Numerical Solution and Multiple Scenario Investigation of Linear Quadratic Differential Games

By
KENNETH R. DEAL

FACULTY OF BUSINESS
McMASTER UNIVERSITY
HAMILTON, ONTARIO

Working Paper Series, No. 157
February, 1980

Numerical Solution and Multiple Scenario Investigation

of Linear Quadratic Differential Games

Kenneth R. Deal

McMaster University

Abstract

A numerical solution procedure is fully described for a linear-quadratic nonzero-sum differential game in advertising. An experimental design is used as a vehicle for investigating multiple competitive scenarios and for tactical planning within encounter situations. Specific results are provided.

Mailing Address: Faculty of Business
McMaster University
1280 Main Street West
Hamilton, Ontario, Canada

Telephone: 416-525-9140, ext. 4434

This research was supported by the National Research Council of Canada Grant No. A9270 and Social Sciences and Humanities Research Council of Canada Grant No. 451-790561.

I. INTRODUCTION

The portrayal and investigation of management and economic problems as differential games has become increasingly popular in recent years[1], [2], [3], [7]. In developing a differential game model in these areas, researchers often anticipate the extension of their models to more realistically portray the commercial or economic situation being studied. Typically, these model expansions would result in significantly increasing the complexity of the models and most likely making analytical solutions extremely difficult. To broaden the applicability of differential games to more practically-oriented problems, numerical solution procedures are necessary. A survey of numerical solutions to zero-sum and nonzero-sum differential games and a call for additional research in this area was presented by Tabak[9].

This paper describes a numerical investigation of a linear-quadratic nonzero-sum differential game in advertising. The basic solution procedure is based on an iterative technique proposed by Starr [8] and Holt and Mukundan[4]. Numerical stability and convergence to Nash Equilibrium Solutions are investigated. Practical advertising interpretation of the results are also examined. To allow for an ordered exploration of the marketing consequences of the solutions, an experimental model of advertising encounters was developed and executed.

The problem treated is that of two firms competing over a finite planning horizon, each attempting to maximize its two-part performance criterion solely through the control of the magnitude and timing of its advertising expenditures. The Nash Equilibrium Solution definition was selected for these non-cooperative encounters.

The model of the problem is

$$\max_{u_1} J_1 = w_1 \frac{x_1(t_f)}{x_1(t_f) + x_2(t_f f} + \int_{t_0}^{t_f} \{c_1 x_1(t) - u_1^2(t)\}\, dt$$

$$\max_{u_2} J_2 = w_2 \frac{x_2(t_f)}{x_1(t_f) + x_2(t_f)} + \int_{t_0}^{t_f} \{c_2 x_2(t) - u_2^2(t)\}\, dt \qquad (1)$$

subject to:

$$\dot{x}_1(t) = -a_1 x_1(t) + b_1 u_1(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\}$$

$$\dot{x}_2(t) = -a_2 x_2(t) + b_2 u_2(t) \left\{\frac{M - x_1(t) - x_2(t)}{M}\right\} \qquad (2)$$

$$x_1(t_0) = x_{10}$$

← initial conditions

$$x_2(t_0) = x_{20}$$

$$u_1(t) \geq 0$$

$\leftarrow$ constraints

$$u_2(t) \geq 0$$

$$x_1(t) \geq 0$$

$$x_2(t) \geq 0$$

$$x_1(t) + x_2(t) \leq M$$

where: $x_i(t)$ = sales for brand i at time t

$u_i(t)$ = advertising for brand i at time t

w_i = weighting factor for performance index

t_0 = initial time of the planning horizon

t_f = terminal time of the planning horizon

a_i = the sales decay constant

b_i = the sales response constant

c_i = the net revenue coefficient

M = the total potential market size

The first two expressions represent the performance indices. The system dynamics are presented in (2) followed by the initial conditions, non-negativity constraints and a constraint on total market size. The objective of each firm is to determine the magnitude and timing of its advertising expenditures in order to maximize a weighted combination of total profit measured over the finite planning period and market share evaluated at the end of the planning period. Because of the relative shortness of the planning period, the current-value formulation was not necessary. The "profit" part of the performance index represents the characteristic of diminishing returns to scale of advertising expenditures through the squaring of the advertising term. Although this effect could be represented by other expressions, the one chosen is reasonable and allows for solution through an alteration of methodology developed for linear-quadratic differential games.

The system dynamics (2) reflects the phenomenon of sales decay in the absence of advertising through the use of the term $-a_i x_i(t)$. The right-most part of (2), $b_i u_i(t) \left[\frac{M - x_1(t) - x_2(t)}{M}\right]$, represents the direct response of the rate of sales to advertising expenditures and the saturation effect as total sales approach the market poential, $M. The saturation effect portrays the diminishing returns to cumulative advertising expenditures. Over time, advertising acts to increase the sales of the firms through the direct response of sales to advertising and through the carry-over effect. As sales increase, the size of the remaining unsold market decreases. Greater effort is needed to bring these late adopters into the purchasing mode. Hence, the dollar effectiveness of advertising diminishes in these highly sold markets.

The value of this model lies in its embodiment of many of the essential aspects of macro advertising decisions, in its ability to be expanded to encompass other aspects of advertising and other marketing control variables, and in its conduciveness to solution.

Several construction-motivated assumptions were necessary in order to obtain a relatively simple but sensitive initial model. To coincide with the continuous nature of the model, it is assumed that the advertising is done in relatively continuous-appearance media such as radio, television, and newspapers. It is further assumed that the brands engage in advertising competition...price, distribution, sales and quality aspects are either identical for the competing firms or do not affect sales and market share. Further, it is assumed that the advertising that is done has the sole effect of distributing (or redistributing) sales from a market of fixed size between the firms...advertising does not influence the total size of the potential market $M, in this model. It is assumed that advertising affects sales but that the reverse effect of sales directly influencing advertising does not exist.

II. A NUMERICAL ALGORITHM

The approach here will be to employ a numerical algorithm to obtain a solution to the previously mentioned class of problems from marketing. The technique used is called the "Ping-Pong" algorithm and is based on the work of Holt and Mukundan[4] and Starr[8].

After specifying the necessary conditions, there are 2 system equations ($\dot{x}$'s) with 2 given initial conditions and 4 costate equations (λ's) with their 4 terminal conditions and the 2 equations for the control variables (u's).

The Ping-Pong Algorithm is initiated by guessing values for each competitor's controls for each discrete instant of time during the planning horizon.

The n system equations are functions of the state variables and the control variables and depend on the given initial conditions for solution. Since the control variables are now parameters and known, this set of differential equations can be solved forward in time by utilizing a numerical procedure such as the Runge-Kutta-Simpson Fourth Order Method. At each forward step of the solution, the values of $x_i(t)$ are calculated and stored.

When the end of the planning horizon is reached, the $x_i(t_f)$'s are obtained. It is at this point that the values can be calculated for the set of boundary conditions that are specified at the terminal time. These terminal conditions for the costate equations are obtained from:

$$\lambda_i(t_f) = K_{i\underline{x}}(x(t_f), t_f), \; i = 1,\ldots,N.$$

Also, during the forward pass, the algorithm has been integrating the performance indices using Simpson's 1/3 Rule. At the terminal time, then, the value of the performance index is known for the guessed values of the control functions.

It is now possible to solve the costate equations backward in time, using as initial conditions the terminal costate conditions calculated at the end of the forward pass.

At each successive time increment of the backward pass, the appropriate values of the costate variables are calculated. Also, a new set of $u_i(t)$'s is found. When back at the initial time, there is a revised time sequence of values for the control functions. These are used in the next forward pass, the objective of which is to evaluate the effect of these new controls on the performance index. A stopping criterion is calculated at the end of each forward pass. If there has not been a significant change in the values of the performance indices since the last iteration, the algorithm is terminated.

When the iterative procedure has been terminated, the solution is checked against the Nash equilibrium property. The control values for each of the players, in turn, are perturbed by a small constant while the opponents' Nash controls are maintained. The performance index of the player who changed from the Nash strategy should not have improved as a result of the change away from the Nash path.

III. RESOLUTION OF THE ALGORITHM

Although the area of differential games has existed since 1954[5] there is still a great deal of research to be done in the development of general solution techniques. In many situations, a heuristic approach is followed to obtain a solution method that conforms to the necessary and sufficient conditions. The method developed here is somewhat representative of this type of approach.

As in the development of all solution algorithms, and especially heuristics, testing must be performed to determine the accuracy of the algorithm in obtaining solutions to the model. In turn, further testing is then executed to resolve the faithfulness of the solution and the model in typifying the actual physical phenomenon.

The validity and usefulness of the algorithm for solving differential games must be justified on two main points:

1) the numerical algorithm exhibits stability and realistic convergence properties while obtaining Nash Equilibrium Solutions; and

2) the solutions make logical sense from a marketing standpoint.

The usefulness of the algorithm as an advertising planning device and as a vehicle for aiding in the development of marketing theory crucially depends upon satisfactory performance on these two criteria.

Numerical Verification

Before the algorithm can be investigated to determine the consistency of its results with reasonable marketing interpretation, there must be justification that the numerical procedure is valid. This numerical validity will be interpreted as the exhibition of

1) numerical stability; and

2) practical convergence to Nash Equilibrium Solutions.

Numerical Stability. A critical consideration in verifying numerical solutions to sets of differential equations is that the numerical procedure be such that no instability exists in the solution. For the Runge-Kutta method of solution, this instability occurs when the time step, h, exceeds a critical value. As long as h is kept "adequately small", there is no fear of instability developing[6]. However, smaller time steps represent longer computing time ... possibly longer than is necessary.

In the normal operation of this algorithm, a time step of h = .05 has been used. There has been no indication of instability in any of the competitive encounters investigated to date while using this time step. The table below presents the results that were obtained for a given set of parameter values when time steps of h = .100, .050 and .025 were used to cover the same fixed length planning horizon. These results are typical for other solutions.

TABLE I

Parameter Values

a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
.05	.45	1.10	2.50	.40	.40	250	25	.05M	2.00M

Results

	Combined Performance Index Brand #		Profit Brand #		Market Share Brand #	
h	1	2	1	2	1	2
.100	$813.74	$60.29	$790.23	$.56	.94	.06
.050	813.71	60.87	790.22	.67	.94	.06
.025	813.70	61.16	790.21	.73	.94	.06

The numerical solution mechanics were tested to determine its consistency with Richardson's extrapolation. Richardson's extrapolation can be used to obtain an improved solution to a problem when the chosen numerical technique has provided results that are near convergence. First, two numerical solutions are found using time increments of different sizes. Then, an improved approximation is found by extrapolating the two solutions back to the point where the time increment is zero. The justification for the extrapolation is based on the linearity of the discretization error with respect to the time increment, when that increment is sufficiently small.

Assume that two solutions were found using time increments h_1 and h_2 using a fourth order method that resulted in a remainder of fifth order. These solutions are labelled $x_1(h_1;T)$ and $x_2(h_2;T)$, respectively. Then, the improved approximation, i.e., the solution that is found by extrapolating back to h=0 is:

$$x_{extrap} = \frac{x_1(h_1;T).h_2^5 - x_2(h_2;T)\cdot h_1^5}{h_2^5 - h_1^5}$$

when $h_2 = 2h_1$ the above can be reduced to:

$$x_{extrap} = \frac{2^5 x_1(h_1;T) - x_2(h_2;T)}{2^5-1}$$

The following results were obtained for the terminal value of the state variable $x_1(t_f)$ where t_f=10, for the parameter values presented in Table I and for successively smaller time increments:

TABLE II

Index	h	x(h;T=10)	h^5
1	.10	154.0497	$10{,}000{,}000 \times 10^{-12}$
2	.05	154.0382	$312{,}500 \times 10^{-12}$
3	.025	154.0325	$9{,}766 \times 10^{-12}$
4	.0125	154.0296	305×10^{-12}
5	.00625	154.0282	9.5×10^{-12}

The use of Richardson's extrapolation on pairs of the above solutions provides the following improved approximations:

TABLE III

Index Numbers of Points Used for the Extrapolation	Revised Approximation x_{extrap}	Relative Differences Between the Approximations
1 & 2	154.0378	
		.000036
2 & 3	154.0323	
		.000018
3 & 4	154.0295	
		.000009
4 & 5	154.02815	

The magnitude of the errors of the approximations are very small as can be seen from Table II. Further, the magnitude of the error obtained when using h=.10 is 1,052,631 times the error resulting from the use of h=.00625. However, Table III shows that the resulting relative differences between the extrapolated approximations is negligible.

From these results, the linearity assumption of Richardson's extrapolation is justified. This in turn provides evidence that the operating value of the time increment, h=.05, is small enough to provide acceptable convergence and stability.

Convergence to Nash Equilibrium Solutions

As mentioned earlier, the Nash Equilibrium Solution definition is that which is used here. Of course, after prescribing such a solution the algorithm must be capable of providing results that are consistent with the Nash definition.

The numerical algorithm iterates until the "Nash Equilibrium Solution" is reached. Operationally, the iteration is continued until a stopping rule is satisfied in that there is no significant further improvement in the performance index values. Consequently, although the exact Nash solution may not be obtained, there should be confidence that the achieved solution is within an arbitrary distance of the exact solution.

Initially, the value of the algorithm for obtaining Nash solutions was checked by taking increasingly larger perturbations around one, and then the other, brand's control path and reevaluating the performance index values each time. As required, the brand that deviated from the Nash control path did not improve its position. Also, it was generally not possible to predict the direction of movement on the non-deviating firm's performance index. This firm might benefit or be hurt by the other firm's deviation.

A competitive encounter between two firms with the moderately different parameters of:

a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
.05	.15	1.10	1.50	.40	.40	250	150	.05M	.50M

is used here to illustrate the efficacy of the model in obtaining Nash Equilibrium solutions. This is one of the worst cases that resulted. The Nash results and the values resulting from perturbing the Nash path are tabled below:

	Performance Index	
	Brand 1	Brand 2
Nash Equilibrium Solution	\$808.684031	\$359.194112
Brand 1's Advertising:		
Increased by .1%	808.684032	359.193762
Decreased by .1%	808.684026	359.194461
Increased by .5%	808.684007	359.192363
Decreased by .5%	808.683978	359.195860
Increased by 1%	808.683908	359.190615
Decreased by 1%	808.683850	359.197609
Brand 2's Advertising:		
Increased by .1%	808.683771	359.194105
Decreased by .1%	808.684291	359.194112
Increased by .5%	808.682731	359.194006
Decreased by .5%	808.685331	359.194040
Increased by 1%	808.681432	359.193721
Decreased by 1%	808.686631	359.193790

By investigation of this table, it can be seen that the Nash requirement is generally met for all the perturbations. However, the first perturbation requires additional explanation. The performance index obtainable by Brand 1 if its advertising expenditures are everywhere increased by .1% is $0.000001 greater than the Nash value. This is a trivially small value in magnitude which fades to even less significance when considered relative to the stopping rule. The stopping rule (previously mentioned in conjunction with the numerical algorithm) is such that the algorithm is terminated if there is less than a .001% improvement in the performance indices. The relative difference between the Nash performance value for Brand 1 and the slight improvement that results from the .1% perturbation amounts to only a 0.0000001% improvement. This will be taken as adequately small for purposes of justifying that the obtained solution is in fact a Nash solution.

After the initial confirmation of the algorithm, a simpler Nash check was continued in conjunction with the more extensive market foray testing of the algorithm. This check involved only one perturbation of the Nash control path of each of the brands separately and checking on the effect of this change on the performance index of the deviating firm. Again, the model qualified according to this test.

Convergence for a numerical solution procedure basically implies two things:

1) the algorithm consistently iterates towards a definable solution, and
2) after reaching the solution, the algorithm will not again diverge from the solution if the iterations are continued.

In all runs to date, both of these criteria have been met. However, there can be significant differences in the number of iterations to termination. Generally, the more "mild" the parameters of an encounter and the more alike the competitors are, the faster will be convergence.

Justification of the Marketing Common Sense of the Results of the Algorithm

A useful way of evaluating a new analytical procedure is to test it on a problem that has a known solution or on one whose solution can be easily hypothesized. The testing of the differential game model was completed partially by using this approach. The basic model presented previously is realistic enough to be rich in vital characteristics of the advertising problem while at the same time being simple enough to allow for speculation of advertising strategies that would follow from the model.

Consequently, four hypotheses, and some special cases, were advanced from the model. The hypotheses are concerned with the main parameters of the model: the sales decay parameter, a_i; the sales response parameter, b_i; total industry sales level, $x_1(t)+x_2(t)$; and the weighting of the performance indices. The algorithm was then tested against each of the hypothesized solutions. Because of the targeting of this article on the numerical solution procedure, these hypotheses and the results will not be presented here. The interested reader is referred to[2]. An experimental procedure for investigating these solutions is presented below.

IV. THE EXPERIMENTAL DESIGN

It is relatively easy to hypothesize general forms of advertising strategy from the model when variation in only one variable is considered. However, it becomes substantially more difficult to determine these paths of advertising expenditures when variation is allowed in more than one parameter at a time. Also, it would be particularly laborious to obtain the actual numerical values for the advertising expenditures. The numerical algorithm described in Section III provides a means for solving these problems.

An experimental design has been used to gain information on the advertising strategies that would be followed under various competitive situations. The experimental design approach has been used for the following reasons:

1. to gain a wider breadth of general information from the runs of the algorithm than by solely using a step-wise investigation of interesting and important competitive situations;
2. the information is obtained in such a manner that the interpretation of cases of various degrees of similarity is facilitated;
3. the record keeping of the model investigation is more efficient.

Macro-Usage of the Experimental Design

A factorial type design has been utilized so that each possible competitive combination of parameters, objectives, and initial conditions can be investigated. A specific example is used here to illustrate the manner in which this design is used in conjunction with the algorithm.

Suppose that it is desired to generate results for the model where the parameters a_1, a_2, b_1 and b_2 are allowed to assume each of two values. For this situation, the full factorial treatment would provide for 2^4 or 16 competitive forays. However, when redundant pairings are deleted, the number reduces to 10. The results of the design for one particular comparison can be shown as

b_1	b_2	a_1 / a_2	.05 / .05	.05 / .45	.45 / .05	.45 / .45
1.10	1.10		.	.	x	.
1.10	2.50		.	.	x	.
2.50	1.10		x	.	x	x
2.50	2.50		.	.	x	.

where c_i = .40; $x_i(0)$ = 25; and w_i = .00, for i = 1, 2.

The redundant pairings have been crossed out. If the above pairings are extended to allow for variation in the initial sales values at two levels and in the performance index weights at two levels, there are 2^8 or 256 total possible pairings. As illustrated in Figure 1, these can be reduced to 136 distinctly different contests.

Each of the factor combinations in the design represents a competitive meeting of two firms having distinctive characteristics that are conveyed by their model parameters. They interact in a specific market environment, the nature of which is described by the potential market, the initial sales levels, and the form of the model. For example, the following two vastly different competitive cases can be investigated:

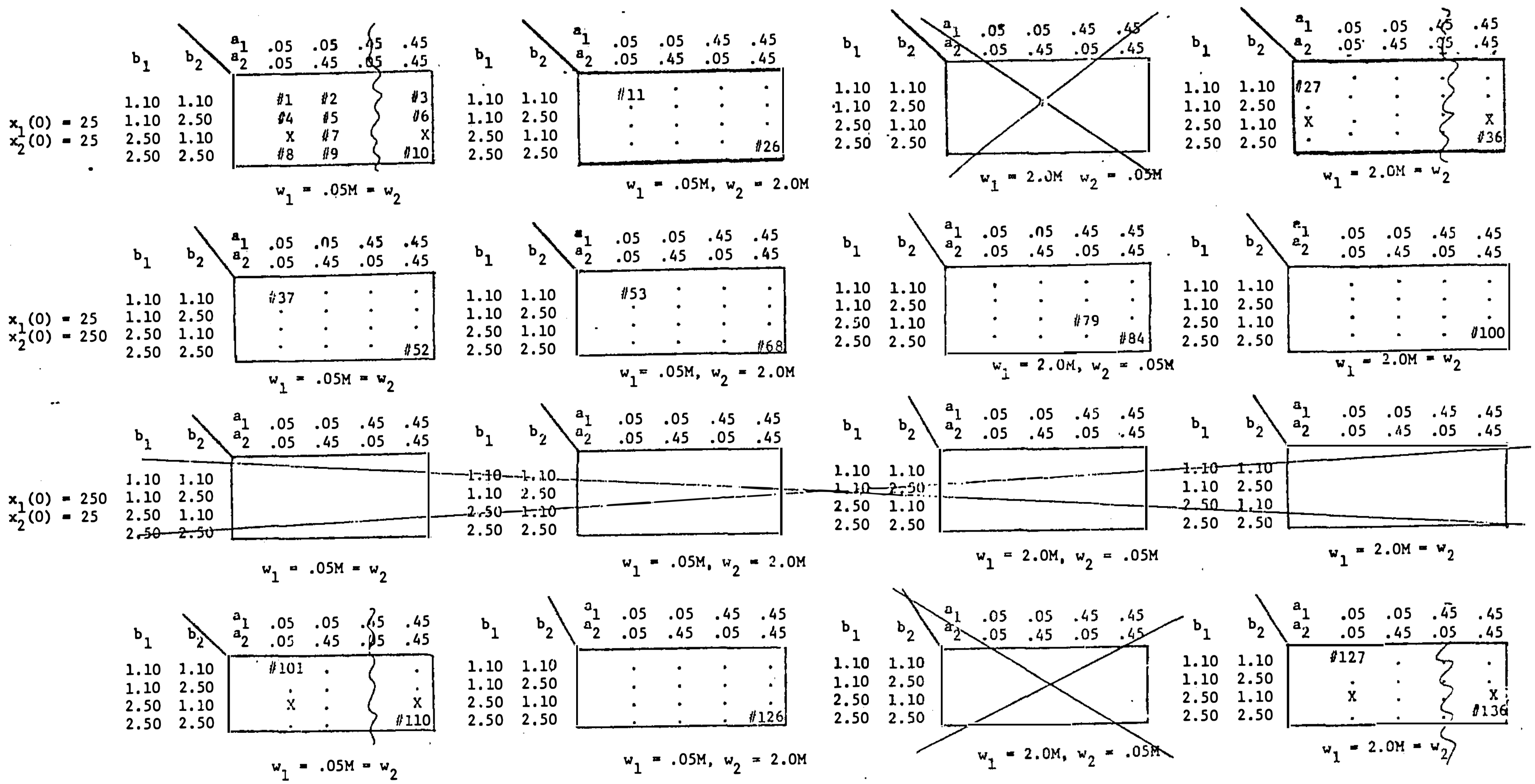
x1(0) = 25
x2(0) = 25
x1(0) = 25
x2(0) = 250
x1(0) = 250
x2(0) = 25
w1 = .05M = w2
w1 = .05M, w2 = 2.0M
w1 = 2.0M, w2 = .05M
w1 = 2.0M = w2
a1
a2
b1
b2

Figure 1

Index	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
# 36	.45	.45	2.50	2.50	.40	.40	25	25	2.0M	2.0M
#101	.05	.05	1.10	1.10	.40	.40	250	250	.05M	.05M

Encounter #36 can be characterized as the meeting of two firms in a relatively young product market (low initial sales levels) where each is offering a fashion-oriented brand. Because of the fashion appeal of the product, there is a fairly high sales response to advertising (indicated by the b_i values). However, the brands are of such a nature that there is a fairly high decay in the sales rate over time as shown by the values of the a_i parameters. The firms are engaged in a highly competitive advertising war where each is intent on gaining a substantial share of the total market by the end of the current planning period. This aspect can be seen by the form of the performance indices where there is a relatively heavy weighting of the terminal market share portion of the performance index. The resulting advertising and sales paths for this contest are plotted in Figures 2.1 and 2.2, respectively.

A contrasting situation appears in encounter #101. The hypothesis in this case would be that there are two relatively practically oriented brands competing for profits in an established older market. There is high saturation of the market with each firm selling to two-fifths of the total market. Profit is the main objective of each firm as can be seen by the relatively light weighting of the terminal market share parts of the performance indices. The "a" and "b" parameters indicate a low degree of sales decay accompanied by a relatively low level of initial sales response to advertising, respectively. These parameter values could

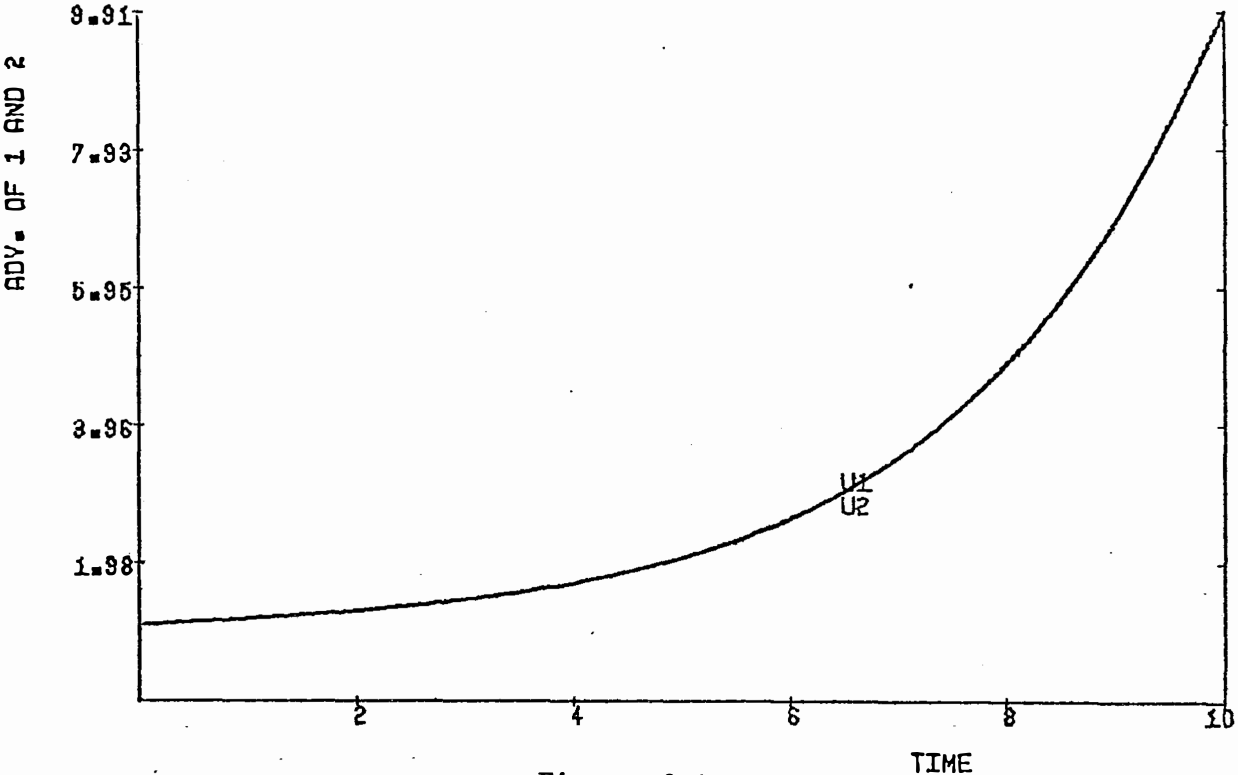

Figure 2.1

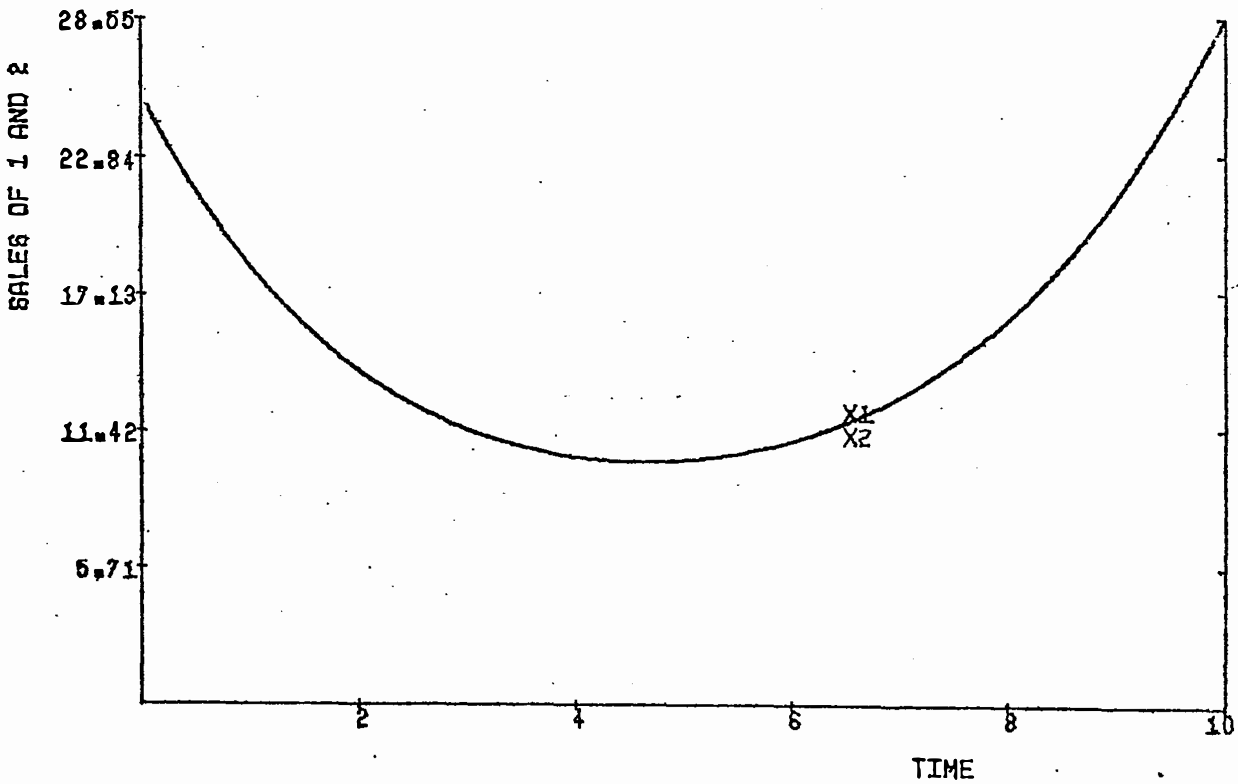

Figure 2.2

indicate the case of older brands that are recognized as having valuable functional product characteristics. Figures 3.1 and Figure 3.2 contain plots of this encounter.

The main value of the experimental design in this research is as a frame of classification and as a convenient device for sampling in order to investigate the effect of varying the parameters of the model on the form (shape or nature) of the advertising strategies. There is no connection here to the statistical uses made of factorial experiments in marketing and survey research. Also, there is no intention of using the design to discover the effect of various model parameters on a specified phenomenon. These effects have been defined in the model. There is no need to discover which advertising strategy is best for a specific market situation--this can be found by use of the numerical algorithm.

There are substantial advantages to broadly surveying the various advertising strategies that can ensue under the multitudinous combinations of parameter values. However, there is additional applicability of the experimental design approach when a particular type of encounter deserves further scrutiny. In these situations experimentation on a micro level can provide information about the possible effects of parameter disturbances on the Brand advertising strategies and performance. This micro-experimentation is the subject of the next section.

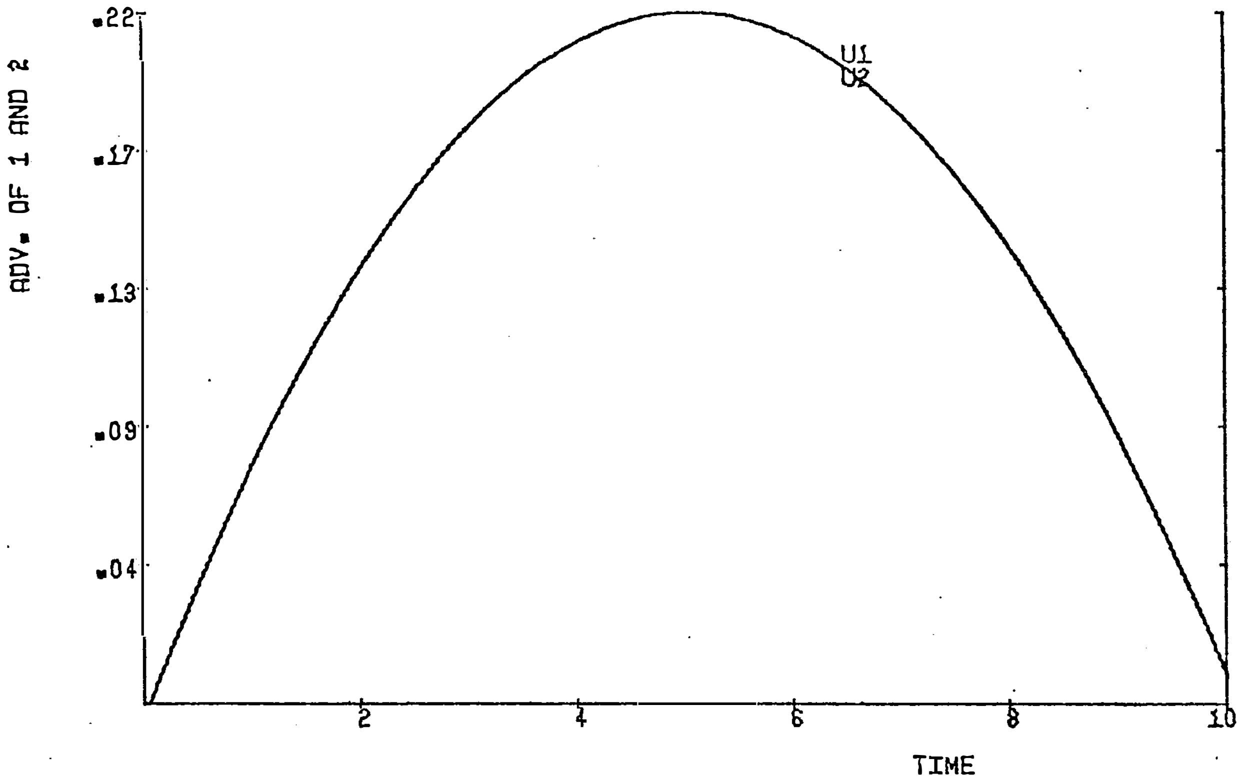

Figure 3.1

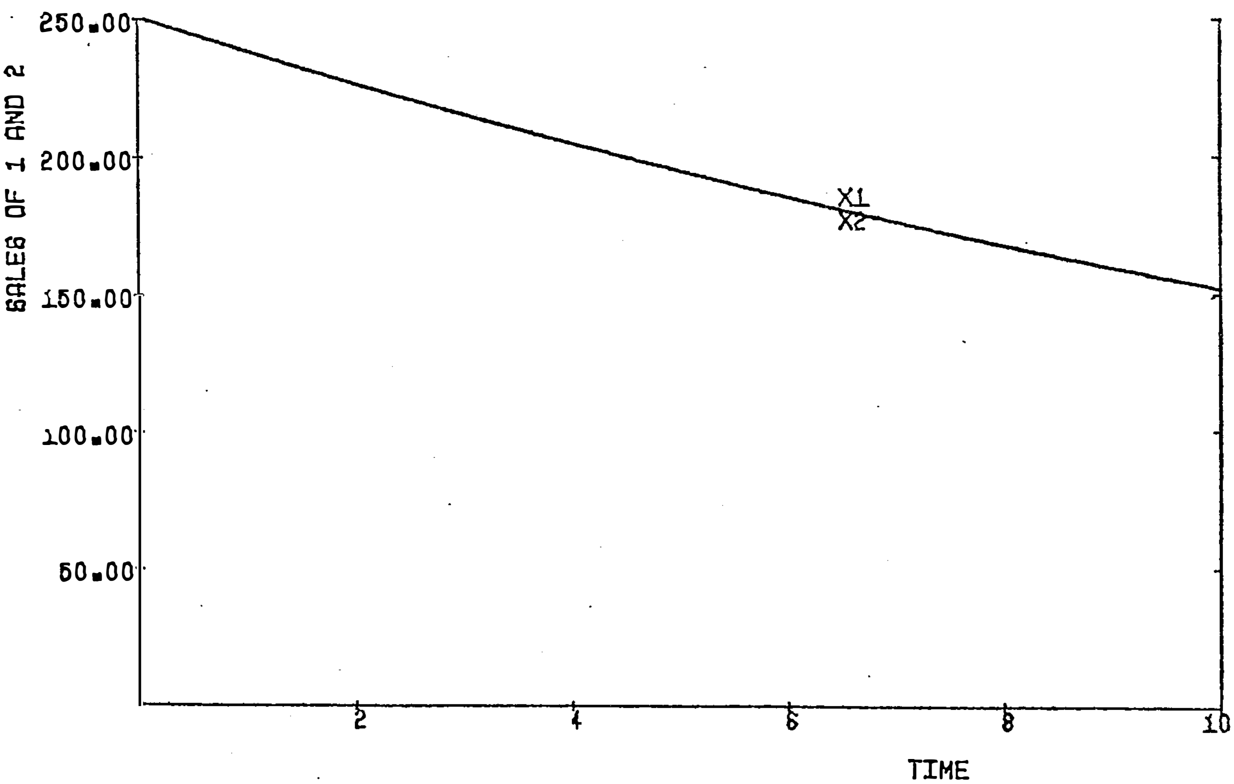

Figure 3.2

Micro Usage of the Experimental Design

The previous section described the use of an experimental design to gain general information concerning advertising strategies over wide ranges of the many parameters of the model. However, after a specific duopoly conflict is identified, it would be more valuable to investigate strategies and resulting performance values when the parameters are varied in a close neighborhood around the existing values. This information would be important when developing advertising strategies to counter possible moves of the competition and also when determining beneficial directions of movement of the parameters.

While investigating various market encounters by using the differential game algorithm in conjunction with the experimental design, one type of competitive situation appeared to be particularly interesting in the strategies that developed. This was the case in which a newer fashion oriented brand attempts to gain a larger share of a market in which the only other competitor is a profit-motivated older brand that possesses a well-recognized functional appeal.

With respect to the experimental design layout of Figure 1, this encounter would coincide with cell #79. The parameter values of the competitors are

a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
.05	.45	1.10	2.50	.40	.40	250	25	.05M	2.0M

and the results are

Combined Performance Index		Profit		Market Share	
Brand #		Brand #		Brand #	
1	2	1	2	1	2
$813.71	$60.87	$790.22	$.67	.94	.06

The graphs of the advertising strategies and the resulting sales paths are displayed in Figures 4.1 and 4.2.

After this case-of-interest is identified, the next step would be to explore strategies that develop as the parameters are perturbed from this reference case. Interest in these results would stem from the necessity of planning for cases in which the parameter values are not as expected. Although the differential games models used here are deterministic, the luxury of perfect information would certainly not exist in actual competitive situations.

The results of the perturbations are obtained by further explorations using the experimental design format. The design would involve running the model for parameter values that vary around those that were first isolated. The competitive parameters that most likely would be candidates for influence by the firm are the sales decay constant, the sales response constant, and the weighting of the performance indices. To investigate the affect of variation of these parameters on Brand 2's performance index and on its advertising strategy, a 2^3 factorial design could be constructed around the "central reference values" isolated earlier. Such a design would take the form illustrated in Figure 5.

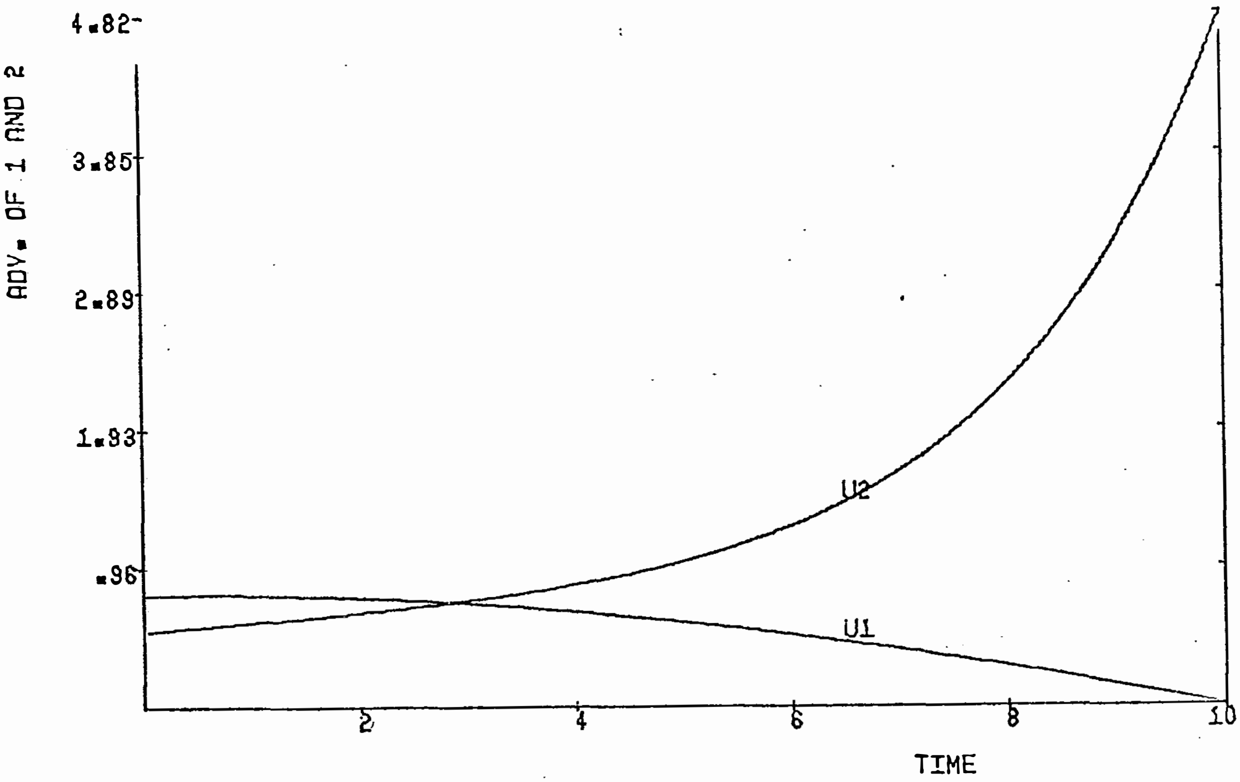

Figure 4.1

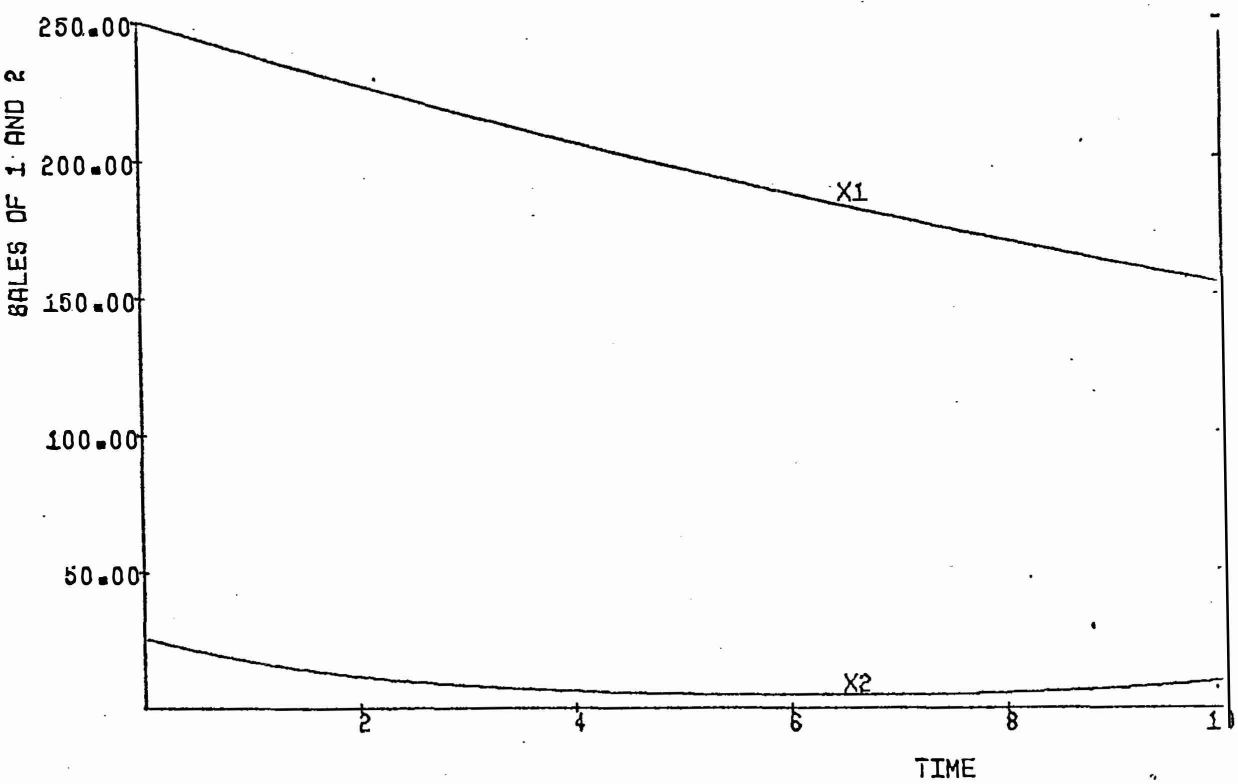

Figure 4.2

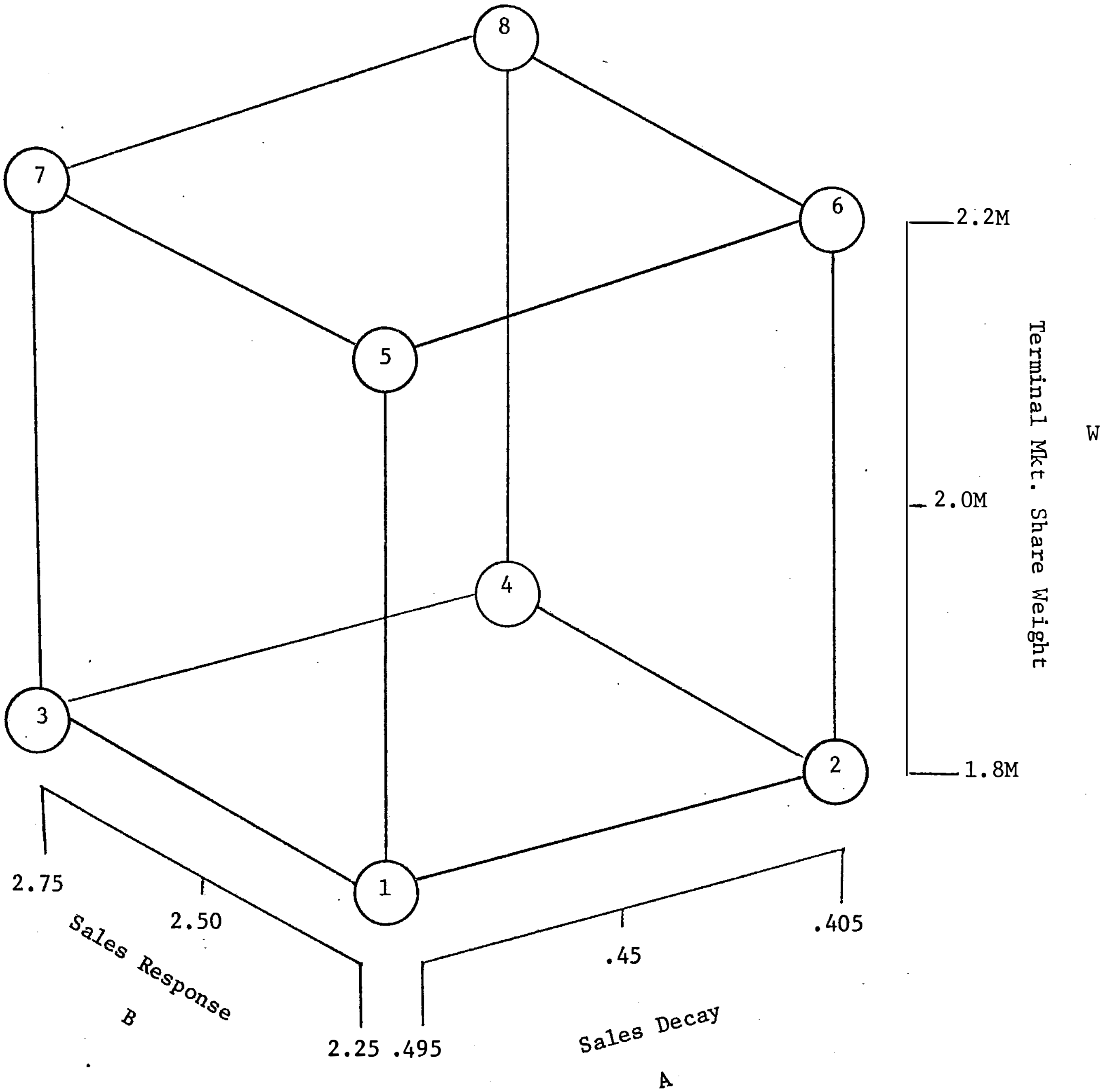
8
7
6
5
4
3
2
1
2.2M
2.0M
1.8M
Terminal Mkt. Share Weight
W
2.75
2.50
2.25
Sales Response
B
.495
.45
.405
Sales Decay
A

Figure 5

Ten percent perturbations in both directions around the central reference values for each of the three key parameters mentioned above was used to construct the experimental design. The parameter sets that are in effect at each of the vertices of the design are given below along with the results.

Parameter Values

Encounter Number	a_1	a_2	b_1	b_2	c_1	c_2	$x_1(0)$	$x_2(0)$	w_1	w_2
1	.05	.495	1.10	2.25	.40	.40	250	25	.05	1.80
2	.05	.405	1.10	2.25	"	"	250	25	.05	1.80
3	.05	.495	"	2.75	"	"	"	"	"	1.80
4	"	.405	"	2.75	"	"	"	"	"	1.80
5	"	.495	"	2.25	"	"	"	"	"	2.20
6	"	.405	"	2.25	"	"	"	"	"	2.20
7	"	.495	"	2.75	"	"	"	"	"	2.20
8	"	.405	"	2.75	"	"	"	"	"	2.20

Results

Encounter Number	Combined Performance Index Brand # 1	Combined Performance Index Brand # 2	Profit Brand # 1	Profit Brand # 2	Market Share Brand # 1	Market Share Brand # 2
1	\$814.17	\$44.73	\$790.24	\$6.10	.96	.04
2	813.89	55.83	790.20	8.62	.95	.05
3	813.74	54.95	790.23	1.32	.94	.05
4	814.40	68.13	790.19	3.70	.93	.07
5	814.00	53.90	790.24	.83	.95	.05
6	813.70	67.00	790.20	.83	.94	.06
7	813.52	67.64	790.23	7.71	.93	.07
8	813.15	83.31	790.19	6.25	.92	.08

The relative results for each of the main effects and the interaction effects for Brand 2 are given below:

Nature Of Effect	Effect
Mean	$61.9362
A	13.2625
B	13.1425
W	12.0525
AB	1.1625
AW	1.1225
BW	1.8825
ABW	.1225

These results indicate that the interaction effect of the parameters is relatively minor as compared to the main effects. In comparing the main effects, a decrease in the sales decay constant has the greatest effect on the performance index for Brand 2. An increase in the sales response constant has the next greatest influence, followed by the weighting term in the performance criterion. These relative results are valid only for the set of competitive parameters and the level of operation that exists in this comparison. Had the level of potential market size, actual sales volumes, or form of the model been different, the results would very likely be different also.

From these response effects, Firm 2 could decide on a forced campaign to move product characteristics and/or image in the most appropriate direction(s). The most beneficial activity would be for Firm 2 to reduce the sales decay constant from its present level. However, a very similar reaction would be obtained by increasing the sales response to advertising, followed by increasing the weighting of the terminal market share part of the performance index. By also considering the form of the model, the firm would probably decide to reduce sales decay. The form of the model dictates that the effect of the sales decay constant becomes reduced as the market becomes more saturated, whereas the sales decay constant maintains its relative effect regardless of the sales level.

The design used in this micro-sense can be used in a number of different ways. For example, the Brand can investigate movement of the parameter values in all possible directions around the key parameters and establish contingent strategies or budgets to cover a wide range of possible competitive situations. Another application would be in helping to determine Brand strategies in response to uncertainties concerning the parameters of the opposition.

For example, the effect of variations in the parameter values of Brand 2 on the performance index values of Brand 1 can be obtained from the results acquired earlier. These effects are listed below:

Nature of Effect	Effect
Mean	$813.6963
A	-.3225
B	-.4875
W	-.2075
AB	-.0325
AW	-.0125
BW	-.0275
ABW	-.0025

Since the interaction effects are relatively negligible as compared to the main effects, attention can be focused chiefly on the effect of Brand 2's sales decay, sales response, and terminal market share weight on Brand 1's performance index. In this encounter, Brand 1's competitive position provides an effective defense against any encroachment on its position made possible by the movement of Brand 2's parameters in the indicated directions. Consequently, although the parameter changes are fairly beneficial to Brand 2, the increase in its performance index is attributable more to attracting new customers than to a transfer of sales from Brand 1 to Brand 2.

V. CONCLUSIONS

Numerical solution techniques are critical to the continuing conceptual development of differential games and their eventual increased application to applied problems in management and other disciplines. This paper has illustrated the application of numerical methods and experimental investigation to obtain solutions and to gain insight to the market environment and opportunities. These procedures and their extensions will become more important when problems such as the one presented here are extended to handle more state and control variables and larger numbers of competitors.

REFERENCES

[1] S. Clemhout, G. Leitmann and H.Y. Wan, "A differential game model of duopoly," Econometrica, vol. 39, Nov. 1971.

[2] K.R. Deal, "Optimizing advertising expenditures in a dynamic duopoly," Operations Research, July-August 1979.

[3] K. Deal, S.P. Sethi and G.L. Thompson, "A bilinear-quadratic differential game in advertising," in Control Theory in Mathematical Economics, P.T. Liu and J.G. Sutinen, Eds. New York: Marcel Dekker, 1979.

[4] D. Holt and R. Mukundan, "A Nash algorithm for a class of non-zero sum differential games," Int. J. of Systems Science, vol. 2, 1972.

[5] R. Isaacs, "Differential games I, II, III and IV," RAND Corp. Research Memos RM-1391, -1399, -1411, -1468 (1954-1956).

[6] Robert L. Ketter and Sherwood P. Prawel, Jr., Modern Methods of Engineering Computation, New York: McGraw Hill, 1969, p. 287.

[7] G. Leitmann and W.E. Schmitendorf, "Profit maximization through advertising: A nonzero sum differential game approach," IEEE Trans. Automat. Contr., vol. AC-23, August 1978.

[8] A.W. Starr, "Non-zero-sum differential games: concepts and models," Div. Eng. and Applied Physics, Harvard Univ., Tech. Rep. 590, 1969.

[9] D. Tabak, "Numerical solutions of differential game problems," Inter. J. of Systems Science, vol. 6, no. 6, 1975.

Faculty of Business
McMaster University

WORKING PAPER SERIES

101. Torrance, George W., "A Generalized Cost-effectiveness Model for the Evaluation of Health Programs," November, 1970.

102. Isbester, A. Fraser and Sandra C. Castle, "Teachers and Collective Bargaining in Ontario: A Means to What End?" November, 1971.

103. Thomas, Arthur L., "Transfer Prices of the Multinational Firm: When Will They be Arbitrary?" (Reprinted from: Abacus, Vol. 7, No. 1, June, 1971).

104. Szendrovits, Andrew Z., "An Economic Production Quantity Model with Holding Time and Costs of Work-in-process Inventory," March, 1974.

111. Basu, S., "Investment Performance of Common Stocks in Relation to their Price-earnings Ratios: A Text of the Efficient Market Hypothesis," March, 1975.

112. Truscott, William G., "Some Dynamic Extensions of a Discrete Location-Allocation Problem," March, 1976.

113. Basu, S. and J.R. Hanna, "Accounting for Changes in the General Purchasing Power of Money: The Impact on Financial Statements of Canadian Corporations for the Period 1967-74," April, 1976. (Reprinted from Cost and Management, January-February, 1976).

114. Deal, K.R., "Verification of the Theoretical Consistency of a Differential Game in Advertising," March, 1976.

114a. Deal, K.R. "Optimizing Advertising Expenditures in a Dynamic Duopoly," March, 1976.

115. Adams, Roy J., "The Canada-United States Labour Link Under Stress," [1976].

116. Thomas, Arthur L., "The Extended Approach to Joint-Cost Allocation: Relaxation of Simplifying Assumptions," June, 1976.

117. Adams, Roy J. and C.H. Rummel, "Worker's Participation in Management in West Germany: Impact on the Work, the Enterprise and the Trade Unions," September, 1976.

118. Szendrovits, Andrew Z., "A Comment on 'Optimal and System Myopic Policies for Multi-echelon Production/Inventory Assembly Systems'," [1976].

119. Meadows, Ian S.G., "Organic Structure and Innovation in Small Work Groups," October, 1976.

120. Basu, S., "The Effect of Earnings Yield on Assessments of the Association Between Annual Accounting Income Numbers and Security Prices," October, 1976.

121. Agarwal, Naresh C., "Labour Supply Behaviour of Married Women - A Model with Permanent and Transitory Variables," October, 1976.

122. Meadows, Ian S.G., "Organic Structure, Satisfaction and Personality," October, 1976.

123. Banting, Peter M., "Customer Service in Industrial Marketing: A Comparative Study," October, 1976. (Reprinted from: European Journal of Marketing, Vol. 10, No. 3, Summer, 1976).

124. Aivazian, V., "On the Comparative-Statics of Asset Demand," August, 1976.

125. Aivazian, V., "Contamination by Risk Reconsidered," October, 1976.

126. Szendrovits, Andrew Z. and George O. Wesolowsky, "Variation in Optimizing Serial Multi-Stage Production/Inventory Systems, March 1977.

127. Agarwal, Naresh C., "Size-Structure Relationship: A Further Elaboration," March 1977.

128. Jain, Harish C., "Minority Workers, the Structure of Labour Markets and Anti-Discrimination Legislation," March, 1977.

129. Adams, Roy J., "Employer Solidarity," March, 1977.

130. Gould, Lawrence I. and Stanley N. Laiken, "The Effect of Income Taxation and Investment Priorities: The RRSP," March 1977.

131. Callen, Jeffrey L., "Financial Cost Allocations: A Game-Theoretic Approach," March 1977.

132. Jain, Harish C., "Race and Sex Discrimination Legislation in North America and Britain: Some Lessons for Canada," May, 1977.

133. Hayashi, Kichiro. "Corporate Planning Practices in Japanese Multinationals." Accepted for publication in the Academy of Management Journal in 1978.

134. Jain, Harish C., Neil Hood and Steve Young, "Cross-Cultural Aspects of Personnel Policies in Multi-Nationals: A Case Study of Chrysler UK", June, 1977.

135. Aivazian, V. and J. L. Callen, "Investment, Market Structure and the Cost of Capital", July, 1977.

136. Adams, R. J., "Canadian Industrial Relations and the German Example", October, 1977.

137. Callen, J. L., "Production, Efficiency and Welfare in the U.S. Natural Gas Transmission Industry", October, 1977.

138. Richardson, A. W. and Wesolowsky, G.O., "Cost-Volume-Profit Analysis and the Value of Information", November, 1977.

139. Jain, Harish C., "Labour Market Problems of Native People in Ontario", December, 1977.

140. Gordon, M.J. and L.I. Gould, "The Cost of Equity Capital: A Reconsideration", January, 1978.

141. Gordon, M.J. and L.I. Could, "The Cost of Equity Capital with Personal Income Taxes and Flotation Costs", January 1978.

142. Adams, R. J., "Dunlop After Two Decades : Systems Theory as a Framework For Organizing the Field of Industrial Relations", January, 1978.

143. Agarwal, N.C. and Jain, H.C., "Pay Discrimination Against Women in Canada: Issues and Policies", February, 1978.

144. Jain, H. C. and Sloane, P.J., "Race, Sex and Minority Group Discrimination Legislation in North America and Britain", March, 1978.

145. Agarwal, N.C., "A Labor Market Analysis of Executive Earnings", June, 1978.

146. Jain, H. C. and Young, A., "Racial Discrimination in the U.K. Labour Market : Theory and Evidence", June, 1978.

147. Yagil, J., "On Alternative Methods of Treating Risk," September 1978.

148. Jain, H. C., "Attitudes toward Communication System: A Comparison of Anglophone and Francophone Hospital Employees," September, 1978

149. Ross, R., "Marketing Through the Japanese Distribution System", November, 1978.

150. Gould, Lawrence I. and Stanley N. Laiken, "Dividends vs. Capital Gains Under Share Redemptions," December, 1978.

151. Gould, Lawrence I. and Stanley N. Laiken, "The Impact of General Averaging on Income Realization Decisions: A Caveat on Tax Deferral," December, 1978.

152. Jain, Harish C., Jacques Normand and Rabindra N. Kanungo, "Job Motivation of Canadian Anglophone and Francophone Hospital Employees", April, 1979.

153. Stidsen, Bent, "Communications Relations", April, 1979.

154. Szendrovits, A. Z. and Drezner, Zvi, "Optimizing N-Stage Production/Inventory Systems by Transporting Different Numbers of Equal-Sized Batches at Various Stages", April, 1979.

155. Truscott, W. G., "Allocation Analysis of a Dynamic Distribution Problem", June, 1979.

156. Hanna, J. R., "Measuring Capital and Income", November, 1979.