INFERENTIAL METHODS FOR EXTREME VALUE REGRESSION MODELS

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Abstract

In this thesis, we consider different inferential methods for the multi-group extreme value regression model and evaluate their relative merits.

First, we derive expressions of estimators of the parameters for the multi-group extreme value regression model using the following methods: (i) best linear unbiased estimation (BLUE), (ii) maximum likelihood estimation (MLE), (iii) approximate maximum likelihood estimation (AMLE), and (iv) large-sample approximation to the best linear unbiased estimation. These derivations are presented for complete samples, progressively Type-II right-censored samples, and its special case – Type-II right-censored samples. Explicit expressions of the estimators' bias (for AMLE), asymptotic (or approximate) variances and covariances are derived as well, for all the methods mentioned above. A proof of the asymptotic normality of the BLUE's of the parameters for the multi-group extreme value regression model is presented. We then compare all these estimation methods for various choices of sample sizes and censoring schemes through a Monte Carlo simulation study.

We also study the confidence interval estimation of these parameters through pivotal quantities and simulate the probability coverages of confidence intervals based on all the methods for various choices of sample sizes and censoring schemes. A comparison of these probability coverages is made as well, and some conclusions are drawn.

We illustrate all these inferential methods through three real-life examples

discussed earlier by Lawless (1982).

Finally, in order to test the validity of the assumption of the extreme value regression model, we extend Tiku and Singh's (1981) method to the multi-group extreme value regression model. We determine the level of significance as well as the power under different alternatives for various choices of sample sizes and censoring schemes through Monte Carlo simulations.

Glossary

MEVR	–	Muliti-group extreme value regression
BLUE	–	Best linear unbiased estimation
MLE	–	Maximum likelihood estimation
AMLE	–	Approximate maximum likelihood estimation
App. BLUE	–	Large-sample approximation to the best linear unbiased estimation
MSE	–	Mean square error
pdf	–	Probability density function
cdf	–	Distribution function
$\Gamma(a)$	–	Complete gamma Function
$B(a, b)$	–	Complete beta Function

Table of Notations

T	–	Lifetime variable
t	–	Realization of T
$f(x)$	–	Probability density function
$F(x)$	–	Distribution function
$F^{-1}(x)$	–	Inverse cumulative probability density function
Y	–	Log (T)
y	–	Log (t)
ν_0, ν_1 and σ	–	Regression parameters
θ	–	Column vector of regression parameters $\begin{pmatrix} \nu_0 \\ \nu_1 \\ \sigma \end{pmatrix}$
X	–	A column vector of covariates
x	–	A column vector of different values of a single covariate
x_l	–	Value of x at l-th level, $l = 1, \ldots, k$
$y_{i:n_l}$	–	i-th ordered observation from a sample of size n_l
$z_{i:n_l}$	–	i-th order statistic in a sample of size n_l from standard extreme value distribution

$\alpha_{i:n_l}$	–	Mean of $z_{i:n_l}$, $E(z_{i:n_l})$
$\beta_{i,i:n_l}$	–	Variance of $z_{i:n_l}$, $\text{var}(z_{i:n_l})$
$\beta_{i,j:n_l}$	–	Covariance of $z_{i:n_l}$ and $z_{j:n_l}$, $\text{cov}(z_{i:n_l}, z_{j:n_l})$
Σ	–	Variance-covariance matrix
Σ'	–	Transpose of Σ
Σ^{-1}	–	Inverse of Σ
Σ_{n_l}	–	Variance-covariance matrix of $z_{i:n_l}$, $i = 1, \ldots n_l$
$y_{i:m_l:n_l}$	–	i-th ordered observation in a progressively Type-II right-censored sample of size m_l from n_l units
$z_{i:m_l:n_l}$	–	i-th order statistic in a progressively Type-II right-censored sample of size m_l from n_l units in standard extreme value distribution
$\Sigma_{m_l:n_l}$	–	Variance-covariance matrix of $z_{i:m_l:n_l}$, $i = 1, \ldots m_l$
L_n	–	L – estimator (a linear function of order statistics of sample size n from a distribution)
$J(u)$	–	Weight function of the L - estimator

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CHAPTER 1

INTRODUCTION

In clinical studies or reliability analysis, one is often interested in obtaining inference on the survival of a patient or the reliability of an equipment at a specified time t_0. In these situations, it is quite common that one or more factors may affect this life-time (which are the covariates) and it is therefore, necessary to use a regression model in order to incorporate these covariates in the statistical analysis.

1.1 The Model

In situations wherein we have enough information to fit an appropriate parametric model for the survival time *T*, a distribution that is found to be useful in many applications is the Weibull distribution (Lawless, 1982; Johnson, Kotz and Balakrishnan, 1994) with density

$$f(t;\alpha,\delta) = \frac{\delta}{\alpha}\left(\frac{t}{\alpha}\right)^{\delta-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\delta}\right], \qquad t \geq 0, \qquad (1.1.1)$$

where $\delta > 0$ and $\alpha > 0$ are the shape and scale parameters, respectively.

Assuming that the vector of covariates $X = (x_1, x_2, \ldots, x_p)'$ affects only the scale parameter α, we have a proportional hazards model for the survival time *T* (Kalbfleisch and Prentice, 1980). Then, considering a logarithmic transformation on the survival time *T*, we have an extreme value distribution for $Y = \log(T)$ with density function

$$f(y \mid \mu(x), \sigma) = \frac{1}{\sigma}\exp[\frac{y-\mu(x)}{\sigma} - \exp(\frac{y-\mu(x)}{\sigma})], \qquad (1.1.2)$$

where $-\infty < y < \infty$, $\mu(x) = \log\alpha(x)$, and $\sigma = 1/\delta$.

From Eq. (1.1.2), we can write y in the location-scale model as

$$y = \mu(x) + \sigma z, \qquad (1.1.3)$$

where the random variable z has a standard extreme value distribution with density function $\exp\{z - e^z\}$, $-\infty < z < \infty$. A simple and useful form for $\mu(x)$ (known as power-rule model) in Eq. (1.1.3) is given by the choice

$$\mu(X) = X'\nu, \qquad (1.1.4)$$

where $X = (x_1, x_2, \ldots, x_p)'$ is a vector of p covariates and $\nu = (\nu_1, \nu_2, \ldots, \nu_p)'$ is a vector of regression parameters.

In this study, we develop statistical inference for the special case of one covariate, i.e. $\mu(x_1) = \nu_0 + \nu_1 x_1$, and assume that independent random samples are taken at each level of x_1. For example, in life-testing experiments, each level of x_1 may correspond to one type of treatment and several patients may be enrolled in each treatment; similarly, in reliability studies, each level of x_1 may correspond to a stress (voltage, load, temperature, etc.) level and several units may be tested under that specific stress level. We call this "data with several observations at each level of x_1" extreme value regression model as **Multi-group Extreme Value Regression** model **(MEVR)**.

For the purpose of simplicity and without loss of generality, we will use x to denote the single covariate, and x_1, x_2, …, x_k to denote the different levels of x throughout this thesis.

1.2 Background and Related Work

Life-time (or failure time) data can be analyzed in a variety of ways. The method of analysis will depend on the assumptions that can reasonably be made. Broadly speaking, there are three approaches to statistical analysis of life-time (or failure time) data – parametric, non-parametric and semi-parametric. The parametric regression models (also known as accelerated failure time models) for life-time (or failure time) data involve two kinds of assumptions: the assumption about the underlying distributional form and the assumption about the form of regression (model form, or form of link function). The proportional hazards regression models (semi-parametric) require the proportionality assumption as well as assumption on the form of regression. The non-parametric (distribution-free) models assume less about the underlying distributions than do the parametric methods.

Most of the methods and techniques associated with parametric regression models assume that the distribution of life-time (or failure time) is an exponential distribution (Cox, 1964; Feigl and Zelen, 1965; Zippin and Armitage, 1966; Glasser, 1967; Cox and Snell, 1968; Sprott and Kalbfleisch, 1969; Prentice, 1973; and Breslow, 1974). The hazard function (exponential failure rate) has been taken to be a linear, reciprocal linear or an exponential function of the covariate. These techniques, however, apply only to situations where one is testing all equipments under the same fixed environmental conditions. When the data are taken over a range of different environmental conditions, the Weibull (Extreme Value) model may be used as it is more flexible and it can be extended to include covariates in different ways.

There are two main approaches to the analysis of the Weibull (Extreme Value) regression model. The first is a least-squares fit and normal theory analysis of variance procedures as applied to the logarithm of the life-time (or failure time). The other approach is based on large-sample maximum likelihood theory.

Nelson and Hahn (1972) suggest a method of simple (but not minimum variance) linear unbiased estimation of the parameters of a linear regression model with censored data on the dependent variables for the special case of one independent variable. This simple method involves obtaining the best linear unbiased estimates of the location and scale parameters of the distribution at each test condition using existing tables of these estimates, and then using these estimates to fit to the data the regression relationship between the independent variable and the location parameter. A weighted regression analysis is required since the variances of the estimates of the location parameters at different test conditions will vary in general due to different sample sizes and different amounts of censoring at each test condition. These authors also provide the best linear unbiased estimation method in the special case when the sample size at each test condition is the same and only the first order statistic is observed.

Prentice and Shillington (1975) present a simple modification to least-squares method for uncensored Weibull data with the aim of producing a computationally simple method for selecting important covariates. The main drawbacks of this method are that it is rather inefficient and also does not apply to censored data.

To improve the efficiency of the least-squares method, Williams (1978) suggests to correct the original survival times for shape and covariate values using estimates from the regression analysis and then handle as if it were a mixed random sample of negative-exponential variables. The method is for the multi-group extreme value regression model, but cannot be used with censored data.

The method of maximum likelihood estimation can be used to get an efficient solution. Pike (1966) analyzed the continuous-carcinogenesis experiments by fitting appropriate Weibull distributions using maximum likelihood estimation method. Peto and Lee (1973) give details of how regression-type arguments can be used in a multi-group experiment to find simple relations between treatment applied to each group and the value of the third (treatment dependent) Weibull coefficient for that group.

Elperin and Gertsbakh (1987) presented results of a Monte Carlo study on the performance of maximum likelihood estimator of the scale parameter σ in the multi-group extreme value regression model with two explanatory variables. Based on large-sample normal approximation via the observed information matrix and the Type-I censoring to an average amount of 0-30%, the estimate of the scale parameter was found to be significantly negatively biased in case of small sample sizes. This resulted in a poor quality of confidence interval for σ and low-level quantiles. It was also shown that a moderate amount of censoring improved the quality of point as well as interval estimation.

Bugaighis (1990) examined the properties of MLE's through simulated biases and mean square errors for the parameters of a multi-group extreme value regression model under Type I censoring. He examined the effects of 1) number of levels of the regressor variable x, 2) censorship time, and 3) sample size.

In the context of the confidence interval estimation and parameter testing procedures for the Weibull (Extreme Value) regression model, McCool (1980) presented the interval and median unbiased point estimators for the shape parameter, stress-life exponent, and a specified percentile at any stress in terms of percentage points of the sampling distributions of the pivotal functions involving the MLE in the multi-group extreme value regression model associated with the power-rule model. A numerical example was also given.

Bugaighis (1993) examined percentiles of three pivotal ratios, associated with the power-rule model, for the parameters of an extreme value regression model based on MLE's. The simulation study revealed that the distributions of these pivotal ratios are closely related to appropriate t and χ^2 distributions.

To examine statistical inference based on the approximate normality of the MLE's, Paula and Rojas (1997) derived the asymptotic normality of the MLE's in the multi-group extreme value regression model associated with the power-rule. They presented the asymptotic null distribution of three asymptotically equivalent statistics for two situations of general one-sided hypotheses, namely, for testing the hypotheses of simple order or simple tree order, for location parameters and scale parameters in g populations and for the intercepts in parallel regression lines.

Vander Wiel and Meeker (1990) compared the performance of confidence intervals between the normal-theory of MLE's and the likelihood ratio based on small-sized censored Weibull regression data. The simulation showed that the likelihood-ratio based confidence intervals have much more symmetric error rates, which are not as extremely anti-conservative as normal-theory intervals are.

Achcar and Damasceno (1996) presented a modified form of re-parameterization proposed by Guerrero and Johnson (1982) to improve the accuracy of the inferences based on the asymptotic normality of MLE's, especially in case of small or moderate sample sizes.

With randomly censored data, Abdelhafez and Thomas (1991) suggested using the bootstrap algorithm of Efron (1979) to construct confidence bands. The validity of the confidence bands in the complete sample case was investigated by these authors.

In addition to the linear estimation methods and MLE's, Nelson (1972) presented graphical methods for analyzing accelerated life test data with the inverse power law model based on multi-group extreme value regression model. The graphical methods are satisfactory for many practical purposes and provide certain information that the analytic methods do not. However, graphical methods do not provide an objective assessment of the accuracy of the information obtained with them, whereas analytic methods do by means of standard errors and confidence intervals.

As we can see from the review above on the multi-group extreme value regression model based on Type-II censored samples, little has been done on the following:

1) The performance of BLUE and MLE,
2) The accuracy of the approximate normality of BLUE and MLE, based on which asymptotic inference may be developed,

3) Tests for the validity of the model against departures from the original assumption of Weibull distribution for the life-times.

1.3 Scope of the Thesis

In Chapter 2, the best linear unbiased estimation of the regression (or location) parameters ν_0, ν_1 and scale parameter σ based on the MEVR model is discussed. In Section 2.2, we first present the basic formulation of the BLUEs of ν_0, ν_1 and σ for the complete sample case, and then derive explicit expressions for the exact variances and covariances of these estimators. The extension of these methods to the case of Type-II right-censored or progressively Type-II right-censored samples are presented as well. Then in Section 2.3, we derive the asymptotic normality of the BLUEs of ν_0, ν_1 and σ, which enable us to obtain statistical inferences for these parameters, such as confidence intervals, hypotheses testing procedures, etc. Finally, in Section 2.4, we conduct a simulation study to evaluate the performance of the BLUEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes. The discussions based on results are presented as well.

In Chapter 3, the maximum likelihood estimation of the regression parameters ν_0, ν_1, and scale parameter σ based on the MEVR model is discussed. In Section 3.2, we present the likelihood equations for the parameters ν_0, ν_1 and σ based on Type-II censored samples as well as expressions for approximate variances and covariances of these estimators. We then derive the asymptotic variances and covariances of these estimators through the expected Fisher information matrix in Section 3.3. Finally, we conduct a simulation study to evaluate the performance of the MLEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes in Section 3.4. The discussions based on results are presented as well.

In Chapter 4, an approximation to the maximum likelihood estimators, which are in closed form, is developed. These estimators can be used as initial guess for the Newton-Raphson procedure to obtain the MLEs discussed in Section 3.2. In Section 4.2, we derive the AMLEs of ν_0, ν_1 and σ for Type-II censored samples as well as describe the procedure to obtain their approximate variances and covariances based on the observed Fisher information matrix. We then derive the asymptotic variances and covariances of these estimators through the expected Fisher information matrix in Section 4.3. In Section 4.4, we derive explicit expressions for the approximate biases of the AMLEs of ν_0, ν_1 and σ. Finally, we conduct a simulation study to evaluate the performance of the AMLEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes in Section 4.5. The discussions based on results are presented as well.

In Chapter 5, the construction of confidence intervals based on the estimators of ν_0, ν_1 and σ (BLUE, MLE and AMLE) is discussed. In constructing the confidence intervals of the regression (or location) and scale parameters, the pivotal quantities based on equivariant estimators play an important role. Therefore, in Sections 5.2 and 5.3, the definition of equivariant estimators, pivotal quantities and a related theorem are presented. Since all the estimators we discussed before are equivariant estimators and approximately normally distributed, we show in Section 5.4 that confidence intervals can be easily constructed through these pivotal quantities. Moreover, we use probability coverages in this section to examine the accuracy of these interval estimation procedures. Finally, we conduct a simulation study to evaluate the performance of the probability coverages of the pivotal quantities based on all these estimators for various choices of sample sizes and censoring schemes in Section 5.5.

In Chapter 6, we first assess the effects of the following five factors on the performance of BLUE, MLE and AMLE of ν_0, ν_1 and σ in Sections 6.2, 6.3 and 6.4, respectively:

1. The number of levels of the regressor variable x,
2. The balanced (equal sized) group sample vs. unbalanced (unequal sized) group sample,
3. The total sample size N,
4. The complete sample vs. Type-II right-censored sample,
5. The degree of censoring.

The assessments are based on estimators' bias, mean square error, variances and probability coverages. We then make comparisons between BLUE, MLE and AMLE based on relative efficiency of the estimators and the accuracy of the normal approximation in terms of probability coverages of intervals based on these estimators in Section 6.5.

In Chapter 7, in order to check the adequacy of models upon which inferences are based, the test of validity of multi-group extreme value regression model is presented. In

Section 7.2, we introduce Tiku's test, which provide a test for an extreme value model for a single group sample. We then extend this test to the multi-group sample situation in Section 7.3. To assess the validity of the assumption of the extreme value regression model and to test for departures from the original assumption of Weibull distribution for life-times, we describe the method of determining the level of significance and the power in Section 7.4. In Section 7.5, we simulate the values of levels of significance under the standard extreme value model, and the values of power under five distributional alternatives for various choices of sample sizes and censoring schemes. Finally, we discuss the simulation results in Section 7.6.

In Chapter 8, we illustrate the BLUE, MLE and AMLE approaches using three real-life examples for both complete as well as the Type-II right-censored samples. We present a detailed illustration of these approaches for the complete sample case in Example 8.2.1. Then we present the analysis and the results for Type-II right-censored samples in Example 8.2.2, and for both complete and Type-II right-censored samples in Example 8.2.3.

In Chapter 9, a large-sample approximation to BLUEs is proposed. As we know, in order to obtain the BLUEs of ν_0, ν_1 and σ in the MEVR model, it is necessary to have means, variances, and covariances of order statistics from the standard extreme value distribution. For large sample sizes (say, $n \geq 30$ or so), the variances and covariances are not readily available for most distributions, including extreme value [see Balakrishnan and Chan (1992a, b)]. Adding to this problem, one also needs to invert a large variance-covariance matrix to derive the BLUE's. Therefore, large-sample approximation to BLUEs is proposed. In Section 9.2, we derive the first-order and second-order approximations for the variance-covariance matrix of order statistics from the standard extreme value distribution using David and Johnson's (1954) approximation. Then, in Section 9.3, we derive an explicit form for the inverse of the variance-covariance matrix of order statistics from the standard extreme value distribution. In order to assess the performance of this first-order approximation and second-order approximation methods as compared to the exact method, we conduct a simulation study and discuss the results in Section 9.4. Finally, in Section 9.5, we illustrate the approximation methods through three real-life examples considered earlier in Chapter 8.

In Chapter 10, we generalize four types of estimation procedures —BLUE, approximate BLUE, MLE, and AMLE — to progressively Type-II right-censored samples for the MEVR model. In Section 10.2, we derive all four types of estimators for ν_0, ν_1 and σ for the MEVR model. We conduct a simulation study in Section 10.3 based on progressively Type-II right-censored two- and four-grouped samples with n = 10, m = 5, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 2$ to evaluate these four types of estimation procedures. A discussion of the simulation results is presented as well. Finally, in Section 10.4, we illustrate these methods of estimation by using a progressively Type-II right-censored sample generated from Example 8.2.1 considered earlier in Chapter 8.

Finally, in Chapter 11, we outline the contributions in this thesis and give suggestions for further research.

1.4 Some Basic Concepts

In this thesis, we are mainly concerned with statistical inference for the multi-group extreme value regression model. In the following subsections, we first describe some basic statistical concepts, which will be used throughout this thesis.

1.4.1 Censoring

Data are defined as singly censored if the values of observations on one of the tails of the distribution are not known, and are doubly censored if the values of observations on both tails are not observed. Life test data are frequently singly censored on the right; that is, the failure times of unfailed units are known only to be beyond their current running times. This would be the case, for example, in a life test if all units are placed on test at the same time and all unfailed units have, as a result, accumulated the same running time at the time of analysis. Instrumentation data may be doubly censored; that is, observations may be beyond the scale of measurement at either tail of the distribution.

Censored data are defined to have Type-I censoring if censored observations occur only at specified values of the time. Such censoring results, for example, in life testing when all units are put on test at the same time and the data are collected and analyzed at a specified point in time. For life data, this is called "time censoring". In this type of censoring, the censoring values are fixed and the number of censored observations is random.

Censored data are defined to have Type-II censoring if the number of censored observations is specified and their censored values are random. Such censoring results, for example, in life testing when all units are put on test at the same time and the testing is terminated when a specified number of units have failed. For life data, this is called "failure censoring".

Data are defined to be progressively censored if the censored values and uncensored values are intermixed. Assume the following general censoring scheme: censoring times, $T_1, \cdots, T_{m-1}$, are fixed such that at these times, $R_1, \cdots, R_{m-1}$ surviving units are randomly removed (censored) from the test, respectively. The experiment terminates at time T_m with R_m being the number of surviving units at that time. This is called as "progressive Type-I right-censoring". Assume the following general censoring scheme: m censoring times are fixed and n units are placed on test at time zero. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the second observed failure, R_2 surviving units are randomly removed from the test at random. This process continues until, at the time of the m-th observed failure, the remaining $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ units are all removed from the experiment. This is called as progressive "Type-II right-censoring".

1.4.2 Order Statistics

Suppose that $X_1, \cdots, X_n$ are n independent and identically distributed random variables. The corresponding order statistics are the X_i's arranged in non-decreasing order. The smallest of the X_i's is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$, …, and finally the largest is denoted by $X_{n:n}$, Thus $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, and $X_{i:n}$ is called the "i-th order statistic", $i = 1, 2, \ldots, n$.

Order statistics and functions of order statistics play a very important role in statistical theory and methodology and, as a result, order statistics and their moments have received a great deal of attention during the past 75 years or so. Order statistics may, in some situations like life-testing experiments that we described earlier, arise in a natural way. In some other situations, sample observations may be deliberately ordered and analysis may then based on order statistics due to considerations of robustness. Many robust estimation procedures based on censored samples have been developed by using the theory of order statistics; see, for example, Andrews et al. (1972), David (1981), and Tiku, Tan and Balakrishnan (1986).

In addition to statistical analysis based on censored data and robust inference, there are a number of other areas where order statistics have found important applications, such as outlier detection, reliability studies, quality control, ranking and selection methodology, goodness-of-fit techniques, and characterization problems. The eight-volume bibliography by Harter (1983 – 1993), the books by David (1981), Arnold, Balakrishnan and Nagaraja (1992) and Castillo (1988), and the two-volumes by Balakrishnan and Rao (1998a, b) will illustrate many of these applications quite well.

CHAPTER 2

BEST LINEAR UNBIASED ESTIMATION (BLUE)

2.1 Introduction

In this chapter, the best linear unbiased estimation of the regression (location) parameters ν_0, ν_1, and scale parameter σ for the MEVR model is discussed. In Section 2.2, we first present the basic formulation of the BLUEs of ν_0, ν_1 and σ for complete sample, and then derive explicit expressions for the exact variances and covariances of these estimators. We extend these methods to the case of Type-II right-censored and progressively Type-II right-censored samples as well in this section. Then in Section 2.3, we prove the asymptotic normality of the BLUEs of ν_0, ν_1 and σ, which will enable us to develop statistical inferences for these parameters, such as confidence intervals, hypotheses testing procedures, etc. Finally, we conduct a simulation study to examine the performance of the BLUEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes in Section 2.4.

2.2 Complete Sample

Suppose that observations $y_{1:n_l} \leq y_{2:n_l} \leq \ldots \leq y_{n_l:n_l}$ denote a complete sample taken on n_l individuals at the single regressor x_l from the l-th group, for $l = 1, \ldots, k$, from the extreme value population with a location parameter $\mu(x)$ and a constant scale parameter σ. Then the model can be written in the form

$$y_{i:n_l} = \mu(x) + \sigma z_{i:n_l} = \nu_0 + \nu_1 x_l + \sigma z_{i:n_l}, \quad i = 1, \ldots, n_l, \quad l = 1, \ldots, k, \quad \sigma > 0,$$

where ν_0 and ν_1 are the regression (or location) parameters, and $z_{i:n_l}$ ($1 \le i \le n_l$) are the order statistics from a sample of size n_l from the standard extreme value distribution with density function $\exp\{z - e^z\}$, $-\infty < z < \infty$.

Given $E(z_{i:n_l}) = \alpha_{i:n_l}$ $(1 \le i \le n_l)$ and covariance $Cov\,(z_{i:n_l}, z_{j:n_l}) = \beta_{i,j:n_l}$ $(1 \le i \le j \le n_l)$, it is easy to note that

$$E(y_{i:n_l}) = \nu_0 + \nu_1 x_l + \sigma \alpha_{i:n_l}, \qquad 1 \le i \le n_l,$$

and

$$Cov\,(y_{i:n_l}, y_{j:n_l}) = \sigma^2 \beta_{i,j:n_l}, \qquad 1 \le i \le j \le n_l.$$

Denote

$$Y = [y_{1:n_1}, \ldots, y_{n_1:n_1}, y_{1:n_2}, \ldots, y_{n_2:n_2}, \ldots, y_{1:n_k}, \ldots, y_{n_k:n_k}]'_{N\times 1}, \tag{2.2.1}$$

$$X = [x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_k, \ldots, x_k]'_{N\times 1}, \tag{2.2.2}$$

$$\alpha = [\alpha_{1:n_1}, \ldots, \alpha_{n_1:n_1}, \alpha_{1:n_2}, \ldots, \alpha_{n_2:n_2}, \ldots, \alpha_{1:n_k}, \ldots, \alpha_{n_k:n_k}]'_{N\times 1}, \tag{2.2.3}$$

$$1 = [1, \ldots, 1]'_{N\times 1}, \tag{2.2.4}$$

$$W = [1 \;\; X \;\; \alpha]_{N\times 3}, \tag{2.2.5}$$

$$\theta = [\nu_0 \;\; \nu_1 \;\; \sigma]'_{3\times 1}, \tag{2.2.6}$$

$$\Sigma_{n_l} = \begin{bmatrix} \beta_{1,1:n_l} & \beta_{1,2:n_l} & \cdots & \beta_{1,n_l:n_l} \\ \beta_{1,2:n_l} & \beta_{2,2:n_l} & \cdots & \beta_{2,n_l:n_l} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,n_l:n_l} & \beta_{2,n_l:n_l} & \cdots & \beta_{n_l,n_l:n_l} \end{bmatrix}_{n_l \times n_l}, \qquad (2.2.7)$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{n_1} & 0 & \ldots & 0 \\ 0 & \Sigma_{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Sigma_{n_k} \end{bmatrix}_{N \times N}, \qquad (2.2.8)$$

where $N = \sum_{i=1}^{k} n_l$. We may then write

$$E(Y) = W\theta$$

and

$$Var\,(Y) = \sigma^2 \Sigma.$$

Thus, the generalized variance is given by

$$S = (Y - W\theta)'\Sigma^{-1}(Y - W\theta) = Y'\Sigma^{-1}Y - 2\theta' W'\Sigma^{-1}Y + \theta' W'\Sigma^{-1}W\theta.$$

By minimizing this expression of the generalized variance with respect to θ and solving the following equation

$$\frac{\partial S}{\partial \theta} = -2W'\Sigma^{-1}Y + 2W'\Sigma^{-1}W\theta = 0,$$

we derive the BLUE of θ to be

$$\theta^* = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}Y, \qquad (2.2.9)$$

and its mean and variance-covariance matrix as

$$E(\theta^*) = (W'\Sigma^{-1}W)^{-1}W'\Sigma^{-1}E(Y) = \theta, \tag{2.2.10}$$

and

$$Cov\,(\theta^*) = \sigma^2(W'\Sigma^{-1}W)^{-1}. \tag{2.2.11}$$

Using the special symbol $\begin{Vmatrix} \alpha & A & \Phi \\ \beta & B & \Lambda \\ \gamma & C & \Psi \end{Vmatrix}$ to denote the $(n \times n)$ matrix given by the expression $\alpha(B\Psi - C\Lambda) - \beta(A\Psi - C\Phi) + \gamma(A\Lambda - B\Phi)$, where α, β and γ are $n \times 1$ vectors, A, B and C are scales and Φ, Λ and Ψ are $1 \times n$ vectors, we may write explicit expressions of the BLUEs $v_0{}^*$, $v_1{}^*$ and σ^* (of v_0, v_1 and σ) as

$$v_0{}^* = X'\Delta_{v_0}Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} a_{i:n_l}\, y_{i:n_l}\,, \tag{2.2.12}$$

$$v_1{}^* = X'\Delta_{v_1}Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} b_{i:n_l}\, y_{i:n_l}\,, \tag{2.2.13}$$

and

$$\sigma^* = X'\Delta_{\sigma}Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} c_{i:n_l}\, y_{i:n_l}\,, \tag{2.2.14}$$

where

$$\Delta_{v_0} = \frac{\delta_{v_0}}{\delta}, \qquad \Delta_{v_1} = \frac{\delta_{v_1}}{\delta}, \qquad \Delta_{\sigma} = \frac{\delta_{\sigma}}{\delta}, \tag{2.2.15}$$

$$\delta = \det\begin{vmatrix} 1'\Sigma^{-1}1 & X'\Sigma^{-1}1 & \alpha'\Sigma^{-1}1 \\ 1'\Sigma^{-1}X & X'\Sigma^{-1}X & \alpha'\Sigma^{-1}X \\ 1'\Sigma^{-1}\alpha & X'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1}\alpha \end{vmatrix}, \tag{2.2.16}$$

$$\delta_{v_0} = \begin{Vmatrix} \Sigma^{-1}1 & \alpha'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & \alpha'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & \alpha'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix}, \tag{2.2.17}$$

$$\delta_{v_1} = \begin{Vmatrix} \Sigma^{-1}1 & 1'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & 1'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & 1'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix} \tag{2.2.18}$$

and

$$\delta_{\sigma} = \begin{Vmatrix} \Sigma^{-1}1 & X'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & X'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & X'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix}. \tag{2.2.19}$$

Furthermore, explicit expressions of the exact variances and covariances of the estimators v_0^*, v_1^* and σ^* are derived from (2.2.11) as

$$Var(v_0^*) = \frac{(X'\Sigma^{-1}X)(\alpha'\Sigma^{-1}\alpha) - (X'\Sigma^{-1}\alpha)^2}{\delta}\sigma^2, \tag{2.2.20}$$

$$Var(v_1^*) = \frac{(1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2}{\delta}\sigma^2, \tag{2.2.21}$$

$$Var(\sigma^*) = \frac{(1'\Sigma^{-1}1)(X'\Sigma^{-1}X) - (1'\Sigma^{-1}X)^2}{\delta}\sigma^2, \tag{2.2.22}$$

$$Cov(v_0^*, v_1^*) = \frac{(X'\Sigma^{-1}\alpha)(\alpha'\Sigma^{-1}1) - (1'\Sigma^{-1}X)(\alpha'\Sigma^{-1}\alpha)}{\delta}\sigma^2, \tag{2.2.23}$$

$$Cov(v_0^*, \sigma^*) = \frac{(1'\Sigma^{-1}X)(X'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)(X'\Sigma^{-1}X)}{\delta}\sigma^2, \tag{2.2.24}$$

and

$$Cov(v_1^*, \sigma^*) = \frac{(X'\Sigma^{-1}1)(\alpha'\Sigma^{-1}1) - (1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}X)}{\delta}\sigma^2. \tag{2.2.25}$$

In the case of Type-II right-censored or progressively Type-II right-censored samples, all the above formulas for the BLUEs are the same, except the means, variances and covariances of the standard extreme value order statistics should be replaced by the corresponding values for the Type-II right-censored or progressively Type-II right-censored order statistics.

2.3 Asymptotic Normality of the BLUEs of ν_0, ν_1 and σ

Before proceeding to the proof of the asymptotic normality of the BLUEs of ν_0, ν_1 and σ, we need to introduce some basic concepts.

2.3.1 *L*-statistic and Its Asymptotic Properties

Suppose $a_{i,n}$'s form a (double) sequence of constants. The statistic

$$L_n = \sum_{i=1}^{n} a_{i,n} X_{i:n}$$

is called a *L*-statistic. When used as an estimator, it is often referred to as an *L*-estimator. The exact distribution of L_n is difficult to obtain in general except when $a_{i,n} = 0$ for all but a few *i*.

When $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denotes the order statistics of a sample from cdf F and $a_{i,n}$ is of the form $J\left(\frac{i}{n+1}\right)\Big/ n$, where $J(u)$, $0 \le u \le 1$, is the associated weight function, L_n can be expressed as

$$L_n = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right)X_{i:n} . \qquad (2.3.1.1)$$

The asymptotic normality of L_n can be established by imposing conditions on the weight function (see Stigler, 1974, Mason, 1981, David, 1981, Section 9.6, and Arnold, Balakrishnan and Nagaraja, 1992, Section 8.6).

Define

$$\mu(J,F) = \int xJ[F(x)]dF(x) \qquad (2.3.1.2)$$

and

$$\sigma^2(J,F) = 2\int\int_{-\infty<x<y<\infty} J(F(x))J(F(y))\{F(x)(1-F(y))\}dxdy . \qquad (2.3.1.3)$$

Then, we have the following theorem (Mason, 1981).

Theorem 2.3.1 *Let the weight function J(u) be bounded and be continuous at every discontinuity point of* $F^{-1}(u)$. *Then* $\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}dF^{-1}(u) < \infty$ *implies:*

$$\sqrt{n}(L_n - \mu(J,F)) \xrightarrow{d} N(0,\sigma^2(J,F)),$$

where L_n, $\mu(J,F)$ *and* $\sigma^2(J,F)$ *are given by* (2.3.1.2) – (2.3.1.3), *respectively.*

2.3.2 Asymptotic Normality of the BLUE of ν_0

Since the derivation procedures for the asymptotic normality of BLUEs of ν_0, ν_1 and σ are all similar, we will present here only the derivation of the asymptotic normality for the BLUE of ν_0^* (of ν_0).

Under the MEVR model, we can express the L-statistic for BLUE v_0^* (of v_0) as

$$_{v_0}L_N = v_0^* = X'\Delta_{v_0} Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} a_{i:n_l} y_{i:n_l} ,$$

where Y, X, and Δ_{v_0} are as defined in (2.2.1), (2.2.2) and (2.2.15), respectively, and $N = \sum_{l=1}^{k} n_l$.

There are three conditions that need to be satisfied which are:

1. The weight function $J(u)$ is bounded.
2. The weight function $J(u)$ is continuous at every discontinuity point of $F^{-1}(u)$.
3. $\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\, dF^{-1}(u) < \infty$.

The proof of the first condition is presented in Appendix. Since there is no discontinuity point in $F^{-1}(u)$ in the case of extreme value distribution, Condition 2 is automatically satisfied. Now, we prove Condition 3 as follows.

Condition 3: $\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\, dF^{-1}(u) < \infty$.

Proof: For the standard extreme value distribution with density function $\exp\{z-e^z\}$, $-\infty < z < \infty$, we have $F^{-1}(u) = \log(-\log(1-u)$ and hence

$$\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\, dF^{-1}(u) = -\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}} \frac{1}{(1-u)\log(1-u)} du .$$

Expanding the functions $-\log(1-u)$ in a power-series as

$$-\log(1-u) = u + \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4} \cdots = u(1 + \frac{u}{2} + \frac{u^2}{3} + \frac{u^3}{4} \cdots) > u ,$$

it can be seen that

$$-\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\frac{1}{(1-u)\log(1-u)}du < \int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\frac{1}{(1-u)u}du = B(\frac{1}{2},\frac{1}{2}) = \pi.$$

Therefore, the condition $\int_0^1 u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}dF^{-1}(u) < \pi < \infty$ is satisfied and the asymptotic normality of ${}_{v_0}L_N$ is established.

It should be mentioned that we have already obtained the mean and variance of the BLUE v_0^* of v_0 from expressions in (2.2.9) and (2.2.11), and hence it is not necessary to derive them again using the formulas in (2.3.1.2) and (2.3.1.2) in Theorem 2.3.1.

Proceeding in a manner similar to the one as we did for BLUE v_0^* of v_0, it is easy to show the asymptotic normality of BLUEs v_1^* and σ^* of v_1 and σ.

2.4 Simulations and Results

In the simulation study, we took $v_0 = 0, v_1 = 1$ and $\sigma = 1$ and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored samples. In order to study the BLUEs in the MEVR model, we performed the simulations based on 10,000 Monte Carlo runs for each of the following cases:

1. Complete samples

two groups: n = [6 6(1)10], [7 7(1)10], [8 8(1)10], [9 9(1)10], [10 10], [15 15(5)20] and [20 20] .

four groups: $n = [6\times2\ 6(1)10\times2]$, $[7\times2\ 7(1)10\times2]$, $[8\times2\ 8(1)10\times2]$, $[9\times2\ 9(1)10\times2]$, $[10\ 10\ 10\ 10]$, $[15\times2\ 15(5)20\times2]$ and $[20\ 20\ 20\ 20]$.

2. Type-II right-censored samples

two groups: s = [4 4(1)0], [3 3(1)0], [2 2(1)0] and [1 1(1)0] from n = [10 10] and [5 5(1)0] from n = [20 20].

four groups: $s = [4\times2\ 4(1)0\times2]$, $[3\times2\ 3(1)0\times2]$, $[2\times2\ 2(1)0\times2]$ and $[1\times2\ 1(1)0\times2]$ from n = [10 10 10 10] and $[5\times2\ 5(1)0\times2]$ from n = [20 20 20 20].

We first generated order statistics from the standard extreme value sample $z_{i:n_l}$, $i = 1,\ldots, n_l$, $l = 1,\ldots, k$, and then using the model $y_{i:n_l} = \nu_o + \nu_1 x_l + \sigma z_{i:n_l}$, transformed the sample into $y_{i:n_l}$, $i = 1,\ldots, n_l$, $l = 1,\ldots, k$. Using the formula in (2.2.9), we obtained the values of BLUEs ν_0^*, ν_1^* and σ^*. Based on 10,000 runs, we determined the values of (1) $MSE(\nu_0^*)/\sigma^2$, (2) $MSE(\nu_1^*)/\sigma^2$, (3) $MSE(\sigma^*)/\sigma^2$, (4) $Var(\nu_0^*)/\sigma^2$, (5) $Var(\nu_1^*)/\sigma^2$, (6) $Var(\sigma^*)/\sigma^2$, (7) $Cov(\nu_0^*,\nu_1^*)/\sigma^2$, (8) $Cov(\nu_0^*,\sigma^*)/\sigma^2$ and (9) $Cov(\nu_1^*,\sigma^*)/\sigma^2$. To make comparison with the simulated results, the exact values of (4) – (9) were computed as well by formula in (2.2.11). These results are presented in Tables 2.4.1 – 2.4.12.

From these tables, we observe the following points. The variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a increase in N. In addition, with the same N, the variances of BLUEs for ν_0 and ν_1 tend to be smaller in value in the case of more balanced groups than among the less balanced groups. Moreover, for the same N, the variance of BLUE of σ tends to be the same in the complete samples and increase with the more balanced groups in the Type-II censored samples. The variances of all the estimators tend to increase with increasing amounts of censoring. This is true for both two- and four-levels of x.

Since the BLUEs for ν_0, ν_1 and σ are all unbiased, the simulated mean square errors are almost identical to the simulated variances. The close agreement between the simulated variances and the exact variances of BLUEs of ν_0, ν_1 and σ should be noted as well.

Table 2.4.1 BLUE procedure: Simulated MSE for two-grouped complete sample

$[n_1\ n_2]$	$MSE(v_0^*)/\sigma^2$	$MSE(v_1^*)/\sigma^2$	$MSE(\sigma^*)/\sigma^2$
[6 6]	0.0919	0.3640	0.0679
[6 7]	0.0896	0.3389	0.0584
[6 8]	0.0828	0.3110	0.0551
[6 9]	0.0800	0.2973	0.0503
[6 10]	0.0753	0.2872	0.0459
[7 7]	0.0803	0.3085	0.0533
[7 8]	0.0760	0.2849	0.0513
[7 9]	0.0716	0.2756	0.0465
[7 10]	0.0701	0.2637	0.0433
[8 8]	0.0711	0.2724	0.0470
[8 9]	0.0673	0.2555	0.0431
[8 10]	0.0629	0.2431	0.0392
[9 9]	0.0639	0.2386	0.0411
[9 10]	0.0598	0.2327	0.0379
[10 10]	0.0543	0.2125	0.0356
[15 15]	0.0376	0.1386	0.0229
[15 20]	0.0324	0.1246	0.0191
[20 20]	0.0284	0.1031	0.0162

Table 2.4.2 BLUE procedure: Simulated variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
[6 6]	0.0919	0.3640	0.0679	-0.0016	-0.0154	-0.0004
[6 7]	0.0896	0.3389	0.0584	-0.0134	-0.0137	-0.0028
[6 8]	0.0828	0.3110	0.0551	-0.0228	-0.0145	-0.0007
[6 9]	0.0800	0.2972	0.0503	-0.0326	-0.0141	-0.0048
[6 10]	0.0753	0.2872	0.0459	-0.0351	-0.0124	-0.0032
[7 7]	0.0803	0.3085	0.0533	0.0015	-0.0137	-0.0000
[7 8]	0.0760	0.2849	0.0513	-0.0108	-0.0139	0.0018
[7 9]	0.0715	0.2756	0.0465	-0.0195	-0.0134	0.0013
[7 10]	0.0701	0.2637	0.0433	-0.0264	-0.0114	-0.0017
[8 8]	0.0711	0.2724	0.0470	0.0004	-0.0133	-0.0003
[8 9]	0.0673	0.2555	0.0431	-0.0082	-0.0125	0.0004
[8 10]	0.0629	0.2431	0.0392	-0.0126	-0.0118	-0.0013
[9 9]	0.0639	0.2386	0.0411	-0.0002	-0.0122	-0.0001
[9 10]	0.0598	0.2326	0.0379	-0.0071	-0.0115	0.0010
[10 10]	0.0543	0.2125	0.0356	-0.0012	-0.0106	0.0008
[15 15]	0.0376	0.1386	0.0229	-0.0001	-0.0079	0.0004
[15 20]	0.0324	0.1246	0.0191	-0.0092	-0.0063	-0.0008
[20 20]	0.0284	0.1031	0.0162	0.0005	-0.0057	0.0003

Table 2.4.3 BLUE procedure: Exact variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
[6 6]	0.0956	0.3674	0.0660	0.0000	-0.0157	0.0000
[6 7]	0.0885	0.3392	0.0597	-0.0138	-0.0149	-0.0015
[6 8]	0.0833	0.3185	0.0545	-0.0239	-0.0141	-0.0023
[6 9]	0.0792	0.3027	0.0501	-0.0317	-0.0134	-0.0029
[6 10]	0.0759	0.2902	0.0464	-0.0379	-0.0126	-0.0032
[7 7]	0.0815	0.3109	0.0545	0.0000	-0.0143	0.0000
[7 8]	0.0762	0.2901	0.0502	-0.0101	-0.0136	-0.0009
[7 9]	0.0722	0.2742	0.0464	-0.0179	-0.0129	-0.0015
[7 10]	0.0689	0.2617	0.0432	-0.0241	-0.0123	-0.0019
[8 8]	0.0710	0.2693	0.0465	0.0000	-0.0130	0.0000
[8 9]	0.0669	0.2534	0.0432	-0.0078	-0.0125	-0.0006
[8 10]	0.0637	0.2409	0.0404	-0.0139	-0.0119	-0.0011
[9 9]	0.0629	0.2375	0.0404	0.0000	-0.0119	0.0000
[9 10]	0.0597	0.2250	0.0380	-0.0061	-0.0114	-0.0004
[10 10]	0.0565	0.2124	0.0358	0.0000	-0.0110	0.0000
[15 15]	0.0374	0.1389	0.0227	0.0000	-0.0078	0.0000
[15 20]	0.0326	0.1211	0.0191	-0.0088	-0.0067	-0.0004
[20 20]	0.0280	0.1032	0.0166	0.0000	-0.0060	0.0000

Table 2.4.4 BLUE procedure: Simulated MSE for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$MSE(v_0^*)/\sigma^2$	$MSE(v_1^*)/\sigma^2$	$MSE(\sigma^*)/\sigma^2$
[6 6 6 6]	0.0473	0.3390	0.0324
[6 6 7 7]	0.0411	0.3070	0.0301
[6 6 8 8]	0.0406	0.2876	0.0268
[6 6 9 9]	0.0382	0.2705	0.0249
[6 6 10 10]	0.0375	0.2660	0.0237
[7 7 7 7]	0.0410	0.2795	0.0275
[7 7 8 8]	0.0385	0.2610	0.0254
[7 7 9 9]	0.0360	0.2501	0.0234
[7 7 10 10]	0.0339	0.2324	0.0218
[8 8 8 8]	0.0356	0.3426	0.0235
[8 8 9 9]	0.0338	0.2209	0.0215
[8 8 10 10]	0.0320	0.2142	0.0203
[9 9 9 9]	0.0309	0.2154	0.0204
[9 9 10 10]	0.0304	0.2079	0.0187
[10 10 10 10]	0.0276	0.1881	0.0177
[15 15 15 15]	0.0184	0.1260	0.0114
[15 15 20 20]	0.0161	0.1112	0.0097
[20 20 20 20]	0.0140	0.0929	0.0082

Table 2.4.5 BLUE procedure: Simulated variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
[6 6 6 6]	0.0473	0.3390	0.0324	-0.0014	-0.0073	-0.0004
[6 6 7 7]	0.0411	0.3070	0.0301	-0.0088	-0.0076	0.0011
[6 6 8 8]	0.0406	0.2876	0.0268	-0.0128	-0.0074	-0.0008
[6 6 9 9]	0.0382	0.2705	0.0249	-0.0180	-0.0067	-0.0012
[6 6 10 10]	0.0375	0.2660	0.0236	-0.0221	-0.0067	-0.0029
[7 7 7 7]	0.0410	0.2795	0.0275	0.0002	-0.0076	-0.0014
[7 7 8 8]	0.0385	0.2611	0.0254	-0.0066	-0.0064	-0.0004
[7 7 9 9]	0.0360	0.2501	0.0234	-0.0099	-0.0064	-0.0012
[7 7 10 10]	0.0339	0.2324	0.0218	-0.0148	-0.0064	-0.0013
[8 8 8 8]	0.0356	0.3426	0.0235	-0.0009	-0.0065	-0.0002
[8 8 9 9]	0.0338	0.2208	0.0215	-0.0043	-0.0061	-0.0014
[8 8 10 10]	0.0320	0.2142	0.0203	-0.0076	-0.0060	-0.0013
[9 9 9 9]	0.0309	0.2154	0.0204	-0.0006	-0.0057	0.0001
[9 9 10 10]	0.0304	0.2080	0.0187	-0.0020	-0.0056	-0.0003
[10 10 10 10]	0.0276	0.1881	0.0177	0.0007	-0.0054	-0.0007
[15 15 15 15]	0.0184	0.1260	0.0114	-0.0002	-0.0042	0.0005
[15 15 20 20]	0.0161	0.1112	0.0097	-0.0058	-0.0032	-0.0003
[20 20 20 20]	0.0140	0.0929	0.0082	0.0002	-0.0030	0.0000

Table 2.4.6 BLUE procedure: Exact variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
[6 6 6 6]	0.0478	0.3333	0.0330	0.0000	-0.0078	0.0000
[6 6 7 7]	0.0442	0.3072	0.0299	-0.0082	-0.0075	-0.0009
[6 6 8 8]	0.0414	0.2874	0.0273	-0.0142	-0.0071	-0.0014
[6 6 9 9]	0.0392	0.2718	0.0251	-0.0188	-0.0067	-0.0017
[6 6 10 10]	0.0374	0.2590	0.0232	-0.0223	-0.0064	-0.0019
[7 7 7 7]	0.0407	0.2820	0.0273	0.0000	-0.0072	-0.0000
[7 7 8 8]	0.0381	0.2629	0.0251	-0.0061	-0.0068	-0.0006
[7 7 9 9]	0.0360	0.2478	0.0232	-0.0107	-0.0065	-0.0009
[7 7 10 10]	0.0342	0.2356	0.0216	-0.0143	-0.0062	0.0216
[8 8 8 8]	0.0355	0.2443	0.0232	0.0000	-0.0065	0.0000
[8 8 9 9]	0.0334	0.2297	0.0216	-0.0046	-0.0062	-0.0004
[8 8 10 10]	0.0318	0.2179	0.0202	-0.0083	-0.0059	-0.0006
[9 9 9 9]	0.0315	0.2155	0.0202	0.0000	-0.0060	0.0000
[9 9 10 10]	0.0298	0.2040	0.0190	0.2040	-0.0003	0.0190
[10 10 10 10]	0.0282	0.1927	0.0179	0.0000	-0.0055	0.0000
[15 15 15 15]	0.0187	0.1260	0.0113	0.0000	-0.0039	0.0000
[15 15 20 20]	0.0163	0.1093	0.0096	-0.0053	-0.0034	-0.0002
[20 20 20 20]	0.0140	0.0936	0.0083	0.0000	-0.0030	0.0000

Table 2.4.7 BLUE procedure: Simulated MSE for two-grouped Type-II censored sample

$[s_1 \quad s_2]$	$MSE(v_0^*)/\sigma^2$	$MSE(v_1^*)/\sigma^2$	$MSE(\sigma^*)/\sigma^2$
4 4	0.1099	0.3581	0.0820
4 3	0.0916	0.3329	0.0738
4 2	0.0822	0.3283	0.0642
4 1	0.0760	0.3249	0.0580
4 0	0.0721	0.3110	0.0505
3 3	0.0773	0.3089	0.0654
3 2	0.0727	0.2839	0.0601
3 1	0.0684	0.2756	0.0541
3 0	0.0653	0.2714	0.0480
2 2	0.0663	0.2681	0.0542
2 1	0.0630	0.2494	0.0487
2 0	0.0612	0.2409	0.0432
1 1	0.0606	0.2360	0.0443
1 0	0.0589	0.2264	0.0390
*5 5	0.0348	0.1374	0.0280
*5 0	0.0302	0.1233	0.0212

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 2.4.8 BLUE procedure: Simulated variances and covariances for two-grouped Type-II censored sample

$[s_1 \ \ s_2]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*, v_1^*)/\sigma^2$	$Cov(v_0^*, \sigma^*)/\sigma^2$	$Cov(v_1^*, \sigma^*)/\sigma^2$
4 4	0.1099	0.3580	0.0820	-0.0024	0.0371	0.0007
4 3	0.0915	0.3329	0.0738	-0.0187	0.0256	-0.0150
4 2	0.0822	0.3283	0.0642	-0.0275	0.0148	-0.0260
4 1	0.0761	0.3249	0.0580	-0.0383	0.0084	-0.0345
4 0	0.0721	0.3110	0.0505	-0.0388	0.0032	-0.0348
3 3	0.0773	0.3089	0.0654	-0.0017	0.0150	0.0015
3 2	0.0727	0.2839	0.0601	-0.0094	0.0092	-0.0121
3 1	0.0684	0.2756	0.0540	-0.0200	0.0021	-0.0196
3 0	0.0653	0.2714	0.0480	-0.0233	-0.0021	-0.0263
2 2	0.0662	0.2682	0.0542	-0.0000	0.0036	-0.0004
2 1	0.0630	0.2494	0.0487	-0.0058	-0.0022	-0.0078
2 0	0.0612	0.2409	0.0432	-0.0128	-0.0049	-0.0164
1 1	0.0605	0.2360	0.0443	-0.0006	-0.0064	-0.0002
1 0	0.0590	0.2265	0.0390	-0.0057	-0.0089	-0.0072
*5 5	0.0348	0.1374	0.0280	-0.0002	0.0025	-0.0001
*5 0	0.0302	0.1233	0.0212	-0.0078	-0.0030	-0.0093

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 2.4.9 BLUE procedure: Exact variances and covariances for two-grouped Type-II censored sample

$[s_1 \; s_2]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*, v_1^*)/\sigma^2$	$Cov(v_0^*, \sigma^*)/\sigma^2$	$Cov(v_1^*, \sigma^*)/\sigma^2$
4 4	0.1072	0.3637	0.0829	0.0000	0.0367	0.0000
4 3	0.0925	0.3391	0.0735	-0.0190	0.0250	-0.0152
4 2	0.0829	0.3258	0.0652	-0.0303	0.0160	-0.0257
4 1	0.0765	0.3190	0.0575	-0.0369	0.0090	-0.0329
4 0	0.0722	0.3162	0.0500	-0.0404	0.0034	-0.0375
3 3	0.0808	0.3082	0.0661	0.0000	0.0156	0.0000
3 2	0.0732	0.2900	0.0593	-0.0117	0.0085	-0.0111
3 1	0.0683	0.2795	0.0529	-0.0189	0.0029	-0.0193
3 0	0.0651	0.2741	0.0464	-0.0231	-0.0016	-0.0252
2 2	0.0670	0.2676	0.0537	0.0000	0.0026	0.0000
2 1	0.0631	0.2536	0.0484	-0.0074	-0.0019	-0.0086
2 0	0.0607	0.2455	0.0430	-0.0118	-0.0055	-0.0153
1 1	0.0599	0.2366	0.0441	0.0000	-0.0057	0.0000
1 0	0.0580	0.2258	0.0395	-0.0045	-0.0086	-0.0070
*5 5	0.0348	0.1382	0.0275	0.0000	0.0024	0.0000
*5 0	0.0306	0.1249	0.0207	-0.0075	-0.0028	-0.0093

Asterisk denotes censoring is from n = *[20 20]*, otherwise is from *[10 10]*

Table 2.4.10 BLUE procedure: Simulated MSE for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$MSE(v_0^*)/\sigma^2$	$MSE(v_1^*)/\sigma^2$	$MSE(\sigma^*)/\sigma^2$
4 4 4 4	0.0544	0.3265	0.0410
4 4 3 3	0.0455	0.3078	0.0360
4 4 2 2	0.0410	0.2900	0.0324
4 4 1 1	0.0380	0.2870	0.0285
4 4 0 0	0.0354	0.2787	0.0247
3 3 3 3	0.0397	0.2733	0.0343
3 3 2 2	0.0353	0.2698	0.0298
3 3 1 1	0.0338	0.2520	0.0263
3 3 0 0	0.0320	0.2462	0.0228
2 2 2 2	0.0334	0.2386	0.0269
2 2 1 1	0.0329	0.2317	0.0236
2 2 0 0	0.0297	0.2158	0.0213
1 1 1 1	0.0299	0.2127	0.0220
1 1 0 0	0.0289	0.2033	0.0200
*5 5 5 5	0.0169	0.1246	0.0138
*5 5 0 0	0.0151	0.1136	0.0103

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 2.4.11 BLUE procedure: Simulated variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
4 4 4 4	0.0544	0.3265	0.0410	0.0016	0.0186	0.0003
4 4 3 3	0.0455	0.3079	0.0360	-0.0118	0.0127	-0.0101
4 4 2 2	0.0410	0.2900	0.0324	-0.0188	0.0081	-0.0157
4 4 1 1	0.0379	0.2871	0.0285	-0.0212	0.0038	-0.0170
4 4 0 0	0.0354	0.2787	0.0247	-0.0241	0.0016	-0.0222
3 3 3 3	-0.0397	0.2732	0.0344	-0.0010	0.0078	-0.0004
3 3 2 2	0.0353	0.2698	0.0298	-0.0057	0.0044	-0.0072
3 3 1 1	0.0338	0.2521	0.0263	-0.0107	0.0011	-0.0116
3 3 0 0	0.0320	0.2462	0.0228	-0.0149	-0.0004	-0.0160
2 2 2 2	0.0334	0.2386	0.0269	-0.0014	0.0016	0.0006
2 2 1 1	0.0329	0.2317	0.0236	-0.0049	-0.0009	-0.0049
2 2 0 0	0.0297	0.2158	0.0213	-0.0065	-0.0027	-0.0099
1 1 1 1	0.0299	0.2127	0.0220	0.0003	-0.0022	-0.0000
1 1 0 0	0.0289	0.2033	0.0200	-0.0020	-0.0046	-0.0041
*5 5 5 5	0.0169	0.1246	0.0138	-0.0003	0.0008	-0.0005
*5 5 0 0	0.0151	0.1136	0.0103	-0.0045	-0.0014	-0.0050

Asterisk denotes censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 2.4.12 BLUE procedure: Exact variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$	$Cov(v_0^*,v_1^*)/\sigma^2$	$Cov(v_0^*,\sigma^*)/\sigma^2$	$Cov(v_1^*,\sigma^*)/\sigma^2$
4 4 4 4	0.0536	0.3299	0.0414	0.0000	0.0184	0.0000
4 4 3 3	0.0461	0.3065	0.0367	-0.0113	0.0124	-0.0091
4 4 2 2	0.0411	0.2920	0.0324	-0.0179	0.0078	-0.0152
4 4 1 1	0.0377	0.2827	0.0284	-0.0216	0.0041	-0.0192
4 4 0 0	0.0355	0.2764	0.0245	-0.0233	0.0011	-0.0216
3 3 3 3	0.0404	0.2796	0.0330	0.0000	0.0078	0.0000
3 3 2 2	0.0365	0.2624	0.0296	-0.0070	0.0042	-0.0066
3 3 1 1	0.0340	0.2512	0.0263	-0.0112	0.0013	-0.0115
3 3 0 0	0.0324	0.2440	0.0230	-0.0136	-0.0011	-0.0148
2 2 2 2	0.0335	0.2428	0.0269	0.0000	0.0013	0.0000
2 2 1 1	0.0315	0.2296	0.0242	-0.0044	-0.0010	-0.0051
2 2 0 0	0.0303	0.2210	0.0214	-0.0070	-0.0029	-0.0091
1 1 1 1	0.0299	0.2146	0.0220	0.0000	-0.0028	0.0000
1 1 0 0	0.0290	0.2045	0.0197	-0.0027	-0.0043	-0.0042
*5 5 5 5	0.0174	0.1254	0.0137	0.0000	0.0012	0.0000
*5 5 0 0	0.0152	0.1120	0.0103	-0.0044	-0.0015	-0.0055

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

CHAPTER 3

MAXIMUM LIKELIHOOD ESTIMATION (MLE)

3.1 Introduction

In this chapter, the maximum likelihood estimation of the regression (location) parameters ν_0, ν_1 and scale parameter σ of the MEVR model is discussed. In Section 3.2, we present the likelihood equations for the parameters ν_0, ν_1 and σ based on Type-II right-censored samples as well as the procedure to obtain approximate variances and covariances of the MLEs from the observed Fisher information matrix. We also derive the asymptotic variances and covariances of these estimators from the expected Fisher information matrix in Section 3.3. Finally, we conduct a simulation study to evaluate the performance of the MLEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes in Section 3.4.

3.2 Type-II Right-censored Sample

Suppose that observations are taken on n_l individuals at a single regressor x_l, for l = $1, \ldots, k$, and we allow the sample to be Type-II right-censored, meaning that only the first $n_l - s_l$ ordered values $y_{1:n_l} \le y_{2:n_l} \le \ldots \le y_{n_l - s_l : n_l}$ out of the total of n_l observations are observed. The corresponding likelihood function for the model in extreme value form is

$$\prod_{l=1}^{k}\frac{n_l!}{s_l!}\left\{\prod_{i=1}^{n_l-s_l}\frac{1}{\sigma}\exp\left[\frac{y_{i:n_l}-v_0-v_1x_l}{\sigma}-\exp\left(\frac{y_{i:n_l}-v_0-v_1x_l}{\sigma}\right)\right]\right\}\left\{\exp\left[-\exp\left(\frac{y_{n_l-s_l:n_l}-v_0-v_1x_l}{\sigma}\right)\right]\right\}^{s_l}$$

Dropping the proportionality constant $\prod n_l!/s_l!$, we can take the log-likelihood function to be

$$\log L(v_0,v_1,\sigma)=-\log\sigma\sum_{l=1}^{k}A_l+\sum_{l=1}^{k}\sum_{i=1}^{n_l-s_l}\left\{\frac{y_{i:n_l}-v_0-v_1x_l}{\sigma}-\exp\left(\frac{y_{i:n_l}-v_0-v_1x_l}{\sigma}\right)\right\}$$
$$-\sum_{l=1}^{k}s_l\left[\exp\left(\frac{y_{i:n_l}-v_0-v_1x_l}{\sigma}\right)\right],$$

where $A_l=n_l-s_l$. Let $z_{i:n_l}=(y_{i:n_l}-v_0-v_1x_l)/\sigma$; then, the partial derivatives of $\log L$ are given by

$$\frac{\partial\log L}{\partial v_0}=-\frac{1}{\sigma}\sum_{l=1}^{k}\left\{-s_l\exp(z_{n_l-s_l:n_l})+\sum_{i=1}^{n_l-s_l}(1-\exp(z_{i:n_l}))\right\},\tag{3.2.1}$$

$$\frac{\partial\log L}{\partial v_1}=-\frac{1}{\sigma}\sum_{l=1}^{k}\left\{X_l\left[-s_l\exp(z_{n_l-s_l:n_l})+\sum_{i=1}^{n_l-s_l}(1-\exp(z_{i:n_l}))\right]\right\},\tag{3.2.2}$$

and

$$\frac{\partial\log L}{\partial\sigma}=-\frac{1}{\sigma}\sum_{l=1}^{k}\left\{A_l-s_lz_{n_l-s_l:n_l}\exp(z_{n_l-s_l:n_l})+\sum_{i=1}^{n_l-s_l}[z_{i:n_l}(1-\exp(z_{i:n_l}))]\right\}.\tag{3.2.3}$$

The MLEs $\hat{v}_0$, $\hat{v}_1$ and $\hat{\sigma}_0$ (of v_0, v_1 and σ) can be obtained by simultaneously solving the equations $\partial\log L/\partial v_0=0$, $\partial\log L/\partial v_1=0$ and $\partial\log L/\partial\sigma=0$. Since these three equations cannot be solved analytically, numerical methods must be employed. Newton-Raphson or some other iterative procedure can be applied with no trouble.

The approximate variance-covariance matrix can be obtained by inverting the observed Fisher information matrix I_0 evaluated at the MLEs of ν_0, ν_1 and σ. The observed Fisher information matrix I_0 is of the form

$$I_0 = -\begin{pmatrix} \frac{\partial^2 \log L}{\partial \nu_0^{\,2}} & \frac{\partial^2 \log L}{\partial \nu_0 \partial \nu_1} & \frac{\partial^2 \log L}{\partial \nu_0 \partial \sigma} \\ \frac{\partial^2 \log L}{\partial \nu_0 \partial \nu_1} & \frac{\partial^2 \log L}{\partial \nu_1^{\,2}} & \frac{\partial^2 \log L}{\partial \nu_1 \partial \sigma} \\ \frac{\partial^2 \log L}{\partial \nu_0 \partial \sigma} & \frac{\partial^2 \log L}{\partial \nu_1 \partial \sigma} & \frac{\partial^2 \log L}{\partial \sigma^2} \end{pmatrix}_{(\hat{\nu}_0, \hat{\nu}_1, \hat{\sigma})}, \tag{3.2.4}$$

where

$$\frac{\partial^2 \log L}{\partial \nu_0^{\,2}} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ -s_l \exp(z_{n_l - s_l : n_l}) - \sum_{i=1}^{n_l - s_l} \exp(z_{i:n_l}) \right\}, \tag{3.2.5}$$

$$\frac{\partial^2 \log L}{\partial \nu_1^{\,2}} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ x_l^{\,2} \left[-s_l \exp(z_{n_l - s_l : n_l}) - \sum_{i=1}^{n_l - s_l} \exp(z_{i:n_l}) \right] \right\}, \tag{3.2.6}$$

$$\frac{\partial^2 \log L}{\partial \sigma^2} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \{ A_l - s_l (2 + z_{n_l - s_l : n_l}) z_{n_l - s_l : n_l} \exp(z_{n_l - s_l : n_l})$$
$$+ \sum_{i=1}^{n_l - s_l} [z_{i:n_l} (2 - (2 + z_{i:n_l}) \exp(z_{i:n_l}))] \}, \tag{3.2.7}$$

$$\frac{\partial^2 \log L}{\partial \nu_0 \partial \nu_1} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ x_l \left[-s_l \exp(z_{n_l - s_l : n_l}) - \sum_{i=1}^{n_l - s_l} \exp(z_{i:n_l}) \right] \right\}, \tag{3.2.8}$$

$$\frac{\partial^2 \log L}{\partial \nu_0 \partial \sigma} = \frac{1}{\sigma^2} \sum_{l=1}^{k} [-s_l (1 + z_{n_l - s_l : n_l}) \exp(z_{n_l - s_l : n_l})$$
$$+ \sum_{i=1}^{n_l - s_l} (1 - (1 + z_{i:n_l}) \exp(z_{i:n_l}))], \tag{3.2.9}$$

and

$$\frac{\partial^2 \log L}{\partial \nu_1 \partial \sigma} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \{ x_l [-s_l (1 + z_{n_l - s_l : n_l}) \exp(z_{n_l - s_l : n_l})$$
$$+ \sum_{i=1}^{n_l - s_l} (1 - (1 + z_{i:n_l}) \exp(z_{i:n_l}))] \} . \tag{3.2.10}$$

3.3 Asymptotic Variances and Covariances

The asymptotic variances and covariances can also be obtained by inverting the expected Fisher information matrix I. The expected Fisher information matrix I is of the form

$$I = -\begin{pmatrix} E(\frac{\partial^2 \log L}{\partial v_0^2}) & E(\frac{\partial^2 \log L}{\partial v_0 \partial v_1}) & E(\frac{\partial^2 \log L}{\partial v_0 \partial \sigma}) \\ E(\frac{\partial^2 \log L}{\partial v_0 \partial v_1}) & E(\frac{\partial^2 \log L}{\partial v_1^2}) & E(\frac{\partial^2 \log L}{\partial v_1 \partial \sigma}) \\ E(\frac{\partial^2 \log L}{\partial v_0 \partial \sigma}) & E(\frac{\partial^2 \log L}{\partial v_1 \partial \sigma}) & E(\frac{\partial^2 \log L}{\partial \sigma^2}) \end{pmatrix}, \tag{3.3.1}$$

where

$$E(\frac{\partial^2 \log L}{\partial v_0^2}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{-s_l E[\exp(z_{n_l-s_l:n_l})] - \sum_{i=1}^{n_l-s_l} E[\exp(z_{i:n_l})]\right\}, \tag{3.3.2}$$

$$E(\frac{\partial^2 \log L}{\partial v_1^2}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{x_l^2\left[-s_l E[\exp(z_{n_l-s_l:n_l})] - \sum_{i=1}^{n_l-s_l} E[\exp(z_{i:n_l})]\right]\right\}, \tag{3.3.3}$$

$$E(\frac{\partial^2 \log L}{\partial \sigma^2}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\{A_l - 2s_l E[z_{n_l-s_l:n_l}\exp(z_{n_l-s_l:n_l})] - s_l E[z_{n_l-s_l:n_l}^2 \exp(z_{n_l-s_l:n_l})]$$

$$+\sum_{i=1}^{n_l-s_l}[2E(z_{i:n_l}) - 2E[z_{i:n_l}\exp(z_{i:n_l})] + E[z_{i:n_l}^2\exp(z_{i:n_l})]]\}, \tag{3.3.4}$$

$$E(\frac{\partial^2 \log L}{\partial v_0 \partial v_1}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{x_l\left[-s_l E[\exp(z_{n_l-s_l:n_l})] - \sum_{i=1}^{n_l-s_l} E[\exp(z_{i:n_l})]\right]\right\}, \tag{3.3.5}$$

$$E(\frac{\partial^2 \log L}{\partial v_0 \partial \sigma}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\{-s_l E[\exp(z_{n_l-s_l:n_l})] - s_l E[z_{n_l-s_l:n_l}\exp(z_{n_l-s_l:n_l})]$$

$$+\sum_{i=1}^{n_l-s_l}[1 - E[\exp(z_{i:n_l})] - E[z_{i:n_l}\exp(z_{i:n_l})]]\Big\}, \tag{3.3.6}$$

$$E\left(\frac{\partial^2 \log L}{\partial v_1 \partial \sigma}\right) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\Big\{x_l[-s_l E[\exp(z_{n_l-s_l:n_l})] - s_l E[z_{n_l-s_l:n_l}\exp(z_{n_l-s_l:n_l})]$$

$$+\sum_{i=1}^{n_l-s_l}[1 - E[\exp(z_{i:n_l})] - E(z_{i:n_l}\exp(z_{i:n_l}))]]\Big\}, \tag{3.3.7}$$

and $E(z_{i:n}) = \alpha_{i:n}$ is the mean of the i-th order statistic in a sample of size n from the standard extreme value distribution.

To evaluate the expected information matrix in (3.3.1), we need the exact value of $E[z_{i:n}^k \exp(z_{i:n})]$, for k = 0, 1, 2. Along the lines of the evaluation of the asymptotic variances and covariances of order statistics in the extreme value distribution [see Lieblein (1953); Balakrishnan and Chan (1992a)], we derive the expression of $E[z_{i:n}^k \exp(z_{i:n})]$ as follows:

Consider the density function of $z_{i:n}$ $(1 \le i \le n)$ given by

$$f_{i:n}(z) = \frac{n!}{(i-1)!(n-i)!}\{F(z)\}^{i-1}\{1-F(z)\}^{n-i} f(z), \qquad -\infty < z < \infty. \tag{3.3.8}$$

Then, we have

$$E\left[z_{i:n}^k \exp(z_{i:n})\right] = \int_{-\infty}^{\infty} z^k \exp(z) f_{i:n}(z)dz, \qquad 1 \le i \le n. \tag{3.3.9}$$

Moreover, we can express

$$E[z_{n-i+1:n}^k \exp(z_{n-i+1:n})] = \frac{n!}{(i-1)!(n-i)!}\sum_{r=0}^{n-i}(-1)^r \begin{bmatrix} n-i \\ r \end{bmatrix} \int_{-\infty}^{\infty} z^k e^z e^{z-(i+r)e^z} dz, \qquad 1 \le i \le n \tag{3.3.10}$$

By considering the integral

$$g_k(c) = \int_{-\infty}^{\infty} z^k e^{2z-ce^z} dz$$

and setting $v = e^z$, we get

$$g_k(c) = \int_0^{\infty} (\log v)^k v e^{-cv} dv$$

which, for non-negative integers k, may be written as

$$g_k(c) = \frac{\partial^k}{\partial t^k} \int_0^\infty v^t e^{-cv} dv \,|_{t=1}$$

$$= \frac{\partial^k}{\partial t^k} \{\Gamma(t+1) c^{-(t+1)}\} \,|_{t=1}; \qquad (3.3.11)$$

here, $\Gamma(\cdot)$ denotes the gamma function. The functions $g_0(c)$, $g_1(c)$ and $g_2(c)$ needed for the computation of $E[z_{i:n}^k \exp(z_{i:n})]$, for k = 0, 1, 2, may be derived from (3.3.11) to be

$$g_0(c) = \frac{1}{c^2}\Gamma(2) = \frac{1}{c^2},$$

$$g_1(c) = \frac{1}{c^2}[\Gamma'(2) + \Gamma(2)\log c] = \frac{1}{c^2}(1 - \gamma + \log c),$$

and

$$g_2(c) = \frac{1}{c^2}[\Gamma(2)(\log c)^2 + 2\Gamma'(2)\log c + \Gamma''(2)]$$

$$= \frac{1}{c^2}[(\log c)^2 + 2(1-\gamma)\log c + \gamma^2 - 2\gamma + \frac{\pi^2}{6}],$$

where $\gamma = 0.5772$ is Euler's constant.

By using the above expressions of $g_0(c)$, $g_1(c)$ and $g_2(c)$, we can compute $E[z_{i:n}^k \exp(z_{i:n})]$, for k = 0, 1, 2, from (3.3.10), and obtain the asymptotic variances and covariances of MLEs $\hat{v}_0$, $\hat{v}_1$ and $\hat{\sigma}_0$ (of v_0, v_1 and σ) from (3.3.1).

Of course, the results for the complete sample situation may simply be deduced from the above formulas by setting $s_l = 0$ for l = 1,..., k.

3.4 Simulations and Results

In the simulation study, we took $v_0 = 0$, $v_1 = 1$ and $\sigma = 1$ and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored sample. In order to study the MLEs in the MEVR model, we performed the simulations based on 10,000 Monte Carlo runs for each of the following cases:

1. Complete samples

two groups: n = [6 6(1)10], [7 7(1)10], [8 8(1)10], [9 9(1)10], [10 10], [15 15(5)20] and [20 20] .

four groups: n = [6×2 6(1)10×2], [7×2 7(1)10×2], [8×2 8(1)10×2], [9×2 9(1)10×2], [10 10 10 10], [15×2 15(5)20×2] and [20 20 20 20] .

2. Type-II right-censored samples

two groups: s = [4 4(1)0], [3 3(1)0], [2 2(1)0] and [1 1(1)0] from n = [10 10] and [5 5(1)0] from n = [20 20].

four groups: s = [4×2 4(1)0×2], [3×2 3(1)0×2], [2×2 2(1)0×2] and [1×2 1(1)0×2] from n = [10 10 10 10] and [5×2 5(1)0×2] from n = [20 20 20 20].

We generated order statistics from the standard extreme value sample $z_{i:n_l}$, $i = 1,\dots, n_l$, $l = 1,\dots, k$, and then using the model $y_{i:n_l} = \nu_o + \nu_1 x_l + \sigma z_{i:n_l}$, transformed the sample into $y_{i:n_l}$, $i = 1,\dots,n_l$, $l = 1,\dots, k$. Upon simultaneously solving the equations $\partial \log L/\partial \nu_0 = 0$, $\partial \log L/\partial \nu_1 = 0$ and $\partial \log L/\partial \sigma = 0$, we obtained the values of MLEs $\hat{\nu}_0$, $\hat{\nu}_1$ and $\hat{\sigma}_0$. Based on 10,000 samples, we determined the values of (1) $Bias(\hat{\nu}_0)/\sigma$, (2) $Bias(\hat{\nu}_1)/\sigma$, (3) $Bias(\hat{\sigma})/\sigma$, (4) $MSE(\hat{\nu}_0)/\sigma^2$, (5) $MSE(\hat{\nu}_1)/\sigma^2$, (6) $MSE(\hat{\sigma})/\sigma^2$, (7) $Var(\hat{\nu}_0)/\sigma^2$, (8) $Var(\hat{\nu}_1)/\sigma^2$, (9) $Var(\hat{\sigma})/\sigma^2$, (10) $Cov(\hat{\nu}_0,\hat{\nu}_1)/\sigma^2$, (11) $Cov(\hat{\nu}_0,\hat{\sigma})/\sigma^2$, and (12) $Cov(\hat{\nu}_1,\hat{\sigma})/\sigma^2$. The asymptotic values of (7) – (12) were also computed by inverting the expected information matrix I in (3.3.1). These results are presented in Tables 3.4.1 – 3.4.12.

From these tables, we observe the following points. The variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a major increase in N. In addition, with the same N, the variances of MLEs of ν_0 and ν_1 tend to be smaller in value in the more balanced groups than among the less balanced groups. Moreover, with the same N, the variance of MLE of σ tends to be the same in the complete samples and does not exhibit any clear patterns in the Type-II censored samples. The variances of all the estimators tend to increase with increasing amounts of censoring. This is true for both two- and four-levels of x.

The simulated mean square errors are very close to the simulated variances, but not identical, which means that the biases of the estimators are negligible. Moreover, the agreements tend to increase with increase in the total sample size N. This simply means that the MLEs of ν_0, ν_1 and σ all become almost unbiased as the total sample size N become large.

Table 3.4.1 MLE procedure: Simulated MSE for two-grouped complete sample

$[n_1\ n_2]$	$MSE(\hat{v}_0)/\sigma^2$	$MSE(\hat{v}_1)/\sigma^2$	$MSE(\hat{\sigma})/\sigma^2$
[6 6]	0.1017	0.3703	0.0635
[6 7]	0.0903	0.3347	0.0599
[6 8]	0.0867	0.3156	0.0539
[6 9]	0.0811	0.2909	0.0496
[6 10]	0.0783	0.2816	0.0460
[7 7]	0.0821	0.3086	0.0531
[7 8]	0.0790	0.2862	0.0502
[7 9]	0.0756	0.2733	0.0466
[7 10]	0.0723	0.2589	0.0438
[8 8]	0.0740	0.2714	0.0455
[8 9]	0.0693	0.2475	0.0430
[8 10]	0.0661	0.2370	0.0398
[9 9]	0.0662	0.2320	0.0402
[9 10]	0.0618	0.2237	0.0379
[10 10]	0.0580	0.2082	0.0361
[15 15]	0.0389	0.1369	0.0230
[15 20]	0.0322	0.1179	0.0192
[20 20]	0.0282	0.1031	0.0169

Table 3.4.2 MLE procedure: Simulated variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
[6 6]	0.0983	0.3703	0.0496	0.0016	-0.0182	0.0011
[6 7]	0.0876	0.3346	0.0466	-0.0120	-0.0161	-0.0004
[6 8]	0.0845	0.3154	0.0436	-0.0225	-0.0163	-0.0002
[6 9]	0.0790	0.2899	0.0405	-0.0321	-0.0141	-0.0017
[6 10]	0.0759	0.2805	0.0380	-0.0366	-0.0132	-0.0023
[7 7]	0.0795	0.3086	0.0430	0.0009	-0.0145	-0.0004
[7 8]	0.0771	0.2862	0.0408	-0.0091	-0.0144	0.0002
[7 9]	0.0738	0.2729	0.0384	-0.0183	-0.0135	-0.0020
[7 10]	0.0712	0.2578	0.0361	-0.0245	-0.0136	-0.0027
[8 8]	0.0723	0.2715	0.0370	0.0021	-0.0139	-0.0004
[8 9]	0.0678	0.2475	0.0363	-0.0081	-0.0138	-0.0004
[8 10]	0.0646	0.2368	0.0335	-0.0143	-0.0117	-0.0002
[9 9]	0.0650	0.2320	0.0343	0.0013	-0.0133	0.0001
[9 10]	0.0607	0.2237	0.0324	-0.0072	-0.0118	0.0011
[10 10]	0.0571	0.2082	0.0307	0.0020	-0.0122	0.0006
[15 15]	0.0385	0.1369	0.0209	0.0001	-0.0083	-0.0005
[15 20]	0.0319	0.1179	0.0176	-0.0086	-0.0067	0.0003
[20 20]	0.0280	0.1031	0.0157	-0.0003	-0.0065	0.0001

Table 3.4.3 MLE procedure: Asymptotic variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
[6 6]	0.0924	0.3333	0.0507	-0.0000	-0.0214	0.0000
[6 7]	0.0857	0.3095	0.0468	-0.0119	-0.0198	0.0000
[6 8]	0.0807	0.2917	0.0434	-0.0208	-0.0184	0.0000
[6 9]	0.0767	0.2778	0.0405	-0.0278	-0.0171	-0.0000
[6 10]	0.0735	0.2667	0.0380	-0.0333	-0.0161	0.0000
[7 7]	0.0792	0.2857	0.0434	-0.0000	-0.0184	0.0000
[7 8]	0.0742	0.2679	0.0405	-0.0089	-0.0171	-0.0000
[7 9]	0.0703	0.2540	0.0380	-0.0159	-0.0161	-0.0000
[7 10]	0.0671	0.2429	0.0358	-0.0214	-0.0151	0.0000
[8 8]	0.0693	0.2500	0.0380	-0.0000	-0.0161	-0.0000
[8 9]	0.0654	0.2361	0.0358	-0.0069	-0.0151	-0.0000
[8 10]	0.0623	0.2250	0.0338	-0.0125	-0.0143	0.0000
[9 9]	0.0616	0.2222	0.0338	0.0000	-0.0143	-0.0000
[9 10]	0.0585	0.2111	0.0320	-0.0056	-0.0135	0.0000
[10 10]	0.0554	0.2000	0.0304	0.0000	-0.0129	0.0000
[15 15]	0.0370	0.1333	0.0203	0.0000	-0.0086	0.0000
[15 20]	0.0323	0.1167	0.0174	-0.0083	-0.0073	0.0000
[20 20]	0.0277	0.1000	0.0152	0.0000	-0.0064	0.0000

Table 3.4.4 MLE procedure: Simulated MSE for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$MSE(\hat{v}_0)/\sigma^2$	$MSE(\hat{v}_1)/\sigma^2$	$MSE(\hat{\sigma})/\sigma^2$
[6 6 6 6]	0.0470	0.3175	0.0291
[6 6 7 7]	0.0445	0.3007	0.0263
[6 6 8 8]	0.0403	0.2829	0.0242
[6 6 9 9]	0.0385	0.2636	0.0216
[6 6 10 10]	0.0375	0.2542	0.0207
[7 7 7 7]	0.0416	0.2775	0.0241
[7 7 8 8]	0.0382	0.2623	0.0224
[7 7 9 9]	0.0351	0.2441	0.0211
[7 7 10 10]	0.0347	0.2263	0.0193
[8 8 8 8]	0.0358	0.2370	0.0212
[8 8 9 9]	0.0334	0.2257	0.0191
[8 8 10 10]	0.0323	0.2170	0.0181
[9 9 9 9]	0.0318	0.2141	0.0182
[9 9 10 10]	0.0293	0.2058	0.0173
[10 10 10 10]	0.0277	0.1887	0.0164
[15 15 15 15]	0.0184	0.1227	0.0108
[15 15 20 20]	0.0165	0.1063	0.0092
[20 20 20 20]	0.0139	0.0928	0.0079

Table 3.4.5 MLE procedure: Simulated variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
[6 6 6 6]	0.0464	0.3176	0.0255	0.0004	-0.0098	0.0000
[6 6 7 7]	0.0439	0.3007	0.0232	-0.0089	-0.0092	-0.0002
[6 6 8 8]	0.0399	0.2829	0.0215	-0.0118	-0.0082	-0.0005
[6 6 9 9]	0.0379	0.2635	0.0197	-0.0177	-0.0075	0.0001
[6 6 10 10]	0.0370	0.2541	0.0190	-0.0220	-0.0075	0.0005
[7 7 7 7]	0.0410	0.2775	0.0217	0.0007	-0.0083	0.0001
[7 7 8 8]	0.0377	0.2622	0.0201	-0.0041	-0.0078	-0.0013
[7 7 9 9]	0.0347	0.2440	0.0192	-0.0106	-0.0077	0.0007
[7 7 10 10]	0.0343	0.2263	0.0176	-0.0124	-0.0067	-0.0005
[8 8 8 8]	0.0353	0.2370	0.0193	0.0004	-0.0079	-0.0006
[8 8 9 9]	0.0330	0.2257	0.0174	-0.0027	-0.0067	-0.0011
[8 8 10 10]	0.0320	0.2170	0.0167	-0.0079	-0.0066	-0.0004
[9 9 9 9]	0.0315	0.2141	0.0167	0.0002	-0.0070	-0.0002
[9 9 10 10]	0.0290	0.2058	0.0159	-0.0037	-0.0063	0.0004
[10 10 10 10]	0.0274	0.1887	0.0151	0.0011	-0.0060	0.0006
[15 15 15 15]	0.0183	0.1227	0.0102	-0.0005	-0.0043	-0.0004
[15 15 20 20]	0.0164	0.1063	0.0088	-0.0051	-0.0035	-0.0006
[20 20 20 20]	0.0138	0.0928	0.0076	0.0002	-0.0030	0.0002

Table 3.4.6 MLE procedure: Asymptotic variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
[6 6 6 6]	0.0462	0.3024	0.0253	0.0000	-0.0107	-0.0000
[6 6 7 7]	0.0428	0.2804	0.0234	-0.0071	-0.0099	0.0000
[6 6 8 8]	0.0402	0.2634	0.0217	-0.0124	-0.0092	0.0000
[6 6 9 9]	0.0380	0.2498	0.0203	-0.0165	-0.0086	-0.0000
[6 6 10 10]	0.0363	0.2386	0.0190	-0.0197	-0.0080	0.0000
[7 7 7 7]	0.0396	0.2592	0.0217	-0.0000	-0.0092	-0.0000
[7 7 8 8]	0.0371	0.2427	0.0203	-0.0053	-0.0086	-0.0000
[7 7 9 9]	0.0350	0.2296	0.0190	-0.0095	-0.0080	-0.0000
[7 7 10 10]	0.0333	0.2188	0.0179	-0.0127	-0.0076	0.0000
[8 8 8 8]	0.0346	0.2268	0.0190	-0.0000	-0.0080	-0.0000
[8 8 9 9]	0.0327	0.2140	0.0179	-0.0042	-0.0076	-0.0000
[8 8 10 10]	0.0311	0.2036	0.0169	-0.0075	-0.0071	00000
[9 9 9 9]	0.0308	0.2016	0.0169	0.0.000	-0.0071	-0.0000
[9 9 10 10]	0.0292	0.1914	0.0160	-0.0033	-0.0068	0.0000
[10 10 10 10]	0.0277	0.1814	0.0152	0.0000	-0.0064	0.0000
[15 15 15 15]	0.0185	0.1209	0.0101	0.0000	-0.0043	0.0000
[15 15 20 20]	0.0161	0.1054	0.0087	-0.0050	-0.0037	0.0000
[20 20 20 20]	0.0139	0.0907	0.0076	0.0000	-0.0032	0.0000

Table 3.4.7 MLE procedure: Simulated MSE for two-grouped Type-II censored sample

$[s_1 \ \ s_2]$	$MSE(\hat{v}_0)/\sigma^2$	$MSE(\hat{v}_1)/\sigma^2$	$MSE(\hat{\sigma})/\sigma^2$
4 4	0.1194	0.3653	0.0803
4 3	0.1018	0.3322	0.0726
4 2	0.0889	0.3116	0.0658
4 1	0.0824	0.3112	0.0570
4 0	0.0761	0.3149	0.0496
3 3	0.0885	0.3069	0.0655
3 2	0.0771	0.2907	0.0590
3 1	0.0721	0.2716	0.0524
3 0	0.0688	0.2711	0.0470
2 2	0.0719	0.2651	0.0531
2 1	0.0663	0.2497	0.0485
2 0	0.0609	0.2462	0.0432
1 1	0.0609	0.2344	0.0445
1 0	0.0602	0.2246	0.0392
*5 5	0.0361	0.1393	0.0273
*5 0	0.0314	0.1274	0.0209

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 3.4.8 MLE procedure: Simulated variances and covariances for two-grouped Type-II censored sample

$[s_1 \; s_2]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
4 4	0.0986	0.3653	0.0601	-0.0029	0.0219	-0.0006
4 3	0.0875	0.3306	0.0549	-0.0143	0.0138	-0.0097
4 2	0.0793	0.3088	0.0521	-0.0264	0.0079	-0.0198
4 1	0.0747	0.3055	0.0454	-0.0316	0.0042	-0.0261
4 0	0.0704	0.3072	0.0408	-0.0362	-0.0001	-0.0298
3 3	0.0791	0.3069	0.0514	-0.0004	0.0071	0.0003
3 2	0.0717	0.2903	0.0467	-0.0101	0.0025	-0.0093
3 1	0.0672	0.2680	0.0425	-0.0160	-0.0012	-0.0151
3 0	0.0651	0.2682	0.0394	-0.0212	-0.0041	-0.0224
2 2	0.0674	0.2651	0.0438	0.0012	-0.0003	0.0003
2 1	0.0634	0.2496	0.0404	-0.0034	-0.0045	-0.0086
2 0	0.0589	0.2448	0.0362	-0.0115	-0.0075	-0.0134
1 1	0.0590	0.2344	0.0370	0.0004	-0.0073	-0.0003
1 0	0.0586	0.2239	0.0331	-0.0041	-0.0094	-0.0055
*5 5	0.0344	0.1393	0.0245	-0.0001	0.0012	-0.0009
*5 0	0.0308	0.1267	0.0190	-0.0071	-0.0030	-0.0091

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 3.4.9 MLE procedure: Asymptotic variances and covariances for two-grouped Type-II censored sample

$[s_1 \ s_2]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
4 4	0.0880	0.3333	0.0624	0.0000	0.0171	0.0000
4 3	0.0793	0.3114	0.0568	-0.0138	0.0103	-0.0104
4 2	0.0734	0.2982	0.0515	-0.0226	0.0049	-0.0183
4 1	0.0695	0.2904	0.0464	-0.0281	0.0006	-0.0242
4 0	0.0669	0.2865	0.0409	-0.0312	-0.0030	-0.0285
3 3	0.0719	0.2857	0.0521	0.0000	0.0047	0.0000
3 2	0.0670	0.2693	0.0477	-0.0090	0.0002	-0.0082
3 1	0.0638	0.2589	0.0432	-0.0147	-0.0034	-0.0146
3 0	0.0618	0.2530	0.0384	-0.0182	-0.0064	-0.0197
2 2	0.0628	0.2500	0.0439	0.0000	-0.0036	0.0000
2 1	0.0601	0.2372	0.0401	-0.0058	-0.0066	-0.0067
2 0	0.0585	0.2292	0.0359	-0.0094	-0.0091	-0.0123
1 1	0.0578	0.2222	0.0369	0.0000	-0.0092	0.0000
1 0	0.0565	0.2121	0.0333	-0.0036	-0.0112	-0.0058
*5 5	0.0333	0.1333	0.0246	0.0000	0.0005	0.0000
*5 0	0.0299	0.1203	0.0188	-0.0067	-0.0038	-0.0083

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 3.4.10 MLE procedure: Simulated MSE for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$MSE(\hat{v}_0)/\sigma^2$	$MSE(\hat{v}_1)/\sigma^2$	$MSE(\hat{\sigma})/\sigma^2$
4 4 4 4	0.0504	0.3178	0.0355
4 4 3 3	0.0443	0.2913	0.0314
4 4 2 2	0.0405	0.2877	0.0284
4 4 1 1	0.0378	0.2758	0.0252
4 4 0 0	0.0348	0.2696	0.0217
3 3 3 3	0.0391	0.2733	0.0294
3 3 2 2	0.0356	0.2593	0.0265
3 3 1 1	0.0351	0.2534	0.0240
3 3 0 0	0.0327	0.2373	0.0200
2 2 2 2	0.0334	0.2415	0.0243
2 2 1 1	0.0309	0.2226	0.0218
2 2 0 0	0.0305	0.2203	0.0197
1 1 1 1	0.0297	0.2096	0.0198
1 1 0 0	0.0296	0.2060	0.0178
*5 5 5 5	0.0173	0.1254	0.0130
*5 5 0 0	0.0153	0.1074	0.0097

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 3.4.11 MLE procedure: Simulated variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
4 4 4 4	0.0458	0.3178	0.0306	0.0009	0.0097	-0.0008
4 4 3 3	0.0408	0.2911	0.0276	-0.0105	0.0061	-0.0069
4 4 2 2	0.0384	0.2865	0.0255	-0.0154	0.0030	-0.0107
4 4 1 1	0.0361	0.2736	0.0224	-0.0182	0.0008	-0.0145
4 4 0 0	0.0336	0.2662	0.0197	-0.0190	-0.0010	-0.0175
3 3 3 3	0.0368	0.2733	0.0262	-0.0007	0.0026	-0.0019
3 3 2 2	0.0340	0.2590	0.0237	-0.0069	0.0006	-0.0060
3 3 1 1	0.0339	0.2527	0.0216	-0.0115	-0.0011	-0.0097
3 3 0 0	0.0319	0.2361	0.0181	-0.0123	-0.0032	-0.0114
2 2 2 2	0.0324	0.2415	0.0222	-0.0006	-0.0016	-0.0012
2 2 1 1	0.0301	0.2224	0.0195	-0.0038	-0.0026	-0.0027
2 2 0 0	0.0300	0.2201	0.0182	-0.0067	-0.0046	-0.0085
1 1 1 1	0.0294	0.2096	0.0180	0.0005	-0.0039	-0.0002
1 1 0 0	0.0291	0.2058	0.0166	-0.0027	-0.0055	-0.0036
*5 5 5 5	0.0169	0.1254	0.0123	-0.0005	0.0000	0.0005
*5 5 0 0	0.0151	0.1072	0.0093	-0.0040	-0.0017	-0.0046

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 3.4.12 MLE procedure: Asymptotic variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_0,\hat{v}_1)/\sigma^2$	$Cov(\hat{v}_0,\hat{\sigma})/\sigma^2$	$Cov(\hat{v}_1,\hat{\sigma})/\sigma^2$
4 4 4 4	0.0440	0.3024	0.0312	0.0000	0.0085	0.0000
4 4 3 3	0.0396	0.2818	0.0284	-0.0082	0.0051	-0.0062
4 4 2 2	0.0365	0.2680	0.0257	-0.0134	0.0023	-0.0109
4 4 1 1	0.0344	0.2586	0.0230	-0.0165	0.0000	-0.0142
4 4 0 0	0.0331	0.2522	0.0201	-0.0181	-0.0019	-0.0165
3 3 3 3	0.0359	0.2592	0.0261	0.0000	0.0023	0.0000
3 3 2 2	0.0335	0.2438	0.0238	-0.0054	0.0001	-0.0049
3 3 1 1	0.0318	0.2331	0.0215	-0.0087	-0.0018	-0.0087
3 3 0 0	0.0307	0.2259	0.0190	-0.0107	-0.0034	-0.0116
2 2 2 2	0.0314	0.2268	0.0219	0.0000	-0.0018	0.0000
2 2 1 1	0.0300	0.2148	0.0200	-0.0035	-0.0033	-0.0040
2 2 0 0	0.0292	0.2065	0.0179	-0.0056	-0.0046	-0.0073
1 1 1 1	0.0289	0.2016	0.0185	0.0000	-0.0046	0.0000
1 1 0 0	0.0283	0.1921	0.0167	-0.0022	-0.0056	-0.0035
*5 5 5 5	0.0167	0.1209	0.0123	0.0000	0.0002	0.0000
*5 5 0 0	0.0149	0.1080	0.0093	-0.0039	-0.0019	-0.0049

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

CHAPTER 4

APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION (AMLE)

4.1 Introduction

In this chapter, an closed form approximation to the maximum likelihood estimators, which are in closed form, is developed. These estimators can be used as initial guess for the Newton-Raphson procedure to obtain the MLEs discussed in Section 3.2. In Section 4.2, we derive the AMLEs of ν_0, ν_1 and σ based on Type-II right-censored samples as well as the procedure to obtain their approximate variances and covariances based on the observed Fisher information matrix. We then derive the asymptotic variances and covariances of these estimators through the expected Fisher information matrix in Section 4.3. In Section 4.4, we derive explicit expressions for the approximate biases of the AMLEs of ν_0, ν_1 and σ. Finally, we conduct a simulation study to evaluate the performance of the AMLEs of ν_0, ν_1 and σ for various choices of sample sizes and censoring schemes in Section 4.5.

4.2 Type-II Right-censored Sample

Suppose that observations are taken on n_l individuals at a single regressor x_l, for $l = 1, \ldots, k$, and that we allow the sample to be Type-II right-censored, meaning that only the first $n_l - s_l$ ordered values $y_{1:n_l} \le y_{2:n_l} \le \ldots \le y_{n_l - s_l:n_l}$ out of the total of n_l observations are observed. The corresponding likelihood function for the model in extreme value form is

$$\prod_{l=1}^{k} \frac{n_l!}{s_l!} \left\{ \prod_{i=1}^{n_l - s_l} \frac{1}{\sigma} \exp\left[\frac{y_{i:n_l} - \nu_0 - \nu_1 x_l}{\sigma} - \exp\left(\frac{y_{i:n_l} - \nu_0 - \nu_1 x_l}{\sigma} \right) \right] \right\} \left\{ \exp\left[-\exp\left(\frac{y_{n_l - s_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma} \right) \right] \right\}^{s_l}$$

Dropping the proportionality constant $\prod n_l!/s_l!$, we can take the log-likelihood function as

$$\log L(\nu_0, \nu_1, \sigma) = -\log\sigma \sum_{l=1}^{k} A_l + \sum_{l=1}^{k} \sum_{i=1}^{n_l - s_l} \left\{ \frac{y_{i:n_l} - \nu_0 - \nu_1 x_l}{\sigma} - \exp\left(\frac{y_{i:n_l} - \nu_0 - \nu_1 x_l}{\sigma} \right) \right\}$$
$$- \sum_{l=1}^{k} s_l \left[\exp\left(\frac{y_{i:n_l} - \nu_0 - \nu_1 x_l}{\sigma} \right) \right],$$

where $A_l = n_l - s_l$. Denoting $z_{i:n_l} = (y_{i:n_l} - \nu_0 - \nu_1 x_l)/\sigma$, the likelihood equations in (3.2.1) – (3.2.3) do not admit explicit solutions, as we noted earlier in Chapter 3. However, by expanding the functions $f(z_{n_l - s_l:n_l})/\{1 - F(z_{n_l - s_l:n_l})\}$ and $f'(z_{i:n_l})/f(z_{i:n_l})$ in a Taylor series around the points $F^{-1}(p_{n_l - s_l:n_l}) = \ln(-\ln q_{n_l - s_l i:n_l})$ and $F^{-1}(p_{i:n_l}) = \ln(-\ln q_{i:n_l})$, respectively, we may approximate these functions by [see David (1981) and Arnold and Balakrishnan (1989) for reasoning; see also Balakrishnan and Varadan (1991)]

$$\frac{f(z_{n_l - s_l:n_l})}{1 - F(z_{n_l - s_l:n_l})} \approx 1 - \alpha_{n_l - s_l:n_l} + \beta_{n_l - s_l:n_l} z_{n_l - s_l:n_l}$$

and

$$\frac{f'(z_{i:n_l})}{f(z_{i:n_l})} \approx \alpha_{i:n_l} - \beta_{i:n_l} z_{i:n_l},$$

where

$$p_{n_l-s_l:n_l} = \frac{n_l - s_l}{n_l + 1},\ q_{n_l-s_l:n_l} = 1 - p_{n_l-s_l:n_l},$$

$$p_{i:n_l} = \frac{i}{n_l + 1},\ q_{i:n_l} = 1 - p_{i:n_l},$$

$$\alpha_{n_l-s_l:n_l} = 1 + \ln q_{n_l-s_l:n_l}\{1 - \ln(-\ln q_{n_l-s_l:n_l})\},$$

$$\beta_{n_l-s_l:n_l} = -\ln q_{n_l-s_l:n_l},$$

$$\alpha_{i:n_l} = 1 + \ln q_{i:n_l}\{1 - \ln(-\ln q_{i:n_l})\}$$

and

$$\beta_{i:n_l} = -\ln q_{i:n_l}.$$

It is easy to see that $\beta_{n_l-s_l:n_l} > 0$ and $\beta_{i:n_l} > 0$ for $i = 1, 2, \ldots, n_l - s_l$-1.

By making use of the above linear approximations, we obtain the approximate log-likelihood equations as

$$\frac{\partial \log L}{\partial v_0} \approx \frac{\partial \log L^*}{\partial v_0} = -\frac{1}{\sigma}\sum_{l=1}^{k}\{-s_l(1-\alpha_{n_l-s_l:n_l} + \beta_{n_l-s_l:n_l} z_{n_l-s_l:n_l}) + \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \beta_{i:n_l} z_{i:n_l})\} = 0 \tag{4.2.1}$$

$$\frac{\partial \log L}{\partial v_1} \approx \frac{\partial \log L^*}{\partial v_1} = -\frac{1}{\sigma}\sum_{l=1}^{k}\{x_l[-s_l(1-\alpha_{n_l-s_l:n_l} + \beta_{n_l-s_l:n_l} z_{n_l-s_l:n_l}) + \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \beta_{i:n_l} z_{i:n_l})]\} = 0 \tag{4.2.2}$$

and

$$\frac{\partial \log L}{\partial \sigma} \approx \frac{\partial \log L^{*}}{\partial \sigma} = -\frac{1}{\sigma}\sum_{l=1}^{k}\{A_l - s_l z_{n_l - s_l:n_l}(1-\alpha_{n_l - s_l:n_l} + \beta_{n_l - s_l:n_l} z_{n_l - s_l:n_l}) + \sum_{i=1}^{n_l - s_l} z_{i:n_l}(\alpha_{i:n_l} - \beta_{i:n_l} z_{i:n_l})\} = 0. \tag{4.2.3}$$

Upon solving equations (4.2.1) – (4.2.3), we derive the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ (of ν_0, ν_1 and σ) as

$$\tilde{\nu}_0 = a\tilde{\sigma} + b, \tag{4.2.4}$$

$$\tilde{\nu}_1 = c\tilde{\sigma} + d, \tag{4.2.5}$$

and

$$\tilde{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{4.2.6}$$

where

$$a = \Delta_a / \Delta, \quad b = \Delta_b / \Delta, \quad c = \Delta_c / \Delta, \quad d = \Delta_d / \Delta, \tag{4.2.7}$$

$$\Delta = \det \begin{vmatrix} \sum_{l=1}^{k}\left[s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right] & \sum_{l=1}^{k}\left[x_l\left(s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right)\right] \\ \sum_{l=1}^{k}\left[x_l\left(s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right)\right] & \sum_{l=1}^{k}\left[x_l^2\left(s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right)\right] \end{vmatrix}, \tag{4.2.8}$$

$$\Delta_a = \det \begin{vmatrix} \sum_{l=1}^{k}\left[s_l(1-\alpha_{n_l - s_l:n_l}) - \sum_{i=1}^{n_l - s_l}\alpha_{i:n_l}\right] & \sum_{l=1}^{k}\left[x_l\left(s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right)\right] \\ \sum_{l=1}^{k}\left\{x_l\left[s_l(1-\alpha_{n_l - s_l:n_l}) - \sum_{i=1}^{n_l - s_l}\alpha_{i:n_l}\right]\right\} & \sum_{l=1}^{k}\left[x_l^2\left(s_l \beta_{n_l - s_l:n_l} + \sum_{i=1}^{n_l - s_l}\beta_{i:n_l}\right)\right] \end{vmatrix}, \tag{4.2.9}$$

$$\Delta_b = \det \begin{vmatrix} \sum_{l=1}^{k}\left[s_l\beta_{n_l-s_l:n_l}y_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}y_{i:n_l}\right] & \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] \\ \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l}y_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}y_{i:n_l}\right)\right] & \sum_{l=1}^{k}\left[x_l^2\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] \end{vmatrix}, \tag{4.2.10}$$

$$\Delta_c = \det \begin{vmatrix} \sum_{l=1}^{k}\left[s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right] & \sum_{l=1}^{k}\left[s_l(1-\alpha_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}\alpha_{i:n_l}\right] \\ \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] & \sum_{l=1}^{k}\left\{x_l\left[s_l(1-\alpha_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}\alpha_{i:n_l}\right]\right\} \end{vmatrix}, \tag{4.2.11}$$

$$\Delta_d = \det \begin{vmatrix} \sum_{l=1}^{k}\left[s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right] & \sum_{l=1}^{k}\left[s_l\beta_{n_l-s_l:n_l}y_{n_l-s_n:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}y_{i:n_l}\right] \\ \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] & \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l}y_{n_l-s_n:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}y_{i:n_l}\right)\right] \end{vmatrix}, \tag{4.2.12}$$

$$A = \sum_{l=1}^{k} A_l \,, \tag{4.2.13}$$

$$\begin{aligned} B = \sum_{l=1}^{k}\{&-s_l[y_{n_l-s_l:n_l} - (b+dx_l)][1-\alpha_{n_l-s_l:n_l} - \beta_{n_l-s_l:n_l}(a+cx_l)] \\ &+ \sum_{i=1}^{n_l-s_l}[y_{i:n_l} - (b+dx_l)][\alpha_{i:n_l} + \beta_{i:n_l}(a+cx_l)]\}, \end{aligned} \tag{4.2.14}$$

and

$$C = \sum_{l=1}^{k}\left\{-s_l\beta_{n_l-s_l:n_l}[y_{n_l-s_l:n_l} - (b+dx_l)]^2 + \sum_{i=1}^{n_l-s_l} - \beta_{i:n_l}[y_{i:n_l} - (b+dx_l)]^2\right\}. \tag{4.2.15}$$

It should be mentioned here that upon solving Eq. (4.2.3) for σ, we obtain a quadratic equation in σ which has two roots; however, one estimator is admissible since $A > 0$, $\beta_{n_l-s_l:n_l} > 0$ and $\beta_{i:n_l} > 0$ (for $i = 1, 2, \ldots, n_l - s_l - 1$) and hence $C < 0$.

When all the groups are of the same size, we have $c = 0$ in the expression (4.2.5). This indicates that the asymptotic covariance between $\tilde{\nu}_1$ and $\tilde{\sigma}$ is equal to zero.

The approximate variances and covariances can be obtained by inverting the observed Fisher information matrix I_0^* evaluated at the AMLEs $\tilde{v}_0$, $\tilde{v}_1$ and $\tilde{\sigma}$ (of v_0, v_1 and σ). The observed Fisher information matrix I_0^* is of the form

$$I_0^* = -\begin{pmatrix} \frac{\partial^2 \log L^*}{\partial v_0^2} & \frac{\partial^2 \log L^*}{\partial v_0 \partial v_1} & \frac{\partial^2 \log L^*}{\partial v_0 \partial \sigma} \\ \frac{\partial^2 \log L^*}{\partial v_0 \partial v_1} & \frac{\partial^2 \log L^*}{\partial v_1^2} & \frac{\partial^2 \log L^*}{\partial v_1 \partial \sigma} \\ \frac{\partial^2 \log L^*}{\partial v_0 \partial \sigma} & \frac{\partial^2 \log L^*}{\partial v_1 \partial \sigma} & \frac{\partial^2 \log L^*}{\partial \sigma^2} \end{pmatrix}_{(\tilde{v}_0, \tilde{v}_1, \tilde{\sigma})}, \tag{4.2.16}$$

where

$$\frac{\partial^2 \log L^*}{\partial v_0^2} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ -s_l \beta_{n_l - s_l : n_l} - \sum_{i=1}^{n_l - s_l} \beta_{i:n_l} \right\}, \tag{4.2.17}$$

$$\frac{\partial^2 \log L^*}{\partial v_1^2} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ x_l^2 \left[-s_l \beta_{n_l - s_l : n_l} - \sum_{i=1}^{n_l - s_l} \beta_{i:n_l} \right] \right\}, \tag{4.2.18}$$

$$\begin{aligned} \frac{\partial^2 \log L^*}{\partial \sigma^2} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \{ & A_l - 2s_l (1 - \alpha_{n_l - s_l : n_l}) z_{n_l - s_l : n_l} + 3 s_l \beta_{n_l - s_l : n_l} z^2_{n_l - s_l n_l} \\ & + \sum_{i=1}^{n_l - s_l} (2\alpha_{i:n_l} z_{i:n_l} - 3\beta_{i:n_l} z^2_{i:n_l}) \}, \end{aligned} \tag{4.2.19}$$

$$\frac{\partial^2 \log L}{\partial v_0 \partial v_1} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \left\{ x_l \left[-s_l \beta_{n_l - s_l : n_l} - \sum_{i=1}^{n_l - s_l} \beta_{i:n_l} \right] \right\}, \tag{4.2.20}$$

$$\begin{aligned} \frac{\partial^2 \log L}{\partial v_0 \partial \sigma} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \{ & -s_l (1 - \alpha_{n_l - s_l : n_l} + 2\beta_{n_l - s_l : n_l} z_{n_l - s_l : n_l}) \\ & + \sum_{i=1}^{n_l - s_l} (\alpha_{i:n_l} - 2\beta_{i:n_l} z_{i:n_l}) \}, \end{aligned} \tag{4.2.21}$$

and

$$\begin{aligned} \frac{\partial^2 \log L}{\partial v_1 \partial \sigma} = \frac{1}{\sigma^2} \sum_{l=1}^{k} \{ & x_l [-s_l (1 - \alpha_{n_l - s_l : n_l} + 2\beta_{n_l - s_l : n_l} z_{n_l - s_l : n_l}) \\ & + \sum_{i=1}^{n_l - s_l} (\alpha_{i:n_l} - 2\beta_{i:n_l} z_{i:n_l})] \}. \end{aligned} \tag{4.2.22}$$

4.3 Asymptotic Variances and Covariances

The asymptotic variances and covariances can be obtained by inverting the expected Fisher information matrix I^*. The expected Fisher information matrix I^* is of the form

$$I^* = -\begin{pmatrix} \frac{\partial^2 \log L^*}{\partial v_0^2} & \frac{\partial^2 \log L^*}{\partial v_0 \partial v_1} & E(\frac{\partial^2 \log L^*}{\partial v_0 \partial \sigma}) \\ \frac{\partial^2 \log L^*}{\partial v_0 \partial v_1} & \frac{\partial^2 \log L^*}{\partial v_1^2} & E(\frac{\partial^2 \log L^*}{\partial v_1 \partial \sigma}) \\ E(\frac{\partial^2 \log L^*}{\partial v_0 \partial \sigma}) & E(\frac{\partial^2 \log L^*}{\partial v_1 \partial \sigma}) & E(\frac{\partial^2 \log L^*}{\partial \sigma^2}) \end{pmatrix}, \tag{4.3.1}$$

where

$$\frac{\partial^2 \log L^*}{\partial v_0^2} = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{-s_l\beta_{n_l-s_l:n_l} - \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right\}, \tag{4.3.2}$$

$$\frac{\partial^2 \log L^*}{\partial v_1^2} = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{x_l^2\left[-s_l\beta_{n_l-s_l:n_l} - \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right]\right\}, \tag{4.3.3}$$

$$E(\frac{\partial^2 \log L^*}{\partial \sigma^2}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\{A_l - 2s_l(1-\alpha_{n_l-s_l:n_l})E(z_{n_l-s_l:n_l}) + 3s_l\beta_{n_l-s_l:n_l}E(z^2_{n_l-s_l:n_l}) + \sum_{i=1}^{n_l-s_l}[2\alpha_{i:n_l}E(z_{i:n_l}) - 3\beta_{i:n_l}E(z^2_{i:n_l})]\}, \tag{4.3.4}$$

$$\frac{\partial^2 \log L^*}{\partial v_0 \partial v_1} = \frac{1}{\sigma^2}\sum_{l=1}^{k}\left\{x_l\left[-s_l\beta_{n_l-s_l:n_l} - \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right]\right\}, \tag{4.3.5}$$

$$E(\frac{\partial^2 \log L^*}{\partial v_0 \partial \sigma}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\{-s_l[1-\alpha_{n_l-s_l:n_l} + 2\beta_{n_l-s_l:n_l}E(z_{n_l-s_l:n_l})] + \sum_{i=1}^{n_l-s_l}[\alpha_{i:n_l} - 2\beta_{i:n_l}E(z_{i:n_l})]\} \tag{4.3.6}$$

and

$$E(\frac{\partial^2 \log L^*}{\partial v_1 \partial \sigma}) = \frac{1}{\sigma^2}\sum_{l=1}^{k}\{x_l[-s_l[1-\alpha_{n_l-s_l:n_l} + 2\beta_{n_l-s_l:n_l}E(z_{n_l-s_l:n_l})] + \sum_{i=1}^{n_l-s_l}[\alpha_{i:n_l} - 2\beta_{i:n_l}E(z_{i:n_l})]]\}, \tag{4.3.7}$$

where $E(z_{i:n}) = \alpha_{i:n}$ and $E(z_{i:n}^2) = \beta_{i,i:n} + \alpha_{i:n}^2$ (see Chapter 2, Section 2.2) are the first and second moments of the i-th order statistic in a sample of size n from the standard extreme value distribution.

4.4 Approximate Bias of the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$

The approximate bias of the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ can be obtained as

$$Bias(\tilde{\nu}_0) = E(\tilde{\nu}_0) - \nu_0 = aE(\tilde{\sigma}) + E(b) - \nu_0 = ak\sigma + \nu_0 + l_1\sigma - \nu_0 = (ak + l_1)\sigma, \qquad (4.4.1)$$

$$Bias(\tilde{\nu}_1) = E(\tilde{\nu}_1) - \nu_1 = cE(\tilde{\sigma}) + E(d) - \nu_1 = ck\sigma + \nu_1 + \sigma l_2 - \nu_1 = (ck + l_2)\sigma, \qquad (4.4.2)$$

and

$$\begin{aligned} Bias(\tilde{\sigma}) = E(\tilde{\sigma}) - \sigma &\approx \frac{-E(B) + \sqrt{E(B^2) - 4AE(C)}}{2A} - \sigma \\ &= \frac{-l_B\sigma + \sqrt{l_{B^2}\sigma^2 - 4Al_C\sigma^2}}{2A} - \sigma = \sigma(k-1), \end{aligned} \qquad (4.4.3)$$

where

$$l_1 = \frac{\det\begin{bmatrix} K\sum_{l=1}^{k}\left[s_l(1-\alpha_{n_l-s_l:n_l} + \alpha_{n_l-s_l:n_l}\beta_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \alpha_{i:n_l}\beta_{i:n_l})\right] & \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] \\ K\sum_{l=1}^{k}\left\{x_l\left[s_l(1-\alpha_{n_l-s_l:n_l} + \alpha_{n_l-s_l:n_l}\beta_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \alpha_{i:n_l}\beta_{i:n_l})\right]\right\} & \sum_{l=1}^{k}\left[x_l^2\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] \end{bmatrix}}{\Delta}$$

$$l_2 = \frac{\det\begin{bmatrix} K\sum_{l=1}^{k}\left\{x_l\left[s_l(1-\alpha_{n_l-s_l:n_l} + \alpha_{n_l-s_l:n_l}\beta_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \alpha_{i:n_l}\beta_{i:n_l})\right]\right\} & \sum_{l=1}^{k}\left[x_l\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right)\right] \\ K\sum_{l=1}^{k}\left[s_l(1-\alpha_{n_l-s_l:n_l} + \alpha_{n_l-s_l:n_l}\beta_{n_l-s_l:n_l}) - \sum_{i=1}^{n_l-s_l}(\alpha_{i:n_l} - \alpha_{i:n_l}\beta_{i:n_l})\right] & \sum_{l=1}^{k}\left(s_l\beta_{n_l-s_l:n_l} + \sum_{i=1}^{n_l-s_l}\beta_{i:n_l}\right) \end{bmatrix}}{\Delta}$$

$k = \dfrac{-l_B + \sqrt{l_{B^2} - 4Al_C}}{2A}$, $A = \sum_{l=1}^{k} A_l$, $l_B = E(B)$, $l_{B^2} = E(B^2)$ and $l_C = E(C)$.

4.4.1 Derivation of l_B

To compute l_B, we need to express B in (4.2.14) as

$$B = \left\{ \sum_{l=1}^{k} \sum_{h=1}^{n_l} \omega_{h:n_l} [y_{h:n_l} - (b + dx_l)] \right.,$$

where $$\omega_{h:n_l} = \begin{cases} \alpha_{h:n_l} + \beta_{h:n_l}(a + cx_l) & \text{if } \; h \le n_l - s_l \\ \alpha_{n_l - s_l:n_l} + \beta_{n_l - s_l:n_l}(a + cx_l) - 1 \;\text{ and }\; y_{h:n_l} = y_{n_l - s_l:n_l} & \text{if } \; h > n_l - s_l \end{cases}$$

Replacing $y_{i:n_l}$ by $v_0 + v_1 x_l + \sigma z_{i:n_l}$ in formulas (4.2.7), (4.2.8), (4.2.10) and (4.2.12), we obtain

$$b = v_0 + \left[\left(\frac{\gamma}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l} - \left(\frac{q}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} \right] \sigma$$

and

$$d = v_1 + \left[\left(\frac{p}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} - \left(\frac{q}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l} \right] \sigma,$$

where $p = \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l}$, $q = \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l$, $r = \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l^2$ and $\delta = pr - q^2$.

Furthermore, we get

$$\begin{aligned} & y_{h:n_l} - b - dx_l \\ &= \left\{ z_{h:n_l} - \left(\frac{\gamma}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l} + \left(\frac{q}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} - \left(\frac{p x_l}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} + \left(\frac{q x_l}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l} \right\} \sigma \\ &= S_{h:n_l} \sigma, \end{aligned} \qquad (4.4.1.1)$$

where

$$S_{h:n_l} = z_{h:n_l} - \left(\frac{\gamma}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l} + \left(\frac{q}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} - \left(\frac{p x_l}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} x_l z_{i:n_l} + \left(\frac{q x_l}{\delta}\right) \sum_{l=1}^{k} \sum_{i=1}^{n_l} \beta_{i:n_l} z_{i:n_l}.$$

Hence, we have

$$l_B = \sigma \sum_{l=1}^{k} \sum_{h=1}^{n_l} \omega_{h:n_l} E(S_{h:n_l}). \qquad (4.4.1.2)$$

Rewriting $S_{h:n_l}$ as

$$S_{h:n_{l*}} = z_{h:n_{l*}} + \sum_{l \neq l*}^{k} \sum_{i=1}^{n_l} \left\{ \beta_{i:n_l} \left[\frac{q(x_{l*} + x_l) - p x_{l*} x_l - r}{\delta} \right] z_{i:n_l} \right\} + \sum_{i=1}^{n_{l*}} \beta_{i:n_{l*}} \left(\frac{2q x_{l*} - p x_{l*}^2 - r}{\delta} \right) z_{i:n_{l*}}$$

$$= z_{h:n_{l*}} + \sum_{l \neq l*}^{k} \sum_{i=1}^{n_l} (U_{i:n_l} z_{i:n_l}) + \sum_{i=1}^{n_{l*}} V_{i:n_{l*}} z_{i:n_{l*}},$$

we obtain

$$E(S_{h:n_l}) = \alpha_{h:n_l} + \sum_{l \neq l*}^{k} \sum_{i=1}^{n_l} (U_{i:n_l} \alpha_{i:n_l}) + \sum_{i=1}^{n_{l*}} V_{i:n_{l*}} \alpha_{i:n_{l*}}, \tag{4.4.1.3}$$

where $U_{i:n_l} = \beta_{i:n_l} \left[\frac{q(x_{l^*} + x_l) - p x_{l^*} x_l - r}{\delta} \right]$ and $V_{i:n_{l*}} = \beta_{i:n_{l*}} \left(\frac{2q x_{l*} - p x_{l*}^2 - r}{\delta} \right)$ are the constant coefficients, $l*$ indicates the specific group where the observation $z_{h:n_{l^*}}$ comes from, and $\alpha_{i:n}$ is the mean of the i-th order statistic from a sample of size n from the standard extreme value distribution

By using the formulas in (4.4.1.2) and (4.4.1.3), we can obtain the value of l_B.

4.4.2 Derivation of l_{B^2}

From the expression in (4.4.1.1), and denoting $R_{h:n_l} = \omega_{h:n_l} S_{h:n_l}$, we have

$$l_{B^2} = \sigma^2 E\left[(\sum_{l=1}^{k} \sum_{h=1}^{n_l} R_{h:n_l})^2 \right].$$

The value of $E\left[(\sum_{l=1}^{k} \sum_{h=1}^{n_l} R_{h:n_l})^2 \right]$ can then be computed as

$$E\left[(\sum_{l=1}^{k} \sum_{h=1}^{n_l} R_{h:n_l})^2 \right] = Var(\sum_{l=1}^{k} \sum_{i=1}^{n_l} R_{h:n_l}) + \left[E(\sum_{l=1}^{k} \sum_{i=1}^{n_l} R_{h:n_l}) \right]^2$$

$$= \vec{1}'_{1\times A} \Sigma \vec{1}_{A\times 1} + \vec{E}(R_{h:n_l})' \vec{1}_{A\times A} \vec{E}(R_{h:n_l}),$$

where

$$\vec{E}(R_{h:n_l}) = \left[E(R_{1:n_1}), \cdots, E(R_{n_1:n_1}) \;\; E(R_{1:n_2}), \cdots, E(R_{n_2:n_2}) \cdots\cdots E(R_{1:n_k}), \cdots, E(R_{n_k:n_k}) \right]'_{1\times A},$$

$$\bar{\Sigma}_{n_l} = \begin{bmatrix} \bar{\beta}_{1,1:n_l} & \bar{\beta}_{1,2:n_l} & \cdots & \bar{\beta}_{1,n_l:n_l} \\ \vdots & \bar{\beta}_{2,2:n_l} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \bar{\beta}_{n_l,n_l:n_l} \end{bmatrix}_{n_l \times n_l},$$

$$\bar{\Sigma} = \begin{bmatrix} \bar{\Sigma}_{n_1} & 0 & \cdots & 0 \\ 0 & \bar{\Sigma}_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\Sigma}_{n_k} \end{bmatrix}_{A \times A},$$

and

$$\bar{\beta}_{h,j:n_l} = Cov\left(R_{h:n_l}, R_{j:n_l}\right), \quad h,\ j = 1,\ \ldots,\ n_l.$$

From the fact that $E(R_{h:n_l}) = \omega_{h:n_l} E(S_{h:n_l})$ and $\bar{\beta}_{h,j:n_l} = \omega_{h:n_l}\omega_{j:n_l} Cov(S_{h:n_l}, S_{j:n_l})$, we derive the $Cov(S_{h:n_l}, S_{j:n_l})$ as follows. Once again, to distinguish the group to which the h-th and j-th observations belong from the rest of the groups, we denote that specific group as $l*$. Then,

$$\begin{aligned} & Cov(S_{h:n_{l*}}, S_{j:n_{l*}}) \\ &= Cov(z_{h:n_{l*}}, z_{j:n_{l*}}) + Cov(z_{h:n_{l*}}, \sum_{i=1}^{n_{l*}} (V_{i:n_{l*}} z_{i:n_{l*}}) + Cov(z_{j:n_{l*}}, \sum_{i=1}^{n_{l*}} (V_{i:n_{l*}} z_{i:n_{l*}}) + Var(\sum_{l=1}^{k}\sum_{i=1}^{n_l} U_{i:n_l}, z_{h:n_l}) \\ &= \beta_{h,j:n_{l*}} + \sum_{i=1}^{n_{l*}} (V_{i:n_{l*}} \beta_{h,i:n_{l*}}) + \sum_{i=1}^{n_{l*}} (V_{i:n_{l*}} \beta_{j,i:n_{l*}}) + \vec{U}\Sigma\vec{U}, \end{aligned}$$

where $\beta_{h,j:n_{l*}}$ is the covariance between h and j-th order statistics from a sample of size n_{l*} from the standard extreme value distribution, Σ is as defined in (2.2.8), and

$$\vec{U} = \begin{bmatrix} U_{1:n_1}, \cdots, U_{n_1:n_1} & U_{1:n_2}, \cdots, U_{n_2:n_2} \cdots\cdots U_{1:n_k}, \cdots, U_{n_k:n_k} \end{bmatrix}'_{1 \times A}.$$

4.4.3 Derivation of l_C

To compute l_C, we need to express C in (4.2.15) as

$$C = \sum_{l=1}^{k}\sum_{h=1}^{n_l} \beta_{h:n_l} [y_{h:n_l} - (b + dx_l)]^2,$$

where
$$\begin{cases} \beta_{h:n_l} = \beta_{h:n_l} \quad and \quad y_{h:n_l} = y_{h:n_l} & if \quad h \le n_l - s_l \\ \beta_{h:n_l} = \beta_{n_l - s_l : n_l} \quad and \quad y_{h:n_l} = y_{n_l - s_l : n_l} & if \quad h > n_l - s_l \end{cases} \tag{4.4.3.1}$$

From the expressions in (4.4.1.1) and (4.4.3.1), we have

$$l_C = \sigma^2 \sum_{l=1}^{k} \sum_{h=1}^{n_l} \beta_{h:n_l} E(S_{h:n_l}^2).$$

To evaluate l_C, we need the value of $E(S_{h:n_l}^2)$ which can be computed as

$$E(S_{h:n_l}^2) = Var(S_{h:n_l}) + [E(S_{h:n_l})]^2.$$

The value of $[E(S_{h:n_l})]^2$ can be computed from (4.4.1.3), and

$$Var(S_{h:n_l}) = Var(z_{h:n_l}) + Var(\sum_{l \ne l^*} \sum_{i=1}^{n_l} U_{i:n_l} z_{i:n_l}) + Var(\sum_{i=1}^{n_{l^*}} V_{i:n_{l^*}} z_{i:n_{l^*}}) + 2Cov(z_{h:n_l}, \sum_{i=1}^{n_{l^*}} (V_{i:n_{l^*}} z_{h,i:n_{l^*}})$$
$$= \beta_{h,h:n_l} + Q'\Sigma Q + 2\sum_{i=1}^{n_{l^*}} (V_{i:n_{l^*}} \beta_{h,i:n_{l^*}});$$

here Σ is as defined in (2.2.8), Q is a vector that consists of all $U_{i:n_l}$'s for $i = 1, \ldots, n_l$, $l = 1, \ldots, k$, and the $U_{i:n_l}$'s will be replaced by $V_{h:n_{l^*}}$'s when $l = l^*$.

All the above results for AMLE in the complete sample situation may simply be deduced from the above formulas by setting $s_l = 0$ for $l = 1, \ldots, k$.

4.5 Simulations and Results

In the simulation study, we took $v_0 = 0, v_1 = 1$ and $\sigma = 1$ and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored sample. In order to study the AMLEs in the MEVR model, we performed the simulations based on 10,000 Monte Carlo runs for each of the following cases:

1. Complete samples

two groups: n = [6 6(1)10], [7 7(1)10], [8 8(1)10], [9 9(1)10], [10 10], [15 15(5)20] and [20 20] .

four groups: n = [6×2 6(1)10×2], [7×2 7(1)10×2], [8×2 8(1)10×2], [9×2 9(1)10×2], [10 10 10 10], [15×2 15(5)20×2] and [20 20 20 20] .

2. Type-II right-censored samples

two groups: s = [4 4(1)0], [3 3(1)0], [2 2(1)0] and [1 1(1)0] from n = [10 10] and [5 5(1)0] from n = [20 20].

four groups: s = [4×2 4(1)0×2], [3×2 3(1)0×2], [2×2 2(1)0×2] and [1×2 1(1)0×2] from n = [10 10 10 10] and [5×2 5(1)0×2] from n = [20 20 20 20].

We generated order statistics from the standard extreme value sample $z_{i:n_l}$, $i=1,\dots, n_l$, $l = 1,\dots, k$, and then using the model $y_{i:n_l} = \nu_o + \nu_1 x_l + \sigma z_{i:n_l}$ transformed the sample into $y_{i:n_l}$, $i=1,\dots, n_l$, $l = 1,\dots, k$. Upon using the formulas in (4.2.4) – (4.2.6), we obtained the values of AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$. Based on 10,000 runs, we determined the values of (1) $Bias(\tilde{\nu}_0)/\sigma$, (2) $Bias(\tilde{\nu}_1)/\sigma$, (3) $Bias(\tilde{\sigma})/\sigma$, (4) $MSE(\tilde{\nu}_0)/\sigma^2$, (5) $MSE(\tilde{\nu}_1)/\sigma^2$, (6) $MSE(\tilde{\sigma})/\sigma^2$, (7) $Var(\tilde{\nu}_0)/\sigma^2$, (8) $Var(\tilde{\nu}_1)/\sigma^2$, (9) $Var(\tilde{\sigma})/\sigma^2$, (10) $Cov(\tilde{\nu}_0,\tilde{\nu}_1)/\sigma^2$, (11) $Cov(\tilde{\nu}_0,\tilde{\sigma})/\sigma^2$, and (12) $Cov(\tilde{\nu}_1,\tilde{\sigma})/\sigma^2$. The approximate values of (1) – (3) and (7) – (12) were also computed by the formulas in (4.4.1) – (4.4.3) and the inverse of (4.3.1), respectively. These results are presented in Tables 4.5.1 – 4.5.12.

From these tables, we observe the following points. The approximate biases tend to increase in AMLEs of ν_0 and ν_1 and decrease in σ with increasing number of levels of x when the groups are of the same size in the complete samples. The biases of all estimators tend to decrease with increasing number of levels of x when the amounts of censoring are of the same size in the Type-II censored samples. The biases of all estimators tend to decrease with a major increase in N. In addition, with the same N, the biases of all estimators tend to decrease in the more balanced groups than among the less balanced groups and moreover for σ, the biases tend to be the same in the complete samples. The biases of AMLEs in ν_0 and σ tend to increase whereas decrease in ν_1 with increasing amounts of censoring in the Type-II censored samples. This is true in case of both two- and four-levels of x.

The approximate variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a major increase in N, except for σ in the Type-II censored samples. In addition, with the same N, expect for σ, the variances tend to have smaller values in the more balanced groups than in the less balanced groups; moreover for σ, the variances tend to be the same or decrease with the more balanced groups in the complete samples and increase with the more balanced groups in the Type-II censored samples. The variances of AMLEs in ν_0 and ν_1 tend to increase whereas decrease for σ with increasing amounts of censoring. This is true in case of both two- and four-levels of x.

It should be mentioned here that such approximate (modified) maximum likelihood estimators have been derived for a wide range of other distributions. Interested readers may refer to Tiku, Tan and Balakrishnan (1986) and Balakrishnan and Cohen (1991) for a detailed description of these estimators.

Since the estimators $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ are in closed form, they are used as initial guess for the Newton-Raphson procedure to obtain the MLEs discussed in Section 3.2. It is found that these approximations were very good as the convergence occurred within ten iterations in all cases examined here.

Table 4.5.1 AMLE procedure: Approximate Bias, Simulated Bias and MSE for two-grouped complete sample

$[n_1\ n_2]$	Approximate			Simulated					
	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$MSE(\tilde{v}_0)/\sigma^2$	$MSE(\tilde{v}_1)/\sigma^2$	$MSE(\tilde{\sigma})/\sigma^2$
[6 6]	-0.1498	0.0000	-0.0672	-0.1460	0.0077	-0.0897	0.1240	0.3756	0.0621
[6 7]	-0.1400	0.0222	-0.0621	-0.1349	0.0216	-0.0840	0.1123	0.3394	0.0574
[6 8]	-0.1327	0.0393	-0.0579	-0.1262	0.0426	-0.0806	0.1040	0.3236	0.0536
[6 9]	-0.1269	0.0527	-0.0542	-0.1202	0.0689	-0.0728	0.0971	0.3057	0.0494
[6 10]	-0.1223	0.0637	-0.0510	-0.1192	0.0579	-0.0703	0.0936	0.2966	0.0463
[7 7]	-0.1301	0.0000	-0.0578	-0.1278	0.0049	-0.0787	0.0998	0.3083	0.0528
[7 8]	-0.1227	0.0170	-0.0541	-0.1199	0.0163	-0.0710	0.0948	0.2930	0.0491
[7 9]	-0.1168	0.0306	-0.0509	-0.1118	0.0342	-0.0697	0.0878	0.2814	0.0456
[7 10]	-0.1121	0.0415	-0.0481	-0.1041	0.0438	-0.0673	0.0837	0.2597	0.0437
[8 8]	-0.1151	0.0000	-0.0508	-0.1078	0.0088	-0.0693	0.0871	0.2746	0.0450
[8 9]	-0.1091	0.0135	-0.0480	-0.1032	0.0144	-0.0653	0.0802	0.2530	0.0423
[8 10]	-0.1044	0.0245	-0.0455	-0.1013	0.0215	-0.0608	0.0771	0.2421	0.0394
[9 9]	-0.1031	0.0000	-0.0455	-0.1002	-0.0069	-0.0596	0.0763	0.2435	0.0402
[9 10]	-0.0983	0.0110	-0.0433	-0.0911	0.0142	-0.0562	0.0713	0.2215	0.0371
[10 10]	-0.0934	0.0000	-0.0413	-0.0851	0.0057	-0.0572	0.0649	0.2103	0.0357
[15 15]	-0.0643	0.0000	-0.0267	-0.0621	0.0031	-0.0375	0.0418	0.1408	0.0219
[15 20]	-0.0567	0.0175	-0.0233	-0.0546	0.0189	-0.0322	0.0362	0.1219	0.0187
[20 20]	-0.0488	0.0000	-0.0207	-0.0470	-0.0003	-0.0289	0.0301	0.1042	0.0162

Table 4.5.2 AMLE procedure: Simulated variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
[6 6]	0.1027	0.3756	0.0541	-0.0040	-0.0237	0.0015
[6 7]	0.0941	0.3390	0.0504	-0.0119	-0.0209	-0.0017
[6 8]	0.0881	0.3218	0.0471	-0.0272	-0.0209	-0.0009
[6 9]	0.0826	0.3010	0.0441	-0.0309	-0.0191	0.0017
[6 10]	0.0794	0.2933	0.0414	-0.0410	-0.0178	0.0016
[7 7]	0.0835	0.3083	0.0466	0.0010	-0.0192	-0.0007
[7 8]	0.0804	0.2928	0.0441	-0.0101	-0.0186	-0.0007
[7 9]	0.0753	0.2803	0.0408	-0.0201	-0.0170	-0.0011
[7 10]	0.0728	0.2578	0.0392	-0.0263	-0.0168	0.0012
[8 8]	0.0755	0.2746	0.0402	-0.0012	-0.0180	-0.0002
[8 9]	0.0695	0.2528	0.0380	-0.0039	-0.0161	-0.0014
[8 10]	0.0669	0.2416	0.0357	-0.0133	-0.0154	0.0000
[9 9]	0.0662	0.2435	0.0367	0.0008	-0.0153	-0.0011
[9 10]	0.0630	0.2213	0.0340	-0.0085	-0.0147	0.0012
[10 10]	0.0576	0.2103	0.0325	-0.0001	-0.0143	0.0003
[15 15]	0.0380	0.1408	0.0205	-0.0004	-0.0088	-0.0007
[15 20]	0.0332	0.1216	0.0177	-0.0091	-0.0072	0.0008
[20 20]	0.0279	0.1042	0.0154	-0.0006	-0.0065	0.0007

Table 4.5.3 AMLE procedure: Asymptotic variances and covariances for two-grouped complete sample

$[n_1\ n_2]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
[6 6]	0.0983	0.3924	0.0476	0.0000	0.0033	0.0000
[6 7]	0.0906	0.3622	0.0432	-0.0153	0.0017	-0.0024
[6 8]	0.0849	0.3400	0.0404	-0.0265	0.0008	-0.0040
[6 9]	0.0806	0.3231	0.0379	-0.0350	0.0000	-0.0051
[6 10]	0.0772	0.3099	0.0357	-0.0417	-0.0005	-0.0058
[7 7]	0.0829	0.3316	0.0395	0.0000	0.0005	0.0000
[7 8]	0.0773	0.3093	0.0371	-0.0112	-0.0003	-0.0016
[7 9]	0.0730	0.2923	0.0350	-0.0197	-0.0009	-0.0027
[7 10]	0.0697	0.2789	0.0332	-0.0264	-0.0014	-0.0036
[8 8]	0.0717	0.2868	0.0350	0.0000	-0.0011	0.0000
[8 9]	0.0675	0.2697	0.0332	-0.0085	-0.0016	-0.0012
[8 10]	0.0641	0.2562	0.0315	-0.0152	-0.0020	-0.0020
[9 9]	0.0633	0.2525	0.0315	0.0000	-0.0021	0.0000
[9 10]	0.0599	0.2389	0.0300	-0.0067	-0.0024	-0.0009
[10 10]	0.0566	0.2254	0.0286	0.0000	-0.0027	0.0000
[15 15]	0.0372	0.1461	0.0196	0.0000	-0.0036	0.0000
[15 20]	0.0325	0.1270	0.0169	-0.0094	-0.0036	-0.0008
[20 20]	0.0278	0.1078	0.0149	0.0000	-0.0035	0.0000

Table 4.5.4 AMLE procedure: Approximate Bias, Simulated Bias and MSE for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	Approximate			Simulated					
	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Bias(\tilde{v}_0)/\sigma$	$Bias(\tilde{v}_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$MSE(\tilde{v}_0)/\sigma^2$	$MSE(\tilde{v}_1)/\sigma^2$	$MSE(\tilde{\sigma})/\sigma^2$
[6 6 6 6]	-0.1625	0.0000	-0.0196	-0.1570	0.0058	-0.0307	0.0758	0.3319	0.0291
[6 6 7 7]	-0.1519	0.0259	-0.0181	-0.1505	0.0288	-0.0285	0.0751	0.3279	0.0286
[6 6 8 8]	-0.1434	0.0456	-0.0169	-0.1393	0.0451	-0.0292	0.0729	0.3211	0.0280
[6 6 9 9]	-0.1365	0.0610	-0.0159	-0.1350	0.0570	-0.0243	0.0709	0.3149	0.0272
[6 6 10 10]	-0.1307	0.0733	-0.0150	-0.1285	0.0658	-0.0250	0.0691	0.3085	0.0265
[7 7 7 7]	-0.1416	0.0000	-0.0169	-0.1426	0.0020	-0.0265	0.0685	0.3064	0.0263
[7 7 8 8]	-0.1334	0.0199	-0.0159	-0.1291	0.0187	-0.0263	0.0674	0.3026	0.0259
[7 7 9 9]	-0.1268	0.0356	-0.0150	-0.1248	0.0391	-0.0232	0.0663	0.2974	0.0254
[7 7 10 10]	-0.1212	0.0482	-0.0143	-0.1176	0.0537	-0.0225	0.0650	0.2926	0.0249
[8 8 8 8]	-0.1255	0.0000	-0.0151	-0.1245	0.0046	-0.0225	0.0641	0.2891	0.0246
[8 8 9 9]	-0.1190	0.0158	-0.0143	-0.1165	0.0197	-0.0237	0.0630	0.2853	0.0243
[8 8 10 10]	-0.1136	0.0286	-0.0136	-0.1123	0.0331	-0.0216	0.0619	0.2805	0.0239
[9 9 9 9]	-0.1127	0.0000	-0.0137	-0.1086	-0.0015	-0.0202	0.0608	0.2765	0.0236
[9 9 10 10]	-0.1074	0.0129	-0.0131	-0.1020	0.0093	-0.0209	0.0597	0.2723	0.0232
[10 10 10 10]	-0.1023	0.0000	-0.0126	-0.1043	-0.0020	-0.0175	0.0587	0.2682	0.0229
[15 15 15 15]	-0.0708	0.0000	-0.0070	-0.0701	0.0037	-0.0135	0.0243	0.1240	0.0107
[15 15 20 20]	-0.0622	0.0205	-0.0064	-0.0598	0.0241	-0.0127	0.0204	0.1099	0.0092
[20 20 20 20]	-0.0540	0.0000	-0.0059	-0.0527	0.0043	-0.0106	0.0173	0.0929	0.0079

Table 4.5.5 AMLE procedure: Simulated variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(\tilde{V}_0)/\sigma^2$	$Var(\tilde{V}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{V}_0,\tilde{V}_1)/\sigma^2$	$Cov(\tilde{V}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{V}_1,\tilde{\sigma})/\sigma^2$
[6 6 6 6]	0.0528	0.3383	0.0286	-0.0004	-0.0145	0.0010
[6 6 7 7]	0.0494	0.3117	0.0261	-0.0112	-0.0126	0.0024
[6 6 8 8]	0.0433	0.2864	0.0241	-0.0140	-0.0112	0.0004
[6 6 9 9]	0.0410	0.2754	0.0223	-0.0198	-0.0107	0.0013
[6 6 10 10]	0.0402	0.2603	0.0207	-0.0228	-0.0098	0.0012
[7 7 7 7]	0.0437	0.2905	0.0239	0.0022	-0.0118	-0.0011
[7 7 8 8]	0.0406	0.2685	0.0219	-0.0089	-0.0103	0.0019
[7 7 9 9]	0.0405	0.2450	0.0204	-0.0118	-0.0099	0.0012
[7 7 10 10]	0.0367	0.2374	0.0187	-0.0161	-0.0093	0.0008
[8 8 8 8]	0.0378	0.2476	0.0206	-0.0006	-0.0099	0.0011
[8 8 9 9]	0.0349	0.2366	0.0194	-0.0055	-0.0090	-0.0006
[8 8 10 10]	0.0339	0.2133	0.0184	-0.0072	-0.0090	0.0008
[9 9 9 9]	0.0337	0.2167	0.0185	0.0003	-0.0092	0.0006
[9 9 10 10]	0.0316	0.2062	0.0174	-0.0023	-0.0084	0.0002
[10 10 10 10]	0.0304	0.1985	0.0166	-0.0004	-0.0079	-0.0000
[15 15 15 15]	0.0194	0.1240	0.0105	0.0000	-0.0050	-0.0001
[15 15 20 20]	0.0168	0.1093	0.0090	-0.0054	-0.0043	0.0004
[20 20 20 20]	0.0145	0.0929	0.0078	0.0001	-0.0037	0.0002

Table 4.5.6 AMLE procedure: Asymptotic variances and covariances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
[6 6 6 6]	0.0492	0.3560	0.0238	0.0000	0.0016	0.0000
[6 6 7 7]	0.0452	0.3280	0.0216	-0.0091	0.0009	-0.0014
[6 6 8 8]	0.0422	0.3068	0.0202	-0.0158	0.0003	-0.0024
[6 6 9 9]	0.0399	0.2900	0.0189	-0.0207	0.0000	-0.0030
[6 6 10 10]	0.0380	0.2763	0.0179	-0.0245	-0.0003	-0.0034
[7 7 7 7]	0.0415	0.3008	0.0198	0.0000	0.0002	0.0000
[7 7 8 8]	0.0386	0.2802	0.0186	-0.0067	0.0002	-0.0010
[7 7 9 9]	0.0364	0.2640	0.0175	-0.0118	-0.0005	-0.0016
[7 7 10 10]	0.0346	0.2509	0.0166	-0.0157	-0.0007	-0.0021
[8 8 8 8]	0.0359	0.2602	0.0175	0.0000	-0.0005	0.0000
[8 8 9 9]	0.0337	0.2444	0.0166	-0.0051	-0.0008	-0.0007
[8 8 10 10]	0.0320	0.2317	0.0157	-0.0091	-0.0010	-0.0012
[9 9 9 9]	0.0316	0.2290	0.0157	0.0000	-0.0010	0.0000
[9 9 10 10]	0.0299	0.2166	0.0150	-0.0040	-0.0012	-0.0005
[10 10 10 10]	0.0283	0.2044	0.0143	0.0000	-0.0014	0.0000
[15 15 15 15]	0.0186	0.1325	0.0098	0.0000	-0.0018	0.0000
[15 15 20 20]	0.0162	0.1146	0.0085	-0.0056	-0.0018	-0.0005
[20 20 20 20]	0.0139	0.0978	0.0074	0.0000	-0.0018	0.0000

Table 4.5.7 AMLE procedure: Approximate Bias, Simulated Bias and MSE for two-grouped Type-II censored sample

	Approximate			Simulated					
$[s_1 \ s_2]$	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\sigma)/\sigma$	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$MSE(\tilde{v}_0)/\sigma^2$	$MSE(\tilde{v}_1)/\sigma^2$	$MSE(\tilde{\sigma})/\sigma^2$
4 4	-0.1490	0.0000	-0.0849	-0.1617	0.0060	-01080	0.1242	0.3678	0.0770
4 3	-0.1334	0.0237	-0.0748	-0.1405	0.0349	-0.0998	0.1069	0.3374	0.0698
4 2	-0.1225	0.0390	-0.0663	-0.1266	0.0406	-0.0868	0.0956	0.3181	0.0638
4 1	-0.1156	0.0476	-0.0594	-0.1216	0.0536	-0.0797	0.0896	0.3132	0.0554
4 0	-0.1135	0.0490	-0.0558	-0.1158	0.0530	-0.0725	0.0844	0.3173	0.0489
3 3	-0.1199	0.0000	-0.0664	-0.1233	0.0019	-0.0867	0.0941	0.3104	0.0632
3 2	-0.1107	0.0157	-0.0593	-0.1049	0.0151	-0.0815	0.0825	0.2920	0.0570
3 1	-0.1051	0.0248	-0.0535	-0.1054	0.0420	-0.0724	0.0788	0.2741	0.0510
3 0	-0.1038	0.0264	-0.0507	-0.1041	0.0270	-0.0655	0.0770	0.2723	0.0459
2 2	-0.1028	0.0000	-0.0533	-0.1022	-0.0054	-0.0679	0.0781	0.2673	0.0516
2 1	-0.0981	0.0093	-0.0484	-0.0950	0.0019	-0.0636	0.0729	0.2517	0.0476
2 0	-0.0973	0.0108	-0.0463	-0.0906	0.0143	-0.0634	0.0681	0.2467	0.0422
1 1	-0.0941	0.0000	-0.0442	-0.0882	-0.0000	-0.0629	0.0680	0.2364	0.0435
1 0	-0.0937	0.0013	-0.0427	-0.0991	0.0133	-0.0585	0.0683	0.2272	0.0385
*5 5	-0.0546	0.0000	-0.0299	-0.0900	-0.0013	-0.0581	0.0383	0.1414	0.0270
*5 0	-0.0508	0.0070	-0.0252	-0.0565	-0.0015	-0.0401	0.0339	0.1251	0.0207

Asterisk denotes censoring is from n = *[20 20]*, otherwise is from *[10 10]*

Table 4.5.8 AMLE procedure: Simulated variances and covariances for two-grouped Type-II censored sample

$[s_1 \ \ s_2]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
4 4	0.0981	0.3678	0.0653	0.0036	0.0214	-0.0002
4 3	0.0871	0.3362	0.0598	0.0155	0.0132	-0.0108
4 2	0.0796	0.3165	0.0563	0.0274	0.0067	-0.0220
4 1	0.0748	0.3104	0.0491	0.0318	0.0026	-0.0284
4 0	0.0711	0.3145	0.0436	0.0360	-0.0019	-0.0325
3 3	0.0789	0.3105	0.0557	0.0007	0.0057	0.0004
3 2	0.0715	0.2918	0.0503	0.0101	0.0010	-0.0098
3 1	0.0677	0.2723	0.0458	0.0158	-0.0032	-0.0172
3 0	0.0662	0.2716	0.0416	0.0202	-0.0060	-0.0239
2 2	0.0677	0.2673	0.0470	0.0012	-0.0022	0.0002
2 1	0.0639	0.2517	0.0436	0.0028	-0.0066	-0.0092
2 0	0.0599	0.2465	0.0382	0.0104	-0.0096	-0.0139
1 1	0.0602	0.2364	0.0395	0.0003	-0.0095	-0.0005
1 0	0.0600	0.2270	0.0351	0.0034	-0.0118	-0.0059
*5 5	0.0351	0.1414	0.0254	-0.0008	0.0010	0.0004
*5 0	0.0312	0.1250	0.0195	-0.0077	-0.0040	-0.0085

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 4.5.9 AMLE procedure: Asymptotic variances and covariances for two-grouped Type-II censored sample

$[s_1 \ s_2]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
4 4	0.1091	0.3562	0.0536	0.0000	0.0328	0.0000
4 3	0.0956	0.3339	0.0491	-0.0174	0.0249	-0.0103
4 2	0.0862	0.3208	0.0449	-0.0286	0.0186	-0.0177
4 1	0.0796	0.3132	0.0409	-0.0356	0.0136	-0.0230
4 0	0.0752	0.3094	0.0373	-0.0395	0.0096	-0.0263
3 3	0.0842	0.3073	0.0452	0.0000	0.0182	0.0000
3 2	0.0763	0.2908	0.0416	-0.0114	0.0129	-0.0077
3 1	0.0709	0.2806	0.0382	-0.0188	0.0087	-0.0134
3 0	0.0674	0.2750	0.0350	-0.0232	0.0054	-0.0174
2 2	0.0697	0.2714	0.0386	0.0000	0.0084	0.0000
2 1	0.0651	0.2589	0.0356	-0.0075	0.0048	-0.0059
2 0	0.0622	0.2516	0.0328	-0.0121	0.0020	-0.0103
1 1	0.0612	0.2445	0.0331	0.0000	0.0017	0.0000
1 0	0.0587	0.2356	0.0307	-0.0047	-0.0007	-0.0045
*5 5	0.0355	0.1390	0.0228	0.0000	0.0040	0.0000
*5 0	0.0309	0.1265	0.0180	-0.0076	-0.0005	-0.0074

Asterisk denotes censoring is from n *=[20 20]*, otherwise is from *[10 10]*

Table 4.5.10 AMLE procedure: Approximate Bias, Simulated Bias and MSE for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	Approximate			Simulated					
	$Bias(\tilde{v}_0)/\sigma$	$Bias(\tilde{v}_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Bias(\tilde{v}_0)/\sigma$	$Bias(v_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$MSE(\tilde{v}_0)/\sigma^2$	$MSE(\tilde{v}_1)/\sigma^2$	$MSE(\tilde{\sigma})/\sigma^2$
4 4 4 4	-0.1254	0.0000	-0.0225	-0.1315	-0.0072	-0.0370	0.0641	0.3263	0.0353
4 4 3 3	-0.1174	0.0155	-0.0187	-0.1235	0.0231	-0.0304	0.0567	0.2998	0.0315
4 4 2 2	-0.1123	0.0245	-0.0157	-0.1125	0.0324	-0.0236	0.0517	0.2967	0.0287
4 4 1 1	-0.1099	0.0279	-0.0138	-0.1110	0.0322	-0.0257	0.0496	0.2783	0.0256
4 4 0 0	-0.1115	0.0252	-0.0153	-0.1137	0.0327	-0.0250	0.0478	0.2744	0.0223
3 3 3 3	-0.1104	0.0000	-0.0158	-0.1140	-0.0006	-0.0258	0.0508	0.2759	0.0287
3 3 2 2	-0.1061	0.0092	-0.0134	-0.1073	0.0078	-0.0227	0.0470	0.2634	0.0259
3 3 1 1	-0.1042	0.0129	-0.0120	-0.1054	0.0138	-0.0208	0.0465	0.2501	0.0233
3 3 0 0	-0.1058	0.0096	-0.0136	-0.1040	0.0066	-0.0212	0.0435	0.2416	0.0204
2 2 2 2	-0.1023	0.0000	-0.0114	-0.1011	0.0011	-0.0183	0.0433	0.2462	0.0244
2 2 1 1	-0.1007	0.0037	-0.0104	-0.1008	-0.0061	-0.0213	0.0418	0.2222	0.0219
2 2 0 0	-0.1023	-0.0002	-0.0121	-0.1006	0.0029	-0.0171	0.0417	0.2208	0.0200
1 1 1 1	-0.0993	0.0000	-0.0095	-0.0952	-0.0051	-0.0181	0.0401	0.2099	0.0204
1 1 0 0	-0.1009	-0.0043	-0.0112	-0.1024	-0.0073	-0.0162	0.0412	0.2082	0.0183
*5 5 5 5	-0.0529	0.0000	-0.0065	-0.1003	0.0010	-0.0192	0.0197	0.1260	0.0128
*5 5 0 0	-0.0533	-0.0007	-0.0072	-0.0522	0.0043	-0.0105	0.0186	0.1121	0.0095

Asterisk denotes censoring is from n *=[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 4.5.11 AMLE procedure: Simulated variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
4 4 4 4	0.0468	0.3262	0.0340	-0.0009	0.0081	-0.0006
4 4 3 3	0.0415	0.2993	0.0306	-0.0083	0.0045	-0.0040
4 4 2 2	0.0390	0.2957	0.0281	-0.0149	0.0011	-0.0126
4 4 1 1	0.0373	0.2773	0.0250	-0.0185	-0.0010	-0.0149
4 4 0 0	0.0349	0.2734	0.0217	-0.0191	-0.0032	-0.0178
3 3 3 3	0.0378	0.2759	0.0280	-0.0002	0.0012	0.0001
3 3 2 2	0.0355	0.2634	0.0254	-0.0056	-0.0007	-0.0047
3 3 1 1	0.0354	0.2499	0.0228	-0.0102	-0.0023	-0.0099
3 3 0 0	0.0326	0.2416	0.0200	-0.0098	-0.0045	-0.0119
2 2 2 2	0.0331	0.2462	0.0240	0.0005	-0.0031	0.0020
2 2 1 1	0.0317	0.2222	0.0214	-0.0040	-0.0049	-0.0054
2 2 0 0	0.0316	0.2208	0.0197	-0.0058	-0.0061	-0.0072
1 1 1 1	0.0311	0.2099	0.0201	-0.0004	-0.0057	0.0003
1 1 0 0	0.0307	0.2081	0.0181	-0.0021	-0.0069	-0.0043
*5 5 5 5	0.0170	0.1260	0.0127	0.0000	-0.0003	0.0002
*5 5 0 0	0.0155	0.1121	0.0094	-0.0038	-0.0021	-0.0051

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

Table 4.5.12 AMLE procedure: Asymptotic variances and covariances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_0,\tilde{v}_1)/\sigma^2$	$Cov(\tilde{v}_0,\tilde{\sigma})/\sigma^2$	$Cov(\tilde{v}_1,\tilde{\sigma})/\sigma^2$
4 4 4 4	0.0546	0.3231	0.0268	0.0000	0.0164	0.0000
4 4 3 3	0.0477	0.3021	0.0245	-0.0104	0.0124	-0.0061
4 4 2 2	0.0428	0.2885	0.0223	-0.0170	0.0091	-0.0105
4 4 1 1	0.0393	0.2794	0.0203	-0.0209	0.0065	-0.0135
4 4 0 0	0.0370	0.2737	0.0184	-0.0231	0.0044	-0.0154
3 3 3 3	0.0421	0.2788	0.0226	0.0000	0.0091	0.0000
3 3 2 2	0.0381	0.2633	0.0208	-0.0068	0.0064	-0.0046
3 3 1 1	0.0353	0.2529	0.0191	-0.0112	0.0042	-0.0080
3 3 0 0	0.0335	0.2465	0.0174	-0.0137	0.0025	-0.0103
2 2 2 2	0.0348	0.2461	0.0193	0.0000	0.0042	0.0000
2 2 1 1	0.0325	0.2345	0.0178	-0.0045	0.0024	-0.0035
2 2 0 0	0.0310	0.2272	0.0164	-0.0072	0.0010	-0.0061
1 1 1 1	0.0306	0.2217	0.0166	0.0000	0.0008	0.0000
1 1 0 0	0.0294	0.2135	0.0153	-0.0028	-0.0003	-0.0027
*5 5 5 5	0.0177	0.1261	0.0114	0.0000	0.0020	0.0000
*5 5 0 0	0.0154	0.1137	0.0090	-0.0045	-0.0003	-0.0044

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

CHAPTER 5

INTERVAL ESTIMATION AND PROBABILITY COVERAGE

5.1 Introduction

In this chapter, for the MEVR model, the construction of confidence intervals for the parameters ν_0, ν_1 and σ using the BLUE, MLE and AMLE of these parameters are discussed. In constructing the confidence intervals for the location and scale parameters, the pivotal quantities based on equivariant estimators play an important role. Therefore, in Sections 5.2 and 5.3, the definition of equivariant estimators, pivotal quantities and a related theorem are presented. Since all the estimators we discussed before are equivariant estimators and approximately normally distributed, we show in Section 5.4 that the confidence intervals can be easily constructed from these pivotal quantities which are also approximately normally distributed. Moreover, we use the probability coverages in this section to examine the accuracy of these interval estimation procedures (i.e. the accuracy of the normal approximation in terms of probability coverages for all these estimators). Finally, we conduct a simulation study to evaluate the performance of the probability coverages of all these estimators for various choices of sample sizes and censoring schemes in Section 5.5.

5.2 Equivariant Estimators

In constructing confidence intervals or conducting tests of hypotheses for the location and scale parameters, pivotal quantities based on equivariant estimators play an important role.

Consider a Type-II right-censored sample

$$y_{1:n} \le y_{2:n} \le \ldots \le y_{n-s:n} \tag{5.2.1}$$

from a location-scale family with density $f(y;\mu,\sigma)=\frac{1}{\sigma}g\left(\frac{y-\mu}{\sigma}\right)$, $-\infty < y < \infty$.

Suppose that $\tilde{\tilde{\mu}}=\tilde{\tilde{\mu}}(y_{1:n},\ldots,y_{n-s:n})$ and $\tilde{\tilde{\sigma}}=\tilde{\tilde{\sigma}}(y_{1:n},\ldots,y_{n-s:n})$ form a pair of estimators of μ and σ which have the property that for any real constant c ($-\infty < y < \infty$) and d ($d>0$),

$$\tilde{\tilde{\mu}}(dy_{1:n}+c,\ldots,dy_{n-s:n}+c)=d\tilde{\tilde{\mu}}(y_{1:n},\ldots,y_{n-s:n})+c \tag{5.2.2}$$

and

$$\tilde{\tilde{\sigma}}(dy_{1:n}+c,\ldots,dy_{n-s:n}+c)=d\tilde{\tilde{\sigma}}(y_{1:n},\ldots,y_{n-s:n}). \tag{5.2.3}$$

Then, $\tilde{\tilde{\mu}}$ and $\tilde{\tilde{\sigma}}$ are termed as equivariant estimators of μ and σ (Zacks, 1971; Lawless, 1982).

The requirements (5.2.2) and (5.2.3) are natural ones for estimators of location and scale parameters and most, if not all, of the common estimators satisfy them.

It has been proved that the BLUEs, MLEs and AMLEs we have discussed in previous chapters are all equivariant estimators (Lawless, 1982; Chan, 1993).

5.3 Pivotal Quantities

The following theorem is very useful in constructing confidence intervals for the location and scale parameters.

***Theorem* 5.3.1** *Let $\tilde{\tilde{\mu}}$ and $\tilde{\tilde{\sigma}}$ be equivariant estimators, based on a Type-II right-censored sample given in* (5.2.1). *Then*

1. $Z_1 = (\tilde{\tilde{\mu}} - \mu)/\tilde{\tilde{\sigma}}$ *and* $Z_2 = \tilde{\tilde{\sigma}}/\sigma$ *are pivotal (parameter free) quantities.*

2. *The quantities* $a_i = (y_{i:n} - \tilde{\tilde{\mu}})/\tilde{\tilde{\sigma}}$, $i = 1,\ldots,n\text{-}s$, *form a set of ancillary statistics, only n-s-2 of which are functionally independent.*

For a proof of this theorem, see Lawless (1982).

Since the BLUE, MLE and AMLE are all equivariant, Z_1 and Z_2 based on them are pivotal quantities by Theorem 5.3.1. Hence, if we know the distribution of Z_1 and Z_2, then the construction of confidence intervals for μ and σ is straightforward. For example, if we know the values of a_{z_1} and b_{z_1} such that

$$\Pr\left(a_{z_1} < Z_1 < b_{z_1}\right) = p,$$

then the confidence interval for μ with confidence $100p\%$ is simply

$$\left[\tilde{\tilde{\mu}} - b_{z_1}\tilde{\tilde{\sigma}}, \tilde{\tilde{\mu}} - a_{z_1}\tilde{\tilde{\sigma}}\right].$$

Unfortunately, the distribution of Z_1 and Z_2 are not mathematically tractable expect in few cases (e.g., the pivotal quantities based on the MLE's in the case of exponential distribution or from a complete sample from normal distribution). Therefore, one may obtain the distribution of Z_1 and Z_2 by either simulations or approximations.

5.4 Probability Coverages

In the MEVR model, we have the asymptotic normality of the estimators of ν_0, ν_1 and σ based on the methods of BLUE, MLE and AMLE. Therefore, the asymptotic distributions of the following pivotal quantities

$$p_1 = \frac{\tilde{\tilde{\nu}}_0 - \nu_0}{\tilde{\tilde{\sigma}}\sqrt{V_{11}}} \quad p_2 = \frac{\tilde{\tilde{\nu}}_1 - \nu_1}{\tilde{\tilde{\sigma}}\sqrt{V_{22}}} \quad p_3 = \frac{\tilde{\tilde{\sigma}} - \sigma}{\tilde{\tilde{\sigma}}\sqrt{V_{33}}} \tag{5.4.1}$$

are standard normal, where V_{11}, V_{22} and V_{33} are the corresponding exact values of $Var(\tilde{\tilde{\nu}}_0)/\sigma^2$, $Var(\tilde{\tilde{\nu}}_1)/\sigma^2$ and $Var(\tilde{\tilde{\sigma}})/\sigma^2$, respectively. Through Monte Carlo simulations, the percentage points of these pivotal quantities can be determined. These simulated percentage points will allow us to construct confidence intervals for the parameters ν_0, ν_1 and σ. For example, if $p_{1,\alpha/2}$ and $p_{1,1-\alpha/2}$ denote the lower and upper percentage points determined through simulation for the pivotal quantity p_1, then $[\tilde{\tilde{\nu}}_0 - \tilde{\tilde{\sigma}} p_{1,1-\alpha/2}\sqrt{V_{11}},\ \tilde{\tilde{\nu}}_0 + \tilde{\tilde{\sigma}} p_{1,\alpha/2}\sqrt{V_{11}}]$ will form a 100(1-α)% confidence interval for ν_0 when σ is unknown.

To examine the accuracy of these interval estimation procedures, we simulated the probability coverages of these approximate confidence intervals (which we naturally expect to be approximately 95%) through the values of

$$\Pr(-1.96 \le p_i \le 1.96) \qquad \text{for } i = 1, 2, 3.$$

5.5 Simulation Results

In the simulation study, we took $\nu_0 = 0$, $\nu_1 = 1$ and $\sigma = 1$, and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored sample. The process is based on 10,000 Monte Carlo runs. We simulated the probability coverages of intervals based on BLUEs, MLEs and AMLEs for each of following cases.

1. Complete samples

two groups: n = [6 6(1)10], [7 7(1)10], [8 8(1)10], [9 9(1)10], [10 10], [15 15(5)20] and [20 20] .

four groups: n = [6×2 6(1)10×2], [7×2 7(1)10×2], [8×2 8(1)10×2], [9×2 9(1)10×2], [10 10 10 10], [15×2 15(5)20×2] and [20 20 20 20] .

2. Type-II right-censored samples

two groups: s = [4 4(1)0], [3 3(1)0], [2 2(1)0] and [1 1(1)0] from n = [10 10] and [5 5(1)0] from n = [20 20].

four groups: s = [4×2 4(1)0×2], [3×2 3(1)0×2], [2×2 2(1)0×2] and [1×2 1(1)0×2] from n = [10 10 10 10] and [5×2 5(1)0×2] from n = [20 20 20 20].

These results are presented in Tables 5.5.1 - 5.5.4.

Table 5.5.1 Simulated probability coverages for two-grouped complete samples

$[n_1\ n_2]$	BLUE			MLE			AMLE		
	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
[6 6]	92.77	92.02	91.37	88.48	86.77	79.14	80.47	90.91	74.72
[6 7]	92.88	92.81	92.14	88.61	88.14	80.17	79.97	90.87	75.22
[6 8]	92.65	92.90	91.79	89.77	88.44	80.90	81.88	90.75	76.95
[6 9]	93.03	92.67	92.18	90.49	89.50	82.21	82.70	91.04	77.72
[6 10]	93.31	93.37	92.19	89.43	89.34	82.79	83.62	91.37	79.56
[7 7]	93.20	92.60	92.22	89.95	88.71	81.25	81.90	91.69	76.09
[7 8]	92.90	92.91	92.32	90.27	89.55	81.98	82.52	91.34	77.49
[7 9]	92.89	93.01	92.79	90.26	89.41	82.96	83.59	91.72	79.29
[7 10]	93.07	93.08	92.12	90.77	89.77	82.98	84.00	91.31	79.11
[8 8]	93.10	93.38	92.01	90.34	89.24	83.31	84.22	92.14	78.27
[8 9]	92.94	92.74	92.36	90.43	89.85	83.65	84.55	92.07	79.89
[8 10]	93.30	92.73	92.70	90.78	90.48	84.43	84.49	92.36	79.84
[9 9]	93.73	93.31	92.62	90.86	90.36	84.75	84.61	91.66	80.18
[9 10]	93.53	93.10	92.65	91.04	90.54	84.76	85.22	92.90	81.20
[10 10]	93.49	93.47	93.07	92.03	90.60	85.47	85.69	92.23	81.59
[15 15]	94.13	93.86	93.41	92.63	92.52	88.62	89.07	93.43	86.45
[15 20]	94.45	93.85	93.70	93.26	93.25	89.25	87.76	94.04	87.43
[20 20]	94.21	94.21	93.92	93.70	92.99	89.74	90.88	93.99	88.17

Table 5.5.2 Simulated probability coverages for four-grouped complete samples

$[n_1\ n_2\ n_3\ n_4]$	BLUE			MLE			AMLE		
	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
[6 6 6 6]	94.03	93.18	93.69	92.23	91.27	86.71	68.77	93.41	87.07
[6 6 7 7]	93.97	93.70	93.21	92.34	91.96	87.71	70.43	93.55	87.19
[6 6 8 8]	94.03	93.91	93.49	92.82	91.33	87.75	70.75	93.70	88.02
[6 6 9 9]	94.10	94.09	93.61	92.47	92.17	89.16	72.16	93.56	88.34
[6 6 10 10]	93.95	93.82	93.21	93.19	92.65	89.34	73.16	93.36	88.16
[7 7 7 7]	93.63	93.72	93.71	92.65	92.03	88.64	71.25	93.83	87.87
[7 7 8 8]	93.77	94.22	93.52	92.12	91.48	89.25	71.95	94.03	88.24
[7 7 9 9]	94.14	93.83	93.88	93.68	92.08	88.92	73.84	94.26	88.12
[7 7 10 10]	94.06	94.19	93.80	93.14	92.70	89.77	74.67	94.10	88.65
[8 8 8 8]	93.90	93.75	93.45	92.98	92.33	88.98	73.46	93.68	88.04
[8 8 9 9]	93.87	94.46	93.89	93.31	92.66	89.93	74.78	94.55	88.64
[8 8 10 10]	94.32	94.40	93.73	92.43	93.18	89.89	75.39	94.58	89.33
[9 9 9 9]	94.23	94.31	93.90	93.12	92.72	89.81	75.88	94.20	89.23
[9 9 10 10]	94.02	93.96	94.12	93.24	93.05	90.32	75.19	94.18	89.66
[10 10 10 10]	94.55	94.51	94.22	93.29	93.18	90.30	77.51	94.94	89.93
[15 15 15 15]	94.65	95.05	94.15	94.23	93.69	91.97	82.31	94.72	91.49
[15 15 20 20]	94.92	94.18	94.29	94.13	93.81	92.38	84.06	94.75	91.37
[20 20 20 20]	94.78	94.76	94.69	94.40	93.89	92.47	84.82	95.05	92.48

Table 5.5.3 Simulated probability coverages for two-grouped Type-II censored samples

	BLUE			MLE			AMLE		
$[s_1\ s_2]$	v_0	v_1	σ	v_0	v_1	σ	v_0	v_1	σ
4 4	91.68	91.88	91.40	81.56	86.09	75.88	76.21	88.41	69.49
4 3	92.46	92.52	91.57	83.37	86.68	77.85	77.43	88.89	70.89
4 2	93.10	92.23	91.84	85.69	88.06	79.16	80.00	89.10	73.32
4 1	93.41	92.67	92.29	86.99	88.23	80.87	82.27	89.55	75.46
4 0	93.20	93.52	92.46	88.36	88.50	82.48	82.66	89.97	77.50
3 3	92.99	92.02	91.71	85.86	87.76	79.76	80.48	89.21	73.28
3 2	92.84	92.43	92.12	87.49	88.46	80.17	81.88	90.42	75.82
3 1	93.02	92.99	92.31	88.08	88.77	81.81	82.66	90.95	77.49
3 0	92.92	93.20	92.13	88.96	89.54	82.89	83.68	90.38	77.84
2 2	92.50	92.65	91.99	87.86	88.79	82.06	83.11	91.13	77.09
2 1	93.31	92.96	92.27	89.16	89.95	83.13	83.94	91.15	78.73
2 0	93.25	93.71	92.55	90.86	89.83	84.08	84.70	92.01	79.54
1 1	92.90	93.14	92.72	90.40	89.85	82.92	85.33	91.98	80.33
1 0	93.18	93.42	92.88	91.03	90.50	84.57	85.30	92.48	80.51
*5 5	93.85	93.81	93.17	91.53	91.66	87.96	88.67	92.98	84.90
*5 0	94.49	94.42	93.64	92.36	92.17	89.12	90.00	93.32	87.10

Asterisk denotes censoring is from n = *[20 20]*, otherwise is from n=*[10 10]*.

Table 5.5.4 Simulated probability coverages for four-grouped Type-II censored samples

	BLUE			MLE			AMLE		
$[s_1\ s_2\ s_3\ s_4]$	v_0	v_1	σ	v_0	v_1	σ	v_0	v_1	σ
4 4 4 4	93.11	93.72	92.96	89.00	91.04	85.78	78.50	92.24	84.98
4 4 3 3	93.58	93.56	93.31	90.00	91.83	87.22	79.07	92.83	86.32
4 4 2 2	93.81	93.79	93.21	90.41	91.78	88.17	78.76	92.58	87.12
4 4 1 1	94.05	93.24	93.24	91.07	91.77	88.24	78.53	93.25	87.67
4 4 0 0	94.01	94.30	93.70	92.58	91.83	89.01	77.72	93.31	87.77
3 3 3 3	94.15	93.75	92.76	90.98	91.75	87.49	79.19	92.69	86.99
3 3 2 2	94.03	93.19	93.28	91.77	91.77	88.05	79.97	93.32	87.77
3 3 1 1	93.91	93.95	93.26	91.39	91.37	88.31	78.98	93.67	88.01
3 3 0 0	94.04	94.15	93.88	92.43	92.38	89.70	77.97	93.56	88.19
2 2 2 2	93.95	94.03	93.65	92.11	91.86	88.86	80.08	93.70	88.49
2 2 1 1	93.57	94.04	93.70	92.76	92.85	88.97	79.44	93.79	88.64
2 2 0 0	94.30	94.40	93.86	92.69	92.54	89.55	78.58	93.92	89.25
1 1 1 1	94.36	94.14	93.92	92.74	92.67	89.56	79.46	93.86	88.83
1 1 0 0	94.40	94.45	93.69	93.05	92.48	90.35	78.12	93.91	89.06
*5 5 5 5	94.71	94.2	94.53	93.06	93.68	91.69	86.48	94.48	91.51
*5 5 0 0	94.61	94.27	94.46	93.84	94.34	92.27	85.47	94.71	91.52

Asterisk denotes censoring is from n = *[20 20 20 20]*, otherwise is from n=*[10 10 10 10]*.

CHAPTER 6

COMPARISON WITHIN AND BETWEEN BLUE, MLE AND AMLE

6.1 Introduction

In this chapter, we first assess the effects of the following five factors on the performance of BLUE, MLE and AMLE for ν_0, ν_1 and σ in Sections 6.2, 6.3 and 6.4, respectively:

1. The number of levels of the regressor variable x,
2. The balanced (equal sized) group sample vs. unbalanced (unequal sized) group sample,
3. The total sample size N,
4. The complete sample vs. Type-II right-censored sample,
5. The degrees of censoring.

The assessments are based on estimators' bias, mean square error, variances and probability coverages.

We then make comparisons in Section 6.5 between BLUE, MLE and AMLE based on the relative efficiency of the estimators and the accuracy of the normal approximation in terms of probability coverages.

The numbers of levels of x are chosen as two and four (which correspond to the two-grouped and four-grouped samples, respectively) in this study; the values of x are equally spaced between –0.5 and 0.5. Any equally spaced values of x can be produced through appropriate linear transformations. In this study, we have x = [-0.5 0.5] for two-grouped sample and x = [-0.5 -0.16 0.16 0.5] for four-grouped sample. In the complete sample case, both equal sized and unequal sized group samples are used in the study, and the group size differences between (or among the four-grouped sample) the unbalanced groups are set from 1 to 5. The single group size n of the balanced groups is chosen as 6, 7, 8, 9, 10, 15 and 20 for both two-grouped and four-grouped samples. With the scheme of [n; n(1)10] for n < 10, (or [n; n(5)20] for n=15) in two-grouped sample and the scheme [n; n; n(1)10; n(1)10] for n < 10, (or [n; n; n(5)20; n(5)20] for n = 15) in four-grouped sample, the unbalanced groups are produced as shown in Tables 6.2.1 and 6.2.2 in Section 6.2.

In the Type-II right-censored sample, we took either 10 or 20 as the complete sample size for each group, and adopt all samples of various group sizes in the complete sample case to form samples of various Type-II right-censored samples. In other words, with the same number of groups and group sizes (does not matter balanced or unbalanced), the complete sample has the complete sample in each group whereas the Type-II censored sample has the censored sample either censored from n = 10 or n = 20. For example, [7 9] in the complete sample case, means a two-grouped complete samples of size 7 and 9; and in the Type-II censored sample, means a two-grouped censored samples of size 7 and 9 censored from the complete samples of size 10 and 10. Such a choice of censoring spans the severe, moderate, and light censoring situations. It should be noted that in the Type-II censored samples, if $n > 10$, the censoring is from the complete sample of size 20, otherwise is from size 10.

Without loss of generality, we set the true parameter values of ν_0, ν_1 and σ as 0, 1 and 1, respectively.

6.2 Assessment of BLUEs

In this section, we use the exact variances and the simulated probability coverages to assess the performance of the BLUEs of ν_0, ν_1 and σ. We rearrange the exact variances according to increasing order of the total sample size N in Tables 6.2.1 - 6.2.4 for both two- and four-grouped samples as well as complete and Type-II right-censored samples.

Table 6.2.1 BLUEs: Exact variances for two-grouped complete sample

$[n_1\ n_2]$		$Var(\nu_0^*)/\sigma^2$	$Var(\nu_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$
[6 6]		0.0956	0.3674	0.0660
[6 7]		0.0885	0.3392	0.0597
N=14	[6 8]	0.0833	0.3185	0.0545
	[7 7]	0.0815	0.3109	0.0545
N=15	[6 9]	0.0792	0.3027	0.0501
	[7 8]	0.0762	0.2901	0.0502
N=16	[6 10]	0.0759	0.2902	0.0464
	[7 9]	0.0722	0.2742	0.0464
	[8 8]	0.0710	0.2693	0.0465
N=17	[7 10]	0.0689	0.2617	0.0432
	[8 9]	0.0669	0.2534	0.0432
N=18	[8 10]	0.0637	0.2409	0.0404
	[9 9]	0.0629	0.2375	0.0404
[9 10]		0.0597	0.2250	0.0380
[10 10]		0.0565	0.2124	0.0358
[15 15]		0.0374	0.1389	0.0227
[15 20]		0.0326	0.1211	0.0191
[20 20]		0.0280	0.1032	0.0166

Table 6.2.2 BLUEs: Exact variances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$		$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$
[6 6 6 6]		0.0478	0.3333	0.0330
[6 6 7 7]		0.0442	0.3072	0.0299
N=28	[6 6 8 8]	0.0414	0.2874	0.0273
	[7 7 7 7]	0.0407	0.2820	0.0273
N=30	[6 6 9 9]	0.0392	0.2718	0.0251
	[7 7 8 8]	0.0381	0.2629	0.0251
N=32	[6 6 10 10]	0.0374	0.2590	0.0232
	[7 7 9 9]	0.0360	0.2478	0.0232
	[8 8 8 8]	0.0355	0.2443	0.0232
N=34	[7 7 10 10]	0.0342	0.2356	0.0216
	[8 8 9 9]	0.0334	0.2297	0.0216
N=36	[8 8 10 10]	0.0318	0.2179	0.0202
	[9 9 9 9]	0.0315	0.2155	0.0202
[9 9 10 10]		0.0298	0.2040	0.0190
[10 10 10 10]		0.0282	0.1927	0.0179
[15 15 15 15]		0.0187	0.1260	0.0113
[15 15 20 20]		0.0163	0.1093	0.0096
[20 20 20 20]		0.0140	0.0936	0.0083

Table 6.2.3 BLUEs: Exact variances for two-grouped Type-II censored sample

$[s_1\ \ s_2]$		$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$
4 4		0.1072	0.3637	0.0829
4 3		0.0925	0.3391	0.0735
N=14	4 2	0.0829	0.3258	0.0652
	3 3	0.0808	0.3082	0.0661
N=15	4 1	0.0765	0.3190	0.0575
	3 2	0.0732	0.2900	0.0593
N=16	4 0	0.0722	0.3162	0.0500
	3 1	0.0683	0.2795	0.0529
	2 2	0.0670	0.2676	0.0537
N=17	3 0	0.0651	0.2741	0.0464
	2 1	0.0631	0.2536	0.0484
N=18	2 0	0.0607	0.2455	0.0430
	1 1	0.0599	0.2366	0.0441
1 0		0.0580	0.2258	0.0395
*5 5		0.0348	0.1382	0.0275
*5 0		0.0306	0.1249	0.0207

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 6.2.4 BLUEs: Exact variances for four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$		$Var(v_0^*)/\sigma^2$	$Var(v_1^*)/\sigma^2$	$Var(\sigma^*)/\sigma^2$
4 4 4 4		0.0536	0.3299	0.0414
4 4 3 3		0.0461	0.3065	0.0367
N=28	4 4 2 2	0.0411	0.2920	0.0324
	3 3 3 3	0.0404	0.2796	0.0330
N=30	4 4 1 1	0.0377	0.2827	0.0284
	3 3 2 2	0.0365	0.2624	0.0296
N=32	4 4 0 0	0.0355	0.2764	0.0245
	3 3 1 1	0.0340	0.2512	0.0263
	2 2 2 2	0.0335	0.2428	0.0269
N=34	3 3 0 0	0.0324	0.2440	0.0230
	2 2 1 1	0.0315	0.2296	0.0242
N=36	2 2 0 0	0.0303	0.2210	0.0214
	1 1 1 1	0.0299	0.2146	0.0220
1 1 0 0		0.0290	0.2045	0.0197
*5 5 5 5		0.0174	0.1254	0.0137
*5 5 0 0		0.0152	0.1120	0.0103

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

The variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a increase in N. In addition, with the same N, the variances of BLUEs for v_0 and v_1 tend to be smaller in value in the case of more balanced groups than among the less balanced groups. Moreover, for the same N, the variance of BLUE of σ tends to be the same in the complete samples and increase with the more balanced groups in the Type-II censored samples. The variances of all the estimators tend to increase with increasing amounts of censoring. This is true for both two- and four-levels of x.

According to the results in Tables 5.5.1 - 5.5.4, the simulated probability coverages of all the estimators tend to increase with increasing number of levels of x when the groups (or the amounts of censoring in the Type-II censored samples) are of the same size. The simulated probability coverages of all estimators tend to increase with a major increase in N. Moreover, with the same N, the simulated probability coverages of BLUEs for ν_0, ν_1 and σ do not exhibit a clear pattern between the more balanced and the less balanced groups. The simulated probability coverages of all the estimators tend to decrease with increasing amounts of censoring. This fact is more obvious in the case of two-levels of x.

Since the BLUEs for ν_0, ν_1 and σ are all unbiased, the simulated mean square errors are almost identical to the simulated variances. The close agreement between the simulated variances and the exact variances of BLUEs of ν_0, ν_1 and σ should be noted as well.

6.3 Assessment of MLEs

Similar to the BLUEs, we use the asymptotic variances of the MLEs of ν_0, ν_1 and σ in Tables 6.3.1 - 6.3.4 and the simulated probability coverages (in Tables 5.5.1 - 5.5.4) to assess the performance of the MLEs of ν_0, ν_1 and σ.

Table 6.3.1 MLEs: Asymptotic variances for two-grouped complete sample

$[n_1\ n_2]$		$Var(\hat{\nu}_0)/\sigma^2$	$Var(\hat{\nu}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$
[6 6]		0.0924	0.3333	0.0507
[6 7]		0.0857	0.3095	0.0468
N=14	[6 8]	0.0807	0.2917	0.0434
	[7 7]	0.0792	0.2857	0.0434
N=15	[6 9]	0.0767	0.2778	0.0405
	[7 8]	0.0742	0.2679	0.0405
N=16	[6 10]	0.0735	0.2667	0.0380
	[7 9]	0.0703	0.2540	0.0380
	[8 8]	0.0693	0.2500	0.0380
N=17	[7 10]	0.0671	0.2429	0.0358
	[8 9]	0.0654	0.2361	0.0358
N=18	[8 10]	0.0623	0.2250	0.0338
	[9 9]	0.0616	0.2222	0.0338
[9 10]		0.0585	0.2111	0.0320
[10 10]		0.0554	0.2000	0.0304
[15 15]		0.0370	0.1333	0.0203
[15 20]		0.0323	0.1167	0.0174
[20 20]		0.0277	0.1000	0.0152

Table 6.3.2 MLEs: Asymptotic variances for four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$		$Var(\hat{\nu}_0)/\sigma^2$	$Var(\hat{\nu}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$
[6 6 6 6]		0.0462	0.3024	0.0253
[6 6 7 7]		0.0428	0.2804	0.0234
N=28	[6 6 8 8]	0.0402	0.2634	0.0217
	[7 7 7 7]	0.0396	0.2592	0.0217
N=30	[6 6 9 9]	0.0380	0.2498	0.0203
	[7 7 8 8]	0.0371	0.2427	0.0203
N=32	[6 6 10 10]	0.0363	0.2386	0.0190
	[7 7 9 9]	0.0350	0.2296	0.0190
	[8 8 8 8]	0.0346	0.2268	0.0190
N=34	[7 7 10 10]	0.0333	0.2188	0.0179
	[8 8 9 9]	0.0327	0.2140	0.0179
N=36	[8 8 10 10]	0.0311	0.2036	0.0169
	[9 9 9 9]	0.0308	0.2016	0.0169
[9 9 10 10]		0.0292	0.1914	0.0160
[10 10 10 10]		0.0277	0.1814	0.0152
[15 15 15 15]		0.0185	0.1209	0.0101
[15 15 20 20]		0.0161	0.1054	0.0087
[20 20 20 20]		0.0139	0.0907	0.0076

Table 6.3.3 MLEs: Asymptotic variances for two-grouped Type-II censored sample

$[s_1 \; s_2]$		$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$
4 4		0.0880	0.3333	0.0624
4 3		0.0793	0.3114	0.0568
N=14	4 2	0.0734	0.2982	0.0515
	3 3	0.0695	0.2904	0.0464
N=15	4 1	0.0669	0.2865	0.0409
	3 2	0.0719	0.2857	0.0521
N=16	4 0	0.0670	0.2693	0.0477
	3 1	0.0638	0.2589	0.0432
	2 2	0.0618	0.2530	0.0384
N=17	3 0	0.0628	0.2500	0.0439
	2 1	0.0601	0.2372	0.0401
N=18	2 0	0.0585	0.2292	0.0359
	1 1	0.0578	0.2222	0.0369
1 0		0.0565	0.2121	0.0333
*5 5		0.0333	0.1333	0.0246
*5 0		0.0299	0.1203	0.0188

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 6.3.4 MLEs: Asymptotic variances for four-grouped Type-II censored sample

$[s_1 \; s_2 \; s_3 \; s_4]$		$Var(\hat{v}_0)/\sigma^2$	$Var(\hat{v}_1)/\sigma^2$	$Var(\hat{\sigma})/\sigma^2$
4 4 4 4		0.0440	0.3024	0.0312
4 4 3 3		0.0396	0.2818	0.0284
N=28	4 4 2 2	0.0365	0.2680	0.0257
	3 3 3 3	0.0359	0.2592	0.0261
N=30	4 4 1 1	0.0344	0.2586	0.0230
	3 3 2 2	0.0335	0.2438	0.0238
N=32	4 4 0 0	0.0331	0.2522	0.0201
	3 3 1 1	0.0318	0.2331	0.0215
	2 2 2 2	0.0314	0.2268	0.0219
N=34	3 3 0 0	0.0307	0.2259	0.0190
	2 2 1 1	0.0300	0.2148	0.0200
N=36	2 2 0 0	0.0292	0.2065	0.0179
	1 1 1 1	0.0289	0.2016	0.0185
1 1 0 0		0.0283	0.1921	0.0167
*5 5 5 5		0.0167	0.1209	0.0123
*5 5 0 0		0.0149	0.1080	0.0093

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

The variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a major increase in N. In addition, with the same N, the variances of MLEs of ν_0 and ν_1 tend to be smaller in value in the more balanced groups than among the less balanced groups. Moreover, with the same N, the variance of MLE of σ tends to be the same in the complete samples and does not exhibit any clear patterns in the Type-II censored samples. The variances of all the estimators tend to increase with increasing amounts of censoring. This is true for both two- and four-levels of x.

The simulated probability coverages of all the estimators tend to increase with increasing number of levels of x when the groups (or the number of observations in the Type-II censored sample) are of the same size. The simulated probability coverages of all the estimators tend to increase with a major increase in N. Moreover, with the same N, the simulated probability coverages of MLEs of ν_0 and ν_1 do not exhibit a clear pattern between the more balanced and the less balanced groups. The simulated probability coverages of all the estimators present a strong tendency to decrease while the amounts of censoring increase in the two-levels of x. This fact is also seen in most of the cases with four-levels of x.

The simulated mean square errors are very close to the simulated variances, but not identical, which means that the biases of the estimators are negligible. Moreover, the agreements tend to increase with increase in the total sample size N. This simply means that the MLEs of ν_0, ν_1 and σ all become almost unbiased as the total sample size N become large.

6.4 Assessment of AMLEs

To assess the performance of the AMLEs of ν_0, ν_1 and σ, we rearrange the approximate biases and variances in an increasing order of the total sample size N in Tables 6.4.1 - 6.4.4 for both two- and four-grouped samples as well as the complete and the Type-II censored samples.

Table 6.4.1 AMLEs: Approximate bias and variances for two-grouped complete sample

$[n_1\ n_2]$		$Bias(\tilde{\nu}_0)/\sigma$	$Bias(\nu_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Var(\tilde{\nu}_0)/\sigma^2$	$Var(\tilde{\nu}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$
[6 6]		-0.1498	0.0000	-0.0672	0.0983	0.3924	0.0476
[6 7]		-0.1400	0.0222	-0.0621	0.0906	0.3622	0.0432
N=14	[6 8]	-0.1327	0.0393	-0.0579	0.0849	0.3400	0.0404
	[7 7]	-0.1301	0.0000	-0.0578	0.0829	0.3316	0.0395
N=15	[6 9]	-0.1269	0.0527	-0.0542	0.0806	0.3231	0.0379
	[7 8]	-0.1227	0.0170	-0.0541	0.0773	0.3093	0.0371
N=16	[6 10]	-0.1223	0.0637	-0.0510	0.0772	0.3099	0.0357
	[7 9]	-0.1168	0.0306	-0.0509	0.0730	0.2923	0.0350
	[8 8]	-0.1151	0.0000	-0.0508	0.0717	0.2868	0.0350
N=17	[7 10]	-0.1121	0.0415	-0.0481	0.0697	0.2789	0.0332
	[8 9]	-0.1091	0.0135	-0.0480	0.0675	0.2697	0.0332
N=18	[8 10]	-0.1044	0.0245	-0.0455	0.0641	0.2562	0.0315
	[9 9]	-0.1031	0.0000	-0.0455	0.0633	0.2525	0.0315
[9 10]		-0.0983	0.0110	-0.0433	0.0599	0.2389	0.0300
[10 10]		-0.0934	0.0000	-0.0413	0.0566	0.2254	0.0286
[15 15]		-0.0643	0.0000	-0.0267	0.0372	0.1461	0.0196
[15 20]		-0.0567	0.0175	-0.0233	0.0325	0.1270	0.0169
[20 20]		-0.0488	0.0000	-0.0207	0.0278	0.1078	0.0149

Table 6.4.2 AMLEs: Approximate bias and variances for four-grouped complete sample

N	$[n_1\ n_2\ n_3\ n_4]$	$Bias(\tilde{v}_0)/\sigma$	$Bias(\tilde{v}_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$
	[6 6 6 6]	-0.1625	0.0000	-0.0196	0.0492	0.3560	0.0238
	[6 6 7 7]	-0.1519	0.0259	-0.0181	0.0452	0.3280	0.0216
N=28	[6 6 8 8]	-0.1434	0.0456	-0.0169	0.0422	0.3068	0.0202
	[7 7 7 7]	-0.1416	0.0000	-0.0169	0.0415	0.3008	0.0198
N=30	[6 6 9 9]	-0.1365	0.0610	-0.0159	0.0399	0.2900	0.0189
	[7 7 8 8]	-0.1334	0.0199	-0.0159	0.0386	0.2802	0.0186
N=32	[6 6 10 10]	-0.1307	0.0733	-0.0150	0.0380	0.2763	0.0179
	[7 7 9 9]	-0.1268	0.0356	-0.0150	0.0364	0.2640	0.0175
	[8 8 8 8]	-0.1255	0.0000	-0.0151	0.0359	0.2602	0.0175
N=34	[7 7 10 10]	-0.1212	0.0482	-0.0143	0.0346	0.2509	0.0166
	[8 8 9 9]	-0.1190	0.0158	-0.0143	0.0337	0.2444	0.0166
N=36	[8 8 10 10]	-0.1136	0.0286	-0.0136	0.0320	0.2317	0.0157
	[9 9 9 9]	-0.1127	0.0000	-0.0137	0.0316	0.2290	0.0157
	[9 9 10 10]	-0.1074	0.0129	-0.0131	0.0299	0.2166	0.0150
	[10 10 10 10]	-0.1023	0.0000	-0.0126	0.0283	0.2044	0.0143
	[15 15 15 15]	-0.0708	0.0000	-0.0070	0.0186	0.1325	0.0098
	[15 15 20 20]	-0.0622	0.0205	-0.0064	0.0162	0.1146	0.0085
	[20 20 20 20]	-0.0540	0.0000	-0.0059	0.0139	0.0978	0.0074

Table 6.4.3 AMLEs: Approximate bias and variances for two-grouped Type-II censored sample

$[s_1 \ s_2]$		$Bias(\tilde{v}_0)/\sigma$	$Bias(\tilde{v}_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$
4 4		-0.1490	0.0000	-0.0849	0.0987	0.3562	0.0258
4 3		-0.1334	0.0237	-0.0748	0.0900	0.3330	0.0275
N=14	4 2	-0.1225	0.0390	-0.0663	0.0833	0.3182	0.0284
	3 3	-0.1199	0.0000	-0.0664	0.0816	0.3073	0.0294
N=15	4 1	-0.1156	0.0476	-0.0594	0.0782	0.3092	0.0283
	3 2	-0.1107	0.0157	-0.0593	0.0753	0.2904	0.0305
N=16	4 0	-0.1135	0.0490	-0.0558	0.0745	0.3043	0.0271
	3 1	-0.1051	0.0248	-0.0535	0.0705	0.2796	0.0304
	2 2	-0.1028	0.0000	-0.0533	0.0693	0.2714	0.0316
N=17	3 0	-0.1038	0.0264	-0.0507	0.0673	0.2735	0.0290
	2 1	-0.0981	0.0093	-0.0484	0.0650	0.2588	0.0315
N=18	2 0	-0.0973	0.0108	-0.0463	0.0622	0.2513	0.0300
	1 1	-0.0941	0.0000	-0.0442	0.0612	0.2445	0.0314
1 0		-0.0937	0.0013	-0.0427	0.0587	0.2355	0.0299
*5 5		-0.0546	0.0000	-0.0299	0.0355	0.1390	0.0228
*5 0		-0.0508	0.0070	-0.0252	0.0309	0.1265	0.0180

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 6.4.4 AMLEs: Approximate bias and variances for four-grouped Type-II censored sample

	$[s_1\ s_2\ s_3\ s_4]$	$Bias(\tilde{v}_0)/\sigma$	$Bias(\tilde{v}_1)/\sigma$	$Bias(\tilde{\sigma})/\sigma$	$Var(\tilde{v}_0)/\sigma^2$	$Var(\tilde{v}_1)/\sigma^2$	$Var(\tilde{\sigma})/\sigma^2$
	4 4 4 4	-0.1254	0.0000	-0.0225	0.0494	0.3231	0.0129
	4 4 3 3	-0.1174	0.0155	-0.0187	0.0499	0.3015	0.0137
N=28	4 4 2 2	-0.1123	0.0245	-0.0157	0.0414	0.2867	0.0142
	3 3 3 3	-0.1104	0.0000	-0.0158	0.0408	0.2788	0.0147
N=30	4 4 1 1	-0.1099	0.0279	-0.0138	0.0387	0.2767	0.0141
	3 3 2 2	-0.1061	0.0092	-0.0134	0.0376	0.2630	0.0152
N=32	4 4 0 0	-0.1115	0.0252	-0.0153	0.0367	0.2703	0.0134
	3 3 1 1	-0.1042	0.0129	-0.0120	0.0351	0.2522	0.0151
	2 2 2 2	-0.1023	0.0000	-0.0114	0.0347	0.2461	0.0158
N=34	3 3 0 0	-0.1058	0.0096	-0.0136	0.0334	0.2454	0.0144
	2 2 1 1	-0.1007	0.0037	-0.0104	0.0325	0.2344	0.0157
N=36	2 2 0 0	-0.1023	-0.0002	-0.0121	0.0310	0.2270	0.0150
	1 1 1 1	-0.0993	0.0000	-0.0095	0.0306	0.2217	0.0157
	1 1 0 0	-0.1009	-0.0043	-0.0112	0.0294	0.2135	0.0150
	*5 5 5 5	-0.0529	0.0000	-0.0065	0.0177	0.1261	0.0114
	*5 5 0 0	-0.0533	-0.0007	-0.0072	0.0154	0.1137	0.0090

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

The approximate biases tend to increase in AMLEs of ν_0 and ν_1 and decrease in σ with increasing number of levels of x when the groups are of the same size in the complete samples. The biases of all estimators tend to decrease with increasing number of levels of x when the amounts of censoring are of the same size in the Type-II censored samples. The biases of all estimators tend to decrease with a major increase in N. In addition, with the same N, the biases of all estimators tend to decrease in the more balanced groups than among the less balanced groups and moreover for σ, the biases tend to be the same in the complete samples. The biases of AMLEs in ν_0 and σ tend to increase whereas decrease in ν_1 with increasing amounts of censoring in the Type-II censored samples. This is true in case of both two- and four-levels of x.

The approximate variances of all the estimators tend to decrease with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. The variances of all the estimators tend to decrease with a major increase in N, except for σ in the Type-II censored samples. In addition, with the same N, expect for σ, the variances tend to have smaller values in the more balanced groups than in the less balanced groups; moreover for σ, the variances tend to be the same or decrease with the more balanced groups in the complete samples and increase with the more balanced groups in the Type-II censored samples. The variances of AMLEs in ν_0 and ν_1 tend to increase whereas decrease for σ with increasing amounts of censoring. This is true in case of both two- and four-levels of x.

From Tables 5.5.1 - 5.5.4, we observe that the simulated probability coverages of all the estimators tend to increase, except for ν_0 in the complete samples, with increasing number of levels of x when the groups (or the number of observations in the Type-II censored samples) are of the same size. Due to the larger bias of AMLE of ν_0 in the four-grouped sample case, the values of simulated probability coverages are smaller as compared to the two-grouped sample case in the complete sample cases when the groups are of the same size. The simulated probability coverages of all the estimators tend to decrease with a major increase in N. Moreover, with the same N, the simulated probability coverages of AMLEs of ν_0 and ν_1 do not exhibit any clear pattern between the more balanced and the less balanced groups. The simulated probability coverages of all the estimators, except for AMLE of ν_0 in case of four-levels of x, reveal a strong tendency to decrease when the amounts of censoring increase.

6.5 Comparisons between BLUE, MLE and AMLE

In terms of the simplicity of the estimation procedure, BLUE requires minimal derivation to obtain formulas of the estimators and are easy to program to obtain the estimates by the use of computer software, such as, Matlab, Minitab, etc. However, BLUE are less efficient as compared to MLE. Moreover, its use is restricted to situations when the sample size n does not exceeds 30, since it is necessary to have the means, variances, and covariances of order statistics from the standard extreme value distribution and which are not available for n beyond 30.

The AMLE are explicit estimators (unlike the MLE) and do not need the construction of any special tables (unlike the BLUE). However, it involves somewhat complicated derivation and calculation if the model involves multi-groups and includes many unknown parameters. In addition, it is more biased and less efficient than the other two procedures in most of the cases.

The MLE process high efficiency and are also approximately unbiased. Nevertheless, the lack of closed-form solutions of MLEs requires the use of iterative methods to obtain the estimates. Moreover, it faces convergence problem, as the convergence is often quite slow.

In the following, we compare the BLUEs, MLEs and AMLEs in terms of relative efficiency and the accuracy of the normal approximation in terms of probability coverages in more detail.

6.5.1 Relative Efficiency

Relative efficiencies to MLEs of BLUEs and AMLEs can be calculated by the formulas that have been presented earlier in (4.3.1) - (4.3.7), (2.2.20) - (2.2.25), and (3.3.1) - (3.3.7), respectively. A summary of these efficiency ratios are displayed in Tables of 6.5.1.1 - 6.5.1.4.

Table 6.5.1.1 Relative efficiency of BLUEs and AMLEs from two-grouped complete sample

$[n_1\ n_2]$		BLUEs			AMLEs		
		ν_0^*	ν_1^*	σ^*	$\tilde{\nu}_0$	$\tilde{\nu}_1$	$\tilde{\sigma}$
[6 6]		0.9665	0.9072	0.7682	0.9400	0.8494	1.0651
[6 7]		0.9684	0.9124	0.7839	0.9459	0.8545	1.0833
N=14	[6 8]	0.9688	0.9159	0.7963	0.9505	0.8579	1.0743
	[7 7]	0.9718	0.9189	0.7963	0.9554	0.8616	1.0987
N=15	[6 9]	0.9684	0.9177	0.8084	0.9516	0.8598	1.0686
	[7 8]	0.9738	0.9235	0.8068	0.9599	0.8661	1.0916
N=16	[6 10]	0.9684	0.9190	0.8190	0.9521	0.8606	1.0644
	[7 9]	0.9737	0.9263	0.8190	0.9630	0.8690	1.0857
	[8 8]	0.9761	0.9283	0.8172	0.9665	0.8717	1.0857
N=17	[7 10]	0.9739	0.9282	0.8287	0.9627	0.8709	1.0783
	[8 9]	0.9776	0.9317	0.8287	0.9689	0.8754	1.0783
N=18	[8 10]	0.9780	0.9340	0.8366	0.9719	0.8782	1.0730
	[9 9]	0.9793	0.9356	0.8366	0.9731	0.8800	1.0730
[9 10]		0.9799	0.9382	0.8421	0.9766	0.8836	1.0667
[10 10]		0.9805	0.9416	0.8492	0.9788	0.8873	1.0629
[15 15]		0.9893	0.9597	0.8943	0.9946	0.9124	1.0357
[15 20]		0.9908	0.9637	0.9110	0.9938	0.9189	1.0296
[20 20]		0.9893	0.9690	0.9157	0.9964	0.9276	1.0201

Table 6.5.1.2 Relative efficiency of BLUEs and AMLEs from four-grouped complete sample

$[n_1\ n_2\ n_3\ n_4]$		BLUEs			AMLEs		
		ν_0^*	ν_1^*	σ^*	$\tilde{\nu}_0$	$\tilde{\nu}_1$	$\tilde{\sigma}$
[6 6 6 6]		0.9665	0.9073	0.7667	0.9390	0.8494	1.0630
[6 6 7 7]		0.9683	0.9128	0.7826	0.9469	0.8549	1.0833
N=28	[6 6 8 8]	0.9710	0.9165	0.7949	0.9526	0.8585	1.0743
	[7 7 7 7]	0.9730	0.9191	0.7949	0.9542	0.8617	1.0960
N=30	[6 6 9 9]	0.9694	0.9191	0.8088	0.9524	0.8614	1.0741
	[7 7 8 8]	0.9738	0.9232	0.8088	0.9611	0.8662	1.0914
N=32	[6 6 10 10]	0.9706	0.9212	0.8190	0.9553	0.8636	1.0615
	[7 7 9 9]	0.9722	0.9266	0.8190	0.9615	0.8697	1.0857
	[8 8 8 8]	0.9746	0.9284	0.8190	0.9638	0.8716	1.0857
N=34	[7 7 10 10]	0.9737	0.9287	0.8287	0.9624	0.8721	1.0783
	[8 8 9 9]	0.9790	0.9316	0.8287	0.9703	0.8756	1.0783
N=36	[8 8 10 10]	0.9780	0.9344	0.8366	0.9719	0.8787	1.0764
	[9 9 9 9]	0.9778	0.9355	0.8366	0.9747	0.8803	1.0764
[9 9 10 10]		0.9799	0.9382	0.8421	0.9766	0.8837	1.0667
[10 10 10 10]		0.9823	0.9414	0.8492	0.9788	0.8875	1.0629
[15 15 15 15]		0.9893	0.9595	0.8938	0.9946	0.9125	1.0306
[15 15 20 20]		0.9877	0.9643	0.9063	0.9938	0.9197	1.0235
[20 20 20 20]		0.9929	0.9690	0.9157	1.0000	0.9274	1.0270

Table 6.5.1.3 Relative efficiency of BLUEs and AMLEs from two-grouped Type-II censored sample

$[s_1\ s_2]$		BLUEs			AMLEs		
		v_0^*	v_1^*	σ^*	$\tilde{v}_0$	$\tilde{v}_1$	$\tilde{\sigma}$
4 4		0.8209	0.9164	0.7527	0.8066	0.9357	1.1642
4 3		0.8573	0.9183	0.7728	0.8295	0.9326	1.1568
N=14	4 2	0.8854	0.9153	0.7899	0.8515	0.9296	1.1470
	3 3	0.8601	0.9422	0.7020	0.8731	0.9272	1.1345
N=15	4 1	0.8745	0.8981	0.7113	0.8896	0.9260	1.0965
	3 2	0.9822	0.9852	0.8786	0.8539	0.9297	1.1527
N=16	4 0	0.9280	0.8517	0.9540	0.8781	0.9261	1.1466
	3 1	0.9341	0.9263	0.8166	0.8999	0.9227	1.1309
	2 2	0.9224	0.9454	0.7151	0.9169	0.9200	1.0971
N=17	3 0	0.9647	0.9121	0.9461	0.9010	0.9211	1.1373
	2 1	0.9525	0.9353	0.8285	0.9232	0.9162	1.1264
N=18	2 0	0.9638	0.9336	0.8349	0.9405	0.9110	1.0945
	1 1	0.9649	0.9391	0.8367	0.9444	0.9088	1.1148
1 0		0.9741	0.9393	0.8430	0.9625	0.9003	1.0847
*5 5		0.9569	0.9645	0.8945	0.9380	0.9590	1.0789
*5 0		0.9771	0.9632	0.9082	0.9676	0.9510	1.0444

Asterisk denotes censoring is from n =*[20 20]*, otherwise is from *[10 10]*

Table 6.5.1.4 Relative efficiency of BLUEs and AMLEs from four-grouped Type-II censored sample

$[s_1\ s_2\ s_3\ s_4]$		BLUEs			AMLEs		
		v_0^*	v_1^*	σ^*	$\tilde{v}_0$	$\tilde{v}_1$	$\tilde{\sigma}$
4 4 4 4		0.8209	0.9166	0.7536	0.8059	0.9359	1.1642
4 4 3 3		0.8590	0.9194	0.7738	0.8302	0.9328	1.1592
N=28	4 4 2 2	0.8881	0.9178	0.7932	0.8528	0.9289	1.1525
	3 3 3 3	0.8886	0.9270	0.7909	0.9135	0.9277	1.2857
N=30	4 4 1 1	0.9125	0.9148	0.8099	0.9297	0.9448	1.2500
	3 3 2 2	0.9178	0.9291	0.8041	0.7957	0.8745	1.0531
N=32	4 4 0 0	0.9324	0.9124	0.8204	0.8688	0.9578	0.9663
	3 3 1 1	0.9353	0.9279	0.8175	0.9008	0.9217	1.1257
	2 2 2 2	0.9373	0.9341	0.8141	0.9373	0.9201	1.2586
N=34	3 3 0 0	0.9475	0.9258	0.8261	0.8822	0.9179	0.9845
	2 2 1 1	0.9524	0.9355	0.8264	0.9231	0.9160	1.1236
N=36	2 2 0 0	0.9637	0.9344	0.8364	0.9419	0.9089	1.0915
	1 1 1 1	0.9666	0.9394	0.8409	0.9444	0.9093	1.1145
1 1 0 0		0.9759	0.9394	0.8477	0.9626	0.8998	1.0915
*5 5 5 5		0.9598	0.9641	0.8978	0.9435	0.9588	1.0789
*5 5 0 0		0.9803	0.9643	0.9029	0.9675	0.9499	1.0333

Asterisk denotes censoring is from n =*[20 20 20 20]*, otherwise is from *[10 10 10 10]*

The relative efficiency of BLUEs and AMLEs of ν_0, ν_1 and σ do not appear to be affected by changes in the number of levels of x. This is true for both complete and Type-II censored samples.

The relative efficiencies of BLUEs and AMLEs in all cases tend to increase (or be more closer to 1 for the AMLE of σ) with a major increase in N in both complete and Type-II censored samples. In addition, with the same N, the relative efficiency of BLUEs of all parameters and AMLEs of ν_0 and ν_1 tend to have higher value in the more balanced groups than the less balanced groups. On the other hand, the relative efficiency of the AMLE of σ with the same N tends to be the same or less closer to 1 in the more balanced groups of the complete samples and do not exhibit any clear pattern in the censored sample.

The relative efficiency of the AMLE of ν_1 has higher values in Type-II censored samples than the complete samples. Moreover, the relative efficiency of the BLUEs and the AMLEs of ν_0 and ν_1 tend to increase with decreasing amounts of censoring.

The relative efficiency of the AMLE of σ does not exhibit any clear pattern to changes in both complete and Type-II censored samples.

Overall, the BLUEs and the AMLEs of ν_0 and ν_1 are almost as efficient as the MLEs, especially for large sample size N. The AMLE of σ has higher efficiency as compared to the BLUE in both two- and four-grouped samples in the complete sample case.

6.5.2 Accuracy of the Normal Approximation

In Chapter 2, we derived the asymptotic normality of the BLUEs of ν_0, ν_1 and σ. With the large-sample approximation, we also have the asymptotic normality of the MLEs and the AMLEs of ν_0, ν_1 and σ. Based on the simulated probability coverages presented in Tables 5.5.1 - 5.5.4 of

$$\Pr(-1.96 \le p_i \le 1.96) \qquad i = 1,\ 2,\ 3,$$

(which are expected to be approximately 95%) based on BLUEs, MLEs and AMLEs of ν_0, ν_1 and σ, we assess the accuracy of the normal approximation as follows.

Overall, the simulated probability coverages of the BLUEs of ν_0, ν_1 and σ are the closest to 95 % not only for the two- and four-grouped samples, but also in the complete as well as Type-II censored samples.

For both MLEs and AMLEs, the values of the simulated probability coverages increase tremendously (close to 95%) as the total sample size N increases.

The simulated probability coverages of the AMLE of ν_1 appear to have higher values (close to 95%) whereas of ν_0 and σ have lower values as compared to the corresponding MLE.

Hence, in terms of the simulated probability coverages, BLUEs exhibit the best results than the MLEs and the AMLEs.

CHAPTER 7

TEST OF VALIDITY OF MULTI-GROUP EXTREME VALUE REGRESSION MODEL

7.1 Introduction

It is important to check the adequacy of models upon which inferences are based. In this chapter, a test of validity of the multi-group extreme value regression model is discussed. In Section 7.2, we introduce Tiku's procedure, which provides a test for an extreme value model for a single-group sample. We then extend Tiku's test to the multi-group sample situation in Section 7.3. To assess the validity of multi-group extreme value regression model and to test against departures from the original assumption of Weibull distribution of life-times, we explain the determination of the level of significance and the power of this test, respectively, in Section 7.4. In Section 7.5, we simulate the value of level of significance under the standard extreme value model, and the values of the power under five alternatives to the extreme value regression model for various choices of sample sizes and censoring schemes. Finally, we discuss the simulation results in Section 7.6.

Without loss of generality, we consider Type-II censored samples here. Suppose that $y_{1:n_l} \leq y_{2:n_l} \leq \ldots \leq y_{n_l - s_l:n_l}$ are the $n_l - s_l$ smallest observations in a random sample of size n_l from the extreme value population with a location parameter $\nu_0 + \nu_1 x_l$ and a constant scale parameter σ, for l = 1,...,k. Let $z_{i:n_l} = \left(y_{i:n_l} - \nu_0 - \nu_1 x_l\right)/\sigma$; then the sample $z_{1:n_l} \le z_{2:n_l} \le \ldots \le z_{n_l - s_l : n_l}$, for l = 1,...,k, are the first $n_l - s_l$ order statistics from a standard extreme value distribution with density $\exp\{z - e^z\}$, $-\infty < z < \infty$. Assume that $\tilde{\tilde{\nu}}_0$, $\tilde{\tilde{\nu}}_1$ and $\tilde{\tilde{\sigma}}$ are estimators (from any of the three methods of estimation) of the unknown parameters ν_0, ν_1 and σ, respectively, and let $\tilde{\tilde{z}}_{i:n_l} = \left(y_{i:n_l} - \tilde{\tilde{\nu}}_o - \tilde{\tilde{\nu}}_1 x_l\right)/\tilde{\tilde{\sigma}}$. Then the main problem here is to test the hypotheses

$H_0 : \tilde{\tilde{z}}_{i:n_l}$'s are from standard extreme value distribution

vs.

$H_1 : \tilde{\tilde{z}}_{i:n_l}$'s are not from standard extreme value distribution.

7.2 Tiku's Test for a Single Group Sample

To test H_0 against H_1 in a single group sample, one of the tests available in the literature is by Tiku and Singh (1981). They applied the results of Tiku (1980b) to propose a test for an extreme value model that can accommodate Type-II censored data. This procedure appears to have reasonably good power against certain types of alternatives. Let $z_{(i)}$ represent the i-th order statistic from a random sample of size n from the standard extreme value distribution. Let $x_{(1)} \le \ldots \le x_{(r)}$ be the r smallest observations in a random sample of size n from the distribution under study, and define the normalized spacing

$$s_i = \frac{x_{(i+1)} - x_{(i)}}{E(z_{(i+1)} - z_{(i)})} \quad i = 1,\ldots,r\text{-}1.$$

The statistic proposed by Tiku and Singh (1981) is

$$T = \frac{2\sum_{i=1}^{r-2}(r-i-1)s_i}{(r-2)\sum_{i=1}^{r-1}s_i}.$$

Large or small values of T provide evidence against the extreme value model. Let the numerator and the denominator of T be denoted by W_1 and W_2. For large n, the mean and variance of the null distribution of T are then approximated by

$$E(T) \approx 1$$

and

$$Var(T) \approx \frac{Var(W_1)}{E(W_1)^2} + \frac{Var(W_2)}{E(W_2)^2} - \frac{2Cov(W_1, W_2)}{E(W_1)E(W_2)}.$$

Tiku and Singh showed that for $n \geq 20$, the approximation $T \sim N\{1, Var(T)\}$ provides a very good approximation to the null distribution of T.

7.3 Test for the Multi-group Sample

In order to test H_0 against H_1 in the case of MEVR model, we define the normalized spacing as

$$s_{i:n_l} = \frac{\tilde{\tilde{z}}_{i+1:n_l} - \tilde{\tilde{z}}_{i:n_l}}{E\left(z_{(i+1)} - z_{(i)}\right)} \qquad i = 1, \ldots, n_l - s_l - 1, \quad l = 1, \ldots, k,$$

and the statistic for the l-th group as

$$T_l = \frac{2\sum_{i=1}^{n_l - s_l - 2}(n_l - s_l - i - 1)s_{i:n_l}}{(n_l - s_l - 2)\sum_{i=1}^{n_l - s_l - 1} s_{i:n_l}} = \frac{W_{1:n_l}}{W_{2:n_l}}, \; l = 1,\ldots,k.$$

Just as in the case of single group, for large n, the mean and variance of the null distribution of T_l are approximated by

$$E(T_l) \approx 1, \qquad \text{for } l = 1,\ldots,k, \tag{7.3.1}$$

and

$$Var(T_l) \approx \frac{Var(W_{1:n_l})}{E(W_{1:n_l})^2} + \frac{Var(W_{2:n_l})}{E(W_{2:n_l})^2} - \frac{2Cov(W_{1:n_l}, W_{2:n_l})}{E(W_{1:n_l})E(W_{2:n_l})}, \quad \text{for } l = 1,\ldots,k. \tag{7.3.2}$$

The normality approximation yields $T_l \sim N\{1, Var(T_l)\}$, for $l = 1,\ldots,k$.

Define the combined test statistic for the multi-group as

$$T^* = \frac{\sum_{l=1}^{k} T_l \big/ Var(T_l)}{\sum_{l=1}^{k} 1 \big/ Var(T_l)}; \tag{7.3.3}$$

then we have

$$E(T^*) \approx 1, \tag{7.3.4}$$

$$Var(T^*) = \frac{1}{\sum_{l=1}^{k} 1 \big/ Var(T_l)}, \tag{7.3.5}$$

and hence its null distribution is approximated by

$$T^* \sim N\left(1, \quad \frac{1}{\sum_{l=1}^{k} \frac{1}{Var(T_l)}}\right). \tag{7.3.6}$$

Large or small values of T^* provide evidence against the null hypothesis H_0.

7.4 Level of Significance and Power of the Test

To examine the efficiency of the proposed test procedure, we examine the level of significance under the extreme value model and the power under some alternatives.

7.4.1 Level of Significance

Suppose the upper and the lower $\alpha/2$ percentage points of T^* are determined from (7.3.6) in advance for a multi-group sample with specified group size n_l and censoring values s_l, for $l = 1,\ldots,k$. For each Monte Carlo run, k groups of censored observations $y_{1:n_l} \le y_{2:n_l} \le \ldots \le y_{n_l - s_l:n_l}$, for $l = 1,\ldots,k$, are generated from the extreme value population. After calculating the estimates $\tilde{\tilde{v}}_0$, $\tilde{\tilde{v}}_1$ and $\tilde{\tilde{\sigma}}$ (from one of the three procedures), T^* statistic is computed. This computed value of T^* will fall either inside or outside the critical values. If there are m Monte Carlo runs in total, the level of significance is determined as the proportion of times that T^* falls outside the critical values.

7.4.2 Power

Under the alternative distribution, the normalized spacing becomes

$$ {}_a s_{i:n_l} = \frac{{}_a \tilde{\tilde{z}}_{i+1:n_l} - {}_a \tilde{\tilde{z}}_{i:n_l}}{E(z_{(i+1)} - z_{(i)})}, \quad i = 1,\ldots, n_l - s_l - 1, \quad l = 1,\ldots,k. $$

The statistic for the single group is

$$ {}_a T_l = \frac{2 \sum_{i=1}^{n_l - s_l - 2} (n_l - s_l - i - 1)\, {}_a s_{i:n_l}}{(n_l - s_l - 2) \sum_{i=1}^{n_l - s_l - 1} s_{i:n_l}} = \frac{{}_a W_{1:n_l}}{W_{2:n_l}}, \; l = 1,\ldots,k, $$

and its mean and variance become

$$ E({}_a T_l) \approx \frac{E({}_a W_{1:n_l})}{E(W_{2:n_l})} \neq 1, \; l = 1,\ldots,k, $$

and

$$ Var({}_a T_l) \approx \left[\frac{E({}_a W_{1:n_l})}{E(W_{2:n_l})}\right]^2 \left[\frac{Var({}_a W_{1:n_l})}{E({}_a W_{1:n_l})^2} + \frac{Var(W_{2:n_l})}{E(W_{2:n_l})^2} - \frac{2Cov({}_a W_{1:n_l}, W_{2:n_l})}{E({}_a W_{1:n_l}) E(W_{2:n_l})}\right], \; l = 1,\ldots,k. $$

The combined test statistic then becomes

$$ {}_a T^* = \frac{\sum_{l=1}^{k} {}_a T_l \big/ Var(T_l)}{\sum_{l=1}^{k} 1 \big/ Var(T_l)}. $$

The index "a" stands for "computed under alternative distribution". Similar to what was done earlier in determining the level of significance, we can now compute the proportion of times that ${}_a T^*$ falls outside the critical values to determine the power.

It should be mentioned here that the simulation results are invariant to the estimation method employed. This fact is obvious by examining the normalized spacing

$s_{i:n_l} = \dfrac{\tilde{\tilde{z}}_{i+1:n_l} - \tilde{\tilde{z}}_{i:n_l}}{E(z_{(i+1)} - z_{(i)})}$, where the estimators of $\tilde{\tilde{v}}_0$, and $\tilde{\tilde{v}}_1$ get canceled out, while the scale invariance is clearly evident from the ratio form. Therefore, the validity examination of the MEVR model under various estimation methods turns out to be simply examining the validity of the sample itself.

7.5 Simulations and Results

In the simulation study, we took $\nu_0 = 0, \nu_1 = 1$ and $\sigma = 1$, and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored sample. Based on 10,000 Monte Carlo runs, we simulated the level of significance under the standard extreme value model and the power under five alternatives: Normal(0,1), Lognormal(0,1), Gamma(2,1) Gamma(4,1) and Gamma(6,1), at 5% and 10% significance levels in the following cases:

1. Complete sample

two groups: n = [6 6], [6 10], [8 8], [8 10], [10 10], [15 15], [15 20] and [20 20].

four groups: n = [6 6 6 6], [6 6 10 10], [8 8 8 8], [8 8 10 10], [10 10 10 10], [15 15 15 15], [15 15 20 20] and [20 20 20 20].

2. Type-II censored sample

two groups: s = [4 4], [4 0], [2 2] and [2 0] from n = [10 10] and [5 5] and [5 0] from n = [20 20].

four groups: s = [4 4 4 4], [4 4 0 0], [2 2 2 2] and [2 2 0 0] from n = [10 10 10 10] and

[5 5 5 5] and [5 5 0 0] from n = [20 20 20 20].

These results are presented in Tables 7.5.1 – 7.5.8.

The histograms and normal p-p plots for the simulated values of T^* based on 10,000 runs are constructed for n = [6 6], [15, 15], [6 6 6 6] and [15 15 15 15] for the compete samples; and for s = [4 4] from [10 10], [5 5] from [20 20], [4 4 4 4] from [10 10 10 10] and [5 5 5 5] from [20 20 20 20] for the Type-II censored samples. These results are presented in Figures 7.5.1 – 7.5.8.

7.6 Discussion

With regard to the following aspects

1) The complete sample vs. the Type-II censored sample

2) The degree of censoring

3) The group size n (or total sample size N)

4) The two-grouped vs. the four-grouped sample (or the number of levels of x),

we assess the simulated value of the level of significance under the standard extreme value model and the power under five alternatives: Normal(0,1), Lognormal(0,1), Gamma(2,1) Gamma(4,1) and Gamma(6,1), at pre-fixed 5% and 10% significance levels.

It is expected that the simulated value of the level of significance under the standard extreme value model are close to 5% (or 10 %), at the pre-fixed 5% (or 10%) significance levels. The simulated value of the level of significance from the complete sample seems to be close to the pre-fixed 5% (or 10%) as compared to the Type-II censored sample. The simulated value of the level of significance does not appear to be affected by the total sample size N as well as the number of levels of x.

The simulated means and variances of the T^* statistic defined in (7.3.3) are almost identical to the expected values that are computed by the expressions in (7.3.4) and (7.3.5). The results do not appear to be affected by any of the factors mentioned above.

The histograms and normal p-p plots show that the normal distribution provides a very good approximation to the null distribution of T^* statistic for the MEVR model regardless of the sample size, the number of levels of x, and the type of the samples (complete or censored).

With the same total sample size N, the value of power from the complete sample is much higher than that from the Type-II censored sample. This is true under all five alternative distributions. The value of power increases dramatically with a major increase in the total sample size N and this is true for both complete and Type-II censored samples. When the groups are of the same size, the four-grouped sample tends to have higher values of power as compared to the two-grouped sample. This is also true for both complete and Type-II censored samples. In the Type-II censored samples case, the value of power appears to be smaller when the degree of censoring is higher. With the large group size, say $n \geq 10$, the powers from all types of samples (doesn't matter complete or censored, two-grouped or four-grouped) are close to 100% under the alternative distributions considered except for the Normal (0, 1).

Table 7.5.1 Simulation results: Level of significance from extreme value complete samples

Complete Sample (n)	Significance level	
	5%	10%
[6 6]	3.86%	8.49%
[6 10]	4.58%	9.61%
[8 8]	4.11%	8.90%
[8 10]	4.71%	9.43%
[10 10]	4.42%	9.37%
[15 15]	4.66%	9.37%
[15 20]	5.29%	10.20%
[20 20]	4.63%	9.40%
[6 6 6 6]	3.78%	8.23%
[6 6 10 10]	4.02%	9.09%
[8 8 8 8]	4.24%	8.94%
[8 8 10 10]	4.68%	9.35%
[10 10 10 10]	4.17%	8.80%
[15 15 15 15]	4.70%	9.42%
[15 15 20 20]	4.60%	9.37%
[20 20 20 20]	4.40%	8.97%

Table 7.5.2 Simulation results: Level of significance from extreme value Type-II censored samples

Censoring Scheme (s)	Significance level	
	5%	10%
[4 4]	3.53%	7.96%
[4 0]	4.18%	8.62%
[2 2]	4.09%	8.75%
[2 0]	4.38%	9.57%
*[5 5]	4.46%	8.82%
*[5 0]	4.51%	9.29%
[4 4 4 4]	3.27%	7.26%
[4 4 0 0]	4.21%	8.92%
[2 2 2 2]	4.14%	8.43%
[2 2 0 0]	4.17%	8.83%
*[5 5 5 5]	4.62%	8.90%
*[5 5 0 0]	4.44%	9.16%

Asterisk denotes censoring is from n = [20 20] or [20 20 20 20], otherwise is from [10 10] or [10 10 10 10],

Table 7.5.3 Results for "T* - statistics" from extreme value complete samples

Complete Sample (n)	Approximated Mean	Simulated Mean	Approximated Variance	Simulated Variance
[6 6]	1.0000	0.9890	0.0393	0.0353
[6 10]	1.0000	0.9914	0.0247	0.0232
[8 8]	1.0000	0.9919	0.0248	0.0231
[8 10]	1.0000	0.9920	0.0208	0.0194
[10 10]	1.0000	0.9907	0.0180	0.0177
[15 15]	1.0000	0.9953	0.0106	0.0104
[15 20]	1.0000	0.9951	0.0087	0.0083
[20 20]	1.0000	0.9970	0.0075	0.0072
[6 6 6 6]	1.0000	0.9884	0.0197	0.0175
[6 6 10 10]	1.0000	0.9925	0.0123	0.0115
[8 8 8 8]	1.0000	0.9895	0.0124	0.0116
[8 8 10 10]	1.0000	0.9925	0.0104	0.0097
[10 10 10 10]	1.0000	0.9921	0.0090	0.0086
[15 15 15 15]	1.0000	0.9936	0.0053	0.0051
[15 15 20 20]	1.0000	0.9943	0.0044	0.0043
[20 20 20 20]	1.0000	0.9965	0.0037	0.0035

Table 7.5.4 Results for "T* - statistics" from extreme value Type-II censored samples

Censoring Scheme (s)	Approximated Mean	Simulated Mean	Approximated Variance	Simulated Variance
[4 4]	1.0000	0.9956	0.0456	0.0399
[4 0]	1.0000	0.9921	0.0258	0.0243
[2 2]	1.0000	0.9965	0.0275	0.0248
[2 0]	1.0000	0.9942	0.0217	0.0204
*[5 5]	1.0000	0.9991	0.0121	0.0115
*[5 0]	1.0000	0.9970	0.0092	0.0090
[4 4 4 4]	1.0000	0.9972	0.0228	0.0198
[4 4 0 0]	1.0000	0.9936	0.0129	0.0121
[2 2 2 2]	1.0000	0.9953	0.0137	0.0130
[2 2 0 0]	1.0000	0.9944	0.0109	0.0101
*[5 5 5 5]	1.0000	0.9970	0.0061	0.0056
*[5 5 0 0]	1.0000	0.9957	0.0046	0.0044

Asterisk denotes censoring is from n = [20 20] or [20 20 20 20], otherwise is from [10 10] or [10 10 10 10],

Table 7.5.5 Simulation results: values of the power at 5% significance level from complete samples

Complete Sample (n)	Alternatives				
	Normal (0, 1)	Lognormal (0, 1)	Gamma (2, 1)	Gamma (4, 1)	Gamma (6, 1)
[6 6]	10.37%	70.92%	39.48%	28.53%	24.32%
[6 10]	16.31%	89.85%	63.52%	48.85%	42.56%
[8 8]	16.11%	89.40%	62.85%	48.04%	41.30%
[8 10]	18.95%	94.19%	71.59%	56.39%	49.50%
[10 10]	22.64%	96.62%	78.65%	63.52%	57.48%
[15 15]	38.58%	99.87%	95.96%	88.57%	83.03%
[15 20]	46.49%	100.00%	98.45%	93.69%	89.47%
[20 20]	53.66%	99.99%	99.45%	96.63%	94.16%
[6 6 6 6]	17.29%	93.72%	69.33%	52.50%	45.45%
[6 6 10 10]	29.16%	99.33%	90.86%	78.57%	71.84%
[8 8 8 8]	29.74%	99.28%	90.48%	78.60%	70.62%
[8 8 10 10]	35.12%	99.87%	95.45%	85.33%	79.37%
[10 10 10 10]	41.77%	99.95%	97.34%	90.71%	85.62%
[15 15 15 15]	66.61%	100.00%	99.93%	99.33%	98.35%
[15 15 20 20]	76.99%	100.00%	100.00%	99.87%	99.56%
[20 20 20 20]	83.20%	100.00%	100.00%	99.96%	99.89%

Table 7.5.6 Simulation results: values of the power at 5% significance level from Type-II censored samples

Censoring Scheme (s)	Alternatives				
	Normal (0, 1)	Lognormal (0, 1)	Gamma (2, 1)	Gamma (4, 1)	Gamma (6, 1)
[4 4]	4.56%	32.30%	19.17%	11.81%	9.69%
[4 0]	13.15%	83.58%	55.87%	40.62%	34.73%
[2 2]	9.78%	70.91%	45.05%	28.59%	24.42%
[2 0]	15.49%	89.98%	66.04%	48.90%	41.44%
*[5 5]	19.34%	97.15%	83.12%	61.49%	52.88%
*[5 0]	37.32%	99.81%	96.72%	87.91%	82.40%
[4 4 4 4]	7.74%	61.38%	39.62%	22.86%	18.69%
[4 4 0 0]	23.06%	98.36%	85.52%	69.91%	62.09%
[2 2 2 2]	17.06%	94.95%	76.52%	55.62%	46.23%
[2 2 0 0]	29.56%	99.61%	91.95%	79.08%	71.31%
*[5 5 5 5]	37.15%	99.97%	98.79%	90.65%	84.37%
*[5 5 0 0]	65.44%	100.00%	99.98%	99.43%	98.41%

Asterisk denotes censoring is from n = [20 20] or [20 20 20 20], otherwise is from [10 10] or [10 10 10 10],

Table 7.5.7 Simulation results: values of the power at 10% significance level from complete samples

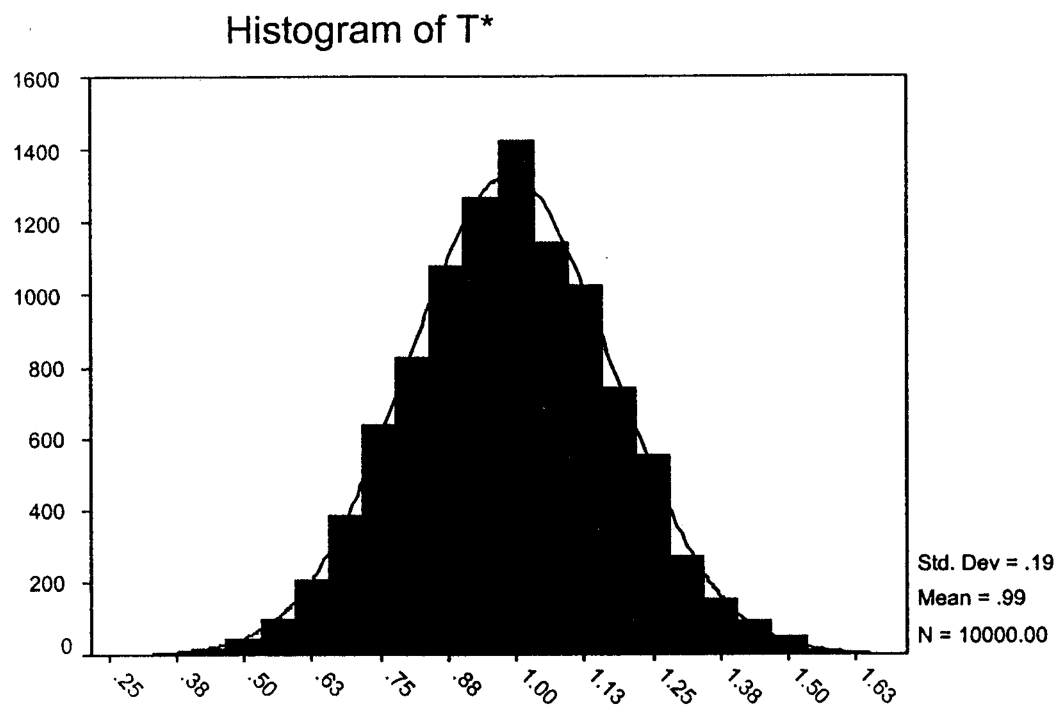
Complete Sample (n)	Alternatives				
	Normal (0, 1)	Lognormal (0, 1)	Gamma (2, 1)	Gamma (4, 1)	Gamma (6, 1)
[6 6]	18.47%	79.90%	53.06%	41.16%	36.94%
[6 10]	26.42%	93.91%	75.57%	61.99%	55.26%
[8 8]	25.91%	93.80%	74.62%	60.88%	55.09%
[8 10]	29.21%	96.99%	81.57%	68.70%	62.16%
[10 10]	34.13%	98.25%	86.96%	75.04%	68.92%
[15 15]	51.84%	99.97%	98.23%	93.61%	89.86%
[15 20]	60.07%	100.00%	99.28%	96.77%	94.23%
[20 20]	66.39%	100.00%	99.77%	98.36%	97.05%
[6 6 6 6]	27.62%	96.49%	79.89%	65.14%	58.82%
[6 6 10 10]	42.25%	99.64%	95.20%	86.60%	81.85%
[8 8 8 8]	42.06%	99.70%	94.95%	86.52%	80.75%
[8 8 10 10]	48.35%	99.95%	97.73%	91.54%	87.60%
[10 10 10 10]	55.61%	99.98%	98.75%	94.93%	91.82%
[15 15 15 15]	78.54%	100.00%	99.96%	99.77%	99.28%
[15 15 20 20]	85.38%	100.00%	100.00%	99.96%	99.83%
[20 20 20 20]	90.12%	100.00%	100.00%	99.98%	99.97%

Table 7.5.8 Simulation results: values of the power at 10% significance level from Type-II censored samples

Censoring	Alternatives				
Scheme (s)	Normal (0, 1)	Lognormal (0, 1)	Gamma (2, 1)	Gamma (4, 1)	Gamma (6, 1)
[4 4]	9.65%	46.66%	31.54%	21.54%	18.26%
[4 0]	21.91%	90.10%	68.33%	53.40%	47.63%
[2 2]	17.75%	81.22%	59.42%	41.73%	36.80%
[2 0]	25.70%	94.06%	77.37%	62.28%	54.60%
*[5 5]	30.43%	98.63%	90.35%	74.38%	66.00%
*[5 0]	50.02%	99.93%	98.46%	93.15%	89.54%
[4 4 4 4]	14.54%	73.65%	54.29%	35.20%	29.50%
[4 4 0 0]	34.74%	99.26%	91.69%	80.41%	74.01%
[2 2 2 2]	27.92%	97.68%	85.76%	68.53%	60.23%
[2 2 0 0]	42.09%	99.90%	95.67%	86.96%	81.59%
*[5 5 5 5]	50.75%	100.00%	99.54%	95.54%	91.31%
*[5 5 0 0]	76.97%	100.00%	100.00%	99.81%	99.29%

Asterisk denotes censoring is from n = [20 20] or [20 20 20 20], otherwise is from [10 10] or [10 10 10 10],

Figure 7.5.1 Complete sample of $n = [6\ 6]$



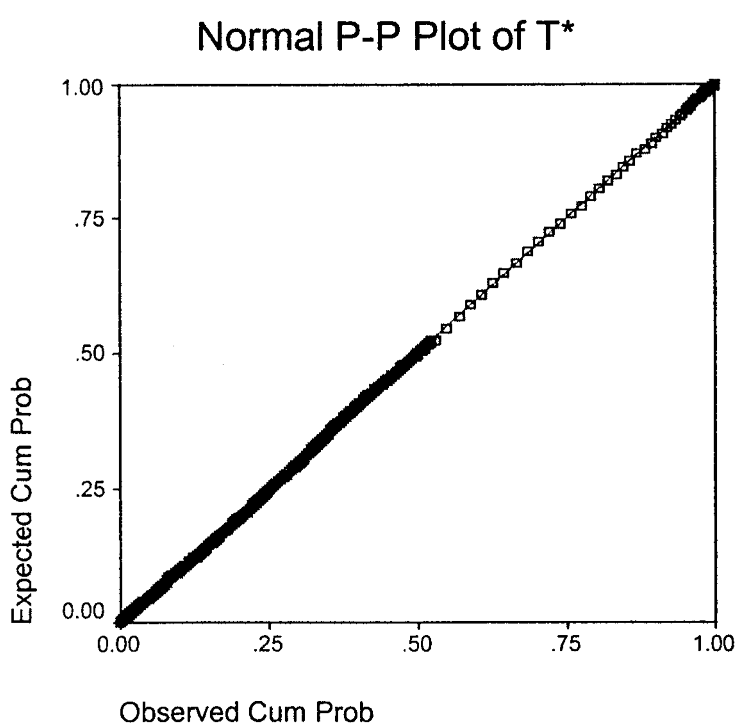
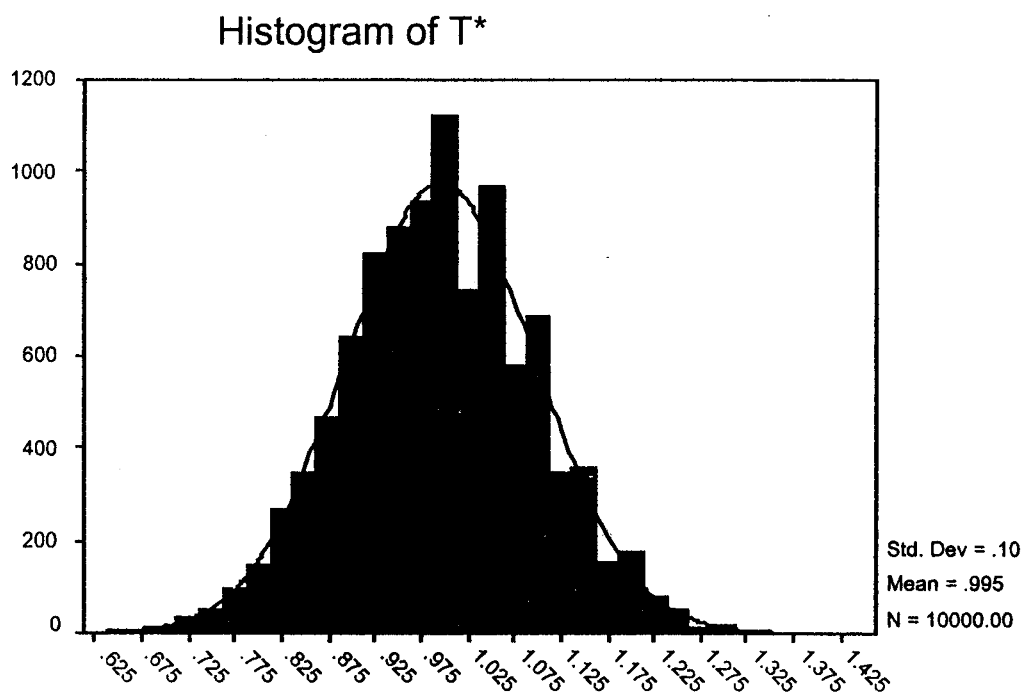


Figure 7.5.2 Complete sample of $n = [15\ 15]$



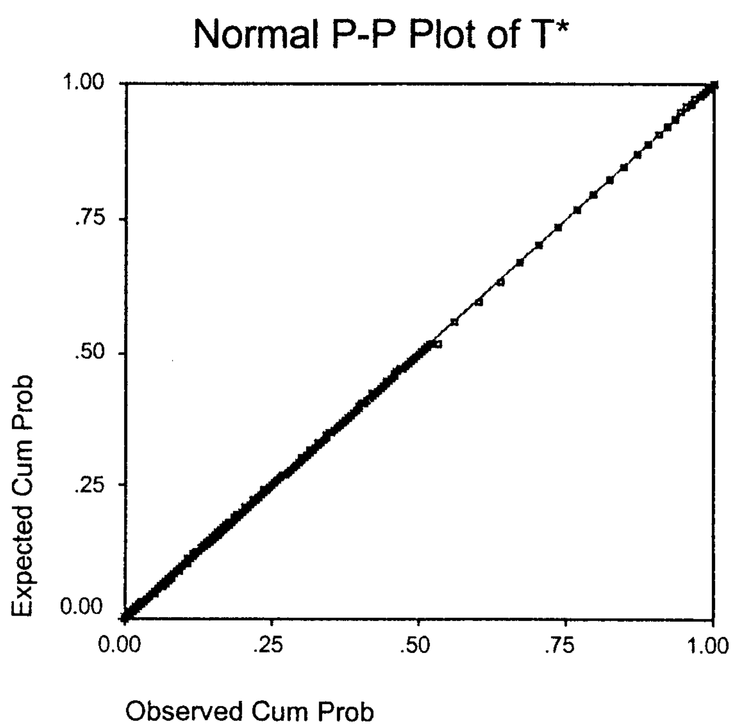
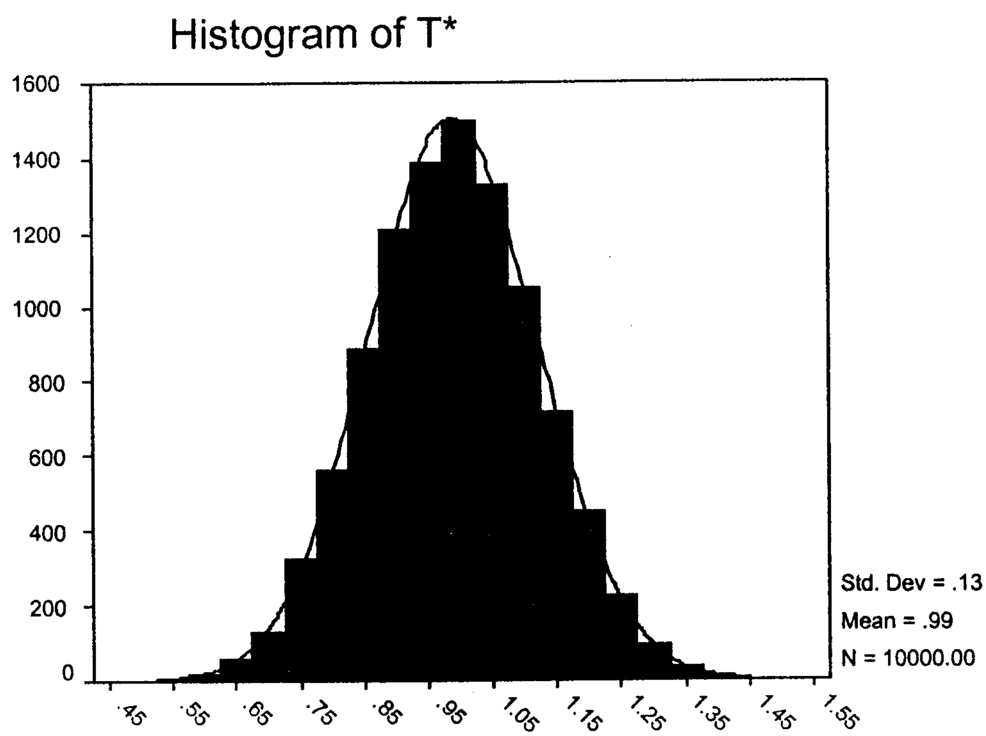


Figure 7.5.3 Complete sample of $n = [6\ 6\ 6\ 6]$



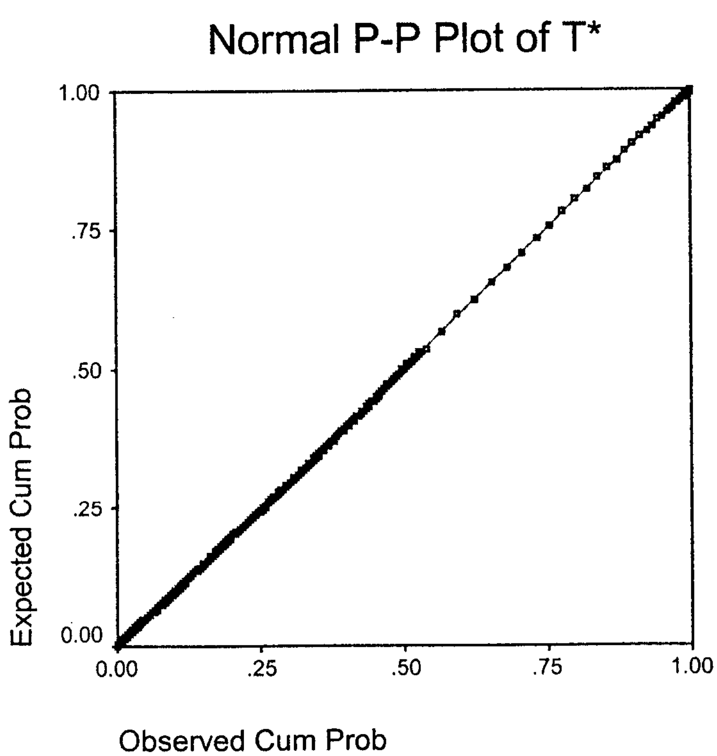
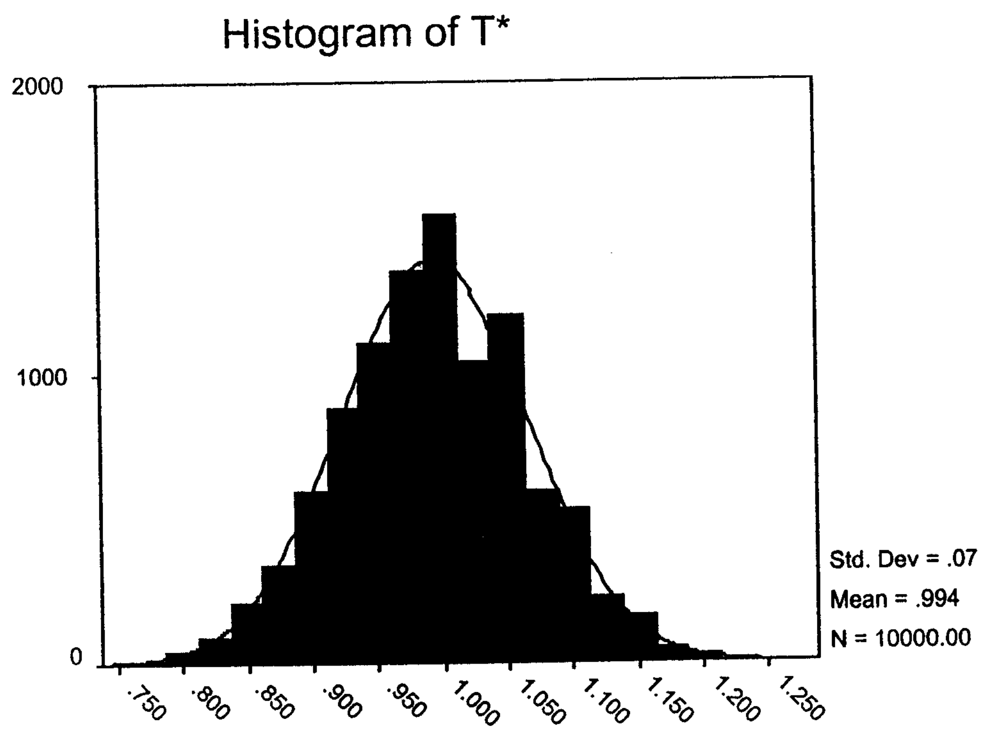


Figure 7.5.4 Complete sample of $n = [15\ 15\ 15\ 15]$



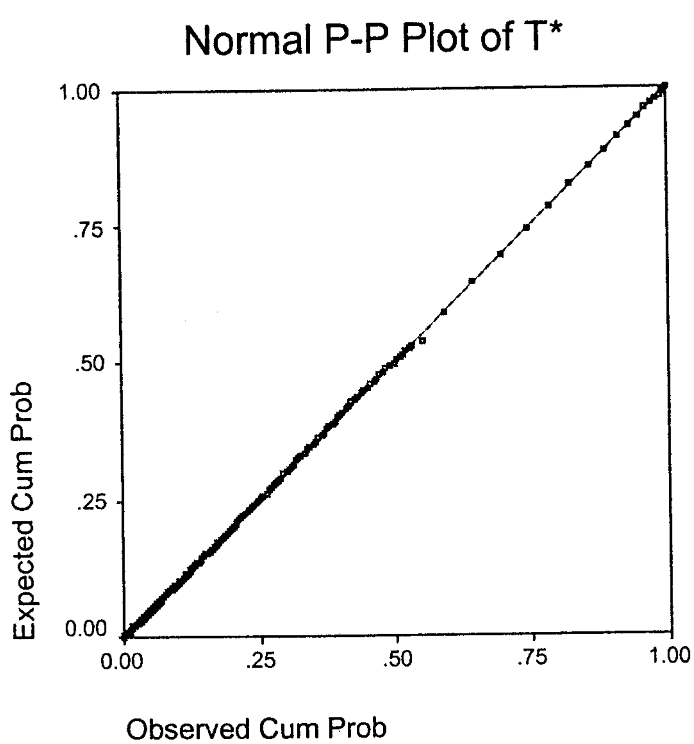


Figure 7.5.5 Type-II censored sample of $s = [4\ 4]$ from $n = [10\ 10]$

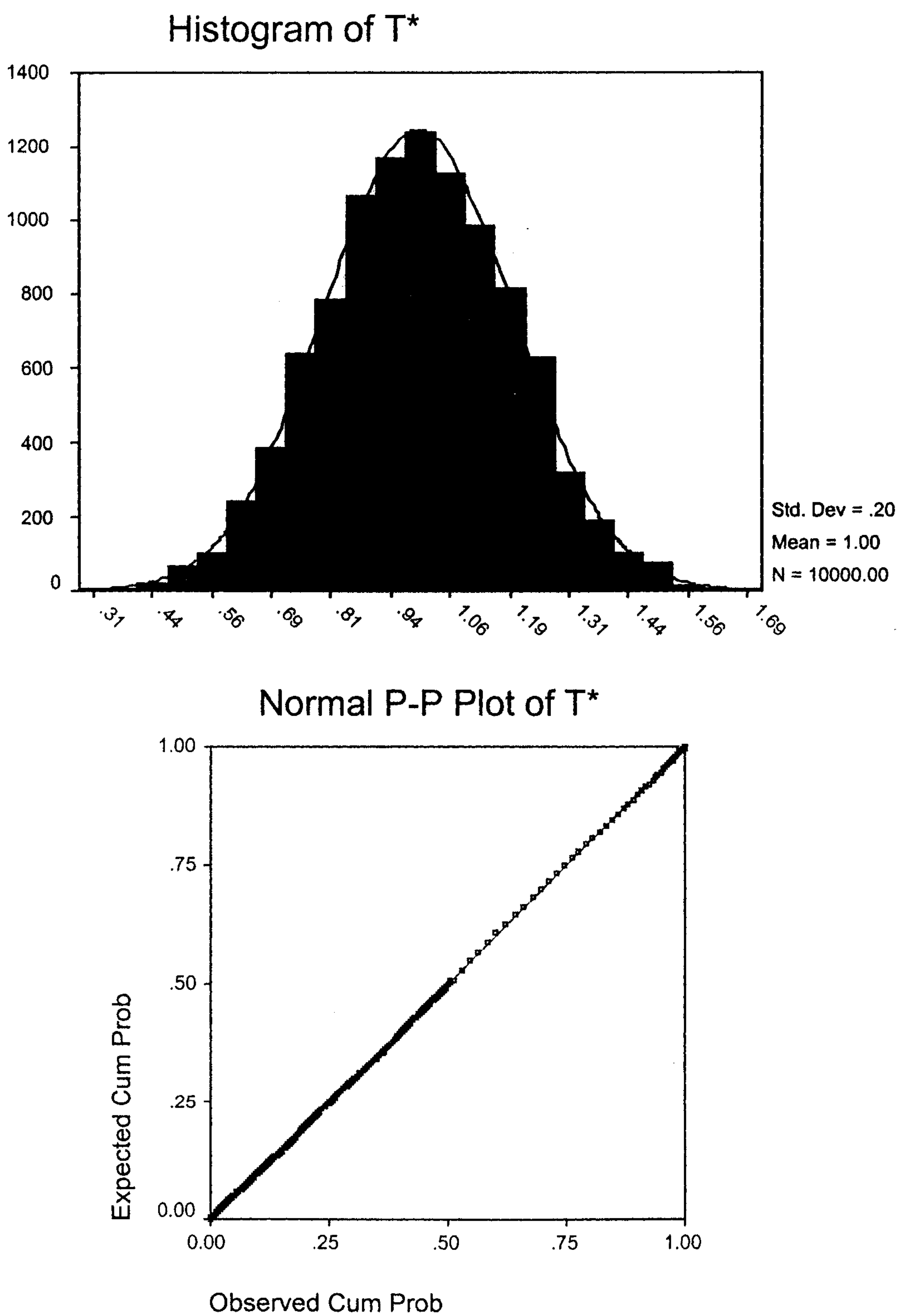
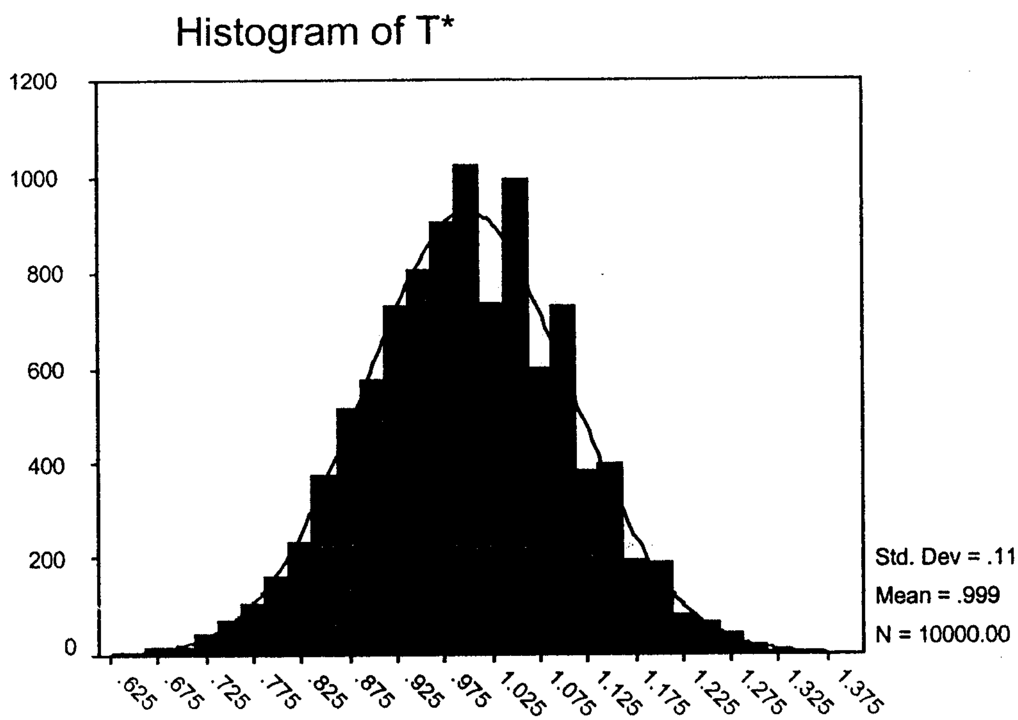


Figure 7.5.6 Type-II censored sample of s = [5 5] from n =[20 20]



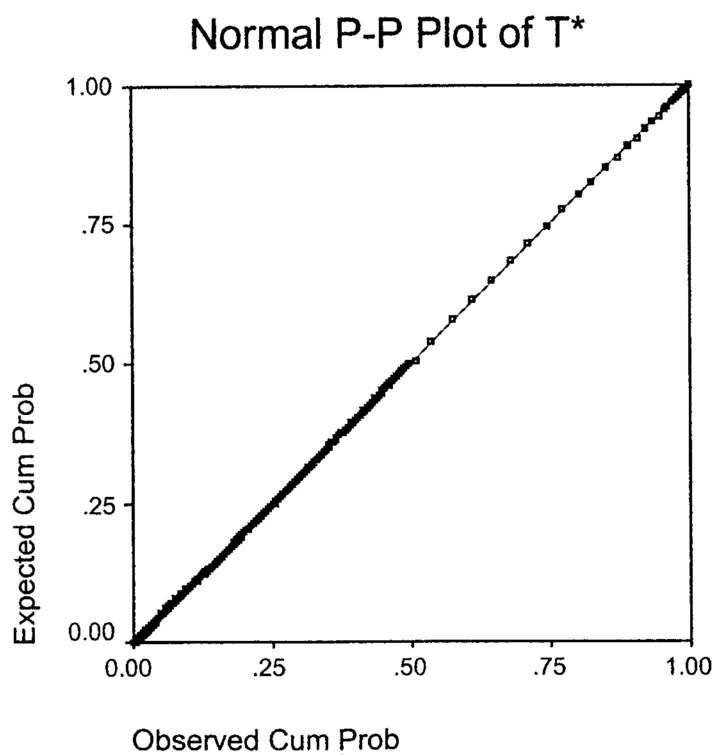
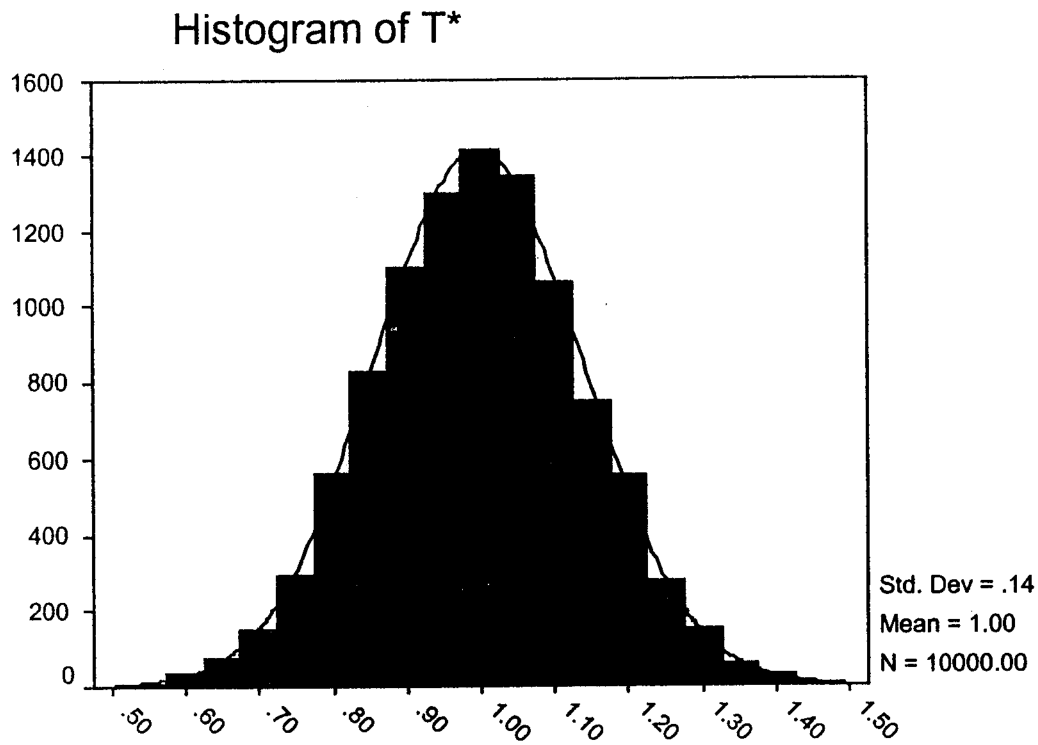


Figure 7.5.7 Type-II censored sample of $s = [4\ 4\ 4\ 4]$ from $n = [10\ 10\ 10\ 10]$



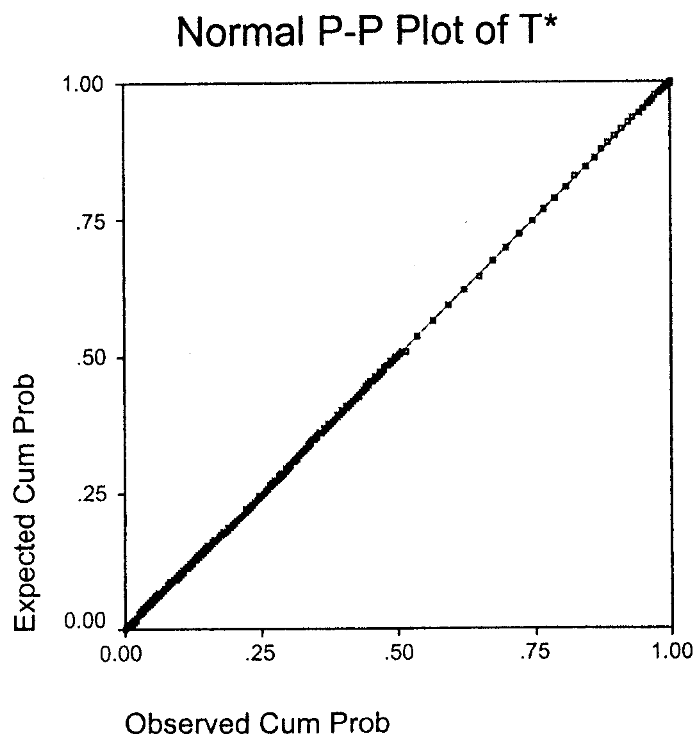
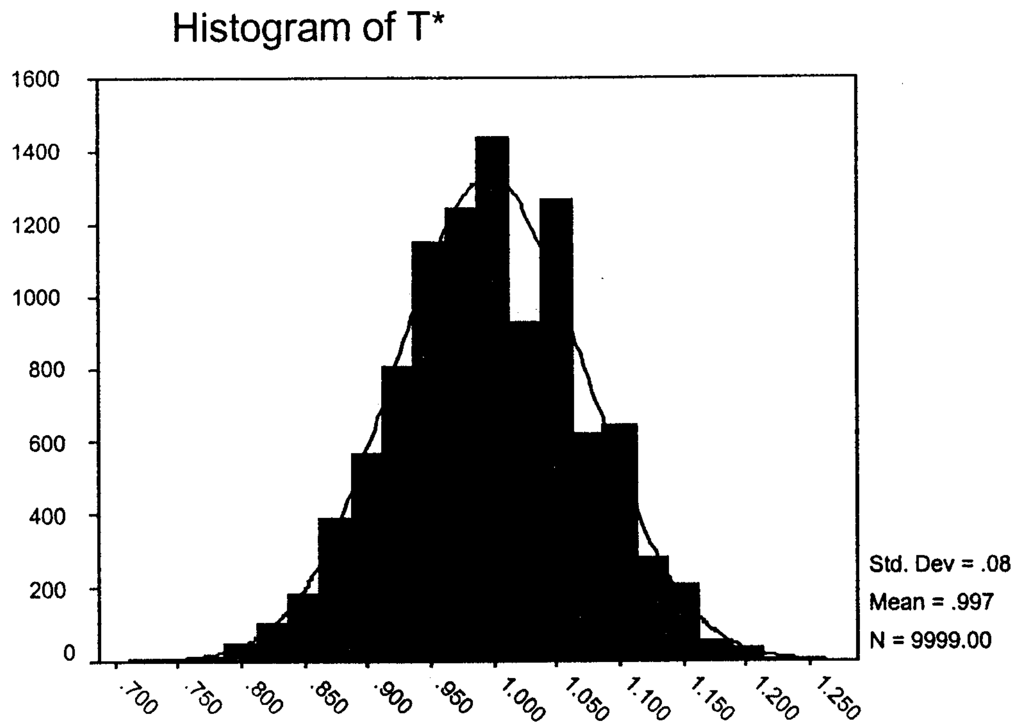
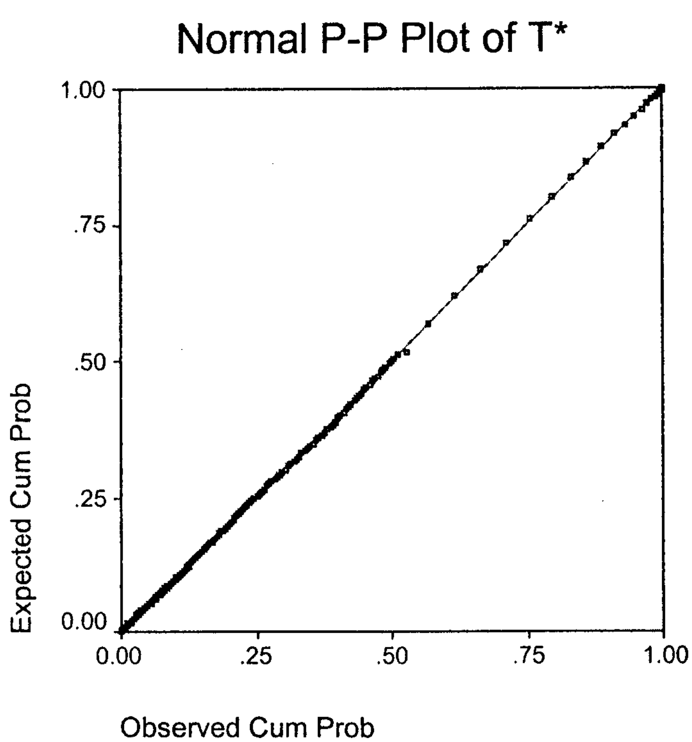


Figure 7.5.8 Type-II censored sample of $s = [5\ 5\ 5\ 5]$ from $n = [20\ 20\ 20\ 20]$





CHAPTER 8

ILLUSTRATIVE EXAMPLES

8.1 Introduction

In this chapter, we illustrate the BLUE, MLE and AMLE methods using three real-life examples including both complete as well as Type-II right-censored samples. We present a detailed illustration of these approaches for the complete sample case in Example 8.2.1. Then we present the analysis and the results for Type-II right-censored sample in Example 8.2.2, and for both complete and Type-II right-censored sample in Example 8.2.3.

8.2 Examples

Example 8.2.1: Nelson (1970) has presented data on the time to breakdown of a type of electrical insulating fluid subject to a constant voltage stress. The data, shown in Table 8.2.1.1, are breakdown times for seven groups of specimens, each group involving a different voltage level. A model suggested by engineering considerations is that, for a fixed voltage, time to breakdown has a Weibull distribution. Furthermore, distributions corresponding to different voltage levels are thought to differ only with respect to their scale parameters through power model $\alpha = cV^{p}$, and the shape parameter δ being the same for different levels. The data are uncensored, and times to break down are given in minutes and voltage levels are given in kV.

Table 8.2.1.1 Insulating Fluid Failure Data

Voltage Level (kV)	n_l	Breakdown Times (min)
26	3	5.79, 1579.52, 2323.7
28	5	68.85, 426.07, 110.29, 108.29, 1067.6
30	11	17.05, 22.66, 21.02, 175.88, 139.07, 144.12, 20.46, 43.40, 194.90, 47.30, 7.74
32	15	0.40, 82.85, 9.88, 89.29, 215.10, 2.75, 0.79, 15.93, 3.91, 0.27, 0.69, 100.58, 27.80, 13.95, 53.24
34	19	0.96, 4.15, 0.19, 0.78, 8.01, 31.75, 7.35, 6.50, 8.27, 33.91, 32.52, 3.16, 4.85, 2.78, 4.67, 1.31, 12.06, 36.71, 72.89
36	15	1.97, 0.59, 2.58, 1.69, 2.71, 25.50, 0.35, 0.99, 3.99, 3.67, 2.07, 0.96, 5.35, 2.90, 13.77
38	8	0.47, 0.73, 1.40, 0.74, 0.39, 1.13, 0.09, 2.38

Lawless (1982) presented two formal tests, viz. likelihood ratio test and Bartlett's test, to assess the hypothesis: $H_0 : \delta_1 = \cdots = \delta_7$. The significance levels under H_0 from the likelihood ratio test and Bartlett's test are 0.14 and 0.22, respectively. Therefore, there is enough evidence to assume the equality of the shape parameters.

Using the test proposed in the last chapter, we examine the extreme value distribution assumption for this multi-grouped sample. A p-value of 0.0728 does not give enough evidence to reject the null hypothesis if we use the traditional 5% level of significance.

Let $x_l = \log V_l$ and $\alpha_l = \exp(\nu_0 + \nu_1 x_l)$, where $\nu_0 = \log c$ and $\nu_1 = p$. With $\sigma = 1/\delta$ and the log lifetime $y_{i:n_l} = \log t_{i:n_l}$, this is of the form of the MEVR model with

$$\mu_l = \nu_0 + \nu_1 x_l, \; l = 1, \ldots, 7,$$

and

$$y_{i:n_l} = \nu_0 + \nu_1 x_l + \sigma z_{i:n_l}, \qquad i = 1, \ldots n_l, \qquad l = 1, \ldots, 7,$$

where z has a standard extreme value distribution with density $\exp\{z - e^z\}$, $-\infty < z < \infty$.

Based on the observations $y_{i:n_l}$ and x_l, and the means, variances and covariances of the order statistics $z_{i:n_l}$, $i = 1, \ldots, n_l$, $l = 1, \ldots, 7$, we determine the BLUEs ν_0^*, ν_1^* and σ^* from formula (2.2.9), and their variances and covariances from (2.2.11).

From Eqs. (3.2.1) – (3.2.3), we apply the Newton-Raphson iterative procedure and obtain the MLEs $\hat{\nu}_0$, $\hat{\nu}_1$ and $\hat{\sigma}$. The asymptotic variances and covariances are computed by inverting the expected information Matrix I that is presented in (3.3.1).

For the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$, we use the explicit expressions in (4.2.4) – (4.2.6) to compute the values of the estimates and obtain their asymptotic variances and covariances by inverting the expected information Matrix I^* that is presented in (4.3.1). We determine the approximate bias for $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ as well by expressions in (4.4.1) – (4.4.3).

Finally, we apply the asymptotic normality of the BLUEs, MLEs and AMLEs and use the pivotal quantities in (5.4.1) in order to compute the 95% confidence intervals for ν_0, ν_1 and σ.

We have presented all these results in Tables 8.2.1.2 - 8.2.1.5.

Table 8.2.1.2 Estimates from the complete sample for Example 8.2.1

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	65.8483	-18.0101	1.3413
MLE	64.8472	-17.7296	1.2877
AMLE	63.5906	-17.3992	1.3158

Table 8.2.1.3 Asymptotic variances and covariances from the complete sample for Example 8.2.1

Asymptotic variances and covariances $/\sigma^2$								
BLUE			MLE			AMLE		
19.0421	-5.4410	0.0088	17.2443	-4.9285	-0.0034	20.1903	-5.7685	0.0120
-5.4410	1.5559	-0.0034	-4.9285	1.4098	0.0000	-5.7685	1.6493	-0.0036
0.0088	-0.0034	0.0093	-0.0034	0.0000	0.0080	0.0120	-0.0036	0.0044

* Values for BLUE are exact.

Table 8.2.1.4 Approximate bias of AMLEs from the complete sample for Example 8.2.1

Approximate bias of AMLEs $/\sigma$		
ν_0	ν_1	σ
-1.1877	0.3125	0.0012

Table 8.2.1.5 95% confidence interval from the complete sample for Example 8.2.1

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	50.4609	81.2357	-22.4085	-13.6117	1.00124	1.68136
MLE	51.3903	78.3041	-21.5798	-13.8620	0.99701	1.57839
AMLE	47.1551	77.6507	-21.4447	-12.7287	1.09191	1.54209

LL denotes lower limit and UL denotes upper limit of the 95% confidence interval.

Example 8.2.2: Stone (1978) has reported an experiment in which specimens of solid epoxy electrical insulation were studied in an accelerated voltage life test. In all, 20 specimens were tested at each of three voltage levels: 52.5, 55.0 and 57.5 kilovolts. Failure times, in minutes, for the insulation specimens are given in Table 8.2.2.1. Asterisk denotes a censored observation.

Table 8.2.2.1 Failure Times for Epoxy Insulation Specimens at Three Voltage levels

Voltage (kV)	Failure times (min)
52.5	4690, 740, 1010, 1190, 2450, 1390, 350, 6095, 3000, 1458, 6200*, 550, 1690, 745, 1225, 1480, 245, 600, 246, 1805
55.0	258, 114, 312, 772, 498, 162, 444, 1464, 132, 1740*, 1266, 300, 2440*, 520, 1240, 2600*, 222, 144, 745, 396
57.5	510, 1000*, 252, 408, 528, 690, 900*, 714, 348, 546, 174, 696, 294, 234, 288, 444, 390, 168, 558, 288

We assume this to be a Type-II right-censored sample and that the different voltage levels differ only with respect to their scale parameters through the power law model $\alpha = cV^{p}$ (correspond to $\mu(x) = \nu_0 + \nu_1 x$ in the MEVR model, where $\mu(x) = \log(\alpha)$, $x = \log(V)$, $\nu_0 = \log c$ and $\nu_1 = p$). Using Bartlett's test, we first examine the equality of the shape parameters among all three groups. The significance level under H_0 is 0.0082. We then test H_0 between the first two groups, last two groups, and the first and last groups. The significance levels turn out to be 0.0063, 0.0516 and 0.0397, respectively. It appears that only groups 2 and 3 show no significant evidence against the equality of shape parameters if we use the usual 5% level of significance.

By using the test proposed in the last chapter, we examine MEVR model hypothesis. The p-value equals 0.1032, which indicates that the extreme value model is plausible to this two-grouped sample.

Based on these two-grouped Type-II right-censored sample with r = [3 2], we present the results of the analysis in Tables 8.2.2.2 - 8.2.2.5.

Table 8.2.2.2 Estimates from the Type-II right-censored sample for Example 8.2.2

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	52.7001	-11.4702	0.6700
MLE	54.3099	-11.8694	0.6583
AMLE	50.5854	-10.9542	0.6726

Table 8.2.2.3 Asymptotic variances and covariances from the Type-II right-censored sample for Example 8.2.2

Asymptotic variances and covariances/σ^2								
BLUE*			MLE			AMLE		
972.5061	-241.3027	0.1690	941.3632	-233.5779	0.1492	990.8492	-245.8538	0.1283
-241.3027	59.8750	-0.0427	-233.5779	57.9589	-0.0380	-245.8538	61.0042	-0.0320
0.1690	-0.0427	0.0215	0.1492	-0.0380	0.0196	0.1283	-0.0320	0.0172

*Values for BLUE are exact.

Table 8.2.2.4 Approximate bias of AMLEs from the Type-II right-censored sample for Example 8.2.2

Approximate bias of AMLEs/σ		
ν_0	ν_1	σ
-0.2079	0.0398	-0.0231

Table 8.2.2.5 95% confidence interval from the Type-II right-censored sample for Example 8.2.2

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	25.2621	80.1381	-18.2783	-4.6621	0.54099	0.79901
MLE	28.2494	80.3704	-18.3358	-5.4030	0.53939	0.77721
AMLE	22.4666	78.2884	-17.8399	-3.9889	0.53321	0.76579

LL denotes lower limit and UL denotes upper limit of the 95% confidence interval.

Example 8.2.3: In Table 8.2.3.1, McCool (1980) has given the failure times for hardened steel specimens in a rolling constant fatigue test; 10 independent observations were taken at each of 4 values of contact stress. Engineering background suggests that at stress level s, failure time should have approximately a Weibull distribution with a scale parameter α related to s by a power law relationship $\alpha = cs^p$ (correspond to $\mu(x) = \nu_0 + \nu_1 x$ in the MEVR model, where $\mu(x) = \log(\alpha)$, $x = \log(s)$, $\nu_0 = \log c$ and $\nu_1 = p$), and with a shape parameter δ that is independent of s.

Table 8.2.3.1 Failure Times for Steel Specimens at Four stress Levels

Stress ($psi^2 \div 10^6$)	Ordered Failure Times
0.87	1.67, 2.20, 2.51, 3.00, 2.90, 4.70, 7.53, 14.70, 27.8, 37.4
0.99	0.80, 1.00, 1.37, 2.25, 2.95, 3.70, 6.07, 6.65, 7.05, 7.37
1.09	0.012, 0.18, 0.2, 0.24, 0.26, 0.32, 0.32, 0.42, 0.44, 0.08
1.18	0.073, 0.098, 0.117, 0.135, 0.175, 0.262, 0.270, 0.350, 0.386, 0.456

In order to present the procedures for both complete sample and Type-II right-censored sample, we have used the censoring scheme s = [2 1 4 3] to the complete sample in order to get a Type-II right-censored sample. The significance level of the test for the equality of shape parameters for complete and Type-II right-censored samples, based on Bartlett's test, turn out to be 0.21 and 0.74, respectively.

The p-values of the test for the MEVR model for complete and Type-II right-censored samples turn out to be 0.5782 and 0.3382, respectively. Therefore, the MEVR model is suitable for complete sample as well as Type-II right-censored sample.

We have presented the results of these analyses for complete sample in Tables 8.2.3.2 - 8.2.3.5.

Table 8.2.3.2 Estimates from the complete sample for Example 8.2.3

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	0.7321	-13.7518	0.7862
MLE	0.7842	-13.8635	0.8634
AMLE	0.6108	-13.5491	0.8892

Table 8.2.3.3 Asymptotic variances and covariances from the complete sample for Example 8.2.3

Asymptotic variances and covariances$/\sigma^2$								
BLUE*			MLE			AMLE		
0.0296	-0.0526	-0.0055	0.0290	-0.0495	-0.0064	0.0297	-0.0558	-0.0008
-0.0526	2.0548	0.0000	-0.0495	1.9344	0.0000	-0.0558	2.1797	0.0000
-0.0055	0.0000	0.0179	-0.0064	0.0000	0.0152	-0.0008	0.0000	0.0089

*Values for BLUE are exact.

Table 8.2.3.4 Approximate bias of AMLEs from the complete sample for Example 8.2.3

Approximate bias of AMLEs$/\sigma$		
ν_0	ν_1	σ
-0.1027	0.0000	-0.0111

Table 8.2.3.5 95% confidence interval from the complete sample for Example

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	0.5237	0.9405	-15.4884	-12.0152	0.62411	0.94829
MLE	0.5354	1.0330	-15.8956	-11.8314	0.68326	1.04354
AMLE	0.2410	0.7752	-15.8371	-11.2611	0.73190	1.02430

LL denotes lower limit and UL denotes upper limit of the 95% confidence interval.

We have presented the results of the analyses for Type-II right-censored sample in Tables 8.2.3.6-8.2.3.9.

Table 8.2.3.6 Estimates from the Type-II right-censored sample for Example 8.2.3

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	0.7830	-12.3971	0.8583
MLE	0.8394	-12.5250	0.9309
AMLE	0.6916	-12.2568	0.9375

Table 8.2.3.7 Asymptotic variances and covariances from the Type-II right-censored sample for Example 8.2.3

Asymptotic variances and covariances/σ^2								
BLUE*			MLE			AMLE		
0.0368	-0.0392	0.0029	0.0340	-0.0416	-0.0008	0.0380	-0.0417	0.0040
-0.0392	2.8245	0.0299	-0.0416	2.6187	0.0221	-0.0417	2.8219	0.0151
0.0029	0.0299	0.0285	-0.0008	0.0221	0.0231	0.0040	0.0151	0.0145

* Values for BLUE are exact.

Table 8.2.3.8 Approximate bias of AMLEs from the Type-II right-censored sample for Example 8.2.3

Approximate bias of AMLEs/σ		
ν_0	ν_1	σ
-0.1063	-0.0443	-0.0129

Table 8.2.3.9 95% confidence interval from the Type-II right-censored sample for Example 8.2.3

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	0.5060	1.0600	-14.8237	-9.9705	0.61454	1.10206
MLE	0.5262	1.1526	-15.2736	-9.7764	0.67275	1.18905
AMLE	0.2495	0.9211	-15.1949	-9.4073	0.71716	1.13204

LL denotes lower limit and UL denotes upper limit of the 95% confidence interval.

CHAPTER 9

LARGE-SAMPLE APPROXIMATION TO BLUEs

9.1 Introduction

To obtain the BLUEs of ν_0, ν_1 and σ in the MRVR model, we require means, variances, and covariances of order statistics from the standard extreme value distribution. For large sample sizes (say, $n \geq 30$ or so), the variances and covariances are not readily available for most distributions, including extreme value [see Balakrishnan and Chan (1992a, b)]. One more difficulty involved with BLUEs is the necessity to invert a large variance-covariance matrix. In this chapter, we therefore propose a large-sample approximation to BLUEs. In Section 9.2, we derive the first-order and second-order approximations for the variance-covariance matrix of order statistics from the standard extreme value distribution using David and Johnson's (1954) approximation. Then, in Section 9.3, we derive an explicit form for the inverse of the variance-covariance matrix of order statistics from the standard extreme value distribution. In order to assess the performance of the first-order and second-order approximation methods as compared to the exact method, we conduct a simulation study and discuss the results in Section 9.4. Finally, in Section 9.5, we illustrate the first-order and second-order approximation methods through the three real-life examples considered earlier in Chapter 8.

9.2 David and Johnson's Approximation

Express $u = F(x)$ as the probability integral from a population with pdf $f(x)$ and cdf $F(x)$. It transforms the order statistics $X_{i:n}$ into uniform order statistics $U_{i:n}$ for $i = 1, 2, \ldots, n$. Hence, by inverting the above transformation, we get for $1 \le i \le n$

$$X_{i:n} = F^{-1}(U_{i:n}) = \xi(U_{i:n}),$$

which, when expanded in a Taylor series around $E(U_{i:n}) = i/(n+1) = p_i$, gives

$$X_{i:n} = \xi_i + \xi_i'(U_{i:n} - p_i) + \frac{1}{2}\xi_i''(U_{i:n} - p_i)^2 + \cdots; \tag{9.2.1}$$

here, ξ_i denotes $\xi(p_i)$, ξ_i' denotes $\frac{d}{du}\xi(u)\big|_{u=p_i}$, and similarly ξ_i'', ξ_i''', ... denote successive derivatives of $\xi(u)$ evaluated at $u = p_i$. Then, by taking expectation on both sides of (9.2.1) and by using the expressions of the central moments of uniform order statistics (see Balakrishnan and Cohen, 1991, Section 3.4), we obtain

$$\alpha_{i:n} = E(X_{i:n}) \approx \xi_i + \frac{p_i q_i \xi_i''}{2(n+2)} + \frac{p_i q_i}{(n+2)^2}\left(\frac{(q_i - p_i)\xi_i'''}{3} + \frac{p_i q_i \xi_i^{iv}}{8}\right), \tag{9.2.2}$$

$$\beta_{i,i:n} = Var(X_{i:n}) \approx \frac{p_i q_i}{(n+2)}\xi_i'^2 + \frac{p_i q_i}{(n+2)^2}\left\{2(q_i - p_i)\xi_i'\xi_i'' + p_i q_i\left[\xi_i'\xi_i''' + \frac{\xi_i''^2}{2}\right]\right\}, \tag{9.2.3}$$

and

$$\beta_{i,j:n} = Cov(X_{i:n}, X_{j:n})$$
$$\approx \frac{p_i q_j}{n+2}\xi_i'\xi_j' + \frac{p_i q_j}{(n+2)^2}\left[(q_i - p_i)\xi_i''\xi_j' + (q_j - p_j)\xi_i'\xi_j'' + \frac{p_i q_i \xi_i'''\xi_j' + p_j q_j \xi_i'\xi_j''' + p_i q_j \xi_i''\xi_j''}{2}\right], \tag{9.2.4}$$

where $q_i = 1 - p_i$, $1 \le i \le j \le n$.

In this study, we compare the BLUEs based on i) the exact values, ii) only the first term in (9.2.3) and (9.2.4) (termed as "**first-order approximation**"), and iii) first two terms in (9.2.3) and (9.2.4) (termed as "**second-order approximation**") for the means and variances and covariances of extreme value order statistics. In the case of the first-order approximation, an explicit expression for the inverse of the variance-covariance matrix can also be derived.

9.3 Σ^{-1} from First-order Approximation

Denote $a_i = \frac{p_i}{\sqrt{n+2}}\xi_i'$ and $b_j = \frac{(1-p_j)}{\sqrt{n+2}}\xi_j'$; we then have $\beta_{i,i:n} \approx a_i b_i$ and $\beta_{i,j:n} \approx a_i b_j$. By using the following Lemma, we can express the inverse of the variance-covariance matrix Σ^{-1} using first-order approximation in an explicit form.

Lemma Let $C = (c_{ij})$ be a $k \times k$ nonsingular symmetric matrix with $c_{ij} = a_i b_j$, $i \le j$. Then C^{-1} is a symmetric matrix, and for $i \le j$, its (i, j) th element is given by

$$c^{ij} = \begin{cases} -(a_{i+1}b_i - a_i b_{i+1})^{-1}, & j = i+1,\ i = 1,\dots,\ k-1, \\ \dfrac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}, & i = j = 2,\dots,\ k-1, \\ a_2[a_1(a_2 b_1 - a_1 b_2)]^{-1}, & i = j = 1, \\ b_{k-1}[b_k(a_k b_{k-1} - a_{k-1}b_k)]^{-1}, & i = j = k, \\ 0, & j > i+1. \end{cases}$$

The lemma follows by a direct manipulation of the fact that $CC^{-1} = \mathrm{I}$; see, for example, Graybill (1983).

In the case of the standard extreme value model, $F(x) = 1 - \exp\{-e^x\}$, $-\infty < x < \infty$; hence, we have

$$\xi_i = F^{-1}(p_i) = \ln(-\ln(1 - p_i)),$$

$$\xi_i' = -\frac{1}{(1 - p_i)\ln(1 - p_i)}.$$

Then,

$$a_{i:n} = -\frac{p_i}{\sqrt{n+2}(1 - p_i)\ln(1 - p_i)}$$

and

$$b_{i:n} = -\frac{1}{\sqrt{n+2}\ln(1 - p_i)}.$$

It then follows that

$$\alpha_{i:n} \approx \xi_i = \ln\left(-\ln\left(1 - \frac{i}{n+1}\right)\right),$$

$$\beta_{i,i:n} \approx a_{i:n} b_{i:n} = \frac{p_i}{(n+2)(1 - p_i)[\ln(1 - p_i)]^2},$$

and

$$\beta_{i,j:n} \approx a_{i:n} b_{j:n} = \frac{p_i}{(n+2)(1 - p_i)\ln(1 - p_i)\ln(1 - p_j)}, \; i \le j.$$

Thus, according to the lemma, we obtain

$$\beta^{i,j:n} = \begin{cases} -(n+1)(n+2)q_i q_{i+1} \ln q_i \ln q_{i+1}, & j = i+1, i = 1,\dots,k-1, \\ 2(n+1)(n+2)q_i^2 (\ln q_i)^2, & i = j = 2,\dots,k-1, \\ 2(n+1)(n+2)q_1^2 (\ln q_1)^2, & i = j = 1, \\ (n+1)(n+2)q_k q_{k-1} (\ln q_k)^2, & i = j = k, \\ 0, & j > i+1. \end{cases}$$

where $\beta^{i,j:n}$ is the (*i,j*)th element of Σ^{-1} based on the first-order approximation.

9.4 Simulation and Discussion

When we apply these first-order approximation and second-order approximation to get the BLUEs, the expression (2.2.10) turns out to be

$$\begin{aligned} E(\theta^*) &= ({}_aW'{}_a\Sigma^{-1}{}_aW)^{-1}{}_aW'{}_a\Sigma^{-1}E(Y) \\ &= ({}_aW'{}_a\Sigma^{-1}{}_aW)^{-1}{}_aW'{}_a\Sigma^{-1}W\theta \neq \theta \end{aligned} \tag{9.4.1}$$

where "*a*" stands for the approximation used, and consequently the estimators are biased. The Monte Carlo simulations also reveal this fact and it is shown that the bias is especially higher in the estimator of σ. Since the means of the extreme value order statistics for lager sample sizes are easily obtained which is not the case for the variances and covariances (Balakrishnan and Chan, 1992), we make use of the exact means in these two approximation approaches. Using expression (2.2.11), we can compute the exact variances and covariances of the estimators based on first-order and second-order approximations.

In the simulation study, we took $\nu_0 = 0$, $\nu_1 = 1$ and $\sigma = 1$, and x = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two- or four-grouped samples, respectively. We use n to denote the vector of the multi-group sizes for the cases of complete sample and s to denote the vector of the multi-group censoring schemes for the cases of Type-II right-censored sample. Based on Monte Carlo process, for both exact and approximate methods, we simulated the probability coverages, bias, MSE, variances and covariances of the BLUEs for the following cases. We computed the exact and approximate variances and covariances of these estimators as well.

1. Complete sample

two groups: n = [6 6], [8 8], [10 10], [15 15], [15 20] and [20 20] .

four groups: n = [6 6 6 6], [6 6 10 10], [8 8 10 10], [15 15 15 15], [15 15 20 20] and [20 20 20 20] .

2. Type-II censored sample

two groups: s = [4 4], [2 2] from n = [10 10] and [5 5] and [5 0] from n = [20 20].

four groups: s = [4 4 4 4], [4 4 0 0] and [2 2 0 0] from n = [10 10 10 10] and [5 5 5 5], and [5 5 0 0] from n = [20 20 20 20].

These results are presented in Tables 9.4.1.1 – 9.4.1.20.

We compare the first-order and second-order approximation methods with the exact method of BLUE based on simulated probability coverages, bias, MSE, variances and covariances, and the exact variances and covariances computed from formula (2.2.11).

The results are quite surprising and interesting. Overall, the results from the first-order and second-order approximation methods are almost identical as the results from the exact method of BLUE. Of course, the second-order approximation method provides a closer result to the exact method as compared to the first-order approximation method, but this improvement is very slight. However, the improvement comes at the cost of having to numerically invert a large variance-covariance matrix.

Therefore, we recommend the use of the first-order approximation to the BLUE even for moderate sample sizes as the resulting estimates are very close to the exact BLUEs and also the computation is quite easy as the numerical inversion of the variance-covariance matrix is avoided completed.

Table 9.4.1.1 Simulated probability coverages for two-grouped complete samples

Probability coverages		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1$	$n_2]$	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	92.38	91.56	91.42	92.38	91.53	91.58	92.35	91.59	91.65
8	8	93.13	92.71	92.25	93.16	92.77	92.24	93.19	92.74	92.41
10	10	93.96	93.70	93.10	93.90	93.76	93.12	93.81	93.69	93.22
15	15	94.14	93.75	93.53	94.16	93.78	93.51	94.04	93.77	93.63
15	20	94.70	94.28	94.21	94.74	94.29	94.24	94.66	94.34	94.22
20	20	94.61	94.18	93.81	94.59	94.17	93.78	94.49	94.17	93.72

Table 9.4.1.2 Simulated bias for two-grouped complete samples

Bias		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1$	$n_2]$	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	-0.0033	-0.0026	0.0023	-0.0032	-0.0027	0.0019	-0.0030	-0.0028	0.0013
8	8	-0.0016	0.0100	-0.0021	-0.0016	0.0101	-0.0022	-0.0016	0.0101	-0.0025
10	10	0.0011	-0.0004	0.0004	0.0011	-0.0004	0.0004	0.0011	-0.0001	0.0003
15	15	-0.0018	-0.0002	-0.0005	-0.0018	-0.0002	-0.0005	-0.0017	-0.0001	-0.0006
15	20	-0.0013	-0.0052	0.0007	-0.0013	-0.0052	0.0008	-0.0014	-0.0052	0.0011
20	20	0.0030	-0.0073	-0.0027	0.0030	-0.0073	-0.0027	0.0030	-0.0072	-0.0026

Table 9.4.1.3 Simulated MSE for two-grouped complete samples

MSE/σ^2		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1$	$n_2]$	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	0.0970	0.3777	0.0664	0.0971	0.3782	0.0665	0.0973	0.3795	0.0683
8	8	0.0702	0.2742	0.0457	0.0702	0.2744	0.0459	0.0703	0.2747	0.0474
10	10	0.0559	0.2117	0.0361	0.0559	0.2117	0.0362	0.0560	0.2121	0.0373
15	15	0.0368	0.1373	0.0224	0.0368	0.1373	0.0225	0.0369	0.1373	0.0231
15	20	0.0320	0.1204	0.0190	0.0320	0.1204	0.0190	0.0321	0.1205	0.0196
20	20	0.0277	0.1047	0.0165	0.0277	0.1048	0.0165	0.0278	0.1047	0.0170

Table 9.4.1.4 Simulated variances and covariances for two-grouped complete samples

Variances and covariances[#]		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
n_1	n_2									
6	6	0.0970	0.0046	-0.0149	0.0971	0.0046	-0.0150	0.0973	0.0047	-0.0156
6	6	0.0046	0.3778	0.0012	0.0046	0.3782	0.0011	0.0047	0.3796	0.0008
6	6	-0.0149	0.0012	0.0664	-0.0150	0.0011	0.0665	-0.0156	0.0008	0.0683
8	8	0.0703	0.0017	-0.0122	0.0702	0.0017	-0.0122	0.0703	0.0016	-0.0126
8	8	0.0017	0.2741	-0.0008	0.0017	0.2743	-0.0008	0.0016	0.2746	-0.0008
8	8	-0.0122	-0.0008	0.0457	-0.0122	-0.0008	0.0459	-0.0126	-0.0008	0.0474
10	10	0.0559	0.0005	-0.0111	0.0559	0.0005	-0.0111	0.0560	0.0004	-0.0113
10	10	0.0005	0.2118	-0.0009	0.0005	0.2117	-0.0008	0.0004	0.2121	-0.0007
10	10	-0.0111	-0.0009	0.0361	-0.0111	-0.0008	0.0362	-0.0113	-0.0007	0.0373
15	15	0.0368	0.0007	-0.0081	0.0368	0.0007	-0.0081	0.0369	0.0007	-0.0083
15	15	0.0007	0.1373	-0.0009	0.0007	0.1373	-0.0009	0.0007	0.1373	-0.0008
15	15	-0.0081	-0.0009	0.0224	-0.0081	-0.0009	0.0225	-0.0083	-0.0008	0.0231
15	20	0.0320	-0.0081	-0.0070	0.0320	-0.0081	-0.0070	0.0321	-0.0081	-0.0072
15	20	-0.0081	0.1204	-0.0004	-0.0081	0.1204	-0.0004	-0.0081	0.1205	-0.0004
15	20	-0.0070	-0.0004	0.0190	-0.0070	-0.0004	0.0190	-0.0072	-0.0004	0.0196
20	20	0.0277	0.0002	-0.0064	0.0277	0.0002	-0.0064	0.0278	0.0002	-0.0065
20	20	0.0002	0.1047	0.0003	0.0002	0.1047	0.0003	0.0002	0.1047	0.0002
20	20	-0.0064	0.0003	0.0165	-0.0064	0.0003	0.0165	-0.0065	0.0002	0.0170

"#" Denotes the variances and covariances are expressed in the form of $\left| \begin{matrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{matrix} \right| \Big/ \sigma^2$ within the sample

Table 9.4.1.5 Results computed from formulas for variances and covariances in two-grouped complete samples

Variances and Covariances[#]		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
n_1	n_2									
6	6	0.0956	0.0000	-0.0157	0.0956	0.0000	-0.0158	0.0958	0.0000	-0.0164
6	6	0.0000	0.3674	0.0000	0.0000	0.3676	0.0000	0.0000	0.3683	0.0000
6	6	-0.0157	0.0000	0.0660	-0.0158	0.0000	0.0663	-0.0164	0.0000	0.0684
8	8	0.0710	0.0000	-0.0130	0.0710	0.0000	-0.0131	0.0712	0.0000	-0.0136
8	8	0.0000	0.2693	0.0000	0.0000	0.2694	0.0000	0.0000	0.2697	0.0000
8	8	-0.0130	0.0000	0.0465	-0.0131	0.0000	0.0466	-0.0136	0.0000	0.0481
10	10	0.0565	0.0000	-0.0110	0.0565	0.0000	-0.0110	0.0567	0.0000	-0.0114
10	10	0.0000	0.2124	0.0000	0.0000	0.2125	0.0000	0.0000	0.2127	0.0000
10	10	-0.0110	0.0000	0.0358	-0.0110	0.0000	0.0359	-0.0114	0.0000	0.0370
15	15	0.0374	0.0000	-0.0078	0.0374	0.0000	-0.0078	0.0375	0.0000	-0.0080
15	15	0.0000	0.1389	0.0000	0.0000	0.1390	0.0000	0.0000	0.1390	0.0000
15	15	-0.0078	0.0000	0.0227	-0.0078	0.0000	0.0227	-0.0080	0.0000	0.0234
15	20	0.0326	-0.0088	-0.0067	0.0326	-0.0088	-0.0068	0.0327	-0.0088	-0.0070
15	20	-0.0088	0.1211	-0.0004	-0.0088	0.1211	-0.0004	-0.0088	0.1211	-0.0003
15	20	-0.0067	-0.0004	0.0191	-0.0068	-0.0004	0.0192	-0.0070	-0.0003	0.0197
20	20	0.0280	0.0000	-0.0060	0.0280	0.0000	-0.0060	0.0280	0.0000	-0.0062
20	20	0.0000	0.1032	0.0000	0.0000	0.1032	0.0000	0.0000	0.1032	0.0000
20	20	-0.0060	0.0000	0.0166	-0.0060	0.0000	0.0166	-0.0062	0.0000	0.0170

"#" Denotes the variances and covariances are expressed in the form of

$$\left| \begin{matrix} Var(v_0^*) & Cov(v_0^*,v_1^*) & Cov(v_0^*,\sigma^*) \\ Cov(v_0^*,v_1^*) & Var(v_1^*) & Cov(v_1^*,\sigma^*) \\ Cov(v_0^*,\sigma^*) & Cov(v_1^*,\sigma^*) & Var(\sigma^*) \end{matrix} \right| \Big/ \sigma^2$$

within the sample

Table 9.4.1.6 Simulated probability coverages for four-grouped complete samples

Probability coverages				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1\ n_2\ n_3\ n_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	6	6	93.64	93.39	93.16	94.05	93.22	93.69	93.87	93.14	93.54
6	6	10	10	94.16	94.05	93.55	93.86	94.29	93.56	93.90	94.14	93.71
8	8	10	10	94.07	94.64	94.07	94.28	94.91	93.95	94.32	94.94	94.06
15	15	15	15	94.64	94.72	94.23	94.46	94.63	94.23	94.47	94.53	94.30
15	15	20	20	94.54	94.74	94.87	94.63	94.50	95.25	94.68	94.57	95.19
20	20	20	20	94.71	94.29	94.74	95.06	94.84	94.68	95.11	94.85	94.65

Table 9.4.1.7 Simulated bias for four-grouped complete samples

Bias				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1\ n_2\ n_3\ n_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	6	6	-0.0024	-0.0128	0.0015	-0.0021	-0.0027	0.0011	-0.0020	-0.0029	0.0008
6	6	10	10	0.0004	0.0013	0.0012	0.0010	-0.0013	-0.0008	0.0010	-0.0015	-0.0008
8	8	10	10	0.0018	-0.0026	-0.0011	-0.0031	-0.0039	0.0013	-0.0031	-0.0039	0.0013
15	15	15	15	0.0016	0.0082	-0.0019	0.0015	0.0019	-0.0018	0.0015	0.0019	-0.0018
15	15	20	20	-0.0013	0.0049	0.0008	0.0000	0.0011	0.0009	0.0000	0.0010	0.0009
20	20	20	20	0.0002	-0.0045	0.0001	0.0014	-0.0049	-0.0022	0.0014	-0.0049	-0.0023

Table 9.4.1.8 Simulated MSE for four-grouped complete samples

MSE/σ^2				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[n_1\ n_2\ n_3\ n_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
6	6	6	6	0.0480	0.3305	0.0334	0.0473	0.3392	0.0325	0.0474	0.3398	0.0334
6	6	10	10	0.0376	0.2602	0.0235	0.0376	0.2589	0.0233	0.0377	0.2594	0.0240
8	8	10	10	0.0319	0.2132	0.0201	0.0313	0.2079	0.0201	0.0314	0.2080	0.0208
15	15	15	15	0.0183	0.1233	0.0114	0.0184	0.1273	0.0114	0.0185	0.1273	0.0117
15	15	20	20	0.0163	0.1084	0.0093	0.0164	0.1083	0.0094	0.0164	0.1083	0.0097
20	20	20	20	0.0141	0.0957	0.0081	0.0136	0.0906	0.0082	0.0136	0.0906	0.0084

Table 9.4.1.9 Simulated variances and covariances for four-grouped complete samples

Variances and Covariances[#]				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
n_1	n_2	n_3	n_4									
6	6	6	6	0.0480	0.0013	-0.0078	0.0473	-0.0014	-0.0074	0.0474	-0.0013	-0.0077
6	6	6	6	0.0013	0.3304	-0.0002	-0.0014	0.3392	-0.0005	-0.0013	0.3398	-0.0007
6	6	6	6	-0.0078	-0.0002	0.0334	-0.0074	-0.0005	0.0325	-0.0077	-0.0007	0.0334
6	6	10	10	0.0376	-0.0229	-0.0066	0.0376	-0.0225	-0.0066	0.0377	-0.0227	-0.0069
6	6	10	10	-0.0229	0.2602	-0.0013	-0.0225	0.2590	-0.0017	-0.0227	0.2595	-0.0016
6	6	10	10	-0.0066	-0.0013	0.0235	-0.0066	-0.0017	0.0233	-0.0069	-0.0016	0.0240
8	8	10	10	0.0319	-0.0077	-0.0061	0.0313	-0.0072	-0.0062	0.0314	-0.0072	-0.0064
8	8	10	10	-0.0077	0.2132	-0.0011	-0.0072	0.2080	-0.0017	-0.0072	0.2080	-0.0015
8	8	10	10	-0.0061	-0.0011	0.0201	-0.0062	-0.0017	0.0201	-0.0064	-0.0015	0.0208
15	15	15	15	0.0183	0.0002	-0.0040	0.0184	-0.0008	-0.0041	0.0185	-0.0009	-0.0042
15	15	15	15	0.0002	0.1233	-0.0006	-0.0008	0.1273	0.0004	-0.0009	0.1273	0.0005
15	15	15	15	-0.0040	-0.0006	0.0114	-0.0041	0.0004	0.0114	-0.0042	0.0005	0.0117
15	15	20	20	0.0163	-0.0057	-0.0033	0.0164	-0.0053	-0.0034	0.0164	-0.0053	-0.0034
15	15	20	20	-0.0057	0.1084	-0.0004	-0.0053	0.1083	-0.0006	-0.0053	0.1083	-0.0006
15	15	20	20	-0.0033	-0.0004	0.0093	-0.0034	-0.0006	0.0094	-0.0034	-0.0006	0.0097
20	20	20	20	0.0141	0.0003	-0.0030	0.0136	0.0002	-0.0030	0.0136	0.0002	-0.0031
20	20	20	20	0.0003	0.0957	0.0002	0.0002	0.0906	-0.0003	0.0002	0.0906	-0.0003
20	20	20	20	-0.0030	0.0002	0.0081	-0.0030	-0.0003	0.0082	-0.0031	-0.0003	0.0084

"#" Denotes the variances and covariances are expressed in the form of $\begin{vmatrix} Var(v_0^*) & Cov(v_0^*,v_1^*) & Cov(v_0^*,\sigma^*) \\ Cov(v_0^*,v_1^*) & Var(v_1^*) & Cov(v_1^*,\sigma^*) \\ Cov(v_0^*,\sigma^*) & Cov(v_1^*,\sigma^*) & Var(\sigma^*) \end{vmatrix} \Big/ \sigma^2$ within the sample

Table 9.4.1.10 Results computed from formulas for variances and covariances in four-grouped complete samples

Variances and Covariances[#] $[n_1\ n_2\ n_3\ n_4]$				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
6	6	6	6	0.0478	0.0000	-0.0078	0.0478	0.0000	-0.0079	0.0479	0.0000	-0.0082
6	6	6	6	0.0000	0.3333	0.0000	0.0000	0.3334	0.0000	0.0000	0.3341	0.0000
6	6	6	6	-0.0078	0.0000	0.0330	-0.0079	0.0000	0.0332	-0.0082	0.0000	0.0342
6	6	10	10	0.0374	-0.0223	-0.0064	0.0374	-0.0223	-0.0064	0.0376	-0.0224	-0.0067
6	6	10	10	-0.0223	0.2590	-0.0019	-0.0223	0.2591	-0.0019	-0.0224	0.2595	-0.0018
6	6	10	10	-0.0064	-0.0019	0.0232	-0.0064	-0.0019	0.0233	-0.0067	-0.0018	0.0241
8	8	10	10	0.0318	-0.0083	-0.0059	0.0318	-0.0083	-0.0060	0.0319	-0.0083	-0.0062
8	8	10	10	-0.0083	0.2179	-0.0006	-0.0083	0.2179	-0.0006	-0.0083	0.2182	-0.0006
8	8	10	10	-0.0059	-0.0006	0.0202	-0.0060	-0.0006	0.0203	-0.0062	-0.0006	0.0209
15	15	15	15	0.0187	0.0000	-0.0039	0.0187	0.0000	-0.0039	0.0188	0.0000	-0.0040
15	15	15	15	0.0000	0.1260	0.0000	0.0000	0.1260	0.0000	0.0000	0.1261	0.0000
15	15	15	15	-0.0039	0.0000	0.0113	-0.0039	0.0000	0.0114	-0.0040	0.0000	0.0117
15	15	20	20	0.0163	-0.0053	-0.0034	0.0163	-0.0053	-0.0034	0.0163	-0.0053	-0.0035
15	15	20	20	-0.0053	0.1093	-0.0002	-0.0053	0.1093	-0.0002	-0.0053	0.1094	-0.0002
15	15	20	20	-0.0034	-0.0002	0.0096	-0.0034	-0.0002	0.0096	-0.0035	-0.0002	0.0098
20	20	20	20	0.0140	0.0000	-0.0030	0.0140	0.0000	-0.0030	0.0140	0.0000	-0.0031
20	20	20	20	0.0000	0.0936	0.0000	0.0000	0.0936	0.0000	0.0000	0.0937	0.0000
20	20	20	20	-0.0030	0.0000	0.0083	-0.0030	0.0000	0.0083	-0.0031	0.0000	0.0085

"#" Denotes the variances and covariances are expressed in the form of $\left.\begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \middle/ \sigma^2 \right.$ within the sample

Table 9.4.1.11 Simulated probability coverages for two-grouped Type-II censored samples

Probability coverages		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2]$		ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	92.06	91.63	90.95	92.12	91.57	90.97	92.10	91.53	91.00
2	2	92.95	92.53	92.25	92.94	92.56	92.19	92.74	92.59	92.25
*5	5	93.86	93.97	94.06	93.91	93.95	93.94	93.89	93.97	93.81
*5	0	94.20	94.01	93.78	94.15	94.04	93.77	94.18	94.01	93.74

"*" Denotes the censoring is from n = *[20 20]*, otherwise is from *[10 10]*.

Table 9.4.1.12 Simulated bias for two-grouped Type-II censored samples

Bias		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2]$		ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	-0.0034	0.0060	0.0018	-0.0034	0.0061	0.0017	-0.0035	0.0063	0.0015
2	2	0.0011	0.0021	-0.0006	0.0011	0.0022	-0.0008	0.0010	0.0023	-0.0012
*5	5	-0.0025	0.0018	0.0003	-0.0025	0.0017	0.0004	-0.0025	0.0017	0.0005
*5	0	0.0032	-0.0114	-0.0019	0.0032	-0.0114	-0.0019	0.0032	-0.0113	-0.0018

"*" Denotes the censoring is from n = *[20 20]*, otherwise is from *[10 10]*.

Table 9.4.1.13 Simulated MSE for two-grouped Type-II censored samples

MSE/σ^2		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2]$		ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	0.1081	0.3671	0.0822	0.1081	0.3671	0.0824	0.1088	0.3674	0.0853
2	2	0.0664	0.2679	0.0538	0.0664	0.2680	0.0539	0.0664	0.2684	0.0559
*5	5	0.0342	0.1357	0.0266	0.0342	0.1357	0.0267	0.0342	0.1357	0.0276
*5	0	0.0299	0.1274	0.0208	0.0299	0.1274	0.0208	0.0299	0.1275	0.0213

"*" Denotes the censoring is from n = *[20 20]*, otherwise is from *[10 10]*.

Table 9.4.1.14 Simulated variances and covariances for two-grouped Type-II censored samples

Variances and covariances[#]		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
s_1	s_2									
4	4	0.1081	-0.0027	0.0377	0.1081	-0.0027	0.0378	0.1088	-0.0027	0.0391
4	4	-0.0027	0.3671	-0.0007	-0.0027	0.3671	-0.0007	-0.0027	0.3674	-0.0008
4	4	0.0377	-0.0007	0.0822	0.0378	-0.0007	0.0824	0.0391	-0.0008	0.0853
2	2	0.0664	0.0007	0.0023	0.0664	0.0007	0.0024	0.0664	0.0006	0.0026
2	2	0.0007	0.2679	-0.0002	0.0007	0.2680	-0.0001	0.0006	0.2685	0.0002
2	2	0.0023	-0.0002	0.0538	0.0024	-0.0001	0.0539	0.0026	0.0002	0.0559
*5	5	0.0342	-0.0005	0.0025	0.0342	-0.0005	0.0025	0.0342	-0.0005	0.0026
*5	5	-0.0005	0.1357	0.0008	-0.0005	0.1357	0.0008	-0.0005	0.1357	0.0009
*5	5	0.0025	0.0008	0.0267	0.0025	0.0008	0.0267	0.0026	0.0009	0.0276
*5	0	0.0299	-0.0074	-0.0031	0.0298	-0.0074	-0.0031	0.0299	-0.0074	-0.0032
*5	0	-0.0074	0.1273	-0.0100	-0.0074	0.1273	-0.0100	-0.0074	0.1274	-0.0102
*5	0	-0.0031	-0.0100	0.0208	-0.0031	-0.0100	0.0208	-0.0032	-0.0102	0.0213

"*" Denotes the censoring is from n = *[20 20]*, otherwise is from *[10 10]*.

"#" Denotes the variances and covariances are expressed in the form of

$$\begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \Big/ \sigma^2$$

within the sample

Table 9.4.1.15 Results computed from formulas for variances and covariances in two-grouped Type-II censored samples

Variances and covariances[#]		Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2]$										
4	4	0.1072	0.0000	0.0367	0.1072	0.0000	0.0369	0.1079	0.0000	0.0383
4	4	0.0000	0.3637	0.0000	0.0000	0.3637	0.0000	0.0000	0.3641	0.0000
4	4	0.0367	0.0000	0.0829	0.0369	0.0000	0.0832	0.0383	0.0000	0.0864
2	2	0.0670	0.0000	0.0026	0.0670	0.0000	0.0026	0.0670	0.0000	0.0028
2	2	0.0000	0.2676	0.0000	0.0000	0.2676	0.0000	0.0000	0.2679	0.0000
2	2	0.0026	0.0000	0.0537	0.0026	0.0000	0.0539	0.0028	0.0000	0.0560
*5	5	0.0348	0.0000	0.0024	0.0348	0.0000	0.0024	0.0348	0.0000	0.0025
*5	5	0.0000	0.1382	0.0000	0.0000	0.1382	0.0000	0.0000	0.1383	0.0000
*5	5	0.0024	0.0000	0.0275	0.0024	0.0000	0.0275	0.0025	0.0000	0.0285
*5	0	0.0306	-0.0075	-0.0028	0.0306	-0.0075	-0.0028	0.0306	-0.0074	-0.0029
*5	0	-0.0075	0.1249	-0.0093	-0.0075	0.1249	-0.0093	-0.0074	0.1251	-0.0096
*5	0	-0.0028	-0.0093	0.0207	-0.0028	-0.0093	0.0207	-0.0029	-0.0096	0.0213

"*" Denotes the censoring is from n = *[20 20]*, otherwise is from *[10 10]*.

"#" Denotes the variances and covariances are expressed in the form of $\left. \begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \middle/ \sigma^2 \right.$ within the sample

Table 9.4.1.16 Simulated probability coverages for four-grouped Type-II censored samples

Probability coverages				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2\ s_3\ s_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	4	4	93.63	93.56	92.66	93.62	93.53	92.62	93.59	93.50	92.88
4	4	0	0	94.41	93.93	93.31	94.39	93.99	93.21	94.35	93.98	93.14
2	2	0	0	94.08	93.55	93.67	94.06	93.65	93.63	94.00	93.66	93.41
*5	5	5	5	94.73	94.68	94.16	94.64	94.69	94.17	94.54	94.61	94.26
*5	5	0	0	94.63	94.06	94.68	94.60	94.07	94.69	94.60	94.07	94.65

"*" Denotes the censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*.

Table 9.4.1.17 Simulated bias for four-grouped Type-II censored samples

Bias				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2\ s_3\ s_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	4	4	0.0024	-0.0056	0.0034	0.0023	-0.0056	0.0033	0.0022	-0.0057	0.0031
4	4	0	0	0.0047	-0.0022	-0.0012	0.0047	-0.0023	-0.0012	0.0047	-0.0024	-0.0013
2	2	0	0	-0.0011	0.0001	-0.0015	-0.0011	0.0001	-0.0016	-0.0011	0.0003	-0.0018
*5	5	5	5	-0.0015	0.0011	-0.0009	-0.0015	0.0011	-0.0008	-0.0015	0.0011	-0.0007
*5	5	0	0	-0.0003	-0.0034	-0.0005	-0.0003	-0.0034	-0.0004	-0.0003	-0.0034	-0.0003

"*" Denotes the censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*.

Table 9.4.1.18 Simulated MSE for four-grouped Type-II censored samples

MSE/σ^2				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2\ s_3\ s_4]$				ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
4	4	4	4	0.0536	0.3317	0.0428	0.0536	0.3318	0.0429	0.0541	0.3324	0.0445
4	4	0	0	0.0345	0.2773	0.0250	0.0345	0.2773	0.0251	0.0346	0.2783	0.0261
2	2	0	0	0.0301	0.2239	0.0213	0.0302	0.2240	0.0213	0.0302	0.2244	0.0221
*5	5	5	5	0.0173	0.1211	0.0135	0.0173	0.1211	0.0136	0.0173	0.1210	0.0141
*5	5	0	0	0.0153	0.1138	0.0099	0.0152	0.1138	0.0100	0.0153	0.1138	0.0103

"*" Denotes the censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*.

Table 9.4.1.19 Simulated variances and covariances for four-grouped Type-II censored samples

Variances And covariances[#]				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
$[s_1\ s_2\ s_3\ s_4]$												
4	4	4	4	0.0536	-0.0003	0.0191	0.0536	-0.0002	0.0192	0.0541	-0.0002	0.0200
4	4	4	4	-0.0003	0.3317	0.0006	-0.0002	0.3318	0.0007	-0.0002	0.3324	0.0008
4	4	4	4	0.0191	0.0006	0.0428	0.0192	0.0007	0.0429	0.0200	0.0008	0.0445
4	4	0	0	0.0345	-0.0244	0.0016	0.0345	-0.0243	0.0016	0.0345	-0.0243	0.0015
4	4	0	0	-0.0244	0.2774	-0.0241	-0.0243	0.2774	-0.0242	-0.0243	0.2783	-0.0249
4	4	0	0	0.0016	-0.0241	0.0250	0.0016	-0.0242	0.0252	0.0015	-0.0249	0.0261
2	2	0	0	0.0302	-0.0064	-0.0028	0.0302	-0.0064	-0.0029	0.0302	-0.0063	-0.0030
2	2	0	0	-0.0064	0.2239	-0.0096	-0.0064	0.2240	-0.0097	-0.0063	0.2245	-0.0101
2	2	0	0	-0.0028	-0.0096	0.0213	-0.0029	-0.0097	0.0213	-0.0030	-0.0101	0.0221
*5	5	5	5	0.0173	0.0003	0.0011	0.0173	0.0003	0.0011	0.0173	0.0003	0.0012
*5	5	5	5	0.0003	0.1211	-0.0006	0.0003	0.1211	-0.0006	0.0003	0.1211	-0.0007
*5	5	5	5	0.0011	-0.0006	0.0135	0.0011	-0.0006	0.0136	0.0012	-0.0007	0.0141
*5	5	0	0	0.0153	-0.0044	-0.0016	0.0153	-0.0044	-0.0016	0.0153	-0.0044	-0.0016
*5	5	0	0	-0.0044	0.1138	-0.0047	-0.0044	0.1138	-0.0047	-0.0044	0.1138	-0.0048
*5	5	0	0	-0.0016	-0.0047	0.0100	-0.0016	-0.0047	0.0100	-0.0016	-0.0048	0.0103

"*" Denotes the censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*.

"#" Denotes the variances and covariances are expressed in the form of $\begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \Big/ \sigma^2$ within the sample

Table 9.4.1.20 Results computed from formulas for variances and covariances in four-grouped Type-II censored samples

Variances And covariances# $[s_1\ s_2\ s_3\ s_4]$				Exact BLUE			Second-order approximation to BLUE			First-order approximation to BLUE		
4	4	4	4	0.0536	0.0000	0.0184	0.0536	0.0000	0.0184	0.0540	0.0000	0.0192
4	4	4	4	0.0000	0.3299	0.0000	0.0000	0.3299	0.0000	0.0000	0.3303	0.0000
4	4	4	4	0.0184	0.0000	0.0414	0.0184	0.0000	0.0416	0.0192	0.0000	0.0432
4	4	0	0	0.0355	-0.0233	0.0011	0.0355	-0.0233	0.0011	0.0355	-0.0233	0.0012
4	4	0	0	-0.0233	0.2764	-0.0216	-0.0233	0.2765	-0.0217	-0.0233	0.2775	-0.0225
4	4	0	0	0.0011	-0.0216	0.0245	0.0011	-0.0217	0.0246	0.0012	-0.0225	0.0254
2	2	0	0	0.0303	-0.0070	-0.0029	0.0303	-0.0070	-0.0029	0.0303	-0.0070	-0.0030
2	2	0	0	-0.0070	0.2210	-0.0091	-0.0070	0.2210	-0.0091	-0.0070	0.2214	-0.0094
2	2	0	0	-0.0029	-0.0091	0.0214	-0.0029	-0.0091	0.0215	-0.0030	-0.0094	0.0222
*5	5	5	5	0.0174	0.0000	0.0012	0.0174	0.0000	0.0012	0.0174	0.0000	0.0013
*5	5	5	5	0.0000	0.1254	0.0000	0.0000	0.1254	0.0000	0.0000	0.1254	0.0000
*5	5	5	5	0.0012	0.0000	0.0137	0.0012	0.0000	0.0138	0.0013	0.0000	0.0142
*5	5	0	0	0.0152	-0.0044	-0.0015	0.0152	-0.0044	-0.0015	0.0152	-0.0044	-0.0015
*5	5	0	0	-0.0044	0.1120	-0.0055	-0.0044	0.1120	-0.0055	-0.0044	0.1121	-0.0057
*5	5	0	0	-0.0015	-0.0055	0.0103	-0.0015	-0.0055	0.0103	-0.0015	-0.0057	0.0106

"*" Denotes the censoring is from n = *[20 20 20 20]*, otherwise is from *[10 10 10 10]*.

"#" Denotes the variances and covariances are expressed in the form of

$$\left| \begin{matrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{matrix} \right| \Big/ \sigma^2$$

within the sample

9.5 Illustrative Examples

With the three examples considered earlier in Chapter 8, we will illustrate here the usefulness and efficiency of the approximate BLUEs developed in this chapter.

Example 8.2.1 revisited: In this case, by using the formulas in (2.2.9), (9.2.3) and (9.2.4), we find the values of the approximate BLUEs of parameters ν_0, ν_1 and σ as follows (the exact values of the BLUEs taken from Table 8.2.1.2 are also presented here for comparison purposes):

Table 9.5.1 BLUEs and the approximate BLUEs for Example 8.2.1

Estimation procedure	Estimates ν_0	ν_1	σ
BLUE	65.8483	-18.0101	1.3413
First order approximation to BLUE	65.4331	-17.8905	1.3211
Second order approximation to BLUE	65.7310	-17.9766	1.3376

From the formulas in (2.2.11), (9.2.3) and (9.2.4), we find the values of variances and covariances as follows (the exact variances and covariances of the BLUEs taken from Table 8.2.1.3 are also presented here for comparison purposes):

Table 9.5.2 Approximate variances and covariances of BLUEs and the approximate BLUEs for Example 8.2.1

Approximate variances and covariances/σ^2								
BLUE*			First order approximation to BLUE			Second order approximation to BLUE		
19.0421	-5.4410	0.0088	15.2989	-4.3763	0.0106	18.367	-5.2497	0.0104
-5.4410	1.5559	-0.0034	-4.3763	1.2529	-0.0036	-5.2497	1.5017	-0.0038
0.0088	-0.0034	0.0093	0.0106	-0.0036	0.007	0.0104	-0.0038	0.0088

*Values for BLUE are exact.

We presented the 95% confidence interval for the parameters based on these approximate BLUEs as well here (the 95% confidence interval of the BLUEs taken from Table 8.2.1.5 are also presented here for comparison purposes):

Table 9.5.3 95% confidence intervals based on BLUEs and the approximate BLUEs for Example 8.2.1

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	50.4609	81.2357	-22.4085	-13.6117	1.00124	1.68136
First order approximation to BLUE	54.3661	75.3283	-20.7264	-14.7328	1.0620	1.5135
Second order approximation to BLUE	50.7021	80.7599	-22.2739	-13.6793	1.00863	1.66657

Example 8.2.2 revisited: Similar to the last example, we will illustrate here the usefulness and efficiency of the approximate BLUEs in the following tables for Example 8.2.2:

Table 9.5.4 BLUEs and the approximate BLUEs for Example 8.2.2

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	52.7001	-11.4702	0.6700
First order approximation to BLUE	51.7989	-11.2454	0.6376
Second order approximation to BLUE	52.5241	-11.4262	0.6631

Table 9.5.5 Approximate variances and covariances of BLUEs and the approximate BLUEs for Example 8.2.2

Asymptotic variances and covariances $/\sigma^2$								
BLUE*			First order approximation to BLUE			Second order approximation to BLUE		
972.5061	-241.3027	0.1690	899.641	-223.223	0.1400	967.9942	-240.183	0.1642
-241.3027	59.8750	-0.0427	-223.223	55.3887	-0.0353	-240.183	59.5972	-0.0414
0.1690	-0.0427	0.0215	0.1400	-0.0353	0.0175	0.1642	-0.0414	0.0209

* Values for BLUE are exact.

Table 9.5.6 95% confidence intervals based on BLUEs and the approximate BLUEs for Example 8.2.2

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	25.2621	80.1381	-18.2783	-4.6621	0.54099	0.79901
First order approximation to BLUE	27.8995	75.6983	-17.1755	-5.3153	0.5322	0.7430
Second order approximation to BLUE	25.7108	79.3374	-18.0793	-4.7731	0.5385	0.7877

Example 8.2.3 revisited: Once again, we illustrate the usefulness and efficiency of the approximate BLUEs for the complete sample from Example 8.2.3 in the following tables:

Table 9.5.7 BLUEs and the approximate BLUEs for Example 8.2.3 in complete sample

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	0.7321	-13.7518	0.7862
First order approximation to BLUE	0.7355	-13.7151	0.7743
Second order approximation to BLUE	0.7328	-13.7325	0.7828

Table 9.5.8 Approximate variances and covariances of BLUEs and the approximate BLUEs for Example 8.2.3 in complete sample

Asymptotic variances and covariances $/\sigma^2$								
BLUE*			First order approximation to BLUE			Second order approximation to BLUE		
0.0296	-0.0526	-0.0055	0.0252	-0.0457	-0.0036	0.0290	-0.0520	-0.0050
-0.0526	2.0548	0.0000	-0.0457	1.7862	0.0000	-0.0520	2.0317	.00000
-0.0055	0.0000	0.0179	-0.0036	0.0000	0.0130	-0.0050	0.0000	0.0170

* Values for BLUE are exact.

Table 9.5.9 95% confidence intervals based on BLUEs and the approximate BLUEs for Example 8.2.3 in complete sample

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	0.5237	0.9405	-15.4884	-12.0152	0.62411	0.94829
First order approximation to BLUE	0.5490	0.9220	-15.2856	-12.1446	0.6403	0.9083
Second order approximation to BLUE	0.5283	0.9373	-15.4444	-12.0206	0.6262	0.9394

The results corresponding to the Type-II censored sample situation in Example 8.2.3 are as follows:

Table 9.5.10 BLUEs and the approximate BLUEs for Example 8.2.3 in Type-II censored sample

Estimation procedure	Estimates		
	ν_0	ν_1	σ
BLUE	0.7830	-12.3971	0.8583
First order approximation to BLUE	0.7785	-12.4058	0.8284
Second order approximation to BLUE	0.7818	-12.4007	0.8496

Table 9.5.11 Approximate variances and covariances of BLUEs and the approximate BLUEs for Example 8.2.3 in Type-II censored sample

Asymptotic variances and covariances $/\sigma^2$								
BLUE*			First order approximation to BLUE			Second order approximation to BLUE		
0.0368	-0.0392	0.0029	0.0316	-0.0347	0.0028	0.0361	-0.0388	0.0029
-0.0392	2.8245	0.0299	-0.0347	2.4083	0.0209	-0.0388	2.7694	0.0278
0.0029	0.0299	0.0285	0.0028	0.0209	0.0198	0.0029	0.0278	0.0265

* Values for BLUE are exact.

Table 9.5.12 95% confidence intervals based on BLUEs and the approximate BLUEs for Example 8.2.3 in Type-II censored sample

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
BLUE	0.5060	1.0600	-14.8237	-9.9705	0.61454	1.10206
First order approximation to BLUE	0.5394	1.0176	-14.4931	-10.3185	0.6391	1.0177
Second order approximation to BLUE	0.5130	1.0506	-14.7551	-10.0463	0.6193	1.0799

CHAPTER 10

GENERALIZATION TO PROGRESSIVELY TYPE-II RIGHT-CENSORED SAMPLES

10.1 Introduction

Progressive censoring is used in certain life and fatigue test situation. The unfailed items removed from test may be examined for deterioration or used for some other experimentation. In other applications, it may be desirable to have rapid completion of the tests for many items and yet have some extreme life spans represented in the data; see, for example, Balakrishnan and Aggarwala (2000). We consider the progressively Type-II right-censored samples in this chapter.

It is assumed that items are randomly sampled from a population whose failure times T have a two-parameter Weibull distribution with cdf

$$F_T(t) = 1 - \exp\{-(\frac{t}{\alpha})^{\delta}\}, \qquad t \geq 0$$

where $\delta > 0$ and $\alpha > 0$. The natural logarithm of failure time, $Y = \log(T)$, is then known to have an extreme-value distribution with cdf

$$F_Y(y) = 1 - \exp\{-\exp[\frac{y-\mu(x)}{\sigma}]\}, \quad -\infty < y < \infty,$$

where the location parameter $\mu(x) = \log\alpha(x)$ and scale parameter $\sigma = 1/\delta$.

Let $y_{1:m_l:n_l} \le y_{2:m_l:n_l} \le ... \le y_{m_l:m_l:n_l}$ represent the logarithms of the m_l ordered observed failure times $t_{1:m_l:n_l} \le t_{2:m_l:n_l} \le ... \le t_{m_l:m_l:n_l}$ from a sample of n_l units in l-th group (l = 1,..., k), which are all place on a life test at the same time and under the same condition with the single covariate x_l. For Type-II progressive right censoring, a pre-specified number of units $r_{i:m_l:n_l}$ ($r_{i:m_l:n_l} \ge 0$) are removed from test (censored) at the failure times $t_{i:m_l:n_l}$, i = 1, 2, ... , m_l, with $m_l + \sum_{i=1}^{m_l} r_{i:m_l:n_l} = n_l$ for l = 1,..., k. For the MEVR model, the progressively Type-II right-censored data can be expressed as

$$y_{i:m_l:n_l} = \mu(x) + \sigma z_{i:m_l:n_l} = v_o + v_1 x_l + \sigma z_{i:m_l:n_l}, \quad i = 1, ..., m_l, \quad l = 1, ..., k, \ \sigma > 0,$$

where v_0 and v_1 are the regression (location) parameters, and $z_{i:m_l:n_l}$ ($1 \le i \le m_l$) form the progressively Type-II right-censored sample of size m_l from a sample of size n_l from the standard extreme value distribution with density $\exp\{z - e^z\}$, $-\infty < z < \infty$. The density function of $y_{1:m_l:n_l}, \quad \cdots, \quad y_{m_l:m_l:n_l}$, for l = 1,..., k, is then given by

$$f(y_{1:m_l:n_l}, \quad \cdots, \quad y_{m_l:m_l:n_l}; v_0, \ v_1, \ \sigma)$$
$$= (\prod_{i=1}^{m_l} g_{i:m_l:n_l})\sigma^{-m_l} \exp\{\sum_{i=1}^{m_l}[\frac{y_{i:m_l:n_l} - v_0 - v_1 x_l}{\sigma} - (r_{i:m_l:n_l} + 1)\exp(\frac{y_{i:m_l:n_l} - v_0 - v_1 x_l}{\sigma})]\}, \tag{10.1.1}$$

where $g_{i:m_l:n_l} = \sum_{j=i}^{m_l}(r_{i:m_l:n_l} + 1), \quad i = 1, \cdots, m_l$, is the number of units remaining on test immediately preceding the i-th failure.

In this chapter, we generalize four types of estimation procedures, the BLUE, the approximate BLUE, the MLE and the AMLE, to the progressively Type-II right-censored for the MEVR model. In Section 10.2, we derive all four types of estimators for ν_0, ν_1 and σ for the MEVR model under progressively Type-II right-censored samples. We conduct a simulation study in Section 10.3 based on progressively Type-II right-censored two- and four-grouped samples with n = 10, m = 5, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 2$ to evaluate all four types of estimation procedures. A discussion of the simulation results is presented as well. Finally, in Section 10.4, we illustrate the methods of approximate BLUE, the MLE and the AMLE through a progressively Type-II right-censored sample generated from a real-life Example 8.2.1 considered earlier in Chapter 8.

10.2 Estimation Procedures for Progressively Type-II Right-censored Sample for the MEVR Model

10.2.1 BLUE

Given $E(z_{i:m_l:n_l}) = \alpha_{i:m_l:n_l}$ $(1 \le i \le m_l)$ and covariance $Cov\,(z_{i:m_l:n_l}, z_{j:m_l:n_l}) = \beta_{i,j:m_l:n_l}$ $(1 \le i \le j \le m_l)$, it is easy to note that

$$E(y_{i:m_l:n_l}) = \nu_o + \nu_1 x_l + \sigma\alpha_{i:m_l:n_l},\ \ 1 \le i \le m_l,$$

and

$$Cov\,(y_{i:m_l:n_l}, y_{j:m_l:n_l}) = \sigma^2 \beta_{i,j:m_l:n_l},\ 1 \le i \le j \le m_l,$$

Denote

$$Y = [y_{1:m_1:n_1}, \ldots, y_{m_1:m_1:n_1}, y_{1:m_2:n_2}, \ldots, y_{m_2:m_2:n_2}, \ldots, y_{1:m_k:n_k}, \ldots, y_{m_k:m_k:n_k}]'_{N\times 1} \tag{10.2.1.1}$$

$$X = [x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_k, \ldots, x_k]'_{N\times 1} \tag{10.2.1.2}$$

$$\alpha = [\alpha_{1:m_1:n_1}, \ldots, \alpha_{m_1:m_1:n_1}, \alpha_{1:m_2:n_2}, \ldots, \alpha_{m_2:m_2:n_2}, \ldots, \alpha_{1:m_k:n_k}, \ldots, \alpha_{m_k:m_k:n_k}]'_{N\times 1} \tag{10.2.1.3}$$

$$1 = [1 \ldots 1]'_{N\times 1} \tag{10.2.1.4}$$

$$W = [1 \;\; X \;\; \alpha]_{N\times 3} \tag{10.2.1.5}$$

$$\theta = [\nu_0 \;\; \nu_1 \;\; \sigma]'_{3\times 1} \tag{10.2.1.6}$$

$$\Sigma_{m_l:n_l} = \begin{bmatrix} \beta_{1,1:m_l:n_l} & \beta_{1,2:m_l:n_l} & \cdots & \beta_{1,m_l:m_l:n_l} \\ \beta_{1,2:m_l:n_l} & \beta_{2,2:m_l:n_l} & \cdots & \beta_{2,m_l:m_l:n_l} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1,m_l:m_l:n_l} & \beta_{2,m_l:m_l:n_l} & \cdots & \beta_{m_l,m_l:m_l:n_l} \end{bmatrix}_{m_l\times m_l} \tag{10.2.1.7}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{m_1:n_1} & 0 & \ldots & 0 \\ 0 & \Sigma_{m_2:n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Sigma_{m_k:n_k} \end{bmatrix}_{N\times N}, \tag{10.2.1.8}$$

where $N = \sum_{i=1}^{k} m_l$. We may then write

$$E(Y) = W\theta$$

and

$$Var(Y) = \sigma^2\Sigma.$$

Thus the generalized variance is given by

$$S = (Y - W\theta)'\Sigma^{-1}(Y - W\theta) = Y'\Sigma^{-1}Y - 2W'\Sigma^{-1}Y\theta + \theta'W'\Sigma^{-1}W\theta.$$

By minimizing the expression of the generalized variance with respect to θ and solving the following equation

$$\frac{\partial S}{\partial \theta} = -2W'\Sigma^{-1}Y + 2W'\Sigma^{-1}W\theta = 0,$$

we have the BLUE of θ as

$$\theta^* = (\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{\Sigma}^{-1}Y, \tag{10.2.1.9}$$

and its mean and variance-covariance matrix as

$$E(\theta^*) = (\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{\Sigma}^{-1}E(Y) = \theta, \tag{10.2.1.10}$$

and

$$Cov\,(\theta^*) = \sigma^2(\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{W})^{-1}. \tag{10.2.1.11}$$

Once again, using the special symbol $\begin{Vmatrix} \alpha & A & \Phi \\ \beta & B & \Lambda \\ \gamma & C & \Psi \end{Vmatrix}$ presented earlier in section 2.2, we derive the explicit expressions of BLUEs $v_0{}^*$, $v_1{}^*$ and σ^* (of v_0, v_1 and σ) under the progressively Type-II right-censored sample as

$$v_0^* = X'\mathbf{\Delta}_{v_0}Y = \sum_l^k \sum_i^{m_l} \mathbf{a}_{i:m_l:n_l} y_{i:m_l:n_l}, \tag{10.2.1.12}$$

$$v_1^* = X'\mathbf{\Delta}_{v_1}Y = \sum_l^k \sum_i^{m_l} \mathbf{b}_{i:m_l:n_l} y_{i:m_l:n_l}, \tag{10.2.1.13}$$

and

$$\sigma^* = X'\mathbf{\Delta}_{\sigma}Y = \sum_l^k \sum_i^{m_l} \mathbf{c}_{i:m_l:n_l} y_{i:m_l:n_l}. \tag{10.2.1.14}$$

where

$$\mathbf{\Delta}_{v_0} = \frac{\boldsymbol{\delta}_{v_0}}{\boldsymbol{\delta}}, \qquad \mathbf{\Delta}_{v_1} = \frac{\boldsymbol{\delta}_{v_1}}{\boldsymbol{\delta}}, \qquad \mathbf{\Delta}_{\sigma} = \frac{\boldsymbol{\delta}_{\sigma}}{\boldsymbol{\delta}}, \tag{10.2.1.15}$$

$$\boldsymbol{\delta} = \det\begin{vmatrix} 1'\mathbf{\Sigma}^{-1}1 & X'\mathbf{\Sigma}^{-1}1 & \boldsymbol{\alpha}'\mathbf{\Sigma}^{-1}1 \\ 1'\mathbf{\Sigma}^{-1}X & X'\mathbf{\Sigma}^{-1}X & \boldsymbol{\alpha}'\mathbf{\Sigma}^{-1}X \\ 1'\mathbf{\Sigma}^{-1}\boldsymbol{\alpha} & X'\mathbf{\Sigma}^{-1}\boldsymbol{\alpha} & \boldsymbol{\alpha}'\mathbf{\Sigma}^{-1}\boldsymbol{\alpha} \end{vmatrix}, \tag{10.2.1.16}$$

$$\delta_{v_0} = \begin{Vmatrix} \Sigma^{-1}1 & \alpha'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & \alpha'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & \alpha'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix}, \tag{10.2.1.17}$$

$$\delta_{v_1} = \begin{Vmatrix} \Sigma^{-1}1 & 1'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & 1'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & 1'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix} \tag{10.2.1.18}$$

and

$$\Sigma_{\sigma} = \begin{Vmatrix} \Sigma^{-1}1 & X'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & X'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & X'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix}. \tag{10.2.1.19}$$

Furthermore, explicit expressions of the exact variances and covariances of the estimators v_0*, v_1* and $\sigma*$ are derived as

$$Var(v_0^*) = \frac{(X'\Sigma^{-1}X)(\alpha'\Sigma^{-1}\alpha) - (X'\Sigma^{-1}\alpha)^2}{\delta}\sigma^2, \tag{10.2.1.20}$$

$$Var(v_1^*) = \frac{(1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)^2}{\delta}\sigma^2, \tag{10.2.1.21}$$

$$Var(\sigma^*) = \frac{(1'\Sigma^{-1}1)(X'\Sigma^{-1}X) - (1'\Sigma^{-1}X)^2}{\delta}\sigma^2, \tag{10.2.1.22}$$

$$Cov(v_0^*, v_1^*) = \frac{(X'\Sigma^{-1}\alpha)(\alpha'\Sigma^{-1}1) - (1'\Sigma^{-1}X)(\alpha'\Sigma^{-1}\alpha)}{\delta}\sigma^2, \tag{10.2.1.23}$$

$$Cov(v_0^*, \sigma^*) = \frac{(1'\Sigma^{-1}X)(X'\Sigma^{-1}\alpha) - (1'\Sigma^{-1}\alpha)(X'\Sigma^{-1}X)}{\delta}\sigma^2, \tag{10.2.1.24}$$

and

$$Cov(v_1^*, \sigma^*) = \frac{(X'\Sigma^{-1}1)(\alpha'\Sigma^{-1}1) - (1'\Sigma^{-1}1)(\alpha'\Sigma^{-1}X)}{\delta}\sigma^2. \qquad (10.2.1.25)$$

10.2.2 Approximate BLUE

To obtain the BLUEs of v_0, v_1 and σ under the progressively Type-II right-censored sample for the MRVR model, we require means, variances, and covariances of the corresponding progressively censored order statistics from the standard extreme value distribution. These values are not readily available except for $2 \le n \le 9$ (Mann, 1970) and the special case of $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 2$ (Thomas and Wilson, 1972). Recently, an efficient algorithm for computing the moments of order statistics from progressively censored samples has been proposed by Balakrishnan, Childs and Chandrasekar, (2001). But, there are no published values yet for these moments. In the following, we consider the first-order approximation to the moments of progressively Type-II right-censored ordered statistics (see Balakrishnan and Aggarwala, 2000) and use them to derive the approximate BLUE.

Suppose the progressively Type-II right-censored order statistics of size m, with censoring scheme as $r_1, \cdots, r_m$, have come from a sample of size n from the *Uniform*(0,1) distribution. For convenience in notation, let us denote them by $U_{1:m:n}^{(r_1,\cdots,r_m)}$, $U_{2:m:n}^{(r_1,\cdots,r_m)}, \cdots, U_{m:m:n}^{(r_1,\cdots,r_m)}$. Then we readily have their joint density function to be

$$f(u_1, u_2, \cdots, u_m) = c\prod_{i=1}^{m}(1-u_i)^{r_i}, \qquad 0 < u_1 < \cdots < u_m < 1, \qquad (10.2.2.1)$$

where c is the normalizing constant. It has been established that the random variables [see Balakrishnan and Sandhu (1995)]

$$V_i = \frac{1 - U^{(r_1,\cdots,r_m)}_{m-i+1:m:n}}{1 - U^{(r_1,\cdots,r_m)}_{m-i:m:n}}, \qquad i = 1,\cdots,m-1, \text{ and } \quad V_m = 1 - U^{(r_1,\cdots,r_m)}_{1:m:n} \qquad (10.2.2.2)$$

are all mutually independent, and further that

$$W_i = V_i^{i+r_m+r_{m-1}+\cdots+r_{m-i+1}}, \qquad i = 1,\ 2,\cdots,m \qquad (10.2.2.3)$$

are all independently and identically distributed as $Uniform(0,1)$.

From (10.2.2.2), we can readily write

$$U^{(r_1,\cdots,r_m)}_{i:m:n} \stackrel{d}{=} 1 - \prod_{j=m-i+1}^{m} V_j, \qquad i = 1,\ 2,\cdots,m, \qquad (10.2.2.4)$$

where V_j's are independently distributed as $Bata(j + \sum_{k=m-j+1}^{m} r_k, 1)$. Using this result, we obtain the following explicit expressions for means, variances and covariances of progressively Type-II right-censored order statistics from the $Uniform(0,1)$ distribution:

$$E(U^{(r_1,\cdots,r_m)}_{i:m:n}) = \Pi_i = 1 - b_i, \qquad i = 1,\ 2,\cdots,m, \qquad (10.2.2.5)$$

$$Var(U^{(r_1,\cdots,r_m)}_{i:m:n}) = k_i b_i, \qquad i = 1,\ 2,\cdots,m, \qquad (10.2.2.6)$$

and

$$Cov(U^{(r_1,\cdots,r_m)}_{i:m:n}, U^{(r_1,\cdots,r_m)}_{j:m:n}) = k_i b_j, \qquad 1 \le i < j \le m \qquad (10.2.2.7)$$

where, for $i = 1,\ 2,\cdots,m,$

$$k_i = \prod_{k=1}^{i} \left\{ \frac{m-k+2+r_k+r_{k+1}+\cdots+r_m}{m-k+3+r_k+r_{k+1}+\cdots+r_m} \right\} - \prod_{k=1}^{i} \left\{ \frac{m-k+1+r_k+r_{k+1}+\cdots+r_m}{m-k+2+r_k+r_{k+1}+\cdots+r_m} \right\} \qquad (10.2.2.8)$$

and

$$b_i = \prod_{k=1}^{i} \left\{ \frac{m-k+1+r_k+r_{k+1}+\cdots+r_m}{m-k+2+r_k+r_{k+1}+\cdots+r_m} \right\}. \tag{10.2.2.9}$$

We shall now use these expressions to get first-order approximations to the means, variances and covariances of progressively Type-II right-censored order statistics from an arbitrary continuous distribution $F(\cdot)$. From the inverse probability integral transformation, we readily have the relationship

$$Y_{i:m:n}^{(r_1,\cdots,r_m)} \stackrel{d}{=} F^{-1}(U_{i:m:n}^{(r_1,\cdots,r_m)}), \tag{10.2.2.10}$$

where $F^{-1}(\cdot)$ is the inverse cumulative distribution function of the lifetime distribution from which the progressively censored sample has come from. Expanding the function on the right hand side of (10.2.2.10) in a Taylor series around $E(U_{i:m:n}^{(r_1,\cdots,r_m)}) = \Pi_i$ and then taking expectation and retaining only the first term, we obtain the approximation

$$E(Y_{i:m:n}^{(r_1,\cdots,r_m)}) \approx F^{-1}(\Pi_i), \qquad i = 1,\ 2,\cdots,m \tag{10.2.2.11}$$

where Π_i is as given in (10.2.2.5). Proceeding similarly, we obtain the approximations

$$Var(Y_{i:m:n}^{(r_1,\cdots,r_m)}) \approx \left\{F^{-1^{(1)}}(\Pi_i)\right\}^2 k_i b_i, \qquad i = 1,\ 2,\cdots,m \tag{10.2.2.12}$$

and

$$Cov(Y_{i:m:n}^{(r_1,\cdots,r_m)}, Y_{j:m:n}^{(r_1,\cdots,r_m)}) \approx F^{-1^{(1)}}(\Pi_i) F^{-1^{(1)}}(\Pi_j) k_i b_j, \qquad i < j, \tag{10.2.2.13}$$

where k_i and b_i are as given in (10.2.2.8) and (10.2.2.9), respectively, and $F^{-1^{(1)}}(u) = \frac{d}{du} F^{-1}(u)$. This type of a Taylor series approximation has been utilized by Balakrishnan and Rao (1997) for developing best linear unbiased prediction under progressively Type-II censored samples.

In the case of the standard extreme value model with $F(x)=1-\exp\{-\exp(x)\}$, $-\infty < x < \infty$, we obtain the approximation

$$E(Y_{i:m:n}^{(r_1,\cdots,r_m)}) \approx \ln[-\ln(1-\Pi_i)], \qquad i=1,\ 2,\cdots,m \qquad (10.2.2.14)$$

where Π_i is as given in (10.2.2.5). Proceeding similarly, we obtain the approximations

$$Var(Y_{i:m:n}^{(r_1,\cdots,r_m)}) \approx \left\{\frac{1}{(1-\Pi_i)\ln(1-\Pi_i)}\right\}^2 k_i b_i, \qquad i=1,\ 2,\cdots,m \quad (10.2.2.15)$$

and

$$Cov(Y_{i:m:n}^{(r_1,\cdots,r_m)}, Y_{j:m:n}^{(r_1,\cdots,r_m)}) \approx \frac{1}{(1-\Pi_i)\ln(1-\Pi_i)} \frac{1}{(1-\Pi_j)\ln(1-\Pi_j)} k_i b_j,\ i<j. \ (10.2.2.16)$$

It should be noted that when we apply the first-order approximation obtained from (10.2.2.14)- (10.2.2.16) to obtain the BLUEs, the expression (10.2.1.9) turns out to be

$$\begin{aligned} E(\theta^*) &= ({}_aW'{}_a\Sigma^{-1}{}_aW)^{-1}{}_aW'{}_a\Sigma^{-1}E(Y) \\ &= ({}_aW'{}_a\Sigma^{-1}{}_aW)^{-1}{}_aW'{}_a\Sigma^{-1}W\theta \neq \theta \end{aligned}$$

where "a" stands for the approximation approach, and consequently the estimator is biased.

10.2.3MLE

The corresponding likelihood function for the density function expressed in (10.1.1) is

$$L=\prod_{l=1}^{k}\left\{(\prod_{i=1}^{m_l} g_{i:m_l:n_l})\sigma^{-m_l}\exp\left[\sum_{i=1}^{m_l}[\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma}-(r_{i:m_l:n_l}+1)\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})]\right]\right\}$$

where $g_{i:m_l:n_l}=\sum_{j=i}^{m_l}(r_{i:m_l:n_l}+1), \quad i=1,\ \cdots,\ m_l$, is the number of units remaining on test immediately preceding the i-th failure.

Dropping the proportionality constant $\prod_{i=1}^{m_l} g_{i:m_l:n_l}$, we can take the log-likelihood function as

$$\ln L = \sum_{l=1}^{k}\{-m_l \ln\sigma + \sum_{i=1}^{m_l}[\frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma} - (r_{i:m_l:n_l} + 1)\exp(\frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma})]\}.$$

The log-likelihood equations for ν_0, ν_1 and σ become

$$\frac{\partial \ln L}{\partial \nu_0} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 - (r_{i:m_l:n_l} + 1)\exp(\frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma})\} = 0, \qquad (10.2.3.1)$$

$$\frac{\partial \ln L}{\partial \nu_1} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l[1 - (r_{i:m_l:n_l} + 1)\exp(\frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma})]\} = 0, \qquad (10.2.3.2)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 + \frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma}[1 - (r_{i:m_l:n_l} + 1)\exp(\frac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma})]\} = 0, \qquad (10.2.3.3)$$

respectively.

The MLEs $\hat{\nu}_0$, $\hat{\nu}_1$ and $\hat{\sigma}_0$ (of ν_0, ν_1 and σ) for progressively Type-II right-censored samples can be obtained by simultaneously solving the equations (10.2.3.1) – (10.2.3.3). Since these three equations cannot be solved analytically, numerical method must be employed. Newton-Raphson or some other iterative procedure can be applied once again.

The approximate variance-covariance matrix can be obtained by inverting the observed Fisher information matrix I_0 valuated at the MLEs of ν_0, ν_1 and σ.

The observed Fisher information matrix I_0 is of the form

$$\mathrm{I}_0 = -\begin{pmatrix} \frac{\partial^2 \ln L}{\partial {v_0}^2} & \frac{\partial^2 \ln L}{\partial v_0 \partial v_1} & \frac{\partial^2 \ln L}{\partial v_0 \partial \sigma} \\ \frac{\partial^2 \ln L}{\partial v_0 \partial v_1} & \frac{\partial^2 \ln L}{\partial {v_1}^2} & \frac{\partial^2 \ln L}{\partial v_1 \partial \sigma} \\ \frac{\partial^2 \ln L}{\partial v_0 \partial \sigma} & \frac{\partial^2 \log L}{\partial v_1 \partial \sigma} & \frac{\partial^2 \ln L}{\partial \sigma^2} \end{pmatrix}_{(\hat{v}_0, \hat{v}_1, \hat{\sigma})}, \tag{10.2.3.4}$$

where

$$\frac{\partial^2 \ln L}{\partial {v_0}^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{(r_{i:m_l:n_l}+1)\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\}, \tag{10.2.3.5}$$

$$\frac{\partial^2 \ln L}{\partial {v_1}^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l^2(r_{i:m_l:n_l}+1)\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\}, \tag{10.2.3.6}$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma}[-2+(r_{i:m_l:n_l}+1)$$

$$\times(2+\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})]-1\} \tag{10.2.3.7}$$

$$\frac{\partial^2 \ln L}{\partial v_0 \partial v_1} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l(r_{i:m_l:n_l}+1)\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\}, \tag{10.2.3.8}$$

$$\frac{\partial^2 \ln L}{\partial v_0 \partial \sigma} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{(r_{i:m_l:n_l}+1)(1+\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})-1\}, \tag{10.2.3.9}$$

and

$$\frac{\partial^2 \ln L}{\partial v_1 \partial \sigma} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l[(r_{i:m_l:n_l}+1)(1+\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})\exp(\frac{y_{i:m_l:n_l}-v_0-v_1x_l}{\sigma})-1]\}. \tag{10.2.3.10}$$

10.2.4 AMLE

Denoting $z_{i:m_l:n_l} = \dfrac{y_{i:m_l:n_l} - \nu_0 - \nu_1 x_l}{\sigma}$, the corresponding likelihood function for the density function in (10.1.1) can be expressed as

$$L = \prod_{l=1}^{k}\left\{(\prod_{i=1}^{m_l} g_{i:m_l:n_l})\sigma^{-m_l} \exp\left[\sum_{i=1}^{m_l}[z_{i:m_l:n_l} - (r_{i:m_l:n_l} + 1)\exp(z_{i:m_l:n_l})]\right]\right\}$$

where $g_{i:m_l:n_l} = \sum_{j=i}^{m_l}(r_{i:m_l:n_l} + 1)$, $i = 1, \cdots, m_l$, is the number of units remaining on test immediately preceding the i-th failure.

Dropping the proportionality constant $\prod_{i=1}^{m_l} g_{i:m_l:n_l}$, we can take the log-likelihood function as

$$\ln L = \sum_{i=1}^{k}\{-m_l \ln\sigma + \sum_{i=1}^{m_l}[z_{i:m_l:n_l} - (r_{i:m_l:n_l} + 1)\exp(z_{i:m_l:n_l})]\}.$$

The log-likelihood equations for ν_0, ν_1 and σ become

$$\frac{\partial \ln L}{\partial \nu_0} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 - (r_{i:m_l:n_l} + 1)\exp(z_{i:m_l:n_l})\} = 0, \quad (10.2.4.1)$$

$$\frac{\partial \ln L}{\partial \nu_1} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l[1 - (r_{i:m_l:n_l} + 1)\exp(z_{i:m_l:n_l})]\} = 0, \quad (10.2.4.2)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 + z_{i:m_l:n_l}[1 - (r_{i:m_l:n_l} + 1)\exp(z_{i:m_l:n_l})]\} = 0, \quad (10.2.4.3)$$

respectively.

The likelihood equations in (10.2.4.1) – (10.2.4.3) do not admit explicit solutions. However, by expanding the function $\exp(z_{i:m_l:n_l})$ in a Taylor series around the point $F^{-1}(p_{i:m_l:n_l}) = \ln(-\ln q_{i:m_l:n_l})$, we may approximate this function by

$$\exp(z_{i:m_l:n_l}) \approx 1 - \alpha_{i:m_l:n_l} + \beta_{i:m_l:n_l} z_{i:m_l:n_l}.$$

where

$$p_{i:m_l:n_l} = 1 - b_i, \tag{10.2.4.4}$$

$$q_{i:m_l:n_l} = 1 - p_{i:m_l:n_l} = b_i,$$

$$\alpha_{i:m_l:n_l} = 1 + \ln q_{i:m_l:n_l}\{1 - \ln(-\ln q_{i:m_l:n_l})\},$$

$$\beta_{i:m_l:n_l} = -\ln q_{i:m_l:n_l},$$

and b_i is as defined in (10.2.2.9).

It is easy to see that $\beta_{i:m_l:n_l} > 0$.

By making use of the above linear approximations, we obtain the approximate log-likelihood equations as

$$\frac{\partial \ln L}{\partial v_0} \approx \frac{\partial \ln L^*}{\partial v_0} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 - (r_{i:m_l:n_l} + 1)(1 - \alpha_{i:m_l:n_l} + \beta_{i:m_l:n_l} z_{i:m_l:n_l})\} = 0, \tag{10.2.4.5}$$

$$\frac{\partial \ln L}{\partial v_1} \approx \frac{\partial \ln L^*}{\partial v_1} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{x_l[1 - (r_{i:m_l:n_l} + 1)(1 - \alpha_{i:m_l:n_l} + \beta_{i:m_l:n_l} z_{i:m_l:n_l})]\} = 0, \tag{10.2.4.6}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \approx \frac{\partial \ln L^*}{\partial \sigma} = -\frac{1}{\sigma}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{1 + z_{i:m_l:n_l}[1 - (r_{i:m_l:n_l} + 1)(1 - \alpha_{i:m_l:n_l} + \beta_{i:m_l:n_l} z_{i:m_l:n_l})]\} = 0. \tag{10.2.4.7}$$

Upon solving equations (10.2.4.5) – (10.2.4.7), we derive the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ (of ν_0, ν_1 and σ) based on progressively Type-II right-censored sample as

$$\tilde{\nu}_0 = a\tilde{\sigma} + b, \tag{10.2.4.8}$$

$$\tilde{\nu}_1 = c\tilde{\sigma} + d, \tag{10.2.4.9}$$

and

$$\tilde{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{10.2.4.10}$$

where

$$a = \Delta_a / \Delta, \quad b = \Delta_b / \Delta, \quad c = \Delta_c / \Delta, \quad d = \Delta_d / \Delta, \tag{10.2.4.11}$$

$$\Delta = \det \begin{vmatrix} \sum_{l=1}^{k}\sum_{i=1}^{m_l}[(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \\ \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l^2(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \end{vmatrix}, \tag{10.2.4.12}$$

$$\Delta_a = -\det \begin{vmatrix} \sum_{l=1}^{k}\sum_{i-1}^{m_l}[1-(r_{i:m_l:n_l}+1)(1-\alpha_{i:m_l:n_l})] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \\ \sum_{l=1}^{k}\sum_{i-1}^{m_l}\{x_l[1-(r_{i:m_l:n_l}+1)(1-\alpha_{i:m_l:n_l})\} & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l^2(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \end{vmatrix}, \tag{10.2.4.13}$$

$$\Delta_b = \det \begin{vmatrix} \sum_{l=1}^{k}\sum_{i=1}^{m_l}[(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}y_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \\ \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}y_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l^2(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] \end{vmatrix}, \tag{10.2.4.14}$$

$$\Delta_c = -\det \begin{vmatrix} \sum_{l=1}^{k}\sum_{i=1}^{m_l}[(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i-1}^{m_l}[1-(r_{i:m_l:n_l}+1)(1-\alpha_{i:m_l:n_l})] \\ \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i-1}^{m_l}\{x_l[1-(r_{i:m_l:n_l}+1)(1-\alpha_{i:m_l:n_l})]\} \end{vmatrix}, \tag{10.2.4.15}$$

$$\Delta_d = \det \begin{vmatrix} \sum_{l=1}^{k}\sum_{i=1}^{m_l}[(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}y_{i:m_l:n_l}] \\ \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}] & \sum_{l=1}^{k}\sum_{i=1}^{m_l}[x_l(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}y_{i:m_l:n_l}] \end{vmatrix}, \tag{10.2.4.16}$$

$$A = \sum_{l=1}^{k} m_l , \tag{10.2.4.17}$$

$$B = \sum_{l=1}^{k}\sum_{i=1}^{m_l}[y_{i:m_l:n_l} - (b+dx_l)]\{1-(r_{i:m_l:n_l}+1)[1-\alpha_{i:m_l:n_l} - \beta_{i:m_l:n_l}(a+cx_l)]\} \tag{10.2.4.18}$$

and

$$C = \sum_{l=1}^{k}\sum_{i=1}^{m_l} -(r_{i:m_l:n_l}+1)\beta_{i:m_l:n_l}[y_{i:m_l:n_l} - (b+dx_l)]^2 . \tag{10.2.4.19}$$

It should be mentioned here that upon solving Eq. (10.2.4.7) for σ, we obtain a quadratic equation in σ which has two roots; however, one of them drops out since $A > 0$, $\beta_{n_l - s_l:n_l} > 0$ and $\beta_{i:n_l} > 0$, and hence $C < 0$.

When all the groups are of the same size, we have $c = 0$ in the expression (10.2.4.15).

The approximate variances and covariances can be obtained by inverting the observed Fisher information matrix I_0^* evaluated at the AMLEs $\tilde{\nu}_0$, $\tilde{\nu}_1$ and $\tilde{\sigma}$ (of ν_0, ν_1 and σ). The observed Fisher information matrix I_0^* is of the form

$$I_0^* = - \begin{pmatrix} \frac{\partial^2 \ln L^*}{\partial \nu_0^2} & \frac{\partial^2 \ln L^*}{\partial \nu_0 \partial \nu_1} & \frac{\partial^2 \ln L^*}{\partial \nu_0 \partial \sigma} \\ \frac{\partial^2 \ln L^*}{\partial \nu_0 \partial \nu_1} & \frac{\partial^2 \ln L^*}{\partial \nu_1^2} & \frac{\partial^2 \ln L^*}{\partial \nu_1 \partial \sigma} \\ \frac{\partial^2 \ln L^*}{\partial \nu_0 \partial \sigma} & \frac{\partial^2 \ln L^*}{\partial \nu_1 \partial \sigma} & \frac{\partial^2 \ln L^*}{\partial \sigma^2} \end{pmatrix}_{(\tilde{\nu}_0, \tilde{\nu}_1, \tilde{\sigma})} , \tag{10.2.4.20}$$

where

$$\frac{\partial^2 \ln L^*}{\partial {v_0}^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}(1+r_{i:m_l:n_l})\beta_{i:m_l:n_l}, \tag{10.2.4.21}$$

$$\frac{\partial^2 \ln L^*}{\partial {v_1}^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}x_l^2(1+r_{i:m_l:n_l})\beta_{i:m_l:n_l}, \tag{10.2.4.22}$$

$$\frac{\partial^2 \ln L^*}{\partial \sigma^2} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}\{-1+2[r_{i:m_l:n_l}-\alpha_{i:m_l:n_l}(1+r_{i:m_l:n_l})]z_{i:m_l:n_l}+3\beta_{i:m_l:n_l}z^2_{i:m_l:n_l}\}, \tag{10.2.4.23}$$

$$\frac{\partial^2 \ln L^*}{\partial v_0 \partial v_1} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum_{i=1}^{m_l}x_l(1+r_{i:m_l:n_l})\beta_{i:m_l:n_l}, \tag{10.2.4.24}$$

$$\frac{\partial^2 \ln L}{\partial v_0 \partial \sigma} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum\{(1+r_{i:m_l:n_l})(1-\alpha_{i:m_l:n_l}+2\beta_{i:m_l:n_l}z_{i:m_l:n_l})-1\}, \tag{10.2.4.25}$$

and

$$\frac{\partial^2 \ln L}{\partial v_0 \partial \sigma} = -\frac{1}{\sigma^2}\sum_{l=1}^{k}\sum\{x_l[(1+r_{i:m_l:n_l})(1-\alpha_{i:m_l:n_l}+2\beta_{i:m_l:n_l}z_{i:m_l:n_l})-1]\}. \tag{10.2.4.26}$$

10.3 Simulation and Discussion

In the simulation study, we use the special example of n = 10, m = 5, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 2$ presented by Thomas and Wilson (1972) as a each group to form the two- and four-grouped samples. We took $v_0 = 0$, $v_1 = 1$ and $\sigma = 1$, and X = [-0.5, 0.5] or [-0.5, -0.16, 0.16, 0.5] for two or four-grouped samples, respectively.

In the case of BLUE, both exact and the biased first-order approximate are presented. We also performed a simulation for the unbiased first-order approximate method, in which the approximate means in expression (10.2.1.9) are replaced by the exact values. The assessments are based on the estimators' probability coverages, bias, mean square error, variances and covariances.

Similar assessments are made for MLE and AMLE.

All the simulation results are based on 10,000 Monte Carlo runs. These results are presented in Tables 10.3.1 – 10.3.9.

Since the estimators of ν_0, ν_1 and σ from the biased approximate method to BLUE are considerably biased, the comparison of the probability coverages is made only between the exact BLUE and the unbiased approximate method to BLUE. The results from the approximate method to BLUE are in good agreement with the exact BLUE. The exact BLUE has a value closer to the 95% for probability coverages as compared to the unbiased approximate method.

The MLEs of ν_0 and σ are highly biased when the total sample size N is small. The biases are decreased dramatically as the total sample size N increases.

Overall, as we expected, the BLUE's turn out to be best in terms of probability coverages by having values closer to 95%, and the MLE's turn out to be the best in terms of variances and mean square errors.

Compared with the initial guess of $\nu_0 = 0, \nu_1 = 1$ and $\sigma = 1$ (the true values of the parameter set in the simulation), use of the AMLE estimators dramatically increased the convergence success rate from about 54% to 99% for the Newton-Raphson procedure for determining the MLEs. It is also found that the AMLEs were very good in improving the speed of convergence as the convergence occurred within ten iterations in all the cases examined here.

Table 10.3.1 BLUE: Simulated probability coverages based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

Probability coverages	Exact BLUE			First-order approximate BLUE (unbiased)			First-order approximate BLUE (biased)		
$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
5 5	90.84	90.08	82.29	89.38	88.30	85.82	93.81	92.75	95.61
5 5 5 5	92.60	92.07	84.64	90.30	90.13	86.58	94.78	94.48	93.87

Table 10.3.2 BLUE: Simulated bias based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

Bias/σ	Exact BLUE			First-order approximate BLUE (unbiased)			First-order approximate BLUE (biased)		
$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$	ν_0	ν_1	σ	ν_0	ν_1	σ	ν_0	ν_1	σ
5 5	-0.0019	-0.012	0.0021	0.0039	-0.0005	0.0030	0.0268	-0.0097	0.1933
5 5 5 5	-0.0004	0.0015	0.0015	-0.0002	-0.0049	0.0009	0.0221	-0.0008	0.1856

Table 10.3.3 BLUE: Simulated MSE based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

MSE/σ^2				Exact BLUE			First-order approximate BLUE (unbiased)			First-order approximate BLUE (biased)		
$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				v_0	v_1	σ	v_0	v_1	σ	v_0	v_1	σ
5		5		0.1394	0.4559	0.1141	0.1338	0.4606	0.0888	0.1384	0.4526	0.1572
5	5	5	5	0.0677	0.4134	0.0586	0.0671	0.3980	0.0452	0.0683	0.4060	0.0942

Table 10.3.4 BLUE: Simulated variances and covariances# based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

Variances and covariances#				Exact BLUE			First-order approximate BLUE (unbiased)			First-order approximate BLUE (biased)		
$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				v_0	v_1	σ	v_0	v_1	σ	v_0	v_1	σ
5		5		0.1394	0.0014	0.0530	0.1338	-0.0013	0.0439	0.1377	0.0016	0.0553
5		5		0.0014	0.4558	0.0014	-0.0013	0.4607	0.0002	0.0016	0.4526	0.0015
5		5		0.0530	0.0014	0.1142	0.0439	0.0002	0.0888	0.0553	0.0015	0.1199
5	5	5	5	0.0677	0.0017	0.0275	0.0671	0.0004	0.0223	0.0678	-0.0002	0.0277
5	5	5	5	0.0017	0.4135	-0.0003	0.0004	0.3980	0.0000	-0.0002	0.4060	-0.0017
5	5	5	5	0.0275	-0.0003	0.0586	0.0223	0.0000	0.0452	0.0277	-0.0017	0.0598

"#" Denotes the variances and covariances are expressed in the form of $\left| \begin{matrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{matrix} \right| \Big/ \sigma^2$ within the sample

Table 10.3.5 MLE: Simulated probability coverages, bias and MSE based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				Probability coverages			Bias/σ			MSE/σ^2		
				v_0	v_1	σ	v_0	v_1	σ	v_0	v_1	σ
5		5		72.60	76.34	68.03	-0.1731	-0.0064	-0.1481	0.1501	0.4461	0.0832
5	5	5	5	82.89	86.00	79.05	-0.0833	-0.0066	-0.0726	0.0641	0.3821	0.0361

Table 10.3.6 MLE: Simulated variances and covariances# based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				Variances and covariances#		
				v_0	v_1	σ
5		5		0.1201	0.0002	0.0240
5		5		0.0002	0.4461	0.0002
5		5		0.0240	0.0002	0.0613
5	5	5	5	0.0572	0.0032	0.0112
5	5	5	5	0.0032	0.3821	0.0002
5	5	5	5	0.0112	0.0002	0.0312

"#" Denotes the variances and covariances are expressed in the form of

$$\begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \Big/ \sigma^2 \text{ within the sample}$$

Table 10.3.7 AMLE: Simulated bias and MSE based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				Bias/σ			MSE/σ^2		
				v_0	v_1	σ	v_0	v_1	σ
5		5		-0.2136	0.0020	-0.1072	0.1675	0.4538	0.0796
5	5	5	5	-0.1822	-0.0016	-0.0302	0.0897	0.4162	0.0355

Table 10.3.8 AMLE: Simulated variances and covariances# based on Type-II progressively right-censored sample with each group $n = 10$, $m = 5$, $r_1 = r_3 = r_4 = 0$, $r_2 = 3$ and $r_5 = 3$.

$[m_1\ m_2]$ or $[m_1\ m_2\ m_3\ m_4]$				Variances and covariances#		
				v_0	v_1	σ
5		5		0.1218	0.0002	0.0214
5		5		0.0002	0.4538	0.0027
5		5		0.0214	0.0027	0.0681
5	5	5	5	0.0565	0.0041	0.0566
5	5	5	5	0.0041	0.4162	-0.0008
5	5	5	5	0.0078	-0.0022	0.0070

"#" Denotes the variances and covariances are expressed in the form of

$$\begin{vmatrix} Var(v_0^*) & Cov(v_0^*, v_1^*) & Cov(v_0^*, \sigma^*) \\ Cov(v_0^*, v_1^*) & Var(v_1^*) & Cov(v_1^*, \sigma^*) \\ Cov(v_0^*, \sigma^*) & Cov(v_1^*, \sigma^*) & Var(\sigma^*) \end{vmatrix} \Bigg/ \sigma^2 \text{ within the sample}$$

10.4 Illustrative Example

We illustrate three estimation methods of the approximate BLUE, the MLE and the AMLE, to the progressively Type-II right-censored sample for the MEVR model here from real-life Example 8.2.1.

In order to generate the progressively Type-II right-censored sample from Example 8.2.1, we have used the following censoring scheme.

Group (l)	n_l	m_l	Censoring Scheme s_l
1	3	3	[0, 0, 0]
2	5	4	[0, 0, 0, 1]
3	11	6	[0, 2, 0, 0, 2, 1]
4	15	7	[0, 1, 0, 2, 0, 0, 0]
5	19	9	[4, 0, 0, 3, 0, 0, 0, 0, 3]
6	15	8	[0, 0, 3, 0, 0, 0, 3, 1]
7	8	5	[0, 0, 0, 0, 3]

By using the formulas in (10.2.1.9), (10.2.2.14) - (10.2.2.16), (10.2.3.1) – (10.2.3.3) and (10.2.4.8) – (10.2.4.10), we find the values of the approximate BLUEs, MLEs and AMLEs of parameters ν_0, ν_1 and σ, respectively, as follows:

Table 10.4.1 Estimates for progressively Type-II right-censored sample from Example 8.2.1

Estimation procedure	Estimates		
	ν_0	ν_1	σ
Approximate BLUE	61.2096	-16.7002	1.5133
MLE	61.2474	-16.7146	1.3215
AMLE	58.7554	-16.0558	1.3675

From the formulas in (10.2.1.11), (10.2.2.14) - (10.2.2.16), (10.2.3.4) and (10.2.4.20), we find the values of approximate variances and covariances of the approximate BLUEs, MLEs and AMLEs of ν_0, ν_1 and σ, respectively, as follows:

Table 10.4.2 Approximate variances and covariances for progressively Type-II right-censored sample from Example 8.2.1

Approximate variances and covariances/σ^2								
Approximate BLUE			MLE			MLE		
21.1296	-6.0669	-0.0449	24.3203	-6.9745	-0.0665	27.9636	-8.0126	-0.0426
-6.0669	1.7439	0.0148	-6.9745	2.0021	0.0196	-8.0126	2.2982	0.0135
-0.0449	0.0148	0.0184	-0.0665	0.0196	0.0138	-0.0426	0.0135	0.0132

As the AMLEs are biased and the expected values of these biases are not available, we presented the 95% confidence interval only for the approximate BLUEs and MLEs of ν_0, ν_1 and σ here as follows:

Table 10.4.3 95% confidence interval of the approximate BLUEs and MLEs for progressively Type-II right-censored sample from Example 8.2.1

Estimation procedure	95% confidence interval					
	LL(ν_0)	UL(ν_0)	LL(ν_1)	UL(ν_1)	LL(σ)	UL(σ)
Approximate BLUE	40.5781	81.8412	-22.6274	-10.7731	0.9047	2.1218
MLE	44.3671	78.1277	-21.5578	-11.8713	0.9191	1.7239

CHAPTER 11

CONTRIBUTIONS AND SUGGESTIONS FOR FURTHER RESEARCH

11.1 Contributions

In this thesis, we have presented different inferential methods for the parameters in a multi-group extreme value regression model based on complete sample, progressively Type-II right-censored sample and its special case – Type-II right-censored sample, and have evaluated the relative merits of these methods. We have also developed a large-sample approximation to BLUEs, which will be particularly useful when the means, variances and covariances of order statistics from the standard extreme value distribution are not readily available (in large samples, say, $n \geq 30$ or so). To check the adequacy of models upon which inferences are based on, a test of validity of the multi-group extreme value regression model is discussed as well. A list of the contributions in this thesis are given below:

1. We have used the best linear unbiased estimation method to derive expressions of estimators of the regression parameters for the multi-group extreme value regression model. The proof of the asymptotic normality of the BLUEs of these parameters is presented as well. We have also conducted a simulation study to

evaluate the performance of the BLUEs of these parameters for various choices of sample sizes and censoring schemes.

2. To obtain the maximum likelihood estimation of the regression parameters, we have derived the likelihood equations of the regression parameters for the multi-group extreme value regression model. The approximate and the asymptotic variances and covariances of these estimators are also derived through the observed and expected Fisher information matrix, respectively. In addition, we have conducted a simulation study to evaluate the performance of these MLEs for various choices of sample sizes and censoring schemes.

3. Since the maximum likelihood estimators of the regression parameters are not in closed form, we have developed approximate maximum likelihood estimators of these regression parameters. The approximate and the asymptotic variances and covariances of these estimators are derived through the observed as well as expected Fisher information matrix, respectively. We have also derived explicit expressions for the approximate biases of these estimators. A simulation study to evaluate the performance of the AMLEs of these parameters for various choices of sample sizes and censoring schemes has been conducted as well.

4. We have discussed confidence intervals based on estimators from the three different estimation methods mentioned above. We have used probability coverages to examine the accuracy of the interval estimation procedures. We have also conducted a simulation study to evaluate the probability coverages of the

pivotal quantities based on all these estimators for various choices of sample sizes and censoring schemes.

5. We have assessed the performance of BLUE, MLE and AMLE for the regression parameters with respect to the following factors:
   - The number of levels of the regressor variable x,
   - The balanced (equal sized) group sample vs. unbalanced (unequal sized) group sample,
   - The total sample size N,
   - The complete sample vs. Type-II right-censored sample,
   - The degrees of censoring.

   We have also made comparisons between BLUE, MLE and AMLE based on the relative efficiency of the estimators and the accuracy of the normal approximation in terms of probability coverages of intervals of these estimators.

6. We have extended Tiku's test to the multi-group sample situation to check the adequacy of models upon which inferences are based on. We have also described an approximate method of determining the level of significance and the power of this test procedure. Further, we have simulated the values of levels of significance under the standard extreme value model, and the values of power under five distributional alternatives for various choices of sample sizes and censoring schemes.

7. We have developed a large-sample approximation to BLUEs for the cases when the mean, variances and covariances of order statistics from the standard extreme

value distribution are not readily available (say, $n \geq 30$ or so). For this propose, we have considered the first-order and second-order approximations for the variance-covariance matrix of order statistics from the standard extreme value distribution using David and Johnson's (1954) approximation. A simulation study has been conducted as well in order to assess the performance of these two approximation methods as compared to the exact method.

8. All the estimations methods mentioned above have been generalized to progressively Type-II right-censored samples. To evaluate all different types of estimation procedures, a simulation study has been conducted based on progressively Type-II right-censored two- and four-grouped samples.

9. All these inferential procedures have been illustrated through the real-life examples.

11.2 Suggestions for Further Research

In this thesis, we have studied different inferential methods for the multi-group extreme value regression model, and evaluated their relative merits. The developments of the thesis have brought out some more problems that are worth considering for future research. Of special interest among these are the following problems:

1. In this thesis, we have developed statistical inference for the special case of one covariate, i.e. $\mu(x) = \nu_0 + \nu_1 x$. It will naturally be of interest to develop statistical inference for the case of two or more covariates in the multi-group extreme value regression model.

2. We have considered the linear function of covariate, i.e., $\mu(x) = \nu_0 + \nu_1 x$ (power rule model), as the form of regression (form of link function) throughout this thesis. It will also be of great interest to obtain statistical inference based on the following link functions:

   - the reciprocal linear (Arrhenius model)
   - an exponential function
   - polynomial function.

3. In Chapter 7, we have proposed a goodness-of-fit test for the multi-group sample situation to check the adequacy of the extreme value regression. This method does not allow a comparison between the different estimation methods since the estimators all get cancelled out in the expression of the test statistics. Therefore, it will be of interest to develop so other goodness-of-fit tests, which will not only test the validity of the multi-group extreme value regression model but also provide a comparison between the different estimation methods used to estimate the underlying parameters.

APPENDIX

Proof of condition 1: The weight function $J(u)$ of the estimator $v_0^* = X'\Delta_{v_0}Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} a_{i:n_l} y_{i:n_l}$ is bounded.

Proof: For the purpose of simplicity and without any loss of generality, we consider two-grouped sample here ($N = n_1 + n_2$).

We present the proof in three parts as follows:

Part (A):

Prove $\delta > CN^3$, where C is a constant and $N = n_1 + n_2$.

Part (B):

Prove $| X'\delta_{v_0} | < cN^2 \langle 1 |_{1\times N}$ where c is a constant, "$\langle 1 |_{1\times N}$" is a row vector of 1 of size N and $N = n_1 + n_2$.

Part (C):

Prove weight function J(u) in $v_0^* = X'\Delta_{v_0}Y = \sum_{l=1}^{k}\sum_{i=1}^{n_l} a_{i:n_l} y_{i:n_l}$ is bounded.

Part (A)

From (2.2.16), we have $\delta = \det\begin{vmatrix} 1'\Sigma^{-1}1 & X'\Sigma^{-1}1 & \alpha'\Sigma^{-1}1 \\ 1'\Sigma^{-1}X & X'\Sigma^{-1}X & \alpha'\Sigma^{-1}X \\ 1'\Sigma^{-1}\alpha & X'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1}\alpha \end{vmatrix}_{N\times N}$.

Adopt the "Dirac" symbols, i.e., $\langle x|=$ row vector, $|x\rangle=$ column vector, $\langle x|\alpha\rangle \equiv \sum_i x_i\alpha_i \equiv \langle\alpha|x\rangle$ and $|x\rangle\langle\alpha|$=matrix, etc., and define $M = \Sigma^{-1}$. Then, we can express

$$\begin{aligned}\delta &= \langle\alpha|M|a\rangle\langle 1|M|1\rangle\langle x|M|x\rangle - \langle\alpha|M|a\rangle\langle 1|M|x\rangle^2 \\ &+ \langle 1|M|a\rangle\langle x|M|1\rangle\langle\alpha|M|x\rangle - \langle 1|M|a\rangle\langle\alpha|M|1\rangle\langle x|M|x\rangle \\ &+ \langle x|M|a\rangle\langle 1|M|x\rangle\langle\alpha|M|1\rangle - \langle x|M|a\rangle\langle 1|M|1\rangle\langle\alpha|M|x\rangle,\end{aligned} \tag{A.1}$$

$$\langle x| \equiv [x_1,\ldots,x_1,x_2,\ldots,x_2]_{1\times N},$$

$$\langle\alpha| \equiv [\alpha_{1:n_1},\ldots,\alpha_{n_1:n_1},\alpha_{1:n_2},\ldots,\alpha_{n_2:n_2}]_{1\times N},$$

$$M \equiv \begin{vmatrix} M_1 & \\ & M_2 \end{vmatrix}_{N\times N},$$

$$\langle\alpha|M|\alpha\rangle = \langle\alpha_1|M_1|\alpha_1\rangle + \langle\alpha_2|M_2|\alpha_2\rangle,$$

$$\langle 1|M|x\rangle = x_1\langle 1|M_1|1\rangle + x_2\langle 1|M_2|1\rangle,$$

$$\langle 1|M|1\rangle = \langle 1|M_1|1\rangle + \langle 1|M_2|1\rangle,$$

$$\langle x|M|\alpha\rangle = x_1\langle 1|M_1|\alpha_1\rangle + x_2\langle 1|M_2|\alpha_2\rangle,$$

$$\langle 1|M|\alpha\rangle = \langle 1|M_1|\alpha_1\rangle + \langle 1|M_2|\alpha_2\rangle,$$

and

$$\langle x|M|x\rangle = x_1^2\langle 1|M_1|1\rangle + x_1^2\langle 1|M_2|1\rangle.$$

Since the inverse of the covariances matrix M_1 (or M_2) is positive definite, we can express M_1 as $M_1 \equiv C_1^{\tau}\Lambda_1 C_1$, where C_1 is an orthogonal matrix, τ indicates transpose, Λ_1 is a diagonal matrix with $\Lambda_{1(ii)} = \lambda_i > 0$, for i = 1, …, n_1 and λ_i's are the eigen values of M_1.

Define new vectors:

$$\langle \alpha | c^\tau \sqrt{\lambda_i} \equiv \langle \tilde{\alpha} |, \qquad \left(\sqrt{\lambda_i} C | \alpha \rangle \equiv | \tilde{\alpha} \rangle \right),$$

and

$$\langle 1 | C^\tau \sqrt{\lambda_i} \equiv \langle \tilde{1} |, \qquad \left(\sqrt{\lambda_i} C | 1 \rangle \equiv | \tilde{1} \rangle \right).$$

Writing $\langle M_1 \rangle = \langle 1 | M_1 | 1 \rangle$ and $\langle M_2 \rangle = \langle 1 | M_2 | 1 \rangle$, we can express

$$\langle M \rangle \equiv \langle \tilde{1} | \tilde{1} \rangle = (\text{length of } \tilde{1})^2,$$

$$\langle \alpha | M | \alpha \rangle \equiv \langle \tilde{\alpha} | \tilde{\alpha} \rangle = (\text{length of } \tilde{\alpha})^2$$

and

$$\langle \alpha | M | 1 \rangle \equiv \langle \tilde{\alpha} | \tilde{1} \rangle = (\text{length of } \tilde{1})(\text{length of } \tilde{1}) \cos\theta(\tilde{1}, \tilde{\alpha}) = \sqrt{\langle \tilde{\alpha} | \tilde{\alpha} \rangle \langle \tilde{1} | \tilde{\alpha} \rangle} \cos\theta(\tilde{1}, \tilde{\alpha}).$$

Let $\lambda_{\min}^{(l)}$ and $\lambda_{\max}^{(l)}$ be the minimum and maximum of the eigen values of the matrix M_l, respectively, for l = 1, 2, and the symbol (l) denotes the value is from l-th group. We then have

$$\lambda_{\min}^{(l)} n_l = \lambda_{\min}^{(l)} \langle 1 | C_l^\tau C_l | 1 \rangle \le \langle M_l \rangle = \langle 1 | C_l^\tau \Lambda_l C_l | 1 \rangle \le \lambda_{\max}^{(l)} \langle 1 | C_l^\tau C_l | 1 \rangle = \lambda_{\max}^{(l)} n_l, \; l = 1, 2.$$

i.e.

$$\lambda_{\min}^{(l)} n_l \le \langle M_l \rangle \le \lambda_{\max}^{(l)} n_l.$$

Similarly, we can get

$$\alpha_{\min}^{(l)\,2} \lambda_{\min}^{(l)} n_l \le \langle \alpha_l | M_l | \alpha_l \rangle \le \alpha_{\max}^{(l)\,2} \lambda_{\max}^{(l)} n_l,$$

$$\alpha_{\min}^{(l)} \lambda_{\min}^{(l)} n_l \le \langle 1 | M_l | \alpha_l \rangle \le \alpha_{\max}^{(l)} \lambda_{\max}^{(l)} n_l ,$$

$$x_l^{\,2} \lambda_{\min}^{(l)} n_l \le \langle x_l | M_l | x_l \rangle \le x_l^{\,2} \lambda_{\max}^{(l)} n_l ,$$

$$x_l \lambda_{\min}^{(l)} n_l \le \langle 1 | M_l | x_l \rangle \le x_l \lambda_{\max}^{(l)} n_l ,$$

and

$$x_{\min}^{(l)} \alpha_{\min}^{(l)} \lambda_{\min}^{(l)} n_l \le \langle x_l | M_l | \alpha_l \rangle \le x_{\max}^{(l)} \alpha_{\max}^{(l)} \lambda_{\max}^{(l)} n_l .$$

Therefore, for all the terms with the positive sign in (A.1), we can always minimize them by the values of $x_{\min}^{(l)}$, $\alpha_{\min}^{(l)}$ and $\lambda_{\min}^{(l)}$, and vase versa, maximize all the terms with the negative sign by the values of $x_{\max}^{(l)}$, $\alpha_{\max}^{(l)}$ and $\lambda_{\max}^{(l)}$. For example, we have $\langle \alpha | M | \alpha \rangle \langle 1 | M | 1 \rangle \langle x | M | x \rangle > x_{\min}^2 \alpha_{\min}^2 \lambda_{\min}^3 N^3$ for the first term in (A.1) and $\langle \alpha | M | \alpha \rangle \langle 1 | M | x \rangle^2 < x_{\max}^2 \alpha_{\max}^2 \lambda_{\max}^3 N^3$ for the second term. And moreover, we obtain $\langle \alpha | M | \alpha \rangle \langle 1 | M | 1 \rangle \langle x | M | x \rangle - \langle \alpha | M | \alpha \rangle \langle 1 | M | x \rangle^2 > \left(x_{\min}^2 \alpha_{\min}^2 \lambda_{\min}^3 - x_{\max}^2 \alpha_{\max}^2 \lambda_{\min}^3 \right) N^3$, where $x_{\min} = \min(x_{\min}^{(1)}, x_{\min}^{(2)})$, $\alpha_{\min} = \min(\alpha_{\min}^{(1)}, \alpha_{\min}^{(2)})$, $\lambda_{\min} = \min(\lambda_{\min}^{(1)}, \lambda_{\min}^{(2)})$, and the maximums of x, α and λ are defined in the similar manner.

Hence, we can always find a value C such that

$$\delta > CN^3$$

where C is a function of the constants of $x_{\min}$, $\alpha_{\min}$, $\lambda_{\min}$, $x_{\max}$, $\alpha_{\max}$ and $\lambda_{\max}$ which has considered all the terms in (A.2).

Furthermore, we can prove $\delta > 0$. Express the first two terms, second two terms and third two terms of δ as

$$\delta_1 = (x_1 - x_2)^2 \left(\langle \alpha_1 \mid M_1 \mid \alpha_1 \rangle \langle M_1 \rangle \langle M_1 \rangle + \langle \alpha_2 \mid M_2 \mid \alpha_2 \rangle \langle M_2 \rangle \langle M_2 \rangle \right),$$

$$\delta_2 = \langle 1 \mid M \mid \alpha \rangle [x_1 x_2 (\langle M_1 \rangle \langle 1 \mid M_2 \mid \alpha_2 \rangle + \langle M_2 \rangle \langle 1 \mid M_1 \mid \alpha_1 \rangle) - x_1^2 \langle M_1 \rangle \langle 1 \mid M_2 \mid \alpha_2 \rangle - x_2^2 \langle M_2 \rangle \langle 1 \mid M_1 \mid \alpha_1 \rangle],$$

and

$$\delta_3 = \langle x \mid M \mid \alpha \rangle [x_1 (\langle M_1 \rangle \langle 1 \mid M_2 \mid \alpha_2 \rangle - \langle M_2 \rangle \langle 1 \mid M_1 \mid \alpha_1 \rangle) + x_2 (\langle M_2 \rangle \langle 1 \mid M_1 \mid \alpha_1 \rangle - \langle M_1 \rangle \langle 1 \mid M_2 \mid \alpha_2 \rangle)],$$

respectively. We can write δ as

$$\delta = (x_1 - x_2)^2 \left\{ \langle M_1 \rangle \left(\langle M_2 \rangle \langle \alpha_2 \mid M_2 \mid \alpha_2 \rangle - \langle 1 \mid M_2 \mid \alpha_2 \rangle^2 \right) + \langle M_2 \rangle \left(\langle M_1 \rangle \langle \alpha_1 \mid M_1 \mid \alpha_1 \rangle - \langle 1 \mid M_1 \mid \alpha_1 \rangle^2 \right) \right\}.$$

Since

$$\langle M \rangle \langle \alpha \mid M \mid \alpha \rangle - \langle 1 \mid M \mid \alpha \rangle^2 \equiv \langle \tilde{1} \mid \tilde{1} \rangle \langle \tilde{\alpha} \mid \tilde{\alpha} \rangle \left(1 - \cos^2 \theta(\tilde{1}, \tilde{\alpha}) \right) = \langle \tilde{1} \mid \tilde{1} \rangle \langle \tilde{\alpha} \mid \tilde{\alpha} \rangle \sin^2 \theta(\tilde{1}, \tilde{\alpha}),$$

and $\alpha \neq$ constant implies $\theta \neq 0$, i.e. $\sin\theta(\tilde{1}, \tilde{\alpha}) > 0$, we obtain

$$\langle M \rangle \langle \alpha \mid M \mid \alpha \rangle - \langle 1 \mid M \mid \alpha \rangle^2 \equiv \langle M \rangle \langle \alpha \mid M \mid \alpha \rangle \sin^2 \theta(\tilde{1}, \tilde{\alpha}) > 0.$$

Therefore we have $\delta > 0$.

Part (B):

From (2.2.17), we have $\delta_{v_0} = \begin{Vmatrix} \Sigma^{-1}1 & \alpha'\Sigma^{-1}1 & 1'\Sigma^{-1} \\ \Sigma^{-1}X & \alpha'\Sigma^{-1}X & X'\Sigma^{-1} \\ \Sigma^{-1}\alpha & \alpha'\Sigma^{-1}\alpha & \alpha'\Sigma^{-1} \end{Vmatrix}$. Similar to the expression of δ given in (A.1), we have

$$\begin{aligned} X'\delta_{v_0} &= \langle \alpha \mid M \mid a \rangle \langle x \mid M \mid x \rangle \langle 1 \mid M - \langle \alpha \mid M \mid a \rangle \langle x \mid M \mid 1 \rangle \langle x \mid M \\ &+ \langle 1 \mid M \mid a \rangle \langle \alpha \mid M \mid x \rangle \langle x \mid M - \langle 1 \mid M \mid a \rangle \langle x \mid M \mid x \rangle \langle \alpha \mid M \\ &+ \langle x \mid M \mid a \rangle \langle 1 \mid M \mid x \rangle \langle \alpha \mid M - \langle x \mid M \mid a \rangle \langle \alpha \mid M \mid x \rangle \langle 1 \mid M \end{aligned} \qquad \text{(A.2)}$$

Follow the similar procedures as we did in Part (A) for δ and denote the first two terms of $X'\delta_{v_0}$ as $X'\delta_{v_0}(1,\ 2)$, we have

$$\left(x_{\min}^2\alpha_{\min}^2\lambda_{\min}^3 - x_{\max}^2\alpha_{\max}^2\lambda_{\max}^3\right)N^2\langle 1|_{1\times N} < X'\delta_{v_0}(1,\ 2) < \left(x_{\max}^2\alpha_{\max}^2\lambda_{\max}^3 - x_{\min}^2\alpha_{\min}^2\lambda_{\min}^3\right)N^2\langle 1|_{1\times N},$$

where "$\langle 1|_{1\times N}$" is a row vector of 1 of size N. Therefore, we can always find a c such that

$$|X'\delta_{v_0}| < cN^2\langle 1|_{1\times N}$$

where c is a function of the constants of $x_{\min}$, $\alpha_{\min}$, $\lambda_{\min}$, $x_{\max}$, $\alpha_{\max}$ and $\lambda_{\max}$ which has considered all the terms in (A.2).

Part (C):

Since each element of $\left|\frac{X'\delta_{v_0}}{\delta}\right| < \frac{c}{CN}$, and the $J(u)$ function is defined as each element of the row vector $N\frac{X'\delta_{v_0}}{\delta}$. Therefore we have $|J(u)| < \frac{c}{C}$, i.e., $J(u)$ is bounded.

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