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REFUELING STRATEGIES TO MAXIMIZE THE OPERATIONAL RANGE WITH A NON IDENTICAL VEHICLE FLEET

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Abstract

Recently, Mehrez et al. (Mehrez and Stern [1983, 1985], Melkman et al. [1986]) have studied a vehicle fleet refueling problem that arises in military applications and is aimed to maximize the operational range of the fleet. More specifically, they investigated the problem of maximizing the range of the last vehicle from a fleet of n vehicles by employing a sequential refueling chain strategy. The strategy of maximizing the range of the last vehicle is an important criterion to be considered under war conditions. This problem has its own elegant solution which demonstrates how a specific military Operations Research problem may reveal interesting results due to its unique structure. The approach recommended here to solve the problem indicates that numerical computations rather than an analytical approach may result in knowing less about the problem solution. The purpose of this paper is: (i) To construct an ordering rule for n = 2, which contradicts the conjecture of Mehrez et al. that even for the case of n = 2 a simple ordering rule does not exist. (ii) To suggest a recursive procedure which requires only O(n) calculations to solve the linear programming problem of maximizing the operational range for a given refueling chain. (iii) To suggest a new approach, which is based on the derivation of supply and demand curves for each refueling operation, to solve scheduling problems. It is shown how the analysis of these curves provides important information regarding the nature of the optimal solution which was treated by Mehrez et al. for some special cases of fleet configurations. The analysis supports the idea of solving the problem of determining the optimal refueling chain by a enumerative search for n sufficiently small. Finally,

for n = 2 and 3, an analysis is shown by which inferior refueling chains may be eliminated for the vehicle fleet refueling problem.

1. Introduction

This article addresses a problem of vehicle refueling strategies which was discussed by Mehrez et al. [1983]. These authors considered the problem of a self contained mission requiring round trip travel from a common origin. More specifically, they investigated the problem of maximizing the range of the last vehicle from a fleet of n vehicles by employing a sequential refueling chain strategy. In such a strategy the last vehicle to be refueled is the next vehicle to transfer its fuel. Actually, Mehrez et al. dealt with four types of fleet configurations: (1) identical vehicles, (2) vehicles with identical fuel consumption rates but different fuel capacities, (3) vehicles with the same fuel capacity but different fuel consumption rates, and (4) vehicles with both different fuel capacities and different fuel consumption rates.

Mehrez et al. provided the optimal strategy for the first three types of fleet configurations under the so called pure refueling chain. This strategy is based on the following conditions: All vehicles leave the origin simultaneously with tanks at full capacity. After a vehicle transfers fuel it must immediately return to the origin, leaving remaining vehicles proceeding away from the origin. Each transfer operation involves only one vehicle transferring fuel. It follows that there are n-1 ordered refueling operations, after which the last vehicle to receive fuel reaches its maximal range and returns to the origin.

Mehrez et al. were unable to analyze and solve the fourth type of fleet configuration with both different fuel capacities and different fuel consumption rates. This configuration leads to a mixed non-linear integer programming formulation which is difficult to solve by routine methods. The main purpose of this paper is to analyze this problem. In the next section, the problem is formulated and a counter example is provided to reject theorem 5 (p. 337) of Mehrez et al. that the optimal basic solution does not have basic slack variables. In section three, a simple case of $n = 2$ is studied and an ordering rule is constructed for this case. This rule contradicts the conjecture of Mehrez et al. that even for the case of $n = 2$ a simple ordering rule does not exist. However, it is shown that this rule does not hold for general case of $n > 2$. In section four, given a fixed refueling chain, the problem is reduced into a linear programming form. It is shown that, in spite of the linear programming structure, the problem can be solved recursively by a procedure which requires only $O(n)$ calculations. In section five, a pairwise interchange rule is suggested to improve a given refueling chain. This rule is shown to produce a local optimal solution for the general case. An optimal algorithm, which is based on the linear programming solution for a fixed refueling chain and a enumerative search procedure, is also suggested. Computer simulation is conducted to compare this algorithm with a heuristic for problems of small size ($n = 6$). In section six, it is shown for $n = 2, 3$ that it will never be optimal to locate a vehicle with larger capacity and lower fuel consumption rate before a vehicle with smaller capacity and higher fuel consumption rate. A conjecture that has not been rejected by computer simulation has been formulated for the general case. Finally, in section seven, future research is suggested.

2. The Problem Formulation

For the reader's convenience, the problem of maximizing the range of the last vehicle, given a fleet size n with a pure refueling chain, is reformulated by employing the notation used in Mehrez et al. [1983].

It is assumed that each vehicle in the fleet is assigned to a position in a refueling chain. $M = \{ m \}$ is the set of all refueling chains. A representative refueling chain is given by $m = (m_1, \ldots, m_n)$ where m is a permutation of integers $1, \ldots, n$. For a given m, the vehicles are indexed according to the fuel transfer order in the chain. In this chain, vehicle m_1 is active only in the first refueling operation, transferring fuel to vehicle m_2. Vehicle m_2 is active in two refueling operations: receiving fuel in the first operation and transferring fuel in the second operation. Vehicles $m_3, \ldots, m_{n-1}$ are also active in exactly two operations. The last vehicle in the chain, vehicle m_n, is active in receiving fuel from the m_{n-1} vehicle in the (n-1)th operation. Each vehicle m_k has a different fuel capacity C_{m_k}, measured in units of fuel, such as liters or gallons. Each vehicle m_k has also a different fuel consumption rate q_{m_k}, the amount of fuel per distance traveled. To formularize the problem the following additional terms are defined:

d_{m_k} : the maximum distance that the vehicle m_k can travel without refueling. $d_{m_k} = C_{m_k}/q_{m_k}$.

R_k : the distance from the point where the (k-1)th refueling operation ends to the start of the kth refueling operation (k=1,2,...,n-1). The zero th refueling operation occurs at the origin point of all vehicle departure and is not considered part of the refueling chain.

R_n : the distance from the (n-1)th (the last), refueling operation to the farthest point reached by the last vehicle before it commences its return to the origin point.

U_k : the amount of fuel transferred from the vehicle m_k to the vehicle m_{k+1} during the kth refueling operation.

Z(m) : the distance traveled by the last vehicle in the chain m from the origin to its turnaround point, i.e. the maximum distance reached by the last vehicle in the fleet. $Z(m) = R_1 + R_2 + \ldots + R_n$.

The problem P1 is to maximize Z(m). Further development of P1 leads to the following mixed non-linear integer programming problem:

Maximize

$$Z(m) = \sum_{k=1}^{n} R_k \quad (1)$$

Subject to

$$\sum_{i=1}^{n} X_{ik} = 1 \qquad k = 1,\ldots,n \quad (2)$$

$$\sum_{k=1}^{n} X_{ik} = 1 \qquad i = 1,\ldots,n \quad (3)$$

$$X_{ik} = 0,\ 1 \qquad i = 1,\ldots,n,\ k = 1,\ldots n \quad (4)$$

$$\left(\sum_{i=1}^{n} X_{ik}C_i\right) + U_{k-1} - U_k - 2\left(\sum_{j=1}^{k} R_j\right)\left(\sum_{i=1}^{n} X_{ik}q_i\right) \geq 0, \qquad k = 1,\ldots,n \quad (5)$$

$$\left(\sum_{j=1}^{k} R_j\right)\left(\sum_{i=1}^{n} X_{ik+1}q_i\right) - U_k \geq 0, \qquad k = 1,\ldots,n-1 \quad (6)$$

$$\sum_{i=1}^{n} X_{ik}C_i - U_{k-1} \geq 0, \qquad k = 1,\ldots,n \quad (7)$$

$$U_0 = 0,\ U_n = 0,\ U_k \geq 0, \qquad k = 1,\ldots,n-1 \quad (8)$$

$R_k \geq 0,$ $k = 1,\ldots,n$ (9)

The constraints (2)-(4) identify a permutation m for which the ith vehicle is assigned to the kth position. The constraint (5) states that the vehicle transferring fuel reserves an amount needed for the return trip to the origin. The constrain (6) guarantees that the amount of fuel transferred in the kth refueling operation will not exceed the amount of fuel the vehicle has burned up to the point of the kth operation. The constraint (7) guarantees that each vehicle receives fuel less than or equal to its capacity. The constraints (8)-(9) are trivially implied by the structure of the problem.

The problem Pl consists of n^2 0-1 variables and 2n-1 continuous variables which incorporated into a set of non-linear constraints. To solve such a problem we first note that for a given chain m, the sub-problem of maximizing the distance traveled by the last vehicle can be reduced into a linear programming structure which will be derived in section four. Mehrez et al. comment that Pl can be solved by enumerating all the basic solutions corresponding to the n! linear programming problems with non basic slack variables for constraints (5)-(6) (theorem (5) p. 337). Thus, in an optimal solution of Pl, all the fuel is consumed. While this theorem holds for the three special cases of identical fuel capacities or identical fuel consumption rates or a fleet with only two vehicles (n = 2), it does not hold for the general case as demonstrated by the following counter example with n = 4. The data of the counter example is provided in Table 1.

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Table 1 here

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Actually this example has twenty four chains or solutions. In Table 2 the objective function value is computed for each solution and it is indicated if the solution has non zero slack variables.

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Table 2 here

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Table 2 demonstrates two interesting possible cases. First, an optimal solution may not satisfy the equilibrium condition for which the fuel supply and the fuel demand are equal at the point where fuel transfer operation takes place. Second, an equilibrium solution may not be an optimal solution. More specifically, solution number 24 and 12 illustrate the first and the second case respectively. Figure 1 and 2 describe the fuel supply demand relationship at the fuel transfer points for the two cases respectively.

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Figure 1 here

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Figure 2 here

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3. The case of $n = 2$

The case of $n = 2$ was partially studied by Mehrez et al. These authors claimed that there is no simple ordering rule for the problem P1. Actually, the approach leading to this statement stems from the basic ideas corresponding to the rule of d_{m_k} in identifying the optimal order. The following lemma shows that the ordering rule for the optimal chain must depend on both d_{m_k} and q_{m_k} of both vehicles.

Lemma 1

For n = 2 the optimal chain is m = $\{m_1, m_2\}$ if and only if

$$\frac{d_{m_2}}{1 + q_{m_2}/(q_{m_1}+q_{m_2})} \geq \frac{d_{m_1}}{1 + q_{m_1}/(q_{m_1}+q_{m_2})} \ . \tag{10}$$

Proof.

Without loss of generality we compare {1,2} and {2,1} assuming that

$$\frac{d_2}{1+q_2/(q_1+q_2)} \geq \frac{d_1}{1+q_1/(q_1+q_2)} \ , \tag{11}$$

thus it is sufficient to show that $Z(1,2) \geq Z(2,1)$.

To develop the optimal order, P1 is reduced into two linear programming problems P2 for m = {1,2} and {2,1}. For m = {1,2}, the problem P2 can be formulated as following:

$$\text{Max } z(m) = R_1 + R_2$$

S.T.

$$C_1 - U_1 - 2 R_1 q_1 \geq 0 \tag{12}$$

$$C_2 + U_1 - 2(R_1+R_2) q_2 \geq 0 \tag{13}$$

$$R_1 q_2 - U_1 \geq 0 \tag{14}$$

$$C_2 - U_1 \geq 0 \tag{15}$$

$$U_1 \geq 0, \ R_1 \geq 0, \ R_2 \geq 0$$

In a similar way P2 can be formulated for m = {2,1}. Constraint (12) implies

$$2R_1 q_1 \leq C_1 - U_1 \tag{16}$$

and (14) implies

$$R_1 \geq \frac{U_1}{q_2} \tag{17}$$

thus combining (16) and (17) results in

$$\frac{2U_1 q_1}{q_2} \leq 2R_1 q_1 \leq C_1 - U_1 \tag{18}$$

or

$$U_1 \leq \frac{C_1 q_2}{q_2 + 2q_1} \tag{19}$$

Combining (19) and (15), we have upper bound of U_1 as

$$U_1 \leq \text{Min}\left(C_2\ ,\ \frac{C_1 q_2}{q_2 + 2q_1}\right) \tag{20}$$

Furthermore (13) implies that

$$R_1 + R_2 \leq \frac{1}{2q_2}\left(C_2 + U_1\right) \tag{21}$$

Combining (20) and (21) provides that

$$R_1 + R_2 \leq \frac{1}{2q_2} C_2 + \frac{1}{2q_2} \text{Min}\left(C_2,\ \frac{C_1 q_2}{q_2 + 2q_1}\right)$$

$$= \frac{1}{2} d_2 + \frac{1}{2} \text{Min}\left(d_2\ ,\ \frac{d_1}{2 + q_2/q_1}\right) \tag{22}$$

Therefore since the left hand side of (22) is the objective function then the inequality can be replaced by equality sign.

$$Z(1,\ 2) = \frac{1}{2} d_2 + \frac{1}{2} \text{Min}\left(d_2\ ,\ \frac{d_1}{2 + q_2/q_1}\right) \tag{23}$$

Similarly,

$$Z(2,\ 1) = \frac{1}{2}\, d_1 + \frac{1}{2}\, \text{Min}\,(d_1\ ,\ \frac{d_2}{2 + q_1/q_2}) \tag{24}$$

To complete the proof, it is sufficient to show that if

$$\frac{d_2}{1 + q_2/(q_1+q_2)} \geq \frac{d_1}{1 + q_1/(q_1+q_2)} \tag{25}$$

then $Z(1,2) \geq Z(2,1)$, or

$$\frac{1}{2}\, d_2 + \frac{1}{2}\, \text{Min}\,(d_2,\ \frac{d_1}{2+q_2/q_1}) \geq \frac{1}{2}\, d_1 + \frac{1}{2}\, \text{Min}\,(d_1,\ \frac{d_2}{2+q_1/q_2}). \tag{26}$$

Since (25) implies that

$$d_2 \geq d_1\, \frac{1+q_2/(q_1+q_2)}{1+q_1/(q_1+q_2)} = d_1\, \frac{2q_2+q_1}{2q_1+q_2} \geq \frac{d_1}{2 + q_2/q_1} \tag{27}$$

therefore by (23) and (27)

$$Z(1,2) = \frac{1}{2}\,(d_2 + \frac{d_1}{2+q_2/q_1}). \tag{28}$$

Furthermore, from (25) we have

$$d_2 + \frac{d_1}{2+q_2/q_1} \geq d_1 + \frac{d_2}{2+q_1/q_2} \tag{29}$$

since

$$(d_2 + \frac{d_1}{2+q_2/q_1}) - (d_1 + \frac{d_2}{2+q_1/q_2}) = \frac{d_2}{1+q_2/(q_1+q_2)} - \frac{d_1}{1+q_1/(q_1+q_2)} \geq 0$$

Thus by (28), (29) and (23)

$$Z(1,2) = \frac{1}{2}\,(d_2 + \frac{d_1}{2+q_2/q_1}) \geq \frac{1}{2}\,(d_1 + \frac{d_2}{2+q_1/q_2}) \geq Z(2,1).$$

Q.E.D.

4. Formulation of a fixed refueling chain

For a given refueling chain $m = \{m_1, m_2, \ldots, m_n\}$, P1 is reduced into a linear programming problem. Similar to the special case of $n = 2$, the formulation of P1 is reduced into P2 as follows:

$$\text{Max } Z(m) = \sum_{k=1}^{n} R_k$$

S.T.

$$C_{m_k} + U_{k-1} - U_k - 2\left(\sum_{j=1}^{k} R_j\right) q_{m_k} \geq 0, \qquad k = 1,\ldots,n \qquad (30)$$

$$\left(\sum_{j=1}^{k} R_j\right) q_{m_{k+1}} - U_k \geq 0, \qquad k = 1,\ldots,n-1 \qquad (31)$$

$$C_{m_k} - U_{k-1} \geq 0, \qquad k = 1,\ldots,n \qquad (32)$$

$$U_k = 0, \qquad k = 0 \text{ or } k = n \qquad (33)$$

$$U_k \geq 0, \qquad k = 1,\ldots,n-1 \qquad (34)$$

$$R_k \geq 0, \qquad k = 1,\ldots,n \qquad (35)$$

Constraints (30)-(35) are justified according to constraints (5)-(9) of P1. To shorten the paper we skip the explanation.

Mehrez et al. have computed a closed form for special cases of (1) identical vehicles (2) vehicles with identical fuel consumption rates but different fuel capacities, (3) vehicles with the same fuel capacity but different fuel consumption rates. But they were unable to reduce the LP into a recursive computation of O(n) for the general case of vehicles with both different fuel capacities and different fuel consumption rates. In this section we will develop a recursive computation formula for P2. This general formula will be utilized in section 5 to verify the Mehrez et al's formulas for the above specific cases (1)-(3). The basic underlying idea of the development of the recursive relationship between the kth and (k-1)th operation is based on : 1) the maximization of the supply curve of the kth operation under the supply constraint (30) and demand constraints (31)-(32) for the (k-1)th operation. 2) maximizing fuel transferring by minimizing the absolute deviation between supply and demand of each operation. To implement the idea we develop the supply and demand curves for the successive operations recursively. To do so we identify the supply curves which are shown to be linearly piece-wised with two segments. The point of discontinuity of the supply curve will be shown to be the one at which the maximum fuel can be transferred in the previous operation.

The recursive formulas are forward developed. More specifically, the supply-demand recursive formulas are developed for the first and second operations (Lemma 2). Due to a general argument this recursive formula will be held for the third and any successive operations (Lemma 3). The recursive procedure is shown to lead an optimal solution for P2 in Lemma 4. However, the recursive formulas which satisfy constraints (30)-(35) do not provide the values of the decision variables R_k, U_k, $k = 1,\ldots,n-1$. Theorem 1 identifies the optimal solution in terms of these variables by employing a backward recursive procedure. The optimal solution is derived by setting an equality sign for (31). But the optimal solution is not necessarily unique. To verify the feasibility of the solution an induction argument is again repeated.

For notational purpose we denote $R'_k = \sum_{j=1}^{k} R_j$, $k = 1, \ldots n-1$, the fuel transfer point for the kth operation, and R''_k , $k = 1,\ldots,n-1$, the point at which the maximum amount of fuel can be transferred in the kth operation.

<u>Lemma 2</u>.

The amount of fuel transferred in the first and second operation is maximized at R_1'' and R_2'' respectively where

$$R_1'' = \mathrm{Min}\left(\frac{C_{m_2}}{q_{m_2}}, \frac{C_{m_1}}{2q_{m_1} + q_{m_2}}\right)$$

and

$$R_2'' = \mathrm{Min}\left(\frac{C_{m_3}}{q_{m_3}}, \frac{C_{m_2}}{q_{m_2} + q_{m_3}}, \frac{C_{m_2} + q_{m_2}R_1''}{2q_{m_2} + q_{m_3}}\right) \qquad (36)$$

<u>Proof</u>

For the first fuel operation, the supply and demand function can be defined as

$$S_1(R_1') = \begin{cases} C_{m_1} - 2q_{m_1}R_1' & R_1' \le C_{m_1}/2q_{m_1} \\ 0 & R_1' > C_{m_1}/2q_{m_1} \end{cases}$$

$$D_1(R_1') = \begin{cases} q_{m_2} R_1' & R_1' \le C_{m_2}/q_{m_2} \\ 0 & R_1' > C_{m_2}/q_{m_2} \end{cases}$$

Here S_1 indicates how much fuel the vehicle m_1 can transfer to vehicle m_2 at R_1', and D_1 indicates the empty space available for accepting fuel by vehicle m_2 at R_1'. The supply S_1 and demand D_1 are graphically shown in Figure 3 (case 1).

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Figure 3 here

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It is obvious that at any transfer point $R_1' \le C_{m_2} / q_{m_2}$, the amount of fuel that can be transferred is Min $(S_1(R_1'), D_1(R_1'))$. If the demand curve D_1 intersects with the supply curve S_1, the maximum amount of fuel can be transferred at the equilibrium point $R_1'' = C_{m_1}/(2q_{m_1} + q_{m_2})$. If the D_1 curve does not intersect with the S_1 curve, $S_1(R_1')$ always greater than $D_1(R_1')$. The maximum amount of fuel then can be transferred at the point where $D_1(R_1')$ is maximized, i.e. $R_1'' = C_{m_2}/q_{m_2}$ (See Figure 3 (case 2)). Based on these two situations we have

$$R_1'' = \min \left(\frac{C_{m_2}}{q_{m_2}} , \frac{C_{m_1}}{2q_{m_1} + q_{m_2}} \right).$$

From the above analysis, we know that at any point $R_1' \le R_1''$, vehicle m_2's tank can always be filled up, and at any point $R_1' > R_1''$ either it transfers less fuel than that at $R_1' = R_1''$, or it is not feasible.

The next step is to analyze the fuel supply and demand curves in the second operation. The fuel supply in the second operation depends not only on the fuel capacity and the fuel consumption rate of vehicle m_1 and m_2, but also on the first fuel transfer operation. However, there is a maximum supply curve of the combination of m_1 and m_2, with which at any point of the second operation R_2' the supply can be maximized by adjusting the first operation corresponding to the second operation. Under this consideration, the maximum supply curve of the second operation will no longer depend on a prespecified first operation but can be always realized by the selection of the first operation.

For the second operation we have the following supply and demand function:

$$S_2(R_2') = \begin{cases} C_{m_2} - q_{m_2} R_2' & R_2' \le R_1'' \\ C_{m_2} + q_{m_2} R_1'' - 2q_{m_2} R_2' & R_1'' < R_2' \le (C_{m_2} + q_{m_2} R_1'')/2q_{m_2} \\ 0 & R_2' > (C_{m_2} + q_{m_2} R_1'')/2q_{m_2} \end{cases}$$

$$D_2(R_2') = \begin{cases} q_{m_3} R_2' & R_2' \le C_{m_3}/q_{m_3} \\ 0 & R_2' > C_{m_3}/q_{m_3} \end{cases}$$

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Figure 4 here

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Figure 4 represents the maximum supply curve S_2 and the demand curve D_2 for the second operation. It is observed that S_2 is a linear piecewise function with two segments. To the left of the turning point R_1'', S_1 is greater than D_1 and thus the tank of the second vehicle m_2 can always be filled up in the first operation. Therefore the maximum fuel that can be supplied at any point $R_2' \le R_1''$ will be $C_{m_2} - R_2' q_{m_2}$, with the tank of V_{m_2} filled up to C_{m_2} by the first operation at $R_1' = R_2'$ minus $R_2' q_{m_2}$ fuel reserved for vehicle m_2 to go back to the origin. At any point $R_2' > R_1''$, the maximum fuel that can be supplied is $C_{m_2} + q_{m_2} R_1'' - 2q_{m_2} R_2'$ where C_{m_2} is the fuel the vehicle m_2 have from the origin, $q_{m_2}R_1''$ is the maximum fuel received from vehicle m_1 in the first operation at the point $R_1' = R_1''$ and $2q_{m_2}R_2'$ is the amount of fuel used for vehicle m_2 to come to R_2' and reserved to go back to the origin. The supply S_2 will decrease when R_2' increases. At the point

$$R_2' = Z(m_1, m_2) = \frac{C_{m_2} + q_{m_2} R_1''}{2q_{m_2}}$$

S_2 becomes zero. Actually, this point is the solution of the P2 for n = 2, with the first fuel transferred at $R_1' = R_1''$ (see Lemma 1).

Trivially, D_2 is a linear function which is determined by the parameters of the third vehicle. Following the same principle of determining R_1'', we calculate R_2'', the point at which vehicle m_3 can receive the maximum amount of fuel from vehicle m_2. Actually, the calculation of R_2'' depends on the following three cases (see Figure 5):

The first case is where D_2 does not intersect with S_2, i.e. D_2 is always less than S_2. Under this case $R_2'' = \dfrac{C_{m_3}}{q_{m_3}}$ which is the largest distance that vehicle m_3 can travel.

The second case is where D_2 intersects S_2 at R_2'', which is on the left side of R_1''. In this case $R_2'' = \dfrac{C_{m_2}}{q_{m_2} + q_{m_3}} < R_1''$. To meet the demand of vehicle m_3, vehicle m_2 should first fill up its tank by transferring fuel from vehicle m_1 at the same point $R_1' = R_2''$. Since $R_1' < R_1''$, Vehicle m_1 can supply more than that vehicle m_2 can accept. The result is that vehicle m_1 ends its mission at R_1' with surplus fuel wasted.

The third case is where D_2 intersects S_2 to the right of R_1''. In this case $R_2'' = \dfrac{C_{m_2} + q_{m_2}R_1''}{2q_{m_2} + q_{m_3}} > R_1''$. To meet the demand of vehicle m_3, the first operation should take place at $R_1' = R_1''$ in order to maximize the amount fuel transferred from vehicle m_1. It results with all fuel transferred in both operations to the maximum and thus the slack variables of P_2 for the first two operations are zero.

Summarize above three cases we have

$$R_2'' = \text{Min}\left(\frac{C_{m_3}}{q_{m_3}}, \frac{C_{m_2}}{q_{m_2} + q_{m_3}}, \frac{C_{m_2} + q_{m_2}R_1''}{2q_{m_2} + q_{m_3}}\right).$$

To prove the formula, we introduce three lines D_2', S_2' and S_2'' as the linear extension of the demand curve D_2 and two segments of the supply curve S_2 where $D_2' = q_{m_3}R_2'$, $S_2' = C_{m_2} - q_{m_2}R_2'$, and $S_2'' = C_{m_2} + q_{m_2}R_1'' - 2q_{m_2}R_2'$.

Denote $R_a = \dfrac{C_{m_3}}{q_{m_3}}$, $R_b = \dfrac{C_{m_2}}{q_{m_2} + q_{m_3}}$, and $R_c = \dfrac{C_{m_2} + q_{m_2}R_1''}{2q_{m_2} + q_{m_3}}$. Obviously, R_a is the ending point of D_2, R_b is the intersection point of D_2' and S_2', and R_c is the intersection point of D_2' and S_2'' (See Figure 5).

Figure 5 here

We will prove that $R_2'' = \text{Min}\ (R_a, R_b, R_c)$, i.e. (36) hold.

1) If $R_a < R_b$ and $R_a < R_c$, D_2 will not intersect with S_2 so $R_2'' = R_a$ as the first case we have shown before.

2) If $R_b \le R_a$ and $R_b < R_c$, D_2 must intersect with S_2 on the left side of R_1'' , i.e. $R_b < R_1'$. Because $R_b < R_c$ means

$$R_b = \frac{C_{m_2}}{q_{m_2} + q_{m_3}} < \frac{C_{m_2} + q_{m_2} R_1''}{2q_{m_2} + q_{m_3}} = R_c ,$$

we have

$$C_{m_2} (2q_{m_2} + q_{m_3}) < (C_{m_2} + q_{m_2} R_1'') (q_{m_2} + q_{m_3})$$

and thus

$$q_{m_2} C_{m_2} < q_{m_2} R_1'' (q_{m_2} + q_{m_3})$$

and

$$R_b = \frac{C_{m_2}}{q_{m_2} + q_{m_3}} < R_1'' .$$

So this is the above mentioned second case and $R_2'' = R_b$.

3) If $R_c \le R_a$ and $R_c \le R_b$, D_2 must intersect with S_2 on the right side of R_1'' , i.e. $R_c \ge R_1'$. Because $R_c \le R_b$ means

$$R_c = \frac{C_{m_2} + q_{m_2} R''_1}{2\, q_{m_2} + q_{m_3}} \leq \frac{C_{m_2}}{q_{m_2} + q_{m_3}} = R_b ,$$

we have

$$(C_{m_2} + q_{m_2} R''_1)(q_{m_2} + q_{m_3}) \leq C_{m_2} (2q_{m_2} + q_{m_3})$$

and so

$$(q_{m_2} + q_{m_3}) R''_1 \leq C_{m_2} .$$

Adding $q_{m_2} R''_1$ on both sides of the above inequality we obtain

$$(2q_{m_2} + q_{m_3}) R''_1 \leq C_{m_2} + q_{m_2} R''_1 ;$$

thus

$$R''_1 \leq \frac{C_{m_2} + q_{m_2} R''_1}{2q_{m_2} + q_{m_3}} = R_c .$$

So this is the above mentioned third case and $R''_2 = R_c$.

Now, we have shown that $R''_2 = \text{Min}\,(R_a, R_b, R_c)$, i.e. (36) hold.

Q.E.D.

For more general case we have the following lemma:

Lemma 3.

The amount of fuel transferred in the kth operation is maximized at R''_k where

$$R''_k = \text{Min}\,\left(\frac{C_{m_{k+1}}}{q_{m_{k+1}}} , \frac{C_{m_k}}{q_{m_k} + q_{m_{k+1}}} , \frac{C_{m_k} + q_{m_k} R''_{k-1}}{2q_{m_k} + q_{m_{k+1}}}\right), \quad k = 1, \ldots, n-1 \qquad (37)$$

Proof.

In order to prove (37), we show that the aggregated supply curve S_k and demand curve D_k can be represented by

$$S_k(R_k') = \begin{cases} C_{m_k} - q_{m_k} R_k' & R_k' \le R_k'' \\ C_{m_k} + q_{m_k} R_{k-1}'' - 2q_{m_k} R_k' & R_{k-1}'' < R_k' \le (C_{m_k} + q_{m_k} R_{k-1}'')/2q_{m_k} \\ 0 & R_k' > (C_{m_k} + q_{m_k} R_{k-1}'')/2q_{m_k} \end{cases} \quad (38)$$

$$D_k(R_k') = \begin{cases} q_{m_{k+1}} R_k' & R_k' \le C_{m_{k+1}}/q_{m_{k+1}} \\ 0 & R_k' > C_{m_{k+1}}/q_{m_{k+1}} \end{cases}$$

From Lemma 2 we have shown that (37) and (38) hold for k = 2. We can show that (38) holds for k = 3:

$$S_3(R_3') = \begin{cases} C_{m_3} - q_{m_3} R_3' & R_3' \le R_2'' \\ C_{m_3} + q_{m_3} R_2'' - 2q_{m_3} R_3' & R_2'' < R_3' \le (C_{m_3} + q_{m_3} R_2'')/2q_{m_3} \\ 0 & R_3' > (C_{m_3} + q_{m_3} R_2'')/2q_{m_3} \end{cases}$$

$$D_3(R_3') = \begin{cases} q_{m_4} R_3' & R_3' \le C_{m_4}/q_{m_4} \\ 0 & R_3' > C_{m_4}/q_{m_4} \end{cases}$$

Since for any $R_3' < R_2''$, the tank of vehicle m_3 can always be filled up at $R_2' = R_3'$ because the supply S_2 is greater than demand D_2. Therefore vehicle m_3 can supply fuel equal to the capacity C_{m_3} less the amount reserved for going back to the origin, i.e. $C_{m_3} - q_{m_3}R'_3$. For $R'_3 \geq R''_2$ vehicle m_3 should get the maximum fuel at $R'_2 = R''_2$. The fuel available for transfer will be the total fuel received at the origin and the second operation less the fuel used for the round trip up to R'_3 , i.e. $C_{m_3} + q_{m_3}R''_2 - 2q_{m_3}R'_3$. The supply S_3 becomes zero at the point $R'_3 = (C_{m_3}+q_{m_3}R''_2)/2q_{m_3}$, which is the maximum operational range reached by the vehicle m_3, i.e. $Z(m_1,m_2,m_3)$. Because S_3 and D_3 have the same shape as S_2 and D_2, using the same argument in Lemma 2 we can show (37) holds for k = 3, i.e.

$$R''_3 = \text{Min}\left(\frac{C_{m_4}}{q_{m_4}}, \frac{C_{m_3}}{q_{m_3} + q_{m_4}}, \frac{C_{m_3} + q_{m_3}R''_2}{2q_{m_3} + q_{m_4}}\right).$$

Since all C_{m_k}, q_{m_k} are given and R''_k only depends on R''_{k-1}, the above argument can be easily applied to show that (37) and (38) hold for $k = 4, \ldots, n-1$.

Q.E.D.

Lemma 3 provides a computation procedure for R''_k by which the aggregated supply curve of the (n-1)th operation is derived. To show that deriving this curve by lemma 3 will lead into the optimization of P2 , the following lemma is in order.

Lemma 4

P2 will be maximized if R''_{n-1} follows (37) and this will imply that

$$\text{Max } Z(m) = \frac{1}{2} \left(d_{m_n} + R''_{n-1} \right), \tag{39}$$

where $d_{m_n} = C_{m_n} / q_{m_n}$.

Proof.

Following the argument employed in proving Lemma 2 and Lemma 3, it can be shown that S_{n-1} is maximized under the recursive computation of (37). Obviously Z(m) will be maximized if the last vehicle receives the maximum amount of fuel in the (n-1)th operation. This will occur at R''_{n-1} and the amount of fuel received will be $q_{m_n} R''_{n-1}$. In addition to the full capacity of the fuel C_{m_n} received from the origin, the total amount of fuel that can be consumed by the last vehicle is $C_{m_n} + q_{m_n} R''_{n-1}$. The maximum travel distance of the vehicle therefore is

$$\text{Max } Z(m) = \frac{C_{m_n} + q_{m_n} R''_{n-1}}{2q_{m_n}} = \frac{1}{2} \left(d_{m_n} + R''_{n-1} \right),$$

which satisfies the constraint of going back to the origin.

Q.E.D.

It is easily observed through (39) that to calculate the Max Z(m), Lemma 4 does not use a recursive formula based on R'_k , the kth transfer point. Rather, it uses a forward recursive formula based on R''_k, the point at which the maximum fuel can be transferred in the kth operation. It should be mentioned that R'_k is not necessarily equal to R''_k for $k < n-1$ in an optimal solution. In Theorem 1 we derive the solution of R'_k corresponding to the Max Z(m). We employ a backward recursive formula which is shown to generate a feasible solution for P2 by satisfying its constraints. More specifically an induction argument is utilized to compute R'_k and U_k as a function of R''_k. Sufficient conditions are derived to guarantee the optimality. The optimal solution however, is not necessarily unique.

<u>Theorem 1</u>

In order to guarantee that vehicle m_n reaches its maximum distance

Max $Z(m) = \frac{1}{2}\left(\dot{d}_{m_n} + R''_{n-1}\right)$, the fuel transfer operation should be arranged as follows:

1) The kth operation takes place at R'_k where

$$R'_{n-1} = R''_{n-1} \tag{40}$$

$$R'_k = \text{Min}\,(R''_k\ ,\ R'_{k+1}), \qquad k = n-2,\ldots,1 \tag{41}$$

2) The amount of fuel transferred in kth operation is U_k where

$$U_k = q_{m_{k+1}} R'_k\ , \qquad k = n-1,\ldots,1 \tag{42}$$

<u>Proof</u>.

To reach the maximum distance, vehicle m_n should receive the maximum amount of fuel from vehicle m_{n-1} in the (n-1)th operation. Based on Lemma 3, the maximum amount of fuel will be transferred at the point $R'_{n-1} = R''_{n-1}$ and the amount of fuel transferred is $U_{n-1} = q_{m_n} R'_{n-1}$. To guarantee the (n-1)th operation, it requires from (30) that

$$C_{m_{n-1}} + U_{n-2} - q_{m_n} R'_{n-1} - 2q_{m_{n-1}} R'_{n-1} \geq 0,$$

or equivalently,

$$U_{n-2} \geq (q_{m_n} + 2q_{m_{n-1}}) R'_{n-1} - C_{m_{n-1}} \tag{43}$$

where U_{n-2} is the fuel received from the (n-2)th operation. Denote

$$W_{n-2} = (q_{m_n} + 2q_{m_{n-1}}) R'_{n-1} - C_{m_{n-1}} \tag{44}$$

the minimum required fuel transferred in the (n-2)th operation. We will show that

$$W_{n-2} \leq \text{Min}\,(R''_{n-2}, R'_{n-1})\, q_{m_{n-1}}. \tag{45}$$

Notice that (40) and (37) imply that

$$R'_{n-1} \leq R''_{n-1} \leq \frac{C_{m_{n-1}} + q_{m_{n-1}} R''_{n-2}}{2q_{m_{n-1}} + q_{m_n}}, \tag{46}$$

$$R'_{n-1} \leq R''_{n-1} \leq \frac{C_{m_{n-1}}}{q_{m_{n-1}} + q_{m_n}}, \tag{47}$$

$$R'_{n-1} \leq R''_{n-1} \leq \frac{C_{m_n}}{q_{m_n}}. \tag{48}$$

From (44), (46) we have

$$W_{n-2} \le (q_{m_n} + 2q_{m_{n-1}}) \frac{C_{m_{n-1}} + q_{m_{n-1}} R'_{n-2}}{q_{m_n} + 2q_{m_{n-1}}} - C_{m_{n-1}} = q_{m_{n-1}} R''_{n-2} .$$

From (47) and (44) we have

$$C_{m_{n-1}} \ge (q_{m_n} + q_{m_{n-1}}) R'_{n-1}$$

and

$$W_{n-2} \le (q_{m_n} + 2q_{m_{n-1}}) R'_{n-1} - (q_{m_n} + q_{m_{n-1}}) R'_{n-1} \le q_{m_{n-1}} R'_{n-1} .$$

Thus, $W_{n-2} \le \mathrm{Min}\ (R''_{n-2}, R'_{n-1})\ q_{m_{n-1}}$.

We can arrange the (n-2)th operation at the point $R'_{n-2} = \mathrm{Min}\ (R''_{n-2}, R'_{n-1})$ to transfer $U_{n-2} = q_{m_{n-1}} R'_{n-2}$ of fuel. Under this arrangement $U_{n-2} \ge W_{n-2}$. Thus, (30) is satisfied . It is easy to verify that (31)-(34) all hold.

In general, for any $k \ge 1$, to guarantee the kth operation at $R'_k = \mathrm{Min}(R''_k, R'_{k+1})$ to transfer fuel $U_k = q_{m_{k+1}} R'_k$, the constraint (30) requires that in the (k+1)th operation, vehicle m_k receives U_{k-1} , where

$$U_{k-1} \ge W_{k-1} = (q_{m_{k+1}} + 2q_{m_k}) R'_k - C_{m_k} .$$

Since

$$R'_k \le R''_k = \mathrm{Min}\ \left(\frac{C_{m_{k+1}}}{q_{m_{k+1}}} , \frac{C_{m_k}}{q_{m_k} + q_{m_{k+1}}} , \frac{C_{m_k} + q_{m_k} R''_{k-1}}{2q_{m_k} + q_{m_{k+1}}} \right),$$

by the same argument we have $W_{k-1} \le \mathrm{Min}\ (R''_{k-1} , R'_k) q_{m_k}$.

The (k-1)th fuel transfer then can take place at $R'_{k-1} = \text{Min}\ (R''_{k-1}, R'_k)$ and $U_{k-1} = q_{m_k} R'_{k-1}$ of fuel is transferred. Under this arrangement, (30)-(34) are all satisfied by the same argument provided above.

Finally, since $R'_1 \leq R'_2 \leq ,\ldots \leq R'_{n-1}$ and $R_1 = R'_1$, $R_k = R'_k - R'_{k-1}$, $k = 2,\ldots,n-1$, we have $R_k \geq 0$, $k = 1,\ldots, n$, i.e. the constraint (35) is satisfied.

Q.E.D.

Corollary 1.

The solution given by backward recursive formulas (40)-(42) is feasible for P2.

Proof.

The proof is given by following the steps of Theorem 1 to verify that (30)-(35) are all satisfied with the solution provided by (40)-(42).

Q.E.D.

Finally, it is clear from the recursive formula of calculating R''_k and R'_k that P2 can be solved by O(n) operations. The forward recursive calculation of R''_k by Lemma 3 is a linear function of n. The backward recursive calculation of R'_k and U_k by Theorem 1 is also a linear function of n.

5. A pairwise interchange analysis

After solving the P2 for any given refueling chain, the next step is to find what is the optimal refueling chain. One way to approach the optimality is to improve the objective function by pairwise interchange the vehicles in a chain.

Lemma 6 below provides a rule of pairwise interchange adjacent vehicles for improving the performance. The idea behind the rule is based on the recursive formula of $R''_k(m)$, where m is introduced to indicate the corresponding refueling chain m.

Lemma 6

Given two refueling chains $m' = \{m_1,..,m_{k-1}\ m_k,m_{k+1},m_{k+2},..,m_n\}$ and $m'' = \{m_1,..,m_{k-1},m_{k+1},m_k,m_{k+1},m_{k+2},..,m_n\}$ where $1 \le k \le n-1$ is fixed, the following rule holds:

If $R''_{k+1}(m') \ge R''_{k+1}(m'')$ then $Z(m') \ge Z(m'')$.

Proof.

The proof can be derived by using the recursive relations of (37) and the formula of Z(m) given by (39). Since there exists a k for which $R''_{k+1}(m') \ge R''_{k+1}(m'')$ then this relation holds for k+1,...,n-1 by (37) and thus by (39) $Z(m') \ge Z(m'')$.

Q.E.D.

Starting from an initial chain, we can improve the objective function Z(m) by repeatedly applying the pairwise interchange rule to the chain. However this pairwise interchange rule does not guarantee a global optimal solution for P1. In table 3 and 4 we provide an example which supports our conclusion.

Table 3 here

Table 4 here

In Table 4 a local optimal solution was generated by a procedure which employs the idea of Lemma 6. It can be shown that any further improvement of this solution is impossible with respect to the pairwise interchange rule. Some numerical experience that are not reported here demonstrated the fact that the initial solution may have an impact on the number of iterations required to solve the problem heuristically by the pairwise interchange rule or in an optimal way by employing an enumerative search procedure based on the principle of the optimality of dynamic programming.

Computational experience demonstrated that satisfactory results are provided for sample problems by using the following heuristic approach: First, an initial chain is selected by arranging vehicles according to the ascending order of d_{m_k}, i.e select a chain m that satisfies $d_{m_1} \leq d_{m_2} \leq , \ldots , \leq d_{m_n}$. Then, the interchange rule is repeatedly applied to improve the result as long as it is possible. Finally, enumerative search is used to find the optimal solution, which is used to measure the effectiveness of the initial rule and interchange rule.

Table 5 here

A simulation run of 100 observations for $n = 6$ is reported in Table 5. The d_i and the q_i were sampled from statistically independent uniform distribution with d_i uniformly distributed in [1, 100], q_i uniformly distributed in [1 ,100], and $C_i = d_i q_i$. The results indicates that 29% of the initial solutions are optimal and 72% of the interchange solutions are optimal. When the performance is measured by the ratio of the Z(m) of the heuristic solution to the Z(m) of the optimal solution, the average ratio for the initial solution is 0.949 and the average ratio for the interchange solutions is 0.978.

In fact, the initial rule of ordering vehicles by d_{m_k} does generate the optimal solution for the three types of fleet configuration (1)-(3) which are studied by Mehrez et al. The optimal solution can be verified via an alternative algebraic proof by using Lemma 3, Lemma 4, and Lemma 6. The following theorem represents such an approach which is based on analyzing the supply and demand curves.

Theorem 2.

If the vehicles in a fleet have the same fuel consumption rate q:

$$q_1 = q_2 =, \ldots, = q_n = q, \tag{49}$$

or have the same fuel capacity C:

$$C_1 = C_2 = , \ldots = C_n = C , \tag{50}$$

then the refueling chain $m = (m_1, m_2, \ldots, m_n)$ is optimal if

$$d_{m_1} \leq d_{m_2} \leq , \ldots, \leq d_{m_n} , \tag{51}$$

and the maximum operational range is

$$\text{Max } Z(m) = \frac{1}{2}\sum_{k=1}^{n} \frac{d_{m_K}}{3^{n-k}} \tag{52}$$

Proof.

Note first that the fleet configuration of identical vehicles is the special case of the above mentioned cases and thus it is not considered.

Assume that the vehicles in the fleet have the same fuel consumption rate q. The proof consists two steps.

The first step is to prove that a chain $m = \{m_1, m_2, \ldots, m_n\}$ which satisfies (51) is optimal. It can be done by showing that for any two refueling chains $m' = \{m_1, \ldots, m_{k-1}\ m_k, m_{k+1}, m_{k+2}, \ldots, m_n\}$ and $m'' = \{m_1, \ldots, m_{k-1}, m_{k+1}, m_k, m_{k+2}, \ldots, m_n\}$, if $d_{m_{k+1}} \geq d_{m_k}$ then $Z(m') \geq Z(m'')$. To show that $Z(m') \geq Z(m'')$, it is sufficient to verify by Lemma 6 that

$$R''_{k+1}(m') \geq R''_{k+1}(m''). \tag{53}$$

From (37) and (49), R''_k is reduced into

$$R''_k = \text{Min}\ (\ d_{m_{k+1}},\ d_{m_k}/2,\ (d_{m_k} + R''_{k-1})/3\ )$$

To observe (53) we write $R_{k+1}(m')$ and $R_{k+1}(m'')$ as follows:

$$R''_{k+1}(m') = \text{Min}\ (\ d_{m_{k+2}},\ d_{m_{k+1}}/2,\ (d_{m_{k+1}} + R''_k(m')\ )/3\ ) \tag{54}$$

$$R''_{k+1}(m'') = \text{Min}\ (\ d_{m_{k+2}},\ d_{m_k}/2\ ,\ (d_{m_k} + R''_k(m'')\ )/3\ ) \tag{55}$$

Similarly,

$$R''_k\ (m') = \text{Min}\ (\ d_{m_{k+1}},\ d_{m_k}/2\ ,\ (d_{m_k} + R''_{k-1}(m')\ )/3\ ) \tag{56}$$

$$R''_k\ (m'') = \text{Min}\ (\ d_{m_k}\ ,\ d_{m_{k+1}}/2,\ (d_{m_{k+1}} + R''_{k-1}(m'')\ )/3\ ) \tag{57}$$

Substituting $R''_k(m')$ and $R''_k(m'')$ in (54) and (55) with (56) and (57) respectively, we have

$$R''_{k+1}(m') = \text{Min} \left(d_{m_{k+2}},\ d_{m_{k+1}}/2,\ 2d_{m_{k+1}}/3,\ (2d_{m_{k+1}} + d_{m_k})/6,\ (3d_{m_{k+1}} + d_{m_k} + R''_{k-1}(m'))/9 \right) \tag{58}$$

and

$$R''_{k+1}(m'') = \text{Min} \left(d_{m_{k+2}},\ d_{m_k}/2,\ 2d_{m_k}/3,\ (2d_{m_k} + d_{m_{k+1}})/6,\ (3d_{m_k} + d_{m_{k+1}} + R''_{k-1}(m''))/9 \right) \tag{59}$$

Since $R''_{k-1}(m') = R''_{k-1}(m'')$ and $d_{m_{k+1}} \geq d_{m_k}$, it is easy to check that $R''_{k+1}(m') \geq R''_{k+1}(m'')$ by comparing each term of $R''_{k+1}(m')$ with its correspondent. Thus, the pairwise interchange rule can be applied as required for any unordered pair of vehicles to generate an optimal refueling chain with a non-decreasing order of $d_{m_1} \leq d_{m_2} \leq \ldots \leq d_{m_n}$.

The second step is to prove (52). First we show by an induction argument that for the optimal refueling chain, (37) is reduced into:

$$R''_0 = 0,\ R''_k = (d_{m_k} + R''_{k-1})/3 \qquad k = 1, \ldots n-1 \tag{60}$$

From Lemma 2, $R''_1 = \text{Min}(d_{m_2}, d_{m_1}/3)$. Since $d_{m_2} \geq d_{m_1}$ we have $R''_1 = d_{m_1}/3 \leq d_{m_1}/2$. In general, for a given k, assume that $R''_k = (d_{m_k} + R''_{k-1})/3 \leq d_{m_k}/2$. Since $d_{m_k} \leq d_{m_{k+1}} \leq d_{m_{k+2}}$, we have $(d_{m_{k+1}} + R''_k)/3 \leq d_{m_{k+1}}/2 \leq d_{m_{k+2}}$, so $R''_{k+1} = \text{Min}(d_{m_{k+2}}, d_{m_{k+1}}/2, (d_{m_{k+1}} + R''_k)/3) = (d_{m_{k+1}} + R''_k)/3 \leq d_{m_{k+1}}/2$, and thus (60) holds for all k.

Introducing (60) into (39) recursively results

$$Z(m) = \frac{1}{2}(d_{m_n} + R''_{n-1}) = \frac{1}{2}\left(d_{m_n} + \frac{d_{m_{n-1}} + R''_{n-2}}{3}\right) = \ldots = \frac{1}{2}\sum_{k=1}^{n} \frac{d_{m_k}}{3^{n-k}} .$$

We note that above steps can be applied for the case of identical fuel capacity and different fuel consumption rates. We omit the details of the proof.

Q.E.D.

6. Elimination of inferior chains

In this section a rule is studied to identify inferior chains which locate a technologically more advanced vehicle (with larger capacity and lower fuel consumption rate) before a technologically less advanced vehicle (with smaller capacity and higher fuel consumption rate) for the cases of $n = 2$ and $n = 3$. Although this result is quite intuitive, we were unable to show that it holds for the general case. For the special cases we prefer to investigate the problem for the following reasons: 1) logical arguments which are quite useful in analyzing scheduling decisions are not applicable to show the results. Instead a tedious procedure which depends on analyzing the supply demand equations and induction arguments is required to prove the result. 2) Quite often, such as in the group decision making process, consistency may not hold and thus a result which holds for two elements of a set may not hold for the general case. For our problem we will show that the order rule mentioned previously is preserved under the case of $n = 3$. The suggested analysis includes the following steps:

1) The case of $n = 2$ is investigated and it is shown from lemma 1 that the order rule hold for this case.

2) The pairwise interchange rule is employed to show that this rule hold for any $n \geq 2$ with a technologically more advanced vehicle adjacent to a less advanced vehicle.

3) The order rule hold for $n = 3$ when a technologically more advanced vehicle is not adjacent to a less advanced vehicle.

Proposition 1.

For $n = 2$, assume that $C_{m_1} \geq C_{m_2}$ and $q_{m_1} \leq q_{m_2}$ then

$Z(m_2, m_1) \geq Z(m_1, m_2)$.

Proof.

The result is implied directly by lemma 1.

Proposition 2.

Denote $m' = (m_1,..,m_{k-1}, m_k, m_{k+1}, m_{k+2},..,m_n)$, $m" = (m_1,...m_{k-1}, m_{k+1}, m_k\ m_{k+2},..,m_n)$ and assume that $C_{m_k} \leq C_{m_{k+1}}$ and $q_{m_k} \geq q_{m_{k+1}}$ then

$Z(m') \geq Z(m")$.

Proof.

The proof is implied directly by lemma 6 and verifing that $R''_{k+1}(m') \geq R''_{k+1}(m")$ holds.

Proposition 3.

For $n = 3$, assume that $C_1 \geq C_2$ and $q_1 \leq q_2$, then

$Z(2, 3, 1) \geq Z(1, 3, 2)$.

Proof.

Combining Lemma 3 and Lemma 4 results in

$Z(1, 3, 2)$

$$= \frac{1}{2} \text{Min} \left(\frac{C_2}{q_2} + R''_2\right)$$

$$= \frac{1}{2} \text{ Min } (\frac{C_2}{q_2} + \text{Min } (\frac{C_2}{q_2}, \frac{C_3}{q_3+q_2}, \frac{C_3+q_3R_1''}{2q_3+q_2}))$$

$$= \frac{1}{2} \text{ Min } (\frac{2C_2}{q_2}, (\frac{C_2}{q_2} + \frac{C_3}{q_3+q_2}),$$

$$(\frac{C_2}{q_2} + \frac{C_3}{2q_3+q_2} + \frac{q_3}{2q_3+q_2} \text{ Min } (\frac{C_3}{q_3}, \frac{C_1}{2q_1+q_3})))$$

$$= \frac{1}{2} \text{ Min } (\frac{2C_2}{q_2}, (\frac{C_2}{q_2} + \frac{C_3}{q_3+q_2}), (\frac{C_2}{q_2} + \frac{2C_3}{2q_3+q_2}),$$

$$(\frac{C_2}{q_2} + \frac{C_3}{2q_3+q_2} + \frac{q_3C_1}{(2q_3+q_2)(2q_1+q_3)})). \tag{61}$$

Similarly,

$Z(2, 3, 1)$

$$= \frac{1}{2} \text{ Min } (\frac{2C_1}{q_1}, (\frac{C_1}{q_1} + \frac{C_3}{q_3+q_1}), (\frac{C_1}{q_1} + \frac{2C_3}{2q_3+q_1}),$$

$$(\frac{C_1}{q_1} + \frac{C_3}{2q_3+q_1} + \frac{q_3C_2}{(2q_3+q_1)(2q_2+q_3)})). \tag{62}$$

We compare each item of (61) with its equivalent in (62). Clearly, by the assumption $C_1 \geq C_2$ and $q_1 \leq q_2$ we have

$$\frac{2C_1}{q_1} \geq \frac{2C_2}{q_2},$$

$$\frac{C_1}{q_1} + \frac{C_3}{q_3+q_1} \geq \frac{C_2}{q_2} + \frac{C_3}{q_3+q_2},$$

$$\frac{C_1}{q_1} + \frac{2C_3}{2q_3+q_1} \geq \frac{C_2}{q_2} + \frac{2C_3}{2q_3+q_2}, \quad \text{and}$$

$$\frac{C_3}{2q_3+q_1} \geq \frac{C_3}{2q_3+q_2} .$$

Thus it remains to show that

$$\frac{C_1}{q_1} + \frac{q_3 C_2}{(2q_3+q_1)(2q_2+q_3)} \geq \frac{C_2}{q_2} + \frac{q_3 C_1}{(2q_3+q_2)(2q_1+q_3)} \tag{63}$$

Since $\frac{1}{q_1} - \frac{q_3}{(2q_3+q_1)(2q_2+q_3)} \geq 0$ and $C_1 \geq C_2$,

$$\left(\frac{C_1}{q_1} + \frac{q_3 C_2}{(2q_3+q_1)(2q_2+q_3)}\right) - \left(\frac{C_2}{q_2} + \frac{q_3 C_1}{(2q_3+q_2)(2q_1+q_3)}\right)$$

$$= C_1 \left(\frac{1}{q_1} - \frac{q_3}{(2q_3+q_2)(2q_1+q_3)}\right) - C_2 \left(\frac{1}{q_2} - \frac{q_3}{(2q_3+q_1)(2q_2+q_3)}\right)$$

$$\geq C_2 \left(\left(\frac{1}{q_1} - \frac{1}{q_2}\right) + \left(\frac{q_3}{(2q_3+q_1)(2q_2+q_3)} - \frac{q_3}{(2q_3+q_2)(2q_1+q_3)}\right)\right)$$

it is sufficient to show that

$$\left(\frac{1}{q_1} - \frac{1}{q_2}\right) + \left(\frac{q_3}{(2q_3+q_1)(2q_2+q_3)} - \frac{q_3}{(2q_3+q_2)(2q_1+q_3)}\right)$$

$$= (q_2-q_1) \left(\frac{(2q_3+q_1)(2q_2+q_3)(2q_3+q_2)(2q_1+q_3) - 3q_1q_2q_3^2}{q_1q_2(2q_3+q_1)(2q_2+q_3)(2q_3+q_2)(2q_1+q_3)}\right)$$

$$\geq (q_2-q_1) . \frac{16q_1q_2q_3^2 - 3q_1q_2q_3^2}{q_1q_2(2q_3+q_1)(2q_2+q_3)(2q_3+q_2)(2q_1+q_3)} \geq 0 .$$

Q.E.D.

Corollary 2.

For n=3, a refueling chain in which a technologically more advanced vehicle is located before any relatively less advanced vehicle is not optimal.

Proof.

Since there are six possible refueling chains for n = 3 then it is easily checked by proposition 2 and proposition 3 that the Corollary holds.

7. Discussion

This paper accomplished the analysis of maximizing the operational range given a fixed refueling chain. As it often happens in the military scheduling problem, the identification of an optimal ordering rule is tedious even in the case of n = 2. The identification of this rule can be utilized, for example, to solve a problem where 2n vehicles are organized into n teams, each team containing two vehicles and the objective is to maximize the sum of the operational range of all teams. Given this problem, it can be easily shown that by $O(n^2)$ of calculations, the maximum operational range of all possible teams can be obtained. Then, the problem can be converted into a linear assignment problem. Actually, the recursive formulation of R''_k suggests that more complicated rules may be constructed for n sufficiently small. Also, the formulation of a problem which follows according to the lines of the previous problem with each team of three vehicles may result in solving a transportation problem.

Future research may be conducted to investigate the applicability of the recursive procedure derived here for the following problems:

1) An objective function which takes into account a linear weighted sum of both the operational range and the size of the vehicles traveled per

unit of the time. Such an objective function may capture situations in which the fleet travels under threat. The size of the vehicles is an important factor by which threat is reduced. The interesting research question deals with the amount of computations required to solve this problem for a given refueling chain. This problem can be further extended to taking into account the relative importance of the vehicles, while considering their positions in the chain.

2) Another aspect of the problem is to develop supply-demand curves under conditions of uncertainty or under alternative refueling strategies. Such type of extensions were considered by Mehrez et al.

3) An alternative line of research may investigate the value of supply and demand curves to study different scheduling problems. In such problems it may be assumed that the round trip constraint is relaxed. But an alternative objective function may be formulated to analyze strategies which capture the idea of transferring resources from one machine to another. These resources may be fuel, ammunition or any other input. Production rate and capacity of machines may be equivalent to fuel consumption rate and fuel capacity in our problem. To maximize the length of the production period might be equivalent to maximizing the operational range in our problem. The basic question under this settings is concerned with generating recursive procedures for the LP scheduling problem with the order of calculation which is less than the order of the simplex or other LP algorithms.

4) From the view of scheduling theory this paper demonstrates some ideas that might be applicable for other scheduling problems. The basic ideas employ the analysis of refueling supply and demand curves with the principle of pairwise interchange to eliminate inferior solutions from further consideration. Although the reductions are quite expected, we have

not been able to provide the result by logical arguments for the general case. Rather, we had to utilize a tedious algebraic approach, even for $n = 3$. Future research may extend the result and explore optimal refueling rules for $n \geq 3$ or improve heuristics to solve the problem. Clearly, methods such as AHP (Saaty [1980]) that incorporate pairwise analysis of $n = 2$ within the framework of the general case are possible approaches to consider.

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Figure 1

Fuel Supply and Demand for
the Optimal Solution (No. 24)

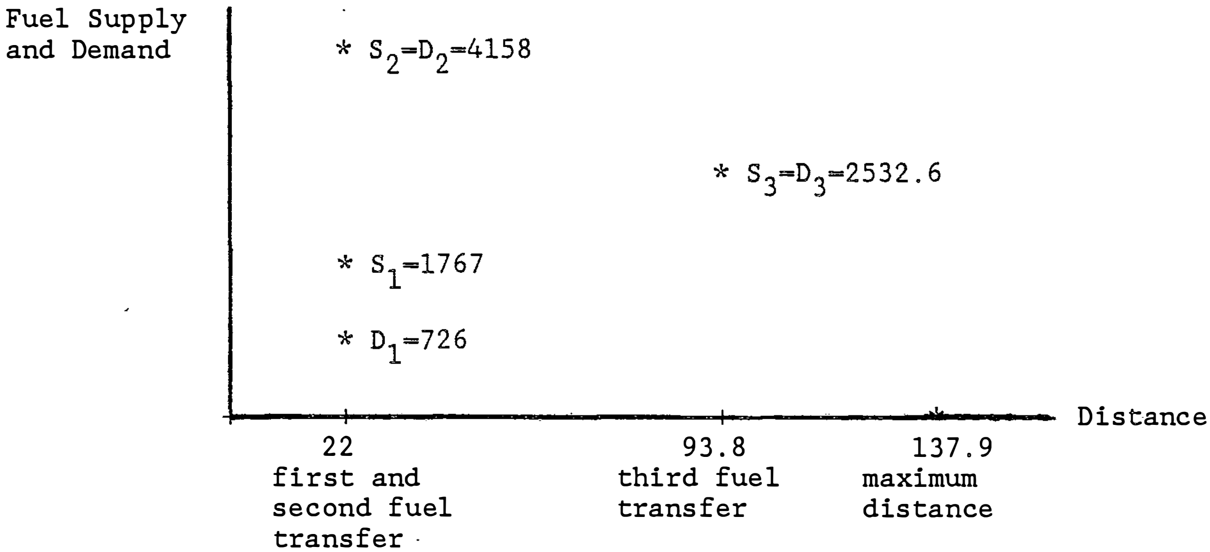


Figure 2

Fuel Supply and Demand for
the Equilibrium Solution (No. 12)

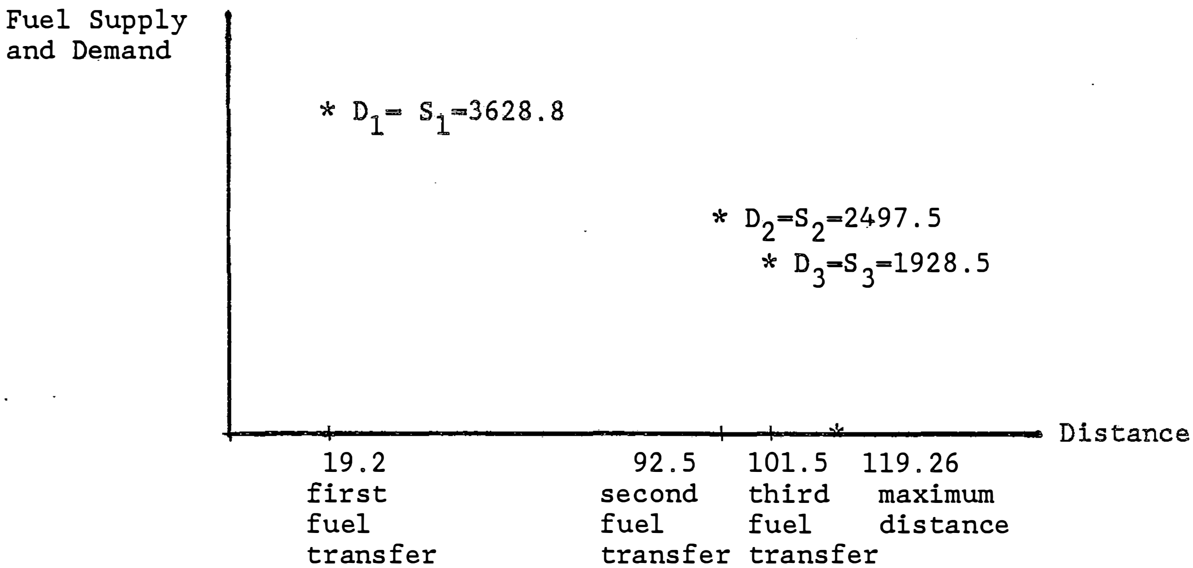
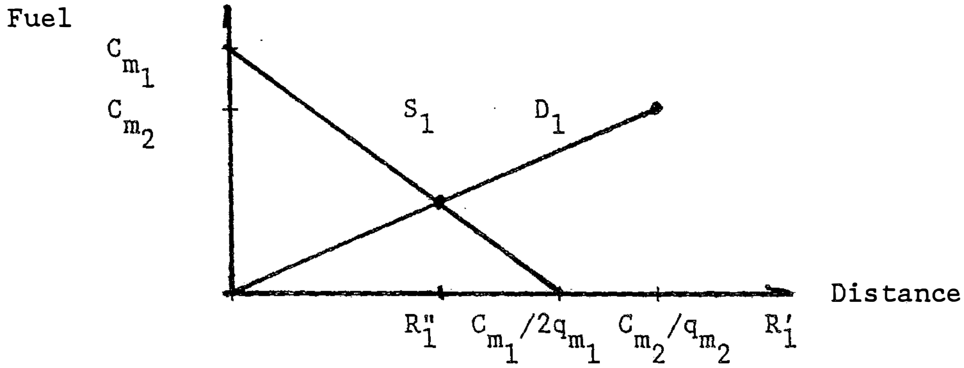


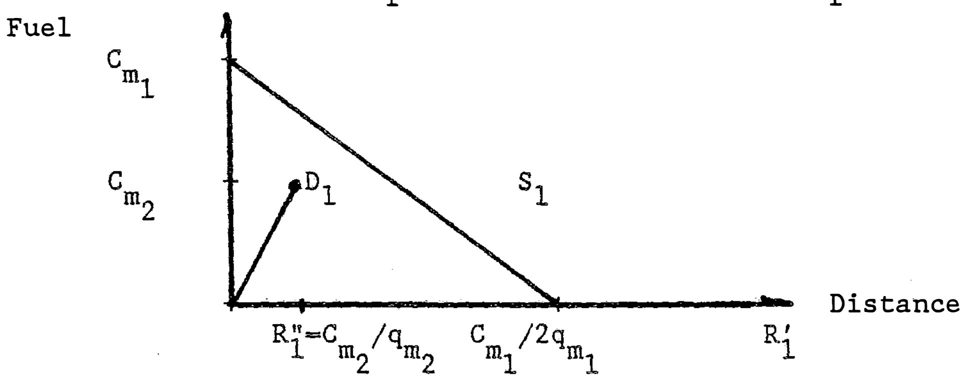
Figure 3

The Fuel Supply and Demand Curve
for the First Operation

Case 1: D_1 intersects with S_1



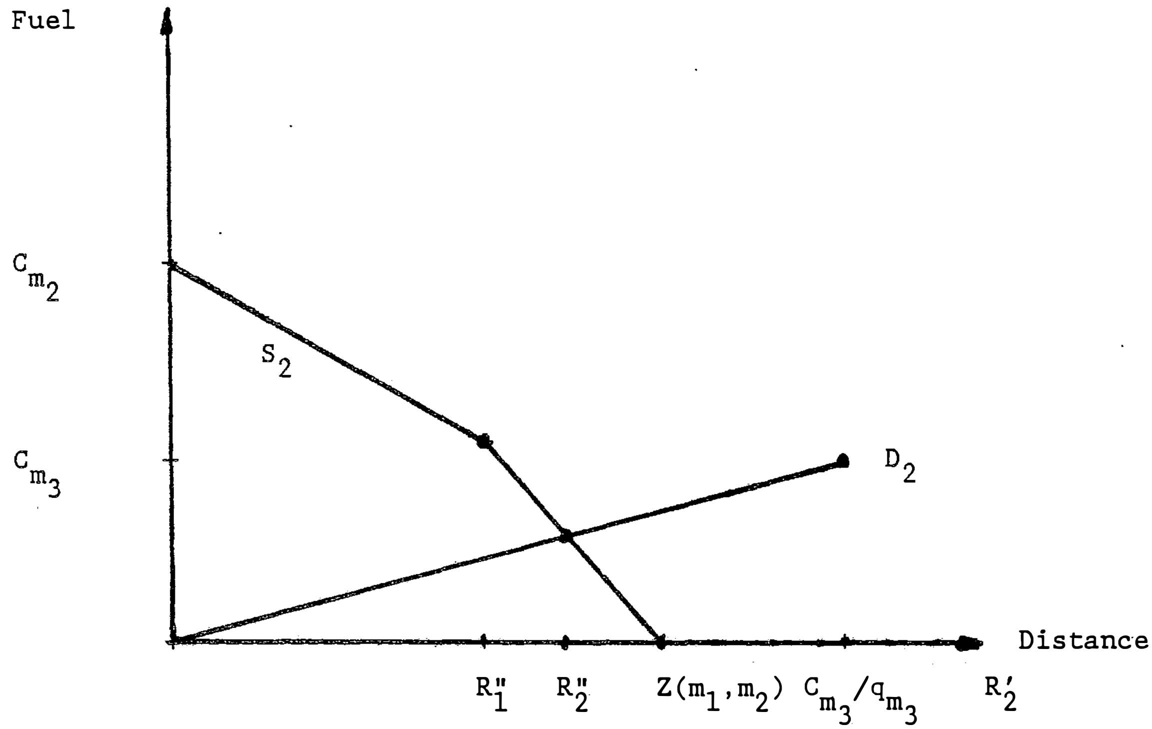
Case 2: D_1 does not intersect with S_1



S_1: fuel supply of vehicle m_1
D_1: fuel demand of vehicle m_2
R_1'': the point where maximum fuel is transferred.

Figure 4

The Supply and Demand Curve
for the Second Operation



S_2: The fuel supply curve in the second operation

D_2: The fuel demand curve in the second operation

R'_1: The maximum fuel transfer point in the first operation

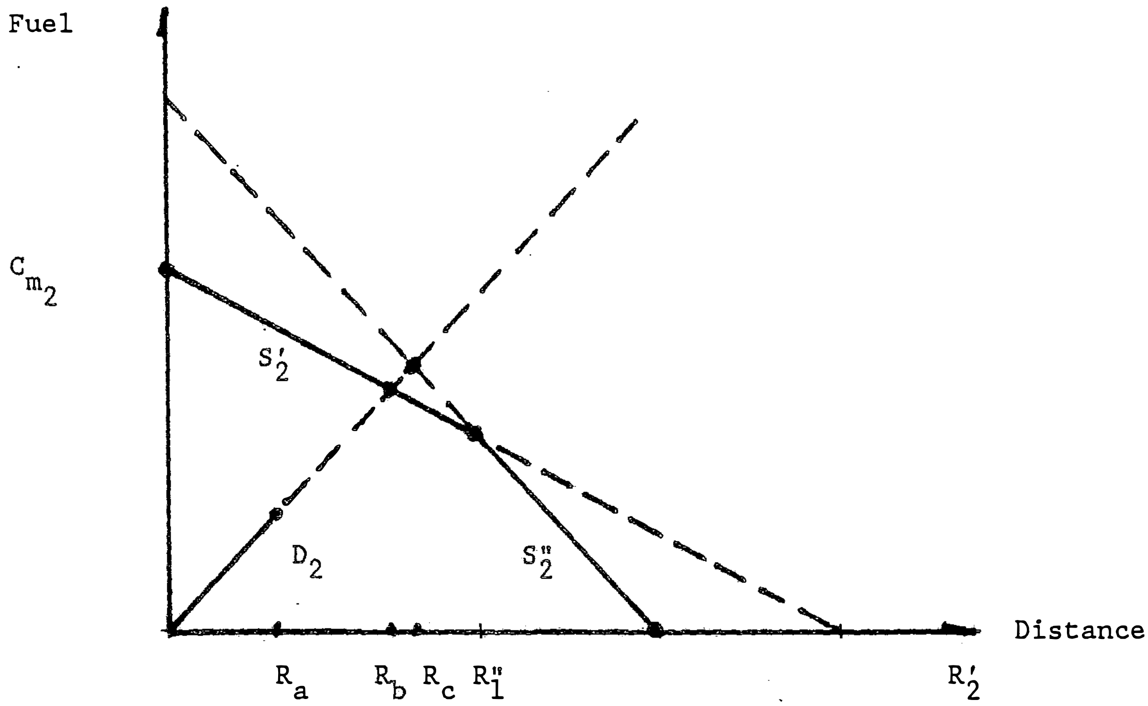
R''_2: The maximum fuel transfer point in the second operation

$Z(m_1,m_2)$: The maximum operation range reached by $\{m_1,m_2\}$ where $S_2 = 0$

Figure 5

The Maximum Fuel Transfer Point
for the Second Operation

Case 1: D_2 does not intersect with S_2



$R_a < R_b$ and $R_a < R_c$ => $R''_2 = R_a$

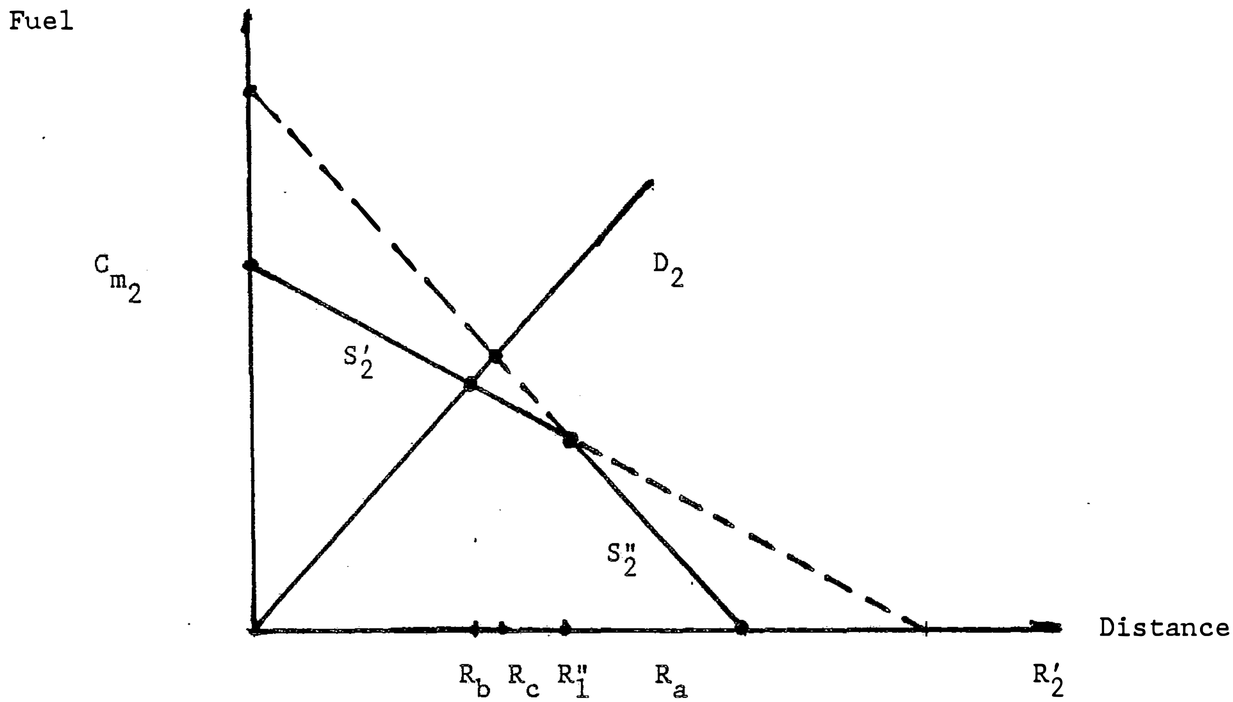
R_a: The ending point of D_2

R_b: The intersection point of D'_2 and S'_2

R_c: The intersection point of D'_2 and S''_2

R''_2: The point where the maximum amount fuel can be transferred

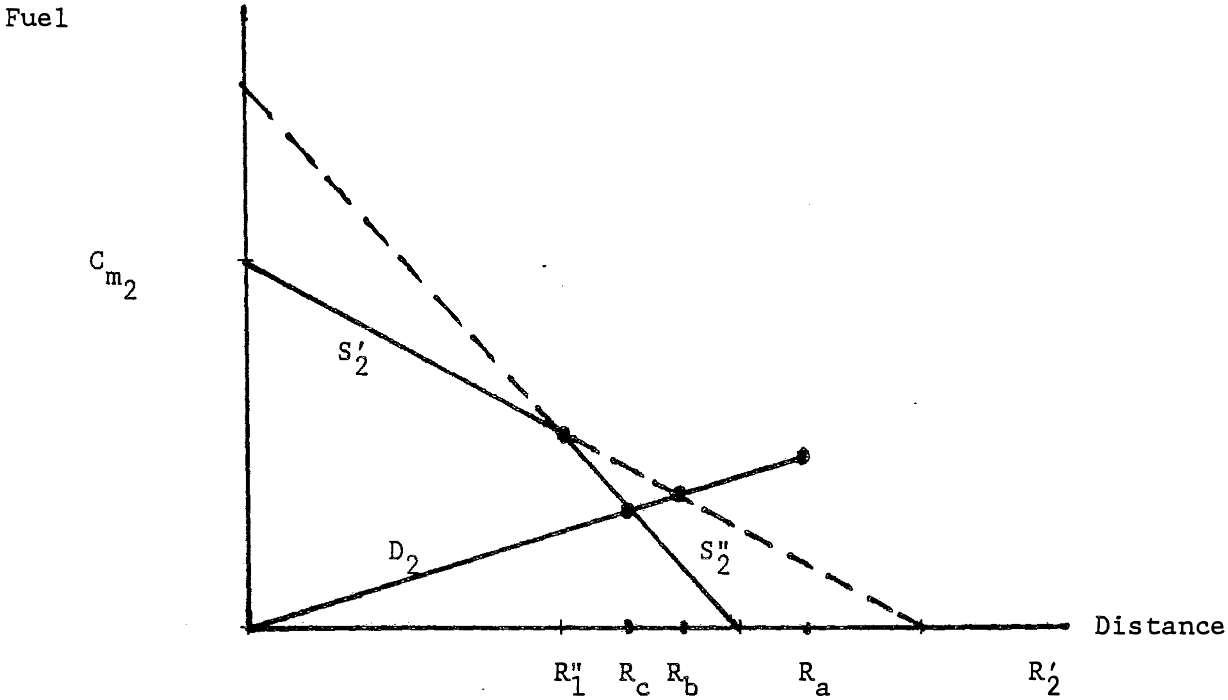
Case 2: D_2 intersects with S_2 on the left side of R_1''



$R_b \le R_c$ and $R_b \le R_a \Rightarrow R_2'' = R_b$

R_a:	The ending point of D_2
R_b:	The intersection point of D_2' and S_2'
R_c:	The intersection point of D_2' and S_2''
R_2'':	The point where the maximum amount fuel can be transferred

Case 3: D_2 intersects with S_2 on the right side of R''_1



$R_c < R_b$ and $R_c \leq R_a \Rightarrow R''_2 = R_c$

R_a: The ending point of D_2

R_b: The intersection point of D'_2 and S'_2

R_c: The intersection point of D'_2 and S''_2

R''_2: The point where the maximum amount fuel can be transferred

Table 1

A Counter Example Data for n = 4

Vehicle	i	1	2	3	4
Fuel Capacity	C_i	4914	2603	4884	33831
Fuel Consum. Rate	q_i	27	19	34	189
Maximum Distance	d_i	182	137	148	179

Table 2

Solutions for n=4

Solution No.	Vehicle Chain (m1,m2,m3,m4)	Objective function value Z(m)	Non-Zero slack variables
1	{3,1,2,4}	95.76	Yes
2	{1,3,2,4}	95.76	Yes
3	{1,4,2,3}	99.03	Yes
4	{4,1,2,3}	99.03	Yes
5	{1,2,3,4}	100.50	Yes
6	{2,1,3,4}	100.50	Yes
7	{3,2,1,4}	100.88	Yes
8	{2,3,1,4}	100.88	Yes
9	{4,2,1,3}	111.02	Yes
10	{4,1,3,2}	113.13	Yes
11	{2,4,1,3}	113.13	Yes
12	{1,4,3,2}	115.01	No
13	{1,3,4,2}	116.35	Yes
14	{3,1,4,2}	116.52	Yes
15	{4,3,1,2}	117.21	Yes
16	{1,2,4,3}	118.03	Yes
17	{3,4,1,2}	119.26	No
18	{3,4,2,1}	119.29	Yes
19	{4,3,2,1}	119.29	Yes
20	{2,1,4,3}	120.39	Yes
21	{4,2,3,1}	126.14	Yes
22	{2,4,3,1}	131.70	Yes
23	{3,2,4,1}	135.69	Yes
24	{2,3,4,1}	137.90	Yes

Table 3

The Data of An Example of Non-optimality for n = 6

i	1	2	3	4	5	6
C_i	9114	3572	18032	7216	12078	7488
q_i	49	19	92	41	99	52
$d_i/2$	93	94	98	88	61	72

i = the vehicle number

C_i = fuel capacity of vehicle i

q_i = fuel consumption rate of vehicle i

$d_i/2$ = the travel range of single vehicle i, which equals to $C_i/2q_i$

Table 4

Solutions for n = 6

Solution Type	Vehicle Chain $\{m_1, m_2, m_3, m_4, m_5, m_6\}$	Z(m)
Initial solution*	{5 , 6 , 4 , 1 , 2 , 3}	113.59
Local optimal solution**	{5 , 6 , 2 , 4 , 3 , 1}	141.71
Grobal optimal solution***	{6 , 4 , 5 , 3 , 1 , 2}	153.76

* The initial solution is based on the increasing order of d_{m_k}.

** The local optimal solution is obtained by all possible pairwise interchange improvement from the initial solution.

*** The global optimal solution is based on enumerating all 6! possible chains.

Table 5

Illustration of the Performance of Different Solution Methods

Solution Method	Percentage* to be optimal	Performance** by average
Ordered by d_i	29%	94.86%
Pairwise interchange method	72%	97.79%
Enumerate method	100%	100.00%

* Percentage of number of cases the method generates the optimal solution.

** Performance measured by the average ratio of the operational range of the solution vs the operational range of the optimal solution.

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