VERIFICATION OF THE THEORETICAL CONSISTENCY OF A DIFFERENTIAL GAME IN ADVERTISING

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A differential game model of dynamic competitive advertising decisions is presented and compared with a set of elasticity relationships that were developed to describe the dynamic interrelationships between firms engaged in advertising competition.

Practitioners and researchers in management and economics have often emphasized the necessity of approaching problems in these disciplines in their most realistic settings. Oscar Morgenstern[1] and Ira Horowitz[2] have appealed to analysts to refrain from being negligent in not considering the two critical aspects of competition and time.

However, previous to the development of differential games by Rufas Isaacs[3] in 1954, there existed no method that could naturally model and solve dynamic competitive problems. Although there has been fairly steady interest in differential games during this time, almost all of the effort expended towards the development

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This research was partially supported by the National Research Council of Canada (Grant Number A9270).

1 Oskar Morgenstern, "Thirteen Critical Points in Contemporary Economic Theory," Journal of Economic Literature, X (December 1972) pp. 1163-1189. (References within this passage have been omitted.)

2 Ira Horowitz, Decision Making and the Theory of the Firm (New York: Holt, Rinehart and Winston, Inc., 1970) p. 333.

3 Rufas Isaacs, "Differential Games I, II, III, IV," RAND Corporation Research Memos RM-1391, 1399, 1411, 1468 (1954-1956).

of solution techniques and real-world applications has been in the fields of electrical and aerospace engineering.

The discipline of differential games is an appropriate answer to Morgenstern's and Horowitz's pleas. In fact, the natural form of a differential game explicitly considers the two critical aspects of an economic model that were mentioned earlier -- competition and dynamics.

In management applications of differential games, each competitor attempts to optimize his criterion of performance by manipulating a set of inputs to a common dynamic economic system. There is the direct recognition of the necessity of operating in an environment in which some of the determinants of an entity's behaviour are controlled by opponents. The dynamic reaction of the market system to changes in the values of the variables controllable by the firms and those that describe the state of the environment must be defined and modelled by each of the competitors. Consequently, rather than ignoring these competitive and dynamic components of economic problems, differential games thrusts the analysis of these elements on the investigator.

It is intended that the construction and manipulation of the following model provides marketing managers and academics with a tool through which a more complete and satisfactory understanding of competitive dynamic marketing systems is obtainable.

1. THE DIFFERENTIAL GAME

The System Dynamics

As stated previously, since advertising effects develop over time, the dynamic aspects of the market environment must be part of any worthwhile model of advertising budgeting.

The time varying aspects that must be treated in the model include the sales response to advertising expenditures, the carry-over effect of past advertising on current sales, and the possibility of diminishing returns to cumulative advertising expenditures. The form of the equations used in this model to describe the firm's sales response to advertising over time is based on the work that was done originally by Vidale and Wolfe.[4] Their empirically justified model relates the rate of change of sales over time to the three factors mentioned above. Although this model does not include all of the determining factors of an advertising decision, it does combine several of the most important of the market factors that influence advertising decisions in a model that is quite realistic while remaining relatively simple mathematically.

A differential equation is used to represent the time rate of change of sales as depending on the carryover effect, conversely represented here by a decay factor, a; the current sales response to advertising, b; and the saturation level of the market, M-x(t):

$$\frac{dx(t)}{dt} = \dot{x}(t) = -ax(t) + bu(t)\ \frac{M-x(t)}{M} \tag{1}$$

Sales at time t is represented by x(t) while advertising expenditures is symbolized by u(t). This states that the change in sales over time increases in response to the effectiveness of advertising expenditures and is decreased by a decay term.[5] The differential equation(s) that

4 M.L. Vidale and H.B. Wolfe, "An Operations-Research Study of Sales Response to Advertising," Operations Research, V (June 1957) pp. 370-81.

5 For a concise discussion of this model, see Philip Kotler, Marketing Decision Making: A Model Building Approach (New York: Holt, Rinehart and Winston, 1971) pp. 131-33.

describe the movement of the system over time are often referred to as the system dynamics.

A drawback of the above model of the environment is that it ignores a critical determinant of market behaviour -- the effect of competition. Economic and management decisions must be made while considering the influence of competition and other external market factors. The integration of competition into the advertising model must accomplish two things:

(1) represent the dynamic sales response of each of the market competitors; and

(2) represent the effect of all competition on each of the competitors.

While the first of these factors forces the time varying description of the competitors, the second demands the explicit recognition of the interactive effects of all market factors.

It will be assumed in the current model that each of the market competitors has essentially the same form for his dynamic equations. However, greatly different competitive characteristics can be portrayed by variation of the values of the sales decay and sales response parameters of the model.

A simplified, but fairly realistic, model of the system dynamics of a duopoly conflict situation is presented by the following extension of the monopoly model of Vidale and Wolfe:

$$\dot{x}_1(t) = -a_1 x_1(t) + b_1 u_1(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\} \tag{2}$$

$$\dot{x}_2(t) = -a_2 x_2(t) + b_2 u_2(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\}$$

where: $x_i(t)$ = sales for brand i at time t

$u_i(t)$ = advertising expenditures for brand i at time t

a_i = the sales decay parameter

b_i = the sales response parameter

M = the total potential market size.

This model is different from the previously presented monopoly model of (1) in that there is an expression for each of the competitors' sales response and in each there is a recognition of the competition's effect. The direct effect of competitive pressure is that as the sales to all other market parties increase, the remaining size of the unsold market decreases and this in turn diminishes the sales effectiveness of succeeding advertising expenditures for the firm. Also, as the total market sales of the duopolists, $x_1(t)+x_2(t)$, approaches the potential market size, M, the effectiveness of advertising expenditures approaches zero. The combination of the sales response parameter with the saturation effect produces the diminishing returns to cumulative advertising expenditures that was mentioned previously.

The Performance Indices

Although the construction of a model of a system is a significantly valuable achievement, a detailed set of operating objectives is necessary in order to use the model for planning. The establishment of the objectives for a firm, or for one of the firm's products, is not a simple task for a manager. Neither is it a task that can be done once and then set aside. The reassessment of goals in response to competitive pressure is a strategic managerial activity.

The goals for a company change over time. As products pass through their life cycle, the goals prescribed for them evolve.[6] Also, the objectives of competing firms do not necessarily take the same form. For example, while one competitor might be solely interested in maximizing his total profits over the coming planning year, his opponent could be concentrating on maximizing his market share by the end of the year. These different objectives could generate significantly different competitive advertising strategies.

These distinctly different performance criteria can be easily and clearly stated. However, it is often the case that a manager might be concerned with doing well on both of these measures. He has a great desire to maximize profits, but he also feels the importance of attaining as dominant a market share as possible. The result is more of a satisficing position with respect to each of the criteria rather than a clearly defined maximization problem. Such a goal orientation generates what will be called a "tempered" performance index. For example, the decision maker's desire to maximize profits must be tempered with his desire to maximize the product's market share by the terminal time of the planning horizon. The resulting strategy would provide for satisfactory performance for each of the goals, but maximization of neither. Rather, the total tempered performance index is maximized.

Although there are many possible objectives that could be used in the advertising model, variants of the above will be the main concentration of this research. The tempered performance index is mathematically represented as follows:

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6 K.R. Deal, "Brand Life Cycle Analysis Using Differential Games," Proceedings of the Annual Conference of the Canadian Association of Administrative Sciences. Edmonton, Alberta, June 2-3, 1975.

$$\max J_i = w_i \frac{x_i(t_f)}{x_i(t_f)+x_j(t_f)} + \int_{t_0}^{t_f} \{c_i x_i(t) - u_i(t)\}dt \tag{3}$$

where: i refers to one of the firms and j to the other

c_i = the net revenue coefficient

w_i = the weighting factor for the performance index

t_0 = initial time of the planning horizon

t_f = terminal time of the planning horizon

To allow for each of the duopolists to have different effective objectives, but to maintain the same basic form for the performance indices, a weighting factor, w_i, is provided as a coefficient on the market share term. The purpose of this is to allow one of the firms to maintain a performance index that places primary emphasis on profits while the other puts major importance on the terminal market share. In the analysis of the model, the weighting of the parts of the performance indices is allowed to vary from the case of total profit orientation to that where primary interest centers on the terminal market share.

The provision for handling the effect of diminishing returns to cumulative advertising expenditures was presented earlier in this research. This roughly corresponds to the phenomenon where advertisements that appear during the initial part of the planning period have a greater marginal effect on sales than do those that have been preceded by a longer history of advertisements. The system dynamics provided for the integration of this concept into the model. A further assumption will be made that there also exists diminishing returns to the scale of advertising expenditures.

Stated very simply, this would imply that if the scale (or magnitude) of advertising were doubled at all points in time, the resulting benefit to the firm would be less than twice as great. This effect is handled in the current research by squaring the advertising term in the performance index:

$$\max_{u_i} J_i = w_i \frac{x_i(t_f)}{x_i(t_f)+x_j(t_f)} + \int_{t_0}^{t_f} \{c_i x_i(t) - u_i^2(t)\}dt \tag{4}$$

where: i represents one player and j the opponent.

There can certainly be arguments over the representation of the scale effects on advertising expenditures. The choice was made to express this through the use of the squared exponent on the advertising term of the performance indices. To generalize the effects of scale, the exponent on the advertising term in the performance index could be expressed as $f_i(u_i(t))$. The case of equation (4) will be taken in this research as a possible expression of scale effects. This arrangement will allow for the solution of the game using an alteration of a solution technique that has been developed for linear-quadratic differential games.[7] The case in which the advertising term is raised only to the first power in the performance indices as in (3), has been treated in an earlier work.[8]

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7 A.W. Starr, "Non-Zero-Sum Differential Games: Concepts and Models," Harvard University, Division of Engineering and Applied Physics, Technical Report No. TR590, 1969.

8 K.R. Deal, "A Differential Games Solution to the Problem of Determining the Optimal Timing of Advertising Expenditures," unpublished Ph.D. Dissertation, State University of New York at Buffalo, 1975.

Synthesis of the Advertising Model

Collecting the parts of the advertising model described above provides the following:

$$\max_{u_1} J_1 = w_1 \frac{x_1(t_f)}{x_1(t_f) + x_2(t_f)} + \int_{t_0}^{t_f} \{c_1 x_1(t) - u_1^2(t)\}\, dt \qquad (5)$$

$$\max_{u_2} J_2 = w_2 \frac{x_2(t_f)}{x_1(t_f) + x_2(t_f)} + \int_{t_0}^{t_f} \{c_2 x_2(t) - u_2^2(t)\}\, dt \qquad (6)$$

subject to:

$$\dot{x}_1(t) = -a_1 x_1(t) + b_1 u_1(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\} \qquad (7)$$

$$\dot{x}_2(t) = -a_2 x_2(t) + b_2 u_2(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\} \qquad (8)$$

$x_1(t_0) = x_{10}$

$x_2(t_0) = x_{20}$ ⟵ initial conditions (9)

$u_1(t) \geqq 0$

$u_2(t) \geqq 0$ ⟵ constraints (10)

$x_1(t) \geqq 0$

$x_2(t) \geqq 0$

$x_1(t) + x_2(t) \leqq M$

The first two expressions represent the performance indices. The two differential equations representing the system dynamics appear next. The initial conditions are then presented, followed by non-negativity constraints on advertising and sales and the constraint that actual sales must not be greater than the potential market size.

Although this representation embodies the most essential ingredients of advertising systems, it is not intended to be an all-encompassing model. Rather, it provides a beginning formulation that hopefully will be extended to cover increasingly realistic marketing cases.

In order to obtain a sensitive but simple beginning model, several construction-motivated assumptions have been necessary. Since the system dynamics are expressed as differential equations and parts of the performance indices are integrals over a time span, it is obvious that the system is being interpreted as operating continuously over time. To coincide with this, it is assumed that the advertising is done in relatively continuous-appearance media such as radio, television, and newspapers. It is further assumed that the brands engage only in advertising competition -- the same price is always charged for the product.

To allow for a simpler formulation of the dynamic equations, two additional assumptions have proven to be helpful. It is assumed that the total size of industry advertising does not influence the size of the potential market. The total potential market size is fixed at $M of sales. Secondly, the assumption is made that advertising affects sales but that the reverse effect, sales directly determining the level of advertising, is non-existent. This problem has been one that has bothered econometricians when constructing distributed lag models of the sales response to advertising.[9] However, since there are situations in which this reverse effect does not appear,[10] it is assumed to be absent from this application.

As stated earlier, the basic framework of a differential game is such that it naturally encompasses the advertising model. The discipline of differential games was developed to model and solve dynamic competitive problems. The essential ingredients are those that were described previously for the duopoly advertising problem. There must be at least two competitors, each of which is intent upon optimizing his performance criterion subject to the laws that describe the movement of the system over time and any constaints and initial or terminal conditions.

Figure 1 on the next page presents a diagram of the framework of the general 2-player differential game. The general model provides a structure that can be altered to fit the specific needs of a problem. A number of assumptions have been utilized in providing for a differential game solution as well as in the construction of the advertising model.

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9 Richard E. Quandt, "Estimating the Effectiveness of Advertising: Some Pitfalls in Econometric Methods," Journal of Marketing Research, I (May 1964) pp. 51-60.

10 Neil E. Beckwith, "Multivariate Analysis of Sales Response of Competing Brands to Advertising," Journal of Marketing Research, IX (May 1972) p. 170.

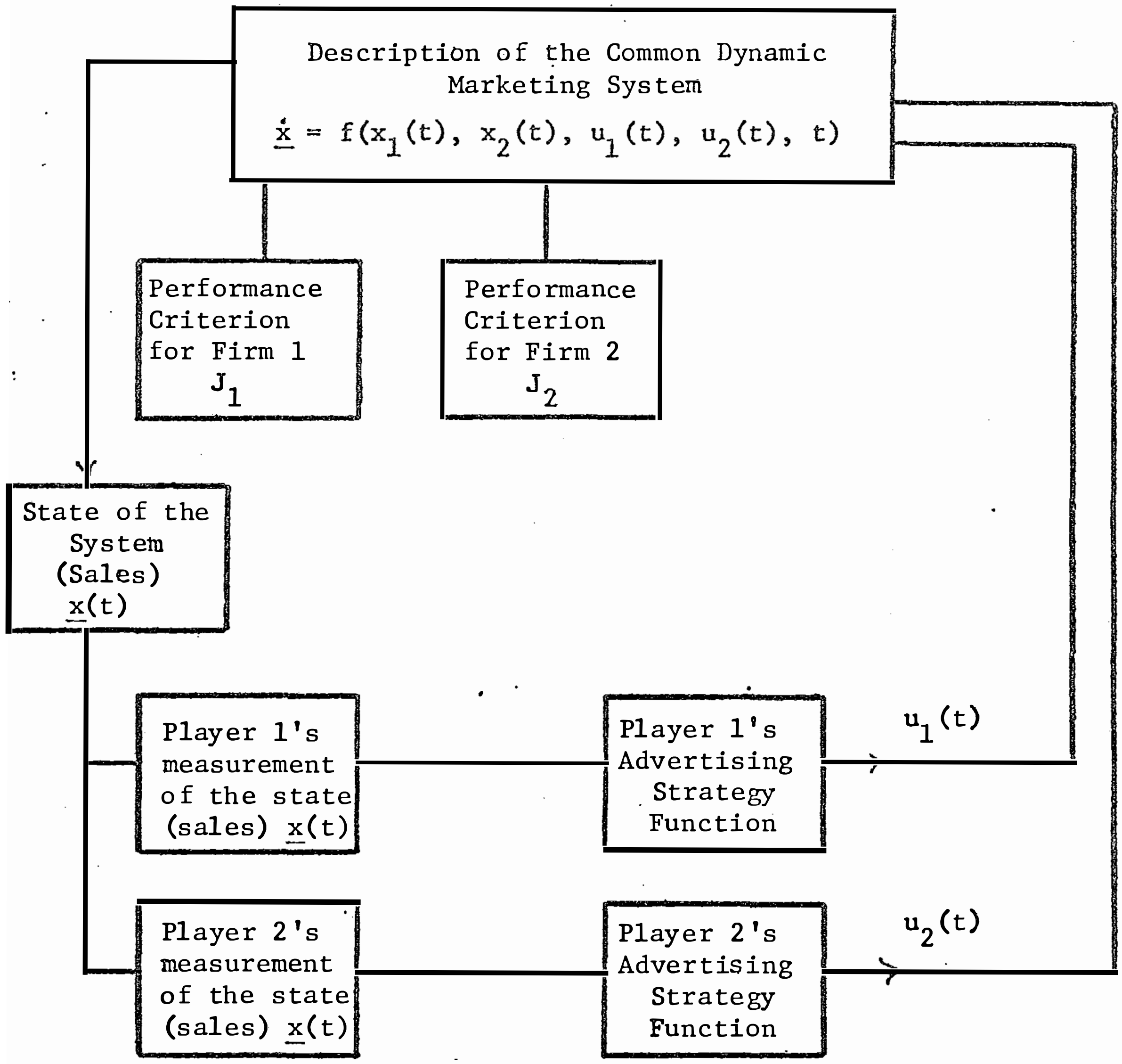

Figure 1

The Two-Firm Advertising Differential Game

It is assumed in this research that each of the firms involved in the conflict acquires perfect knowledge of the model parameters. This initial model makes no provisions for uncertainty. It was felt that it was more important to demonstrate the efficacy of the differential games approach before more complicated representations of the advertising problem are undertaken.

The above model represents a relatively simple formulation of the very complex advertising decision. However, the form of the model is such that it is able to represent characteristics that are of central importance in advertising decision making. The real value of the model lies in its adequacy in an advertising sense and in its approachability for solution by differential games techniques. Once again, it is felt that the main benefit of this research is to establish the promise of differential games in contributing to the fruitful analysis of advertising, marketing and management problems in general -- not in developing an all encompassing advertising model.

2. A NUMERICAL ALGORITHM FOR NASH EQUILIBRIUM SOLUTIONS

The approach used to obtain Nash Equilibrium solutions is to employ an iterative numerical algorithm which is based on the work of Elsner,[11] Holt and Mukundan,[12] and Starr.[13]

11 W.B. Elsner, "Design of a Sub-Optimal Controller for a Class of Systems," unpublished Master's Thesis, Clarkson College of Technology (1966).

12 D. Holt and R. Mukundan, "A Nash Algorithm for a Class of Non-Zero Sum Differential Games," unpublished paper (May 1971).

13 Starr, "Nonzero-sum Differential Games: Concepts and Models."

After specifying the necessary conditions for optimality of the differential game, there are 2 system equations ($\dot{x}$'s) with 2 given initial conditions and 4 costate equations ($\dot{\lambda}$'s) with their 4 terminal conditions and the 2 equations for the control variables (μ's).

The algorithm is initiated by guessing values for each competitor's controls for each discrete instant of time during the planning horizon. The n system equations are functions of the state variables for solution. Since the control variables are now parameters and known, this set of differential equations can be solved forward in time by utilizing a numerical procedure such as the Runge-Kutta-Simpson Fourth Order Method. At each forward step of the solution, the values of $x_i(t)$ are calculated and stored.

When the end of the planning horizon is reached, the $x_i(t_f)$'s are obtained. Also, during the forward pass, the algorithm integrates the performance indices using Simpson's 1/3 Rule. At the terminal time, then, the value of the performance index is known for the guessed values of the control functions.

At this point, numerical values are calculated for the set of boundary conditions that are specified at the terminal time in terms of the costate variables. It is then possible to solve the costate equations backward in time, using the terminal costate conditions calculated at the end of the forward pass as initial conditions.

At each successive time increment of the backward pass, the appropriate values of the costate variables are calculated. Also, a new set of $u_i(t)$'s is found. When back at the initial time, there is a revised time sequence of values for the control functions. These are used in the next forward pass, the objective of which is to evaluate the effect of these new controls on the performance index. A stopping criterion is calculated at the end of each forward pass. If there has not been a significant change in the values of the performance indices since the last iteration, the algorithm is terminated.

When the iterative procedure has been terminated, the solution is checked against the Nash equilibrium property. The control values for each of the players, in turn, are perturbed by a small constant while the opponents' Nash controls are maintained. The performance index of the player who changed from the Nash strategy should not have improved.

Resolution of the Algorithm

Although the area of differential games has existed since 1954, there has not been a great deal of progress made in the development of general solution techniques. In many situations, a heuristic approach is followed to obtain a solution method that conforms to the necessary and sufficient conditions. The method developed here is somewhat representative of this type of approach.

As in the development of all solution algorithms, and especially heuristics, testing must be performed to determine the accuracy of the algorithm in obtaining solutions to the model. In turn, further testing is then executed to resolve the faithfulness of the solution and the model in typifying the actual physical phenomenon.

The validity and usefulness of the algorithm for solving differential games has been justified on two points:

1) that the numerical algorithm exhibits stability and realistic convergence properties while obtaining Nash equilibrium solutions; and

2) that the solutions make logical sense from a marketing standpoint.

Justification for the first of these criteria was obtained through detailed testing of the numerical solution procedure outlined earlier for stability and convergence. These results will not be presented here. A segment of the analysis used to justify the model on the basis of marketing common sense will be presented with the intention of illustrating the results of the model.

A useful way of evaluating a new analytical procedure is to test it on a problem that has a known solution or on one whose solution can be easily hypothesized. The testing of the differential game model was completed partially by using this approach. Fortunately, the basic model presented previously is realistic enough to be rich in vital characteristics of the advertising problem while at the same time being simple enough to allow for speculation of advertising strategies that would follow from the model.

Consequently, four hypotheses, and some special cases, were advanced from the model. The hypotheses are concerned with the main parameters of the model: the sales decay parameter, a_i; the sales response parameter, b_i; total industry sales level, $x_1(t)+x_2(t)$; and the weighting of the performance indicies. The algorithm was then tested against each of the hypothesized solutions. Some of the results will be presented. The hypotheses are as follows.

Hypothesis 1. If both firms are maximizing total profit and have the same parameters and initial conditions, then as both brands' decay constants, a_1 and a_2, decrease in value the competitors can be expected to shift the advertising curve so that more advertising is done initially to take advantage of the carryover effect and relatively less advertising is done towards the end of the period.

Hypothesis 1'. In a profit maximizing situation, if $a_1 < a_2$ then, ceteris paribus, a dollar of advertising for Brand 1 will have a greater long-run effect than will Brand 2's advertising expenditures. The anticipated policy effect would be for firm 1 to advertise more in order to maximize its performance index. This would be particularly true during the beginning of the planning period.

Hypothesis 2. As the sales response constants, b_1 and b_2, decrease in value, with all other parameters identical, the advertising levels decrease for both firms in order that they may maximize their performance indices.

Hypothesis 2'. If $b_1 < b_2$ then, ceteris paribus, player 1 is at a competitive disadvantage. In order to maximize a profit oriented performance criterion, firm 1 would be expected to advertise at a lower rate than would firm 2. (The reason for this is that since firm 1's sales response is lower than firm 2's, there will be less direct benefit to firm 1's advertising expenditures.)

Hypothesis 3. As total industry sales, $x_1(t) + x_2(t)$, apprcaches the total potential market size, M, the immediate effectiveness of each advertising dollar diminishes towards zero. Consequently, as total sales increase, the advertising expenditures would tend to decrease in order to maintain a profit maximizing strategy.

Hypothesis 4. When all other parameters of the models, i.e., aspects of competition, are held constant, as the weights that are placed on the terminal market share part of the performance indices of the two competitors are simultaneously increased, the advertising expenditures pattern changes from that of a profit maximizing firm to that which is mainly concerned with maximizing the terminal market share to the extent that actual losses might be sustained. The nature of the advertising curve under these objectives will change from one that decreases to zero by t_f for profit maximization to one that increases quite sharply towards the terminal time.

Hypothesis 4'. If $w_1 < w_2$, then with all other competitive aspects identical, player 2's advertising will be greater towards the end of the planning horizon and will cause a resulting higher terminal market share for player 2.

To illustrate each hypothesis, the two firms are engaged in four different competitive encounters. The encounters are distinct from each other in the value of the parameter about which that particular hypothesis is concerned. Hypotheses 1, 1', 4 and 4' will be used for illustration.

Hypothesis 1. The parameter values which are used to illustrate Hypothesis 1 are given below.

Parameter Values--Hypothesis 1

Encounter Number	Parameter Values									
	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
1	.01	.01	1.25	1.25	.40	.40	200	200	0	0
2	.07	.07	"	"	"	"	"	"	"	"
3	.10	.10	"	"	"	"	"	"	"	"
4	.20	.20	"	"	"	"	"	"	"	"

The results of the Nash Equilibrium Solution for each encounter as represented by the performance indices is given below.

Performance Indix Values--Hypothesis 1

Encounter Number	Combined Performance Index Brand # 1	Combined Performance Index Brand # 2	Profit Brand # 1	Profit Brand # 2	Market Share Brand # 1	Market Share Brand # 2
1	$762.24	$762.24	$762.24	$762.24	.50	.50
2	576.84	576.84	576.84	576.84	"	"
3	507.40	507.40	507.40	507.40	"	"
4	347.76	347.76	347.76	347.76	"	"

The computer algorithm also provides both numerical and plotter output for the control (advertising) paths and the state (sales) paths for each brand on each encounter. For ease of observation, only the plots will be used to present the results.

The testing of each hypothesis results in two plots -- one for the advertising paths, the other for the sales paths. On each of the plots, eight lines are drawn -- one for each of the two brands on each of the four encounters. The curves are labelled as follows:

iUj = the optimal advertising path for Brand j on encounter i;

iXj = the resulting sales path for Brand j on encounter i.

(Note that the brands' curves for each encounter in the first part of each hypothesis coincide since their parameter values are identical.)

The graphical results of the testing of Hypothesis 1 are presented in Figures 1a and 1b. An explanation of the resulting curves follows the graphs.

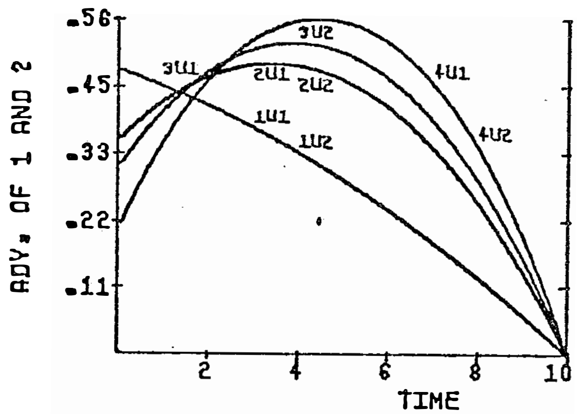

Figure 1a

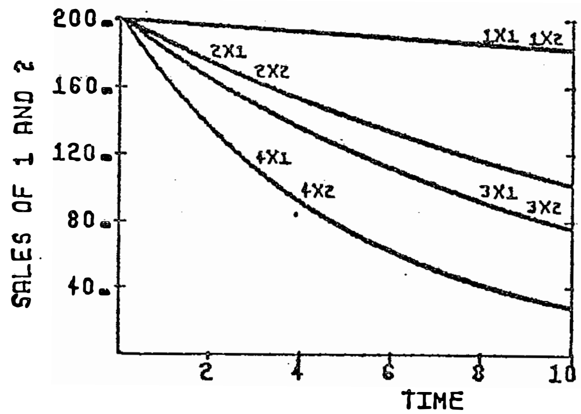

Figure 1b

As the decay constants increase in value for the firms, they find it necessary to increase the magnitude of advertising expenditures as well as to alter the pattern of spending over the time horizon. When the firms are in the "best" situation, i.e., where the decay is small at a_i=.01 and there is very heavy saturation of the market, they can rely substantially on the carryover effect for much of the sales revenue. There simply is not much need for the firms to advertise in this case. The spending pattern here is to have peak advertising during the beginning part of the period and then to decrease expenditures and rely on the carryover effect during the latter part of the period. (Because of the complete profit orientation of the firms, advertising at terminal time is zero.) As can be seen from the diagrams, as the decay constants increase through the values of .07, .10 and .20, the advertising expenditures increase and the peak advertising point occurs later and later during the time horizon. Although the advertising increases in order to maximize profits for each of these later cases, the actual level of sales and profits diminish. The substantial cumulative effect of the decay constant is seen here. The effect of the increase in decay cannot be compensated for (in a profit sense) by increased advertising. The increased advertising has the effect of maximizing profits in successively less profitable situations.

Hypothesis 1'. This hypothesis is concerned with the effect of the variation in one firm's decay parameter . The parameter values are given below.

Parameter Values--Hypothesis 1'

Encounter Number	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
1	.25	.25	1.10	1.10	.40	.40	25	25	.10M	.10M
2	.15	"	"	"	"	"	"	"	"	"
3	.05	"	"	"	"	"	"	"	"	"
4	.01	"	"	"	"	"	"	"	"	"

The optimal values of the performance indices and their constituent parts are given below for each of the four encounters.

Results--Hypothesis 1'

Encounter Number	Combined Performance Index Brand # 1	2	Profit Brand # 1	2	Market Share Brand # 1	2
1	$ 62.32	$ 62.32	$ 37.32	$ 37.32	.50	.50
2	88.01	55.33	55.77	37.57	.64	.36
3	128.39	47.64	87.50	38.53	.82	.18
4	150.51	45.19	106.78	38.92	.87	.13

The optimal advertising paths for each of the four encounters are plotted in Figure 2a while the corresponding sales paths are given in Figure 2b. As the decay constant for firm 1 decreases in

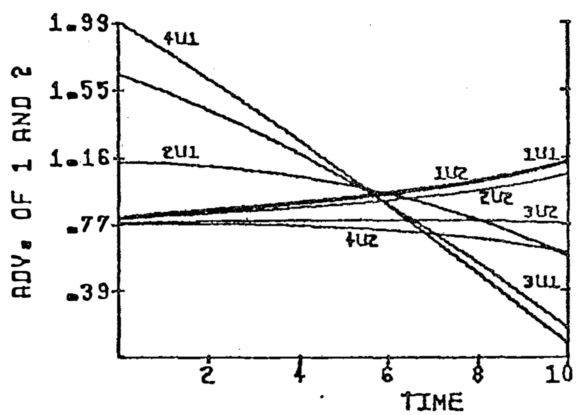

Figure 2a

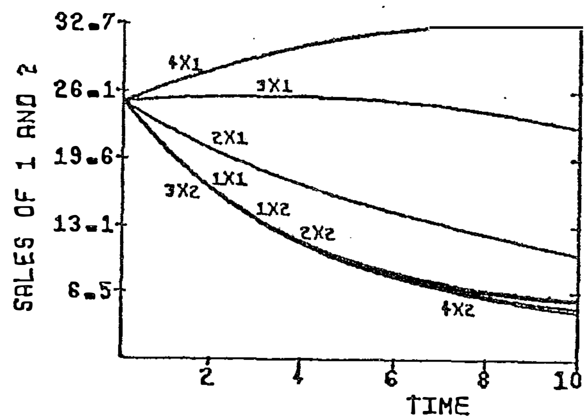

Figure 2b

value, the long-range sales effect of advertising increases. In response to this, firm 1 increases its advertising expenditures during the beginning of the planning horizon in order to build up its stock of sales. Because of the early build up of sales, which will decay more gradually for smaller a_1 values, firm 1 will decrease its advertising during the later part of the planning period. As can be seen in the graph, this strategy gives firm 1 a very substantial advantage, especially when the decay constant has been decreased to .01.

Hypothesis 4. As would be expected, the firm's measure of performance has a significant effect on the timing of advertising expenditures. The parameter values and results of the tests that were done for the verification of this hypothesis are as follows (the graphs of the results are presented in Figures 3a and 3b).

Parameter Values--Hypothesis 4

Encounter Number	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
1	.25	.25	2.50	2.50	.40	.40	25	25	0	0
2	"	"	"	"	"	"	"	"	.50	.50
3	"	"	"	"	"	"	"	"	1.0	1.0
4	"	"	"	"	"	"	"	"	1.5	1.5

Results--Hypothesis 4

Encounter Number	Combined Performance Index Brand # 1	Combined Performance Index Brand # 2	Profit Brand # 1	Profit Brand # 2	Market Share Brand # 1	Market Share Brand # 2
1	\$ 52.60	\$ 52.60	\$ 52.60	\$ 52.60	.50	.50
2	158.40	158.40	33.40	33.40	.50	.50
3	258.47	258.47	8.47	8.47	.50	.50
4	357.13	357.13	-17.87	-17.87	.50	.50

These runs provide for observing the resulting changes in advertising strategies as the firms go from having solely profit oriented P.I.'s to criteria that mainly emphasize high terminal market shares.

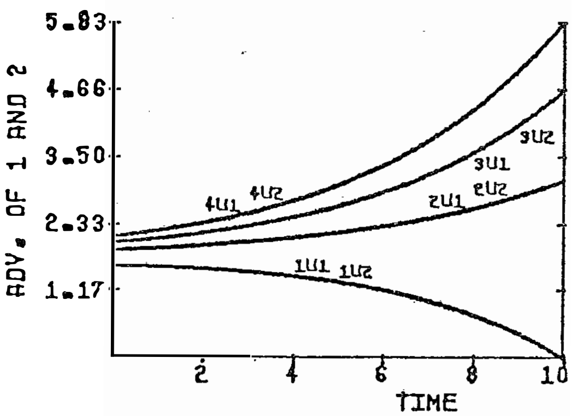

Figure 3a

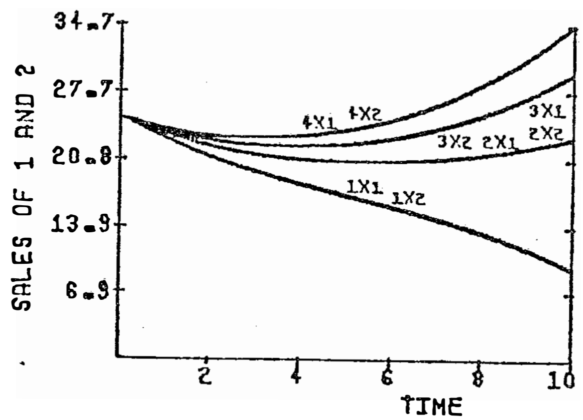

Figure 3b

<u>Hypothesis 4'</u>. The illustrative examples for this hypothesis establish competitive forays between two firms that are identical in every respect except for their performance criteria. The parameter sets are as follows:

Parameter Values

Encounter Number	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
1	.25	.25	2.50	2.50	.40	.40	25	25	.05M	.05M
2	"	"	"	"	"	"	"	"	"	.50M
3	"	"	"	"	"	"	"	"	"	1.00M
4	"	"	"	"	"	"	"	"	"	1.50M

Results

Encounter Number	Combined Performance Index Brand # 1	Combined Performance Index Brand # 2	Profit Brand # 1	Profit Brand # 2	Market Share Brand # 1	Market Share Brand # 2
1	$ 64.32	$ 64.32	$ 51.82	$ 51.82	.50	.50
2	60.41	199.91	51.81	35.94	.34	.66
3	59.08	372.63	51.82	17.86	.29	.71
4	57.75	549.94	51.70	-18.57	.24	.76

The results of these engagements are illustrated in Figures 4a and 4b. The control paths and sales paths are as would be epected As the terminal market share weight increases for firm 2, the advertising magnitude increases and especially becomes monotonically increasing rather than monotonically decreasing as is true for profit maximizing firms. The effect is to influence the sales paths to be correspondingly higher particularly at t_f.

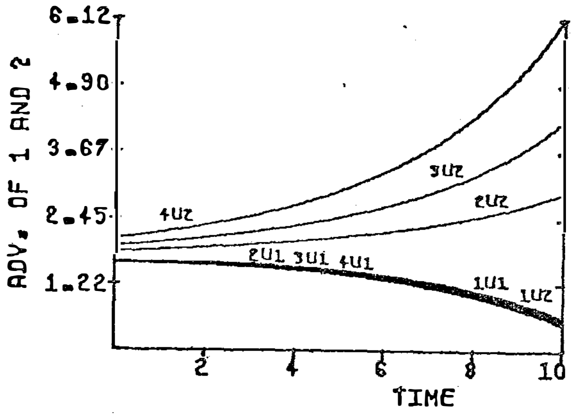

Figure 4a

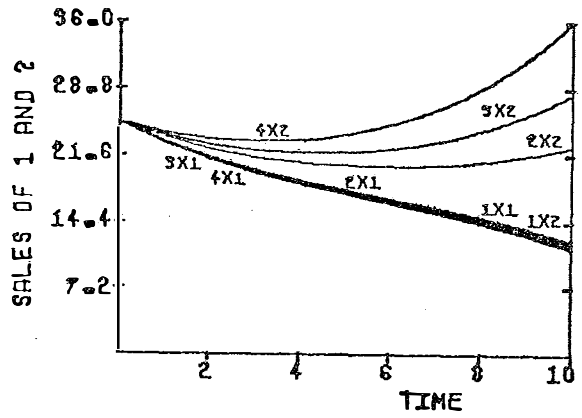

Figure 4b

Testing of the algorithm has continued past this stage to include situations where there are variations in several of the parameters of both brands. This investigation followed an experimental design approach. The detail involved in this experimental approach is too voluminous to be presented here.

3. TELSER'S ELASTICITY RELATIONSHIP

Now that the object of testing has been described, a more theoretically appealing verification of the differential game algorithm can be investigated. It is encouraging when research in relatively untrodden areas proves to be consistent with established theoretical or empirical work. Fortunately, the results of the advertising differential game model have proven to coincide with the elasticity relationships for dynamically competing brands that were developed by Lester Telser in 1962.[14]

14 Lester G. Telser, "Advertising and Cigarettes," Journal of Political Economy, 70 (October 1962) pp. 471-99.

Telser recognized, and was able to explain, the pressures of competition which force interdependent relationships between sales and advertising of the participating firms. His relationship for a duopoly appears as (using notation consistent with that of the preceding model):

$$\gamma_{u_{it}} = (1 - m_{it})(\theta_{it} - \Psi_{it}) \qquad (11)$$
$$= \gamma_{s_{it}} \lambda_{it}$$

where: $m_{it} = x_{it}/(x_{it} + x_{jt})$ = market share of brand i at time t;

$\gamma_{u_{it}} = (dm_{it}/m_{it})/(du_{it}/u_{it})$ = brand i's market share elasticity with respect to brand i's absolute advertising outlay at time t;

$\theta_{it} = (dx_{it}/x_{it})/(du_{it}/u_{it})$ = brand i's absolute sales elasticity with respect to i's absolute advertising outlay at time t;

$\Psi_{it} = (dQ_{it}/Q_{it})/(du_{it}/u_{it})$ = competitor's absolute sales elasticity with respect to i's absolute advertising outlay (Q_{it} represents the competitor's sales at time t);

$\gamma_{s_{it}} = (dm_{it}/m_{it})/(ds_{it}/s_{it})$ = brand i's market share elasticity with respect to i's relative advertising outlay, where s_{it} is the advertising outlay on the i^{th} brand in period t divided by the total of the advertising expenditures on all competing brands in period t, and

$\lambda_{it} = 1 - (dU_{it}/U_{it})/(du_{it}/u_{it})$

= 1 minus the elasticity of response of competitors' advertising, U_{it}, to a change in the given firm's advertising.

Telser and later Lambin,[15] Beckwith[16] and Clarke,[17] were concerned with the empirical estimation and the across-hand comparisons of these elasticities and the elasticity relationships. The estimation of the relationships is not a problem in the simulated context of this model. However, it is instructive to test the coincidence of the output of the differential game model with Telser's elasticity relationship. The terms of equation (11) were calculated at each point during the planning horizon using the "optimal" advertising expenditures and the resulting sales (and market share) values. The relationship was calculated for a wide range of market conditions (represented by variation in the model parameters). In all investigated cases, the equality of equation (11) held with a reasonably small amount of relative error. The error involved in the testing of the elasticity relationships is represented by plotting the relative error between the two sides of equation (11) at each point during the planning horizon. The form of the calculation is

$$[\gamma_{u_{it}} - (1 - m_{it})(\theta_{it} - \Psi_{it})] \div \gamma_{u_{it}}$$

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15 Jean-Jacques Lambin, "Optimal Allocation of Competitive Marketing Efforts: An Empirical Study," The Journal of Business, 43 (1970),,pp. 468-484.

16 Neil E. Beckwith, "Multivariate Analysis of Sales Response of Competing Brands to Advertising," Journal of Marketing Research, IX (May 1972), pp. 168-176.

17 Darral G. Clarke, "Sales-Advertising Cross Elasticities and Advertising Competition," Journal of Marketing Research, X (August 1973), pp. 250-261.

Figure 5 is a plot of this relative error term for that encounter of the testing of Hypothesis 1' where the brands were most different in their parameters, i.e., encounter 4. The relative error values of encounter 4 of Hypothesis 4' are plotted in Figure 6. Although the relative errors are plotted for both competitors, only one line is evident in each of the diagrams since they are the same for both brands.

4. CONCLUSION

The advantage of using Telser's relationship here is that it recognizes the inherent feature of market advertising competition -- that the advertising and sales and, hence, elasticity values will be changing over time. A dynamic optimization model provides these values over the planning period and Telser's relationship provides for a more general and theoretically valid comparison of the resultant advertising strategies.

The correspondence of the differential game model results with Telser's elasticity relationships contributes additional evidence to support the model. The remaining stage of validation lies in empirically testing the model in an existing competitive situation.

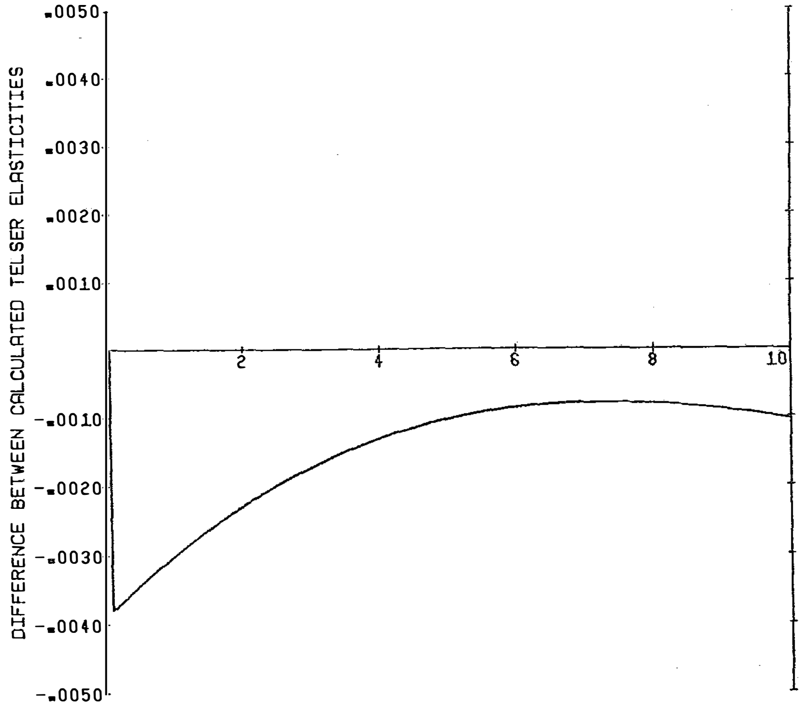

Figure 5

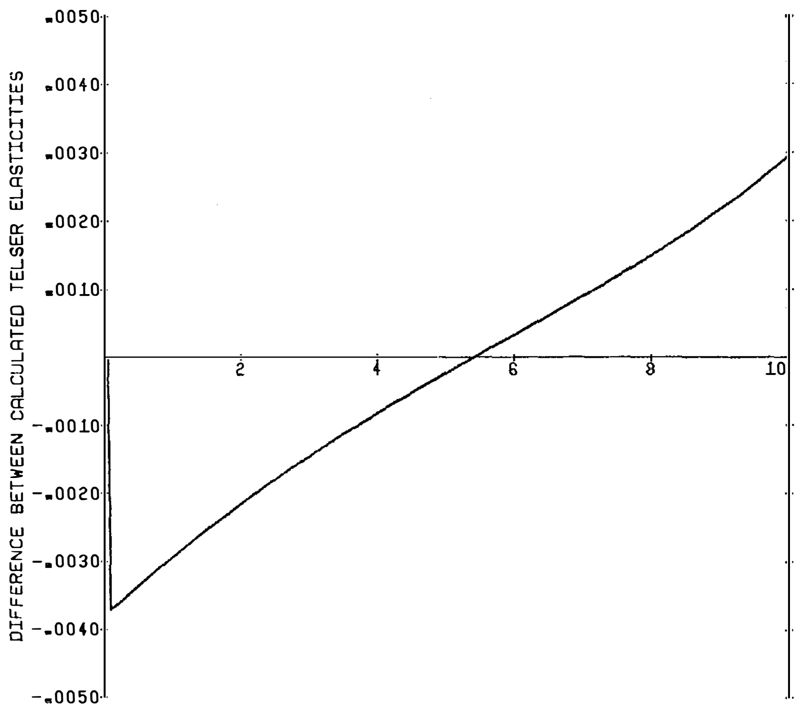

Figure 6