AN OPTIMAL CONTROL APPROACH TO MARKETING – PRODUCTION PLANNING

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ABSTRACT

The problem of finding an optimal advertising and production policy in a firm is analyzed using a recently proposed model of a marketing-production system. First it is shown that the optimal control problem underlying the proposed model is a partially singular control problem. Then, using a reverse time parametric approach, a solution procedure is designed to determine the optimal advertising and production policy for the proposed model. Finally, it is shown that the results deriving from the new model are applicable to problems of capacity expansion in a firm.

1. INTRODUCTION

Marketing and production policies in a firm tend to be interdependent. Marketing policies are normally designed to generate demand for the firm's products whereas production policies are normally designed to meet that demand.

Although marketing and production policies in a firm are interdependent, most of the models of marketing and production planning assume that the decision making in the two areas is separate [e.g., 11, 12, 17, 22, 30]. In recent years, however, models incorporating the interdependencies between the two areas have been proposed [7, 8, 19, 20, 27, 29]. Thomas [27] proposes a linear programming approach to solve the problem. Leitch [20], Bergstorm [7], and Damon and Schramm [8] propose mathematical programming models to address the problem. The problem has also led to models that have a structure of a continuous optimal control problem. Koive et al. [19] propose a stochastic model of a marketing-production system using an arbitrary sales-advertising relationship. Thompson et al. [29] propose an optimal control model for a monopoly. Recently this author proposed [1, 2] an optimal control model where the two functions are represented by empirically derived subsystem models: the HMMS model of production planning [11] and the Vidale-Wolfe model of sales-advertising relationship [30].

In this paper, we consider the optimal control problem underlying the marketing-production model proposed in [1]. First we show that the optimal control problem underlying the new model is a partially singular control problem. Then, using a reverse time parametric approach, we develop a solution procedure for determining the optimal control to the proposed model. The procedure, although described in terms of the problem under consideration, is fairly general and can be applied to other similar singular control problems. Finally, we show that results deriving from the model are applicable to problems of capacity expansions.

2. THE PROPOSED MODEL

The problem of advertising-production planning for a single product can be formulated as [1, 2]:

$$\min_{A,P} J(A,P) = \int_0^T [-(1-q)S + A + C_v P + C_p (P-P^*)^2 + C_I (I - I^*)^2] dt - b_1 I(T) - b_2 S(T) + F \tag{1}$$

$$\text{s.t.} \quad \dot{I} = P - S/C \tag{2}$$

$$\dot{S} = rA(1 - S/M) - \lambda S \tag{3}$$

$$A_{min} \leq A \leq A_{max} \tag{4}$$

$$I(0) = I_0, \; S(0) = S_0 \tag{5}$$

Where

$S(t)$	=	Sales rate at time (t), ($/day).
$I(t)$	=	Level of inventory at time (t), (Units).
$P(t)$	=	Rate of production at time (t), (Units/day).
$A(t)$	=	Rate of advertising expenditure at time t, ($/day).
λ	=	sales decay constant, (/day).
r	=	sales response constant, (/day).

M = saturation level of sales rate, (\$/day).

C = Selling price - assumed to be constant, (\$/unit).

C_v = per unit cost of raw materials, direct labor and other production costs that are proportional to $P(t)$, (\$/unit).

q = fraction reflecting all other variable costs.

$C_p[P(t) - P^*(t)]^2$ = The rate of costs that are related to the deviation of the actual rate of production, $P(t)$, from the desired rate of production, $P^*(t)$ (e.g. undertime-overtime costs), (\$/day).

$C_I[I(t) - I^*(t)]^2$ = Rate of costs associated with the deviation of the actual level of inventory, $I(t)$, from the desired level of inventory $I^*(t)$, (\$/day).

b_1 = value of a unit inventory at $t = T$, (\$/unit).

b_2 = value of a unit sales rate at $t = T$, (\$/unit)

F = All other fixed costs during the planning horizon.

A_{min} = The minimum rate of advertising that the firm can effectively maintain. Assumed to be zero in this paper.

A_{max} = The maximum rate of advertising that the firm can effectively maintain.

I_0 = initial level of inventory.

S_0 = initial sales rate.

Equation (3) above is the Vidale-Wolfe model of sales advertising relationship. The equation was proposed in 1957 [30], and it is empirically validated. The interpretation of the equation is the following: the increase in the rate of sales rate, $\frac{dS}{dt}$, is proportional to the intensity of advertising effort, A, reaching the fraction of potential customers, (1-S/M), less the number of customers that are being lost due to forgetting, λS. Equation (2) is the production-inventory identity. It says that the inventory accumulates at a rate equal to the difference between the production rate and the sales rate.[1] The objective function (1) is the negative of the profit during the planning period plus the value of the ending inventory and the ending sales rate (goodwill).

The model does not include constraints $I(t) \leq 0$ (i.e. no backordering), $P(t)<0$. This is because it is assumed that the presence of the quadratic terms in the objective function would preclude possibility of $I(t)<0$, $P(t)<0$. Moreover, for simplicity, $I^*(t)$ and $P^*(t)$ are assumed to be constants with respect to time.

3. NATURE OF THE OPTIMAL CONTROL PROBLEM UNDERLYING THE PROPOSED MODEL

Let η_1 and η_2 be the adjoint variables associated with constraints (2) and (3) respectively. Then the Hamiltonian for the above problem is

$$\begin{aligned} H(I,S,\eta_1,\eta_2,P,A) = {} & [-(1-q)S + A + C_v P + C_p(P - P^*)^2 \\ & + C_I(I - I^*)^2] + \eta_1[P - S/C] \\ & + \eta_2[rA(1 - S/M) - \lambda S] \end{aligned} \tag{6}$$

[1]Dimension of S(t) is \$/day. Thus, division of S(t) by selling price (C) is necessary for consistent dimensions in the above identity.

It is easy to see that the Hamiltonian is linear with respect to A and quadratic with respect to P. Thus the order of the hessian of H with respect to P and A is one. That is, the optimal control problem under consideration is a partially singular control problem [5].

4. NECESSARY CONDITIONS FOR THE OPTIMAL CONTROL

Using Pontryagin's Maximum Principle [23]:

$$\dot{\eta}_1 = -\frac{\partial H}{\partial I} = -2\, C_I(I - I^*) \tag{7}$$

$$\dot{\eta}_2 = -\frac{\partial H}{\partial S} = (1 - q) + \frac{\eta_1}{C} + \eta_2[\frac{rA}{M} + \lambda] \tag{8}$$

and

$$\eta_1(T) = -b_1 \tag{9}$$

$$\eta_2(T) = -b_2 \tag{10}$$

Furthermore

$$\frac{\partial H}{\partial P} = \eta_1 + 2C_p(P - P^*) + C_v = 0 \tag{11}$$

and, since H is linear with respect to A,

$$A(t) = \begin{cases} A_{max} & H_A < 0 \\ A_s \quad \text{if} & H_A = 0 \\ 0 & H_A > 0 \end{cases} \tag{12}$$

where

$$H_A = \frac{\partial H}{\partial A} = 1 + r\eta_2(1 - \frac{S}{M}) \tag{13}$$

and A_s denotes singular rate of advertising.

From (7) and (11)

$$\dot{\eta}_1 = -2C_I(I - I^*) = -2C_p \frac{dP}{dt}$$

or

$$\frac{dP}{dt} = \frac{C_I}{C_p} (I - I^*) \tag{14}$$

Similarly, from (9) and (11),

$$\eta_1(T) = -2C_p[P(T) - P^*] - C_v = -b_1$$

or

$$P(T) = P^* + [b_1 - C_v]/2C_p \tag{15}$$

Thus equation (11) can be used to eliminate η_1, and equations (14) and (15) can be used in place of equations (7), (9) and (11).

Note that A_s in condition (12) is not characterized yet. Furthermore, given condition (12) and given initial conditions (5) and terminal conditions (9) and (15), the structure of the optimal path for a sufficiently long planning horizon is likely to be one described in Fig. 1. Specifically, optimal advertising policy for the planning period is likely to be non-singular (i.e., $A = A_{max}$ or 0) in the beginning and at the end of the planning period and singular (i.e., $A = A_s$) during the middle of the planning period.

In what follows, we assume that the structure of the optimal path is as described in Fig. 1. Then, in light of this structure, we state the necessary conditions for the stage I and III of the optimal path. We characterize A_s and develop necessary conditions for the stage II of the

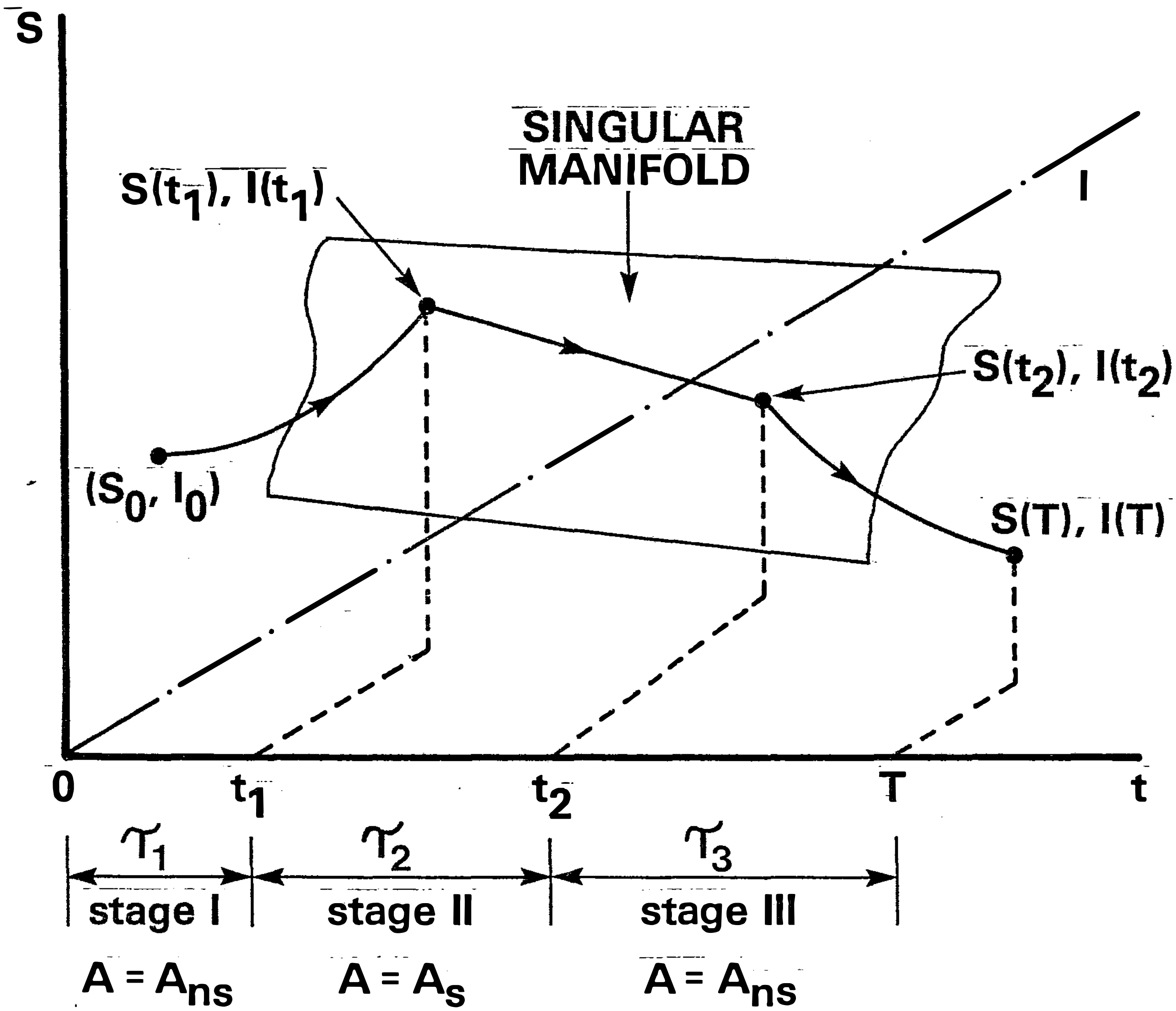

FIGURE 1

Structure of the Optimal Path

path. Finally, we state the necessary conditions for the optimal path at junction times t_1 and t_2.

4.1. Necessary conditions for stage I and III of the Optimal Path

Let A_{ns} denote $A = A_{max}$ or $A = 0$. Then necessary conditions for stage I {i.e. for $t \in [0, t_1)$} in (S, I, η_2, P, A) space are: conditions (2), (3), (8), (14); and the condition

$$A = A_{ns} = \begin{cases} A_{max} & \text{if} \quad H_A < 0 \\ \\ 0 & \text{if} \quad H_A > 0 \end{cases} \tag{16}$$

plus the initial conditions $I(0) = I_0$, $S(0) = S_0$; t_1 is a parameter. Similarly necessary conditions for stage III {i.e., for $t \in (t_2, T]$} are the same as before except that now boundary conditions are terminal conditions $\eta_2(T) = -b_2$, $P(T) = P^* + [b_1 - C_v]/2C_p$; t_2 is a parameter.

4.2 Necessary conditions for stage II of the Optimal Path

In state II, $A = A_s$ and $H_A = 0$. Thus in stage II,

$$H_A = 1 + r\eta_2(1 - \frac{S}{M}) = 0 \tag{17}$$

Differentiating (17) once and substituting for $\dot{\eta}_2$ from (8) and for $\dot{S}$ from (3), we have

$$\frac{d}{dt} H_A = (1 - q) (1 - \frac{S}{M}) + \lambda\eta_2 + \frac{\eta_1}{C} (1 - \frac{S}{M}) = 0 \tag{18}$$

Similarly differentiating (18) once more, and simplifying, it can be shown that

$$A = A_s = \frac{\lambda S}{r(1 - S/M)} - \frac{C_I M}{C\lambda} (I - I^*)(1 - S/M)^2 \qquad (19)$$

Also from (17)

$$\eta_2 = \frac{-1}{r(1 - S/M)} \qquad (20)$$

and from (18) and (11),

$$P = P^* + \frac{C}{2C_p} \{q_v - \frac{\lambda/r}{(1 - S/M)^2}\} \qquad (21)$$

where

$$q_v = 1 - q - C_v/C$$

When $A = A_s$, (3) reduces to

$$\dot{S} = - \frac{rC_I M}{C} (I - I^*)(1 - S/M)^3 \qquad (22)$$

and from (21), (2) reduces to

$$\dot{I} = P^* + \frac{C}{2C_p} \{q_v - \frac{\lambda/r}{(1 - S/M)^2}\} - \frac{S}{C} \qquad (23)$$

Thus the necessary conditions for stage II are conditions (22), (23), (20), (21), (19) and the condition that t_1 and t_2 are parameters.

Note that the equilibrium point of (22) and (23) can be obtained by setting the left hand sides in those equations to zero. Thus the equilibrium point for equation (22) and (23) is

$$I = I^* \qquad (24)$$

and $S = S^*$ where S^* is the solution[2] of

$$P^* + \frac{C}{2C_p} \{q_v - \frac{\lambda/r}{(1 - S/M)^2}\} - \frac{S}{C} = 0 \qquad (25)$$

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[2] It can be shown that only one of the roots of the above cubic would be between 0 and M.

Moreover, from (2) and (3)

$$A = \lambda S^*/[r^*(1 - S^*/M)] \tag{26}$$

$$P = S^*/C \tag{27}$$

It can be shown that the equilibrium point described by equations (24) through (27) above is the long run optimum for the firm. Thus from (1), the profit rate at the long run optimum for the firm is given by:

$$R_e = (1 - q)S^* - A - C_p(P - P^*)^2 - C_v P \tag{28}$$

where A and P are as described in equations (26) and (27) above.

The equilibrium point (I^*,S^*) described above is a saddle point for equations (22) and (23) [9]. The behaviour of the singular subarcs in the neighbourhood of (I^*,S^*) is therefore as described in Fig. 2 [9].

4.3. Necessary conditions for the Optimal Path at Junction Times

The state variables I and S, and the adjoint variables η_1 and η_2 are continuous at the junction points [23]:

Thus, at t_1 and t_2

$$I^- = I^+ \tag{29}$$

$$S^- = S^+ \tag{30}$$

$$\eta_1^- = \eta_1^+ \tag{31}$$

$$\eta_2^- = \eta_2^+ \tag{32}$$

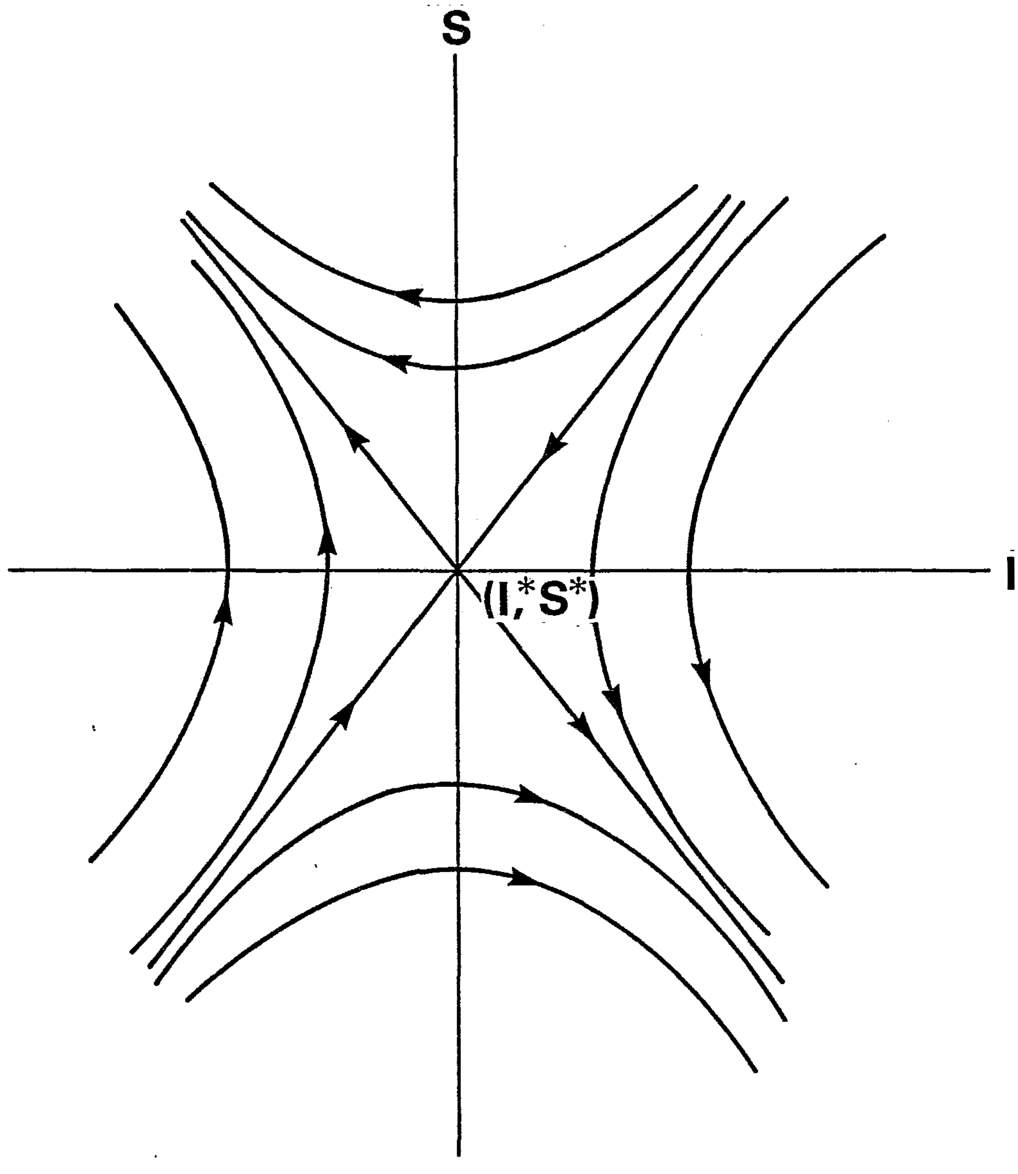

FIGURE 2

Stability of the System Represented by Singular Equations
in the Neighbourhood of the Equilibrium Point (I*,S*)

However, from (11),

$$\eta_1 = -2C_p(P - P^*) - C_v$$

Thus, continuity of η_1 at t_1 and t_2 implies continuity of P at t_1 and t_2.

Hence, at t_1 and t_2,

$$P^- = P^+. \tag{33}$$

5. A PROCEDURE FOR SYNTHESIZING THE OPTIMAL CONTROL FOR A GIVEN PROBLEM

The necessary conditions for the various parts of the optimal path were described in the previous section. In this section, we further examine the optimal path in light of those necessary conditions. First we describe the stage-III boundary value problem that characterizes the stage-III non-singular subarc. Then we describe the admissible set of singular subarcs; i.e. the set of singular subarcs that can be a part of an optimal path. We study the optimality of the stage-I non-singular subarc; and in the final part of this section, present a procedure for synthesizing the optimal control to a given problem.

5.1. Boundary Value Problem for the Stage III Non-Singular Subarc

From (21) and (33)

$$P(t_2^+) = P(t_2^-) = P^* + \frac{C}{2C_p}\{q_v - \frac{\lambda/r}{[1 - S(t_2^-)/M]^2}\} \tag{34}$$

Similarly from (20) and (32),

$$\eta_2(t_2^+) = \eta_2(t_2^-) = -1/\{r[1 - S(t_2^-)/M]\} \tag{35}$$

Thus, for the stage III non-singular subarc, there are three initial conditions [condition (34), condition (35) and the assumption of $S(t_2)$] and two terminal conditions (10) and (15). However the duration of stage-III, $\tau 3 = T3 - t_2$ is free to vary. Thus for a given $S(t_2)$, the stage III non-singular subarc is characterized by differential equations (2), (3), (8) and (14), and the boundary conditions (34), (35), (10) and (15). Computational experience with the model suggests that in general the locus of $[S(t_2), I(t_2)]$ is as described in Fig. 3.

5.2. The Admissible Set of Singular Subarcs

In Fig. 2, the singular subarcs in the neighbourhood of (I*, S*) were described. Given the locus of $[S(t_2), I(t_2)]$ described in Fig. 3 and given that at t_2 the optimal arc is singular the set of singular subarcs that can be part of the optimal solution is (in the reverse-time sense) as shown in Fig. 4. Note that point 'a' in Fig. 4 is called the point of inflection.

5.3. Optimality of the Non-Singular Control for Stage I

It is obvious that optimal control for stage I would be very much a function of the initial conditions I_0, S_0. Because of the partially singular control nature of the problem, it is not possible to study optimality of A_{ns} in the various regions of the I-S plane analytically [14]. Hence a simulation procedure is used in this paper.

First it is noted that, similar to junction time t_2, at t_1

$$P(t_1^-) = P(t_1^+) = P^* + \frac{C}{2C_p}\{q_v - \frac{\lambda/r}{[1 - S(t_1^+)/M]^2}\} \qquad (36)$$

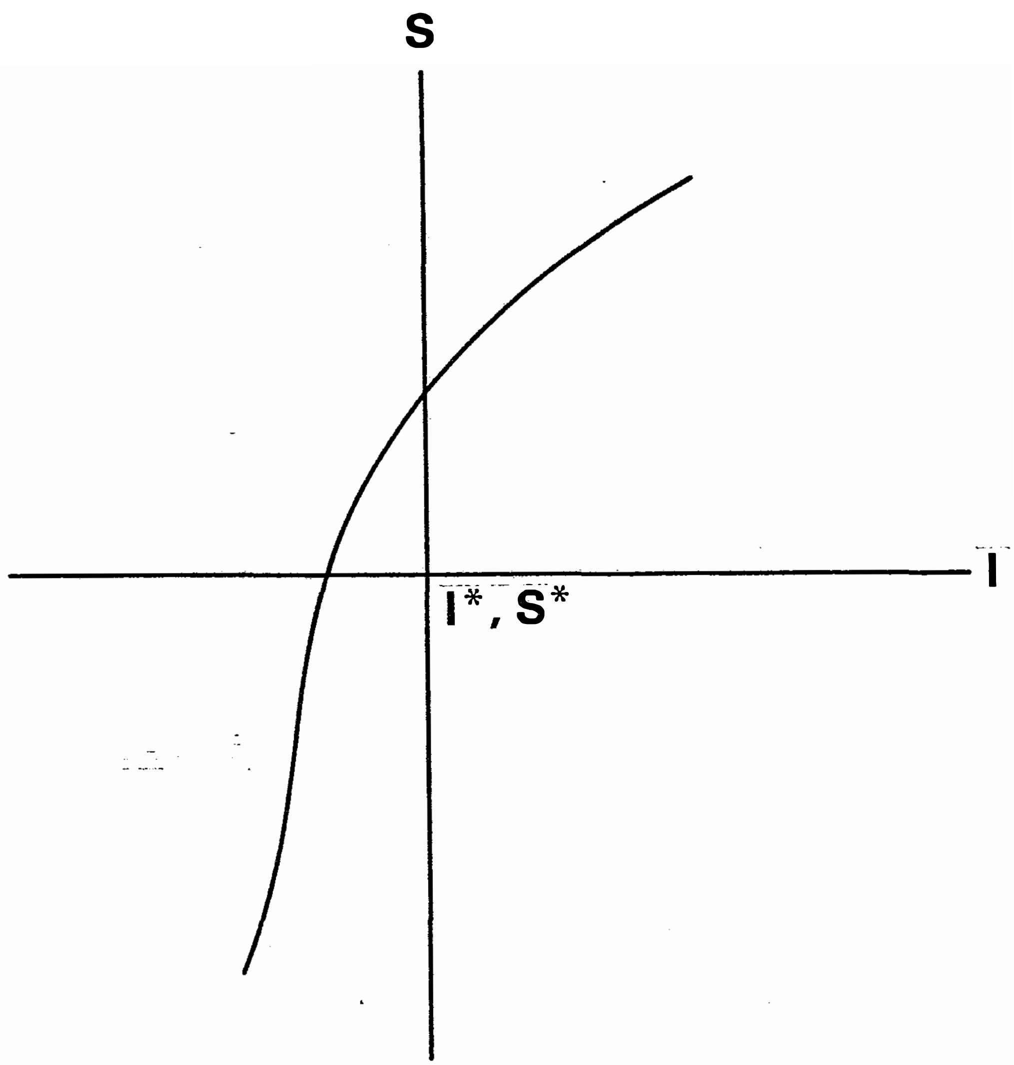

FIGURE 3

The Locus of the Junction Point $[S(t_2), I(t_2)]$ in the I-S Plane

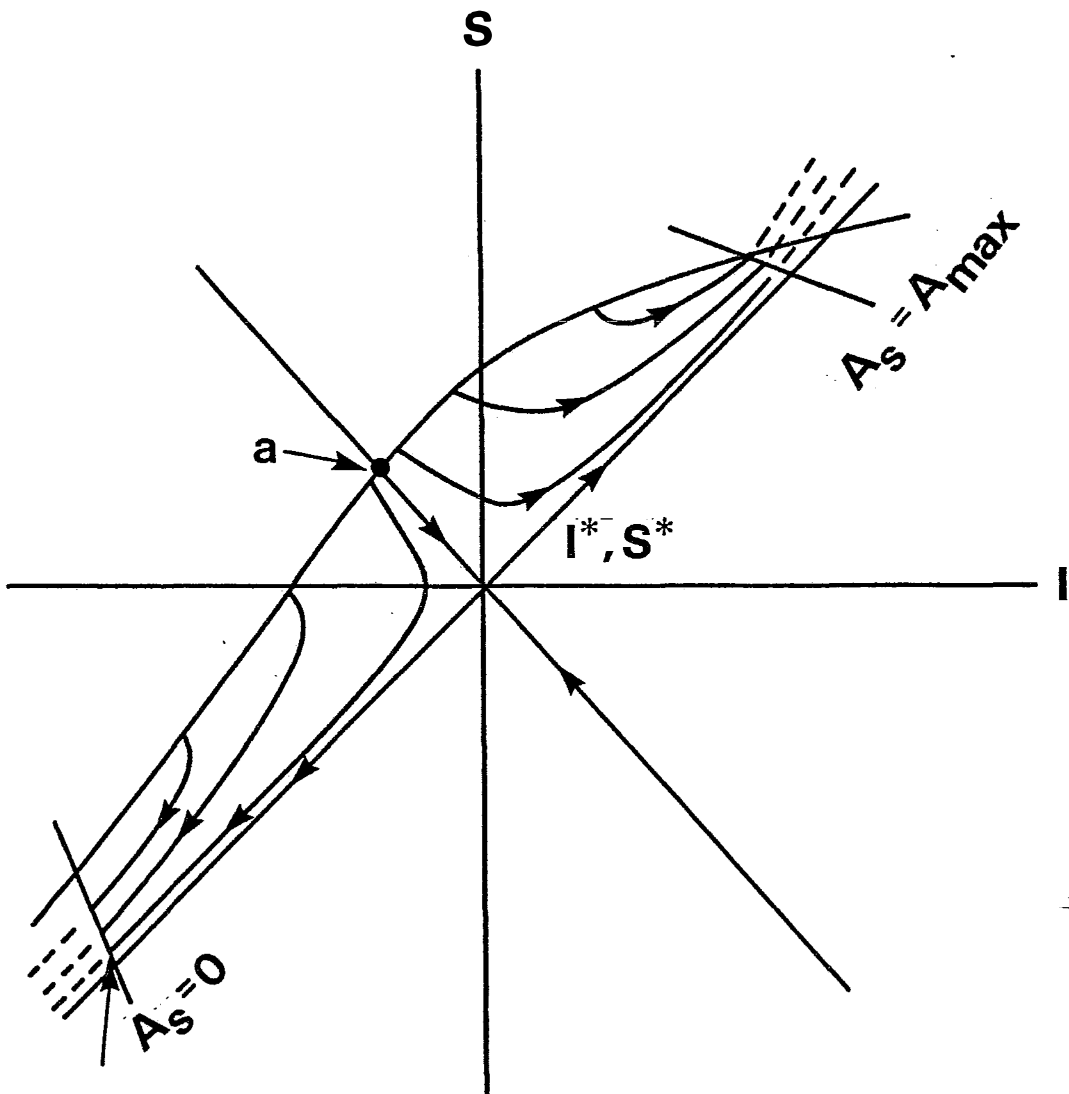

FIGURE 4

Admissible Set of Singular Arcs

$$\eta_2(t_1^-) = \eta_2(t_1^+) = -1/\{r[1 - S(t_1^+)/M]\} \qquad (37)$$

Furthermore, optimality of A_{ns} in stage-I is determined by condition (16). Thus if a point $[S(t_1), I(t_1)]$ is selected on some admissible singular subarc, equations (2), (3), (8) and (14) can be used to simulate (in the reverse-time sense) the non-singular subarc ending at $[S(t_1), I(t_1)]$. Thus, using the simulation procedure, optimality of A_{ns} in the various regions of the I-S plane can be studied empirically. The optimality of A_{ns} in the various regions of I-S plane for a pair of singular subarcs that is symmetric about point 'a' is shown in Fig. 5.

5.4. A Procedure For Finding An Optimal Control to a Given Problem

Given the analysis presented so far, we now describe a procedure for finding an optimal control to a given problem. The procedure is a two phase procedure. Phase I is devoted to determining the specific structure of the optimal control to a given problem. Phase II is devoted to finding the optimal control. The two phases of the procedure are described below.

Phase I

1. Formulate the stage-III boundary value problem described in 5.1. Solve (25) to obtain S^* and for a few values of $S(t_2) > S^*$, sove the stage III boundary value problem described in section 5.1. Plot the locus of $[S(t_2), I(t_2)]$ as described in Fig. 3 and identify point 'a' on the locus.

2. On the locus of $[S(t_2), I(t_2)]$, select two points close but on opposite sides of 'a'. Using the necessary conditions developed in 5.2, develop (in the reverse-time sense) the two singular arcs ending at the two points.

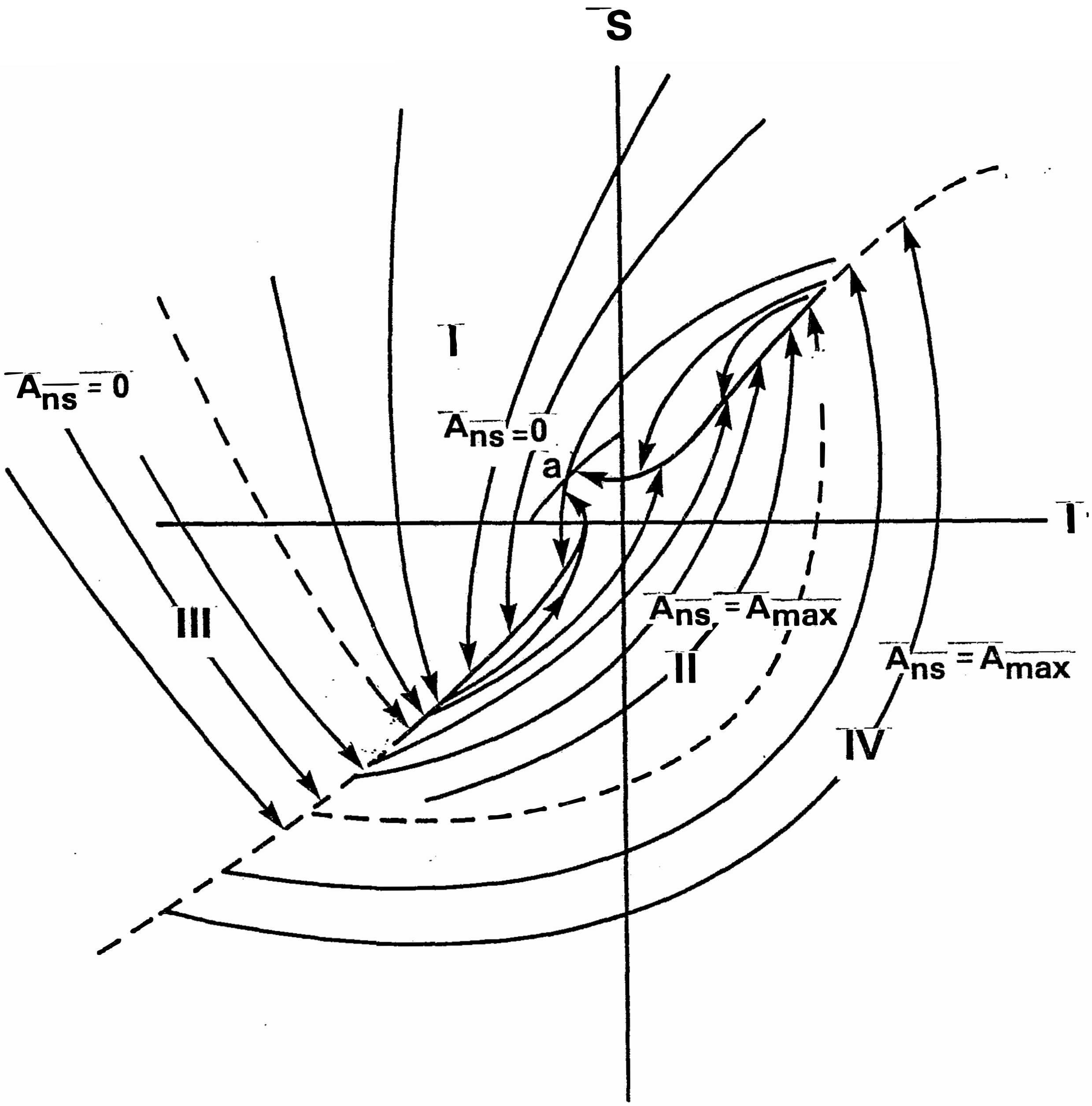

FIGURE 5

Optimality of the Stage I Non-Singular Control in Various
Regions of the I-S Plane

3. Using the simulation procedure described in 6.3, study the optimality of A_{ns} for stage I in the various regions of the I-S plane for the pair of singular arcs developed in step 2. Note the region in which the initial point I_0, S_0 lies.

Phase II

4. Identify the path ending at I_0, S_0 and note the corresponding $S(t_2)$ (see Fig. 6). For that $S(t_2)$, identify the three durations T3, T2 and T1 and record T_c = T3 + T2 + T1.

5. If $T_c > T$, select a new $S(t_2)$ that is farther from S* than the current $S(t_2)$. If $T_c < T$, do the opposite. If $T_c = T$ stop. You have solved the given planning problem.

6. For the new $S(t_2)$, solve the stage-III boundary value problem. Note the new T3. For the new $[S(t_2), I(t_2)]$ develop the singular subarc and using the simulation procedure of subsection 5.3. Identify the stage I non-singular arc that ends at (I_0, S_0). Note the stage II and the stage-I durations (i.e., T2 and T1) and record the new T_c. Go to step 5.

Note that, although the selection of a new $S(t_2)$ in step 5 can be carried out several ways, the following interpolation scheme has been found useful. Let $S^1(t_2)$ and $S^2(t_2)$ be the two previously chosen sales rates so that total durations corresponding to these sales rates, say T_{c_1} and T_{c_2}, are closest to T. Select the new $S(t_2)$ as follows

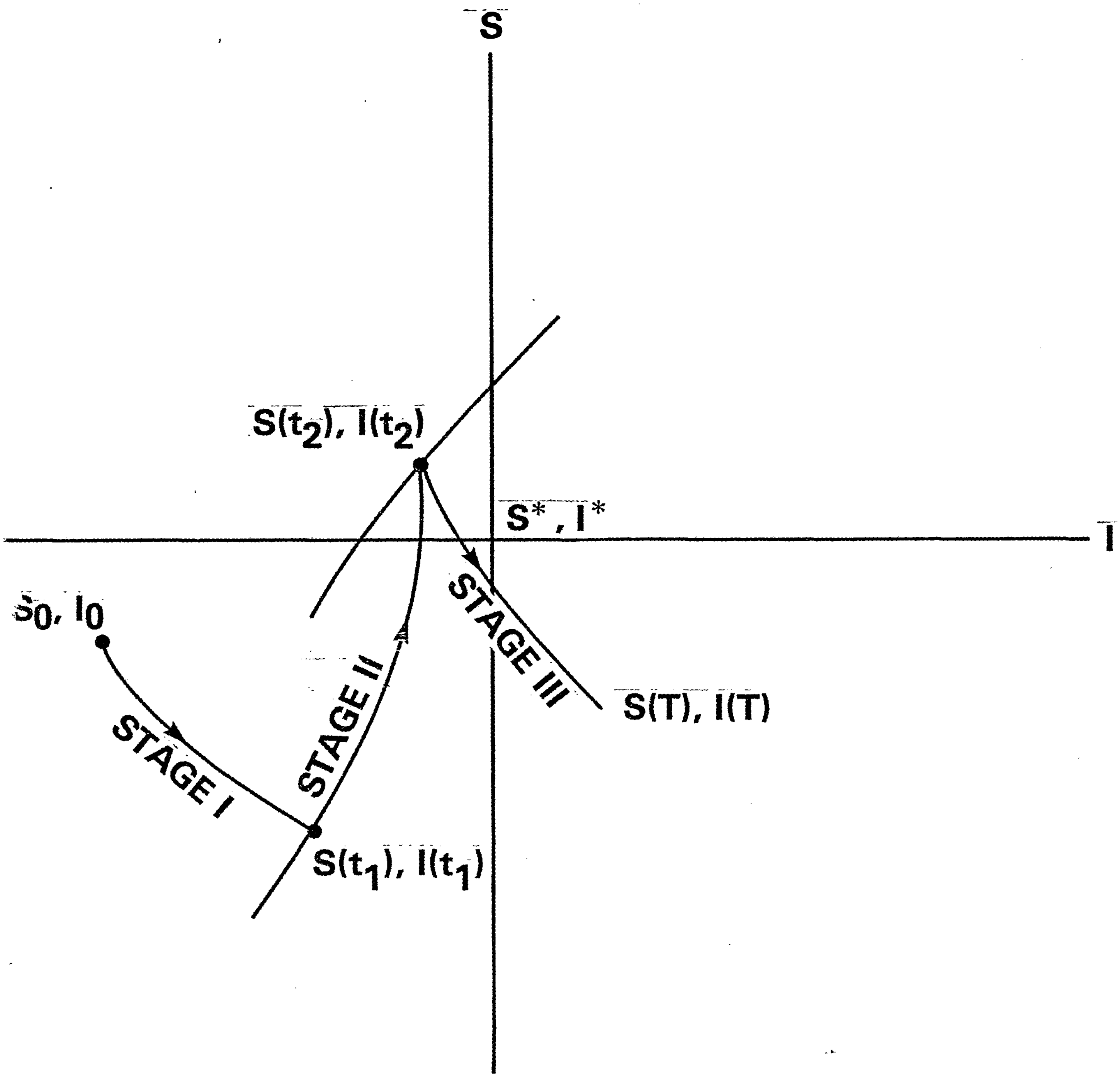

FIGURE 6

Optimal Path for $(S_{0_1} I_0)$

$$S(t_2) = \frac{S^1(t_2)T_{c_2} - S^2(t_2)T_{c_1}}{T_{c_2} - T_{c_1}} + \frac{S^1(t_2) - S^2(t_2)}{T_{c_1} - T_{c_2}} \cdot T$$

6. AN EXAMPLE OF MARKETING-PRODUCTION PLANNING

Example 6.1

Consider a problem where

T	$= 180$ days	P^*	$= 1500$ units
C_p	$= .01$	I^*	$= 15000$ units
C_I	$\doteq .00005$	C	$= \$40$/unit
λ	$= .016$	M	$= \$100000$/day
r	$= .1$	I_0	$\doteq 17000$ units
q	$= .1$	S_0	$= \$40000$/day
C_v	$= 20$	b_1	$= 15,\ b_2 = 6$
q_v	$= (1 - q) - C_v/C$	A_{max}	$= \$20000$/day
	$= 1 - .1 - 20/40$	A_{min}	$= \$0$/day
	$= .4$		

In this example, the long run equilibrium point [as characterized by equations (24) and (25)] is found to be:

$I^* = 15000$

$S^* = 46790.4$ \$/day

Analysis of phase I reveals that in this case the initial point (I_0, S_0) is in region II of Fig. 5. That is, the optimal advertising expenditure for the stage I of the optimal path is $A_{ns} = A_{max}$. The path for $S(t_2) = 47788$ has total duration (T_c) of 132.3 days.

In phase II, it was found that for $S(t_2) = 47788$, the point $[S(t_2),$

$I(t_2)]$ provided by the stage-III boundary value problem was very close to point 'a' described in Fig. 4. That is, the optimal path differed by very little (except for the time spent near the long run optimum) for $T_c > 132.3$. Thus the optimal path for T = 180 is assumed to be similar to the optimal path for $T_c = 132.3$ except that in case of the first, the additional time of 180 - 132.3 = 47.7 days is assumed to be spent near the long run optimum. The optimal path for $T_c = 132.3$ is shown in Fig. 7. Time plots for the optimal for T = 180 (constructed from Fig. 7) are shown in Fig. 8.

7. CAPACITY EXPANSION PROBLEMS

In capacity expansion problems, a firm is typically concerned with the problem of estimating the impact on its profit level if its capacity is changed from a current level, say P_1^*, to some new level, say P_2^*. The planning horizons in capacity expansion problems are, furthermore, long because they are determined by factors such as rate of obsolescence, life of the equipment etc. Hence, in capacity expansion problems, the firm's future profit rates can be estimated using its long run optimal profit rate. Thus, in this paper, the profit rates associated with the level of capacity P* can be estimated using the long run optimal profit rate described in expression (28). The firm can estimate profit rates associated with the current and the proposed level of capacity using expression (28) and use the difference between the two profit rates as an estimate of the incremental cash inflows in its evaluation. The example presented below illustrates the use of this philosophy.

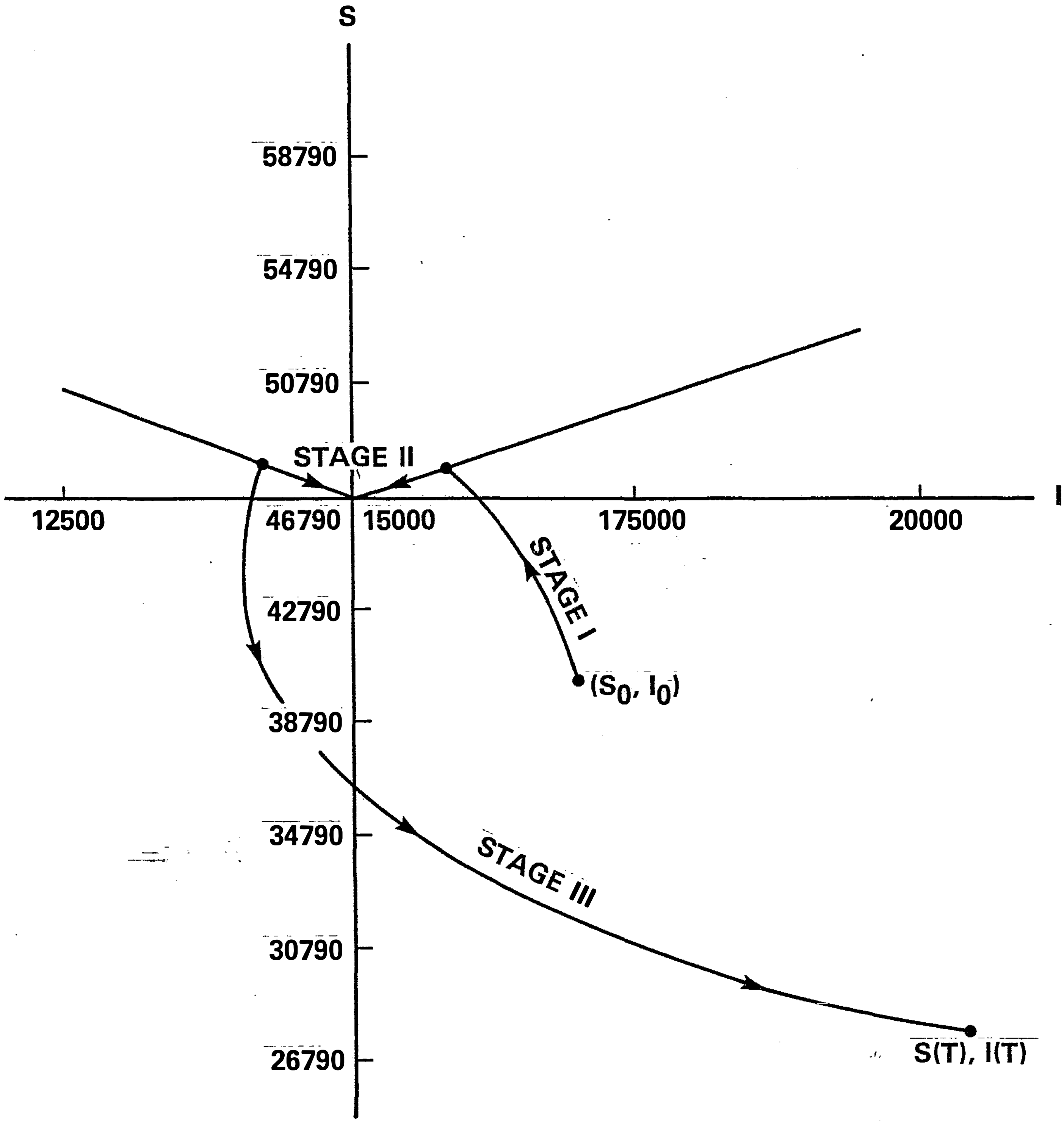

FIGURE 7

Optimal Trajectory Using the Interdependent Approach

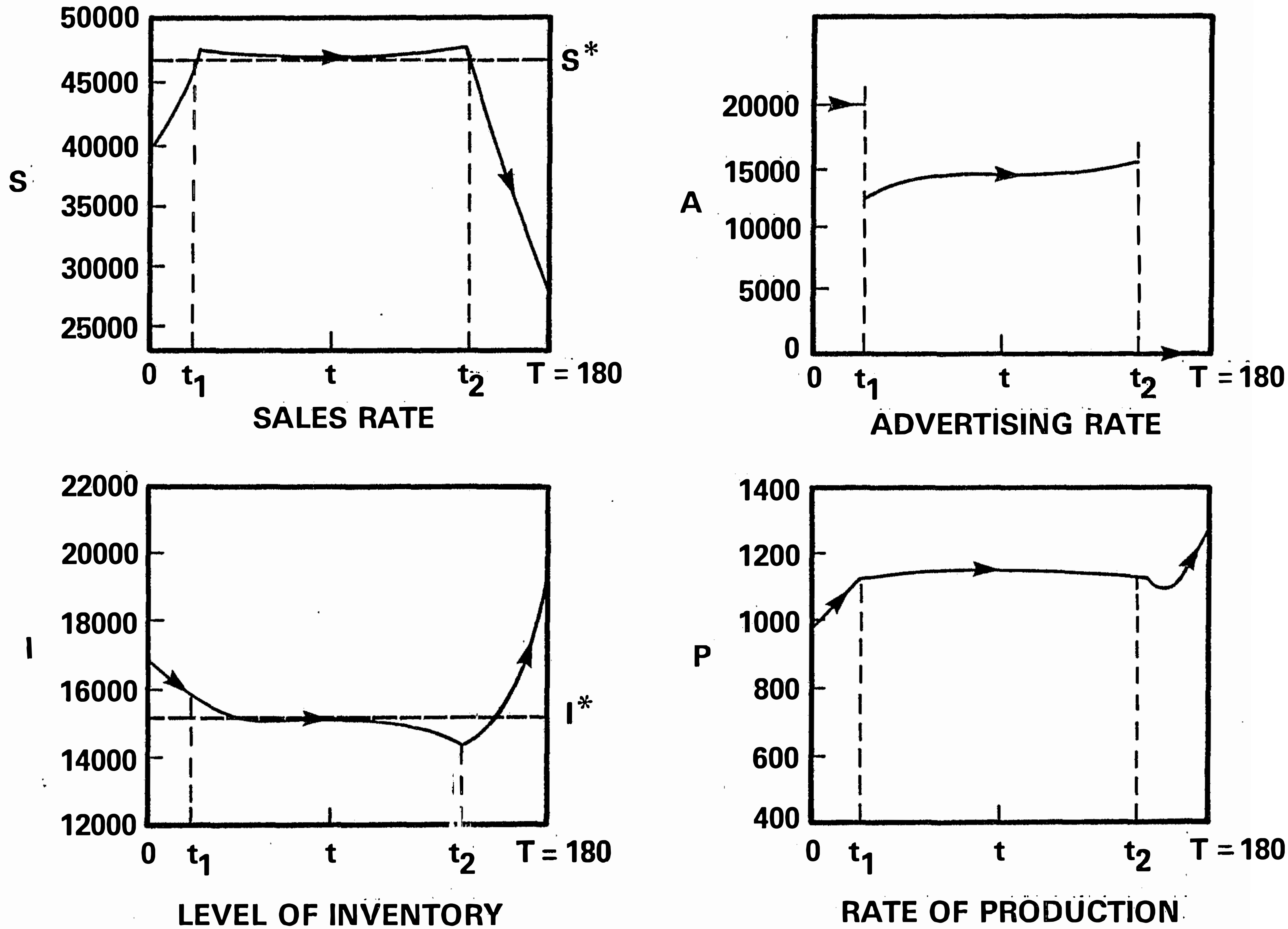

FIGURE 8

Optimal Solution Using the Interdependent Approach

Example 7.1

Consider an example where

$P_1^* = 1000$ units/day	$\lambda = .01$
$C = \$10$/unit	$r = .2$
$M = \$20000$/day	$q = .15$
$C_v = \$5$/unit	$q_v = 1 - q - C_v/C$
$C_p = .01$	$= 1 - .15 - .5 = .35$

In this example, equation (25) provides

$S^* = \$10614$/day

Similarly from (26) and (27)

$A = \$1131.7$/day

$P = 1061.4$ units/day

and from (28), the rate of profit associated with the capacity P_1^* is

$R_e = \$2545.50$/day

Let us say that since $P > P_1^*$, the manufacturer in this case is contemplating to increase the capacity to 1300 units/day, and the cost of the expansion is 200000 \$. Should the manufacturer increase the capacity? Assume that the firm's acceptable rate of return is 12%, and the planning horizon for the problem is 10 years.

For $P_2^* = 1300$, equation (25) now provides

$S^* = \$12813$/day

and (26) and (27) provide

$A = \$1783$/day

$P = 1281.3$ units/day.

Similarly the new rate of profit using equation (28) is

$$R_e = \$2698/\text{day}.$$

Thus, changing capacity from 1000 units/day to 1300 units/day increases daily profit by

$$\$2698 - \$2545.5 = \$152.5$$

and assuming 260 working days, yearly profit increases to

$$\$152.5 \times 260 = \$39650/\text{yr}.$$

The rate of return associated with an annuity of $39650 on an initial investment of $200000 over 10 years is about 15%. Since this rate of return is more than the firm's acceptable rate of return, the firm should expand the capacity.

8. CONCLUSIONS

The interdependencies between marketing and production functions in a firm are explored using an optimal control model of a marketing-production system. The model, which is based upon empirically derived subsystem models, leads to a problem that is partially singular. A procedure is developed to determine the optimal advertising and production policy for the proposed model and it is shown that the results deriving from the model are applicable to capacity expansion problems.

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