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Multi-stage Production with Variable Lot Sizes and Transportation of Partial Lots

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Abstract

This paper describes a model for a multi-stage production/inventory system where lots may be of different sizes. In addition, either completed lots or partial lots, called batches, may be transported to succeeding stages. The model incorporates constraints on lot and batch-sizes and thus provides a rather comprehensive set of possibilities for organizing a production/inventory system. A heuristic solution procedure is developed and is shown to be "close to optimal" by bounding.

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Introduction

When the rate of continuous demand is smaller than the manufacturing rate for a product, intermittent manufacturing in economic lot-sizes is usually justified. Economic lot-sizes are also important when the continuous demand of an assembly line is fed by a part which is manufactured intermittently.

Lot-size models impose a constraint on the scheduling of production facilities which are shared by several products because the manufacturing of lots must be scheduled with priority. Facility scheduling is a lesser problem in single-stage production models than in multi-stage models. Even in the multi-stage case, if a relatively small portion of all products have scheduling priority, there is ample room for manipulating the schedule of the rest of the product line. The key problem is to identify those products which constitute a substantial part of the work-in-process and represent a relatively small portion of the total production capacity. If such products are scheduled according to an apporpriate lot-size model, the process inventory and the total inventory cost can be reduced considerably.

The terminology used in the literature varies substantially. In this paper we call a quantity produced with one set-up at a stage a "lot" and a portion of a lot transported to the next stage a "batch".

Multi-stage production/inventory models have gained increasing attention since an informal survey [1] was presented in the literature in 1972. Deterministic lot-size models for serial and assembly systems represent a variety of process organizations. Two classes of these models, both based on an infinite time horizon, can be distinguished in the literature. One class, which we call "variable lot-size models" [2,3,5,9], allows different and non-increasing lot-sizes across stages. Only complete lots are transported to the next stage and the lot-size of a stage is an integer multiple of the lot-size that follows it. The integrality requirement, except for the special case of infinite production rates, may not be optimal -- but is necessary for the analytical tractability of solution procedures. One variable lot-size model [4] does not have integrality restrictions; it is analytically tractable because any portion of a lot can be transported to the next stage at zero cost. Another class, which we call "batch shipment models" [6,7,8], has uniform lot-sizes at all stages but allows portions of a lot to be transported to the next stage in equal-sized batches at some cost per batch.

In this paper, we present a lot-size model for a single product that is manufactured in a serial system through a large number of stages. Our model allows non-increasing variable lot-sizes across stages and permits batches of equal size rather than entire lots to be transported to the next stage. Transporting batches instead of complete lots may result in higher transportation costs. On the other hand, production at subsequent stages might be scheduled with overlap on the same lot to reduce the size and the cost of the average process inventory. The use of variable lot-sizes balances multiple set-up costs at some stages against the decreased cost of the process inventory. Also, the cost of process inventory is balanced against the cost of transporting batches rather than complete lots. As an additional element of flexibility beyond that in existing models, our model can accommodate constraints on lot-sizes that may result from limited production or storage capacity as well as constraints on batch-sizes that may be caused by limited load-capacity of the transport equipment.

As is done in most existing models, we assume that the lot size of a stage is an integer multiple of the lot size that follows it and that the lot - or batch-size does not have to be an integer (i.e., units of the product are infinitely divisible).

Also, other conventional assumptions are used. Deterministic (constant) demand and production rates, fixed set-up costs and linear inventory-holding costs are assumed over an infinite time horizon. The cost of holding one unit of process inventory is related to the stage which has been completed and is never lower than that for the preceding stage (this may be justified by assuming that value is added to the product at each stage). The unit cost of transportation is related to the load capacity of the transport equipment used at that stage (the load capacity may be different than the batch size). Transportation and set-up times are not considered to be significant and hence are ignored. No backlogging (deliberate shortage) is permitted in the system.

Although the generalizations of previous models that are presented in this paper are straightforward, they add considerable realism to the representation of the process organization. At the same time, they increase substantially the difficulty of solving a traditionally formidable optimization problem.

Constructing the Cost Function

First, we define our notation. The stages in the production system are $i = 1,2,\ldots,n$; the final stage, the one which meets the demand for the finished product, is stage 1. Other symbols are as follows:

D = demand (consumption) rate of the final product (at stage 1);

P_i = production rate at stage i (note that $P_i > D$);

F_i = fixed (set-up) cost per lot at stage i;

T_i = transportation cost of one load from stage i to the next;

g_i = the maximum size of one load (load capacity) transported from the i-th stage to the next;

c_i = unit inventory-holding cost per unit time, at stage i;

Q_i = the lot-size at stage i;

L_i = the maximum lot size permitted at stage i;

S_i = Q_i/Q_{i-1} (note that S_i is required to be integer);

b_i = the number of batches in the lot at stage i (note that b_i is integer);

x_i = Q_1/b_1, the size of batches in the lot at stage i (note that the sizes of the batches are equal).

All parameters above are greater than zero. Also, it should be noted that: i) a non-integer value, A, "rounded-up" to the nearest integer is denoted by $\lceil A \rceil$; ii) the "rounded-down" value is denoted by $\lfloor A \rfloor$; iii) $\lceil \lfloor A \rfloor \rceil$ denotes conventional rounding to the nearest integer; (iv) an integer rounded is the integer. For the convenience of our equations we define:

$Q_0 = 0$, $c_{n+1} = 0$, $P_{n+1} = 0$, $P_0 = D$ and $S_0 = 1$.

To derive the cost function, we start by examining Figure 1.

The output of stage i supplies stage i-1 in b_i batches. The upper dotted line with slope P_i represents the cumulative production at stage i. Thus, the step function immediately below this line is the cumulative output from stage i, that is available, after shipment in batches, at stage i-1. The cumulative output of stage i must be greater than or equal to the cumulative production of the process at stage i-1. The latter is represented by the

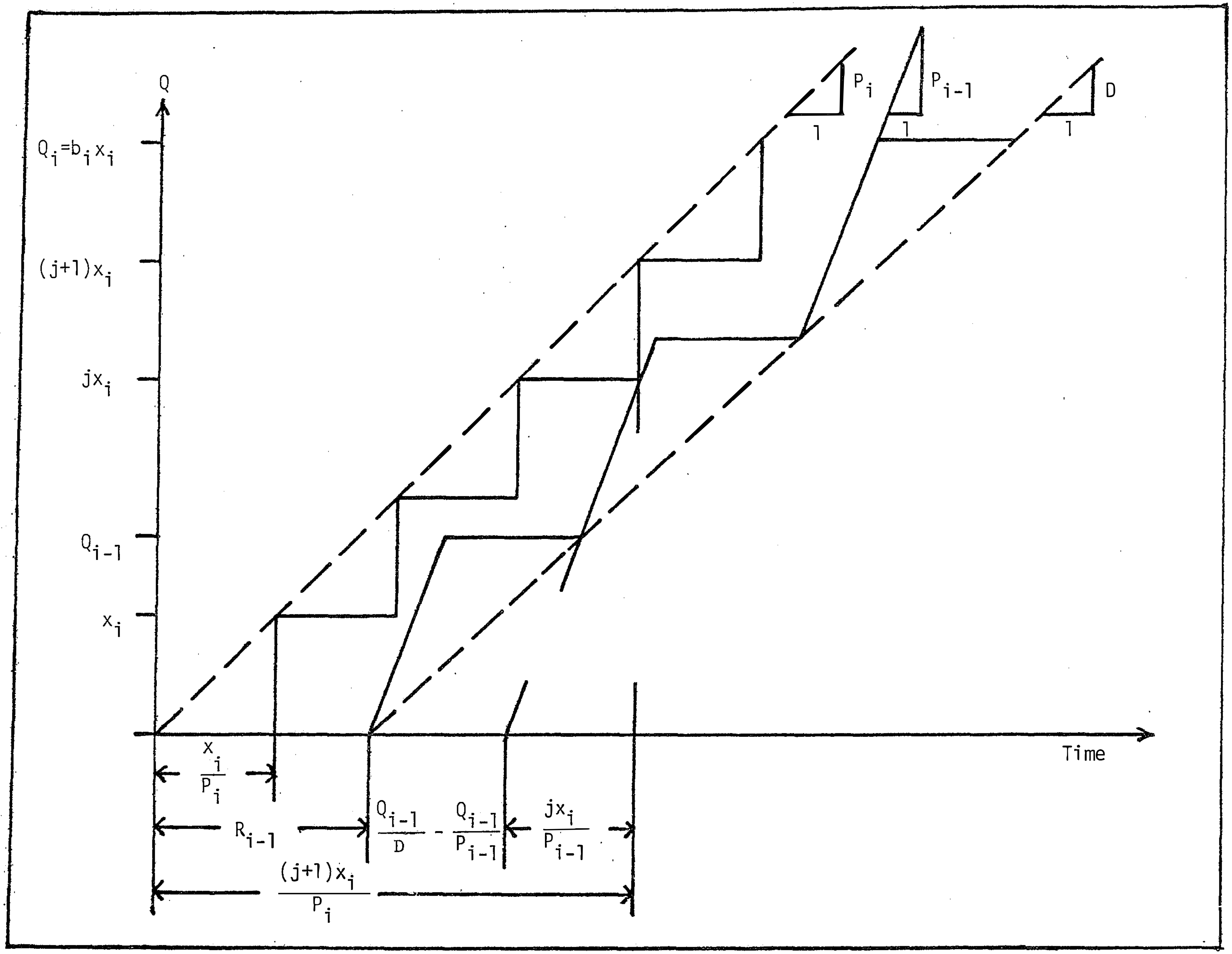

Figure 1 Build-up of Inventory Between Adjacent Stages when $P_i < P_{i-1}$

other step function. Note that the lower dotted line, which has slope D, indicates that the cumulative production at every stage must match the cumulative demand. It can be seen that since Q_i/Q_{i-1} is an integer, the actual inventory at each stage i-1 will cycle back to zero every $(Q_{i-1})/D$ units of time.

Let R_{i-1} be the earliest possible time for the start of production at stage i-1. Clearly, we wish to start as soon as possible to minimize inventory holding costs, but R_{i-1} is constrained by the point where the step functions touch.

First, let us find an analytical expression for R_{i-1}. Denote the batches shipped from stage i with the sequence $j=1,2,\ldots,b_i$. Consider the example in Figure 1. The points $((j+1)x_i/P_i,\ jx_i)$ for $j=0$ to b_i-1 are the right most "corners" of the stage i cumulative inventory available at stage i-1 after shipment in batches. Suppose that the step functions touch at some corner j. This must occur during the production of lot $\lfloor jx_i/Q_{i-1}\rfloor + 1$ at stage i-1.

Therefore, as is seen in Figure 1, to keep stage i-1 production supplied it must be true that

$$R_{i-1} + jx_i/P_{i-1} + \lfloor jx_i/Q_{i-1}\rfloor Q_{i-1}(1/D-1/P_{i-1}) \geq (j+1)x_i/P_i \quad \text{for } 0 \leq j \leq b_i-1.$$

Of course, this will be true if

$$R_{i-1} = \max_{0 \leq j \leq b_i-1} \{ (j+1)x_i/P_i - jx_i/P_{i-1} - \lfloor jx_i/Q_{i-1}\rfloor Q_{i-1}(1/D-1/P_{i-1}) \}.$$

Rearranging the expression for R_{i-1} we obtain:

$$R_{i-1} = x_i/P_i + \max_{0 \leq j \leq b_i-1} \{\theta(j)\} \qquad (1)$$

where $\theta(j) = jx_i(1/P_i-1/P_{i-1}) - \lfloor jx_i/Q_{i-1}\rfloor Q_{i-1}(1/D-1/P_{i-1})$,

$j = 0,1\ldots b_{i-1}$, and j = integer.

One could illustrate graphically that expression (1) is also valid for $P_i \geq P_{i-1}$.

When $P_i \geq P_{i-1}$ the quantity $(1/P_i - 1/P_{i-1})$ is negative and j=0 provides the maximum in (1); thus

$$R_{i-1} = x_i/P_i \qquad \text{for } P_i \geq P_{i-1}.$$

It will be useful to examine the lower bound on R_{i-1}. Naturally, in the case above the lower bound on R_{i-1} is

$$\hat{R}_{i-1} = x_i/P_i \qquad \text{for } P_i \geq P_{i-1}.$$

If $P_i < P_{i-1}$, the expression (1) can be rewritten:

$$R_{i-1} = x_i/P_i + \max\{\theta(\lceil Q_{i-1}/x_i\rceil-1), \max_{j\neq\lceil Q_{i-1}/x_i\rceil-1}\{\theta(j)\}\}$$

and it follows that the lower bound on R_{i-1} is

$$\hat{R}_{i-1} = x_i/P_i + \theta(\lceil Q_{i-1}/x_i\rceil-1).$$

Since $\lfloor(\lceil Q_{i-1}/x_i\rceil-1)x_i/Q_{i-1}\rfloor = 0$ the expression in (1) for the lower bound on R_{i-1} becomes

$$\hat{R}_{i-1} = x_i/P_i + (\lceil Q_{i-1}/x_i\rceil -1)x_i(1/P_i - 1/P_{i-1}) \text{ for } P_i < P_{i-1}.$$

The two lower bounds can be combined in one general expression for $\hat{R}_{i-1}$:

$$\hat{R}_{i-1} = x_i[1/P_i + (\ell_i-1)(1/P_i-1/P_{i-1})] \tag{2}$$

where $\ell_i = \begin{cases} 1 & \text{if } P_i \geq P_{i-1}, \\ \lceil Q_{i-1}/x_i\rceil & \text{if } P_i < P_{i-1}. \end{cases}$

It is important to note that $j = (\lceil Q_{i-1}/x_i\rceil-1)$ means it is the production of the first lot at stage i-1 that determines R_{i-1}. Actually, this rather than the case in Figure 1 is usual. R_{i-1} is not equal to $\hat{R}_{i-1}$ only if D is "nearly equal" to P_i.

Having found the analytical expression for R_{i-1} we can determine the inventory at stage i as shown in Figure 2.

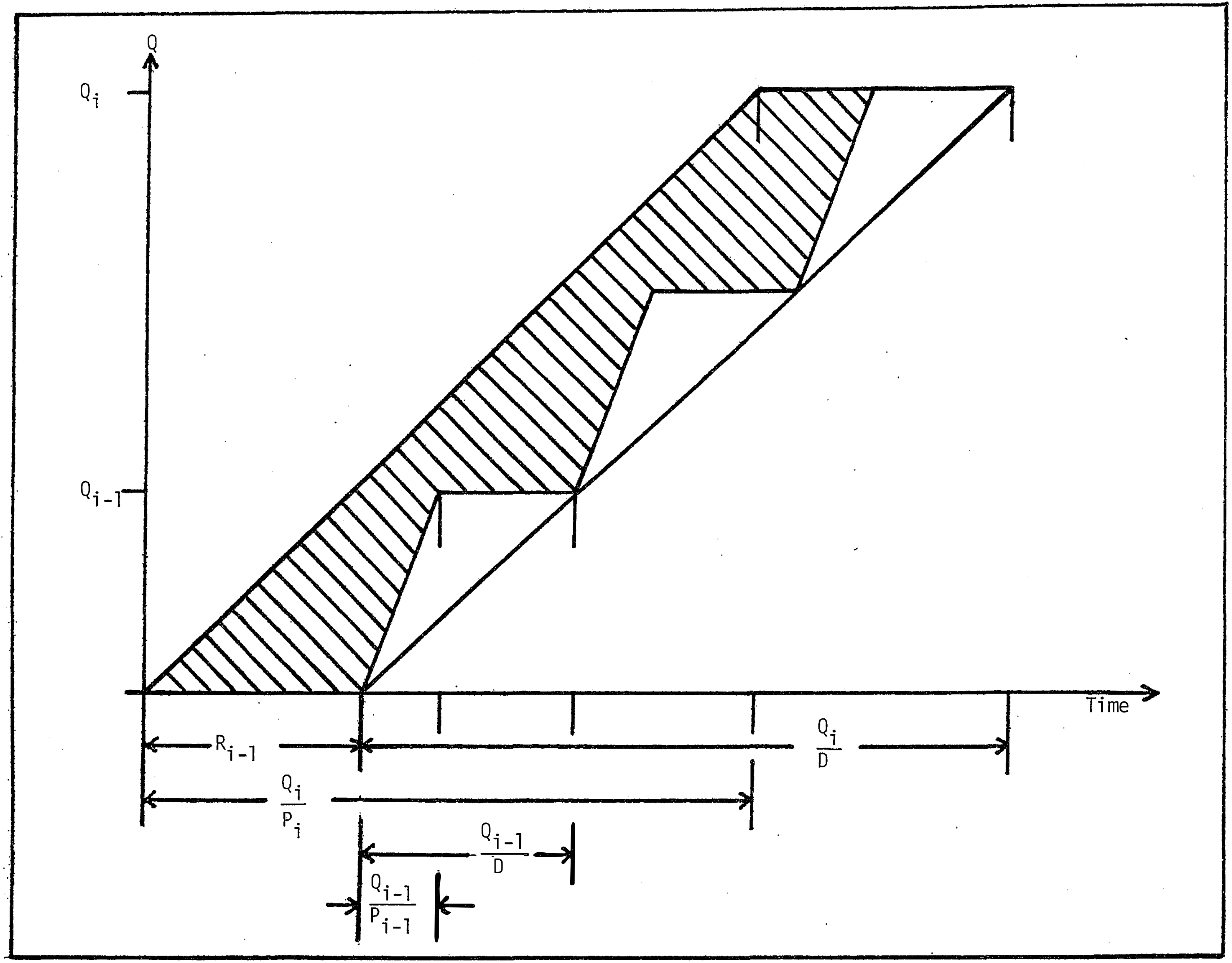

Figure 2 Time-weighted Inventory of Stage i

To find the average inventory-holding cost at stage i, we start by finding the time-weighted inventory (shaded area in Figure 2). This is done by subtracting triangles from a trapezium. Then, we divide the area by Q_i/D to obtain the average inventory and multiply by c_i to obtain the inventory-holding cost per unit time. Thus C_i, the average inventory-holding cost of stage i is

$$C_i = \tfrac{1}{2}c_i D[2R_{i-1} + Q_i(1/D-1/P_i) - Q_{i-1}(1/D-1/P_{i-1})]. \tag{3}$$

Since $P_0 = D$ and $Q_0 = 0$, the expression for C_i holds for all stages.

It is interesting to consider the possibility of using batch-sizes that are larger than the load capacity of the transport equipment, $x_i > g_i$. Each batch would thus require $\lceil x_i/g_i \rceil$ loads and hence the transportation cost per Q_i lot-size would be $T_i\lceil Q_i/x_i\rceil\lceil x_i/g_i\rceil$. If batch-sizes are limited to $x_i \leq g_i$, the transport cost per Q_i lot-size would be $T_i\lceil Q_i/g_i\rceil$ which is obviously smaller. It also can be verified, by examining Figure 1, that increasing the number of batches and hence decreasing the batch-size involves a smaller R_{i-1} (earlier start of production at stage i-1) and thus saves inventory-holding cost. The constraint $x_i \leq g_i$ allows us to simplify the statement of the cost function.

Recall that F_i is the set-up cost per lot at stage i. The sum of the set-up and transportation costs, $(F_i+b_iT_i)$, divided by Q_i/D gives the average fixed cost per unit time.

The total cost of the system can be obtained by summing the fixed and inventory-holding costs:

$$TC = D\sum_{i=1}^{n}\{(F_i+b_iT_i)/Q_i + \tfrac{1}{2}c_i[2R_{i-1} + Q_i(1/D-1/P_i) - Q_{i-1}(1/D-1/P_{i-1})]\}. \tag{4}$$

Since $c_{n+1}=0$, we can rearrange (4) and express the total cost in terms of Q_i. Thus, the optimization problem is as follows:

$$\text{minimize } TC = D\sum_{i=1}^{n} \{F_i/Q_i + Q_i(1/D-1/P_i)(c_i-c_{i+1})/2 + c_iR_{i-1} + T_i/x_i \tag{5}$$

$$\begin{aligned} \text{subject to } \quad & x_i \le g_i && \text{for } i=1,\dots,n, \\ & Q_i \le L_i && \text{for } i=1,\dots,n, \\ & Q_i/Q_{i-1} = S_i = \text{positive integer} && \text{for } i=2,\dots,n, \\ & Q_i/x_i = b_i = \text{positive integer} && \text{for } i=1,\dots,n. \end{aligned}$$

Bounding

In order to obtain a lower bound on the cost of the optimal solution to problem (5) we relax the integrality constraints on S_i, ℓ_i and b_i, replace R_{i-1} with $\hat{R}_{i-1}$ and solve the problem using $x_i = Q_i/b_i$. Thus we have

$$\ell_i^c = \begin{cases} 1 & \text{if } P_i \ge P_{i-1}, \\ Q_{i-1}/x_i & \text{if } P_i < P_{i-1}. \end{cases} \tag{6}$$

Hence

$$\hat{R}_{i-1}^c = x_i[1/P_i + (\ell_i^c-1)(1/P_i-1/P_{i-1})]. \tag{7}$$

Now, we introduce a new indicator variable

$$\delta_i = \begin{cases} 0 & \text{if } P_i \ge P_{i-1} \text{ or } i=n+1, \\ 1 & \text{otherwise} \end{cases} \tag{8}$$

which we can use in (7); therefore

$$\hat{R}_{i-1}^c = (1-\delta_i)x_i/P_i + \delta_i[x_i/P_{i-1} + Q_{i-1}(1/P_i-1/P_{i-1})]. \tag{9}$$

Substituting (9) into (5) and letting $\delta_{i+1} = 0$, we obtain the following total cost expressed in terms of Q_i and x_i:

$$TC^c = D\sum_{i=1}^{n}\{F_i/Q_i + Q_i[(1/D-1/P_i)(c_i-c_{i+1})/2 + \delta_{i+1}c_i(1/P_{i+1}-1/P_i)] \tag{10}$$
$$+ x_i c_i[(1-\delta_i)/P_i + \delta_i/P_{i-1}] + T_i/x_i\}.$$

This can be written as

$$TC^c = D\sum_{i=1}^{n}\{A_i/Q_i + B_iQ_i + H_ix_i + G_i/x_i\} \tag{11}$$

where the A_i, B_i, H_i and G_i coefficients are evident from (10). Since $\hat{R}^c_{i-1} \leq \hat{R}_{i-1}$, $TC^c \leq TC$ for all values of Q_i and x_i.

Now, we have the problem:

$$\text{minimize } TC^c \tag{12}$$
$$\text{subject to} \quad \begin{aligned} Q_i &\leq L_i, \\ x_i &\leq g_i, \\ Q_{i-1} &\leq Q_i, \\ x_i &\leq Q_i, \\ Q_i, x_i &\geq 0. \end{aligned}$$

The solution to this problem provides the lower bound on the solution to (5). Since TC^c is convex, as can be seen from (11), we could use nonlinear programming. An efficient method is given in the appendix.

A Heuristic Solution Procedure

First, we find integers S_i by the following cumulative rounding procedure. Let $Q_1^c,\ldots,Q_n^c$ be the optimal Q_i values found in problem (12), then the S_i's are determined in sequence by:

$$S_k = \left\lceil \frac{Q^c_{k+1}}{Q^c_1(S_1 S_2 \cdots S_{k-1})} \right\rceil \qquad \text{for } k=1,\ldots,n-1 \tag{13}$$

where $S_0=1$.

Since $L_i/(S_1 S_2 \ldots S_{i-1})$ is the maximum allowable Q_1, given the lot size constraint L_i, at stage i,

$$Q_{1u} = \min_{1 \le i \le n} \{L_i/(S_1 S_2 \ldots S_{i-1})\} \tag{14}$$

is the upper bound for Q_1. It can be shown that $Q_1 \le Q_{1u}$ if and only if $Q_i \le L_i$ for $i=1,\ldots,n$. Therefore we choose:

$$Q_1 = \min\{Q_{1u}, Q^c_1\}$$
$$Q_i = Q_1 S_1 \ldots S_{i-1}. \tag{15}$$

Next, we determine the best integer b_i's for the Q_i's found in (15). First we consider the case where $R_{i-1}=\hat{R}_{i-1}$. In equation (5) we convert each x_i to Q_i/b_i; hence each term of the sum now containing b_i can be optimized separately with respect to b_i. Terms not containing b_i (constants) are ignored and we have n problems of the type:

$$\text{minimize } f(b_i) = (T_i/Q_i)b_i + c_i\hat{R}_{i-1}.$$

Substituting (2) this can be written as follows:

$$\text{minimize } f(b_i) = (T_i/Q_i)b_i + (Q_i/b_i)c_i[1/P_i+(\ell_i-1)(1/P_i-1/P_{i-1})] \tag{16}$$

$$\text{subject to } b_i \ge Q_i/g_i,$$

$$\text{where } \ell_i = \begin{cases} 1 & \text{if } P_i \ge P_{i-1}, \\ \lceil b_i/S_i \rceil & \text{if } P_i < P_{i-1}. \end{cases}$$

If $P_i \ge P_{i-1}$, then $\ell_i=1$ and $f(b_i)$ is convex. We find the lowest cost integer on either side of the non-integer solution $b_i = Q_i/(P_i T_i)^{1/2}$. Then, we compare this integer with $\lceil Q_i/g_i \rceil$ and take the larger of the two.

If $P_i<P_{i-1}$, the problem is more difficult. In this case,

$$f(b_i) = (T_i/Q_i)b_i + (Q_i/b_i)c_i[(1/P_i+(r_i+b_i/S_i-1)(1/P_i-1/P_{i-1})] \tag{17}$$

where $r_i = \lceil b_i/S_i \rceil - b_i/S_i$ and $0 \le r_i \le 1$.

Let $y_i = b_i/S_i$, and consider the function

$$h(y_i,r_i) = (u_i+w_ir_i)/y_i + v_iy_i + w_i \tag{18}$$

where $u_i = (Q_i/S_i)(1/P_{i-1})c_i$,

$v_i = T_iS_i/Q_i$,

$w_i = (Q_i/S_i)(1/P_i-1/P_{i-1})c_i$.

Note that $h(y_i,0) \le f(b_i) \le h(y_i,1)$ and $f(b_i) = h(y_i, \lceil y_i \rceil - y_i)$. A "typical" function $h(y_i,r_i)$ is plotted in Figure 3; it is shown there as a continuous function although it exists only when $b_i = y_iS_i$ is an integer. Since S_i is an integer, an integer y_i sets b_i also to an integer.

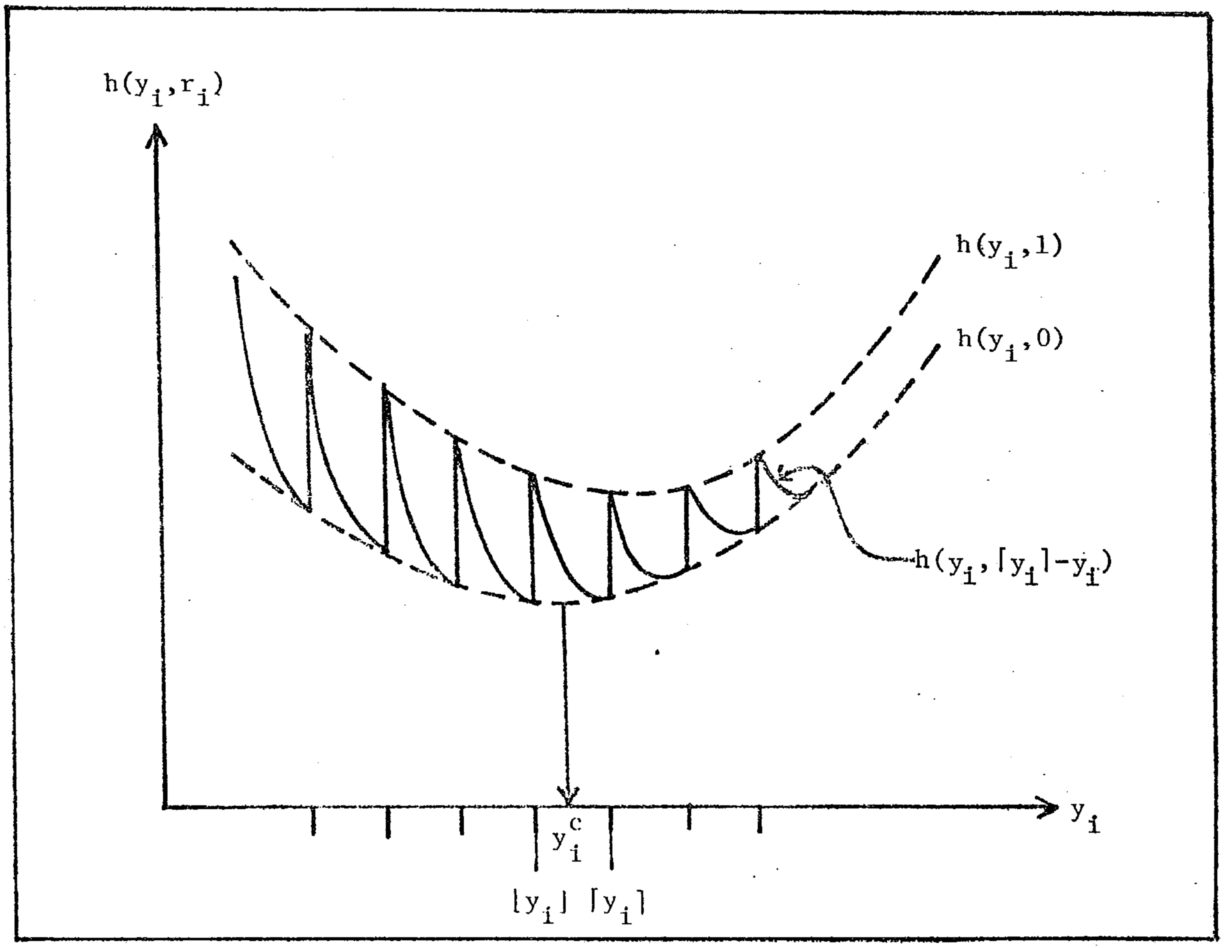

Figure 3 Illustration of a "Typical" Function $h(y_i,r_i)$ in (18)

In Figure 3, y_i^c indicates the minimum of the envelope $h(y_i, 0)$. It can be shown that, when $r_i = 0$, the minimum of function (18) occurs at

$$y_i^c = (u_i/v_i)^{\frac{1}{2}} \tag{19}$$

and that the optimum integer y_i must be in the region $[\lfloor y_i^c \rfloor, \lceil y_i^c \rceil]$.

To find the optimum value of b_i when $\lceil y_i^c \rceil S_i \geq Q_i/g_i$, we need only search for the lowest cost integers in the interval $[\lfloor y_i^c \rfloor S_i \ , \ \lceil y_i^c \rceil S_i]$. If $\lceil y_i^c \rceil S_i \leq Q_i/g_i$ we would search integer b_i's from $\lceil Q_i/g_i \rceil$ to $\lceil \lceil Q_i/g_i \rceil / S_i \rceil S_i$.

Refinements to this procedure are possible. For instance, let y_i^* be the value of y that minimizes $h(y_i, \lceil y_i \rceil - y_i)$, considered as a continuous function. Since $r_i = \lceil y_i^c \rceil - y_i$, it can be shown from (18) that if $\lfloor y_i^c \rfloor \leq y_i^* \leq \lceil y_i^c \rceil$ then y_i^* is given by

$$y_i^* = (u_i + w_i \lceil y_i^c \rceil)/v_i)^{\frac{1}{2}}. \tag{20}$$

Only integer b_i's on either side of $y_i^* S_i$ would be checked if the constraint $b_i \geq Q_i/g_i$ did not interfere. However, unless S_i values are very large such refinements are not necessary.

When $R_{i-1} \neq \hat{R}_{i-1}$ (note that this happens only when $j \neq (\lceil Q_{i-1}/x_i \rceil - 1)$), expression (1) is used for R_{i-1} and problem (16) becomes:

$$\text{minimize } \overline{f}(b_i) = (T_i/Q_i)b_i + c_i \max_{0 \leq j \leq b_{i-1}} \{ jx_i(1/P_i - 1/P_{i-1})$$

$$- \left\lfloor \frac{jx_i}{Q_{i-1}} \right\rfloor (1/D - 1/P_{i-1})\}. \tag{21}$$

Let b_i^* be the optimum number of batches in (16) and b_i^{**} the optimum number of batches in (21). Note that $\overline{f}(b_i^*)$ is an upper bound for $\overline{f}(b_i^{**})$.

As can be seen from (1), $R_{i-1} \geq x_i/P_i$; therefore

$$\overline{f}(b_i) \geq (T_i/Q_i)b_i + (c_iQ_i/P_i)/b_i.$$

We now search for b_i^{**} in the region of $\overline{f}(b_i)$ where

$$(T_i/Q_i)b_i + (c_iQ_i/P_i)/b_i \leq \overline{f}(b_i^*). \tag{22}$$

The resulting range $[b^{**}_{i(min)}, b^{**}_{i(max)}]$ is obtained by setting the preceding inequality (22) to an equality and solving the resulting quadratic equation.

Now that the S_i's and b_i's are known, Q_1 may be updated to it's "current-best" value as follows. Given $S_1 \ldots S_n$ and $b_1 \ldots b_n$ equation (10) can be put in the form:

$$TC = U/Q_1 + VQ_1 \tag{23}$$

$$\text{where} \quad U = D \sum_{i=1}^{n} (F_i + b_iT_i)/(S_1.S_2 \ldots S_{i-1}),$$

$$V = D \sum_{i=1}^{n} S_1.S_2 \ldots S_{i-1} \{ (1/D - 1/P_i)(c_i - c_{i+1})/2 + [\delta_{i+1}(1/P_{i+1} - 1/P_i)$$

$$+ (1-\delta_i)/(P_ib_i) + \delta_i/(P_{i-1}b_i)] c_i \},$$

$\delta_{n+1} = 0$ and δ_i indicator variable is given in (8).

The lot size that minimizes the total cost is expressed by $(U/V)^{\frac{1}{2}}$. Therefore, we can set the new $Q_1 = \min(Q_{1u}, (U/V)^{\frac{1}{2}})$ and the procedure is ready to loop again. It was found empirically that when g_i's are relatively small and binding on the value of x_i, the best value of Q_1 is often an integer multiple of some g_i. Therefore, in addition of the value of Q_1 found above, the closest such integer multiples are checked. Since the cost is non-increasing and the number of possible b_i's is finite the procedure must converge.

Computational Example and Conclusions

Table 1 presents the problem parameters used in the example. The solution to the problem is shown in Table 2.

Table 1 Problem Parameters

i	c_i	P_i	F_i	T_i	g_i	L_i
1	2.5	250000	1.0	.6	250	1500
2	2.4	500000	6.0	.6	250	1500
3	2.2	375000	17.0	.6	250	1500
4	2.1	200000	8.0	2.2	250	1500
5	1.7	150000	26.0	1.3	500	5000
6	1.6	225000	9.0	3.2	500	5000
7	1.5	275000	17.0	1.6	500	5000
8	1.4	125000	24.0	1.0	500	5000
9	.6	175000	46.0	.8	500	5000
10	.4	525000	18.0	1.2	5000	5000
11	.3	800000	16.0	8.0	6000	∞
12	.1	600000	60.0	5.8	6000	∞
Number of stages, n=12			Demand rate, D=60000			

The first two sections of Table 2 show the solutions obtained by the heuristic procedure presented in this paper (without and with constraints). The total costs in each case are denoted by $\hat{TC}$ and the lower bounds on the costs are denoted by TC_*^c. The third section of the table contains results for a typical variable lot-size model [2] when batch shipments are not allowed. The "optimal" total cost here is denoted by TC*. For each of the cases "Δ cost ratio" indicates the percent cost in excess over the lower bound cost.

Table 2 Solution Results

n=12	Unconstrained Q_i and x_i				Constrained $Q_i \le L_i, x_i \le g_i$				Unconstrained $Q_i = x_i$		
i	Q_i	S_i	x_i	b_i	Q_i	S_i	x_i	b_i	Q_i	S_i	b_i
1	1309.526	1	261.905	5	1250.0	1	250.0	5	428.37	3	1
2	1309.526	1	327.381	4	1250.0	1	250.0	5	1285.11	1	1
3	1309.526	1	327.381	4	1250.0	1	250.0	5	1285.11	1	1
4	1309.526	2	654.763	2	1250.0	2	250.0	5	1285.11	1	1
5	2619.052	1	436.502	6	2500.0	1	416.6	6	1285.11	1	1
6	2619.052	1	654.763	4	2500.0	1	500.0	5	1285.11	1	1
7	2619.052	1	523.810	5	2500.0	1	500.0	5	1285.11	1	1
8	2619.052	2	436.509	6	2500.0	2	416.6	6	1285.11	3	1
9	5238.104	1	476.191	12	5000.0	1	500.0	10	3855.33	1	1
10	5238.104	1	1309.526	4	5000.0	1	1250.0	4	3855.33	1	1
11	5238.104	2	5238.104	1	5000.0	2	5000.0	1	3855.33	2	1
12	10476.207	-	5238.104	2	10000.0	-	5000.0	2	7710.66	-	1
	$\hat{TC}$ = \$12265.51				$\hat{TC}$ = \$12515.90				TC* = \$15245.52		
	TC_*^c = \$12212.85				TC_*^c = \$12458.13				TC_*^c = \$15135.91		
	Δ cost-ratio = 0.43%				Δ cost-ratio = 0.46%				Δcost-ratio = 0.72%		

Note that the total costs in Table 2 for both the unconstrained and constrained cases are very close to the lower bound. Considering the fact that the lower bound on cost is a hypothetical result (i.e. very seldom attainable due to integrality requirements), the accuracy of the heuristic procedure is favourably reflected by the example. As a further test, 100 cases of each kind were computed with uniformly randomized input; the results of which are summarized in Table 3.

Table 3 Δ Cost-ratio Percentiles

No. of Cases	Type of cases randomized	Percent of cases ≤ 25	≤ 50	≤ 75	≤ 95	Max.	Min.	Mean
		Δ Cost-ratios (in percentages)						
100	Unconstrained Q_i and x_i	0.31	0.51	1.01	2.23	5.16	0.13	0.77
100	Constrained $Q_i \le L_i$, $x_i \le g_i$	0.92	1.47	1.94	3.88	11.90	0.28	1.80
100	Unconstrained $Q_i = x_i$	0.30	0.70	1.00	1.50	1.60	0.01	0.72

Data used (square brackets denote ranges): n=12, D=60000, F_i=[1.0 - 50.0], T_i=[0.1 - 10.0], c_i=[0.1 - 7.5], P_i=[65000 - 950000], L_i=1500, g_i=[100 - 1000 (in 100's)], for i=1,2,...,n.

The results in Table 3 support confidence in the accuracy of the heuristic procedure. Except for very few cases, the Δ cost-ratio is very moderate. For the unconstrained problem 95 percent of the cases are 2.23 or less percent above the lower bound; for the constrained problem we found this percent to be 3.88 or less. This accuracy is especially remarkable if one considers that even for an optimal solution (for $Q_i=x_i$) 95 percent of the cases are 1.5 percent or less above the lower bound costs.

The computation of the heuristic procedure is rather efficient. Its time was between 0.16 and 0.22 CPU seconds for a large number of 12 stage cases on a CDC6400 computer.

Last but certainly not least, a noteworthy comparison can be made from

Table 2 between the optimal total cost of a variable lot-size model (\$15245.52) and the heuristic total cost (\$12265.51) of the more flexible model presented in this paper. Note that both results include the same transportation cost per shipment whether it is a lot or a batch. The model which accommodates simultaneously both variable lot-size and batch shipment creates its savings by decreasing the set-up and/or the inventory-holding costs. The 24 percent cost savings, in the example, speaks for itself.

Appendix - Minimizing TC^c in Problem (12)

This appendix deals with the solution of problems with the same mathematical form as (12).

Let

$$F_1(Z_1) = \sum_{i=1}^{n} (\alpha_i/z_i + \beta_i z_i) \tag{A1}$$

$$F_2(Z_2) = \sum_{i=n+1}^{2n} (\alpha_i/z_i + \beta_i z_i) \tag{A2}$$

and
$$F(Z) = F_1(Z_1) + F_2(Z_2) \tag{A3}$$

where α_i and β_i are positive constants and where $Z_1 = (z_1,\ldots,z_n)$, $Z_2 = (z_{n+1},\ldots,z_{2n})$, $Z = (z_1,\ldots,z_{2n})$.

Consider the problem:

$$\underset{Z}{\text{minimize}}\ F(Z) \tag{A4a}$$

$$\text{subject to} \quad z_i \leq \gamma_i \quad \text{for } i=1,\ldots,2n; \tag{A4b}$$

$$z_{i-1} \leq z_i \quad \text{for } i=2,\ldots,n; \tag{A4c}$$

$$z_{n+i} \leq z_i \quad \text{for } i=1,\ldots,n. \tag{A4d}$$

We note that problem (A4) is problem (12) when $Z = (Q_1,\ldots,Q_n, x_1,\ldots,x_n)$ and γ_i's are positive constants corresponding to L_i and g_i. We will solve this problem in two phases; the first phase will be to get a good feasible solution (that could be non-optimal) and the second will be to move to an optimum solution.

We begin the first phase by considering the problem:

$$\text{minimize}\ F_1(Z_1) \tag{A5a}$$

$$\text{subject to} \quad z_{i-1} \leq z_i \quad \text{for } i=2,\ldots,n; \tag{A5b}$$

$$z_i \leq \gamma_i \quad \text{for } i=1,\ldots,n. \tag{A5c}$$

In the absence of constraint (A5c) the problem can be easily solved by the collapsing procedure given in [2]. Let this collapsing solution be $Z^c = (z_1^c,\ldots,z_n^c)$. Start with i=n; if there are k+1 values in Z^c such that $z_i^c, z_{i-1}^c,\ldots,z_{i-k}^c > \gamma_i$, set each equal to γ_i. It can be shown that if this process is continued for i = (n-1) to 1, the optimum solution to (A5) is obtained. We call this solution $Z^{(5)} = (z_1^{(5)},\ldots,z_n^{(5)})$.

Consider now the problem

$$\text{Minimize } F_2(Z_2) \tag{A6a}$$

$$\text{subject to } z_{n+i} \le \gamma_{n+i} \text{ for } i=1,\ldots,n. \tag{A6b}$$

The solution is $z_{n+i} = \min\{\gamma_{n+i}, (\alpha_{n+i}/\beta_{n+i})^{\frac{1}{2}}\}$; we will denote each such value by $z_{n+i}^{(6)}$.

We now use the following heuristic to "incorporate" the constraints (A4d) and hence obtain a feasible solution. Find $\max(z_{n+i}^{(6)} - z_i^{(5)})$ (if this is negative, we already have the best solution) at i=i'. Set $\gamma_{i'} = \min\{\gamma_{i'}, \gamma_{n+i'}\}$, $\alpha_{i'} = \alpha_{i'} + \alpha_{n+i'}$, $\beta_{i'} = \beta_{i'} + \beta_{n+i'}$ and re-solve (A5). It is now assumed that $z_{n+i'} = z_{i'}$. If $z_{i'+1} = z_{i'}$ in the actual optimum solution to (A4) and no other constraints of the type (A4d) in that solution are equalities, then it can be shown that we have found that optimum solution by this step. This procedure is continued step by step until $\max(z_{n+i}^{(6)} - z_i^{(5)}) \le 0$; thus, at each step a feasible solution is obtained.

At this stage of the heuristic it is assumed that it is known which of the constraints (A4c) and (A4d) are to be equalities. Were this, in fact, known, then problem (A4) would reduce to the following problem:

$$\text{minimize } F(Z) = \sum_{i=1}^{k} (\tilde{\alpha}_i/\tilde{z}_i + \tilde{\beta}_i\tilde{z}_i) \tag{A7}$$

subject to $\tilde{z}_i \leq \tilde{\gamma}_i$ for i=1,...,k,

where $k \leq 2n$.

Where the constants $\tilde{\alpha}_i$, $\tilde{\beta}_i$ and $\tilde{\gamma}_i$, as well as the variable $\tilde{z}_i$ are obtained by the appropriate grouping of terms. The solution to this problem is

$$\tilde{z}_i^* = \min\{\tilde{\gamma}_i,\ (\tilde{\alpha}_i/\tilde{\beta}_i)^{\frac{1}{2}}\} \tag{A8}$$

We now enter into the second phase. By using the solution of the heuristic in phase one as a starting point we can find the optimum by what is basically a feasible directions method.

Let I(Z) be the set of constraints from among (A4c) and (A4d) holding as equalities at the feasible point Z, and let Z(I) be the optimal solution for the set of equality constraints I, as found from (A8). The principle is as follows. We drop a constraint from I and see if the resulting optimal solution (A8) is feasible with respect to that constraint. If so, we know that at least an infinitesimal move in this direction is feasible and we move in this direction until stopped by a new entering constraint. The process is then repeated. This procedure is summarized in the algorithm below.

Algorithm

Step 1: Set k=1. Obtain $Z^{(1)}$, a feasible starting point by using the heuristic obtained in phase one.

Step 2: Investigate all $j \in I(Z^{(k)})$ in sequence, finding $\overline{Z} = Z(I(Z^{(k)}) - j)$ until $\overline{Z}$ is feasible for constraint j. If no $\overline{Z}$ is feasible for constraint, go to step 5.

Step 3: Find the maximal λ where $(0 \leq \lambda \leq 1)$ for which $Z^{(k+1)} = Z^{(k)} + \lambda(\overline{Z}-Z^{(k)})$ is feasible. If $\lambda = 1$, go to step 2.

Step 4: Find $\overline{Z} = Z(I(Z^{(k+1)}))$. Set k = k+1. Go to step 3.

Step 5: Stop with $Z^{(k)}$ as the optimal solution.

Note that the algorithm always decreases cost and that it will not pass through the same set of equalities twice. There is a finite number of sets of equalities and hence the optimum solution must be reached.

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