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ALLOCATION ANALYSIS OF A DYNAMIC DISTRIBUTION PROBLEM

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ABSTRACT

This paper develops a mathematical program for allocating market demands to facilities, over a multiperiod planning horizon. In the model, demands for the firm's product in a given period are dependent on allocation decisions in the preceding period. This feature is seen as a mechanism for analyzing the timing of market share increases in conjunction with conventional allocation tradeoffs. An example is explored extensively to illustrate the potential applicability of the model, solution approaches for reducing computational requirements are examined, and extensions of the model are presented.

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INTRODUCTION

During the last decade, considerable attention has been given to analytical models for locating distribution facilities and allocating market demands to these facilities, over a multiperiod planning horizon. One of the early works in this field was presented by Ballou [1]. He developed a dynamic programming approach for locating and relocating a warehouse. This research generated further interest with respect to the definition of the state space, and consequent computational feasibility of the approach, for the multifacility case [2,9]. Since that time, numerous dynamic location-allocation problems have been examined in the management science, industrial engineering and other literature [eg. 10,11,12,16]. Also, the early dynamic programming approach for the multifacility problem has been extended and improved [14].

The model in this paper focuses on multiperiod allocation decisions for given facility configurations (ie. numbers, locations and sizes of facilities). The choice of these configurations can be viewed as a separate, but related, issue. Alternately, the allocation model can be considered as a foundation for developing integrated location-allocation formulations to suit particular circumstances. Since the model deals with allocation analysis for assumed location decisions, it is intended for situations in which the choice of facilities to supply markets is controlled by the firm. Thus, the term "facilities" is interpreted more appropriately as plants or warehouses than as retail outlets. In the latter case, there is no allocation question as such; rather, location decisions are of paramount importance (eg. [7,8]).

A general description of the problem treated by the model is as follows. There are 'm' facilities from which to supply a product to 'n' markets over a T-period planning horizon. Each of the facilities has estimated capacities for each period. Demands for the product may vary by market and by period, reflecting changes over time in the distribution of demand across markets. Within this framework, the objective is to allocate quantities of the product from facilities to markets, so as to maximize total contribution (ie. revenue received minus distribution costs incurred) for the firm.[1] To this point, the essence of the problem description applies to other dynamic allocation models in the literature. The distinguishing feature of the model in this research and its potential marketing applicability are discussed now.

Typically, multiperiod allocation models use parameter estimates of a market's demand for the firm's product in each period. These estimates are portions of the market's total demands. Thus, a characteristic of the "demands" in conventional models is that they are independent of the quantities supplied to the market by the firm in previous periods. For example, consider the dynamic transportation problem studied by Bellmore et al [3] and by Szwarc [15]. The dynamic nature of this problem stems from the fact that the timing of shipments is not necessarily coincident with the timing of demands. Since inventories can be accumulated, allocation decisions are related across periods. However, the demand from a market in a period is independent of the portion of the preceding period's demand that is satisfied. Srinivasan and Thompson [13] have presented allocation models for determining optimal growth paths which have been studied subsequently by Fong and Rao [5]. While most of these models are single-period formulations with which growth implications are analyzed parametrically, one multiperiod model is examined. This is the case of "growth paths with prespecified market growth rates". In this model, demand parameters for a market are incremented by a constant proportion across periods. Again, the demand from a market is not dependent on the quantity supplied previously. In contrast, the model presented here is based on the concept that the maximum volume that can be supplied (ie. the firm's demand) in a given period depends on the amount allocated to the market by the firm in the preceding period. This "allocation-dependent demand" notion imparts a dynamic characteristic to the model by establishing an interdependence of allocation decisions across periods.

The practical relevance of allocation-dependent demands can be seen by relating this concept to implications of market share changes. Rapid increases (from one period to the next) in a firm's share of a market can cause substantial reductions in contributions received throughout the market. Such reductions may result from retaliation by competitors and corresponding reactions by the firm (eg. price or service competition in the market). The notion of allocation-dependent demands in the model can be viewed as a mechanism for incorporating upper bounds on increases in market shares, beyond which extreme competitive action is anticipated. By investigating the sensitivity of solutions to variations in maximum share increases and corresponding changes in contributions, the model can assist in developing a strategy for timing penetration into markets.

THE MODEL

The dynamic allocation model is stated mathematically as follows:

(1) Maximize $$Z^T = \sum_{t=1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijt} x_{ijt}$$

subject to:

(2) $$\sum_{j=1}^{n} x_{ijt} \leq a_{it} \qquad \text{for } i = 1,2,\ldots,m;\ t = 1,2,\ldots,T$$

(3) $$\sum_{i=1}^{m} x_{ij1} \leq \lambda_{j1} b_{j0} + d_{j1} \qquad \text{for } j = 1,2,\ldots,n$$

(4) $\sum_{i=1}^{m} (x_{ijt} - \lambda_{jt} x_{ij,t-1}) \leq d_{jt}$ for $j = 1,2,\ldots,n;\ t = 2,\ldots,T$

and, all $x_{ijt} \geq 0$

where: Z^T = present value of total contribution received over the planning horizon

r_{ijt} = present value of contribution received by supplying one unit to market j, in period t, from facility i

a_{it} = number of units available (capacity) at facility i in period t; $a_{it} \geq 0$

b_{j0} = number of units supplied to market j in the period preceding the planning horizon; $b_{j0} \geq 0$

λ_{jt} = fraction of the quantity allocated to market j in period t-1 that can be supplied to j in t; $\lambda_{jt} \geq 0$

d_{jt} = number of units (in addition to the amount derived from λ_{jt}) that can be supplied to market j in period t, regardless of the quantity allocated to j in t-1; $d_{jt} \geq 0$

x_{ijt} = number of units supplied by facility i to market j, in period t.

In this linear program, (1) requires the maximization of total contribution received from all allocations over the planning horizon. Constraint set (2) enforces capacity restrictions at facilities in each period. Upper bounds on quantities that can be supplied to markets in each period are specified by (3) and (4). In order to state the model in standard form, period 1 constraints are specified separately in (3); however, both constraint sets have the same structure when the constant b_{j0} is replaced with $\sum_{i=1}^{m} x_{ij0}$. The "allocation-dependent demand" concept is contained in (3) and (4). This notion and its market share interpretation now can be explored in greater detail by considering the λ_{jt} and d_{jt} parameters in terms of more basic problem data.

Let

(5) $U_{jt} = (S_{j,t-1} + I_{jt})D_{jt}$

where: U_{jt} = upper bound on the quantity supplied by the firm to market j in period t, beyond which average contribution would be reduced

$S_{j,t-1}$ = firm's share of market j in period t-1

I_{jt} = maximum increase in the firm's share of market j between periods t-1 and t, before precipitating significant retaliation by competitors

D_{jt} = total quantity demanded by market j in period t.

The maximum increase in market share, I_{jt}, may consist of either or both of two components: one that is a portion of the previous market share (I'_{jt}); one that is independent of the previous market share (I''_{jt}). Thus, a generally-applicable definition of I_{jt} is:

(6) $I_{jt} = S_{j,t-1}I'_{jt} + I''_{jt}$.

Since $S_{j,t-1} = \sum_{i=1}^{m} x_{ij,t-1}/D_{j,t-1}$, (3) and (4) restrict supply to a maximum of U_{jt} when:

(7) $\lambda_{jt} = (D_{jt}/D_{j,t-1})(1 + I'_{jt})$, and

(8) $d_{jt} = I''_{jt}D_{jt}$.

Considering the fact that I'_{jt} and I''_{jt} are components of a maximum increase in market share, it would be reasonable to set a lower bound of zero on both of these parameters. While such a bound is intuitively appealing, one exception expands the model's flexibility. If $I'_{jt} = -1$, the effect of the previous market share on U_{jt} is removed and any allowable increase in share can be incorporated in I''_{jt}. Since the firm's share of a market can not exceed 100% in any period, there are also upper bounds on the values of these two parameters. By compounding maximum possible shares over periods, rearranging terms and factoring, the following conditions can be derived to specify this requirement.

$$[b_{j0}/D_{j0}][1 + I'_{j1}] + I''_{j1} \leq 1 \quad \text{for } j = 1,2,\ldots,n$$

(9)

$$[b_{j0}/D_{j0}][\prod_{k=1}^{t} (1 + I'_{jk})] + \sum_{k=1}^{t-1} I''_{jk}[\prod_{\ell=k+1}^{t} (1 + I'_{j\ell})] + I''_{jt} \leq 1$$

for $j = 1,2,\ldots,n;\ t=2,\ldots,T$.

AN ILLUSTRATION OF THE MODEL

To demonstrate the broad potential-applicability of the model, an example is given in Table 1. In this table, "basic problem data" includes data that are used directly in the model (r_{ijt} and a_{it}) and raw data (D_{jt}, I'_{jt} and I''_{jt}) from which the "derived model parameters" are developed. The "optimal solution" section of the table includes the maximum total contribution and the allocations required to obtain it (all data rounded to the first decimal place). The following elements of a scenario for this example indicate the great variety of problem characteristics that can be treated by the model.

- Market 1: The firm supplies 20% of this market currently (b_{10}/D_{10}). In any period t, the firm can obtain 20% of the market demand regardless of its share in the previous period ($I'_{1t} = -1.0, \therefore \lambda_{1t} = 0.00$; $I''_{1t} = 0.2$).[2] This capability of obtaining a significant share, at will, is due to the relatively unattractive contributions yielded by the market (all $r_{i1t} << r_{ijt}$, for $j \neq 1$). Within the bounds of a "normal" 20% share, little resistance is expected from competitors.
- Market 2: The firm does not supply this market currently ($b_{20} = 0$). Between any pair of periods, an absolute increase of 10% in market share is feasible

Table 1

EXAMPLE OF THE MODEL

Basic Problem Data

m = 3; n = 5; T = 4

r_{ij1}

i \ j	1	2	3	4	5	a_{i1}
1	2	8	15	4	4	150
2	1	5	12	7	3	25
3	3	7	10	6	5	300

r_{ij2}

i \ j	1	2	3	4	5	a_{i2}
1	2	8	16	4	4	200
2	1	5	13	7	3	50
3	3	7	11	6	5	300

r_{ij3}

i \ j	1	2	3	4	5	a_{i3}
1	2	9	16	4	4	265
2	2	6	13	7	4	100
3	3	8	11	6	6	300

r_{ij4}

i \ j	1	2	3	4	5	a_{i4}
1	3	9	18	4	12	350
2	2	6	14	7	10	200
3	4	8	12	6	9	300

D_{jt}

j \ t	0	1	2	3	4	b_{j0}
1	100	250	325	450	625	20
2	200	230	265	305	350	0
3	600	720	935	1310	1965	60
4	400	300	150	100	50	200
5	325	350	400	600	1000	70

I'_{jt}/I''_{jt}

j \ t	1	2	3	4
1	-1.0/0.2	-1.0/0.2	-1.0/0.2	-1.0/0.2
2	0.0/0.1	0.0/0.1	0.0/0.1	0.0/0.1
3	0.0/0.0	0.0/0.0	0.0/0.0	0.0/0.0
4	0.3/0.0	0.2/0.0	0.1/0.0	0.1/0.0
5	0.2/0.1	0.2/0.1	0.2/0.1	0.0/0.0

Table 1 (continued)

Derived Model Parameters

λ_{jt}

j \ t	1	2	3	4
1	0.00	0.00	0.00	0.00
2	1.15	1.15	1.15	1.15
3	1.20	1.30	1.40	1.50
4	0.98	0.60	0.73	0.55
5	1.29	1.37	1.80	1.67

d_{jt}

j \ t	1	2	3	4
1	50	65	90	125
2	23	27	31	35
3	0	0	0	0
4	0	0	0	0
5	35	40	60	0

Optimal Solution

Z^T = 22,657.3

j \ t	1	2	3	4
1	x_{111}=46.3 x_{311}=3.7	x_{112}=44.3 x_{312}=20.7		
2	x_{121}=23.0	x_{122}=53.5	x_{123}=92.5	x_{324}=59.5
3	x_{131}=72.0	x_{132}=93.6	x_{133}=131.0	x_{134}=196.6
4	x_{241}=25.0 x_{341}=171.0	x_{242}=50.0 x_{342}=67.6	x_{243}=85.8	
5	x_{351}=125.3	x_{352}=211.7	x_{153}=41.5 x_{253}=14.2 x_{353}=300.0	x_{154}=153.4 x_{254}=200.0 x_{354}=240.5

($I'_{2t} = 0.0$; $I''_{2t} = 0.1$). The possible increase in market share is not related to the previous share since the firm's "presence" in the market has no effect on contributions generated by new business. This market is growing at a constant rate of 15% per period ($\lambda_{2t} = D_{2t}/D_{2,t-1} = 1.15$).

- Market 3: The firm supplies 10% of this market currently (b_{30}/D_{30}). Any increase in this share will precipitate substantial competitive retaliation ($I'_{3t} = 0.0$; $I''_{3t} = 0.0$) because contributions from the market are very attractive (all $r_{i3t} >> r_{ijt}$, for $j \neq 3$). It is predicted that this market will grow at an increasing rate over the planning horizon ($\lambda_{3t} > \lambda_{3,t-1} > 1$, with all $I'_{3t} = 0.0$).

- Market 4: The firm's current share of this market is significantly greater than that in any other market ($b_{40}/D_{40} >> b_{j0}/D_{j0}$, for $j \neq 4$). However, it is anticipated that demand in the market will decline greatly over time ($D_{4t} << D_{4,t-1}$). The firm can build on its currently substantial participation in this market without negatively affecting contributions. Market share can be increased to the point where the firm supplies almost the total market in period 4 ($\{b_{40}/D_{40}\}\{\prod_{t=1}^{4} [1 + I'_{4t}]\} \approx 0.94$).

- Market 5: The firm supplies about 22% of this market currently (b_{50}/D_{50}). In each of the first three periods, a 20% increment to the previous market share and an absolute increase of 10% are possible ($I'_{5t} = 0.2$; $I''_{5t} = 0.1$, for $t = 1,2,3$). However, no further increase in share will be possible after period 3 without causing a substantial reduction in average contribution ($I'_{54} = 0.0$; $I''_{54} = 0.0$). The reason for the distinction between periods 1 to 3 and period 4 is that contributions, and thus the market's attractiveness to competitors, are expected to increase dramatically in period 4 (all $r_{i54} >> r_{i5t}$, for $t \neq 4$).

- Facility 1: The firm plans to expand the capacity of this facility at a rate of 33% per period ($a_{1t}/a_{1,t-1} \approx 1.33$, for $t = 2,3,4$). Facility 1 is the preferred supplier of markets 2 and 3 in all periods, due to its proximity to these markets ($r_{12t} > r_{i2t}$; $r_{13t} > r_{i3t}$, for $i \neq 1$).

- Facility 2: This facility is under construction and, considering anticipated start-up difficulties, its effective capacity is limited severely in early periods. An expansion rate of 100% per period is predicted ($a_{2t} = 2a_{2,t-1}$, for $t = 2,3,4$). Facility 2 is the preferred supplier of market 4 (all $r_{24t} > r_{i4t}$, for $i \neq 2$).

- Facility 3: There is no potential for capacity expansion due to space limitations at this site ($a_{3t} = a_{3,t-1}$, for $t = 2,3,4$). Facility 3 is the preferred supplier of market 1 in all periods (all $r_{31t} > r_{i1t}$, for $i \neq 3$), and of market 5, in all but the fourth period ($r_{35t} > r_{i5t}$, for $i \neq 3$ and $t \neq 4$). The opening of a new transportation route to market 5 is expected in period 4. This route will provide greater cost savings for distribution from facilities 1 and 2 than from facility 3 ($[r_{354} - r_{353}] < [r_{i54} - r_{i53}]$, for $i \neq 3$).

The model is a tool for analyzing a variety of tradeoffs to assist in developing a long-term distribution strategy. This analysis includes the normal allocation tradeoffs involved in choosing sources to supply markets while remaining within facility capacity limits. Also, the model examines interactions between these tradeoffs and ones concerned with timing the degree of penetration into various markets. For instance, the merit of supplying a market in periods with small contributions, in order to build a base from which to obtain larger contributions later, is evaluated. An example of this type of evaluation can be seen by referring to the optimal solution in Table 1. The decisions for period 3 are: do not supply market 1 ($\sum_{i=1}^{3} x_{i13}=0$); serve markets 2, 3 and 4 to the maximum degree possible, given the bases for these markets from period 2 ($\sum_{i=1}^{3} x_{ij3} = \lambda_{j3} \sum_{i=1}^{3} x_{ij2} + d_{j3}$, for $j = 2,3,4$); supply market 5 with all remaining capacity ($\sum_{i=1}^{3} \sum_{j=1}^{5} x_{ij3} = \sum_{i=1}^{3} a_{i3}$). An alternative would be to serve market 5 to the extent possible in period 3 ($\lambda_{53} \sum_{i=1}^{3} x_{i52} + d_{53}$), with consequent decreases in the supply to other markets. The benefit of this strategy would be to increase the upper bound on allocations to market 5 in period 4 and thereby take greater advantage of the substantial improvements in unit contributions (all $r_{i54} >> r_{i53}$). The chosen solution indicates that this advantage would be more than offset by reductions required in other contributions.

One of the primary benefits of the model would be provided by sensitivity analysis. This analysis could be performed for alternate capacity expansion plans and on changes in λ_{jt} and d_{jt} parameters. The purpose of the latter type of analysis would be to examine the effects of allowing larger increases in market shares at the expense of reducing average contributions. Such information would be useful in establishing guidelines for a distribution strategy in conjunction with considerations of competitors' possible courses of action.

SOLVING THE MODEL

A variety of commercial codes could be used for practical-sized versions of the model (eg. ℓ.p. routines in the IBM MPSX and the CDC APEX packages). However, under certain circumstances which are specified later, there is an alternative to solving the model directly as it is formulated. The motivation for using this alternate optimizing method would be to reduce computational costs or to permit solution of larger problems with readily available codes. Further, this procedure provides a basis for developing heuristic approaches.

The fundamental notion in this method is that sometimes portions of the model can be solved as single-period problems. If in any period t, all $r_{ijt} \geq 0$ and $\sum_{i=1}^{m} a_{it} \geq \sum_{j=1}^{n} (\lambda_{jt} \sum_{i=1}^{m} x_{ij,t-1} + d_{jt})$, then the optimal solution to the complete model will contain the optimal solution to a subproblem for the single period t. The rationale for this statement is that supplying each market to the extent allowed (ie. $\lambda_{jt} \sum_{i=1}^{m} x_{ij,t-1} + d_{jt}$) is optimal in period t and makes (4) as unrestrictive as possible in period t + 1. Also, as will be shown, such subproblems have the "transportation" structure, and thus can be solved with efficient specialized algorithms. With these concepts in mind, the approach is specified as follows.

Let

(10)

$$Q_{j1} = \lambda_{j1} b_{j0} + d_{j1} \qquad \text{for } j = 1,2,\ldots,n$$

$$Q_{jt} = (\prod_{k=1}^{t} \lambda_{jk}) b_{j0} + \sum_{k=2}^{t} [(\prod_{\ell=k}^{t} \lambda_{j\ell}) d_{j,k-1}] + d_{jt} \qquad \text{for } j = 1,2,\ldots n; \; t = 2,\ldots,T.$$

where: Q_{jt} = maximum quantity that can be allocated to market j in period t, when j is supplied to the extent possible in all preceding periods

The expression in (10) is derived by compounding maximum allocations over periods. Then, a set of consecutive periods, for which single-period subproblems can be used, is defined by:

(11) $$W = \{1,2,\ldots,t' \mid \text{all } r_{ijt} \geq 0 \text{ and } \sum_{i=1}^{m} a_{it} \geq \sum_{j=1}^{n} Q_{jt}, \text{ for } t = 1,2,\ldots,t'\}.$$

For each $t \in W$, the optimal solution to the following transportation problem is part of the optimal solution to the complete model.

(12) Maximize $Z_t = \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijt} x_{ijt}$

subject to:

(13) $\sum_{j=1}^{n} x_{ijt} \leq a_{it}$ for $i = 1,2,\ldots,m$

(14) $\sum_{i=1}^{m} x_{ijt} = Q_{jt}$ for $j = 1,2,\ldots,n$

and, all $x_{ijt} \geq 0$

where: Z_t = present value of the total contribution received in period t.

The multiperiod ℓ.p. for the remainder of the planning horizon (t > t') is:

(15) Maximize $Z_{t'}^{T} = \sum_{t=t'+1}^{T} \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ijt} x_{ijt}$

subject to:

(16) $\sum_{j=1}^{n} x_{ijt} \leq a_{it}$ for $i = 1,2,\ldots,m;\ t = t'+1,\ldots,T$

(17) $\sum_{i=1}^{m} x_{ij,t'+1} \leq \lambda_{j,t'+1} Q_{j,t'} + d_{j,t'+1}$ for $j = 1,2,\ldots,n$

(18) $\sum_{i=1}^{m} (x_{ijt} - \lambda_{jt} x_{ij,t-1}) \leq d_{jt}$ for $j = 1,2,\ldots,n;\ t = t'+2,\ldots,T$

and, all $x_{ijt} \geq 0$

where: $Z_{t'}^{T}$ = present value of the total contribution received in all periods after t'.

It should be noted that, if t' = T-1, there are no constraints in (18) and (15) to (17) is a transportation problem.

The extent of the computational savings with this approach depends on the number of periods in W. When W = {∅}, the method is not applicable;

when $t' \geq T-1$, maximum computational savings are possible. For more likely cases between these extremes, the advantage is in solving t' transportation problems (size $m + n$ by $m \cdot n$) and a linear program (size $[m + n][T - t']$ by $m \cdot n[T - t']$), rather than solving one large linear program (size$[m + n]T$ by $m \cdot n \cdot T$).

The example given in Table 1 is a situation where it is anticipated that capacity increases will not keep pace with growth in market demands. However, excess capacity will exist in early periods. The solution method developed above can be used for this problem as indicated in Table 2. The example was solved using program REGULAR (Simplex) in the MPOS System [4] on a CDC 6400 computer. This program was applied to the complete model and to the three component problems mentioned in Table 2. Data storage weighted by C.P. execution time was used as a measure of computational requirements. Solving the problem as three separate subproblems required 0.24 of the computer resources for solving the complete formulation. A transportation algorithm was not used for a comparison, since savings would be highly dependent on the relative efficiency of the particular codes. However, potential savings are quite substantial since transportation routines can solve problems at least 100 times faster than advanced ℓ.p. codes [6].

The method presented here requires W to begin with period 1 in order to guarantee an optimal solution. Otherwise, the optimal "demands" in (14) are not known for the first transportation problem in the sequence. Nevertheless, the notion of using single-period transportation problems forms a basis for developing heuristic procedures that apply under relaxed conditions. Consider a case in which one or more sets W' exist where:

(19) $W' = \{t'',\ldots,t' \mid \text{all } r_{ijt} \geq 0 \text{ and } \sum_{i=1}^{m} a_{it} \geq \sum_{j=1}^{n} Q_{jt}, \text{ for } t=t'',\ldots t';\ t'' \neq 1\}.$[3]

Table 2

REDUCING COMPUTATIONAL REQUIREMENTS FOR SOLVING THE EXAMPLE

t \ j	Q_{jt} 1	2	3	4	5	$\sum_{j=1}^{5} Q_{jt}$
1	50.0	23.0	72.0	196.0	125.3	466.3
2	65.0	53.5	93.6	117.6	211.7	541.4
3	90.0	92.5	131.0	85.8	441.0	840.3
4	125.0	141.3	196.6	47.2	736.5	1246.6

t	$\sum_{i=1}^{3} a_{it}$
1	475
2	550
3	665
4	850

$\sum_{i=1}^{3} a_{it} \geq \sum_{j=1}^{5} Q_{jt}$ and all $r_{ijt} \geq 0$, for t = 1,2.

$\therefore t' = 2$

(12) to (14) for t = 1: $Z_1 = 3{,}195.20$

(12) to (14) for t = 2: $Z_2 = 3{,}889.84$

(15) to (18) for t > 2: $Z_2^4 = 15{,}572.21$

$22{,}657.25 = Z^4$

Under these circumstances, the model could be solved by alternating between multiperiod linear programs and sequences of transportation problems. The connections would be made by using allocation decisions for the final period of an ℓ.p. to determine maximum quantities that could be supplied in period t''. Then, these quantities would be compounded (using the process underlying (10)) over periods $t'' + 1$ to t' for the remaining transportation problems in the sequence. The link to the next ℓ.p. would be established by compounding maximum allocations one further step into period $t' + 1$.

Given this general structure, the crucial question in formulating a specific procedure is: to what extent should various allocation requirements in the periods $t'' - 1$ be examined in an attempt to find improved solutions? This question can be answered only with reference to particulars of any application. Consideration must be given to potential computational savings relative to available options for optimizing. Also, the effect of solution "quality" on the usefulness of sensitivity analysis should be evaluated.

EXTENSIONS OF THE MODEL

The model (1) to (4) can be extended in a number of ways to increase the scope of its applicability. Two potentially important extensions are given here. First, it may be desirable to specify certain total allocations to each market in the final period of the planning horizon. This requirement would be appropriate when the situation involves a transitional time interval ($1 \leq t \leq T - 1$), followed by an unspecified but lengthy term ($T, T + 1, \ldots$) during which all problem characteristcs are expected to be stationary. In these circumstances, the objective of the analysis would be to develop a strategy to achieve a stated goal for the stationary period, while maximizing contributions over the transitional interval. For this case, the following constraint set would be added to the model.

(20) $\sum_{i=1}^{m} x_{ijT} = b_{jT}$ for $j = 1,2,\ldots,n$

where: b_{jT} = number of units that must be supplied to market j in period T.

Alternately, if the duration of the stationary period can be predicted, the original model can be used by interpreting all r_{ijT} as contributions from allocations over the entire stationary period.[4]

Second, the scope of the model can be increased by generalizing the "allocation-dependent demand" concept. The model assumes that allowable increases in market shares are related only to individual markets and to pairs of consecutive periods. In some applications, it may be appropriate also to restrict increases over groups of markets and/or over longer time intervals. Such restrictions can be treated with the constraint structure of (3) and (4). Considering (3) and (4) as one constraint set (ie. $b_{j0} = \sum_{i=1}^{m} x_{ij0}$), a generalization of this set is:

(21) $\sum_{i=1}^{m} \sum_{j \in J_s} (x_{ij,\ell(p)} - \lambda'_{sp} x_{ij,f(p)}) \leq d'_{sp}$ for $s = 1,2,\ldots,q;\ p = 1,2,\ldots,h$

where: J_s = s'th subset of $\{1,2,\ldots,n\}$

$f(p)$ = first period in the p'th range of periods

$\ell(p)$ = last period in the p'th range of periods

λ'_{sp} = fraction of the quantity allocated to all markets $j \in J_s$ in period $f(p)$, that can be supplied to all $j \in J_s$ in $\ell(p)$; $\lambda'_{sp} \geq 0$

d'_{sp} = number of units (in addition to the amount derived from λ'_{sp}) that can be supplied to all markets $j \in J_s$ in period $\ell(p)$, regardless of the quantity allocated to all $j \in J_s$ in $f(p)$; $d'_{sp} \geq 0$.

Equations (5) to (8) would be generalized accordingly with j replaced by J_s and t, t-1 replaced by $\ell(p)$, $f(p)$. In the original model, (3) and (4) are (21) where $q = n$, $J_s = \{s\}$, $h = T$, $f(p) = p-1$ and $\ell(p) = p$. To add a restriction on share increase over all markets collectively during the entire planning horizon, the structure of (21) would be used with $J_s = \{1,2,\ldots,n\}$, $f(p) = 0$ and $\ell(p) = T$. The flexibility afforded by this generalization can be indicated by noting that subsets J_s need not be mutually exclusive, and that the ranges p can overlap to the extent desired.

CONCLUDING REMARKS

The model presented in this research has been discussed in terms of its potential applicability to actual problems. An additional source of usefulness lies in its richness for demonstrating the general role of mathematical programming in assisting logistics decision-making. In this regard, the development of case material based on the model may be a fruitful direction for further work. The scenario that has been given provides a starting point for such a case; the solution approaches and model extensions might be incorporated in this material as well.

FOOTNOTES

[1] For simplicity, the problem is described in terms of the distribution of a single product, or equivalently a group of completely homogeneous products. However, the model can be given a multiproduct interpretation by associating different products with different markets. In this context, the formulation required homogeneity of products only with respect to their use of facility capacities.

[2] Since all $\lambda_{1t} = 0.00$, maximum amounts that can be supplied to market 1 are independent across periods. If this were true for all markets, the problem would be a special case of the model, requiring a conventional allocation analysis.

[3] For instance, a problem with one W' wherein t' = T may exist when capacity expansion will overtake maximum market growth after t" - 1 periods.

[4] Scott [12] has investigated a problem of sequencing the location (construction) of facilities on a plane, which includes a transitional/stationary period concept. His problem assumes that the stationary period is of such significance that the complete solution should include the optimal solution for period T. With the model (1) to (4), this type of problem characteristic would be represented by exceedingly large r_{ijT} parameters.

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