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Abstract

Lot streaming is the process of splitting a job or lot to allow overlapping between successive operations in a multistage production system. This use of transfer lots usually results in a shorter makespan for the corresponding schedule. In this paper, we study the structural properties of schedules which minimize the makespan for a single job with attached setup times in a flow shop. Although the structure of the optimal schedules is more complex than in the case with no setups [7], using the sructural insights obtained, it is possible to find the optimal solution with s sublots in $O(s)$ time for the three-machine case.

1 Introduction

Lot streaming is the process of using transfer batches to move the processed portion of a production lot to downstream machines so that the makespan of the schedule can be shortened and the work-in-process inventory levels can be lowered. The term was introduced by Reiter[17], but the idea has been considered many times under different names. The increased interest in

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its applications over the last few years is probably due to the fact that it is consistent with the Just-In-Time (JIT) philosophy of making small or single unit sublots and it also agrees with the basic idea of the OPT scheduling package [6], [8].

Szendrovits[18] analyzes the lot streaming problem in a flow shop for a single job with equal sublot sizes. Goyal [9] finds the optimal sublot sizes in Szendrovits' model. Moily[13], Jacobs and Bragg [11], Kulonda [12] and Graves and Kostreva [10] also demonstrate reductions in production time and cost by using transfer lots. Steiner and Truscott [19] find the optimal lot streaming schedules in an open shop with equal size transfer lots and no idling on the machines. Cetinkaya and Gupta [3] analyze the lot streaming problem for a single job in a flow shop with the total flow time criterion.

Most papers on lot streaming consider the objective of minimizing the makespan in an m-machine flow shop where each item is processed on the m machines in the order $1, \ldots, m$. Trietsch, in [20] and [21], and Baker [1] independently develop a conceptual framework for the problem. They present a classification scheme and review the most important results in [22]. Vickson [23] solves the lot streaming problem for multiple jobs in a two-machine flow shop with job setup times and sublot transfer times.

In this paper, we consider the problem of minimizing the makespan by splitting a single job into s sublots in an m-machine flow shop, where the job requires an attached setup on each machine. A setup is *attached* if setting up the machine requires a certain minimal number of items (usually one) to be available, otherwise it is called *detached*. We use the more frequently used assumption of *batch availability*, i.e., items become available for processing at the next machine after the current machine finished processing the last item in their sublot (batch). More formally, we have m machines, denoted by M_1, $M_2, ..., M_m$, the job has positive processing times $p_1, p_2, ..., p_m$ and attached setup times $S_1, S_2, ..., S_m$ on $M_1, M_2, ..., M_m$, respectively. If $x_{i,j}$ $(i = 1, \ldots, m, \ j = 1, \ldots, s)$ is the size of the jth sublot on M_i, then our objective is to find the $x_{i,j}$ values which minimize the makespan. We *assume* that the sublot sizes are normalized to represent the corresponding *proportion* of the job, i.e., $\sum_{j=1}^{s} x_{i,j} = 1$ for $i = 1, ..., m$. Thus, the processing time of sublot j on M_i is $p_i x_{i,j}$. The sublots are *consistent* if $x_{i,j} = x_{i+1,j}$ for $i = 1, \ldots, m-1, \ j = 1, \ldots, s$, otherwise they are *variable*. For consistent sublots we can write x_j instead of $x_{i,j}$.

Another, less frequently studied lot streaming model uses the assumption of *item availability* when individual items become available for processing at the next machine as soon as they are finished on the current machine (unit size transfer lots). Vickson and Alfredsson [24] solve the makespan minimization problem with no setups in the two machine flow shop. The same problem is solved with detached setups in [4] and with attached setups in [2].

Most analytical results using the model with batch availability apply to flow shops with no setups, with the exception of the two-machine case [23]. Baker [1] shows that linear programming can be used to find the consistent sublot sizes which minimize the makespan. As Glass et. al. [7] point out, however, the linear programming approach provides little insight into the structure of the solution which would enable more general models to be solved. Potts and Baker [15] show that for a single job, it is sufficient to consider identical sublot sizes on the first two machines, and on the last two machines. The $m = 2$ case is solved in [15] and in [20]. Glass et. al. [7] develop the solution to minimize the makespan for a single job in a three-stage production process without setup times. Their algorithms compute the minimum makespan in $O(log\ s)$ time for both the flow shop and job shop problem. They also present some structural results for $m > 3$. These results were extended to the case with detached setups in [5].

In this paper, we generalize the results in [7] for flow shops with attached job setups. The presence of setups makes the structure of the optimal solution substantially more complex in most cases. Nevertheless, it is possible to extend most of the results from [7] to our case. For the three-machine flow shop, this leads to an algorithm which finds the optimal schedule in $O(s)$ time.

The paper is organized as follows. Section 2 develops some fundamental structural results and introduces reduced versions of the problem, which are simpler but equivalent to the original one. Section 3 gives a detailed analysis of the three-machine network. Section 4 presents the solution for the problem on three machines. The computation time required is $O(s)$. A summary and conclusions are presented in Section 5.

2 Networks and Fundamental Results

In this section, we study first the relationships between optimal sublot sizes on different machines, which reveal that the sublot sizes could be made *consistent* on the first two machines and the last two machines. By further simplifying the problem, we show that it is equivalent to an alternative problem with no setups on M_1.

Let $C_{i,j}$ denote the completion time of sublot j on machine i ($i = 1, 2, .., m$, $j = 1, 2, ..., s$). The following constraints must be satisfied by any feasible solution.

1) Machine capacity constraints :

$C_{i,j} \geq C_{i,j-1} + p_i\, x_{i,j}$ $\quad (\; i = 1, 2, ..., m, \quad j = 2, ...s \;);$

2) Production constraints :

$C_{i,j} \geq C_{i-1,h(i,j)} + p_i\, x_{i,j}$ $\quad (\, i = 2, ..., m, \quad j = 1, 2, ..., s \;);$

where $h(i, j)$ is the last sublot on machine $i-1$ containing items included in sublot j on machine i;

3) Initialization constraint :

$C_{i,1} \geq \sum_{j=1}^{i} S_j + \sum_{j=1}^{i} p_j x_{j,1}$ $\quad (\, i = 1, 2, ..., m \,).$

4) $x_{j,1} \geq \Delta > 0$ $\quad (\, j = 1, 2, ..., m).$

Constraint (4) reflects the fact that we need at least Δ of the total quantity to be produced for performing the attached setup on each machine.

Theorem 1 *There exists an optimal schedule in which* $x_{1,j} = x_{2,j}$ *and* $x_{m,j} = x_{m-1,j}$ *for* $j = 1, 2, ..., s$.

Proof. Suppose there is an optimal schedule π, in which the sublot sizes on the first two machines, $x_{i,j}$ $i = 1, 2,$ $j = 1, 2, ..., s$ are not consistent . We can construct an alternative schedule π' which is no worse than schedule π with respect to the makespan and has consistent sublot sizes on the first two machines. Let $x'_{i,j}$ denote the size of sublot j on M_i in the schedule π'.

First, keep sublot sizes on $M_2, M_3, ..., M_m$ the same as in the schedule π, i.e., $x'_{i,j} = x_{i,j}$ $\; (\, i = 2, 3, ..., m \;).$

Second, let sublot k be the first sublot such that $x_{1,k} \neq x_{2,k}$, and we reset the sublot sizes of schedule π' on M_1 using the sublots from π:

$$x'_{1,j} = x_{1,j} \quad (\, j = 1, 2, ..., k-1),$$
$$x'_{1,j} = x_{2,j} \quad j = k, ..., s.$$

Let $t_{i,j}$ and $t'_{i,j}$ denote the starting times of the jth ($j = 1, 2, ..., s$) sublot on M_i ($i = 1, 2$) in the two schedules, respectively. We claim that $t_{2,j} \geq t'_{2,j}$ for $j = 1, ..., s$. For $j = 1, ..., k-1$, this is obvious. If $x_{1,k} > x_{2,k}$, then from $x'_{1,k} = x_{2,k}$, it follows that $C'_{1,k} < C_{1,k}$, implying $t'_{2,k} \leq t_{2,k}$. If $x_{1,k} < x_{2,k}$ then $h(2,k) > k$ and $C_{1,h(2,k)} \leq t_{2,k}$ by the production constraints, so enlarging $x_{1,k}$ to $x'_{1,k} = x_{2,k}$ will result in $C'_{1,k} \leq C_{1,h(2,k)} \leq t_{2,k}$. Thus the larger lot size $x'_{1,k}$ will not delay beyond $t_{2,k}$ the start of the kth sublot on M_2, i.e., $t'_{2,k} \leq t_{2,k}$. A similar argument can be used to show inductively that $t_{2,j} \geq t'_{2,j}$ for every $j > k$.

So no sublot starts later in π' than in π on M_2. Since $x_{2,j} = x'_{2,j}$ for $j = 1, .., s$, no sublot finishes later in π' than in π on M_2. So π' is at least as good a schedule as π.

To prove that there is an optimal schedule π'' with consistent sublots on M_{m-1} and M_m, we can make the sublots on M_m the same as on M_{m-1}, i.e., define $x''_{m,j} = x_{m-1,j}$ for $j = 1, ..., s$ and a similar argument to the one above proves that π'' is also optimal. □

Corollary 2 *For the two- and three-machine problem there exists an optimal schedule with consistent sublots.*

Therefore, we can assume consistent sublot sizes in the analysis of the three-machine problem later. We note that Corollary 2 does not extend to the case of four or more machines. This was demonstrated by an example in [15] even without setup times.

Lemma 3 *If $C^1_{\max}$ denotes the optimal makespan for the alternative lot streaming problem $(A1)$ in which the setup time S_1 is set at 0 while other setup times remain the same, then the optimal sublot sizes for $(A1)$ are also optimal for the original problem and $C_{\max} = C^1_{\max} + S_1$.*

Proof. Since the setups are attached, all other machines must wait while the first machine is setting up. Thus the problem is equivalent to an alternative problem which starts processing at S_1 on M_1 with no setup and the setups remain the same on the other machines. It is clear that $C_{\max} = C^1_{\max} + S_1$.□

Motivated by the previous Lemma, we restrict our attention to$(A1)$, the special case of the problem in which $S_1 = 0$. We refer to this as the *reduced* *problem.* If we set $S_i = 0$ for $1 \leq i \leq m$, then this is a problem with no setup times, which we refer to as the *relaxed problem.*

In the remainder of this section we study the structure of optimal solutions for the reduced problem, with the *assumption* that there is an optimal solution with consistent sublots. Following the approach in [14], such a solution can be represented by a network $N(x)$ which contains a vertex for each sublot on every machine (see Fig. 1).

In the network, x_i $(i = 1, 2, ..., s)$ is the ith sublot size. The directed arc from vertex (i, j) to vertex $(i + 1, j)$ $(i < m)$ represents the production constraint that sublot j can be processed on machine $(i + 1)$ only if it is completed on machine i. The directed arc from vertex (i, j) to vertex $(i, j+1)$ $(1 \leq j < s)$ represents the machine capacity constraint that sublot $(j + 1)$ can start on M_i only when the jth sublot is completed on it. The vertex $(1,1)$ has weight of $p_1 x_1$, vertex $(i, 1)$ has weight $p_i x_1 + S_i$, and vertex (i, j) has weight $p_i x_j, 1 \leq i \leq m, 1 < j \leq s$.

Using the network representation, the objective becomes to determine the sublot sizes which minimize the length of the longest path in the network, where the length of any path is the sum of the weights of the vertices on it. Any longest path is referred to as a *critical path.* A subpath of a (critical) path is called a *(critical) segment.* We call sublot j *critical* in $N(x)$ if there is a critical segment containing the arc from (i, j) to $(i + 1, j)$ for some $i \in \{1, 2, ..., m - 1\}$.

We must distinguish two cases, depending on whether $x_1 = \Delta$ or $x_1 > \Delta$.

Theorem 4 *If $x_1 = \Delta$ in an optimal solution with consistent sublots for the m-machine reduced problem with attached setups S_i $(i = 2, ..., s)$, then the problem is equivalent to an alternative m-machine problem with $s - 1$ sublots, detached setups $S_i' = \sum_{j=2}^{i} S_j + \Delta \sum_{j=1}^{i} p_j$ on M_i $(i = 2, ..., s)$ and total lot size $1 - \Delta$.*

Proof. If x_1 is fixed at Δ, then M_2 can start setting up at time Δp_1 and M_i can begin its setup at time $\sum_{j=2}^{i-1} S_j + \Delta \sum_{j=1}^{i-1} p_j$, $2 \leq i \leq s$. Therefore, the problem can be viewed as an m-machine problem with detached setups $\sum_{j=2}^{i} S_j + \Delta \sum_{j=1}^{i} p_j$ on M_i, $s - 1$ sublots and total lot size $1 - \Delta$.□

Corollary 5 *If $x_1 = \Delta$, the three-machine reduced problem with attached setup time S_i on M_i $(i = 2, 3)$ is equivalent to a three-machine problem with*

$s-1$ sublots, detached setup $S_i' = \sum_{j=2}^{i} S_j + \Delta \sum_{j=1}^{i} p_i$ on $M_i (i = 2,3)$ and total lot size $1 - \Delta$.

We note that the three-machine reduced problem with detached setups can be solved efficiently [5].

Since the previous Corollary effectively solves the three-machine problem if $x_1 = \Delta$, from now on, we focus on the reduced problem, with the *assumption* $x_1 > \Delta$.

Theorem 6 *Consider an optimal solution with consistent sublots for the reduced problem on m machines, then every sublot k is critical in the corresponding optimal network for $1 \leq k \leq s$.*

Proof. For any vector $x = (x_1, x_2, x_3, ..., x_s)$ of the sublot sizes, let $L(i, j, i', j', x)$ represent the length of the longest path from vertex (i, j) to vertex (i', j') and $M(x)$ is the length of the longest path from (1,1) to (m, s).

For any machine h and i $(1 \leq h \leq i \leq m)$ and any sublot j, where $1 \leq j \leq s$, let $H(h, i, j, x)$ be the length of the longest path from $(1, 1)$ to (m, s) containing the segment $(h, j) - ... - (i, j)$. Thus,

$$H(h,i,j,x) = L(1,1,h,j-1,x) + x_j \sum_{g=h}^{i} p_g + L(i,j+1,m,s,x) \quad for 1 < j \leq s,$$

assuming that $L(i, s+1, m, s, x) = 0$.
So for any sublot j,

$$M(x) = \max_{1 \leq h \leq i \leq m} \{H(h, i, j, x)\} . \tag{1}$$

Now let $x = (x_1, x_2, ..., x_s)$ be a vector of optimal sublot sizes, which yields the minimum makespan $M(x)$, and suppose that sublot k is non-critical. We construct a vector x' of sublot sizes for which $M(x') < M(x)$, contradicting the assumption that x yields the minimum makespan. Because sublot k is non-critical, each segment $(h, k) - ... - (i, k)$ $(h < i)$ is non-critical and we have

$$M(x) = \max_{1 \leq i \leq m} \{H(i, i, k, x)\}.$$

We define $x' = (x_1', x_2', ..., x_s')$, by $x_j' = (1-\varepsilon) * x_j$ for $1 \leq j \leq k-1$ and $k+1 \leq j \leq s$ and $x_k' = x_k * (1-\varepsilon) + \varepsilon$, with a small positive ε defined below.

From (1), $M(x') = H(h,i,k,x')$ for *some* h and i, satisfying $h \le i$. We show that $H(h,i,k,x') < M(x)$.

Case 1a. $h < i$ and $k > 1$

Let

$$\varepsilon = \min\{[M(x) - \max_{1 \le e < f \le m} (H(e,f,k,x))]/[\sum_{g=1}^{m} p_g + \sum_{j=2}^{m} S_j], x_1 - \Delta\} \qquad (2)$$

Since k is non-critical and $x_1 - \Delta > 0$ by assumption, $\varepsilon > 0$, and note that $\varepsilon < 1$.

If $h = 1$, then the longest path from $(1,1)$ to (m,s) which contains the segment $(h,j)-,\ldots-(i,j)$ must avoid the setups on $M_2, \ldots, M_m$, and therefore,

$$\begin{aligned} H(h,i,k,x') &= (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon)\sum_{g=1}^{i} p_g + \\ &\quad (1-\varepsilon)L(i,k+1,m,s,x) \\ &= (1-\varepsilon)H(h,i,k,x) + \varepsilon \sum_{g=1}^{i} p_g \\ &< H(h,i,k,x) + \varepsilon \sum_{g=1}^{m} p_g \\ &< M(x), \end{aligned}$$

where the last inequality follows from equation (2), which defines ε.

If $h > 1$ and $(h,1)$ is contained in $L(1,1,h,k-1,x)$, then

$$H(h,i,k,x') = (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon) \sum_{g=h}^{i} p_g + (1-\varepsilon)L(i,k+1,m,s,x) + \varepsilon \sum_{j=2}^{h} S_j.$$

If $h > 1$ and $(h,1)$ is not contained in $L(1,1,h,k-1,x)$, then

$$H(h,i,k,x') < (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon) \sum_{g=h}^{i} p_g + (1-\varepsilon)L(i,k+1,m,s,x) + \varepsilon \sum_{j=2}^{h} S_j.$$

Therefore, in either case, for $h > 1$

$$H(h,i,k,x') \le (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon) \sum_{g=h}^{i} p_g +$$

$$
\begin{aligned}
&(1-\varepsilon)L(i,k+1,m,s,x)+\varepsilon\sum_{j=2}^{h}S_j\\
&=(1-\varepsilon)H(h,i,k,x)+\varepsilon\sum_{g=h}^{i}p_g+\varepsilon\sum_{j=2}^{h}S_j\\
&< H(h,i,k,x)+\varepsilon\sum_{g=h}^{m}p_g+\varepsilon\sum_{j=2}^{m}S_j\\
&\le M(x),
\end{aligned}
$$

where the last inequality follows from (2) again.

Case 1b. $h < i$ and $k = 1$

$$
\begin{aligned}
H(h,i,1,x') &= (x_1(1-\varepsilon)+\varepsilon)\sum_{g=1}^{i}p_g+\sum_{j=2}^{i}S_j+(1-\varepsilon)L(i,2,m,s,x)\\
&= (1-\varepsilon)H(h,i,1,x)+\varepsilon\sum_{g=1}^{i}p_g+\varepsilon\sum_{j=2}^{i}S_j\\
&< H(h,i,1,x)+\varepsilon(\sum_{g=1}^{m}p_g+\sum_{j=2}^{m}S_j)\\
&\le M(x),
\end{aligned}
$$

where the last inequality follows from (2) again.

Case 2a. $h = i$ and $k > 1$

Let ε be an arbitrary small number from $(0, x_1 - \Delta)$.

If $h = i = 1$, then

$$
\begin{aligned}
H(h,i,k,x') &= (1-\varepsilon)L(1,1,h,k-1,x)+(x_k(1-\varepsilon)+\varepsilon)p_1\\
&\quad +(1-\varepsilon)L(i,k+1,m,s,x)\\
&= H(h,i,k,x)-\varepsilon(H(h,i,k,x)-p_1). \qquad (3)
\end{aligned}
$$

$H(h,i,k,x)$ should be larger than p_1. Substituting this into (3) and using (1), we get

$$H(h,i,k,x') < H(h,i,k,x) \le M(x).$$

If $h = i \ge 2$ and $(h,1)$ is contained in $L(1,1,h,k-1,x)$, then

$$
\begin{aligned}
H(h,i,k,x') &= (1-\varepsilon)L(1,1,h,k-1,x)+(x_k(1-\varepsilon)+\varepsilon)p_i+\\
&\quad (1-\varepsilon)L(i,k+1,m,s,x)+\varepsilon\sum_{j=2}^{h}S_j.
\end{aligned}
$$

If $h = i \ge 2$ and $(h,1)$ is not contained in $L(1,1,h,k-1,x)$, then

$$H(h,i,k,x') < (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon)p_i + (1-\varepsilon)L(i,k+1,m,s,x) + \varepsilon \sum_{j=2}^{h} S_j.$$

Therefore, in either case, for $h = i \geq 2$

$$\begin{aligned} H(h,i,k,x') &\leq (1-\varepsilon)L(1,1,h,k-1,x) + (x_k(1-\varepsilon)+\varepsilon)p_i \\ &\qquad +(1-\varepsilon)L(i,k+1,m,s,x) + \varepsilon \sum_{j=2}^{h} S_j \\ &= (1-\varepsilon)H(i,i,k,x) + \varepsilon \sum_{j=2}^{h} S_j + \varepsilon p_i \\ &= H(i,i,k,x) - \varepsilon(H(i,i,k,x) - \sum_{j=2}^{h} S_j - p_i). \end{aligned} \tag{4}$$

By considering the path containing the segment $(1,1)-(2,1)-...-(h,1)-(h,2)-...-(h,s)$, we know that $H(i,i,k,x)$ should be longer than $\sum_{j=2}^{h} S_j + p_i$, therefore using this and (1) in (4), we get again

$$H(h,i,k,x') < H(h,i,k,x) \leq M(x).$$

Case 2b. $h = i$ and $k = 1$

In this case, the proof of $H(h,i,1,x') < M(x)$ is identical to the proof for Case 1b.

Thus, we have proved in every case that $M(x') = H(h,i,k,x') < M(x)$, which contradicts the assumption that x yields the minimum makespan. Therefore, sublot k is critical for $1 \leq k \leq s$.□

Corollary 7 *There is a consistent sublot optimal solution for the reduced problem on three machines in which every sublot is critical.*

Corollary 8 *In any consistent sublot optimal solution of the reduced problem on m machines, all the sublot sizes are positive.*

Proof. Let x be the optimal sublot vector and let $k > 1$ be the first sublot with $x_k = 0$. Let sublot j be such that $x_k = x_{k+1} = ... = x_j = 0, j \geq k$.

Case 1. $x_{j+1} \neq 0$

By Theorem 6, there exists h and i $(1 \leq h < i \leq m)$ for which $(h,k) - ... - (i,k)$ is a critical segment for the sublot k, i.e.,

$$M(x) = H(h,i,k,x). \tag{5}$$

Since $x_k = x_{k+1} = ... = x_j = 0$,

$$H(h,i,k,x) = L(1,1,h,k-1,x) + L(i,j+1,m,s,x). \tag{6}$$

There is an alternative path, however, which contains the segment $(h,k) - (h,k+1) - ... - (h,j+1) - (h+1,j+1) - ... - (i,j+1)$ with length $L(1,1,h,k-1,x) + L(h,k,h,j,x) + L(h,j+1,i,j+1,x) + L(i,j+1,m,s,x)$. Using (5) and (6) and the fact that $L(h,j+1,i,j+1,x) > 0$, we see that this alternative path is longer than $M(x)$, a contradiction.

Case 2. $j = s$

According to Theorem 6, there must be a critical segment for sublot k, say $(h,k) - (h+1,k)$. Let the critical path containing $(h,k) - (h+1,k)$ have the length $LEN1$, then since $x_k = ... = x_s = 0$,

$$LEN1 = L(1,1,h,k-1,x)$$

There is an alternative path, however, which coincides with the above critical path until vertex $(h,k-1)$ and contains the segment $(h,k-1) - (h+1,k-1)$. If its length is $LEN2$, then

$$LEN2 = L(1,1,h,k-1,x) + x_{k-1}p_{h+1} > LEN1.$$

This is in contradiction with the assumption that $LEN1$ is the length of a critical path. □

Corollary 9 *There is a consistent sublot optimal solution for the reduced problem on three machines in which all sublot sizes are positive.*

Observation 1 We call (i,l) an *upper critical corner* in the network if $l > 1$ and there is a critical path containing $(i,k) - (i,k+1) - ... - (i,l) - (i+1,l) - ... - (g,l)$, where $k < l$. An upper critical corner has the following property:

$$zS_i + (\sum_{j=k}^{l} x_j)\, p_i + (\sum_{j=i+1}^{g} p_j)\, x_l \geq (\sum_{j=i}^{g} p_j)\, x_k + (\sum_{j=k+1}^{l} x_j)\, p_g + z \sum_{j=i}^{g} S_j,$$

where $z = 1$ if $k = 1$ and $z = 0$ otherwise.

Observation 2 We call (i', k) a *lower critical corner* in the network if $k \geq 1$ and there is a critical segment $(i,k) - ... - (i',k) - (i',k+1) - ... - (i',l)$, where $i' > i$. A lower critical corner (i', k) has the following property :

$$zS_i + (\sum_{j=k}^{l} x_j)\, p_i + (\sum_{j=i+1}^{i'} p_j)\, x_l \leq (\sum_{j=i}^{i'} p_j)\, x_k + (\sum_{j=k+1}^{l} x_j)\, p_{i'} + z\sum_{j=i}^{i'} S_j,$$

where $z = 1$ if $k = 1$ and $z = 0$ otherwise.

Observation 3 We call (i, l) and (i', k) *matching critical corners* in the network if $i < i', 1 \leq k < l$ and both $(i,k) - ... - (i,l) - (i+1,l) - ... - (i',l)$ and $(i,k) - ... - (i',k) - ... - (i',l)$ are critical segments. Matching critical corners (i,l) and (i',k) have the following property:

$$zS_i + (\sum_{j=k}^{l} x_j)\, p_i + (\sum_{j=i+1}^{i'} p_j)\, x_l = (\sum_{j=i}^{i'} p_j)\, x_k + (\sum_{j=k+1}^{l} x_j)\, p_{i'} + z\sum_{j=i}^{i'} S_j,$$

where $z = 1$ if $k = 1$ and $z = 0$ otherwise.

3 Network Structure on Three Machines

By the above analysis, we know that we only need to consider consistent sublots for the problem and every sublot is critical. Similarly to the relaxed problem [7], we have to distinguish three cases, depending on whether $(p_2)^2 < p_1p_3$, $(p_2)^2 = p_1p_3$ or $(p_2)^2 > p_1p_3$.

Case 1 $(p_2)^2 < p_1p_3$

Lemma 10 *If $(p_2)^2 < p_1p_3$, then there is no critical segment of the form $(1,k) - (2,k) - ... - (2,l) - (3,l)$ with $l > k > 1$.*

Proof. If $(1,k) - (2,k) - ... - (2,l) - (3,l)$ was a critical segment, then $(2,k)$ and $(2,l)$ must be lower and upper critical corners, respectively. Using Observation 2 for $(2,k)$ we obtain

$$p_1x_k + p_2\sum_{j=k}^{l} x_j \geq p_1\sum_{j=k}^{l} x_j + p_2x_l,$$

which simplifies to

$$p_2\sum_{j=k}^{l-1} x_j \geq p_1\sum_{j=k+1}^{l} x_j. \tag{7}$$

Using Observation 1 for $(2, l)$, we get $p_2 \sum_{j=k}^{l} x_j + p_3 x_l \geq p_2 x_k + \ p_3 \sum_{j=k}^{l} x_j$, which simplifies to

$$p_2 \sum_{j=k+1}^{l} x_j \geq p_3 \sum_{j=k}^{l-1} x_j. \tag{8}$$

Multiplying (7) and (8) side by side yields a contradiction with the assumption $(p_2)^2 < p_1 p_3$.□

When S_2 and S_3 are "big enough", no horizontal segment on M_1 can be critical (see Remark 1 in Section 4 for this case). Here we *assume* that there is a critical horizontal segment on M_1.

The structure of the critical paths for this case is best described by Figure 2, where heavy lines show critical segments, light lines non-critical segments and dotted lines could be critical or not depending on the actual data. The following theorem summarizes the distinguishing properties of the critical paths.

Theorem 11 *If $(p_2)^2 < p_1 p_3$ and there is a critical horizontal segment on M_1, then there is a $k \in \{2, ..., s\}$ such that*

i) no segment $(1, j) - (2, j)$ is critical for $1 < j < k$;

ii) no segment $(2, j) - (2, j + 1)$ is critical for $k \leq j < s$;

iii) segments $(2,1) - (3,1) - (3,2)$ and $(2, k - 1) - (2, k)$ are critical or not depending on data;

iv) every other 2-node segment is critical.

Proof. Let k be the first sublot which has critical segment $(1, k) - (2, k)$ for $2 \leq k \leq s$. We show that segment $(1, j) - (2, j) - (3, j)$ is critical for $k \leq j \leq s$ and $(2, j - 1) - (2, j)$ is non-critical for $k < j \leq s$.

Consider a $j \in \{k, k+1, ..., s-1\}$ and suppose $(2, j) - (2, j+1)$ was critical. Then there must be a $k_1 \geq j + 1$ such that $(2, j) - ... - (2, k_1) - (3, k_1)$ is critical. Since $(1, k) - (2, k)$ is critical, $(2, j)$ must be reached by a critical segment $(1, k_2) - (2, k_2) - ... - (2, j)$ for some $k \leq k_2 \leq j$. This, however, is in contradiction with Lemma 10.

Since every sublot is critical, at least one of the segments $(1,j)-(2,j)$, $(2,j)-(3,j)$ must be critical for $1 \le j \le s$. Since $(2,j)-(2,j+1)$ cannot be critical for $k \le j \le s-1$, however, the whole segment $(1,j)-(2,j)-(3,j)$ must be critical for $k \le j \le s$.

Since $(1,i)-(2,i)$ is non-critical for $1 < i < k$, $(2,i)-(3,i)$ should be critical. Thus, $(2,i-1)-(2,i)$ should be critical for $1 \le i < k$ to make $(2,i)-(3,i)$ reachable on a critical path.□

We note that when $S_i = 0 (i = 2, ..., m)$, then $k = 1$ and our more complex optimal network structure reduces to the one proved in [7].

Case 2 $(p_2)^2 = p_1 p_3$

When S_2 and S_3 are "big enough", no horizontal segment on M_1 can be critical (see Remark 2 in Section 4 for this case). Here we *assume* that there is a critical horizontal segment on M_1. The optimal network structure is shown in Fig. 3.

Theorem 12 *If $(p_2)^2 = p_1 p_3$ and there is a critical horizontal segment on M_1, then the optimal network can be described as follows:*

i) segment $(2,1)-(3,1)-(3,2)$ is critical if $S_3 > (p_3/p_2)S_2$, segment $(2,1)-(2,2)$ is critical if $S_3 < (p_3/p_2)S_2$ and both segments are critical if $S_3 = (p_3/p_2)S_2$;

ii) all other 2-node segments are critical.

Proof. We show that segment $(1,j)-(2,j)-(3,j)$ is critical for $1 < j \le s$. Since we assumed that there is a critical horizontal segment on M_1, there is a $j > 1$ such that $(1,j)-(2,j)$ is critical. Suppose sublot k is the last sublot which has a critical segment $(1,k)-(2,k)$ for some $k \in \{2, ..., s\}$. According to Theorem 6, $(2,j)-(3,j)$ should be critical for $k < j \le s$, so $(2,j-1)-(2,j)$ should also be critical for $k < j \le s$, as otherwise we could not reach $(2,j)$ on a critical path. Thus $(2,k)$ and $(2,s)$ are lower and upper critical corners, respectively, unless $k = s$. If $k \ne s$, then from Observations 1 and 2, we get

$$p_1 \sum_{i=k+1}^{s} x_i < p_2 \sum_{i=k}^{s-1} x_i \quad \text{and} \quad p_2 \sum_{i=k+1}^{s} x_i \ge p_3 \sum_{i=k}^{s-1} x_i \,.$$

The multiplication of these inequalities yields a contradiction, however, with the assumption of $(p_2)^2 = p_1 p_3$. Therefore $k \notin \{2, ..., s-1\}$. Thus, $k = s$, and $(1, s) - (2, s) - (3, s)$ must be critical.

Now we prove that every segment $(1, j) - (2, j)$ is critical for $1 < j < s$. Suppose there is one, say l, with a non-critical segment $(1, l) - (2, l)$, then $(2, l) - (3, l)$ should be critical by Theorem 6. Therefore, $(3, l)$ is a lower critical corner and $(1, s)$ is an upper critical corner. By using Observation 1 and 2, we get

$$p_1 \sum_{i=l+1}^{s} x_i > p_2 \sum_{i=l}^{s-1} x_i \quad \text{and} \quad p_3 \sum_{i=l}^{s-1} x_i \geq p_2 \sum_{i=l+1}^{s} x_i \ ,$$

which yields a contradiction with the assumption of $(p_2)^2 = p_1 p_3$.

It is clear that $(1, 1) - (2, 1)$ must be critical, otherwise, $(2, 1) - (3, 1)$ cannot be critical, contradicting Theorem 6.

Now we show that $(2, j) - (3, j)$ is critical for $2 \leq j \leq s$.

Suppose sublot k has a non-critical segment $(2, k) - (3, k)$ for some $k \in \{2, ..., s-1\}$. There must be another sublot $k' > k$ such that $(2, k) - (2, k+1) - ... - (2, k') - (3, k')$ is critical. Thus $(2, k)$ is a lower critical corner and $(2, k')$ is an upper critical corner and by Observations 1 and 2, we have

$$p_1 \sum_{i=k+1}^{k'} x_i \leq p_2 \sum_{i=k}^{k'-1} x_i \quad \text{and} \quad p_2 \sum_{i=k+1}^{k'} x_i > p_3 \sum_{i=k}^{k'-1} x_i \ .$$

The multiplication of these inequalities yields a contradiction, however, with the assumption of $(p_2)^2 = p_1 p_3$.

It is obvious that $(2, s) - (3, s)$ is critical.

Because $(1, 1) - (2, 1)$ is critical, at least one of $(2, 1) - (3, 1) - (3, 2)$ and (2,1)-(2,2) should be critical. Suppose (2,1)-(2,2) is critical, then $(2, 1)$ and $(1, 2)$ are matching critical corners. By Observation 3, we have

$$x_1 p_2 + S_2 = x_2 p_1. \tag{9}$$

If $(2, 1) - (3, 1) - (3, 2)$ is not critical, then

$$S_3 + x_1 p_3 < x_2 p_2, \tag{10}$$

which, using (9) and $(p_2)^2 = p_1 p_3$, implies $S_3 < (p_3/p_2) S_2$. Equality holds in (10) if $S_3 = (p_3/p_2) S_2$, i.e., both $(2, 1) - (2, 2)$ and $(2, 1) - (3, 1) - (3, 2)$ are critical in this case.

A similar argument can be used to show that $(2, 1) - (3, 1) - (3, 2)$ is critical and $(2, 1) - (2, 2)$ is not if $S_3 > (p_2/p_1) S_2$.□

We note that when $S_3 = (p_2/p_1)S_2$, which is true if $S_2 = S_3 = 0$, our optimal network structure is the same as the one proved in [7].

Case 3 $(p_2)^2 > p_1 p_3$

When S_2 and S_3 are "big enough", no horizontal segment on M_1 can be critical (see Remark 3 in Section 4 for this case). Here we *assume* that there is a critical horizontal segment on M_1. The optimal network structure is shown in Fig. 4.

Theorem 13 *If $(p_2)^2 > p_1 p_3$, then there exist sublots k and j, with $1 \leq k \leq j \leq k+1$, such that*

i) no segment $(1,i)-(2,i)$ is critical for $k < i \leq s$;

ii) no segment $(2,i)-(3,i)$ is critical for $1 < i < j$;

iii) segments $(2,1)-(3,1)-...-(3,k+1)$ and $(2,k)-(3,k)-(3,k+1)$ are critical or not depending on data;

iv) every other 2-node segment is critical.

Proof. Suppose k is the last sublot with critical segment $(1,k)-(2,k), 1 \leq k \leq s$, then $(2,i)-(3,i)$ is critical by Theorem 6 for $i \in \{k+1,...,s\}$, which implies that $(2,i-1)-(2,i)$ is also critical.

Suppose sublot j is the first sublot with critical segment $(2,j)-(3,j)$. Then $(1,i)-(2,i)$ is critical for $i \in \{2,..,j-1\}$, by Theorem 6, which implies that $(2,i)-(2,i+1)$ is also critical.

We show that $j \geq k$ for $k \geq 2$. Suppose $j < k$, implying that $(1,j)-...-(1,k)-(2,k)$ is a critical segment and $(2,j)-(3,j)-...-(3,k)$ is another critical segment. $(1,k)$ is an upper critical corner and $(3,j)$ is a lower critical corner, so by Observations 1 and 2, we get

$$p_1 \sum_{i=j+1}^{k} x_i \geq p_2 \sum_{i=j}^{k-1} x_i \text{ and } p_3 \sum_{i=j}^{k-1} x_i \geq p_2 \sum_{i=j+1}^{k} x_i,$$

which is in contradiction with the assumption $(p_2)^2 > p_1 p_3$. So we must have $j \geq k$.

Finally, we cannot have $j > k+1$, as this would make sublot $k+1$ noncritical, contradicting Theorem 6.

The resulting network structure is depicted in Fig 4.□

We will refer to sublot k of Theorem 11 and 13 as the *pattern-changing sublot.*

4 Optimal Sublot Sizes on Three Machines

Case 1 $(p_2)^2 < p_1p_3$

Theorem 14 *If $(p_2)^2 < p_1p_3$ and there is a horizontal critical segment on M_1, then each optimal sublot size can determined in constant time for a given pattern-changing sublot k.*

Proof. Let $x = (x_1, x_2, ..., x_s)$ be the optimal sublot sizes. By Theorem 11, we know that $(3,i)$ and $(2,i+1)$ are matching critical corners for $1 < i < k-2$, so we have

$$p_3x_i = p_2x_{i+1} \; for \;\; 1 < i < k-2. \tag{11}$$

$(3,j)$ and $(1,j+1)$ are also matching critical corners for $k < j < s$, so

$$(p_2+p_3)x_j = (p_1+p_2)x_{j+1} \;\; for \; k \le j < s. \tag{12}$$

From (11) we get $x_i = x_2q_2^{i-2}$ for $1 < i < k$, where $q_2 = p_3/p_2$.
From (12) we have $x_j = x_kq^{j-k}$ for $k \le j \le s$, where $q = (p_2+p_3)/(p_1+p_2)$.
Substituting these into $\sum_{i=1}^{s} x_i = 1$, we obtain

$$x_1 + x_2(1 + q_2 + .. + q_2^{k-3}) + x_k(1 + q + ... + q^{s-k}) = 1. \tag{13}$$

Since both segments $(1,1) - ... - (1,k) - (2,k) - (3,k)$ and $(1,1) - (2,1) - (2,2) - (3,2) - ... - (3,k)$ are critical, they should have the same length, i.e.,

$$Ap_1x_2 + (p_1+p_2)x_k = S_2 + p_2x_1 + p_2x_2 + Ap_3x_2, \tag{14}$$

where $A = 1 + q_2 + ... + q_2^{k-3}$.
From (13) and (14) we obtain

$$x_1 = f_1(x_k) \quad and \quad x_2 = f_2(x_k), \tag{15}$$

where $f_1(x_k)$ and $x_2 = f_2(x_k)$ are *linear* functions of x_k.

We now show that one of segments $(2, k-1) - (2, k)$ and $(2, 1) - (3, 1) - (3, 2)$ is critical.

From Theorem 11, we know that path $(1, 1) - ... - (1, s) - (2, s) - (3, s)$ is critical, therefore $M(x) = p_1 + (p_2 + p_3)x_s$. By (12), we can rewrite this equation as

$$M(x) = p_1 + (p_2 + p_3)q^{s-k}x_k. \tag{16}$$

Since $(3, k-1)$ is a lower critical corner, we have $p_3x_{k-1} \geq p_2x_k$. Substituting (11) and (15) into this, we obtain $x_2 = f_2(x_k) \geq x_k/q_2^{k-2}$, which simplifies to

$$x_k \geq C_1, \tag{17}$$

where C_1 is the appropriate constant from solving the inequality.
Since $(2, 2)$ is an upper critical corner, we have $S_3 + p_3x_1 \leq p_2x_2$. Substituting (15), we get $x_1 = f_1(x_k) \leq (p_2x_2 - S_3)/p_3$, which simplifies to

$$x_k \geq C_2, \tag{18}$$

with the appropriate constant C_2.

By (16), in order to minimize $M(x)$, we must make x_k as small as possible subject to (17) and (18). Therefore, one of (17) and (18) must be satisfied as an equality. (If we had $C_1 < 0$ and $C_2 < 0$, then we could minimize (16) by making $x_k = 0$, which contradicts Corollary 8.) This means that one of the segments $(2, k-1) - (2, k)$ and $(2, 1) - (3, 1) - (3, 2)$ must be critical.

Using (15) and the equality version of (17) or (18), whichever is applicable, we can compute x_1, x_2 and x_k, and therefore, each x_i $(i = 1, 2, ..., s)$ in constant time indeed.□

Corollary 15 *If $(p_2)^2 < p_1p_3$, then an optimal solution can be obtained in $O(s)$ time.*

Proof. It is clear that the pattern-changing sublot k can be determined in $O(s)$ time, by simply checking for each possible value the resulting makespan, using (16). □

Remark 1 If no horizontal segment on M_1 is critical, then we show that $(2, 1) - (3, 1) - (3, 2)$ is critical. In this case, $(2, i) - (3, i)$ must be critical for $2 \leq i \leq s$, by Theorem 6, so $(2, j) - (2, j+1)$ should be critical for

$1 \leq j < s$. The critical path $(1,1)-(2,1)-(2,2)-(3,2)-...-(3,s)$ has length

$$\begin{aligned} M(x) &= p_1x_1 + S_2 + p_2x_1 + p_2x_2 + (1-x_1)p_3 \\ &= (p_1+p_2-p_3)x_1 + S_2 + p_2x_2 + p_3 \end{aligned} \tag{19}$$

Using $p_3x_i = p_2x_{i+1}$ for $1 < i < s$ and the fact $\sum_{i=1}^{s} x_i = 1$, we have

$$x_1 = 1 - Cx_2, \tag{20}$$

where $C = 1 + q_2 + ... + q_2^{s-2}$.
Substituting (20) into (19), we obtain

$$M(x) = S_2 + p_1 + p_2 + (p_2 - C(p_1+p_2-p_3))x_2. \tag{21}$$

Since $(2,2)$ is an upper critical corner, we have $S_3 + p_3x_1 \leq p_2x_2$, implying

$$x_2 \geq (S_3 + p_3)/(p_2 + Cp_3). \tag{22}$$

In order to minimize $M(x)$ in (21), we need an x_2 as small as possible, subject to (22). Therefore, the optimal x_2 must satisfy (22) as an equality. (If we had $p_2 - C(p_1+p_2-p_3) \leq 0$, then $x_2 = 1$ would minimize (21), in contradiction with Corollary 8.) Since $(2,1)-(3,1)-(3,2)$ is critical in this case, we can obtain each sublot size in constant time, using (20) and the equality version of (22).□

Example 1 Let p_1=6; p_2=8; p_3=14; s=3 and $\Delta = 1/122$.

Case a. Let $S_2 = S_3$=0.

This is a relaxed case and $k = 1$. The optimal solution is $x_1 = \frac{49}{247}$, $x_2 = \frac{77}{247}$ and $x_3 = \frac{121}{247}$ and $M(x) = 16\frac{192}{247}$. It is clear that segment $(2,1)-(2,2)-(2,3)$ is not critical.

Case b. Let $S_2 = 2, S_3 = 1$.

The solution obtained for Case a. is no longer optimal and sublot 3 becomes the pattern-changing sublot. The optimal solution is $x_1 = \frac{6}{61}$, $x_2 = \frac{20}{61}$, $x_3 = \frac{35}{61}$ and $M(x) = 6+22*\frac{35}{61}$. The length of the longest path containing $(2,2)-(2,3)$ is equal to $M(x)$, but the length of the longest path containing $(2,1)-(3,1)-(3,2)$ is less than $M(x)$. Therefore, segment $(2,2)-(2,3)$ is critical, but segment $(2,1)-(3,1)-(3,2)$ is not.

Case c. Let $S_2 = 2, S_3 = \frac{76}{61}$.
In this case, the optimal sublot sizes remain the same as in previous case. The lengths of the longest paths containing $(2,2)-(2,3)$ and $(2,1)-(3,1)-(3,2)$ are the same, and are equal to the makespan. Therefore, both of these segments are critical.

Case d. Let $S_2 = 3, S_3 = 10$.

Since S_3 is too big, $x_1 = \Delta = 1/122$, and the problem is equivalent to a problem with $s - 1 = 2$ sublots and detached setup times $S_2' = S_2 + (p_1 + p_2)\Delta = 380/122$ and $S_3' = S_2 + S_3 + (p_1 + p_2 + p_3)\Delta = 1614/122$. This class of problems was analyzed in detail in [5]. In Fig. 8 we show an optimal solution. The only critical path is $(1,1)-(2,1)-(3,1)-(3,2)-(3,3)$ and $M(x) = S_2 + S_3 + (p_1 + p_2)\Delta + p_3 = 27\frac{14}{122}$.

Case 2. $(p_2)^2 = p_1 p_3$

Theorem 16 *If $(p_2)^2 = p_1 p_3$ and there is a critical horizontal segment on M_1, then each optimal sublot size can be determined in constant time.*

Proof. By Theorem 12, we know that $(1, i+1)$ and $(2, i)$ are matching critical corners for $1 < i < s$, therefore $p_2 x_i = p_1 x_{i+1}$, which implies

$$x_{i+1} = q^{i-2} x_2 \quad for \;\; 1 < i < s, \tag{23}$$

where $q = (p_2 + p_3)/(p_1 + p_2) = p_2/p_1$ in this case.

If $S_3 < (p_2/p_1)S_2$, then $(2,1)-(2,2)$ is critical by Theorem 12, and $(2,1)$ and $(1,2)$ are matching critical corners, so we have

$$S_2 + p_2 x_1 = p_1 x_2. \tag{24}$$

Combining (23) and (24) with $\sum_{i=1}^{s} x_i = 1$, we can obtain the optimal sublot sizes in constant time from the equations.

If $S_3 \geq (p_2/p_1)S_2$, then $(2,1)-(3,1)-(3,2)$ is critical by Theorem 12, and $(3,1)$ and $(1,2)$ are matching critical corners, so we have

$$S_3 + S_2 + (p_2 + p_3)x_1 = (p_1 + p_2)x_2. \tag{25}$$

Combining (23) and (25) with $\sum_{i=1}^{s} x_i = 1$, we can obtain the optimal sublot sizes in constant time again.□

Remark 2 If no horizontal segment on M_1 is critical, then $(2,i)-(3,i)$ must be critical for $2 \leq i \leq s$ by Theorem 6. Thus, $(2,i)-(2,i+1)$ is critical for $1 \leq i < s$. Applying a similar argument as in Remark 1, we find that $(2,1)-(3,1)-(3,2)$ must be critical. Therefore, we can obtain each optimal sublot size in constant time from the resulting equations again.

Case 3 $(p_2)^2 > p_1p_3$

Theorem 17 *If $(p_2)^2 > p_1p_3$ and there is a critical horizontal segment on M_1, then each optimal sublot size can be obtained in constant time for a given pattern-changing sublot k.*

Proof. By Theorem 13, $(1,i+1)$ and $(2,i)$ are matching critical corners for $1 < i < k$, so we have $p_2x_i = p_1x_{i+1}$, which implies

$$x_i = q_1^{i-2}x_2 \quad for \;\; 1 < i \leq k, \; where \; q_1 = p_2/p_1. \tag{26}$$

$(2,j+1)$ and $(3,j)$ are matching critical corners for $k < j < s$, so we have $p_3x_j = p_2x_{j+1}$, which implies

$$x_{j+1} = q_2^{j-k}x_{k+1} \quad for \;\; k < j < s, \; where \; q_2 = p_3/p_2. \tag{27}$$

We show now that one of $(2,k)-(3,k)-(3,k+1)$ and $(2,1)-(3,1)-(3,2)-\ldots-(3,k+1)$ is critical.
Since $(2,1)$ and $(1,2)$ are matching critical corners, we have

$$S_2 + p_2x_1 = p_1x_2. \tag{28}$$

Substituting (26) and (27) into $\sum_{i=1}^{s} x_i = 1$, we obtain

$$x_1 + A_1x_2 + B_1x_{k+1} = 1, \tag{29}$$

where $A_1 = 1 + q_1 + .. + q_1^{k-2}$ and $B_1 = 1 + q_2 + \ldots + q_2^{s-k-1}$.
From (28) and (29), we get

$$x_1 = g_1(x_{k+1}) \;\; and \;\; x_2 = g_2(x_{k+1}), \tag{30}$$

where $g_1(x_{k+1})$ and $g_2(x_{k+1})$ are *linear* functions of x_{k+1}.

From Theorem 13, we know that segment $(1,1)-(2,1)-...-(2,s)-(3,s)$ is critical. Therefore, $M(x) = p_1x_1 + S_2 + p_2 + p_3x_s$. Substituting (30) and (27), we get

$$M(x) = D + Ex_{k+1}, \tag{31}$$

where D and E are the appropriate constants.

Segment $(2,k) - (2,k+1) - (3,k+1)$ is critical, so by Observation 1, we have $p_3x_k \leq p_2x_{k+1}$. Substituting (26) and (30), we get $x_{k+1} \geq g_2(x_{k+1})q_1^{k-2}q_2$, which simplifies to

$$x_{k+1} \geq F_1, \tag{32}$$

for an appropriate constant F_1.

Since the segment $(2,1) - (2,2) - ... - (2,k+1) - (3,k+1)$ is critical, by Observation 1, we get $A_1p_2x_2 + p_2x_{k+1} \geq S_3 + p_3x_1 + A_1p_3x_2$. Substituting (30), we obtain

$$x_{k+1} \geq F_2, \tag{33}$$

for an appropriate constant F_2.

In order to minimize $M(x)$ in (31), we must minimize x_{k+1} subject to the constraints (32) and (33). It is clear that at least one of (32) and (33) must be satisfied as an equality by the optimal x_{k+1}. (If it was not the case, we could minimize $M(x)$ by letting $x_{k+1} = 0$, if $E \geq 0$, or $x_{k+1} = 1$ otherwise, which contradicts Corollary 8.)

Therefore, x_{k+1} can be obtained from the equality version of (32) or (33), whichever is applicable. Substituting this into (30), (26) and (27), we can compute each x_i in constant time.□

Corollary 18 *If* $(p_2)^2 > p_1p_3$, *then an optimal solution can be obtained in* $O(s)$ *time.*

Proof. It is clear that the pattern changing sublot k can be determined in $O(s)$, by simply checking for each possible value the resulting makespan, using (31). □

Remark 3 If no horizontal segment on M_1 is critical, then $(2,i)-(3,i)$ must be critical for $2 \leq i \leq s$ by Theorem 6. Thus, $(2,i) - (2,i+1)$ is critical for $1 \leq i < s$. Applying a similar argument as in Remark 1, we find that $(2,1)-(3,1)-(3,2)$ must be critical. Therefore, we can obtain each optimal sublot size in constant time from resulting equations again.

5 Summary and Concluding Remarks

We have analyzed the structural properties of lot streaming schedules which minimize the makespan for a single job with attached setup times in an m-machine flow shop. The results of Glass et.al.[7], for the no-setup case, have been extended to the case of attached setups. For $m = 2$ or 3, we have proved that there is always an optimal schedule with consistent sublots. As opposed to the case with detached setups, the attached setups *always* cause an increase in the makespan obtainable without the setups. We have shown that the general problem is equivalent to one in which there is no setup time on the first machine. There are two cases to be distinguished in the solution. If $x_1 = \Delta$, then the problem is equivalent to an alternative m-machine problem with detached setups, $s - 1$ sublots and total lot size $1 - \Delta$, which was solved in [5].

For the more complicated case, when $x_1 > \Delta$, we have proved that every sublot will be critical and will have a positive size. When the setup times on M_2 and M_3 are relatively large, the three-machine problem reduces to a problem on two machines with attached setups. The structure of the optimal network depends on the relative size of the job processing times on the three machines, but there are always some segments in the network whose criticality depends on the actual data values. When $(p_2)^2 = p_1p_3$, the optimal solution is the same as with no setups, but the first sublot may or may not be critical on M_3. When $(p_2)^2 \neq p_1p_3$, the optimal schedule and its structure change substantially in comparison with the no-setup case: The optimal network can be characterized by a *combination of two patterns* with a pattern-changing sublot between them. The optimal sublot sizes follow one *geometric progression* up to the pattern-changing sublot and another one after it. The index of the pattern-changing sublot depends on the setups. The pattern-changing sublot in the optimal solution can be identified in $O(s)$ time, using the explicit formulas obtained for the makespan.

In certain situations it is desirable to have *no-wait* schedules, i.e., to be able to start the processing of each sublot on each machine immediately after it is finished on the preceding machine. It can be easily checked from the structure of the optimal networks that, similarly to the no-setup case [7], the no-wait requirement can be satisfied *without* increasing the length of the optimal schedule.

Sometimes *no-idling* may be required, i.e., each machine should be kept working without any idle time once it starts. (This was called the *contiguity of work* assumption in [18] and in [19].) It is straightforward to transform the optimal schedules into a no-idling schedule on the first and last machine. The situation is more complicated, however, on the second machine. When $(p_2)^2 > p_1p_3$, then every horizontal segment of the network is critical on M_2, and the no-idling requirement is satisfied. When $(p_2)^2 = p_1p_3$ and $S_3 \leq (p_3/p_2)S_2$, then every horizontal segment is critical on M_2, and the no-idling requirement is satisfied again. When $(p_2)^2 = p_1p_3$ and $S_3 > (p_3/p_2)S_2$, or when $(p_2)^2 < p_1p_3$, however, a no-idling schedule will have an increased makespan.

There are many related topics for future investigation. It seems to be natural to try to extend the results in this paper to the case of job shops and open shops. Some of these problems will be studied in the future.

References

[1] Baker, K.R., *"Elements of Sequencing and Scheduling"*, K.R. Baker publ., Hanover, NH 1992.

[2] Baker, K.R., *"Lot Streaming in the Two-Machine Flow Shop with Setup Times"*, Working Paper No. 297, A. Tuck School of Bus. Admin., Dartmouth College, Hanover NH, 1993.

[3] Cetinkaya, F.C., and Gupta, J.N.D., *"Flowshop Lot Streaming to Minimize Total Weighted Flow Time"*, Working Paper No. 94-24, Industrial Engineering, Purdue Univ., West Lafayette, IN, 1994.

[4] Cetinkaya, F.C., and Kayaligil, M.S., "Unit Sized Transfer Batch Scheduling with Setup Times", *Computers and Industrial Engineering,* 22 (1992) 177-183.

[5] Chen, J., and Steiner, G., *"Lot Streaming with Detached Setups in Three-Machine Flow Shops"*, Working paper, School of Business, McMaster University, Hamilton, Ont. Canada, 1994.

[6] Fox, R.E., "OPT - An Answer for America (Part IV)", *Inventories and Production* 3 (1983).

[7] Glass, C.A., Gupta, J.N.D, and Potts, C.N., "Lot Streaming In Three Stage Production Processes", *Europ. J. Oper. Research,* 75 (1994) 378-394.

[8] Goldratt, E., and Fox, R.E., *The Race*, North River Press, Croton-on-Hudson, NY, 1986.

[9] Goyal, S.K., "Note on Manufacturing Cycle Time Determination for a Multi-Stage Economic Production Model", *Management Science* 23 (1976) 332-333; the rejoinder, 334-338.

[10] Graves, S.C., and Kostreva, M., "Overlapping Operations in Material Requirements Planning", *J. Operations Management* 6 (1987) 283-294.

[11] Jacobs, F.R., and Bragg, D.J., "Repetitive Lots: Flow Time Reductions Through Sequencing and Dynamic Batch Sizing", *Decision Sciences* 19 (1988) 281-294.

[12] Kulonda, D.J., "Overlapping Operations - A Step Towards Just-In-Time Production", *Readings in Zero Inventory,* APICS 27th Annual Conf. (1984) 78-80.

[13] Moily, J.P., "Optimal and Heuristic Procedures for Component Lot-Splitting in a Flow Shop", *Management Science* 32 (1986) 113-125.

[14] Monma, C.L., and Rinnooy Kan, A.H.G., "A Concise Survey of Efficiently Solvable Special Cases of the Permutation Flow-Shop Problem", *RAIRO Recherche Opérationelle* 17(1983) 105-119.

[15] Potts, C.N., and Baker, K.R.,"Flow Shop scheduling With Lot Streaming", *Operations Research Letters* 8 (1989) 207 - 303.

[16] Potts, C.N., and Van Wassenhove, L.N., "Integrating Scheduling With Batching and Lot-Sizing : a Review of Algorithms and Complexity", *J. Oper. Res. Soc.* 43 (1992) 395-406.

[17] Reiter, S., "A System for Managing Job Shop Production", *J. of Business,* 34 (1966) 371-393.

[18] Szendrovits, A.Z. "Manufacturing Cycle Time Determination for a Multi-Stage Economic Production Quantity Model", *Management Science,* 22 (1978), 298-308.

[19] Steiner, G., and Truscott, W.G., "Batch Scheduling to Minimize Cycle Time, Flow Time, and Processing Cost", *IIE Transactions,* 23 (1993) 90-97.

[20] Trietsch, D., "*Optimal Transfer Lots For Batch Manufacturing : A Base Case And Extension*", Technical Report NPS-54_87-010, Naval Postgraduate School, Monterey, CA, 1987.

[21] Trietsch, D., " *Polynomial Transfer Lot Sizing Techniques For Batch Processing On Consecutive Machines*", Technical Report NPS-54-89-011, Naval Postgraduate School, Monterey, CA, 1989.

[22] Trietsch, D., and Baker, K.R., "Basic Techniques For Lot Streaming", *Operations Research*, 41 (1993), 1065-1076.

[23] Vickson, R. *"Optimal Lot Streaming For Multiple Products in a Two-Machine Flow Shop"*, Working paper, Department of Management Sciences, Univ. Waterloo, Canada, 1992, to appear *Europ. J. Oper. Res.*

[24] Vickson, R., and Alfredsson, B., "Two and Three Machines Flow Shop Scheduling Problems With Equal Sized Transfer Batches", *Int. J. Production Res.* 30 (1992) 1551-1574.

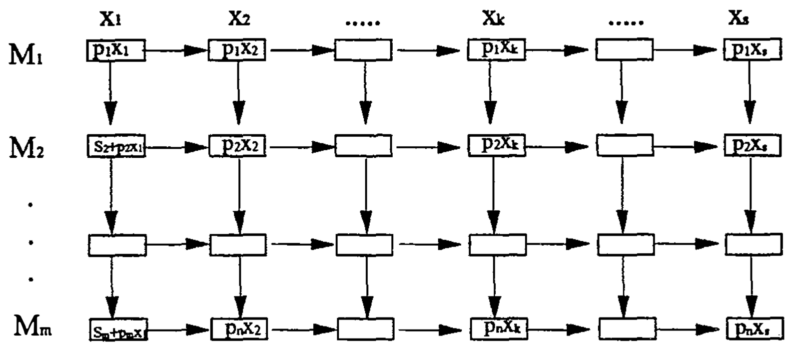

Figure 1: The network for a solution with consistent sublots

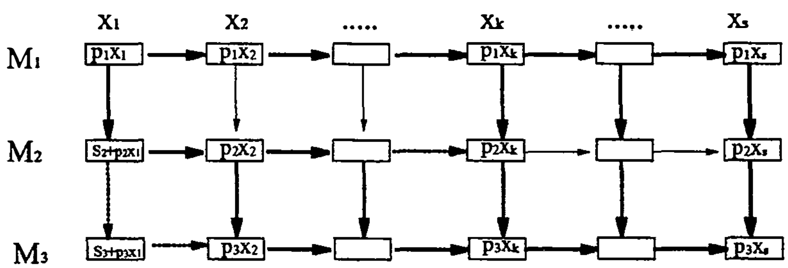

Figure 2: The network structure when $(p_2)^2 < p_1 p_3$

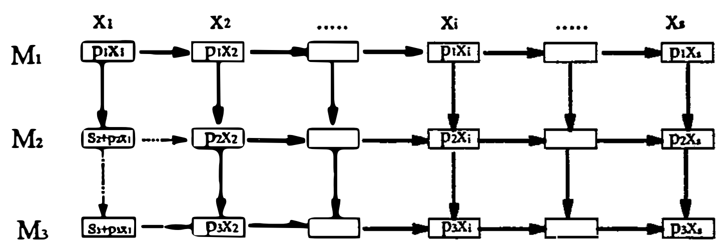

Figure 3: The network structure when $(p_2)^2 = p_1\, p_3$

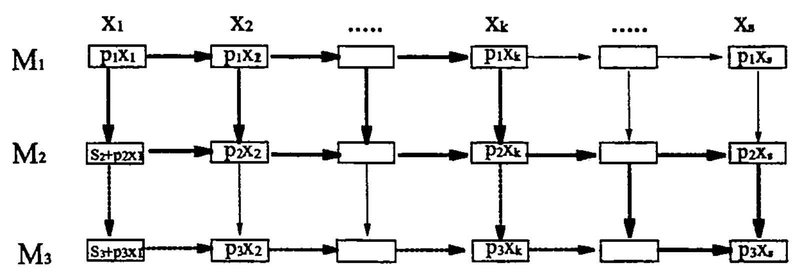

Figure 4: The network structure when $(p_2)^2 > p_1\, p_3$

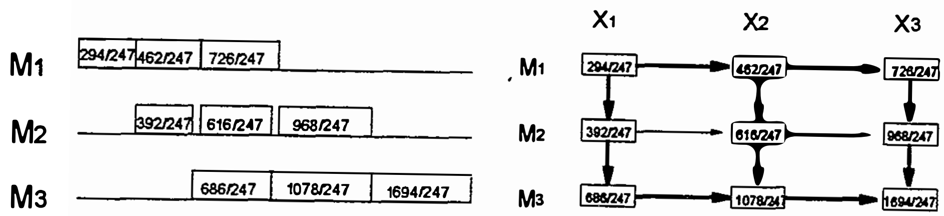

Figure 5: Case a. of Example 1.

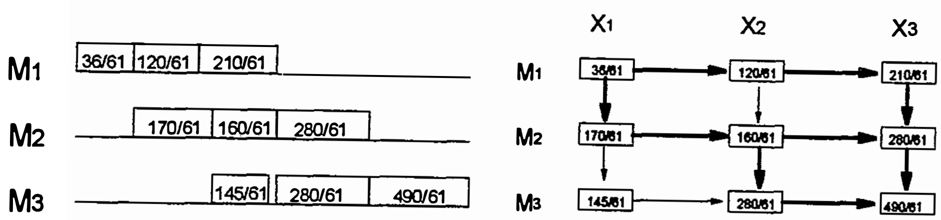

Figure 6: Case b. of Example 1.

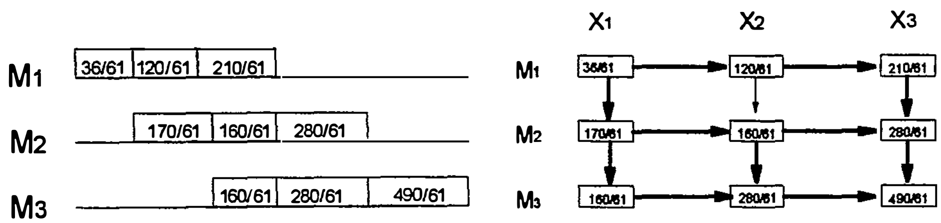

Figure 7: Case c. of Example 1.

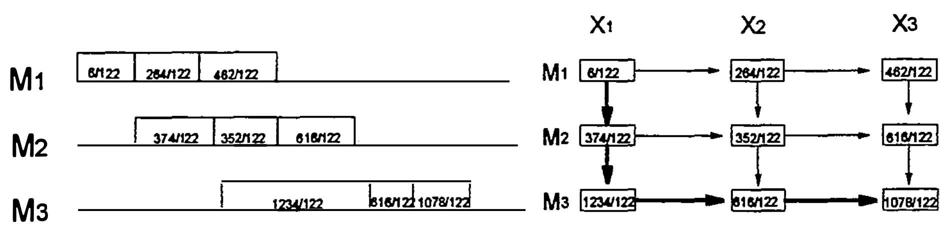

Figure 8: Case d. of Example 1.

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