Hull Properties in Location Problems

A RESEARCH REPORT

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HULL PROPERTIES IN LOCATION PROBLEMS

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Some properties of the solution set for single and multi-facility continuous location problems with ℓ_p distances are given. A set reduction algorithm is developed for problems in k-dimensional space having rectangular distances.

We address the location problem

$$\text{minimize } f(x) = \sum_{i=1}^{m} w_i \ell_p(x-a_i) \tag{1}$$

where m is the number of existing facilities, the w_i, i=1,...,m, are m positive weights, the a_i, i=1,2,...,m, are the locations of m existing facilities, and $p > 1$ is the ℓ_p norm parameter. In this paper we determine the smallest set of points which contains the optimal solution of location problems with ℓ_p distances. This problem has previously been addressed by Kuhn [3] (for Euclidean distances), and by Wendell and Hurter [5]. For the special case where distances are rectangular ($p = 1$), an algorithm is given to determine a least set of points for problems in k-dimensional space. Determining a least set of possible solution points is important when solving location-allocation problems utilizing p-median algorithms as described in [4]. This is due to the property of p-median problems that computation times increase rapidly as a function of the number of possible location sites, making it possible at the present time to solve only relatively modest-sized problems [4].

We use the following notation:

xy denotes the inner product between the vectors x and y,

x^s denotes a vector with components (sign x_i)$|x_i|^s$ for i=1,2,...,k, where x_i for i=1,2,...,k are the components of the vector x and s is a positive real number,

f'(x) denotes the derivative of the function f at the point x,

f'(x;y) denotes the directional derivative of the function f at the point x in the direction y, defined by

$$f'(x;y) = \lim_{h \to o+} \frac{f(x + hy) - f(x)}{h}, \text{ and}$$

$\ell_p(x)$ denotes the ℓ_p norm of the k-vector x, defined by

$$\ell_p(x) = \left(\sum_{i=1}^{k} |x_i|^p\right)^{1/p} .$$

The following observations can be noted.

1. $\ell_p(\cdot)$ is a convex function, because it is a norm
2. $\ell_p'(x) = x^{p-1}[\ell_p(x)]^{1-p}$ for $x \neq 0$ and $p > 1$, from the definition of $\ell_p(\cdot)$ and using the notation x^s
3. The objective function in the location problem is convex, because it is a sum of convex functions (remark 1)
4. The derivative of the objective function, where defined, is given by

$$f'(x) = \sum_{i=1}^{m} w_i(x-a_i)^{p-1}[\ell_p(x-a_i)]^{1-p}, \text{ from remark 2.}$$

5. The directional derivative of the objective function at a point of differentiability x in the direction y is given by

$$f'(x;y) = f'(x)y = \sum_{i=1}^{m} w_i [\ell_p(x-a_i)]^{1-p} y(x-a_i)^{p-1},$$

using the relationship between derivative and directional derivative and remark 4.

6. A point does not solve the location problem if there exists a direction for which the directional derivative of the objective function is negative, because of convexity (remark 3).

We first prove the following property.

<u>Lemma</u>

Let x and y be two 2-vectors, and let s be a positive real number. If $xy < 0$ then $x^s y^s < 0$.

<u>Proof</u>

We are given $x_1y_1 + x_2y_2 < 0$, where $x = (x_1,x_2)$ and $y = (y_1,y_2)$. We must show that

$$[(\text{sign } x_1)|x_1|^s][(\text{sign } y_1)\ |y_1|^s] + [(\text{sign } x_2)\ |x_2|^s][(\text{sign } y_2)\ |y_2|^s]$$
$$= [(\text{sign}(x_1y_1))|x_1y_1|^s] + [(\text{sign}(x_2y_2))|x_2y_2|^s] > 0.$$

If x_1y_1 and x_2y_2 are both non-positive, the result is clear. Otherwise x_1y_1 or x_2y_2 is negative, by the assumption. Without loss of generality we may assume $x_1y_1 < 0$, and thus $|x_2y_2| < |x_1y_1|$. Then it follows that

$$x^s y^s = -|x_1y_1|^s + |x_2y_2|^s < 0,$$

because s is positive.

The following property concerns the optimal solution to problem (1).

Property 1

If x solves the single facility location problem in 2 dimensions, then x lies in the convex hull of the existing facilities.

Proof

Suppose the conclusion of the property were not true. Then there exists a line separating x from the convex hull, and there exists a 2-vector b such that $b(x-a_i) < 0$ for $i=1,2,\dots,m$.

By remark 5 the directional derivative of the objective function in the direction b^{p-1} is

$$f'(x;b^{p-1}) = \sum_{i=1}^{m} w_i [\ell_p(x-a_i)]^{1-p} b^{p-1}(x-a_i)^{p-1}.$$

By the lemma, the directional derivative is negative. Therefore, by remark 6, x does not solve the problem, thus contradicting the assumption of the property.

In a more general setting Wendell and Hurter obtained a slightly weaker result ([5], corollary 4, p. 317). In the context of the problem considered here, they show that an optimal solution exists in the convex hull, but their result does not imply (whereas ours does) that all the optimal solutions must be in the convex hull. Wendell and Hurter ([5], p. 318) note that the result appears to hold for the multi-facility problem, but offer no justification. Our result generalizes easily to the multi-facility location problem:

$$\text{minimize } f(x_1,x_2,\ldots,x_n) = \sum_{j=1}^{n} w_{ij}\ell_p(x_j-a_i) + \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} v_{jk}\ell_p(x_j-x_k)$$

where the notation is analogous to the single-facility location problem. In addition, n is the number of new facilities and v_{jk} for $1 \leq j < k \leq n$, is the non-negative weight between new facilities j and k.

Property 2

The optimal locations of the new facilities in the multi-facility problem in 2 dimensions with the new facilities chained ([1], p. 338) are in the convex hull of the existing facilities.

Proof

The formal proof is identical to the one given by Francis and Cabot ([1], property 2, p. 340) with our Property 1 substituted for Kuhn's Result 1 ([1], p. 337) and will not be repeated.

Informally, consider first the case where no new facilities coincide with each other. If we are given a proposed solution where some new facilities lie outside the convex hull, then there must be a new facility that lies outside the convex hull of the existing and other new facilities. Thus by Property 1 the proposed solution cannot be optimal. For the case where new facilities coincide, the same argument applies when we treat each cluster of coinciding new facilities as one new facility.

The convex hull properties hold for 2-dimensional location problems with ℓ_p norms when $1 < p < \infty$. When $p = 2$ (Euclidean norms) the properties hold for any dimension ([1]). When $p = 1$ (rectangular norm) a stronger property, the rectangular hull property ([4], theorem 2, p. 448) holds for the 2-dimensional case. But for higher dimensions, even the convex hull properties cannot be expected to hold, as illustrated by this single-facility example in 3 dimensions with 4 existing facilities:

i	w_i	a_i	x	$f(x)$	x	$f(x)$
1	2	(0,0,0)	(0,0,0)	11	(1,0,0)	12
2	2	(0,1,1)	(0,0,1)	12	(1,0,1)	13
3	2	(1,1,0)	(0,1,0)	8	(1,1,0)	9
4	1	(1,1,1)	(0,1,1)	9	(1,1,1)	10

The optimal location (0,1,0) of the new facility is outside the convex hull of the existing facilities. A simpler example can be obtained by deleting existing facility 4 above, but in that case the convex hull is 2-dimensional (as in the example given in [2], p. 53).

The algorithm for determining the rectangular hull in two dimensions ([4], p. 443) can be extended to produce a reduced set of points which are candidates for the optimal solution for problems in k-dimensional space. Let the locations of the fixed facilities be given by $a_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$, for $i=1,2,\ldots,m$, the facility location to be determined by $x = (x_1, x_2, \ldots, x_k)$, and let J be the index set $\{1,2,\ldots,m\}$. Let m positive numbers be given by w_i, $i=1,\ldots,m$. We now give a more general version of theorem 1 of [4].

Lemma

Let A and B be subsets of $J = \{1,2,\ldots,m\}$ with no elements in common. Let w_i be positive for $i \varepsilon J$. If $A \cup B \neq J$ then the two inequalities

$$\sum_{i \varepsilon A} w_i \geq \sum_{i \notin A} w_i \quad \text{and} \quad \sum_{i \varepsilon B} w_i \geq \sum_{i \notin B} w_i \tag{2}$$

cannot both hold. ($i \notin A$ refers to the elements of J that do not belong to A).

Proof

From the assumptions on A and B, there exists a non-empty subset of J (call it C), such that A,B, and C form a partition of J. Therefore

$$\sum_{i \varepsilon J} w_i = \sum_{i \varepsilon A} w_i + \sum_{i \varepsilon B} w_i + \sum_{i \varepsilon C} w_i$$

and from the inequalities (2) we have:

$$\sum_{i \varepsilon A} w_i \geq \sum_{i \notin A} w_i = \sum_{i \varepsilon B} w_i + \sum_{i \varepsilon C} w_i \geq \sum_{i \notin B} w_i + \sum_{i \varepsilon C} w_i =$$

$$\left(\sum_{i \varepsilon A} w_i + \sum_{i \varepsilon C} w_i\right) + \sum_{i \varepsilon C} w_i = \sum_{i \varepsilon A} w_i + 2 \sum_{i \varepsilon C} w_i, \text{ or}$$

$$2 \sum_{i \varepsilon C} w_i \leq 0$$

which is false, because C is non-empty and the w_i's are positive.

The single facility location problem in k dimensions with rectangular distances is given by

$$\text{minimize} \sum_{i=1}^{m} w_i \sum_{j=1}^{k} |x_j - a_{ij}|.$$

This problem is decomposable into k separate problems, one along each axis. The x_j-axis problem is given by:

$$\text{minimize} \sum_{i=1}^{m} w_i |x_j - a_{ij}|.$$

It is well known that necessary and sufficient conditions for x_j^* to be an optimal solution for the one-dimensional problem are

$$\sum_{i\epsilon\{i:a_{ij}\leq x_j^*\}} w_i \geq \sum_{i\epsilon\{i:a_{ij}>x_j^*\}} w_i \quad \text{and} \quad \sum_{i\epsilon\{i:a_{ij}\geq x_j^*\}} w_i \geq \sum_{i\epsilon\{i:a_{ij}<x_j^*\}} w_i$$

The k-dimensional problem thus has 2k optimality conditions. This leads to the following method for eliminating points as candidates for the optimal solution of the k-dimensional problem.

Property 3

Let $S_j = \{i:a_{ij} \leq x_j\}$ and $T_j = \{i:a_{ij} \geq x_j\}$ for $j=1,2,\ldots,k$. If any two sets taken from among these 2k sets (call them A and B) have no elements in common, and if $A \cup B \neq \{1,2,\ldots,m\}$, then the point $x = (x_1,x_2,\ldots,x_k)$ does not solve the k-dimensional problem.

Proof:

If x solves the k-dimensional problem, then x_j solves the one-dimensional problem along the jth axis, for $j = 1,2,\ldots,k$. But if the assumptions of the property are satisfied, then by the Lemma it is not possible for all 2k optimality conditions to hold.

Property 4

The locations of the existing facilities cannot be eliminated as candidates for the optimal solution by Property 3.

Proof:

If $x = a_i$, then the 2k sets S_j, T_j for $j = 1,2,\ldots,k$ all have the element i in common.

It is well known that an optimal solution can be found in $R = R_1 \times R_2 \times \ldots \times R_k$, where $R_j = \{a_{1j},a_{2j},\ldots,a_{mj}\}$ for $j = 1,2,\ldots,k$. Each point in R can be checked and may be eliminated as a candidate for the optimal solution by Property 3. The extreme case results in the reduction from m^k candidate points to m candidate points, as exemplified by a problem where $a_i = (i,i,...,i)$ for $i = 1,2,...,m$.

For a more typical case consider the example given previously. The set R contains 8 points, of which 4, by Property 4, cannot be eliminated by Property 3. The remaining 4 points are analyzed in the table, using the notation from Property 3. It is seen that all these remaining points, except the optimal solution, are eliminated.

Table 1

x_1	S_1, T_1	x_2	S_2, T_2	x_3	S_3, T_3	Comment
0	{1,2}, {1,2,3,4}	0	{1}, {1,2,3,4}	1	{1,2,3,4}, {2,4}	eliminated $A = S_2$, $B = T_3$
0	{1,2}, {1,2,3,4}	1	{1,2,3,4}, {2,3,4}	0	{1,3}, {1,2,3,4}	not eliminated
1	{1,2,3,4}, {3,4}	0	{1}, {1,2,3,4}	0	{1,3}, {1,2,3,4}	eliminated $A = T_1$, $B = S_2$
1	{1,2,3,4}, {3,4}	0	{1}, {1,2,3,4}	1	{1,2,3,4}, {2,4}	eliminated (e.g.) $A = S_2$, $B = T_3$

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References

[1] R.L. Francis and A.V. Cabot, "Properties of a Multifacility Location Problem Involving Euclidean Distances", Naval Research Logistics Quarterly, 19, 335-353 (1972)

[2] A.P. Hurter, M.K. Schaefer, and R.E. Wendell, "Solutions of Constrained Location Problems", Management Science, 22, 51-56 (1975)

[3] H.W. Kuhn, "On a Pair of Dual Nonlinear Programs", Chapt. 3 in Nonlinear Programming by J. Abadie, John Wiley & Sons Inc., New York, 1967.

[4] R.F. Love and J.G. Morris, "A Computation Procedure for the Exact Solution of Location-Allocation Problems with Rectangular Distances", Naval Research Logistics Quarterly, 22, 441-453 (1975).

[5] R.E. Wendell and A.P. Hurter, "Location Theory, Dominance, and Convexity", Operations Research, 21, 314-320 (1973).

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