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APPLICATION OF WEIGHTED SUMS OF ORDER *p* TO DISTANCE ESTIMATION

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Working Paper # 427

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Abstract

Distance functions are employed to estimate actual distances in a transportation network. The ℓ_{bp}-norm, which is a weighted sum of order p, is a generalization of the weighted ℓ_p-norm, a well-known distance predicting function. We derive some mathematical properties of the new norm and of a goodness-of-fit criterion. These properties are used to develop computational procedures to determine the best-fitting parameter values of the ℓ_{bp}-norm for a transportation network. We apply the new norm to seventeen geographic regions and find significant improvements in the accuracy of distance estimations over the weighted ℓ_p-norm.

1 Introduction

Distance predicting functions (d.p.f.) are utilized in several applications. In some distribution problems for which only the demand and the general location of the customers are known a d.p.f. may be employed to calculate a predicted travel distance between the depot and the general area (see [5] and [24]). For validating the accuracy of actual road network distance data d.p.f.'s can be used as suggested by Ginsburgh and Hansen [6]. Kolesar, Walker and Hausner [9] incorporated a d.p.f. into a response-time model for emergency vehicles such as fire engines. Klein [8] suggests that a d.p.f. which reflects the nature of a geographic region's road network should be used for constructing Voronoi diagrams of the region. D.p.f.'s appear within the context of larger models such as facilities location and location-allocation problems(see [13]). D.p.f.'s may be used for calculating distances in a Geographic Information System (GIS). As Star and Estes [19] state, distance measurements are of value in many geographic circumstances. D.p.f.'s are being utilized in software packages Roadnet [17] and TruckStops2 [20] since they are much more efficient and comprehensive to use in practice than attempting to assemble large files of distance data.

Love and Morris ([10], [11], [12]) applied several distance norms, including the weighted ℓ_p-norm ($k\ell_p$-norm) to Germany and several regions of the United States. Let $\mathbf{x} = (x_1, x_2)^T$, and $\mathbf{y} = (y_1, y_2)^T$ be any two points in the plane. The $k\ell_p$-norm is given by

$$k\ell_p(\mathbf{x}, \mathbf{y}) = k\left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}, \quad k > 0, \quad p > 0.$$

Love and Morris found that the $k\ell_p$-norm was relatively easy to fit to a geographical region and it has excellent predictive properties. Ward and Wendell [22], [23] have introduced the concept of utilizing block norms as distance predictors. A study by Love and Walker [16] shows that although marginally better results can be obtained by using a block norm with eight parameters than by using the $k\ell_p$-norm, the computational cost of fitting the block norms can be prohibitive. Conversely, the original studies by Love and Morris [10], [11] show that the $k\ell_p$-norm usually gives much superior results compared to other simpler norms such as the weighted Euclidean or weighted rectangular norms. Brimberg and Love [3] introduced a generalized $k\ell_p$-norm in the form of a weighted sum of order p. This is in effect adding a single parameter to the $k\ell_p$-norm since one of the two weights in the sum of order p function replaces the k in the $k\ell_p$-norm. A weighted sum of order p is defined as follows (section 2.10 of [7]):

$$T(\mathbf{y}; \mathbf{b}, p) = \left[\sum_{i=1}^{K} b_i \, y_i{}^p \right]^{1/p}, \quad p \neq 0,$$

where

$$\begin{aligned} \mathbf{y} &= (y_1, \ldots, y_K)^T, \quad y_i > 0, \quad i = 1, \ldots, K, \\ \mathbf{b} &= (b_1, \ldots, b_K)^T, \quad b_i > 0, \quad i = 1, \ldots, K, \end{aligned}$$

The vector $\mathbf{b}$ and the scalar p can be considered as a set of parameter values. If all the weights are equal to one, then T becomes the ordinary sum of order p which is well-known in the literature. Note that the function $T(\mathbf{y}; \mathbf{b}, p)$ has the form of a generalized ℓ_p distance given by

$$\ell_{bp}(\mathbf{x}) = \left[\sum_{i=1}^{K} b_i \, x_i{}^p \right]^{1/p},$$

where $\mathbf{x} = (x_1, \ldots, x_K)^T \in R^K$, p is generally assumed to be greater than zero, and ℓ_{bp} estimates the distance between any two points $\mathbf{y}, \mathbf{z} \in R^K$ such that $\mathbf{x} = \mathbf{y} - \mathbf{z}$. The weights b_i can be used to represent non-symmetric distance irregularities along the axis directions in a location model.

Brimberg and Love [3] derived the properties of the $\ell_{bp}(\mathbf{x})$ function in terms of its parameter p. We employ the 2-dimensional form of the $\ell_{bp}(\mathbf{x})$ function for predicting distances in the plane as follows:

$$\ell_{bp}(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{2} b_i \, |x_i - y_i|^p\right]^{1/p}.$$

It is well-known that the introduction of reference axis rotation brings more accuracy in predicting distances [4]. The reference axes are rotated so that they correspond to the underlying pattern of the road network. Therefore, we include axis rotation, θ, in the above distance prediction model as follows:

$$\ell_{bp\theta}(\mathbf{x}',\mathbf{y}') = \left[\sum_{i=1}^{2} b_i \, |x'_i - y'_i|^p\right]^{1/p}$$

where

$$\mathbf{x}' = (x'_1, x'_2), \;\; \mathbf{y}' = (y'_1, y'_2), \;\; \theta \in [0, \pi/2]$$

and

$$\begin{pmatrix} x'_1 & x'_2 \\ y'_1 & y'_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

In order to determine the parameter values of the distance predicting function for a region, we need a goodness-of-fit criterion. The optimum parameter values of the distance predicting function are determined so that the criterion value is minimized. Four criteria have been used to fit the parameters to a set of data representing a region of interest. Let $d_f(a_i, a_j)$ be the predicted distance between points a_i and a_j by using predicting function f. $A(a_i, a_j)$ is the actual distance between a_i and a_j, and n is the number of points in the data set. The four goodness-of-fit criteria are as follows:

$$
\begin{aligned}
AD_f &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |d_f(a_i, a_j) - A(a_i, a_j)| \\
SD_f &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(d_f(a_i, a_j) - A(a_i, a_j))^2}{A(a_i, a_j)} \\
SN_f &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\frac{d_f(a_i, a_j) - A(a_i, a_j)}{A(a_i, a_j)} \right)^2 \\
NAD_f &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{|d_f(a_i, a_j) - A(a_i, a_j)|}{A(a_i, a_j)}
\end{aligned}
$$

The AD_f and SD_f criteria were introduced by Love and Morris [10], the SN_f criterion by Brimberg [1] and the NAD_f criterion by Love and Walker [14]. The SD_f criterion will be employed in this study to model distances. Two reasons can be given for this choice. First, we know that SD_f is strictly convex in parameter k when used with the $k\ell_p$-norm which is a special case of $\ell_{bp}(\mathbf{x})$, when ${b_1}^{1/p} = {b_2}^{1/p} = k$ [2]. Secondly, SD_f possesses attractive statistical properties as discussed by Love and Morris [10].

In this paper we will first show that $\ell_{bp}(\mathbf{x})$ is a norm. Then we will address the following issues which are related to the use of the ℓ_{bp}-norm as a distance predicting function: review of the properties of the ℓ_{bp}-norm as a function of its parameter p [3]; the behaviour of the ℓ_{bp}-norm as a function of its parameters b_1, b_2 and p; the behaviour of the criterion SD_f as a function of b_1, b_2 and p; computational procedures to find the best parameter values of b_1, b_2 and p. Finally we compare the ℓ_{bp}-norm and the weighted ℓ_p-norm regarding their accuracy in predicting distances in seventeen geographic regions.

2 Properties of the ℓ_{bp}-norm

The following property confirms the use of the $\ell_{bp}(\mathbf{x})$ as a distance estimator.

Property 1 *The $\ell_{bp}(\mathbf{x})$ function is a norm and a convex function.*

Proof We examine the necessary properties for a function to be a norm [18].

(i) It is clear that $\ell_{bp}(\mathbf{x}) > 0\ \forall \mathbf{x} \neq 0$ (positivity), and $\ell_{bp}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

(ii) The homogeneity and the symmetry property are as follows. For any scalar c

$$\begin{aligned}\ell_{bp}(c\mathbf{x}) &= \left(b_1|cx_1|^p + b_2|cx_2|^p\right)^{1/p} \\ &= \left(|c|^p b_1|x_1|^p + |c|^p b_2|x_2|^p\right)^{1/p} \\ &= |c|\ell_{bp}(\mathbf{x}).\end{aligned}$$

(iii) The triangular inequality; for any $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ we show that

$$\ell_{bp}(\mathbf{x}+\mathbf{y}) \leq \ell_{bp}(\mathbf{x}) + \ell_{bp}(\mathbf{y})$$

$$\begin{aligned}\ell_{bp}(\mathbf{x}+\mathbf{y}) &= \left(b_1|x_1+y_1|^p + b_2|x_2+y_2|^p\right)^{1/p} \\ &= \left(|b_1^{1/p}x_1 + b_1^{1/p}y_1|^p + |b_2^{1/p}x_2 + b_2^{1/p}y_2|^p\right)^{1/p} \\ &\leq \left(|b_1^{1/p}x_1|^p + |b_2^{1/p}x_2|^p\right)^{1/p} + \left(|b_1^{1/p}y_1|^p + |b_2^{1/p}y_2|^p\right)^{1/p} \quad \text{(Minkowski Inequality)} \\ &= \ell_{bp}(\mathbf{x}) + \ell_{bp}(\mathbf{y}).\end{aligned}$$

Thus, $\ell_{bp}(\mathbf{x})$ is a norm and is a convex function of the variables x_1 and x_2. The convexity property is useful in the context of using $\ell_{bp}(\mathbf{x})$ in continuous location models. □

We now state a corollary given in Brimberg and Love [3]. This finding is important when the use of the ℓ_{bp}-norm for predicting distances is considered.

Corollary 1 *The ℓ_{bp}-norm is a decreasing function of $p > 0$ for given weights b_1, b_2 and all positive $|x_i - y_i|$, if and only if b_1, $b_2 \geq 1$. Furthermore, the ℓ_{bp}-norm is strictly convex in p under these conditions.*

In predicting distances we would expect to see the coefficients b_1 and b_2 greater than one. This result is potentially useful for the fitting procedures when the *decreasing strictly convex* property is considered. Moreover, since we are expecting both b_1 and b_2 to be greater than one, by utilizing the asymptotic behaviour of the ℓ_{bp}-norm [3] we observe the following limiting cases:

$$\lim_{p\to\infty} \ell_{bp} = \ell_\infty, \quad \lim_{p\to 0^+} \ell_{bp} = \infty$$

We next examine the ℓ_{bp}-norm as a function of its parameters b_1 and b_2. In order to check its convexity properties we need to calculate the following Hessian

$$H_{b_1,b_2} = \begin{bmatrix} \frac{\partial^2 \ell}{\partial {b_1}^2} & \frac{\partial^2 \ell}{\partial b_1 \partial b_2} \\ \frac{\partial^2 \ell}{\partial b_2 \partial b_1} & \frac{\partial^2 \ell}{\partial {b_2}^2} \end{bmatrix}$$

where

$$\frac{\partial \ell_{bp}}{\partial b_1} = \frac{1}{p}[b_1|x_1 - y_1|^p + b_2|x_2 - y_2|^p]^{\frac{1}{p}-1} |x_1 - y_1|^p,$$

$$\frac{\partial \ell_{bp}}{\partial b_2} = \frac{1}{p}[b_1|x_1 - y_1|^p + b_2|x_2 - y_2|^p]^{\frac{1}{p}-1} |x_2 - y_2|^p,$$

$$\frac{\partial^2 \ell_{bp}}{\partial {b_1}^2} = \frac{1-p}{p^2}[b_1|x_1 - y_1|^p + b_2|x_2 - y_2|^p]^{\frac{1}{p}-2} |x_1 - y_1|^{2p},$$

$$\frac{\partial^2 \ell_{bp}}{\partial {b_2}^2} = \frac{1-p}{p^2}[b_1|x_1 - y_1|^p + b_2|x_2 - y_2|^p]^{\frac{1}{p}-2} |x_2 - y_2|^{2p},$$

$$\frac{\partial^2 \ell_{bp}}{\partial b_1 \partial b_2} = \frac{\partial^2 \ell_{bp}}{\partial b_2 \partial b_1} = \frac{1-p}{p^2}[b_1|x_1-y_1|^p + b_2|x_2-y_2|^p]^{\frac{1}{p}-2}\,|x_1-y_1|^p\,|x_2-y_2|^p,$$

and

$$|H_{b_1,b_2}| = \frac{\partial^2 \ell_{bp}}{\partial {b_1}^2}\frac{\partial^2 \ell_{bp}}{\partial {b_2}^2} - \left(\frac{\partial^2 \ell_{bp}}{\partial b_1 \partial b_2}\right)^2.$$

If $p \geq 1$ then $\partial^2 \ell_{bp}/\partial {b_1}^2 \leq 0$, $\partial^2 \ell_{bp}/\partial {b_2}^2 \leq 0$ and $|H_{b_1,b_2}| = 0$.

Therefore, we conclude that for $p \geq 1$ the ℓ_{bp}-norm is concave in its parameters b_1 and b_2. Furthermore, considering, $\partial \ell_{bp}/\partial b_1 \geq 0$ and $\partial \ell_{bp}/\partial b_2 \geq 0$, we see that the ℓ_{bp}-norm is also an increasing function of parameters b_1 and b_2.

3 Properties of the SD_f Function

In this section we derive some useful properties of the SD_f function. These properties will later be utilized in designing computational procedures to determine the best parameter values of the ℓ_{bp}-norm for a given transportation network. First we examine the convexity of SD_f in terms of its parameters b_1 and b_2.

Property 2 *The criterion SD_f is convex in the parameters b_1 and b_2 provided that $p \in [1,2]$.*

Proof Suppose there are n points in our sample set where each point is given by its coordinates (a_{i1}, a_{i2}), $i = 1 \ldots n$, and the actual distance for each pair of points (a_i, a_j) is denoted by A_{ij}. Then we have

$$SD_f = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{A_{ij}}\left(A_{ij} - (b_1|a_{i1}-a_{j1}|^p + b_2|a_{i2}-a_{j2}|^p)^{1/p}\right)^2.$$

Consider one of the terms in this sum, denote it by h_{ij} and also denote $\ell_{bp}(a_i, a_j)$ by ℓ_{bp}.

We examine the convexity of h_{ij} in the parameters b_1 and b_2. The following terms are calculated for the Hessian

$$H_{b_1,b_2} = \begin{bmatrix} \frac{\partial^2 h_{ij}}{\partial {b_1}^2} & \frac{\partial^2 h_{ij}}{\partial b_1 \partial b_2} \\ \\ \frac{\partial^2 h_{ij}}{\partial b_2 \partial b_1} & \frac{\partial^2 h_{ij}}{\partial {b_2}^2} \end{bmatrix}.$$

$$\frac{\partial h_{ij}}{\partial b_1} = -2\,\frac{(A_{ij} - \ell_{bp})\,|a_{i1} - a_{j1}|^p}{A_{ij}\,p\,{\ell_{bp}}^{p-1}},$$

$$\frac{\partial h_{ij}}{\partial b_2} = -2\,\frac{(A_{ij} - \ell_{bp})\,|a_{i2} - a_{j2}|^p}{A_{ij}\,p\,{\ell_{bp}}^{p-1}},$$

$$\frac{\partial^2 h_{ij}}{\partial {b_1}^2} = 2\,\frac{|a_{i1} - a_{j1}|^{2p}}{A_{ij}\,p^2\,{\ell_{bp}}^{2p-1}}(\ell_{bp} + (p-1)(A_{ij} - \ell_{bp})), \text{ and}$$

$$\frac{\partial^2 h_{ij}}{\partial {b_2}^2} = 2\,\frac{|a_{i2} - a_{j2}|^{2p}}{A_{ij}\,p^2\,{\ell_{bp}}^{2p-1}}(\ell_{bp} + (p-1)(A_{ij} - \ell_{bp})).$$

Furthermore,

$$\frac{\partial^2 h_{ij}}{\partial b_1 \partial b_2} = 2\,\frac{|a_{i1} - a_{j1}|^p |a_{i2} - a_{j2}|^p}{A_{ij}\,p^2\,{\ell_{bp}}^{2p-1}}(\ell_{bp} + (p-1)(A_{ij} - \ell_{bp})),$$

which is also equal to $\partial^2 h_{ij}/\partial b_2 \partial b_1$. It can be verified that $|H_{b_1,b_2}| = 0$.

The second derivatives of h_{ij} w.r.t. b_1 and b_2 have a common term given by

$$M_{ij} = (\ell_{bp} + (p-1)(A_{ij} - \ell_{bp})).$$

The sign of M_{ij} must be found in order to see if the diagonal entries in the Hessian are nonnegative. There are two cases to consider:

(i) For the b_1 , b_2 values in which the ℓ_{bp}-norm underpredicts the actual distance A_{ij}, i.e. $A_{ij} \geq \ell_{bp}(i,j)$, M_{ij} is always nonnegative.

(ii) For the b_1 , b_2 values in which the ℓ_{bp}-norm overpredicts the actual distance A_{ij}, i.e. $A_{ij} \leq \ell_{bp}(i,j)$, we proceed as follows;

In order to see if M_{ij} is nonnegative we need to check if the inequality

$$\ell_{bp} > (p-1)|A_{ij} - \ell_{bp}|$$

holds. Rewriting this inequality we have

$$\frac{1}{p-1} > \left|\frac{A_{ij}}{\ell_{bp}} - 1\right|$$

or equivalently by using $A_{ij} \leq \ell_{bp}$ and rearranging , we obtain

$$\frac{A_{ij}}{\ell_{bp}} > \frac{2-p}{1-p} \quad \text{where} \quad \frac{A_{ij}}{\ell_{bp}} \leq 1.$$

The graph corresponding to the right hand side of this inequality is given in Figure 1. The shaded region in this graph represents the area that the above inequality holds, i.e. M_{ij} is nonnegative.

It is apparent from the graph that we must consider the values for $p \in [1, \infty)$. Furthermore, for $p \in [1,2]$ the above inequality always holds and therefore we have

$$\frac{\partial^2 h_{ij}}{\partial {b_1}^2} > 0 \quad \text{and} \quad \frac{\partial^2 h_{ij}}{\partial {b_2}^2} > 0, \quad \text{where} \quad 1 \leq p \leq 2.$$

Hence, h_{ij} is convex in b_1 and b_2 for $p \in [1,2]$. The convexity of SD_f follows from the fact that a sum of convex functions is also a convex function. □

For $p \in (2, +\infty)$ we can argue that in practice there is a good chance for the M_{ij} to be nonnegative. First, although it is possible to have an optimal p value greater than 2 with the ℓ_{bp}-norm this occurs with a p value close to 2, i.e. in a highly Euclidean transportation network.

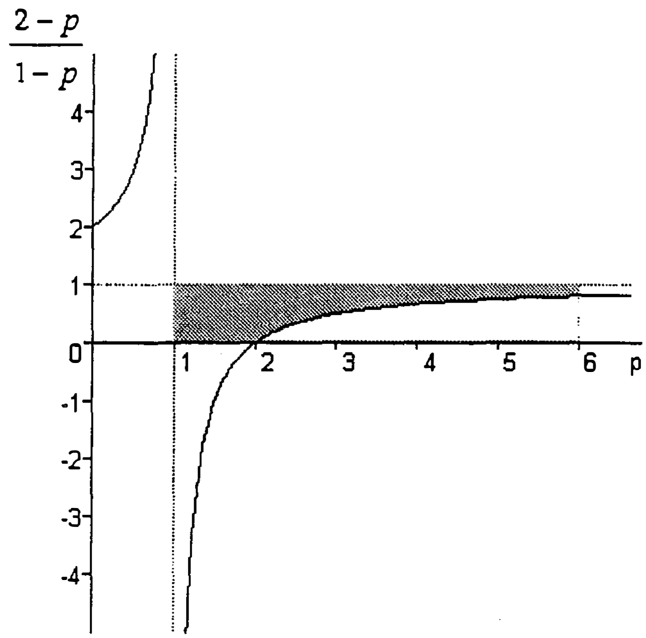


Figure 1: Condition on p

A minimum SD criterion value with a large optimum value of p is closely approximated by another SD criterion value associated with a parameter p value less than 2 with the coordinate axes rotated [21]. Second, for optimal values of the parameters, p, b_1 and b_2, we expect good predictions such that the ratio A_{ij}/ℓ_{bp} is close to 1. Third, since the ℓ_{bp}-norm is a decreasing function of p (Corollary 1), as p increases, ℓ_{bp} will decrease and given that $A_{ij}/\ell_{bp} \leq 1$, where A_{ij} is constant, the A_{ij}/ℓ_{bp} ratio will become closer to 1. It will be more likely to fall into the shaded region. Therefore, for all practical purposes we can assume that SD_f is a convex function of its parameters b_1 and b_2 provided that $p \geq 1$.

Next we examine the behaviour of SD_f in terms of its parameter p.

Property 3 *Consider any term h_{ij} in the sum SD_f as a function of p in the open interval $(0, +\infty)$. There are two cases:*

(i) *If $A_{ij} \le max\{|a_{i1} - a_{j1}|, |a_{i2} - a_{j2}|\} = \ell_\infty(a_i, a_j)$ then h_{ij} is a decreasing strictly convex function of p with a minimum approached asymptotically as $p \to +\infty$.*

(ii) *If $A_{ij} > \ell_\infty(a_i, a_j)$ then h_{ij} is a unimodal function of p with minimum at p^*. Furthermore, h_{ij} is strictly convex over the interval $0 < p \le \mu$ and strictly concave for $\mu \le p$ where μ is the inflection point such that*

$$\frac{\partial^2 h_{ij}(b_1, b_2, \mu)}{\partial p^2} = 0, \quad \mu > p^*.$$

Proof First we obtain the following derivatives with respect to p. Let A and ℓ_{bp} be the actual and the predicted distance respectively for the pair of points (a_i, a_j). Then

$$\begin{aligned}
\ell'_{bp} &= \ell_{bp} K(p) \quad \text{where} \quad K(p) = -\frac{ln(\ell_{bp})}{p} + \frac{\sum_{k=1}^{2} b_k |a_{ik} - a_{jk}|^p \, ln(|a_{ik} - a_{jk}|)}{p(\ell_{bp})^p}, \\
\ell''_{bp} &= \ell_{bp}[K^2(p) + K'(p)], \\
h'_{ij} &= -\frac{2}{A}(A - \ell_{bp})\ell'_{bp}, \quad \text{and} \\
h''_{ij} &= \frac{2}{A}[(\ell'_{bp})^2 - (A - \ell_{bp})\ell''_{bp}].
\end{aligned}$$

Also note that from Corollary 1 we have $\ell'_{bp} < 0$ and $\ell''_{bp} > 0$ for $p \in (0, +\infty)$.

(i) If $A \le \ell_\infty$, we should have $A - \ell_{bp} < A - \ell_\infty \le 0$, so that $h'_{ij} < 0$ and $h''_{ij} > 0$ for $p \in (0, +\infty)$. Clearly then, h_{ij} is a decreasing strictly convex function of p with the minimum $(A - \ell_\infty)^2/A$ approached asymptotically as $p \to +\infty$.

(ii) We next consider the case where $A > \ell_\infty$. In order to see the unimodality of h_{ij} recall that $A = \ell_{bp^*}$ and $\ell'_{bp} < 0$ for $p \in (0, +\infty)$. Thus for $p < p^*$ we have $A - \ell_{bp} < 0$ which implies that $h'_{ij} < 0$, and for $p > p^*$ we have $A - \ell_{bp} > 0$ implying that $h'_{ij} > 0$. Therefore h_{ij} is a unimodal function of p with minimum at p^*. Also utilizing Corollary 1 we note that h_{ij} has a positive vertical asymptote at $p = 0$ and a horizontal asymptote approached from below as

$p \to +\infty$. Hence h_{ij} must have an inflection point μ such that $\partial^2 h_{ij}(b_1, b_2, \mu)/\partial p^2 = 0$. We prove the rest of this case in two parts.

We first consider the $p < p^*$ case where the actual distances are overpredicted. It is readily seen that for $A < \ell_{bp^*}$ we have $h''_{ij} > 0$. Therefore if $p < p^*$ then h_{ij} is a decreasing strictly convex positive function.

Secondly, we consider the case $p > p^*$ where the actual distances are underpredicted, i.e., $A > \ell_{bp^*}$. Substituting ℓ'_{bp} and ℓ''_{bp} into the equation for h''_{ij} and equating to zero we obtain

$$h''_{ij} = \frac{2}{A}\left[(\ell_{bp})^2 K^2(p) - (A - \ell_{bp})\ell_{bp}(K^2(p) + K'(p))\right] = 0,$$

and by rearranging we have

$$\frac{A}{\ell_{bp}} = \frac{2K^2(p) + K'(p)}{K^2(p) + K'(p)}.$$

The inflection point μ must solve this equation. Since $\ell''_{bp} > 0$ we have

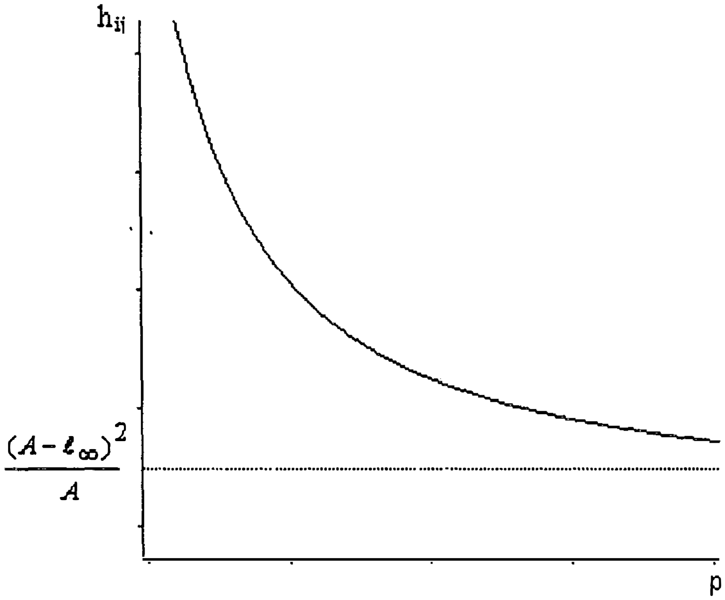
$$2K^2(p) + K'(p) > K^2(p) + K'(p) > 0$$

so that

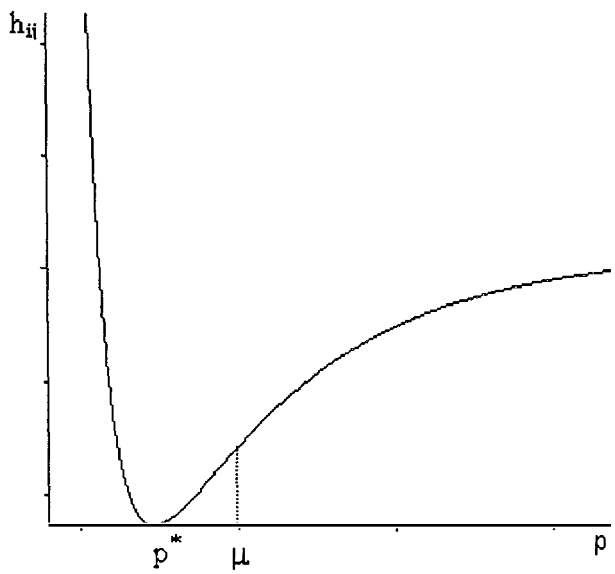
$$\frac{2K^2(p) + K'(p)}{K^2(p) + K'(p)} > 1, \quad \forall p > 0.$$

Hence $A > \ell_{b\mu}$. Finally since $A = \ell_{bp^*}$ and ℓ_{bp} is a decreasing function of p we must have $\mu > p^*$. The general shapes of h_{ij} for varying p are given in Figure 2 for both cases. □

Property 3 shows that h_{ij} is neither a convex nor concave function of p. Since the criterion SD_f is the sum of terms h_{ij} we conclude that SD_f is neither convex nor concave in p. Therefore, in order to determine the best parameter value of p we have to conduct a numerical search over a safe range. The lower bound for such a search range is clearly 1. The upper bound, $\bar{p}$, is chosen by considering the level of rectangularity and nonlinearity in the transportation



i. $A \leq \ell_\infty$



ii. $A > \ell_\infty$

Figure 2: Shape of h_{ij}

network. A study on the directional bias modelled by the ℓ_{bp}-norm [21] reveals that if there exists only one optimal p value greater than 2 this happens in transportation networks which are nearly Euclidean, i.e. p is close to 2, or there exists a high directional nonlinearity, i.e. the b_1 and b_2 values are distinct. If the network has a high level of rectangularity and less pronounced directional nonlinearity then there exists two minimum SD criterion values. One occurs at a p value less than 2 and the other has p value greater than 2 which occurs after rotating the coordinate axes. Therefore, an appropriate upper bound, $\bar{p}$, for the search range of p can be chosen by making a preliminary inspection of the underlying pattern in the network so that the minimum SD criterion value is obtained [21].

4 Computational Procedures

In order to calculate the SD criterion's values and to determine the empirical parameters of the ℓ_{bp}-norm, a *Search-Descent Algorithm* was developed. The best θ and p values were determined by using an incremental *search* procedure and a four-stage incremental search procedure, respectively.

To calculate the best θ value, a two-degree incremental search was conducted on the interval $[0, 90°]$. We chose this interval because the directional bias function $r(\theta)$ of the ℓ_{bp}-norm indicates that the SD_f function is periodic with a period 90° [21]. Therefore a best-fit rotation angle must be encountered in the interval $[0, 90°]$. For each value of θ, parameters b_1, b_2 and p were determined. Once the best θ value, θ_b, was found for the interval, solutions were calculated for $\theta_b - 1$ and $\theta_b + 1$. The best θ value was then chosen from the solutions for $\{\theta_b - 1, \theta_b, \theta_b + 1\}$.

For each value of θ, an initial p value was calculated by conducting a 0.1 increment search on the interval $[1, \bar{p}]$ where $\bar{p} > 2$. Once a p value was found, it was fine-tuned to four decimal places of accuracy. This was accomplished by carrying out three further incremental searches. Each search used an interval centered around the best solution from the previous incremental search with a width equal to twice the increment of the previous search. The increment for the search was one-tenth the size of the increment from the previous incremental search. For example, if the best p value for the first search was 1.6, then the interval width for the second search was $[1.50, 1.70]$ with the increment being 0.01. If the best p value was 1.56 this time, then the interval width for the third search was $[1.550, 1.570]$ with the increment being 0.001.

The best b_1 and b_2 values were calculated by developing a *descent* algorithm given at the end of this section. It is readily seen that if $p \in [1, 2]$ then the SD_f criterion is convex in its parameters b_1 and b_2. Furthermore, for $p > 2$, if the nonnegativity of M_{ij} is satisfied for each pair of points in the sample, then the SD_f criterion is again convex in b_1 and b_2. On the other hand, although SD_f is convex, it is difficult to solve for optimum b_1, b_2 values by using the first order equations. Therefore we employ the descent algorithm which minimizes $SD_f(b_1, b_2)$ for given θ and p values.

Note that the normalized gradient vector $\mathbf{d} = (d_1, d_2)$ is given by

$$\mathbf{d} = \frac{(\nabla SD(b_1), \nabla SD(b_2))}{\| (\nabla SD(b_1), \nabla SD(b_2)) \|}$$

where

$$\nabla SD(b_1) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial h_{ij}}{\partial b_1} \quad \text{and} \quad \nabla SD(b_2) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\partial h_{ij}}{\partial b_2}$$

By considering the insignificance of the time required to evaluate the SD_f we have implemented a Golden Section Method to calculate the step size λ. Clearly SD_f is a unimodal function of (b_1, b_2) for $p \in [1,2]$. For $p \in (2, \bar{p}]$ the convexity of $SD_f(b_1, b_2)$ relies on the nonnegativity of M_{ij}. Therefore when $p > 2$ the Search-Descent Algorithm also checks if the nonnegativity of M_{ij} is satisfied at each iteration point $(p, b_1{}^{(k)}, b_1{}^{(k)})$ for each pair of points (a_i, a_j) contributing an error term in the SD_f.

```
Input:          Sample Data (a_i1, a_i2), i = 1,...,n;
                                A_ij, i = 1,...,n-1, j = 1,...,n.
                Parameters θ and p
                Termination Criteria ε_λ, ε_d.
Output:         Optimum b_1, b_2 and SD criterion value.
Initialize      Set b_1^(1), b_2^(1). Assign a large number to SD^(0).
Iteration       k = 1

Compute     SD^(1)(b_1^(1), b_2^(1))
while       (SD^(k-1) - SD^(k))/SD^(k) > ε_d      do begin
          Compute Normalized Gradients   (d_1^(1), d_2^(1))
          Find λ^(k) such that
             MIN_{λ^(k)≥0} SDλ = SD(b_1^(k) - λ^(k) d_1^(k), b_2^(k) - λ^(k) d_2^(k))
             by using Golden Section Method with ε_λ
          Compute new parameters
             b_1^(k+1) = b_1^(k) - λ^(k) d_1^(k)
             b_2^(k+1) = b_2^(k) - λ^(k) d_2^(k)
          Evaluate   SD^(k+1)(b_1^(k+1), b_2^(k+1))
          k = k + 1
                                                    end.
```

Pseudocode for Descent Algorithm

5 Application Results and Conclusions

In order to model the parameter values of the ℓ_{bp}-norm we used the sample data from seventeen geographic regions presented by Love and Walker [15]. The sample data for each geographic region includes 15 points [locations] based on random selection of *point coordinates* on the map and 105 actual distances corresponding to these points. We applied the Search-Descent Algorithm by using the termination criteria $\varepsilon_\lambda = \varepsilon_d = 0.001$. The search range for parameter p was taken as $[1, 6]$. The reason for choosing a large search interval for p is to find both sets of parameters, if two sets exist, with minimum SD criterion values.

We present the best parameter and SD criterion values for the ℓ_{bp}-norm applied to seventeen regions in Table 1. The percent difference between the criterion values of two ℓ_{bp}-norm best fits, SD_1 and SD_2, are also reported in Table 1. Furthermore the SD $vs.$ θ plots for the regions can be found in the Appendix.

The plots of the b_1/b_2 and the p values for the best parameter values are given in Figures 3 and 4, respectively. The corresponding b_1/b_2 values for the regions can be found in Table 2. $\Delta\tau$, which indicates the existence of dominant directional nonlinearity in a transportation network, is also reported for the regions in Table 2. $\Delta\tau$ is defined as $|\tau_1 - \tau_2|$ where $\tau = max\{b_1, b_2\}/min\{b_1, b_2\}$ and τ_1, τ_2 are for SD_1 and SD_2, respectively. $\Delta\tau$ is also utilized to explain the difference between SD_1 and SD_2 [21].

For comparison purposes we list the minimum SD criterion values for both the ℓ_{bp}-norm (from Table 1) and the weighted ℓ_p-norm where $p \in [1, 2]$ in Table 3. Observe that except in NewYork State where SD has the same value for both norms, the ℓ_{bp}-norm always gives a lower SD criterion value implying that better predictions are obtained. However, the gain in accuracy varies among the regions. The variation is attributed to the underlying pattern of the transportation network. Conclusions can be summarized as follows:

1. In Table 1 we observe that for regions 1, 2, 6, 8, 9, 13, 16, and 17 the difference between the two minimum SD values for the ℓ_{bp}-norm, SD_1 and SD_2, is quite low. A close inspection

		MIN I					MIN II					
No.	Region	θ	b_1	b_2	p	SD_1	θ	b_1	b_2	p	SD_2	$\Delta\%$
1	Australia	42	1.4434	1.4443	2.1894	1158.05	86	1.1945	1.2987	1.7996	1117.89	3.59
2	BC Province	22	2.5828	2.4312	2.7555	1023.19	66	1.3415	1.4859	1.5635	1015.41	0.77
3	Canada	45	1.9325	1.3500	2.2317	497.01	–	–	–	–	–	–
4	France	76	1.1529	1.0819	1.8576	78.87	–	–	–	–	–	–
5	Great Britain	0	1.1116	1.3925	2.0352	172.34	–	–	–	–	–	–
6	NY State	40	1.5710	1.5630	2.6097	164.10	86	1.1303	1.1285	1.5841	159.80	2.69
7	Pennsylvania	10	1.0751	1.1629	1.6589	104.17	44	1.2250	1.4812	2.3892	95.11	9.53
8	United States	0	1.1358	1.1748	1.6956	336.54	42	1.4202	1.4729	2.4485	348.08	3.43
9	Brussels	3	1.2224	1.2020	2.1787	3.60	47	1.0696	1.1177	1.7969	3.47	3.77
10	London Central	26	1.2366	1.6162	2.2708	15.01	–	–	–	–	–	–
11	London North	14	1.0930	1.2953	1.7901	1.36	52	1.3707	1.4841	2.5324	1.71	26.11
12	Los Angeles	0	1.1011	1.3799	1.7750	13.18	54	1.7276	1.5204	2.7420	14.93	13.31
13	NY City	6	1.1673	1.2439	1.7539	13.29	52	1.5099	1.4367	2.3776	13.38	0.70
14	Paris	40	1.3634	1.1440	2.2734	5.84	–	–	–	–	–	–
15	Sydney	8	1.3675	1.1521	1.5571	1.10	54	2.1130	2.1541	3.0007	1.36	23.36
16	Tokyo	13	1.3465	1.3226	2.2249	2.28	57	1.1988	1.1670	1.8260	2.28	0.00
17	Toronto	42	4.1261	4.2427	5.2215	5.06	87	1.0581	1.0275	1.2009	5.06	0.12

Table 1: Best Parameter Values

		MIN I	MIN II	
No.	Region	b_1/b_2	b_1/b_2	$\Delta\tau$
1	Australia	0.9994	0.9198	0.0866
2	BC Province	1.0624	0.9028	0.0453
3	Canada	1.4315	–	–
4	France	1.0656	–	–
5	Great Britain	0.7983	–	–
6	NY State	1.0051	1.0016	0.0035
7	Pennsylvania	0.9245	0.8270	0.1275
8	United States	0.9668	0.9642	0.0028
9	Brussels	1.0170	0.9570	0.0280
10	London Central	0.7651	–	–
11	London North	0.8438	0.9236	0.1024
12	Los Angeles	0.7980	1.1363	0.1169
13	NY City	0.9384	1.0510	0.0147
14	Paris	1.1918	–	–
15	Sydney	1.1870	0.9809	0.1675
16	Tokyo	1.0181	1.0272	0.0092
17	Toronto	0.9725	1.0298	0.0016

Table 2: b_1/b_2 and $\Delta\tau$ Values

No.	Region	SD for ℓ_{bp}	SD for ℓ_p	$\Delta(SD_f)\%$
1	Australia	1117.89	1163.59	3.93
2	BC Province	1015.41	1038.72	2.24
3	Canada	496.88	565.61	12.15
4	France	78.86	92.32	14.58
5	Great Britain	172.34	219.42	21.46
6	NY State	159.80	159.80	0.00
7	Pennsylvania	95.11	107.06	11.16
8	United States	336.53	342.68	1.79
9	Brussels	3.47	3.55	2.25
10	London Central	15.01	16.53	9.20
11	London North	1.36	1.78	23.60
12	Los Angeles	13.17	15.50	15.03
13	NY City	13.29	13.58	2.14
14	Paris	5.84	6.52	10.43
15	Sydney	1.10	1.35	18.52
16	Tokyo	2.28	2.30	0.87
17	Toronto	5.06	5.10	0.78

Table 3: Comparison of ℓ_{bp}-norm and ℓ_p-norm

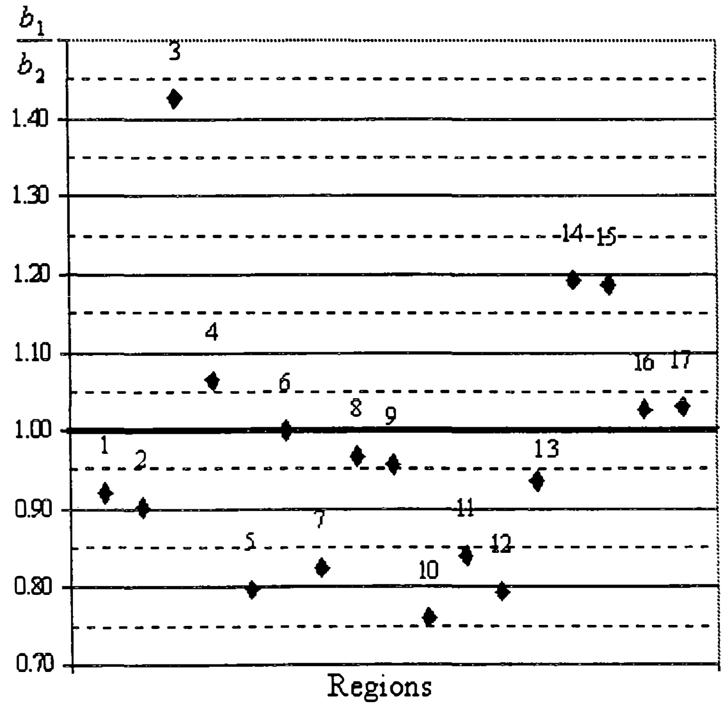


Figure 3: Best-fit b_1/b_2 Plots

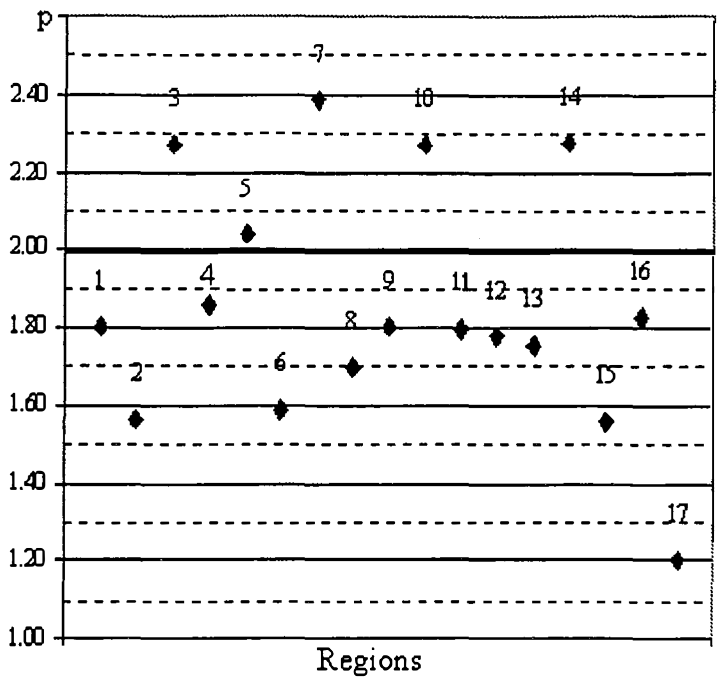


Figure 4: Best-Fit p Plots

of Figures 3 and 4 reveals that the above eight regions have relatively high rectangularity, i.e. p values are not close to 2, and have low directional nonlinearity, i.e. the b_1/b_2 values are close to 1. Low $\Delta\tau$ values show that the nonlinearity in these networks does not have a particular direction. We can say that under these conditions the ℓ_{bp}-norm converges to the weighted ℓ_p-norm. Therefore, similar to the weighted ℓ_p-norm, we have two minimum SD values such that one minimum has an optimum p value where $p \in (1, 2)$ and the other has an optimum p value where $p \in (2, +\infty)$. Moreover, if the b_1/b_2 ratio equals 1 regardless of the axis rotation, we would see these two minimums exactly 45° apart. In the regions mentioned above, since the b_1 and b_2 values are not exactly equal, there is a minor deviation from 45°. In Table 3, we see that the gained accuracy over the weighted ℓ_p-norm is low for these regions. This can again be explained by the similarity of the two norms under the mentioned rectangularity and nonlinearity conditions.

2. In regions 4 and 5 the SD_f function attains its minimum only once. In Figure 4 we see that these regions, France and Great Britain, are highly Euclidean, i.e. p is close to 2. Since the ℓ_2-norm is invariant under axis rotation, in Euclidean transportation networks the variation in $SD(\theta)$ attributed to p is low. Therefore the variation in $SD(\theta)$ is influenced more by the nonlinearity in the transportation network. The study of the directional bias function [21] shows that in such networks $SD(\theta)$ has two bottoms 90° apart. Therefore, for regions 4 and 5, only one minimum is encountered in the $[0, 90°]$ interval. Moreover, since the ℓ_{bp}-norm models the nonlinearity in the transportation network explicitly, it produces more accurate estimations of actual distances. We obtain 14.58% and 21.46% improvements over the best SD values of the weighted ℓ_p-norm for France and Great Britain, respectively.

3. In regions 3, 7, 10, 11, 12, 14, and 15, $SD(\theta)$ either has only one minimum or two minimums with considerably different criterion values.

    i. For regions 3, 10, and 14, in which there is only one minimum SD, the b_1/b_2 values are far away from 1, implying the existence of dominant directional nonlinearity. Although these regions are relatively rectangular the nonlinearity is clearly a dominating characteristic of the regions. Therefore when the ℓ_{bp}-norm is fitted we observe only one minimum $SD(\theta)$ value in the $[0, 90°]$ interval for these three regions. For Canada, London Central and Paris, using the ℓ_{bp}-norm provides significant improvements in the SD criterion values ; 12.15%, 9.20% and 10.43%, respectively.

ii. For the rest of the regions, in which there are two minimum SD values, it is difficult to differentiate a dominating characteristic, i.e., nonlinearity or rectangularity. While the road networks are highly rectangular, relatively high values of $\Delta\tau$ indicates the existence of directional nonlinearity in the transportation networks. In these regions one of the minimum SD criteria gives the optimum parameter values for the region. The corresponding axis rotation provides the best alignment of the coordinate axes with the underlying directional nonlinearity and rectangularity. The percent difference between the two minimum SD criterion values is quite high; 9.53% 26.11%, 13.31% and 23.36% for regions 7, 11, 12 and 15, respectively. However, the inherent nonlinearity in the regions is well captured by the ℓ_{bp}-norm. As a consequence we see, once again, considerably better distance predictions. For these regions, Pennsylvania, London North, Los Angeles and Sydney, the improvements over the SD values given by the weighted ℓ_p-norm are 11.16%, 23.60%, 15.03% and 18.52%, respectively.

As noted earlier, for $p > 2$ the convexity of the SD criterion depends on the nonnegativity of the M_{ij} (shaded region in Figure 1). In our Search-Descent Algorithm we check if this condition is satisfied at any iteration with $p > 2$. Note that since we have 15 locations in a sample for a region, each iteration involves 105 violation checks. We present the results in Table 4. We report the percentage of total number of violations in total number of checks performed in the Search-Descent Algorithm and the maximum number of violations to the condition observed at any iteration. For example, for Canada there exists an iteration in which for 6 pairs of locations out of 105 the condition is violated. We observe that percentages of violations are very low, below 1% in all regions. For Brussels and NY City, although the maximum number of violations in an iteration is 1, the percentage is very close to zero, and for France, Great Britain, United States, London North and Tokyo, no violations are observed. We note that no violations are observed in the vicinity of the parameter values reported in Table 1. We also like to note that the maximum number of violations are observed for iterations with $p \geq 5$. This is an expected result considering the narrowing shaded region for higher values of p in Figure 1. Finally the results given in Table 4 provide additional evidence concerning the

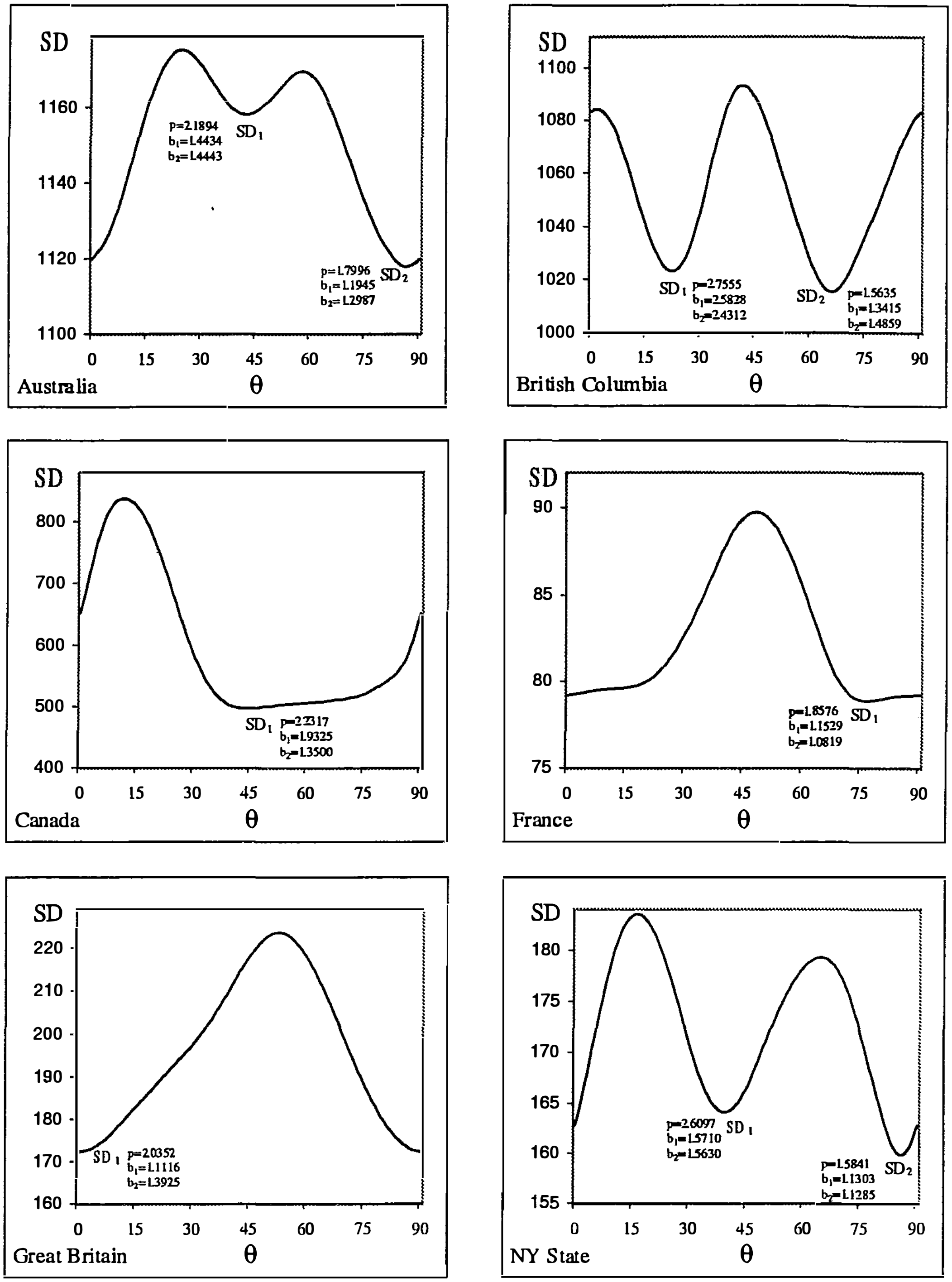
No.	Region	Violations Percentage	Maximum Violations
1	Australia	0.16	3
2	BC Province	0.32	7
3	Canada	0.27	6
4	France	0.00	0
5	Great Britain	0.00	0
6	NY State	0.05	4
7	Pennsylvania	0.23	5
8	United States	0.00	0
9	Brussels	0.00	1
10	London Central	0.38	2
11	London North	0.00	0
12	Los Angeles	0.56	4
13	NY City	0.00	1
14	Paris	0.51	4
15	Sydney	0.01	1
16	Tokyo	0.00	0
17	Toronto	0.17	8

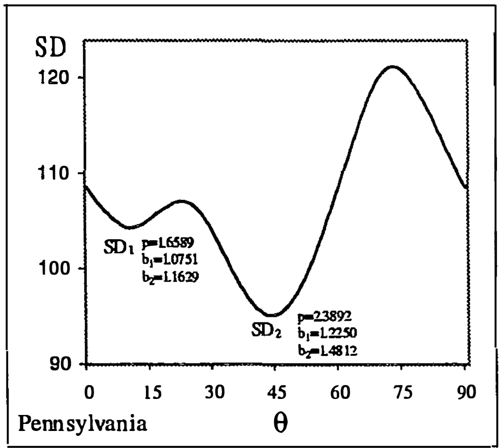
Table 4: Convexity Check Results

validity of the assumption that for all practical purposes the SD_f is a convex function in its parameters b_1 and b_2 for $p \geq 1$.

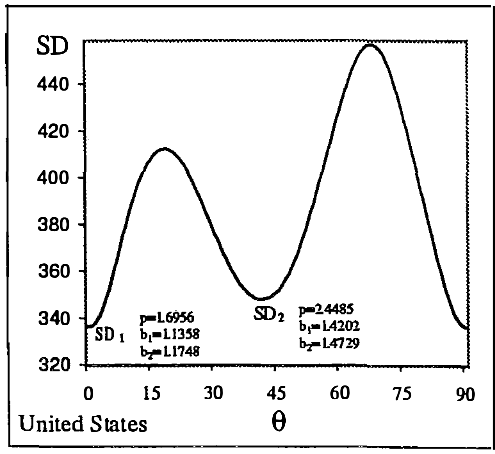
A Appendix

SD vs. θ Plots

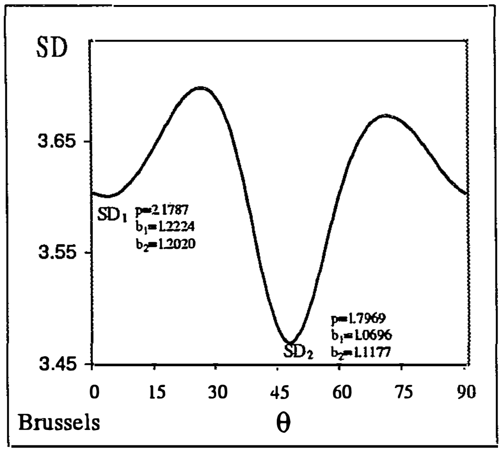




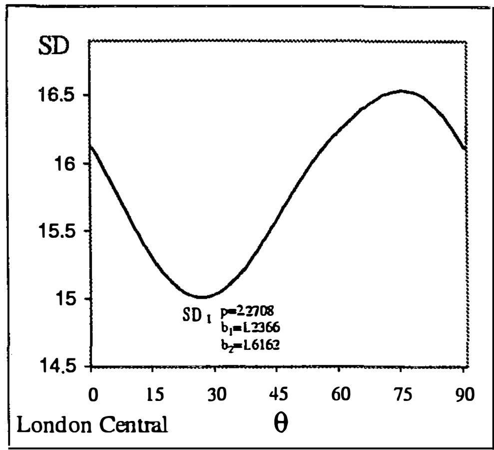
SD
120
110
100
90
SD1 p=1.6589
b1=1.0751
b2=1.1629
SD2 p=2.3892
b1=1.2250
b2=1.4812
0 15 30 45 60 75 90
Pennsylvania
θ



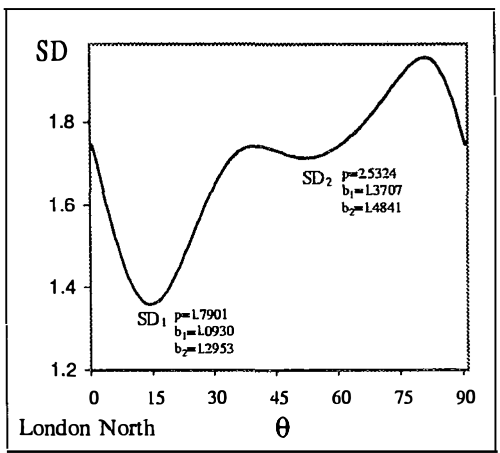
SD
440
420
400
380
360
340
320
SD 1 p=1.6956
b1=1.1358
b2=1.1748
SD2 p=2.4485
b1=1.4202
b2=1.4729
0 15 30 45 60 75 90
United States
θ



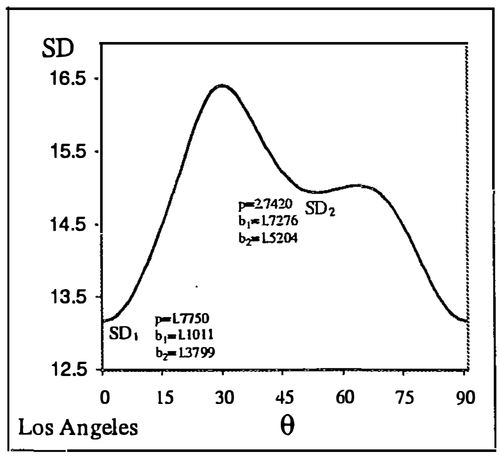
SD
3.65
3.55
3.45
SD1 p=2.1787
b1=1.2224
b2=1.2020
SD2 p=1.7969
b1=1.0696
b2=1.1177
0 15 30 45 60 75 90
Brussels
θ



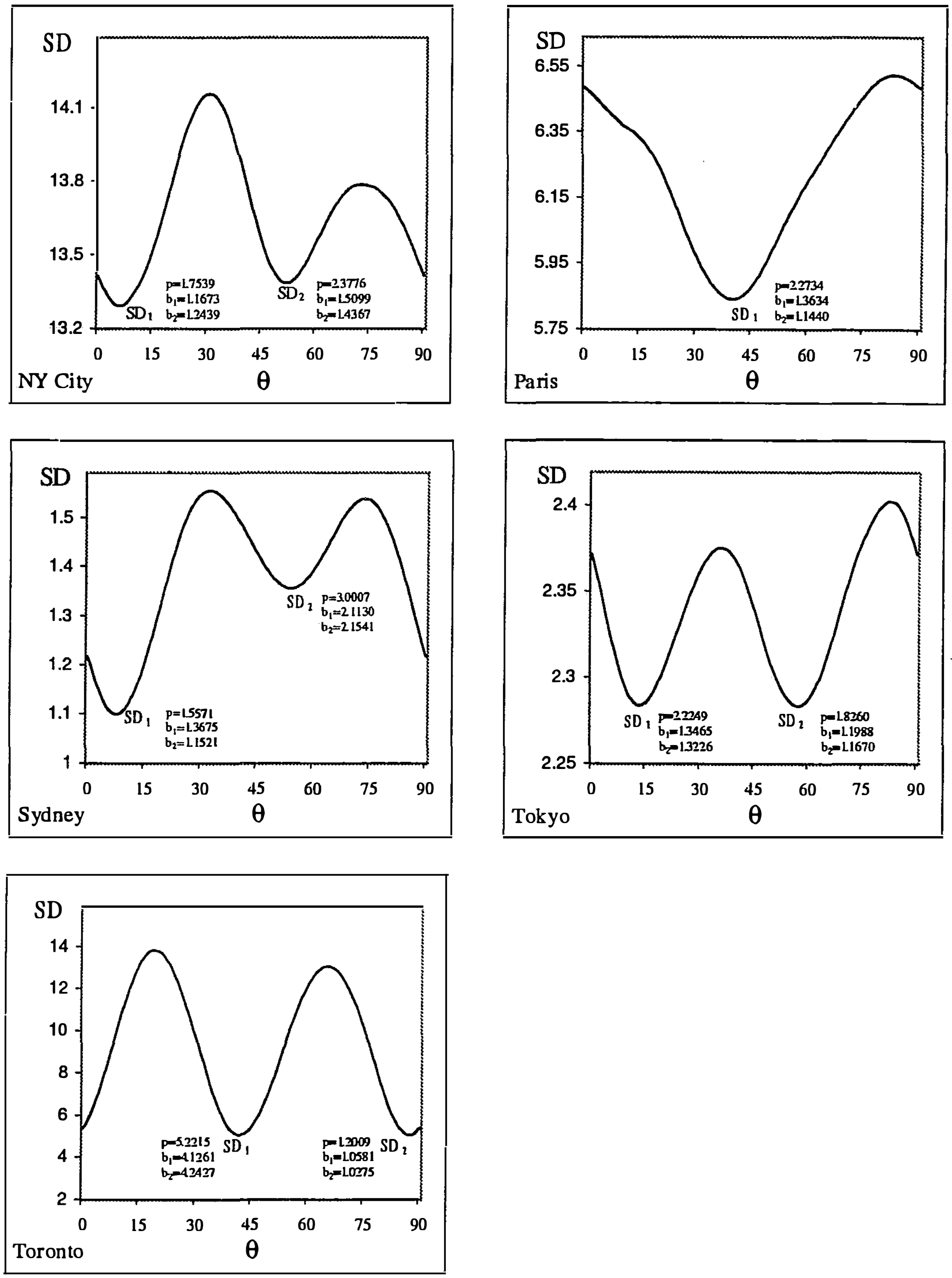
SD
16.5
16
15.5
15
14.5
SD 1 p=2.2708
b1=1.2366
b2=1.6162
0 15 30 45 60 75 90
London Central
θ



SD
1.8
1.6
1.4
1.2
SD2 p=2.5324
b1=1.3707
b2=1.4841
SD1 p=1.7901
b1=1.0930
b2=1.2953
0 15 30 45 60 75 90
London North
θ



SD
16.5
15.5
14.5
13.5
12.5
p=2.7420 SD2
b1=1.7276
b2=1.5204
SD1 p=1.7750
b1=1.1011
b2=1.3799
0 15 30 45 60 75 90
Los Angeles
θ


SD
14.1
13.8
13.5
13.2
0 15 30 45 60 75 90
NY City
θ
SD1
p=1.7539
b1=1.1673
b2=1.2439
SD2
p=2.3776
b1=1.5099
b2=1.4367
SD
6.55
6.35
6.15
5.95
5.75
0 15 30 45 60 75 90
Paris
θ
SD1
p=2.2734
b1=1.3634
b2=1.1440
SD
1.5
1.4
1.3
1.2
1.1
1
0 15 30 45 60 75 90
Sydney
θ
SD1
p=1.5571
b1=1.3675
b2=1.1521
SD2
p=3.0007
b1=2.1130
b2=2.1541
SD
2.4
2.35
2.3
2.25
0 15 30 45 60 75 90
Tokyo
θ
SD1
p=2.2249
b1=1.3465
b2=1.3226
SD2
p=1.8260
b1=1.1988
b2=1.1670
SD
14
12
10
8
6
4
2
0 15 30 45 60 75 90
Toronto
θ
p=5.2215
b1=4.1261
b2=4.2427
SD1
p=1.2009
b1=1.0581
b2=1.0275
SD2

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