat R_n-module (as in Example 2) which insures that $R_{n'}[x]$ is flat over $R_n[x]$ (Proposition 1.31 (d)). Furthermore, $R_{n'}[x]_{(x,\underline{m}_{n'})}$ is a localization of $R_{n'}[x]_{(x,\underline{m}_n)}$ and $R_{n'}[x]_{(x,\underline{m}_n)}$ is flat over $R_n[x]_{(x,\underline{m}_n)}$ (base change). Hence by transitivity of flatness, $S_{n'}$ is a flat S_n-module as required. This completes the proof that S is pseudo-Noetherian.

Now consider the following polynomials defined by iteration:

$$f_o = x$$

$$f_n = y + f_o f_1 \cdots f_{n-1} \qquad (n \in \mathbb{N})$$

Considering these as elements of S, I will represent the ideal of S which has the form $\sum_{j=1}^{\infty} y_j f_1 f_2 \cdots f_j S$. The purpose of the following discussion will be to demonstrate that $Z_S(S/I)$ is not a finite union of prime ideals of S, and hence, that S is not a Z.D. ring.

Claim 1 $y_n f_1 f_2 \cdots f_{n-1}$ is not a member of I for each natural number n.

Proof. Suppose $n = 1$ and $y_1 \in I$. Since $I \subseteq f_1 S$ there exist polynomials h and g in R[x] with $g \notin (\underline{m},x)$, such that $y_1 = f_1(h/g)$. Hence $y_1 g = f_1 h$ in R[x] and if h_o(resp. g_o) represents the constant of h(resp. g), $y_1 g_o = y h_o$. Since g_o is a unit in R, this means y_1 is a multiple of y which is impossible. Therefore $y_1 \notin I$.

Now assume $n > 1$ and $y_n f_1 f_2 \cdots f_{n-1}$ is a member of I. Hence, there exist polynomials h_j ($j=1,\ldots,N$) and g in R[x] with $g \notin (\underline{m},x)$ such that $y_n f_1 f_2 \cdots f_{n-1} = \sum_{j=1}^{N} y_j f_1 f_2 \cdots f_j h_j/g$ or $y_n f_1 f_2 \cdots f_{n-1} g = \sum_{j=1}^{N} y_j f_1 f_2 \cdots f_j h_j$.

By passing to $R/(y_1,\ldots,y_{n-1}) = R'$ and cancelling $f_1' f_2' \cdots f_{n-1}'$, it may be shown that $y_n' g' \in f_n' R'[x]$. Hence $y_n g_o \in yR + \sum_{j=1}^{n-1} y_j R$. This is impossible since g_o is a unit in R. Therefore $y_n f_1 f_2 \cdots f_{n-1} \notin I$.

The above remarks show that $\{f_n | n \in \mathbb{N}\} \subseteq Z_S(S/I)$.

Claim 2 $y \notin Z_S(S/I)$

Proof Suppose to the contrary that $y \in Z_S(S/I)$. Then there exists an

$h \notin I$ such that $yh \in I$. Therefore let $yh = \sum_{n=M_h}^{N_h} y_n f_1 f_2 \cdots f_n h_n/g$ where h_n, $g \in R[x]$, $g \notin (\underline{m}, x)$, M_h is the first natural number n for which $y_n f_1 f_2 \cdots f_n h_n/g \neq 0$, N_h is the last such number and $N_h - M_h$ is minimal with respect to this property. Now for all such representations with $h \notin I$, choose an h for which $N_h - M_h$ is minimal and write $yh = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n/g$. (i)

For this h, $yhg = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n$ and since g is a unit in S (i) might as well be written $yh = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n$ (ii). Now every element r in R may be written uniquely as $r = r_1 + yr_2$ in which the power series r_1 possesses no term divisible by y. The map which associates r_1 with r is a ring endomorphism and its lift, ϕ_y, to $R[x]$ is an endomorphism of $R[x]$. Therefore each polynomial h in $R[x]$ may be written uniquely as $h = \phi_y(h) + yh^y$. Hence, from (ii), $y(h - \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n^y) = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n \phi_y(h_n)$ and $h - \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n^y$ is clearly not in I. This means (i) might as well be written $yh = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n$ with the extra property that

$\phi_y(h_n) = h_n (M \leq n \leq N)$. (iii).

$$\text{Now } yh = y_M f_1 f_2 \cdots f_M h_M + \sum_{n=M+1}^{N} y_n f_1 f_2 \cdots f_n \phi_{y_M}(h_n) + \sum_{n=M+1}^{N} (y_M f_1 f_2 \cdots f_M)(y_n f_{M+1} \cdot f_{M+2} \cdots f_n h_n^{y_M})$$

(The definition of ϕ_{y_M} is analogous to that of ϕ_y). Hence

$$yh = y_M f_1 f_2 \cdots f_M (h_M + \sum_{n=M+1}^{N} y_n f_{M+1} f_{M+2} \cdots f_n h_n^{y_M}) + \sum_{n=M+1}^{N} y_n f_1 f_2 \cdots f_n \phi_{y_M}(h_n)$$

Now for each n $(M+1 \leq n \leq N)$ $f_{M+1}f_{M+2}\cdots f_n = \phi_y(f_{M+1}f_{M+2}\cdots f_n)$

$$+ y(f_{M+1}f_{M+2}\cdots f_n)^y$$

Therefore

$$y(h - y_M f_1 f_2 \cdots f_M \sum_{n=M+1}^{N} y_n (f_{M+1}f_{M+2}\cdots f_n)^y h_n^{y_M})$$

$$= (y_M f_1 f_2 \cdots f_M)[h_M + \sum_{n=M+1}^{N} y_n \phi_y(f_{M+1}f_{M+2}\cdots f_n) h_n^{y_M}]$$

$$+ \sum_{n=M+1}^{N} y_n f_1 f_2 \cdots f_n \phi_{y_M}(h_n) \quad \text{(iv)}.$$

Denote $h - y_M f_1 f_2 \cdots f_M \sum_{n=M+1}^{N} y_n (f_{M+1}f_{M+2}\cdots f_n)^y h_n^{y_M}$ by h' and notice that h' is not a member of I. It will now be shown that the expression in the square brackets is zero.

From (iii) $yh = \sum_{n=M}^{N} y_n f_1 f_2 \cdots f_n h_n$ with $\phi_y(h_n) = h_n$. Applying the ring endomorphism ϕ_y to both sides of this equation one obtains

$0 = \sum_{n=M}^{N} y_n \phi_y(f_1 f_2 \cdots f_n) h_n$ and applying ϕ_{y_M} on this yields

$0 = \sum_{n=M+1}^{N} y_n \phi_y(f_1 f_2 \cdots f_n) \phi_{y_M}(h_n)$.

Hence $\quad 0 = y_M \phi_y(f_1 f_2 \cdots f_M) h_M + \sum_{n=M+1}^{N} y_n \phi_y(f_1 f_2 \cdots f_n) y_M h_n^{y_M}$.

or $\quad 0 = y_M \phi_y(f_1 f_2 \cdots f_M)(h_M + \sum_{n=M+1}^{N} y_n \phi_y(f_{M+1}f_{M+2}\cdots f_n) h_n^{y_M})$.

Thus $\quad h_M + \sum_{n=M+1}^{N} y_n \phi_y(f_{M+1}f_{M+2}\cdots f_n) h_n^{y_M} = 0$ as required.

Now by (iv) and the above remarks,

$$yh' = \sum_{n=M+1}^{N} y_n f_1 f_2 \cdots f_n \phi_{y_M}(h_n)$$

with $h' \notin I$ and $N-(M+1) < N-M$. The existence of this equation contradicts the minimality assumption of (i). Hence $y \notin Z(S/I)$.

Claim 3 $Z_S(S/I)$ is not a finite union of prime ideals of S.

Proof Previously it has been shown that $\{f_n \mid n \in \mathbb{N}\} \subseteq Z_S(S/I)$. Now suppose m and n are two natural numbers with $m > n$. Then if f_m and f_n were both contained in a prime ideal lying $Z_S(S/I)$, $y = f_m - f_0 f_1 \ldots f_n \ldots f_{m-1}$ would also be an element of this prime ideal which is impossible since $y \notin Z_S(S/I)$. Therefore $Z_S(S/I)$ is not a finite union of prime ideals.

This completes the proof that S is a local pseudo-Noetherian ring which is not a Z.D. ring. (The idea for the above proof derives from the proof of a theorem of W. Heinzer and J. Ohm which demonstrates that a ring R is Noetherian if (and only if) R[x] is a Z.D. ring ([14]).)

Z.D. rings were the only non-Noetherian rings the author discovered in the literature for which the pseudo-Noetherian condition had been considered.

The definition of a pseudo-Noetherian ring includes two conditions: first, R must be coherent and second, all non-zero finitely presented modules must be pseudo-Noetherian. The following two examples show that in general, neither condition implies the other.

EXAMPLE 8 A non-coherent ring for which all non-zero modules are pseudo-Noetherian.

Any non-coherent generalized valuation ring will do. For example, let K be a field and Γ^+ the non-negative elements of a simply ordered abelian group Γ with infinitely small positive elements (e.g. any dense subgroup of the reals). Let V represent the totality of all formal sums $v = \sum_{\alpha} k_{\gamma_\alpha} x^{\gamma_\alpha}$ where the summation is taken over all ordinal numbers α less than a fixed ordinal β, where the elements γ_α increase monotonically with α, and where $\gamma_0 \geq 0$. When addition and multiplication are defined in the usual formal manner it is well known that V becomes a valuation domain.

For a fixed $\gamma_1 \in \Gamma^+, \gamma_1 > 0$ represent by I the ideal $\bigcup_{\gamma > \gamma_1} (x^\gamma)$ and let $R = V/I$. Now R is a generalized valuation ring. However if μ represents the canonical map $V \to R$, $\mathrm{ann}_R(\mu(x^{\gamma_1})) = \bigcup_{\gamma > 0}(\mu(x^\gamma))$ is not finitely generated and hence, by Theorem 1.6, R is not coherent. However, since every finitely generated ideal is principal it is obvious that every non-zero module is pseudo-Noetherian.

EXAMPLE 9 A coherent ring for which there exists a non-zero finitely presented module which is not pseudo-Noetherian.

In [34] V. W. Vasconcelos shows that every local ring of global dimension two is a coherent integral domain. It is furthermore proven that if the maximal ideal is principal or not finitely generated, the ring is a valuation domain. Otherwise the maximal ideal is generated by two elements, and the non-finitely generated prime ideals are flat, countably generated, linearly ordered, and contained in any finitely generated prime ideal.

Consider the following example of such a ring. Let A represent the polynomial ring Z[x] localized at the ideal (2,x) and let K denote the quotient field of this Noetherian local ring of global dimension two. Now let R be that subring of the power series ring K[[t]] consisting of all elements with constant term lying in A. In [34] this ring R is shown to be a local ring of global dimension two which is neither a valuation domain nor a Noetherian ring. It will now be proven that R/tR is not a pseudo-Noetherian R-module.

Let $\underline{m}$ be the maximal ideal of A and $\underline{M}$ the maximal ideal of R consisting of all those elements of R with constant term lying in $\underline{m}$. It will first be shown that $\underline{M} = Z_R(R/tR)$. In fact suppose r is any member of $\underline{M}$ with constant $m \in \underline{m}$. Without loss of generality m may be considered non-zero since $tKR \subseteq Z_R(R/tR)$. (One way of seeing this is to observe that tKR is the unique minimal prime ideal containing tR and hence is contained in $Z_R(R/tR)$ by ([17], Theorem 84).) Therefore, let $r' = \frac{1}{m} t$. Since $m \in \underline{m}$, $\frac{1}{m} \notin A$ and $r' \notin tR$. However rr' is obviously a member of tR showing that $r \in Z_R(R/tR)$. Hence $\underline{M} = Z_R(R/tR)$.

Now by previous remarks $\underline{M}$ is finitely generated (by two elements) and therefore, to show R/tR is not a pseudo-Noetherian module, it is only necessary to demonstrate that if $r \in R$ with $r\underline{M} \subseteq tR$, then $r \in tR$. Such an r must have zero constant term since by hypothesis $rm \in tR$ for any non-zero m in $\underline{m}$. If $k = a/b$ ($a,b \in A$) is the coefficient of t in r, $b|am$ for all m in $\underline{m}$ for the same reason. But since A is a UFD, b may be written as a finite product of powers of prime elements and with this it is easy to see that b must divide a and hence $k \in A$ showing that $r \in tR$ as required.

R is therefore a coherent domain for which there exists a non-zero finitely presented module R/tR which is not pseudo-Noetherian. Now consider the semi-direct sum $T = R + R/tR$. (i.e. the additive direct sum for which the rule of multiplication is $(r_1, y_1)(r_2, y_2) = (r_1 r_2, r_1 y_2 + r_2 y_1)$ where $r_i \in R$, $y_i \in R/tR$, $i = 1,2$.) The reader may verify that T is a coherent ring which is not a pseudo-Noetherian module over itself.

Theorem 2.16 demonstrated that a faithfully flat union of pseudo-Noetherian rings is pseudo-Noetherian. But if A is a pseudo-Noetherian subring of a ring R which is a faithfully flat A-module, then R need not be pseudo-Noetherian. Indeed, in the example presented above, R is a faithfully flat extension of the Noetherian local ring $Z[x]_{(2,x)}$.

CHAPTER III

LOCAL PSEUDO-NOETHERIAN RINGS

The theory of Noetherian local rings - in particular that of regular rings - binds together such concepts as R-sequence, depth, and homological dimension. This is illustrated in the following three results which are all stated for a Noetherian local ring (R, $\underline{m}$). (Recall that for such a ring and a finitely generated non-zero R-module M, "$R/\underline{m}$-depth M" is abbreviated "depth M".)

(i) ([17], p. 101) Let I be a proper ideal and M a non-zero finitely generated R-module. Then, the (finite) common length of M-sequences maximal in I is equal to R/I-depth M.

(ii) cf. ([13], Theorem 1.7) depth R = sup {p. dim M|M finitely generated of finite projective dimension} (fPD(R)). Hence, if (R, $\underline{m}$) is regular, depth R = gl. dim R.

(iii) ([17], Theorem 173). If M is a finitely generated non-zero R-module with finite projective dimension, then (depth M) + (p. dim M) = depth R.

This last theorem was also discussed by Auslander and Buchsbaum in [3] where they employed the term "codim M" in place of "depth M". It is the original "codimension theorem" for Noetherian rings.

A careful examination of the above results reveals that the close relationships between the homological and the elemental aspect (dealing with zero-divisors of modules) of the theory actually arises from the fact that R is a pseudo-Noetherian ring. Indeed, results analogous to (i), (ii) and (iii) above may be proven for local pseudo-Noetherian rings and non-zero finitely presented modules. Thus the theory of R-sequences and depth and some of the associated techniques may be extended beyond the class of Noetherian rings.

1. LOCAL PSEUDO-NOETHERIAN RINGS

The following results illustrate how the pseudo-Noetherian condition unites the concepts of R-sequence and depth for finitely presented modules. A major tool in this respect is the following observation due due to Kaplansky.

THEOREM 3.1 ([17], p. 101) Let R be an arbitrary commutative ring and M,N two R-modules. Suppose $x_1,\ldots,x_n$ is a finite M-sequence with $(x_1,\ldots,x_n) N = 0$. Then,

$$\mathrm{Ext}_R^n (N, M) \simeq \mathrm{Hom}_R(N, \frac{M}{(x_1,\ldots,x_n)M})$$

THEOREM 3.2 Assume R is a pseudo-Noetherian ring, I is a finitely generated ideal, and M is a finitely presented R-module with $IM \neq M$. Then the length of any M-sequence maximal in I is equal to $R/_I$ -depth M.

PROOF Let $x_1,\ldots,x_n$ be an M-sequence in I. Then $x_1,\ldots,x_n$ is a maximal sequence in I if and only if $I \subseteq Z(\frac{M}{(x_1,\ldots,x_n)M})$. Equivalently, there exists a non-zero $\bar{m} \in \frac{M}{(x_1,\ldots,x_n)M}$ with $I\bar{m} = 0$. (R is pseudo-Noetherian and $\dfrac{M}{(x_1,\ldots,x_n)M}$ is still a non-zero finitely presented R-module.)

But this in turn is equivalent to the fact that $\mathrm{Hom}_R(R/I, \dfrac{M}{(x_1,\ldots,x_n)M}) \neq 0$ which by Theorem 3.1 means $\mathrm{Ext}_R^n(R/I, M) \neq 0$. Hence $x_1,\ldots,x_n$ is an M-sequence maximal in I if and only if $\mathrm{Ext}_R^n(R/I, M) \neq 0$. This last condition makes no reference to the particular M-sequence being considered. Hence it is now clear that if there exists an M-sequence maximal in I of finite length n, all M-sequences maximal in I must have length n. On the other hand, if there exists one of infinite length, all others must have infinite length.

Now if I contains no M-sequences, then $I \subseteq Z(M)$. Since R is pseudo-Noetherian, this means that $\mathrm{Hom}_R(R/I, M) \neq 0$ and hence, R/I-depth $M = 0$. If $x_1,\ldots,x_n$ is an M-sequence maximal in I for some $n \in N$, $\mathrm{Ext}_R^m(R/I, M) = 0$ $(0 \leq m < n)$ and $\mathrm{Ext}_R^n(R/I, M) \neq 0$ by the above considerations. This means R/I -depth $M = n$. If I contains an infinite M-sequence, R/I -depth $M = \infty$ since $\mathrm{Ext}_R^m(R/I, M) = 0$ for all $m \geq 0$. This concludes the proof of Theorem 3.2.

The corollary below shows that, for local coherent rings, the pseudo-Noetherian condition is in a certain sense exactly what is needed in Theorem 3.2.

<u>COROLLARY 3.3</u> Let R be a local coherent ring. Then the following statements are equivalent:

(a) R is pseudo-Noetherian.

(b) For any non-zero finitely presented module M and any finitely generated proper ideal I, R/I -depth M is the length of any M-sequence maximal in I.

PROOF The theorem has proven that (a) implies (b). Note that $IM \neq M$ by Nakayama's Lemma. Now suppose condition (b) holds. R is coherent by hypothesis. If M is a non-zero finitely presented module and I is a finitely generated ideal with $I \subseteq Z(M)$, R/I -depth M = 0 by (b). This just means $\mathrm{Hom}_R(R/I, M) \neq 0$ or there exists a non-zero m in M with $I \subseteq \mathrm{ann}_R(m)$. Hence all non-zero finitely presented modules are pseudo-Noetherian and this shows R is pseudo-Noetherian.

For the remainder of this chapter considerations will be restricted to local pseudo-Noetherian rings (R, $\underline{m}$) where $\underline{m}$ denotes the unique maximal ideal of R. Let $\mathcal{J}$ represent the directed set of finitely generated ideals contained in $\underline{m}$. Notice that for a non-zero finitely presented module M and ideals I and J in $\mathcal{J}$ with $I \subseteq J$, R/I -depth M $\leq$ R/J -depth M by Theorem 3.2 since any M-sequence maximal in I is still an M-sequence in J.

DEFINITION 3.4 For a local pseudo-Noetherian ring and a non-zero finitely presented module M, $\sup_{I \in \mathcal{J}} R/I$ -depth M will be denoted by "$\mathrm{depth}_R M$" or simply "depth M" where there is no confusion with respect to the ring involved.

Note that for a Noetherian local ring and non-zero finitely generated module M, depth M = $R/\underline{m}$ -depth M since $\underline{m}$ itself is finitely generated. The above definition therefore generalizes M. Auslander's concept of depth and I. Kaplansky's concept of grade for a finitely generated module over a Noetherian local ring.

The next theorem is a sort of globalization of Theorem 3.2 for local pseudo-Noetherian rings.

THEOREM 3.5 Let (R, $\underline{m}$) be a local pseudo-Noetherian ring and M a non-zero finitely presented R-module. Then the lengths of all maximal M-sequences are the same and equal to depth M.

PROOF For an M-sequence s, let $\ell(s)$ denote the length of s. If s is any such M-sequence, $\ell(s) \leq$ depth M. In fact suppose depth M = n < ∞ and s is an M-sequence of length n + 1. If I^s denotes the ideal generated by the elements of s, clearly R/I^s - depth M $\leq$ n by the definition of depth M. Hence, by Theorem 3.2, I^s cannot contain an M-sequence of length n + 1. This is a contradiction and therefore all M-sequences have length less than are equal to depth M.

Now if depth M = 0, $\underline{m} \subseteq Z(M)$ and there is nothing to prove. If depth M = m < ∞ , then by the above remarks all M-sequences have length less than or equal to m. Suppose s is a maximal M-sequence such that $\ell(s)$ = n < m. Now choose an I in $\mathcal{J}$ which contains s and for which R/I -depth M = m. Then by Theorem 3.2 all M-sequences, maximal in I have length m and this contradicts the assumption that s is a maximal M-sequence.

Hence the theorem is proven for the case in which depth $M < \infty$. If depth $M = \infty$ similar techniques easily show that all maximal M-sequences have infinite length.

The following proposition is important in later proofs which involve induction.

PROPOSITION 3.6 Let $(R, \underline{m})$ be a local pseudo-Noetherian ring and M be a non-zero finitely presented R-module. Suppose $x \in \underline{m}$ is a non-zero-divisor for M. Then,

$$\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1.$$

PROOF First, notice $R/(x)$ is again a local pseudo-Noetherian ring (Lemma 2.12) and M/xM is finitely presented both as R-module and $R/(x)$-module. (Lemmas 1.4 and 1.5) Let s be a sequence of elements in $\underline{m} - (x)$, $\bar{s}$ the corresponding sequence in $\underline{m}/(x)$ obtained by applying the canonical map $R \to R/(x)$ to the elements of s, and s' the sequence of elements in $\underline{m}$ manufactured by adjoining x at the beginning of s. It now becomes straightforward to check that $\bar{s}$ is a maximal $R/(x)$-sequence for M/xM if and only if s is a maximal R-sequence for M/xM. Furthermore, these two statements are equivalent to the condition: s' is a maximal R-sequence for M. The application of Theorem 3.5 to this situation completes the proof of this proposition.

For a Noetherian local regular ring it is known that depth R = gl. dim R. The following more general result is also true.

THEOREM 3.7 Let $(R, \underline{m})$ be a local pseudo-Noetherian ring. The following three quantities are equal.

(a) the length of any maximal R-sequence

(b) f. dim R

(c) depth R.

PROOF It will first be shown that f. dim R $\leq$ depth R. This is clear if depth R $= \infty$. Induction on the depth of R will now be used for the other cases. If depth R = 0, R/I-depth R = 0 for all finitely generated ideals I in $\underline{m}$ and hence by Proposition 1.24, f. dim R = 0. For $n \in \mathbb{N}$, assume now that depth R = n and for any other local pseudo-Noetherian ring R' with depth less than n, f. dim R' $\leq$ depth R'. It is necessary to show p. dim M $\leq$ depth R for any finitely presented R-module M with finite projective dimension.

Let M be such a module and assume without loss of generality that p. $\dim_R$ M = m $\geq$ 1. Suppose $0 \to K \to F \to M \to 0$ is a finite presentation for M in R-Mod with F finitely generated free and K finitely generated. p. $\dim_R$ K = m - 1. Since depth R > 0, there exists a non-zero-divisor $x \in \underline{m}$ which is also a non-zero-divisor for the finitely presented module K because K is a submodule of a free module. Hence, by the Third Change of Rings Theorem (Theorem 1.21), m - 1 = p. $\dim_R$ K = p. $\dim_{R/(x)} K/xK$.

Furthermore, by Proposition 3.6, $\operatorname{depth}_{R/(x)} R/(x)$ = n - 1. Since $R/(x)$ is again a local pseudo-Noetherian ring (Lemma 2.12) and K/xK is a finitely presented $R/(x)$-module (Lemma 1.4), m - 1 = p. $\dim_{R/(x)} K/xK \leq$ f.dim $R/(x) \leq \operatorname{depth}_{R/(x)} R/(x)$ = n-1 by the induction hypothesis. Therefore p. dim $M = m \leq n =$ depth R as required showing that f. dim R $\leq$ depth R.

Now if s is any maximal R-sequence f. dim R $\geq \ell(s)$ (the length of s) by Proposition 1.27. Therefore $\ell(s) \leq$ f. dim R $\leq$ depth R. However, by Theorem 3.5 $\ell(s) =$ depth R and all three quantities are accordingly identical. This completes the proof of Theorem 3.7.

Note that the ring in Example 2 of Chapter II has infinite depth because the indeterminates $x_1, x_2, \ldots$ form an infinite R-sequence. The variables $x_1, x_2, \ldots, x_n$ constitute a maximal R-sequence in the ring of Example 4 and this ring therefore has depth n.

The remainder of this section is devoted to an extension to local pseudo-Noetherian rings of the codimension theorem for Noetherian rings. The following three results are useful in Theorem 3.11.

<u>PROPOSITION 3.8</u> Let (R, $\underline{m}$) be a local coherent ring and M a non-zero finitely presented R-module. Suppose $x_1, \ldots, x_n$ is an M-sequence. Then,

$$\text{p. dim}_R \frac{M}{(x_1, \ldots, x_n)M} = \text{p. dim}_R M + n$$

<u>PROOF</u> The proof is exactly parallel to the proof of Theorem 20 (p. 196) of [25]. Here Proposition 1.29 is used in place of Theorem 14 (p. 193) of [25].

<u>LEMMA 3.9</u> Let (R, $\underline{m}$) be a local pseudo-Noetherian ring and M be a non-zero finitely presented R-module. Furthermore, suppose depth R > 0, depth M = 0 and $0 \to K \to F \xrightarrow{\psi} M \to 0$ is a finite presentation of M with F

finitely generated free and K finitely generated. Then if x is a non-zero-divisor in $\underline{m}$, $\text{depth}_{R/(x)} K/xK = 0$.

PROOF Notice first that since depth $R > 0$, there exist non-zero-divisors in $\underline{m}$. For proof of the assertion, it is sufficient, by Definition 3.4, to show that for every finitely generated ideal I with $x \in I \subseteq \underline{m}$, $R/I\text{-depth}_{R/(x)} K/xK = 0$. To do this, observe that depth $M = 0$ and therefore, for such an I, $I \subseteq Z(M)$. Because R is pseudo-Noetherian, there exists a non-zero m in M with $Im = 0$. Let $u \in F$ with $\nu(u) = m$. Now $u \notin K$, $Iu \subseteq K$ and $x \notin Z(K)$. Therefore $xu \notin xK$, $xu \in K$ and $Ixu \subseteq xK$. Let $\mu: K \to K/xK$ be the natural surjection. Then $\mu(xu) \neq 0$ and $I\mu(xu) = 0$. Thus, $\mu(xu)$ induces a non-zero homomorphism in $\text{Hom}_{R/(x)}(R/I, K/xK)$. But this means $R/I\text{-depth}_{R/(x)} K/xK = 0$ as required. (This proof is essentially contained in the proof of Theorem 173 of [17].)

PROPOSITION 3.10 Let $(R, \underline{m})$ be a local pseudo-Noetherian ring and M be a non-zero finitely presented R-module with finite projective dimension. Then,

$$(\text{p. dim } M) + (\text{depth } M) \leq \text{depth } R$$

PROOF If depth $R = \infty$ the result is obvious. Now suppose depth $R = n < \infty$ and let $x_1, \ldots x_m$ be any finite M-sequence. Then by Proposition 3.8 $\text{p. dim}_R \dfrac{M}{(x_1, \ldots, x_m)M} = m + \text{p. dim}_R M$. Hence, $\text{p. dim}_R \dfrac{M}{(x_1, \ldots, x_m)M} < \infty$.

By Lemma 1.5 $\dfrac{M}{(x_1, \ldots, x_m)M}$ is still a finitely presented R-module.

Since f. dim R = depth R = n (Theorem 3.7),

$m + \text{p. dim}_R\, M = \text{p. dim}_R \dfrac{M}{(x_1,\ldots x_m)M} \leq n$. Therefore $m \leq n - \text{p. dim}_R\, M$ and this shows that $n - \text{p. dim}_R\, M$ is a bound on the lengths of all M-sequences. By Theorem 3.5 depth $M \leq n - \text{p. dim } M$ or (p. dim M) + (depth M) $\leq$ depth R as required.

It can now be shown that the inequality of the above proposition is actually an equality.

THEOREM 3.11 Let (R, $\underline{m}$) be a local pseudo-Noetherian ring and M a non-zero finitely presented module of finite projective dimension. Then,

$$(\text{p. dim } M) + (\text{depth } M) = \text{depth } R$$

PROOF Case 1 : depth $R < \infty$

Proceed by induction on depth M which is finite by Proposition 3.10. Suppose depth M = 0 A second induction on depth R is now used to prove depth R = p. dim M in this case.

If depth R = 0, p. dim M = 0 by Proposition 3.10. Now assume depth R = n > 0 and that for any local pseudo-Noetherian ring R' with depth R' < n and any non-zero finitely presented R'-module M' with $\text{p. dim}_{R'}\, M' < \infty$ and $\text{depth}_{R'}\, M' = 0$, depth R' = $\text{p. dim}_{R'}\, M'$.

Let $0 \to K \to F \to M \to 0$ be a finite presentation for M and let $x \in \underline{m}$ be a non-zero divisor as in Lemma 3.9. By that lemma, $\text{depth}_{R/(x)} \dfrac{K}{xK} = 0$ for the finitely presented $R/(x)$ -module K/xK. By the Third Change of Rings Theorem

(Theorem 1.21) $p.\ dim_{R/(x)}\ K/xK = p.\ dim_R\ K < \infty$, and M is not free since depth $M \neq$ depth R. By Proposition 3.6 $depth_{R/(x)}\ R/(x) = depth_R\ R - 1 = n - 1$ and hence, by induction $p.\ dim\ M - 1 = p.\ dim_{R/(x)}\ K/xK = depth_{R/(x)}\ R/(x) =$ $= depth_R\ R - 1$ showing that $p.\ dim_R\ M =$ depth R as required.

Now suppose depth $M = m > 0$ and assume the hypothesis of this theorem is true for all non-zero finitely presented R-modules with finite projective dimension and depth less than m. By Theorem 3.5 there exists $x \in \underline{m}$ which is a non-zero-divisor for M. By Proposition 3.8 $p.\ dim_R\ M/xM = p.\ dim_R\ M + 1 < \infty$ and $depth_R\ M/xM = depth_R\ M - 1 = m - 1$ by Proposition 3.6. Hence, since M/xM is still finitely presented over R, depth $R = (depth_R\ M/xM) + (p.\ dim_R\ M/xM) = (depth_R\ M - 1) + (p.\ dim_R\ M + 1)$ $= (depth_R M) + (p.\ dim_R M)$ as required.

Case 2 : depth $R = \infty$

The first result to be proven here is that depth $M \neq 0$ and to do this a third induction on p. dim M is employed. If p. dim $M = 0$ then depth M = depth $R = \infty$ since M is a free module. Now assume p. dim $M = n > 0$, depth $M = 0$, and that the result holds for all local pseudo-Noetherian rings R' with infinite depth and all non-zero finitely presented R'-modules of finite projective dimension less than n. Let $0 \to K \to F \to M \to 0$ be a finite presentation for M and $x \in \underline{m}$ a non-zero-divisor as in Lemma 3.9. Then $depth_{R/(x)}\ K/xK = 0$ for the finitely presented $R/(x)$ -module K/xK. However $depth_{R/(x)}\ R/(x) = \infty$ by Proposition 3.6 and $p.\ dim_{R/(x)}\ K/xK = p.\ dim_R K = p.\ dim\ M - 1$

by the Third Change of Rings Theorem. This contradicts the induction assumption and hence, depth $M \neq 0$.

Now suppose by way of contradiction that depth $M < \infty$. Since depth $M \neq 0$, there exists $x \in \underline{m}$ which is a non-zero-divisor for M (Theorem 3.5). Hence by Proposition 3.8 $\text{p. dim}_R \, M/xM = \text{p. dim}_R \, M + 1$ and $\text{depth}_R \, M/xM = \text{depth}_R \, M - 1$ by Proposition 3.6. Thus M/xM is a new finitely presented R-module with finite projective dimension but of smaller depth than M. Continuing in this manner, a finitely presented module of finite projective dimension may be found which has zero depth. This is a contradiction to what was proven in the first part of Case 2. Therefore depth $M = \infty$ as required.

2. <u>KRULL DIMENSION</u>

It is well known that the Krull dimension of a regular Noetherian local ring R(K. dim R) is equal to f. dim R (= gl. dim R). The following results illustrate that this equality also exists for faithfully flat directed colimits of regular Noetherian local rings.

<u>LEMMA 3.12</u> Let $(R, \underline{m})$ be a pseudo-Noetherian local ring. Then f. dim R $\leq$ K. dim R.

<u>PROOF</u> By ([17], Theorem 132), K. dim R = rank $\underline{m} \geq n$ if there exists an R-sequence in $\underline{m}$ of length n. Since f. dim R is the length of any maximal R-sequence by Theorem 3.7 it is clear that K. dim R $\geq$ f. dim R.

<u>LEMMA 3.13</u> Let R be a directed union of rings $\{R_d \mid d \in D\}$ over a directed set D - i.e. $R = \bigcup_{d \in D} R_d$. Then K. dim R $\leq \limsup_{d \in D}$ K. dim R_d.

PROOF The result is obvious if $\limsup_{d\epsilon D} K.\dim R_d = \infty$. Suppose, then, that $\limsup_{d\epsilon D} K.\dim R_d = n$ where n is some non-negative integer. Furthermore, assume by way of contradiction that there exists a strictly decreasing sequence of prime ideals $P_{n+1} \supset P_n \supset P_{n-1} \supset \dots \supset P_1 \supset P_o$ of length n + 1. Now let $D_k = \{d \epsilon D \mid P_k \cap R_d = P_{k-1} \cap R_d\}$ for $k = 1,2,\dots,n+1$. Observe that $\bigcup_{k=1}^{n+1} D_k$ is a cofinal subset of D. In fact, suppose $d_o \epsilon D$. There exists a $d_1 \epsilon D$ such that $K.\dim R_d \leq n$ for all d with $d \geq d_1$ since $\limsup_{d\epsilon D} K.\dim R_d = n$. Because D is directed a d_2 may be found with the property that $d_2 \geq d_o$ and $d_2 \geq d_1$. Hence $K.\dim R_{d_2} \leq n$ showing that $d_2 \epsilon \bigcup_{k=1}^{n+1} D_k$. Having affirmed that $\bigcup_{k=1}^{n+1} D_k$ is cofinal in D it is easy to see that there must exist some $k_o (1 \leq k_o \leq n+1)$ such that D_{k_o} is cofinal in D.

To obtain a contradiction it will now be shown that $P_{k_o} \subseteq P_{k_o-1}$. Suppose $p \epsilon P_{k_o}$. Since $R = \bigcup_{d\epsilon D} R_d$, there exists some d' in D such that $p \epsilon P_{k_o} \cap R_{d'}$. However, because D_{k_o} is cofinal in D there exists a $d'' \geq d'$ with $P_{k_o} \cap R_{d''} = P_{k_o-1} \cap R_{d''}$ showing that $p \epsilon P_{k_o-1} \cap R_{d''} \subseteq P_{k_o-1}$. However, P_{k_o-1} was strictly contained in P_{k_o} by hypothesis. This contradiction demonstrates that the original assumption was erroneous and hence, $K.\dim R \leq n$.

The lemmas above are used in the following proposition.

PROPOSITION 3.14 Let R be a faithfully flat directed colimit of Noetherian local regular rings. Then,

$$f.\dim R = K.\dim R$$

PROOF R is coherent by Lemma 1.10 and R is actually a directed union of Noetherian local regular rings by Proposition 1.31 (a). Therefore let $R = \bigcup_{d \in D} R_d$ where D is a directed set and each R_d is a Noetherian local regular ring. Theorem 2.16 insures that R is pseudo-Noetherian and hence, by Lemma 3.12, f. dim R $\leq$ K. dim R. Lemma 3.13 shows that K. dim R $\leq \sup_{d \in D}$ K. dim R_d and, since each R_d is Noetherian local regular, $\sup_{d \in D}$ K. dim R_d = $\sup_{d \in D}$ f. dim R_d. Finally for each $d \in D$, f. dim $R_d \leq$ f. dim R (Proposition 1.34 (b)) showing that $\sup_{d \in D}$ f. dim $R_d \leq$ f. dim R. Thus K. dim R $\leq$ f. dim R and this completes the proof.

Proposition 3.14 indicates there are some pseudo-Noetherian local rings (e.g. Examples 2 and 4 of Chapter II) for which the Krull dimension and the finitistic dimension coincide. However, all regular pseudo-Noetherian local rings do not share this pleasant property. The following example due to B.L. Osofsky ([27]) presents a valuation domain of finite global dimension $n \geq 2$ but infinite Krull dimension.

EXAMPLE ([27], Theorem A, Corollary 2) Assume $2 \leq n < \infty$ and let G represent the additive group of all step functions from Ω_{n-2} to $\mathbf{Z}$ where Ω_{n-2} is the first ordinal of cardinality $\aleph_{n-2}$. Order G lexicographically. For $\gamma < \Omega_{n-2}$ denote by $e(\gamma)$ the characteristic function of $\{\beta \mid \gamma \leq \beta < \Omega_{n-2}\}$ ($e(\gamma) \in G$). Now for a fixed field K, construct the valuation domain of the generalized power series ring in an indeterminate x with respect to well ordered subsets of the totally ordered group G. (See Example 8, Chapter II). In [27] Osofsky proves that this domain has global dimension n.

It will now be shown that the Krull dimension of R is infinity. Let m represent the unique maximal ideal of R. First, notice that if $0 < \gamma < \Omega_{n-2}$, $0 \subset \bigcap_{n\in N} (x^{e(\gamma)})^n \subset \underline{m}$. For suppose $g \in G$ with $g(0) > 0$. Then $ne(\gamma) < g$ for all n in N. Hence $0 \neq x^g \in \bigcap_{n\in N} (x^{e(\gamma)})^n$. Furthermore $x^{e(\gamma)} \in \underline{m}$ but $x^{e(\gamma)} \notin (x^{e(\gamma)})^2$.

Now if I is any ideal in a valuation domain, $\bigcap_{n\in N} I^n$ is a prime ideal ([21], Theorem 5.10). Hence, for any γ with $0 < \gamma < \Omega_{n-2}$, $\bigcap_{n\in N} (x^{e(\gamma)})^n$ is always a prime ideal. Suppose γ_o and γ_1 are two different ordinals strictly between 0 and Ω_{n-2} with $\gamma_o < \gamma_1$. It is clear that $(x^{e(\gamma_o)}) \subseteq (x^{e(\gamma_1)})$ and hence $\bigcap_{n\in N} (x^{e(\gamma_o)})^n \subseteq \bigcap_{n\in N} (x^{e(\gamma_1)})^n$ but since $x^{e(\gamma_o)} \notin \bigcap_{n\in N} (x^{e(\gamma_o)})^n$ and $x^{e(\gamma_o)} \in \bigcap_{n\in N} (x^{e(\gamma_1)})^n$, $\bigcap_{n\in N} (x^{e(\gamma_o)})^n \neq \bigcap_{n\in N} (x^{e(\gamma_1)})^n$. Hence to each γ $(0 < \gamma < \Omega_{n-2})$ there corresponds a prime ideal $\bigcap_{n\in N} (x^{e(\gamma)})^n$. These ideals become strictly larger with increasing γ. This establishes that K.dim R $= \infty$

CHAPTER IV

THE CODIMENSION THEOREM FOR GORENSTEIN DIMENSION

In a paper entitled "Stable Module Theory" ([2]), equipped with the concept of Gorenstein dimension (denoted G-dim), M. Auslander and M. Bridger arrive at the following codimension theorem after lengthy homological considerations.

THEOREM ([2], Theorem 4.13) Let (R, $\underline{m}$) be a local Noetherian ring and M a non-zero finitely generated R-module with G-dim M < ∞. Then,

$$(\text{G-dim } M) + (\text{depth } M) = \text{depth } R.$$

This result is a generalization of the original codimension theorem for Noetherian rings and finitely generated modules (cf. ([17], Theorem 173)) which motivated Theorem 3.11 in Chapter III of this thesis. The present chapter is devoted to a careful examination of [2] with the goal of proving a codimension theorem for local pseudo-Noetherian rings and finitely presented modules of finite Gorenstein dimension. That which follows is a brief description of a few of the ideas and results in [2] of key importance in establishing the aforementioned Theorem 4.13 and of the difficulties which arose, together with the modifications needed for a successful generalization.

"Stable Module Theory" begins with a discussion of general functorial techniques for abelian categories. Such topics as derived functors, projective stability of satellites, and projective stabilization of categories are explored. Following this, however, considerations are usually restricted to the category of finitely generated modules over a Noetherian ring. Many of the results in Chapters 2 and 3 of [2] are homological in nature and it is important to realize that if, in the hypotheses, one substitutes coherent rings for Noetherian, and finitely presented modules for finitely generated, similar conclusions follow by means of analogous proofs. The reason for this is that, in these proofs, finitely generated projective resolutions are needed for the modules under consideration. Such resolutions naturally exist for finitely presented modules over coherent rings. In Chapter 2 functors D and Ω^{i} are introduced and the following two exact sequences of functors become major tools in the calculations of that chapter:

(a) $0 \to \mathrm{Ext}_R^1(D\Omega^k M, \ ) \to \mathrm{Tor}_k^R(M, \ ) \to \mathrm{Hom}_R(\mathrm{Ext}_R^k(M,R), \ ) \to \mathrm{Ext}_R^2(D\Omega^k M, \ )$

(b) $\mathrm{Tor}_2^R(D\Omega^k M, \ ) \to (\mathrm{Ext}_R^k(M,R) \otimes_R \ ) \to \mathrm{Ext}_R^k(M, \ ) \to \mathrm{Tor}_1^R(D\Omega^k M, \ ) \to 0$

(R Noetherian, M finitely generated, $k \geq 0$)

Because these exact sequences are so useful, proofs of their existence will be included here for the case in which R is coherent and M is finitely presented. (Complete proofs are not included in [2]).

With respect to the content of ([2], Chapter 4), the generalization procedure becomes less straightforward. By using primes in $\underline{a}(R)$ Theorem 2.11 (a) has been proven and this theorem is used in the proof of Proposition 4.16 which in turn allows the generalization of important results of ([2], Chapter 4). Moreover, whereas the depth of a finitely generated module over a Noetherian ring is always finite, the depth of a finitely presented module over a coherent ring may be infinite.

This necessitates extra considerations with respect to the final results in which it is preferable to allow rings of infinite depth. With the help of a Third Change of Rings Theorem for Gorenstein dimension, the extended codimension theorem appears here in the following form:

THEOREM (4.26) Let $(R, \underline{m})$ be a local pseudo-Noetherian ring and M a non-zero finitely presented R-module with finite Gorenstein dimension. Then

$$(\text{G-dim } M) + (\text{depth } M) = \text{depth } R.$$

With this result finally established, a few remarks are made concerning the relationship of Gorenstein dimension to weak dimension and FP-injective dimension (absolutely pure dimension).

1. PRELIMINARY RESULTS

Let R be a coherent ring. For an arbitrary R-module X, let $X^* = \text{Hom}_R(X, R)$. If X and Y are two R modules, X and Y are called projectively equivalent if and only if there exist projectives P and Q such that $X \oplus P \simeq Y \oplus Q$ (Equivalently: $\text{Ext}^1_R(X, \ ) \simeq \text{Ext}^1_R(Y, \ )$ as functors). Now let M be a finitely presented R-module and choose for M a finitely generated projective resolution:

$$\cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_o \to M \to 0$$

Then $\ker(P_{i-1} \to P_{i-2}), (i > 1)$, is called the i'th syzygy of M and is denoted $\Omega^i(M)$. $\Omega^o(M) = M$ and $\Omega^1(M) = \ker(P_o \to M)$. Coker $(P_o{}^* \to P_1{}^*)$ is denoted

DM and with this,

$$0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$$

is an exact sequence. Note that DM, $\Omega^i(M)$ ($i \geq 0$) and M* are again finitely presented R-modules. For each such finitely presented M, $\Omega^i M$ and DM are determined up to projective equivalence by the projective equivalence class of M ([2], Corollary 2.3 and remark p. 53). For a discussion of left and right derived functors, denoted L_i and R^i ($i \geq 0$) and left and right satellites S_i and S^i ($i \geq 0$) see ([7], Chapter V, §5) and ([7], Chapter III) respectively. With the help of Remark 1.2 (b) it is not difficult to prove the well known fact that $M^* \otimes_R P \cong \mathrm{Hom}_R(M, P)$ if M is finitely presented and P is projective. Furthermore, for a finitely generated projective P, $P \otimes_R X \cong \mathrm{Hom}_R(P^*, X)$ for any R-module X, the isomorphism being given by the equation $(p \otimes_R x)f = f(p)x$ for $p \in P$, $x \in X$ and $f \in P^*$. The following proposition is needed in the proof of Theorem 4.2.

PROPOSITION 4.1 ([1], Proposition 5.8 (c)) If X is a projective complex in R-Mod and $H^k(X, \ ) \underset{df.}{=} H_k(\mathrm{Hom}_R(X, \ ))$ where H_k is the k'th left homology functor, then the following sequence is exact for any $k \geq 0$:

$$0 \to \mathrm{Ext}^1_R(C_{k-1}, \ ) \to H^k(X, \ ) \to \mathrm{Hom}_R(H_k(X), \ ) \to \mathrm{Ext}^2_R(C_{k-1}, \ )$$

where $C_{k-1} = \mathrm{coker}\,(X_k \to X_{k-1})$. This sequence is functorial in X for X in the category of projective complexes.

THEOREM 4.2 (cf. ([2], Theorem 2.8)) Let R be a coherent ring and M a finitely presented R-module. Then, for any $k \geq 0$ the following exact sequences of functors exist:

(a) $0 \to \mathrm{Ext}^1_R(D\Omega^k M,\ ) \to \mathrm{Tor}^R_k(M,\ ) \to \mathrm{Hom}_R(\mathrm{Ext}^k_R(M,R),\ ) \to \mathrm{Ext}^2_R(D\Omega^k M,\ )$

(b) $\mathrm{Tor}^R_2(D\Omega^k M,\ ) \to (\mathrm{Ext}^k_R(M,R) \otimes_R\ ) \to \mathrm{Ext}^k_R(M,\ ) \to \mathrm{Tor}^R_1(D\Omega^k M,\ ) \to 0$.

PROOF (a) Since M is finitely presented and R is coherent, let M have a finitely generated projective resolution:

$$\cdots \to P_{k+1} \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_o \to M \to 0$$

(P_i ($i \geq 0$) finitely generated projective)

The following complex is obtained by applying $\mathrm{Hom}_R(\ ,R)$ to this resolution.

$$0 \to M^* \to P^*_o \to P^*_1 \to \cdots \to P^*_{k-1} \to P^*_k \to P^*_{k+1} \to \cdots \qquad (1)$$

Consider the projective complex

$$X: \qquad \cdots \to X_{m+1} \to X_m \to X_{m-1} \to \cdots$$

constructed from (1) by dropping M^* and renaming the objects in the complex so that $X_m = \begin{Bmatrix} P^*_{2k-m} & m \leq 2k \\ 0 & m > 2k \end{Bmatrix}$. Now Proposition 4.1 is applied to this finitely generated projective complex.

Up to projective equivalence the following statements are true.

(i) $\Omega^k M = \operatorname{coker}(P_{k+1} \to P_k)$

(ii) $0 \to (\Omega^k M)^* \to P_k^* \to P_{k+1}^* \to D\Omega^k M \to 0$ is exact

(iii) $D\Omega^k M = \operatorname{coker}(P_k^* \to P_{k+1}^*) = \operatorname{coker}(X_k \to X_{k-1}) = C_{k-1}$

Furthermore, it is clear that $H_k(X) = \operatorname{Ext}_R^k(M,R)$. Now consider the functorial complex:

$$\ldots \to \operatorname{Hom}_R(P_{k+1}^*, \ ) \to \operatorname{Hom}_R(P_k^*, \ ) \to \operatorname{Hom}_R(P_{k-1}^*, \ ) \to \ldots \to \operatorname{Hom}_R(P_1^*, \ ) \to \operatorname{Hom}_R(P_o^*, \ ) \to 0 \to 0 \to \ldots \quad (2)$$

But this is just

$$\ldots \to \operatorname{Hom}_R(X_{k-1}, \ ) \to \operatorname{Hom}_R(X_k, \ ) \to \operatorname{Hom}_R(X_{k+1}, \ ) \to \ldots \to \operatorname{Hom}_R(X_{2k-1}, \ ) \to \operatorname{Hom}_R(X_{2k}, \ ) \to 0 \to 0 \to \ldots$$

Since for all $i \geq 0$ $\operatorname{Hom}_R(P_i^*, \ ) \simeq (P_i \otimes_R \ \ )$, (2) may also be written

$$\ldots \to (P_{k+1} \otimes_R \ \ ) \to (P_k \otimes_R \ \ ) \to (P_{k-1} \otimes_R \ \ ) \to \ldots \to (P_1 \otimes_R \ \ ) \to (P_o \otimes_R \ \ ) \to 0 \to 0 \to \ldots,$$

Thus $H^k(X, \ ) = H_k(\operatorname{Hom}_R(X, \ )) = \operatorname{Tor}_k^R(M, \ )$.

Therefore the exact sequence of Proposition 4.1 in this case becomes

$$0 \to \operatorname{Ext}_R^1 (D\Omega^k M, \ ) \to \operatorname{Tor}_k^R (M, \ ) \to \operatorname{Hom}_R(\operatorname{Ext}_R^k(M,R), \ ) \to \operatorname{Ext}_R^2 (D\Omega^k M, \ )$$

as required.

(b) The following exact sequence of functors may be constructed for any covariant half exact functor F of one variable (cf. ([7], p. 105, Exercise 3.)).

$$\cdots \to S_2 S^1 F \to L_o F \to F \to S_1 S^1 F \to 0 .$$

For the purposes of this proof, let $F = \mathrm{Ext}_R^k(M,\ )$, $(k \geq 0)$.

Thus the following sequence is exact:

$$S_2\,\mathrm{Ext}_R^{k+1}(M,\ ) \to L_o(\mathrm{Ext}_R^k(M,\ )) \to \mathrm{Ext}_R^k(M,\ ) \to S_1\mathrm{Ext}_R^{k+1}(M,\ ) \to 0 \qquad (3)$$

<u>Claim 1.</u> $L_o\ (\mathrm{Ext}_R^k(M,\ )) \approx (\mathrm{Ext}_R^k(M,R) \otimes_R\ )$

<u>Proof</u> By the uniqueness property for L_o it is only necessary to show that $(\mathrm{Ext}_R^k(M,R) \otimes_R\ )$ is right exact and that there exists a map ϕ: $(\mathrm{Ext}_R^k(M,R) \otimes_R\ ) \to \mathrm{Ext}_R^k(M,\ )$ which is an isomorphism on projectives. A tensor product is always right exact. Let the following be projective resolutions for M and an arbitrary R-module X:

$$\cdots \to P_{k+1} \xrightarrow{p_k} P_k \xrightarrow{p_{k-1}} P_{k-1} \to \cdots \to P_1 \xrightarrow{p_o} P_o \to M \to 0$$

(P_i $(i \geq 0)$ finitely generated, projective)

$$\cdots \to Q_{k+1} \to Q_k \to Q_{k-1} \to \cdots \to Q_1 \to Q_o \to X \to 0$$

(Q_i $(i \geq 0)$ projective)

Let Q be an arbitrary projective and let $p_k^* = \mathrm{Hom}_R(p_k, R)$. Then the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots \to \mathrm{Hom}_R(P_{k-1}, Q) & \xrightarrow{\mathrm{Hom}_R(p_{k-1},Q)} & \mathrm{Hom}_R(P_k,Q) & \xrightarrow{\mathrm{Hom}_R(p_k,Q)} & \mathrm{Hom}_R(P_{k+1},Q) \to \cdots \\
\wr\wr & & \wr\wr & & \wr\wr \\
\cdots \to P^*_{k-1} \otimes_R Q & \xrightarrow{p^*_{k-1} \otimes_R Q} & P^*_k \otimes_R Q & \xrightarrow{p^*_k \otimes_R Q} & P^*_{k+1} \otimes_R Q \to \cdots
\end{array}$$

where the isomorphisms are given by $(f \otimes_R q)(z) = f(z)q$ in each case for suitably interpreted f and z. The commutativity of this diagram together with the fact that Q is flat insure an isomorphism

$$\mathrm{Ext}^k_R(M, R) \otimes_R Q \to \mathrm{Ext}^k_R(M,Q)$$

since $\mathrm{Ext}^k_R(M,R) \otimes_R Q \simeq \dfrac{(\ker p^*_k) \otimes_R Q}{(\mathrm{im}\ p^*_{k-1}) \otimes_R Q} \simeq \dfrac{\ker(p^*_k \otimes_R Q)}{\mathrm{im}\ (p^*_{k-1} \otimes_R Q)} \simeq \mathrm{Ext}^k_R(M,Q)$.

With such isomorphisms the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Ext}^k_R(M,R) \otimes_R Q_1 & \simeq & \mathrm{Ext}^k_R(M,Q_1) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^k_R(M,R) \otimes_R Q_o & \simeq & \mathrm{Ext}^k_R(M,Q_o) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^k_R(M,R) \otimes_R X & & \mathrm{Ext}^k_R(M,X) \\
\downarrow & & \\
0 & &
\end{array}$$

Thus, it is easy to see there exists a unique map ϕ_X: $\mathrm{Ext}^k_R(M,R) \otimes_R X \to \mathrm{Ext}^k_R(M,X)$ making the bottom square commute. By construction this map is an isomorphism on projectives and this gives the required result.

<u>Claim 2</u> $S_1 \operatorname{Ext}_R^{k+1}(M, \ ) \approx \operatorname{Tor}_1^R(D\Omega^k M, \ )$

<u>Proof</u> It will be shown in general that for a finitely presented module N, $S_1 \operatorname{Ext}_R^1(N, \ ) \approx \operatorname{Tor}_1^R(DN, \ )$. This will give the required result since $S_1 \operatorname{Ext}_R^{k+1}(M, \ ) \approx S_1 \operatorname{Ext}_R^1(\Omega^k M, \ )$. Let X be an arbitrary R-module and let

$$0 \to Y \to P \to X \to 0$$

be an exact sequence of modules with P projective. Then by the properties of the first left satellite,

$$0 \to S_1 \operatorname{Ext}_R^1(N, X) \to \operatorname{Ext}_R^1(N, Y) \to \operatorname{Ext}_R^1(N, P)$$

is exact. Let the following be a finitely generated projective resolution for N :

$$\cdots \to P_3 \to P_2 \to P_1 \to P_o \to N \to 0$$

Then the following diagram commutes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N^* \arrow[r, "\varepsilon"] & P_o^* \arrow[r, "\delta"] \arrow[d, equal] & P_1^* \arrow[r] \arrow[d, equal] & DN \arrow[r] \arrow[d, dashed, "\beta"] & 0 \\
 & Q \arrow[u, dashed, "\alpha"] \arrow[r, "\gamma"] & P_o^* \arrow[r, "\delta"] & P_1^* \arrow[r] & P_2^* &
\end{tikzcd}$$

when Q is a projective making $Q \to P_o^* \to P_1^* \to DN \to 0$ exact. α and β exist because $N^* = \ker(P_o^* \xrightarrow{\delta} P_1^*)$ and $DN = \operatorname{coker}(P_o^* \xrightarrow{\delta} P_1^*)$. Tensoring the above diagram with X the following commutes with the row being exact:

$$
\begin{tikzcd}
N^* \otimes_R X \arrow[r, "\varepsilon \otimes_R X"] & P_o^* \otimes_R X \arrow[r, "\delta \otimes_R X"] & P_1^* \otimes_R X \arrow[r] & DN \otimes_R X \arrow[r] & 0 \\
Q \otimes_R X \arrow[u, "\alpha \otimes_R X"] \arrow[ur, "\gamma \otimes_R X"'] & & & &
\end{tikzcd}
$$

Thus $\mathrm{Tor}_1^R(DN, X) = \dfrac{\ker(\delta \otimes_R X)}{\mathrm{im}(\gamma \otimes_R X)} = \dfrac{\ker(\delta \otimes_R X)}{\mathrm{im}(\varepsilon \otimes_R X)}$. ($\alpha$ epi, implies $\alpha \otimes_R X$ epi)

Using the fact that for any finitely presented R-module M there exists a map $(M^* \otimes_R \ ) \to \mathrm{Hom}_R(M, \ )$ which is an isomorphism on projectives the following diagram commutes with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(N,Y) \arrow[r] & \mathrm{Hom}_R(N,P) \arrow[r] & \mathrm{Hom}_R(N,X) & \\
 & N^* \otimes_R Y \arrow[u] \arrow[r] & N^* \otimes_R P \arrow[u, phantom, "\cong"] \arrow[r] & N^* \otimes_R X \arrow[u] \arrow[r] & 0
\end{tikzcd}
$$

Bringing all of this information together the following diagram is commutative and all columns are exact except the first in which $\ker g \subseteq \mathrm{im}\, f$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(N,Y) \arrow[r] \arrow[d, "f"] & \mathrm{Hom}_R(P_o,Y) \arrow[r, "\ell"] \arrow[d] & \mathrm{Hom}_R(P_1,Y) \arrow[r, "m"] \arrow[d, "k"] & \mathrm{Hom}_R(P_2,Y) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_R(N,P) \arrow[r] \arrow[d, "g"] & \mathrm{Hom}_R(P_o,P) \arrow[r, "j"] \arrow[d, "h"] & \mathrm{Hom}_R(P_1,P) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_2,P) \arrow[d] \\
 & N^* \otimes_R X \arrow[r, "\varepsilon \otimes_R X"] \arrow[d] & P_o^* \otimes_R X \arrow[r, "\delta \otimes_R X"] \arrow[d] & P_1^* \otimes_R X \arrow[r] \arrow[d] & P_2^* \otimes_R X \arrow[d] \\
 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Now define a map $\theta : S_1 \ \mathrm{Ext}_R^1(N,X) \to \mathrm{Tor}_1^R(DN,X)$ as follows:

Let $\bar{u} \in S_1 \ \mathrm{Ext}_R^1(N,X)$ ($\bar{u}$ the coset determined by $u \in \ker m$ where $S_1 \ \mathrm{Ext}_R^1(N,X)$ is treated as a submodule of $\mathrm{Ext}_R^1(N,Y)$).

Since $\bar{u} \in S_1 \ \mathrm{Ext}_R^1 (N, X)$,

$k(u) = j(w)$ for some $w \in \mathrm{Hom}_R(P_o,P)$

Moreover, $h(w) \in \ker(\delta \underset{R}{\otimes} X)$.

Define $\theta(\bar{u}) = h(w) + \mathrm{im}(\varepsilon \underset{R}{\otimes} X) \in \mathrm{Tor}_1^R(DN,X)$

Routine diagram chasing reveals that this map makes sense and is an isomorphism.

Thus $S_1 \ \mathrm{Ext}_R^1(N,X) \simeq \mathrm{Tor}_1^R(DN,X)$ as required.

<u>Claim 3</u> $\quad S_n \ \mathrm{Ext}_R^{k+1} (M, \ ) \simeq \mathrm{Tor}_n^R(D\Omega^k M, \ ) \quad (n \geq 1)$

<u>Proof</u> $\quad S_1 \ \mathrm{Ext}_R^1(N, \ ) \simeq \mathrm{Tor}_1^R(DN, \ )$ and therefore $S_{n-1}S_1 \ \mathrm{Ext}_R^1(N, \ ) \simeq S_{n-1} \ \mathrm{Tor}_1^R(DN, \ )$ for any finitely presented N.

Hence, $S_{n-1}S_1 \ \mathrm{Ext}_R^1(\Omega^k M, \ ) \simeq S_{n-1} \ \mathrm{Tor}_1^R(D\Omega^k M, \ )$

showing that $S_n \ \mathrm{Ext}_R^{k+1}(M, \ ) \simeq \mathrm{Tor}_n^R (D\Omega^k M, \ )$.

The identities of the above three claims applied to (3) show that $\mathrm{Tor}_2^R(D\Omega^k M, \ ) \to (\mathrm{Ext}_R^k(M,R) \underset{R}{\otimes} \ ) \to \mathrm{Ext}_R^k(M, \ ) \to \mathrm{Tor}_1^R(D\Omega^k M, \ ) \to 0$ exists and is exact, thereby completing the proof of Theorem 4.2.

The following corollary may be proven entirely by using the sequences (a) and (b) above. However, a different proof is given below which only needs sequence (a). A functor $F : R\text{-Mod} \to R\text{-Mod}$ is called projectively stable if $F(P) = 0$ for any projective module P. Injective stability of F is defined analogously.

COROLLARY 4.3 Let R be a coherent ring and M a finitely presented R-module. Let k be a non-negative integer. The following conditions are equivalent.

(a) $\mathrm{Ext}_R^k(M,R) = 0$

(b) $\mathrm{Ext}_R^k(M,\ )$ is projectively stable.

(c) $\mathrm{Tor}_k^R(M,\ )$ is injectively stable.

PROOF ((a) implies (b)) Coherence of R may be characterized by the fact that $\mathrm{Ext}_R^n(M,\ )$ commutes with directed colimits for any finitely presented module M and for all $n \geq 0$ (Theorem 1.6(f)). It is well known that any flat module, and therefore any projective module, is a directed colimit of finitely generated free modules. These two statements show that (a) implies (b)

((b) implies (a)). Clear

((b) implies (c)). Suppose $\mathrm{Ext}_R^k(M,\ )$ is projectively stable. Then $\mathrm{Ext}_R^k(M,R) = 0$. From Theorem 4.2, sequence (a), $\mathrm{Tor}_k^R(M,\ ) \cong \mathrm{Ext}_R^1(D\Omega^k M,\ )$.

Therefore $\mathrm{Tor}_k(M,\ )$ is injectively stable.

((c) implies (b)) Let P be a projective module. By the remarks on duality ([7], p. 120),

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}_R^k(M,P), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Tor}_k^R(M, \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})).$$

Now $\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. Therefore $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}_R^k(M,P), \mathbb{Q}/\mathbb{Z}) = 0$ implying $\mathrm{Ext}_R^k(M,P) = 0$. Thus $\mathrm{Ext}_R^k(M, \ )$ is projectively stable.

The cokernel of the map $L_o(\mathrm{Hom}_R(M, \ )) \to \mathrm{Hom}_R(M, \ )$ for an R-module M is called the projective stabilization of $\mathrm{Hom}_R(M, \ )$ and is denoted $\underline{\mathrm{Hom}}_R(M, \ )$. The functor $\underline{\mathrm{Hom}}_R(M, \ )$ is projectively stable and for an R-module N, $\underline{\mathrm{Hom}}_R(M, \ )(N)$ is denoted $(\underline{M},\underline{N})$.

Recall:

$$\left.\begin{array}{ll} (1) & S_i(M \otimes_R \ ) \cong \mathrm{Tor}_i^R(M, \ ), \ (i \geq 0) \\ (2) & S^i \,\mathrm{Hom}_R(M, \ ) \cong \mathrm{Ext}_R^i(M, \ ), \ (i \geq 0) \end{array}\right\} \text{([7], Chapter V, Theorem 6.1)}$$

(3) $S^i \,\mathrm{Hom}_R(M, \ ) \cong S^i \,\underline{\mathrm{Hom}}_R(M, \ )$, $(i > 0)$, cf.([2], Corollary 1.18)

The following proposition is needed in later considerations and the proof is completely analogous to the proof of Proposition 2.11 of [2].

<u>PROPOSITION 4.4</u> Let R be a coherent ring and M a finitely presented module. Then:

(a) For $k, j > 0$ $\quad S_k S^j(\underline{M},\underline{\ }) \cong \mathrm{Tor}_k^R(D\Omega^{j-1}M, \ )$

(b) For $k, j \geq 0$ $\quad S^k S_j(\underline{M},\underline{\ }) \cong S^k \mathrm{Tor}_{j+1}^R(DM, \ )$

When $k > 0$, $S^k \mathrm{Tor}_{j+1}^R(DM, \ ) \cong \mathrm{Ext}_R^k(D\Omega^j DM, \ )$.

2. GORENSTEIN DIMENSION

Gorenstein dimension is defined in terms of the projective or injective stability of certain functors. The definition may at first seem rather complex but it will be seen that there is a much simpler description for modules of finite Gorenstein dimension. Indeed, it is the finitely presented modules of finite dimension which are important in the later theorems.

LEMMA 4.5 Let R be a coherent ring and M a finitely presented R-module. Let k be a non-negative integer. If $\mathrm{Ext}_R^i(\Omega^k M, \ )$ is projectively stable for all $i > 0$ and $\mathrm{Tor}_i^R(D\Omega^k M, \ )$ is injectively stable for all $i > 0$ then, $\mathrm{Ext}_R^i(\Omega^{k+1} M, \ )$ is projectively stable for all $i > 0$ and $\mathrm{Tor}_i^R(D\Omega^{k+1} M, \ )$ is injectively stable for all $i > 0$.

PROOF It is clear that if $\mathrm{Ext}_R^i(\Omega^k M, \ )$ is projectively stable for all $i > 0$ then $\mathrm{Ext}_R^i(\Omega^{k+1} M, \ ) \simeq \mathrm{Ext}_R^{i+1}(\Omega^k M, \ )$ is projectively stable for all $i > 0$.

Now $\mathrm{Tor}_i^R(D\Omega^{k+1} M, \ ) \simeq S_i S^{k+2}(\underline{M}, \_)$ (Proposition 4.4)

$\simeq S_{i-1} S_1 S^1 \ \mathrm{Ext}_R^{k+1}(M, \ )$

$\simeq S_{i-1} \ \mathrm{Ext}_R^{k+1}(M, \ )$ (cf.([2], Corollary 1.16))

since $\mathrm{Ext}_R^{k+1}(M, \ ) \simeq \mathrm{Ext}_R^1(\Omega^k M, \ )$ is projectively stable.

If $i = 1$ $\mathrm{Ext}_R^{k+1}(M, \ )$ is injectively stable in any case.

If $i > 1$ $S_{i-1} \ \mathrm{Ext}_R^{k+1}(M, \ ) \simeq \mathrm{Tor}_{i-1}(D\Omega^k M, \ )$ (Proposition 4.4)

and $\mathrm{Tor}_{i-1}(D\Omega^k M, \ )$ is injectively stable by hypothesis.

Now the following definition may be introduced.

DEFINITION 4.6 Let M be a finitely presented R-module over a coherent ring R. Then:

Gorenstein dimension M = inf { k| $Ext_R^i(\Omega^k M,\ )$ is projectively stable for all $i > 0$ and $Tor_i^R(D\Omega^k M,\ )$ is injectively stable for all $i > 0$}

Denote Gorenstein dimension M by G-dim M.

The following lemma is essentially a repetition of ([2], Corollary 3.6).

LEMMA 4.7 Let R be coherent and M be a finitely presented R-module. Furthermore suppose G-dim M < ∞ . Then:

(a) If $Ext_R^i(M,\ )$ is projectively stable for all $i > 0$, G-dim M = 0

(b) If there exists an $i > 0$ such that $Ext_R^i(M,\ )$ is not projectively stable then G-dim M = sup{t| $Ext_R^t(M,\ )$ is not projectively stable}.

REMARK 4.8 Corollary 4.3 together with Lemma 4.7 show that if M is a finitely presented R-module with G-dim M < ∞ , then:

$$\text{G-dim } M = \sup\{t \mid Ext_R^t(M,R) \neq 0\}.$$

Let k be a positive integer. An R-module M is called k-torsion free if $Ext_R^i(DM,R) = 0$ for all i, $1 \leq i \leq k$. For the proofs of the following results see ([2], Chapter 3).

PROPOSITION 4.9 (cf. ([2], Proposition 3.8)) Let R be coherent and M be a finitely presented R-module. The following statements are equivalent:

(a) G-dim $M = 0$

(b) $\mathrm{Ext}_R^i(M,R) = \mathrm{Ext}_R^i(DM,R) = 0$ for all $i > 0$.

(c) M is reflexive and $\mathrm{Ext}_R^i(M,R) = \mathrm{Ext}_R^i(M^*,R) = 0$ for all $i > 0$

(d) M and DM are both i-torsion free for all $i > 0$.

LEMMA 4.10 (cf. ([2], Lemma 3.10)) Let R be a coherent ring. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely presented R-modules with G-dim $M'' = 0$. Then G-dim $M' = 0$ if and only if G-dim $M = 0$.

THEOREM 4.11 (cf. ([2], Theorem 3.13)) For an integer $k \geq 0$ and a finitely presented module M over a coherent ring R the following statements are equivalent:

(a) G-dim $M \leq k$

(b) G-dim $\Omega^k M = 0$

(c) There exists an exact sequence of finitely presented modules: $0 \to X_k \to \ldots \to X_1 \to X_o \to M \to 0$ with G-dim $X_i = 0$ for $0 \leq i \leq k$.

COROLLARY 4.12 Let R be a coherent ring and M a finitely presented R-module. Then, G-dim $M \leq$ p. dim M.

NOTE: If p. dim $M < \infty$, then G-dim M = p. dim M. This is easy to see if one notices that for a finitely presented module M with projective dimension k over a coherent ring R, $\mathrm{Ext}_R^k(M,R) \neq 0$ (cf.([7], Chapter VI, Exercise 9)). Then G-dim M = k by Remark 4.8.

COROLLARY 4.13 (cf. ([2], Corollary 3.15)) Let R be a coherent ring and M a finitely presented R-module.

(a) If G-dim M < ∞, then G-dim M is the smallest non-negative integer k such that there exists a finitely presented exact sequence $0 \to X_k \to \dots \to X_1 \to X_0 \to M \to 0$ with G-dim $X_i = 0, 0 \leq i \leq k$.

(b) If G-dim M ≠ 0 and $0 \to N \to X \to M \to 0$ is an exact sequence of finitely presented R-modules with G-dim X = 0, then

$$\text{G-dim } N = \text{G-dim } M - 1.$$

3. THE CODIMENSION THEOREM

Recall that for two R-modules M and N $M\text{-depth } N = \inf\{t \mid \mathrm{Ext}_R^t(M,N) \neq 0\}$ and if $(R,\underline{m})$ is a local pseudo-Noetherian ring and M is a finitely presented R-module, $\mathrm{depth}_R M = \mathrm{depth}\, M = \sup_{I \in \mathcal{J}}(R/I\text{-depth } M)$ where $\mathcal{J}$ is the directed set of finitely generated ideals in $\underline{m}$. The following two lemmas will be needed later.

LEMMA 4.14 If M is an R-module and x ε R is both M- and R-regular, then:

(a) $\mathrm{Tor}_i^R(M, R/(x)) = 0$ for all $i > 0$.

(b) $\mathrm{Ext}_R^i(M, R/(x)) \cong \mathrm{Ext}_{R/(x)}^i(M/xM, R/(x))$ for all $i \geq 0$.

PROOF (a) Tensor the exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ with M.

(b) ([7], p. 118, Proposition 4.1.3)

LEMMA 4.15 (cf. ([2], Lemma 4.10)) Let R be a coherent ring and M a finitely presented R-module with G-dim $M < \infty$. Let $x \in R$ be both M- and R-regular. Then, $\text{G-dim}_{R/(x)} M/xM < \infty$.

Let R be a pseudo-Noetherian ring for which every non-zero-divisor is a unit (total quotient ring). If M is a finitely presented R-module, $M = 0$ if and only if $\text{Hom}_R(M,R) = M^* = 0$. In fact, suppose $M^* = 0$. By Theorem 2.11 (a) there is a non-zero-divisor in $\text{ann}_R M$. By hypothesis this means $\text{ann}_R M = R$ and therefore $M = 0$. The following important proposition may now be proven without too much difficulty.

PROPOSITION 4.16 Let R be a pseudo-Noetherian ring for which every non-zero-divisor is a unit. Let M be a finitely presented R-module. Then, G-dim $M = 0$ if and only if G-dim $M < \infty$.

PROOF cf. ([2], Proposition 4.11).

The considerations now become restricted to local rings.

PROPOSITION 4.17 (cf. ([2], Proposition 4.12)) Let $(R,\underline{m})$ be a local pseudo-Noetherian ring with finite depth. Let M be a non-zero finitely presented R-module of finite Gorenstein dimension. The following statements are equivalent.

(a) G-dim $M = 0$

(b) $\text{Tor}_i^R(M,X) = 0$ for all $i > 0$ and every R-module X with p.dim $X < \infty$.

(c) Every R-sequence in $\underline{m}$ is an M-sequence.

(d) depth $M \geq$ depth R.

(e) depth $M =$ depth R.

(f) $\mathrm{Ext}_R^i(M,R) = 0$ for all $i > 0$.

(g) $\mathrm{Ext}_R^i(M,X) = 0$ for all $i > 0$ and every R-module X with p. dim $X < \infty$.

PROOF ((a) implies (b) and (b) implies (c)).

(cf. ([2], Proposition 4.12))

((c) implies (d)).

This is clear because depth M is the length of a maximal M-sequence in $\underline{m}$ and depth R is the length of a maximal R-sequence in $\underline{m}$ (Theorem 3.5).

((d) implies (e)).

(e) is proven by induction on depth R. Therefore, first suppose that depth $R = 0$. The aim is to show that depth $M = 0$. Assume by way of contradiction that $\underline{m}$ contains an M-regular element r. Let I be any finitely generated ideal in $\underline{m}$ containing r. Since depth $R = 0$, $\underline{m} = Z(R)$. Because R is pseudo-Noetherian, there exists $s \neq 0$ in R with $r \in I \subseteq \mathrm{ann}_R(s) = s^o$. Now $R/s^o \simeq sR \subseteq R$ and by applying $\mathrm{Hom}_R(M^*, \ )$ to this, one obtains $0 \to \mathrm{Hom}_R(M^*, R/s^o) \to M^{**}$ is exact. Since $\underline{m} = Z(R)$ and G-dim $M < \infty$, G-dim $M = 0$ by Proposition 4.16. Thus, by Proposition 4.9, $M^{**} \simeq M$. Now r is a non-zero-divisor for M and r annihilates $\mathrm{Hom}_R(M^*, R/s^o)$ forcing $\mathrm{Hom}_R(M^*, R/s^o) = 0$. For each finitely generated ideal $I \subseteq \underline{m}$ with $r \in I$, such an s^o may be constructed with the above properties. Let A represent the set of all these annihilators. Clearly A is a cofinal set in the directed set of all finitely generated ideals in $\underline{m}$ and therefore,

$\varinjlim_{s^o \in A} R/s^o = R/\underline{m}$. Furthermore M^* is finitely presented because M is and R is coherent. As has been mentioned earlier, this means that $\mathrm{Hom}_R(M^*, \ )$ commutes with directed colimits giving $\mathrm{Hom}_R(M^*, R/\underline{m}) = 0$. The natural surjection $M^* \to M^*/\underline{m}M^*$ induces an injection $\mathrm{Hom}_R(M^*/\underline{m}M^*, R/\underline{m}) \to \mathrm{Hom}_R(M^*, R/\underline{m}) = 0$. This gives $M^*/\underline{m}M^* = 0$ and by Nakayama's Lemma $M^* = 0$. By the reflexivity of M, $M = 0$. This is a contradiction to the original hypothesis. Thus, depth $M = 0$ as required.

Now suppose depth $R = n \geq 1$ and the result holds for rings of depth less than n. By assumption depth $M \geq$ depth $R = n$.

<u>Claim</u> There exists $x \in \underline{m}$ which is both M- and R-regular.

<u>Proof</u> Suppose to the contrary that $\underline{m} \subseteq Z(R) \cup Z(M)$. It is easy to see that $Z(R \oplus M) = Z(R) \cup Z(M)$ for the finitely presented R-module $R \oplus M$. Let I be a finitely generated ideal contained in $\underline{m}$. Since R is pseudo-Noetherian and $I \subseteq Z(R \oplus M)$, there exists $(r,m) \in R \oplus M$ with $r \neq 0$ or $m \neq 0$ such that $I(r,m) = 0$. Therefore $Ir = 0$ and $Im = 0$ giving $I \subseteq Z(R)$ or $I \subseteq Z(M)$. If all proper finitely generated ideals are contained in $Z(R)$, $\underline{m} = Z(R)$ and depth $R = 0$. If there exists a proper finitely generated ideal with $I \nsubseteq Z(R)$, then from the above considerations it follows that $I \subseteq Z(M)$ and all finitely generated proper ideals containing I are also contained in $Z(M)$. This means that $\underline{m} = Z(M)$ and depth $M = 0$. Thus, either depth $R = 0$ or depth $M = 0$ contradicting the fact that depth $M \geq$ depth $R \geq 1$. The existence of an $x \in \underline{m}$ which is both M- and R-regular is assured.

For such an $x \in \underline{m}$, Lemma 4.15 shows that $\text{G-dim}_{R/(x)} M/xM < \infty$.

By the induction assumption $\text{depth}_{R/(x)} M/xM = \text{depth}_R R/(x) = \text{depth } R - 1$.

Since $\text{depth}_{R/(x)} M/xM = \text{depth}_R M - 1$ (Proposition 3.6), $\text{depth}_R M = \text{depth } R$ as required.

((e) implies (f))

Suppose (e) holds and again use induction on depth R. If depth R = 0 Proposition 4.16 implies that G-dim M = 0 which in turn implies that $\text{Ext}_R^i(M,R) = 0$ for $i > 0$ (Remark 4.8). Now suppose that depth R ≥ 1 and the result holds for rings of smaller depth. Since depth M = depth R ≥ 1 there exists an $x \in \underline{m}$ as above with $x \notin Z(M)$, and $x \notin Z(R)$.

Depth M = depth R implies $\text{depth}_{R/(x)} M/xM = \text{depth}_{R/(x)} R/(x) = \text{depth } R - 1$. Since $\text{G-dim}_{R/(x)} M/xM < \infty$, $\text{Ext}^i_{R/(x)}(M/xM, R/(x)) = 0$ for all $i > 0$ by the induction hypothesis. Lemma 4.14 illustrates that $\text{Ext}^i_R(M, R/(x)) \cong \text{Ext}^i_{R/(x)}(M/xM, R/(x)) = 0$ for all $i > 0$. From the exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0 \qquad (1)$$

one obtains the following portion of the long exact sequence constructed by applying $\text{Hom}_R(M, \ )$ to (1).

$$\text{Ext}^i_R(M,R) \xrightarrow{x} \text{Ext}^i_R(M,R) \to \text{Ext}^i_R(M,R/(x))$$

For all $i > 0$ the last term is zero giving $\text{Ext}^i_R(M,R) = (x)\ \text{Ext}^i_R(M,R)$.

Now it is not difficult to see that $\mathrm{Ext}_R^i(M,R)$ is finitely presented over R. By Nakayama's Lemma, $\mathrm{Ext}_R^i(M,R) = 0$ for all $i > 0$.

((f) implies (g))

Using Corollary 4.3, this follows in a straightforward manner by using induction on p. dim X.

((g) implies (a))

Clear by Remark 4.8.

This concludes the proof of Proposition 4.17.

REMARK 4.18

(1) In the Noetherian case ([2], Proposition 4.12) the assumption that depth $R < \infty$ is naturally not needed since the depth is always finite. However, this condition is required for the proposition above in the coherent case. Consider the following example.

Let $R = K[[(x_n)_{n \in \mathbb{N}}]]$ be the power series ring over a field K in a countably infinite number of indeterminates indexed by the natural numbers, with the extra condition that only a finite number of indeterminates occur in the expansion of any given element of R. This has previously been shown to be a local pseudo-Noetherian ring with infinite depth (Chapter II, Example 2). $M = R/x_1R$ is a finitely presented R-module with G-dim M = p. dim M = 1 (p. dim $M = 1 < \infty$. Therefore, G-dim M = 1). Furthermore, depth $M = \infty$ since $x_2, x_3, \ldots$ is an infinite M-sequence. Thus, (e) holds in the above proposition for this ring and module but (a) does not.

(2) If the assumption that depth R is finite is deleted in the above proposition and (c) is replaced by:

(c^1) Every finite R-sequence in $\underline{m}$ is an M-sequence,

an examination of the proof reveals that the following implications remain valid:

(a) implies (b) implies (c^1) implies (d) implies (e)

Moreover (a), (f), and (g) are equivalent.

(3) It can be shown (cf. ([2], Theorem 4.13)) that for a finitely presented R-module M of finite Gorenstein dimension over a local pseudo-Noetherian ring (R, $\underline{m}$), the following numbers are equal:

(i) G-dim M

(ii) $\inf\{t \mid \mathrm{Ext}_R^i(M,X) = 0$ for all $i > t$ and all modules X with p. $\dim X < \infty\}$

The following Change of Rings Theorems were proven for the Noetherian case in [2]. The proofs of the generalizations given below are completely analogous.

<u>PROPOSITION 4.19</u> (cf. ([2], Corollary 4.30), Third Change of Rings Theorem for Gorenstein Dimension.) Let (R, $\underline{m}$) be a local coherent ring and M a finitely presented R-module. Let $x \in \underline{m}$ be both M- and R-regular. Then,

$$\mathrm{G\text{-}dim}_R M = \mathrm{G\text{-}dim}_{R/(x)} M/xM .$$

PROPOSITION 4.20 (cf. ([2], Lemma 4.32), First Change of Rings Theorem for Gorenstein Dimension) Let $(R,\underline{m})$ be a coherent local ring and M a finitely presented non-zero R-module. Let x be an R-regular element such that $xM = 0$. Then, if $\text{G-dim}_{R/(x)} M/xM = k < \infty$, $\text{G-dim}_R M = k + 1$.

REMARK 4.21 If $(R, \underline{m})$ is a coherent local ring, M is a finitely presented non-zero R-module with G-dim $M < \infty$, and if $x \in \underline{m}$ is both M- and R-regular, then $\text{G-dim}_R M/xM = \text{G-dim}_R M + 1$. This result is a straightforward application of the above two propositions.

The main goal of this chapter is realized in the following codimension theorem which generalizes ([2] , Theorem 4.13) and Theorem 3.11. The case in which depth R is infinite does not occur in the Noetherian situation, but here it must of course again be considered. This theorem shows that the concept of depth may sometimes be thought of as a codimension with respect to Gorenstein dimension.

THEOREM 4.22 Let $(R, \underline{m})$ be a local pseudo-Noetherian ring, and M a non-zero finitely presented R-module with finite Gorenstein dimension. Then,

$$(\text{G-dim } M) + (\text{depth } M) = \text{depth } R$$

PROOF Case 1 : depth $R < \infty$

Proceed by induction on G-dim M. If G-dim $M = 0$, then depth M = depth R by Proposition 4.17. Now suppose G-dim $M = n > 0$ and the result holds for all finitely presented modules of smaller Gorenstein dimension. Consider the following finite presentation for M:

$$0 \to K \to P \to M \to 0$$

(P finitely generated, projective and K finitely generated)

Now depth M < depth P = depth R. Otherwise, by Proposition 4.17 G-dim M = 0. With this it is easily proven that depth K = depth M + 1. Now G-dim K = n − 1 by Corollary 4.13. By the induction assumption,

$$n - 1 = \text{G-dim } K = \text{depth } R - \text{depth } K = \text{depth } R - \text{depth } M - 1$$

Therefore, n = G-dim M = depth R − depth M as required.

Case 2 : depth $R = \infty$.

The proof to follow is analogous to the corresponding proof of Case 2 of Theorem 3.11. First it will be shown that depth $M \neq 0$. If G-dim M = 0, depth M = depth $R = \infty$ by Remark 4.18 (2). Now assume G-dim M = n > 0 and depth $M' \neq 0$ for all finitely presented non-zero R'-modules M' of finite Gorenstein dimension less than n for any pseudo-Noetherian local ring R' of infinite depth. Let

$$0 \to K \to F \xrightarrow{\nu} M \to 0$$

be a finite presentation of M with F finitely generated free, and K finitely generated. Since depth R > 0, there exists a non-zero divisor x in $\underline{m}$. x is clearly also a K-regular element. Now K/xK is a finitely presented $R/(x)$-module with the property that $\text{G-dim}_{R/(x)} K/xK = \text{G-dim}_R K = n - 1$ (Corollary 4.13 and Proposition 4.19). Furthermore, $R/(x)$ is a local pseudo-Noetherian ring with infinite depth and if depth M = 0 Lemma 3.9 indicates that $\operatorname{depth}_{R/(x)} K/xK = 0$. But then the induction assumption is contradicted, and therefore, depth $M \neq 0$ as required.

Now suppose by way of contradiction that depth $M < \infty$. By the above, depth $M \neq 0$. Furthermore depth $R = \infty$. Thus, by the argument found in Proposition 4.17 ((d) implies (e)) there exists an $x \in \underline{m}$ which is a non-zero-divisor for both M and R. By Remark 4.21 $\text{G-dim}_R M/xM = \text{G-dim } M + 1$. Moreover, depth $M/xM =$ depth $M - 1$ (Proposition 3.6). Thus M/xM is a new finitely presented (non-zero) R-module with finite Gorenstein dimension but of smaller depth than M. Continuing in this manner a finitely presented module with finite Gorenstein dimension may be found which has zero depth. This contradicts what was proven above. Therefore depth $M = \infty$.

The above proof shows that the codimension identity also holds when depth $R = \infty$.

4. GORENSTEIN DIMENSION AND ABSOLUTE PURITY

DEFINITION 4.23 For a ring R, an R-module X is called "absolutely pure" or "FP-injective" if it is a pure submodule of every module which contains it. This condition is equivalent to X being a pure submodule of its injective hull.

DEFINITION 4.24 A module X is said to have FP-injective dimension less than or equal to n if there exists a resolution

$$0 \to X \to A_o \to A_1 \to \dots \to A_{n-1} \to A_n \to 0$$

in which all A_i ($0 \leq i \leq n$) are absolutely pure. The least such integer is called the FP-injective dimension of X and is denoted by "FP-inj. dim X". This could also be called absolutely pure dimension.

The reason the term "FP-injective" is used is that for a coherent ring the above dimension may be checked by considering all finitely presented R-modules in the sense indicated by the following lemma.

<u>LEMMA 4.25</u> Let R be a coherent ring. For an R-module X the following statements are equivalent:

(a) FP-inj. dim $X \leq n$

(b) $\mathrm{Ext}_R^{n+1}(M, X) = 0$ for all finitely presented R-modules M.

This lemma and many more results on absolute purity and FP-injective dimension may be found in [32].

The following proposition extends Proposition 3.5 of [32] and gives a characterization of the FP-injective dimension of the ring itself in terms of the Gorenstein dimension of its finitely presented modules.

<u>PROPOSITION 4.26</u> Let R be a coherent ring. Then,

$$\text{FP-inj. dim } R = \sup\{\text{G-dim } M \mid M \text{ a finitely presented R-module}\}.$$

<u>PROOF</u> Let M be an arbitrary finitely presented R-module. It will be shown that G-dim M $\leq$ FP-inj. dim R. If FP-inj.dim R = ∞ this is clear. Suppose FP-inj. dim R = $n < \infty$. By Theorem 4.11 it will be enough to show G-dim $\Omega^n M = 0$. Now $0 = \mathrm{Ext}_R^{n+1}(M, R) \cong \mathrm{Ext}_R^{i}(\Omega^n M, R)$ for all $i > 0$.

Consider an exact sequence as follows:

$$P_{n+1} \to P_n \to \dots \to P_1 \to P_0 \to \Omega^n M \to 0$$

(P_i $(0 \leq i \leq n+1)$ finitely generated projective)

$\mathrm{Ext}^i_R(\Omega^n M, R) = 0$ for all $i > 0$ implies the sequence below is exact.

$$0 \to (\Omega^n M)^* \to P^*_o \to P^*_1 \to \dots \to P^*_n \to P^*_{n+1}$$

Let $C = \mathrm{coker}\ (P^*_n \to P_{n+1}{}^*)$. Then,

$$0 \to D(\Omega^n M) \to P^*_2 \to \dots \to P^*_n \to P_{n+1}{}^* \to C \to 0 \quad \text{is exact.}$$

Thus, $D(\Omega^n M)$ is projectively equivalent to $\Omega^n C$ and this means $\mathrm{Ext}^i_R(D(\Omega^n M), R) \simeq \mathrm{Ext}^i_R(\Omega^n C, R) \simeq \mathrm{Ext}^{n+i}_R(C, R) = 0$ for all $i > 0$ since FP-inj. dim $R = n$. $\mathrm{Ext}^i_R(\Omega^n M, R) = 0$ and $\mathrm{Ext}^i_R(D(\Omega^n M), R) = 0$ for all $i > 0$ imply that G-dim $\Omega^n M = 0$ by Proposition 4.9. The above considerations show that

$$\sup\{\text{G-dim } M \mid M \text{ finitely presented}\} \leq \text{FP-inj. dim } R.$$

Now if FP-inj. dim $R = 0$, this shows that the supremum is zero. If FP-inj. dim $R = n > 0$ there exists a finitely presented module M_n for which $\mathrm{Ext}^n_R(M_n, R) \neq 0$. But G-dim $M_n \leq n$ and therefore, by Remark 4.8, G-dim $M_n = n$. Thus $n = \sup\{\text{G-dim } M \mid M \text{ finitely presented}\}$. If FP-inj. dim $R = \infty$, there exists a finitely presented R-module M_m for each natural number m such that $\mathrm{Ext}^{m+1}_R(M_m, R) \neq 0$. Hence G-dim $M_m \geq m + 1$. This shows $\infty = \sup\{\text{G-dim } M \mid M \text{ finitely presented}\}$ and the proof of Proposition 4.26 is complete.

Notice that if R is a local pseudo-Noetherian ring, f. dim R = sup{G-dim M|M finitely presented and G-dim M < ∞}. In fact f. dim R does not exceed the supremum because p. dim M = G-dim M if M is finitely presented and p. dim M < ∞. On the other hand, if f. dim R = n < ∞ and M is a finitely presented module with finite Gorenstein dimension, G-dim M $\leq$ f. dim R by Theorem 4.22 and Theorem 3.7.

For a ring R and an R-module X, the character module $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is also an R-module. If R is Noetherian it is known that w. dim $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ = inj. dim X ([10]). For a coherent ring, a similar result holds.

<u>LEMMA 4.27</u> Let R be a coherent ring and X an R-module. Then,

$$\text{w. dim } (\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) = \text{FP-inj. dim } X$$

<u>PROOF</u> FP-inj. dim X $\leq$ n if and only if $\mathrm{Ext}_R^{n+1}(M, X) = 0$ for all finitely presented modules M by Lemma 4.25. This condition is equivalent to: $\mathrm{Tor}^R_{n+1}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}), M) = 0$ for all finitely presented modules M (by the duality on p. 120 of [7]). But this in turn is equivalent to: w. dim $(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \leq n$ (Tor commutes with directed colimits and every module is a directed colimit of finitely presented modules). Therefore FP-inj. dim X $\leq$ n if and only if w. dim $(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})) \leq n$ and the proof of Lemma 4.27 is complete.

The following result now becomes a corollary to Proposition 4.26.

PROPOSITION 4.28 Let R be a coherent ring. The following numbers are identical.

(a) FP-inj. dim R.

(b) sup{G-dim M|M a finitely presented R-module}.

(c) w. dim $(\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$.

The following is also true for a coherent ring R ([32], Theorem 3.3).

$$\text{w. gl. dim } R = \sup\{\text{FP-inj. dim } X \mid X \text{ an } R\text{-module}\}.$$

(The right hand side of this equation might be called the global FP-injective dimension of the ring.) Thus, it is clear that for a coherent ring R, FP-inj. dim R $\leq$ w. gl. dim R. Furthermore, if M is a finitely presented R-module with finite projective dimension, p.dim M = G-dim M $\leq$ FP-inj. dim R (Proposition 4.26). Therefore, for a coherent ring R,

$$\text{f. dim } R \leq \text{FP-inj. dim } R \leq \text{w. gl. dim } R.$$

A Noetherian local ring is called a Gorenstein ring if inj. dim R $< \infty$ and has this property if and only if every finitely generated R-module has finite Gorenstein dimension. ([2], Theorem 4.20). One might ask if the analogous two conditions (in terms of FP-inj. dim R, and finitely presented R-modules) are equivalent in the case of a coherent local ring. The following example illustrates that they are not.

Again let $R = K[[(x_n)_{n \in \mathbb{N}}]]$ be the power series ring in a countably infinite number of indeterminates over a field K, with the extra stipulation that only a finite number of indeterminates occur in the expansion of any given element of R. It has been noted previously (Chapter II, Example 2) that R is a faithfully flat directed union of regular Noetherian local rings. Thus, R is a local coherent regular ring (Proposition 1.36). Moreover w. gl. dim R = ∞ . Because R is regular, all finitely presented R-modules have finite projective dimension and f. dim R = w. gl. dim R. Hence, by Corollary 4.12 all finitely presented R-modules have finite Gorenstein dimension and by the remarks following Proposition 4.28, f. dim R = FP-inj. dim R = w. gl. dim R = ∞ . Thus, R is a coherent local ring with infinite FP-injective dimension but with the property that all finitely presented R-modules have finite Gorenstein dimension.

The following example due to B.L. Osofsky describes a coherent (non-Noetherian) local pseudo-Noetherian ring which possesses finitely presented modules with Gorenstein dimension zero but weak dimension infinity.

<u>EXAMPLE</u> (cf. ([26], p. 378, Example 1)) Let $\mathbb{Z}_{(p)}$ denote the discrete valuation domain consisting of the p-adic integers for some prime p. Let $R = \mathbb{Z}_{(p)} + \mathbb{Z}_{p^\infty}$ represent the semi-direct sum of $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{p^\infty}$ where $\mathbb{Z}_{(p)} + \mathbb{Z}_{p^\infty}$ is an additive direct sum and the rule of multiplication is given by $(\lambda, x)(\mu, y) = (\lambda\mu, \lambda y + \mu x)$ for (λ, x) and (μ, y) in $\mathbb{Z}_{(p)} \times \mathbb{Z}_{p^\infty}$.

In [26], Osofsky shows that any proper ideal of R is either of the form $(p^i,0)R$ for some $i > 0$ or may be identified with an additive subgroup of $\mathbb{Z}_{p^\infty}$. This shows that R is a non-Noetherian valuation ring. (The ideal lattice of $\mathbb{Z}_{(p)}$ is a chain and the subgroups of $\mathbb{Z}_{p^\infty}$ also form a chain.) The maximal ideal of R is $(p,0)R$, the socle of R (the only simple module of R) is identified with the subgroup of $\mathbb{Z}_{p^\infty}$ of order p, and the only non-cyclic ideal of R is $(0,\mathbb{Z}_{p^\infty})$. It is not difficult to see that R is coherent. Because the ideals of R form a chain, the intersection of any two finitely generated ideals is finitely generated. The annihilator of $(p^i,0)R$, $(i>0)$ is the cyclic ideal associated with the subgroup of $\mathbb{Z}_{p^\infty}$ of order p^i and vice-versa. Thus the annihilators of cyclic ideals are cyclic and R is coherent. It is furthermore proven in [26] that R is self-injective. Therefore FP-inj. dim R = 0 and by Proposition 4.26, the Gorenstein dimension of any finitely presented R-module is zero. Moreover w. gl. dim R = ∞ because otherwise w. gl. dim R = f. dim R $\leq$ FP-inj. dim R = 0 by the remarks following Proposition 4.28 and since R is local, this would mean R is a field. Thus, there exist finitely presented modules with infinite weak dimension and zero Gorenstein dimension.

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