# FINANCIAL COST ALLOCATIONS: A GAME-THEORETIC APPROACH



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In two research studies published by the American Accounting Association, Arthur L. Thomas concludes that financial cost allocations are not only arbitrary but also incorrigible, i.e., incapable of verification or refutation by reference to external "real world" phenomena.[1] While maintaining that the logic can be generalized to all financial cost allocations, Thomas particularizes his argument to depreciation allocations. Either depreciation allocations are essentially arbitrary because they have no theoretical justification, Thomas contends, or they are predicated on a net-revenue-contributions (NRC) approach. The latter appears to be justifiable since the resulting allocation pattern follows the expected net revenue contributions of the asset or project to the firm-entity. However, even NRC allocations cannot be justified if the inputs to the revenue generating process interact to produce the revenues of the firm. Unless the inputs operate independently of each other, the allocation of depreciation over time must result in the arbitrary allocation of a joint cost to a specific asset. Since there is no unique and identifiable cause-and-effect relationship between a specific asset and the revenues generated by interacting assets, any and all allocations are equally justifiable and, therefore, incorrigible.[2]

In this paper we hope to demonstrate that, by employing certain elementary game-theoretic concepts, financial accounting allocations need not be arbitrary nor incorrigible even though asset interactions are prevalent. In what follows, Section I illustrates Thomas' argument in a simple depreciation allocation example. Section II introduces the concept of a Shapley value and applies it to our example. Finally, Section III concludes that Shapley values represent a defensible and corrigible cost allocation mechanism.

Section I: The Cost Allocation Problem, An Example

Consider the three sets of cash flows in Table I below which are assumed to eminate from the hypothetical projects A, B and C. Without loss of generality, each project has a three year life and no salvage value at the end of the period. The internal rates of returns of the projects are listed underneath the cash flows. The cash flows for projects A and B assume one or the other project is undertaken but not both. On the other hand, project C represents the simultaneous investment in A and B. Also, C's cash flows are set to reflect the synergetic benefits of project interaction in that C's revenues are greater than and cost less than the sum of the component revenues and costs.

The allocation problem can be illustrated with reference to Table II which lists in columns (1) through (3), the NRC depreciation schedules for projects A, B and C, respectively. Evidently project interaction precludes allocating C's depreciation schedule on the basis of the independent schedules. Columns (1) and (2) of Table II simply do not sum to column (3).

Another seemingly reasonable approach might be to allocate by incremental depreciation charges. The incremental charges allocated to project A, column (4) of Table II, are obtained by subtracting column (2) from (3). Similarly, the incremental depreciation schedule for project B is the difference between columns (3) and (1). But which is the incremental project? If A is assumed to be incremental to B, then column (4) would be allocated to A and column (2) to B. On the other hand, if B is incremental to A, then column (5) is allocated

Table I

Costs, Net Revenues and Internal Rates of Return
for Projects A, B and C

| Year | Project A | Project B | Project C |
|---|---|---|---|
| 0 | -$1,000 | -$500 | -$1,484 |
| 1 | 278 | 94 | 450 |
| 2 | 450 | 250 | 750 |
| 3 | 500 | 250 | 800 |
| IRR | 10% | 8% | 15% |

Table II

Net Revenue Contribution Depreciation Schedules for Projects A, B and C
and Incremental Depreciation Schedules for Projects A and B

| | NRC Depreciation Schedules | | | Incremental Depreciation Schedules | |
|---|---|---|---|---|---|
| Year | (1) Project A | (2) Project B | (3) Project C | (4) Project A | (5) Project B |
| 1 | $178 | $ 54 | $227 | $173 | $ 49 |
| 2 | 367 | 215 | 561 | 346 | 194 |
| 3 | 455 | 231 | 696 | 465 | 241 |

to B and column (1) to A. Accepting one or the other makes the arbitrary ordering of the projects fundamental to the allocation process and, therefore, the allocation is incorrigible. One could just as reasonably allocate one half of C's depreciation schedule to each of the component projects.

It is worth noting that the allocation problem does not depend on accepting simultaneous interacting projects. Any time the firm invests in a new asset, there are bound to be interactions between the firm's capacity to generate revenues and the new asset. Few projects are likely to give the same cashflows when divorced from the remainder of the firm's assets. The question again arises: how much of the firm's revenues should be allocated to the new project and how much to existing assets? What proportion of the firm's depreciation schedule should be allocated to the new project and what proportion to existing assets?

## Section II: Game Theory Concepts

In an interesting and, until recently, neglected paper M. Shubik advocates the potential utility of Shapley values for allocating joint costs in transfer pricing problems. More recently Shapley values were used to analyse merger benefits[4] and public utility pricing schemes.[5] To appreciate the application of Shapley values to financial accounting methodology we require some elementary game-theoretic notions.

The theory of games conceptualizes a measure of interaction between players in a joint venture or coalition in comparison to their effectiveness as individuals. This measure of joint co-operation is called the characteristic function of the game. The characteristic function (V) is defined over all potential coalitions and is assumed to be superadditive. This means that value of the players acting independently cannot be greater than their value in a coalition. In a two person coalition superadditivity is defined by the mathematical relationship

$$V(A,B) \geq V(A)+V(B)$$

where V(A), V(B) and V(A,B) are the characteristic functions of player A, player B and the coalition comprised of A and B, respectively.

Shapley values allocate the benefits of the coalition to each player in a specific and unique manner. Each player is valued by his incremental benefit to the coalition. Since the incremental benefit is not invariant to the order in which the player is presumed to join the coalition, each possible alternative is assumed to be equi-probable and weighted accordingly. For example, in a two player coalition the incremental value of player A is V(A) if A enters the coalition first and V(A,B)-V(B) if B is first. Assuming each occurrence is equally likely, the allocation to player A is

$$S_A = 1/2\ V(A) + 1/2\ [V(A,B)-V(B)]$$

Similarly, B's allocation is

$$S_B = 1/2\ V(B) + 1/2\ [V(A,B)-V(A)]$$

Total coalition benefits are allocated by this technique since

$$S_A + S_B = V(A,B)$$

In a three player coalition the concept is the same but the allocation formula is more complex. If A, B and C are the players then the relevant characteristic functions for all possible coalitions are:

$V(A)$, $V(B)$, $V(C)$, $V(A,B)$, $V(A,C)$, $V(B,C)$ and $V(A,B,C)$. The Shapley value allocations would be:

$$S_A = 1/3\ V(A) + 1/6\ [V(A,B)-V(B)] + 1/6\ [V(A,C)-V(C)] + 1/3\ [V(A,B,C)-V(B,C)]$$

$$S_B = 1/3\ V(B) + 1/6\ [V(A,B)-V(A)] + 1/6\ [V(B,C)-V(C)] + 1/3\ [V(A,B,C)-V(A,C)]$$

$$S_C = 1/3\ V(C) + 1/6\ [V(A,C)-V(A)] + 1/6\ [V(B,C)-V(B)] + 1/3\ [V(A,B,C)-V(A,B)]$$

Again, all coalition benefits are allocated since

$$S_A + S_B + S_C = V(A,B,C)$$

Shapley values can be generalized to an n player coalition. The value of the jth player in an n player coalition is

$$S_j = \sum_{G \subset J} \frac{(n-g)!(g-1)!}{n!} [V(G)-V(G-\{j\})]$$

where $J = \{j: j=1, \ldots, n\}$ is the set of players and G is any subset (coalition) of g players. The incremental benefit conferred on the coalition by player j when he is in the gth position is weighted by the term $\frac{(n-g)!(g-1)!}{n!}$. n! is the total number of possible coalitions while $(g-1)!$ and $(n-g)!$ represent the number of ways of ordering the players in coalition G and J-G, respectively.

If for players and coalitions we substitute the terms projects and firms, the transition to our immediate interest is obvious. Define the characteristic function of a project to be the schedule of cash flows which result from the project. This function is superadditive provided, for each corresponding year, the cashflow from interacting projects is greater than or equal to the sum of cash flows of the separate projects. The cash flows in our example are superadditive.

Shapley values can be applied to our example in one of two equivalent ways. Either the cash flows are allocated initially and then the NRC depreciation schedules are calculated, or the depreciation schedules are determined first and subsequently allocated to the component projects. The first approach is illustrated in Table III. Columns (1) and (2) are the Shapley value allocated cash flows for projects A and B, respectively. The internal rates of return are calculated for these cash flows and then the NRC depreciation schedules. Column (3) gives A's depreciation schedule and column (4) that of B's. Equivalently the Shapley value technique is applied directly to the depreciation schedules in Table II above. For example, A's depreciation schedule [Column (3), Table III] can be determined by multiplying columns (1) and (4) of Table II by .5 and adding the result.

## Section III: The Corrigibility of Shapley Values

Specifying an additional cost allocation procedure, albeit one which takes account of project interaction, does not of itself alleviate the incorrigibility problem. It is not the paucity of allocation procedures which is problematic but the reverse. If the Shapely value approach is valid it must be defensible against all other cost allocation procedures. In other words, the Shapley value technique is corrigible provided it alone satisfies "reasonable" cost allocation criteria and is, therefore, optimal with respect to these criteria.

Table III

Shapley Value Allocated Cash Flows, Internal Rates of Return, and Shapley Value Depreciation Schedules for Projects A and B

| | Shapley Value Cash Flow | | Shapley Value Depreciation Schedules | |
|---|---|---|---|---|
| | (1) | (2) | (3) | (4) |
| Year | Project A | Project B | Project A | Project B |
| 0 | -$992 | -$492 | - | - |
| 1 | 317 | 133 | $175 | $ 52 |
| 2 | 475 | 275 | 357 | 204 |
| 3 | 525 | 275 | 460 | 236 |
| IRR | 14.3% | 16.3% | | |

Mossin has shown three axioms to be necessary and sufficient for the optimality of the Shapley value approach.[6] If these axioms comprise an acceptable cost allocation constitution then Shapley values are the only valid cost allocation procedure. Although these axioms can be described in rather general terms we will state them in the context of depreciation allocations so that they are self-explantory.

Axiom 1. If the firm invests in a non-interactive project, the project's depreciation schedule is a function of its own revenue-cost structure and independent of the firm.

Axiom 2. If the firm invests in two projects which do not interact with each other - although each may interact with the firm - the depreciation schedule allocated to the projects simultaneously is equal to the sum of the depreciation schedules allocated to the separate projects.

Axiom 3. Projects with the same revenue - cost structure are allocated the same depreciation schedules.

## Conclusion

Although Thomas argues in his research studies that much of financial accounting is void of meaningful content, his conclusions are somewhat premature. Using the same cost allocation backdrop as does Thomas, namely, depreciation, we have shown that financial cost allocations need not be arbitrary nor incorrigible. As long as the accounting profession is willing to accept (i) a constitution of three simple cost allocation axioms and (ii) the necessarily concomitant Shapley value allocation procedure, Thomas paints too gloomy a picture.

It would be unfortunate should the accounting profession view the potential application of game theory to accounting to be too esoteric. In the absence of a uniquely justifiable cost allocation scheme, such as Shapley values, Thomas is probably correct.

Footnotes

[1]See Thomas [1969] and Thomas [1964].

[2]These conclusions have not been universally accepted. See Eckel [1976].

[3]See Shubik [1962]. Shapley values were first introduced by Shapley [1953].

[4]See Mossin [1968].

[5]See Littlechild [1970], Loehman and Whinston [1971] and Loehman and Whinston [1974].

[6]See Mossin [1968] for a development of these axioms in the context of merger benefits. The Mossin axioms are similar to those of Shapley [1953]. Loehman and Whinston [1974] have developed another set of axioms which do not presuppose a superadditive characteristic function.

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