Single Machine Scheduling with Precedence Constraints of Dimension 2

By

GEORGE STEINER
Production and Management Science Area

FACULTY OF BUSINESS

McMASTER UNIVERSITY

HAMILTON, ONTARIO

Research and Working Paper Series No. 188
June, 1982

Single Machine Scheduling

with Precedence Constraints of Dimension 2

By George Steiner

Production and Management Science Area
Faculty of Business
McMaster University
Hamilton, Ontario
Canada.

June 1982

Abstract

Consider the set of tasks that are partially ordered by precedence constraints. The tasks are to be sequenced so that a given objective function will assume its optimal value over the set of feasible solutions. A subset of tasks is called feasible, if for every task in the subset, all of its predecessors are also in the subset. We present an efficient dynamic programming solution to the problem, when the constraining partial order has a dimension ≤ 2. This is done by defining a "compact" labeling scheme and a very efficient enumerative procedure for all the feasible subsets. In this process a new characterization is given for 2-dimensional partial orders.

SINGLE MACHINE SCHEDULING

WITH PRECEDENCE CONSTRAINTS OF DIMENSIONS ≤ 2

Consider the set of n jobs to be sequenced for processing by a single machine. The possible sequences may be restricted by precedence constraints represented by a given acyclic digraph $G = (V,A)$ where each node $i \in V$ corresponds to one of the n tasks and the arc $(i,j) \in A$ means that i is a predecessor of j. (If i is a predecessor of j we will also use the notation $i<j$.) These constraints require that a given job i may not be processed until after the processing of all its predecessors has been finished and assume that i is available for processing at any time thereafter. A subset $S \subseteq V$ is called feasible if for every $i \in S$ all the predecessors of i are also in S. Each task $i \in V$ has a given processing time $c(i)$ and the finishing time of the i-th job in a sequence is the sum of the processing times of the first i jobs in the sequence. Let $g(i,t)$ be the cost incurred by job i if it finished at time t, and assume that $g(i,t)$ is non-negative and nondecreasing in t. We assume that this cost is additive i.e. the cost associated with a given feasible sequence is the sum of the costs of the jobs in this sequence. (Such a function is e.g. the tardiness or weighted tardiness, but many other satisfy these general conditions.) The objective is to find an optimal sequence of the n jobs which satisfies the precedence constraints and for which the total cost incurred is minimal.

Baker and Schrage [2] described a dynamic programming algorithm for the problem which outperformed all previously known algorithms on their set of test problems. Burns and Steiner [3] gave some motivations why this algorithm is so effective and presented a modified version of the algorithm for the special case when G is a series-parallel digraph. This modified algorithm used a "compact" labeling scheme by assigning the non-negative integers to feasible subsets in such a way that each of the labels generated belongs to exactly one of the feasible subsets. In this paper we show that by performing the labeling in the "right" sequence the "compactness" property of the labeling can be extended to the class of all precedence graphs with "dimension" less than or equal to two. In addition we show that no further extension of the labeling is possible, without violating the compactness requirement, in fact if the labeling is compact, the precedence graph has to have dimension ≤ 2. This provides a new characterization of partial orders with dimension less than or equal to two. We also define a new family of 2-dimensional digraphs (WGSP) which properly contains the class of general series-parallel digraphs. Using the compact labeling scheme we present a modified version of the dynamic programming algorithm requiring O(Kn) time and O(K) space, where K is the number of feasible subsets in the precedence graph. These bounds of course are still exponential (K can be as large as 2^n), but they are the best obtained so far and in many cases K is substantially smaller than 2^n [cf. 2].

1. The Original Dynamic Programming Algorithm [2]

For a feasible subset $S \subseteq V$ let us define the following:

c(S) = the sum of the processing times of the tasks in S.

R(S) = the set of tasks in S with no successor in S.

f(S) = the cost of the minimum cost sequence of tasks in S.

Then obviously the following DP recursion is valid:

$$f(S) = \min \{f(S\setminus\{i\}) + g(i,c(S)) \mid \text{for all } i \in R(S)\}$$

To minimize the computer storage required and to provide quick access to the f(S) values in the DP tables, Baker and Schrage [2] defined the following labeling scheme for the precedence graph:

Let L(i) be the label assigned to each $i \in V$; b(i) = the sum of labels of previously labeled tasks that are predecessors of i; a(i) = the sum of labels of previously labeled tasks that are successors of i; t(i) = the sum of labels of all tasks labeled prior to i.

Then the labeling can be done by the following algorithm:

Let $t(i) = a(i) = b(i) = 0$ for all $i \varepsilon V$.

For $i = 1$ to n:

let $L(i) = t(i) - a(i) - b(i) + 1$

let $b(j) = b(j) + L(i)$ for every j which has not been labeled yet and $i \prec j$

let $a(j) = a(j) + L(i)$ for every j which has not been labeled yet and $j \prec i$.

let $t(i+1) = t(i) + L(i)$ and if $i = n$ $t(V) = L(V) = t(i) + L(i)$.

Next i.

The labeling scheme can be extended to subsets of V by

$$L(S) = \sum_{i \varepsilon S} L(i) \text{ for every } S \subseteq V.$$

Baker and Schrage have proved that independent of the order of labeling, for every feasible subset $S \subseteq V$ the label $L(S)$ uniquely belongs to S, in the sense that there is no other feasible subset with the same label. In other words the labeling scheme represents a mapping of the feasible subsets into the set of integers between 0 and $L(V)$. We say that this mapping is compact if for any integer k $(0 \leq k \leq L(V))$ there is a feasible subset $S_k \subseteq V$ such that $L(S_k) = k$. We define in general the compact labeling of a digraph:

Let $G = (V,A)$ be an acyclic directed graph on $V = \{1,2,\ldots,n\}$. Let G_k denote the subgraph of G induced by the vertices $\{1,2,\ldots,k\}$ $(1 \leq k \leq n)$. We say that an assignment of labels $L(1), L(2),\ldots,L(n)$ to the vertices of G is a compact labeling of G if and only if for every k $(1 \leq k \leq n)$ $\sum_{i \leq k} L(i)$ = the number of nonempty feasible subsets in G_k.

In the DP algorithm the label $L(S)$ is used to address the feasible subset S and the associated $f(S)$ value. This means that the storage requirements of the algorithm are proportional to the highest label (address) used, which is $L(V)$. Therefore the storage requirements for the DP table highly depend on how close the labeling scheme can get to a compact mapping, i.e., how small L(V) can be for a given precedence graph G = (V,A). Baker and Schrage provide statistics on this for their fairly extensive test problem set and also give a simple example for which the mapping is not compact. They also discuss briefly how the order of labeling the vertices may affect the L(V) value, and mention that in their computer implementation of the algorithm, the tasks were numbered in such an order that the task labeled next was the one that would receive the smallest label if added next. This requires the calculation of a label possibly for every unlabeled node before one can select the next node to be labeled and it will not necessarily result in a compact labeling. In [3] Burns and Steiner replaced this selection rule by a simpler one which resulted in a compact labeling for general series-parallel graphs. It was also shown that this sequencing rule cannot be extended to non-series-parallel graphs without violating the compactness property. In the following development we define a new sequencing rule, which results in a compact labeling for all precedence graphs with dimension ≤ 2.

Another component of the DP algorithm, which facilitates the use of the DP recursion, is an enumerative procedure in which all the feasible subsets S are enumerated in such an order that $S \setminus \{i\}$ is enumerated before S for all $i \in R(S)$ and $S \subseteq V$. Baker and Schrage use for this a standard binary coding procedure. In the subsequent development we show how the labels could be used for a more efficient enumeration scheme.

2. The Labeling of Precedence Relations of Dimension $\leq$ 2.

First we introduce some definitions and known results necessary to understand the development which follows these.

Any directed acyclic graph G = (V,A) induces a partial order $\prec$ on its vertex set V by $u \prec v$, $u,v \in V$ iff there is a directed path from u to v in G.

The transitive closure of G is the directed acyclic graph $G_1 = (V,A_1)$, for which $A \subseteq A_1$ and whenever there is a directed path from u to v in G, $(u,v) \in A_1$. An arc (u,v) of G is called redundant if there is a directed path from u to v in G that does not include the arc (u,v). The transitive reduction of G is the unique directed acyclic graph which contains no redundant arcs and has the same transitive closure as G.

If we consider a set of precedence constraints represented by the directed acyclic graph $G = (V,A)$, this always induces a unique partial order P on V, and if we define $G_1 = (V,A_1)$ s.t. for any $u,v \in V$ $u \prec v$ iff $(u,v) \in A_1$, then G_1 is the transitive closure of G. We will say that P induces G_1. For any directed graph $G = (V,A)$ let $\bar{G} = (V,\bar{A})$ be its undirected version and let $\bar{G}^c$ be the complementary graph of $\bar{G}$. ($\bar{G}^c = (V,\bar{A}^c)$, where the undirected edge $(\overline{x,v}) \in \bar{A}^c$ iff $(\overline{x,v}) \notin \bar{A}$). An undirected graph $\bar{G} = (V,\bar{A})$ is called a comparability graph if there exists a transitive orientation of its edges, i.e., there exists a directed version of $\bar{G}$, $G = (V,A)$ in which if $(u,v) \in A$ and $(v,w) \in A$ then $(u,w) \in A$ also holds for every $u,v,w \in V$.

A partial order on V is called a total order if any two elements of V are comparable. Szpilrajn showed [11] that any partial order is extendable into a total order, and any partial order can be defined as the intersection of several total orders expressed as binary relations. For example, if we consider the partial order induced by the digraph G of Figure 1 on the set $V = \{1,2,3,4\}$, then the digraphs G_1 and G_2 induce total orders on V, which are extensions of the partial order. Considering these orders as binary relations, G induces the relationships $R = \{(1,3), (2,3), (2,4)\}$, G_1 induces $R_1 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$ and G_2 induces $R_2 = \{(2,4), (2,1), (2,3), (4,1), (4,3), (1,3)\}$. Clearly $R = R_1 \cap R_2$.

Dushnik and Miller [5] defined the dimension of a partial order P as the minimum number of total orders such that their intersection is P. Let us denote this number by dim P. According to this the dimension of the partial order induced by the digraph G of Figure 1 is 2. They have also proved the following theorem.

Theorem 1: Let the partial order P induce the digraph G. Then dim P $\leq$ 2 if and only if $\overline{G}^c$ is a comparability graph.

Now let us consider $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, a permutation of the numbers $1,2,\ldots,n$ and let $\pi^{-1}(i)$, denoted shortly by π_i^{-1}, be the position in π where the number i can be found. (E.g., if $\pi = (3,1,4,2)$, then $\pi_4^{-1} = 3$, $\pi_3^{-1} = 1$, etc.) We can construct an undirected graph $\bar{G}[\pi]$ from π in the following way: the vertices of $\bar{G}[\pi]$ are the integer numbers, $1,2,\ldots,n$ and two vertices are joined by an edge if the larger one of them (as numbers) is to the left of the smaller one in π. The graph $\bar{G}[\pi]$ corresponding to the above permutation π is shown in Figure 2. An undirected graph $\bar{G}$ is called a permutation graph if there exists a permutation π such that $\bar{G}$ is isomorphic to $\bar{G}[\pi]$. (Denoted by $\overline{G} \tilde{=} \overline{G}[\pi]$).

Even, Lempel and Pnueli [6] proved the following:

Theorem 2: An undirected graph $\bar{G}$ is a permutation graph if and only if $\bar{G}$ and $\overline{G}^c$ both are comparability graphs.

Combining theorems 1 and 2 we get the following:

Theorem 3: Let P be a partial order with an induced digraph G, then dim P $\leq$ 2 iff $\overline{G}$ is a permutation graph.

In view of the above theorems to determine for a partial order P whether dim P $\leq$ 2, or equivalently whether (for its induced digraph G) $\bar{G}$ is a permutation graph, it is sufficient to check whether $\bar{G}^c$ is transitively orientable. Golumbic [7] has studied this problem and described a polynomial time algorithm, which answers this question and finds a permutation π such that $\bar{G}$ is isomorphic to $\bar{G}[\pi]$ whenever $\bar{G}$ is a permutation graph. If we direct $\bar{G}[\pi]$ so that each edge is directed towards its larger end point, when considering the vertices of $\bar{G}[\pi]$ as integer numbers, and denote this directed graph by $G[\pi]$ then $G \cong G[\pi]$ also holds. The permutation π defines a sequence of the vertices of G, which leads to a compact labeling of the feasible subsets:

Theorem 4: Let $G = (V,A)$ be a directed acyclic graph representing the partial order P for which $\dim P \leq 2$. Assume there exists a permutation π of the nodes of G such that $G[\pi] \cong G^*$, where G^* is the transitive closure of G. Further assume (without the loss of generality) that the nodes of G have been numbered so that the i-th node corresponds to i in π. $(0 \leq i \leq |V|)$.

If the nodes of G are labeled in order of increasing i, using the Baker-Schrage labeling formulae, then the resulting labeling is compact.

Proof: By induction on the number of nodes.

For $|V| = 1$ or 2 the proof is obvious by simple enumeration.

Hypothesis: Let us assume that for any graph with the above properties on less than n nodes $(n > 2)$ the labeling is compact, and let $|V| = n$. Since there is a one-to-one correspondence between the nodes of G and the integer numbers between 1 and n, we will refer to these nodes by using the corresponding integer numbers. Let us define the following subsets of nodes:

$S_k = \{1,2,\ldots,k\} \qquad 1 \leq k \leq n$

$P_k = \{j \mid j \text{ precedes } k \text{ in } G\} \qquad 1 \leq k \leq n$

$Q_k = S_{k-1} \setminus P_k \qquad 2 \leq k \leq n$

We assumed that the labeling occurs in the order 1,2,...,n. This clearly means that if $j \in P_k \Rightarrow j \in S_{k-1}$, because of the direction rule for $G[\pi]$. Applying the Baker-Schrage formulae in this order it follows immediately that $a(k) = 0$ for every k $(1 \leq k \leq n)$ and that $L(k) = t(k) - b(k)+1 = L(Q_k) + 1$.

Let us consider the induced subgraphs $G_k = (S_k,A)$ of G^*. It is clear that each of these subgraphs represents a partial order with dimension less than or equal to two, and the permutation $\pi|k$ induced by π on S_k is such that

$G_k \cong G[\pi|k]$. The labeling L of G is clearly a labeling of each of the G_k-s and it satisfies the assumptions of the theorem. To prove the theorem we shall prove that the labeling L is a unique, compact labeling for each of the graphs G_k. ($1 \leq k \leq n$ and $G_n = G^*$). This is clearly true for G_1 and G_2 and suppose it is true for $G_1, G_2, \ldots, G_{n-1}$. By this hypothesis the number of non-empty feasible subset of G_{n-1} is $L(S_{n-1})$. The uniqueness of the labeling on the feasible subsets follows from the following two observations:

1. If n is an element of any feasible subset T, then $L(T) \geq L(n) + L(P_n) = L(Q_n) + L(P_n)+1 = L(S_{n-1})+1$. Hence no feasible subset containing n has the same label of any feasible subset of G_{n-1}.

2. If T_1 and T_2 are different feasible subsets of G_n, each containing n, then $L(T_1) \neq L(T_2)$. Otherwise, we would have $L(T_1 \setminus \{n\}) = L(T_2 \setminus \{n\})$ contradicting the compactness of the labeling on G_{n-1}.

For the compactness of the labeling on G_n it remains to prove that there are precisely $L(S_n) = L(S_{n-1}) + L(Q_n)+1$ feasible subsets in G_n.

T is a feasible subset of G_n containing n iff $T = \{n\} \cup P_n \cup R$, where $R = \emptyset$ or R is a feasible subset of $G^*(Q_n,A)$. Thus it suffices to prove that $L(Q_n)$ is precisely the number of non-empty feasible subsets of $G^*(Q_n,A)$. We shall go further, by showing that L restricted to Q_n is a compact labeling. For this we note the following two facts about the permutation π:

i) all elements of Q_n precede n in π .

ii) n precedes all elements of P_n in π.

Therefore if $j \in Q_n$ and $i \in S_{j-1} \cap P_n$ then i has precedence over j in G. Hence all elements of P_n, which are labeled <u>before</u> j are predecessors of j. Thus $L(j) = t(j) - b(j) + 1 = [t(j) - L(S_{j-1} \cap P_n)] - [b(j) - L(S_{j-1} \cap P_n)] + 1$ proves that the labels $L(j)$ $(j \in Q_n)$ are exactly the labels we would get if we applied the labeling scheme to the permutation graph $G^*(Q_n,A)$. We can clearly apply the original inductive hypothesis to this graph, and so the compactness of L on $G^*(O_n,A)$ follows.

As an example for performing the labeling calculations for a permutation graph in the order defined by the permutation, consider the graph shown in Figure 3. The labeling calculations are summarized in Table I. Since the total sum of the labels $L(V) = 9$, the graph has precisely 9 non-empty feasible subsets.

In [3] it was proved that labeling the nodes of a series-parallel digraph by the Baker-Schrage formulae will result in a compact labeling, if this was done in a particular sequence, defined there. Since the transitive closure of every series-parallel graph represents a partial order of dimension ≤ 2 (see [9]) theorem 4 defines a new compact labeling sequence for series-parallel graphs and also extends the compactness property beyond this class. Series-parallel graphs have a forbidden subgraph characterization (cf. [9]). Baker, Fishburn and Roberts [1] have shown however that a forbidden subgraph characterization is impossible for precedence graphs of dimension 2. In the following we identify a class of 2-dimensional precedence graphs which properly contains the class of series-parallel digraphs.

Consider the digraph G shown in Figure 3. The subgraph of G induced by $\{2,3,4,5\}$ is the forbidden subgraph for series-parallel graphs, while the subgraph induced by $\{1,3,4,6\}$ is what is known as a directed Wheatstone bridge [4]. G is a permutation graph which is not series-parallel.

Definition WMSP (Wheatstone Minimal Series-Parallel):

i) The directed acyclic graph having a single vertex and no arc is WMSP.

ii) The directed acyclic graph $G[\pi]$ shown in Figure 2 is WMSP.

iii) If $G_1 = (V_1,A_1)$ and $G_2 = (V_2,A_2)$ are WMSP, $V_1 \cap V_2 = \emptyset$, then either one of the following directed acyclic graphs is WMSP too:

a) Parallel Composition: $G_p = (V_1 \cup V_2,\ A_1 \cup A_2)$

b) Series Composition: $G_s = (V_1 \cup V_2,\ A_1 \cup A_2 \cup (O_1 \times I_2))$, where O_1 is the set of exit nodes in G_1 and I_2 in the set of entry nodes in G_2.

Definition WGSP (Wheatstone General Series-Parallel): A directed acyclic graph is WGSP iff its transitive reduction is WMSP.

Theorem 5: If $G_1 = (V_1,A_1)$ is the transitive closure of a WGSP graph, P_1 is the partial order induced by G_1 then $\dim P_1 \leq 2$ or equivalently $\bar{\bar{G}}_1$ is a permutation graph.

Proof: By induction on the number of nodes $n = |V_1|$. For $n = 1,2,3$ it is clear that G must be a GSP graph, therefore $\dim P_1 \leq 2$.

For $n = 4$ a) if G_1 is the graph $G[\pi]$ shown on Figure 2 (or isomorphic to it) then it was shown earlier that its undirected version $\bar{G}[\pi]$ is a permutation graph, i.e., by theorem 3, $\dim P_1 \leq 2$.

b) if G_1 is not isomorphic to the graph $G[\pi]$ of Figure 2, then it is clear that G_1 is series-parallel implying $\dim P_1 \leq 2$.

Hypothesis: Assume that the theorem is true for any WGSP graph on less than n nodes. $(n > 4)$

Let $G_1 = (V_1,A_1)$ be a WGSP graph on n nodes.

a) If G_1 is the parallel composition of two WGSP graphs $G_2 = (V_2,A_2)$ and $G_3 = (V_3,A_3)$ let the partial orders induced by G_2 and G_3 be P_2 and P_3 resp. By the inductive hypothesis $\dim P_2 \leq 2$ and $\dim P_3 \leq 2$. As a result of the parallel composition the nodes of G_2 and G_3 are incomparable in P_1. So if R_2^1 and R_2^2 are two total orders s.t. $R_2^1 \cap R_2^2 = P_2$ and R_3^1 and R_3^2 are two total orders s.t. $R_3^1 \cap R_3^2 = P_3$ then we can define two total orders on V_1:

$$R_1^1 = \{(x,y) \mid (x,y) \in R_2^1 \text{ or } (x,y) \in R_3^1 \text{ or } x \in V_2 \text{ and } y \in V_3\}$$

$$R_1^2 = \{(x,y) \mid (x,y) \in R_2^2 \text{ or } (x,y) \in R_3^2 \text{ or } x \in V_3 \text{ and } y \in V_2\}$$

It is clear that $R_1^1 \cap R_1^2 = P_1$ implying $\dim P_1 \leq 2$.

b) If G_1 is the series composition of two WGSP graphs $G_2 = (V_2,A_2)$ and $G_3 = (V_3,A_3)$.

Let the partial orders induced by G_2 and G_3 be P_2 and P_3 resp. By the inductive hypothesis we have dim $P_2 \leq 2$ and dim $P_3 \leq 2$. As a result of the series composition the nodes of G_2 are all predecessors of every node in G_3. If R_2^1 and R_2^2 are two total orders s.t. $R_2^1 \cap R_2^2 = P_2$ and R_3^1 and R_3^2 are total orders s.t. $R_3^1 \cap R_3^2 = P_3$ then we can define the following total orders on V_1:

$$R_1^1 = \{(x,y) \mid (x,y) \in R_2^1 \text{ or } (x,y) \in R_3^1 \text{ or } x \in V_2 \text{ and } y \in V_3\}$$

$$R_1^2 = \{(x,y) \mid (x,y) \in R_2^2 \text{ or } (x,y) \in R_3^2 \text{ or } x \in V_2 \text{ and } y \in V_3\}$$

It is clear that $R_1^1 \cap R_1^2 = P_1$ implying dim $P_1 \leq 2$._

As an illustration we show one WGSP graph on Figure 4. A somewhat 'loose' definition for the class WGSP could be that its members are GSP graphs with certain nodes substituted by Wheatstone bridges.

A natural question to ask is whether the compactness of the Baker-Schrage labeling system can be extended further to partial orders (precedence graphs) with higher dimension than two. The answer for this is negative, actually the fact that the Baker-Schrage formulae result in a compact labeling implies that the partial order has a dimension ≤ 2. The first proof of this is due to J.B. Orlin [10]. In the following we present the proof of a stronger result, but first we have to review the concepts of basic feasible subsets and basic complements due to Held, Karp and Shareshian [8].

Let P be a partial order $\prec$ on $V = \{1,2,\ldots,n\}$. (In the following development we always assume that $i \prec j$ implies $i < j$.) Let us define the basic feasible subsets in P by $B_k = \{i \mid i=k \text{ or } i \prec k\}$ for $k > 0$ and let B_0 be the empty set. These basic feasible sets determine the sets $\bar{B}_0, \bar{B}_1, \ldots, \bar{B}_n$, called the basic complements by $\bar{B}_k = \{i \mid i < k \text{ and } i \notin B_k\}$. Each $\bar{B}_k$ induces a partial order, which is the restriction of P to the elements of $\bar{B}_k$. If we consider those feasible subsets S in P which contain k as their highest numbered element and the feasible subsets R in the induced partial order on $\bar{B}_k$, then there is a one-to-one correspondence between S and R by $S = R \cup B_k$.

From this follows the following theorem:

Theorem 6 [8]: $[P] = \sum_{i=0}^{n} [\bar{B}_i]$, where [X] denotes the number of feasible subsets (including the empty set) in the partial order induced by the set X.

Lemma 7. Let P be a partial order $\prec$ on $V = \{1,2,\ldots,n\}$ as above and let $G = (V,A)$ be the digraph induced by P. Let L(1), L(2),...,L(n) be a labeling of the vertices of G. Then this labeling is a compact labeling of G if and only if

$$L(k) = [\bar{B}_k] \text{ for every } 1 \leq k \leq n \tag{1}$$

Proof: In one direction the proof is obvious by Theorem 6. For the other direction we use an induction on n, the number of elements. For n = 1 the only non-empty feasible subset in $G_1 = G$ is $\{1\}$ so L(1) = 1, on the other hand $\bar{B}_1 = \emptyset$, so $[\bar{B}_1] = 1$ implying (1).

Hypothesis: Let us assume that for any partial order on less than n elements if L is a compact labeling then (1) is also true. Let us consider then the partial order P on n elements and let P_{n-1} be the partial order induced on $\{1,2,\ldots,n-1\}$. It is clear that the basic complements $\bar{B}_0, \bar{B}_1,\ldots, \bar{B}_{n-1}$ and the partial orders induced by them are identical in P and P_{n-1}. Therefore by Theorem 6

$$[P] = \sum_{i=0}^{n} [\bar{B}_i] = \sum_{i=0}^{n-1} [\bar{B}_i] + [\bar{B}_n] = [P_{n-1}] + [\bar{B}_n]. \tag{2}$$

It is clear that if L(1), L(2),..., L(n) is a compact labeling of P then L(1), L(2),..., L(n-1) is a compact labeling of P_{n-1}, $[P_{n-1}] = 1 + \sum_{i=1}^{n-1} L(i)$ and by the inductive hypothesis we have $L(k) = [\bar{B}_k]$ for $1 \leq k \leq n-1$. On the other hand $[P] = 1 + \sum_{i=1}^{n} L(i)$ from which it follows by (2) that $L(n) = [\bar{B}_n]$ is also true, thus proving the lemma.

Theorem 8: Let P be a partial order $\prec$ on $V = \{1,2,\ldots,n\}$ and let $G = (V,A)$ be the digraph induced by P. Assume that the Baker-Schrage labeling formulae result in a compact labeling $L(1), L(2), \ldots, L(n)$ for G, then

(i) $\dim P \leq 2$

(ii) any compact labeling of G can be generated by the Baker-Schrage formulae.

Proof: We define the following "incomparability" relationship on the elements of V: we say that $i||j$ (read i is incomparable to j) iff $i < j$ but i is unrelated to j in P. $||$ is not a transitive relationship in general, but in view of theorem 1 if $||$ is transitive then $\dim P \leq 2$.

Consider the basic complements $\bar{B}_k$ in P. Since $L(1), \ldots, L(n)$ is a compact labeling, by Lemma 7 we have $L(k) = [\bar{B}_k]$ for $1 \leq k \leq n$. Each $\bar{B}_k$ with the relation $\prec$ is itself a partially ordered set. We define the basic feasible subsets (C_{ki}) and the basic complements $(\bar{C}_{ki})$ in these posets: For each k $(1 \leq k \leq n)$ and $i \in \bar{B}_k$ (i.e., $i||k$) let

$C_{ki} = \{j | j = i \text{ or } j \in \bar{B}_k \text{ and } j \prec i\}$,

$\bar{C}_{ki} = \{j | j < i,\ j \in \bar{B}_k \text{ and } j \notin C_{ki}\}$ and let $C_{k0} = \bar{C}_{k0} = \emptyset$

It is clear that $C_{ki} = \bar{B}_k \cap B_i$ and $\bar{C}_{ki} = \bar{B}_k \cap \bar{B}_i$. Thus applying theorem 6 to the poset $\bar{B}_k$ we get

$$[\bar{B}_k] = [\bar{C}_{k0}] + \sum_{i||k} [\bar{C}_{ki}] = 1 + \sum_{i||k} [\bar{B}_k \cap \bar{B}_i] \tag{3}$$

Consider a feasible subset S in the poset $\bar{B}_k \cap \bar{B}_i$, where $i||k$, and let us "extend" S into $\bar{B}_i$ by $e(S) = \{j | j \in S \text{ or } j \in \bar{B}_i$ and there exists an $\ell \in S$ such that $j \prec \ell$ in $\bar{B}_i\}$. It is clear that $e(S)$ is a feasible subset in $\bar{B}_i$. We claim that e is a mapping of the feasible subsets in $\bar{B}_k \cap \bar{B}_i$ into the set of feasible subsets in $\bar{B}_i$. To prove this, assume the contrary, i.e., there exist two different feasible subsets S_1 and S_2 in $\bar{B}_k \cap \bar{B}_i$ for which $e(S_1) = e(S_2)$. Without the loss of generality we can assume that there is a $j \in S_1 \setminus S_2$, for this j however $j \in e(S_1)$ and $j \notin e(S_2)$, a contradiction. From this it follows that $[\bar{B}_k \cap \bar{B}_i] \leq [\bar{B}_i]$ for every i,k if $i||k$. Substituting this into (3) we get

$$[\bar{B}_k] = 1 + \sum_{i||k} [\bar{B}_k \cap \bar{B}_i] \leq 1 + \sum_{i||k} [\bar{B}_i] \tag{4}$$

Let us assume that $\bar{B}_k \cap \bar{B}_i \subset \bar{B}_i$, i.e., there exists a $j \in \bar{B}_i \setminus \bar{B}_k$. This means that $j||i$, $i||k$ but $j \nless k$, i.e., the relation $||$ is not transitive. In other words $\bar{B}_k \cap \bar{B}_i = \bar{B}_i$ for every $i||k$ if and only if the relation $||$ is transitive. Furthermore if $\bar{B}_k \cap \bar{B}_i \subset \bar{B}_i$, we also have $[\bar{B}_k \cap \bar{B}_i] < [\bar{B}_i]$, since if we consider the smallest (as a number) $j \in \bar{B}_i \setminus \bar{B}_k$ and a subset $S' \subseteq \bar{B}_i$ with highest index j and feasible in $\bar{B}_i$, then clearly there is no feasible subset S in $\bar{B}_k \cap \bar{B}_i$ for which $e(S) = S'$. In summary $[\bar{B}_k \cap \bar{B}_i] = [\bar{B}_i]$ for every $i||k$ if and only if the relation $||$ is transitive, i.e., $\dim P \leq 2$.

Based on Lemma 7 we must have $L(j) = [\bar{B}_j]$ for every $j \in V$ and substituting this into (4) we get

$$L(k) \leq 1 + \sum_{i||k} L(i), \tag{5}$$

and equality holds in (5) only if $\dim P \leq 2$. It is clear that $L(k) = 1 + \sum_{i||k} L(i)$ for every $k \in V$ is identical to the Baker-Schrage labeling formulae, thus proving the theorem.

Besides resulting in a compact labeling for 2-dimensional precedence graphs, the Baker-Schrage labeling scheme uniquely assigns labels to all the feasible subsets, moreover this happens in an additive fashion, i.e., if S_1, S_2, $S \subseteq V$ are feasible subsets for which $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$ then for their labels we have $L(S) = L(S_1) + L(S_2)$. This enables us to define a simple algorithm that could be used to identify the feasible subset S such that $L(S) = k$ for a given integer k ($1 \leq k \leq L(V)$).

<u>Algorithm</u> <u>DECODE</u>

Let $S = \emptyset$

For $i = n$ to 1

If $k > t(i)$ then $S = S \cup \{i\}$ and $k = k - L(i)$

Next i

Theorem 9: Let us assume that the 2-dimensional precedence graph $G = (V,A)$ has been compactly labeled by the Baker-Schrage formulae. Then for any given integer k ($1 \le k \le L(V)$) the Algorithm DECODE identifies the unique feasible subset S in G for which $L(S) = k$ in $O(n)$ times and $O(n)$ space.

Proof: Consider the vertex n which was labeled last. If $n \in S$, then by the feasibility of S all predecessors of n must be in S too, i.e.,

$$L(S) \ge L(n) + b(n) = t(n) - b(n) - a(n) + 1 + b(n) = t(n) + 1, \qquad (6)$$

where the first equality follows from the labeling formulae and the second equality follows from $a(i) = 0$ ($1 \le i \le n$), since we assumed that $i \prec j$ implies $i < j$ for any pair i,j.

On the other hand if n is not in S, then

$$L(S) \le L(1) + L(2) + \dots + L(n-1) = t(n) \qquad (7)$$

Comparing (6) and (7) we get that S contains n if and only if $k > t(n)$ and using this argument in an inductive fashion for the induced subgraphs G_{n-1}, $G_{n-2}, \dots, G_1$ proves the correctness of the decoding algorithm.

Since the only information we need to store for DECODE are the labels $L(i)$, $t(i)$ ($i = 1,2,\dots,n$) the algorithm requires $O(n)$ space indeed. The $O(n)$ time requirement is obvious.

3. The Modified Dynamic Programming Algorithm

Consider a sequencing problem with sequencing function f and with precedence graph $G = (V,A)$ of dimension ≤ 2. Assume that π is a permutation for which $G \cong G[\pi]$ and the graph G has been compactly labeled by the Baker-Schrage formulae. We redefine the DP algorithm for this problem.

Algorithm DYNPRO

For $k = 1$ to $L(V)$

Let $S = \emptyset$, $R(S) = \emptyset$, $c(S) = 0$, $j = n+1$

For $i = n$ to 1

If $k > t(i)$ then $S = S \cup \{i\}$, $k = k-L(i)$ and $c(S) = c(S) + p_i$

otherwise go to next i

If $\pi_i^{-1} < j$ then $R(S) = R(S) \cup \{i\}$ and $j = \pi_i^{-1}$

Next i

$f(S) = \min \{f(S \setminus \{i\}) + q(i,c(S)) \mid i \in R(S)\}$ and let i^* be the index, where the minimum is obtained.

Store f(S) and i^* under the address L(S).

Next k.

Theorem 10: The Algorithm DYNPRO solves the above defined sequencing problem in O(Kn) time and O(K) space, where K is the number of feasible subsets in the precedence graph G.

Proof: Since the labeling formulae assign a compact labeling to G, K = L(V). The Algorithm DECODE is used in DYNPRO to identify the feasible subsets, so based on Theorem 8, this will require O(Kn) time and O(n) space. Within the same loop we use the permutation π to identify the set R(S). The correctness of this method follows from the following argument: For any $i,k \in V$ $i \prec k$ if and only if $i < k$ and $\pi_i^{-1} > \pi_k^{-1}$. Therefore if we identify the elements of R(S) in their decreasing sequence (as numbers), at any point a vertex i is in R(S) if and only if for every k assigned to R(S) up to this point $\pi_i^{-1} < \pi_k^{-1}$. Since j is used in the algorithm to store $\min_k \pi_k^{-1}$ for these $k \in R(S)$, this proves that DYNPRO will indeed identify R(S) in the same loop as S, and this again requires no more than O(Kn) time and O(n) space.

To calculate f(S) for one S by the dynamic programming recursion clearly requires at most O(n) time and O(1) space, and to do this for all feasible S requires then O(Kn) time and O(1) space. For each S we store f(S) and $i^* \in S$ which is the last vertex in the optimal sequence for S, therefore the DP tables require O(K) space.

Once f(V) has been calculated, we can get the optimal sequence, where this value is obtained, by putting the i^* belonging to S = V in the last available position and repeating this for $S \setminus \{i^*\}$ until we reach the empty set. This proves the theorem.

There are special situations where we may be interested only in finding the optimal value $f(V)$ but not the optimal sequence. In this case we need not store the vertices i^* in the above algorithm. Furthermore if $L_{max} = \max\{L(i) | i \varepsilon V\}$, for the DP recursion the $f(S-\{i\})$ values for any S and $i \varepsilon R(S)$ must be stored in one of the L_{max} addresses immediately preceding the address $L(S)$, therefore at any point in the algorithm we need to refer back to at most L_{max} different locations in the DP table. In this case the space requirements of the algorithm can be reduced to $O(L_{max} + n)$.

Acknowledgement

The author gratefully acknowledges Professors R. N. Burns, J. B. Orlin and J. B. Sidney for their helpful comments and useful suggestions for the presentation of this material.

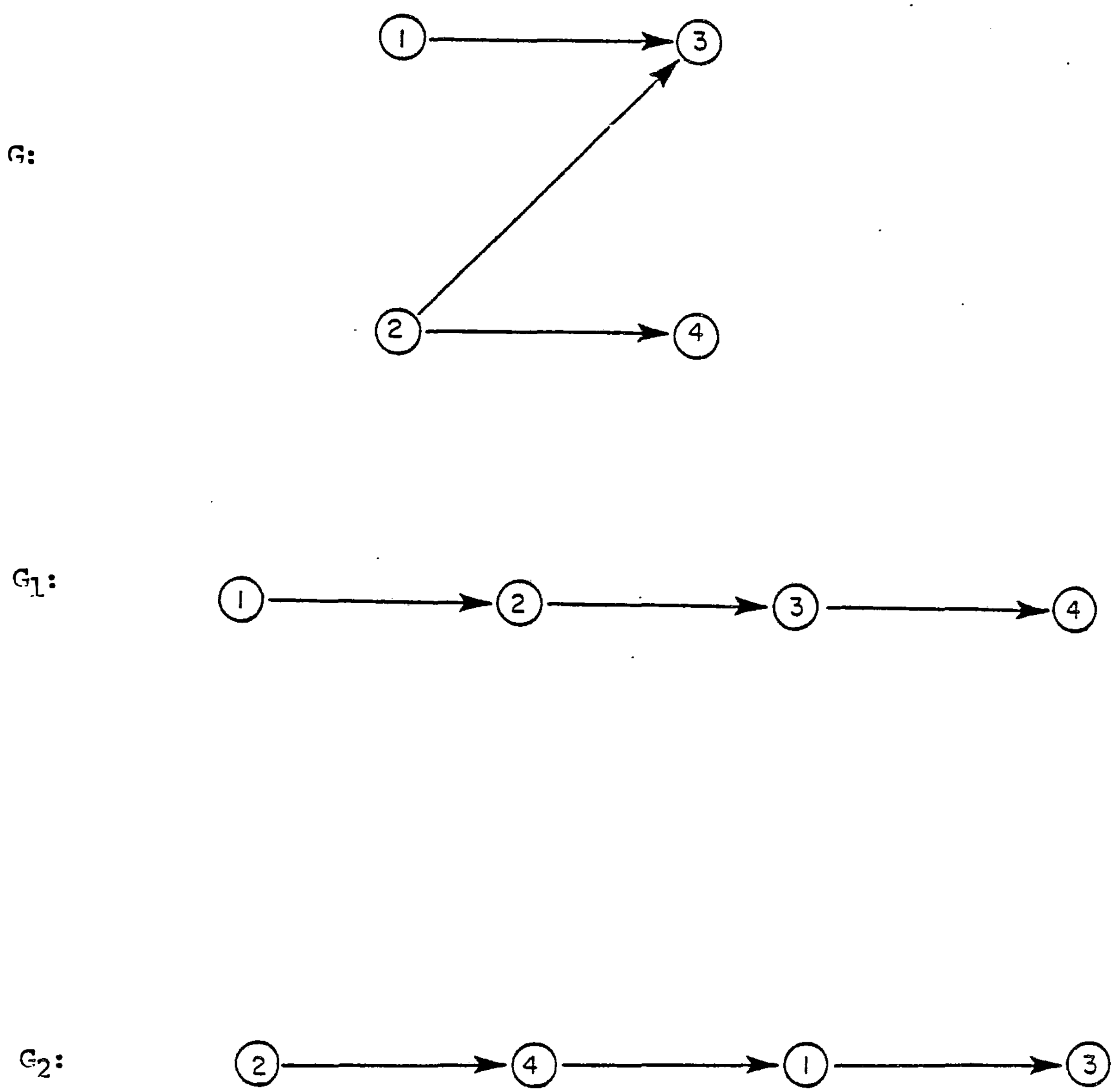
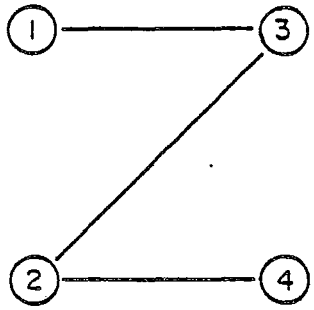


Figure 1

$\overline{G}[\pi]$:
$\pi = (3,1,4,2)$



$G[\pi]$:

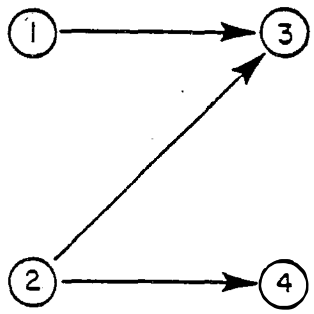


Figure 2

G[π]:

$\pi = (6,4,2,5,3,1)$

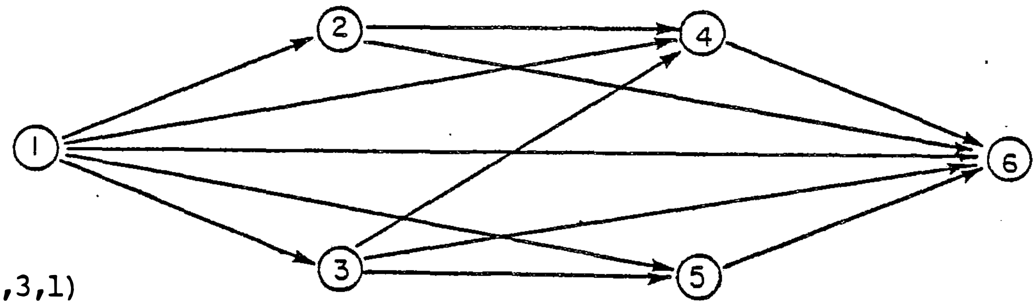


Figure 3

Table I

i	b(i)	L(i)	t(i)
1	0	1	0
2	1	1	1
3	1	2	2
4	4	1	4
5	3	3	5
6	8	1	8

Total Sum of labels 9

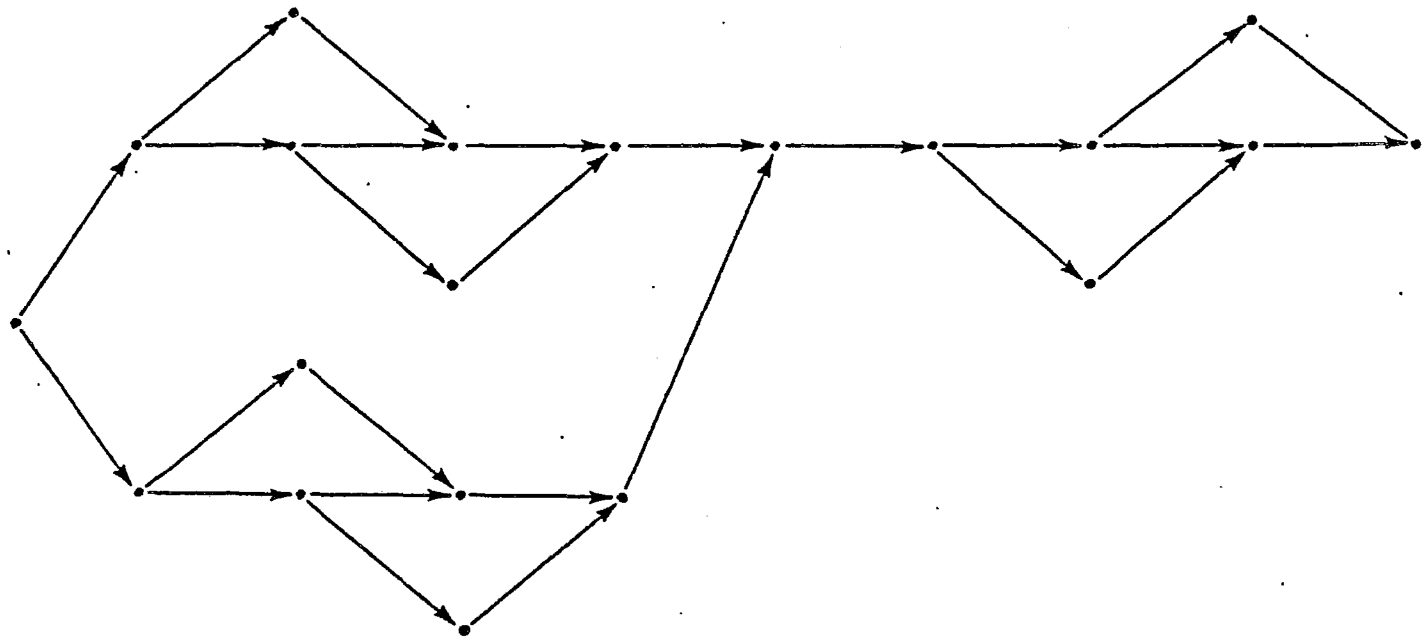


Figure 4

REFERENCES

1. K.A. Baker, P.C. Fishburn, F.S, Roberts: "Partial orders of dimension 2", Networks 2 (1971), 11-28.

2. K.R. Baker, L. Schrage: "Dynamic programming solution for sequencing problems with precedence constraints". Opns. Res. 26, (1978) 444-449.

3. R.N. Burns, G. Steiner: "Single machine scheduling with series-parallel precedence constraints", to appear in Opns. Res.

4. R.J. Duffin: "Topology of series-parallel networks", Amer. J. Math. Anal. Appl. 10 (1965), 303-318.

5. B. Dushnik, E.W. Miller: "Partially ordered sets", Amer. J. Math. 63 (1941), 600-610.

6. S. Even, A. Pnueli, A. Lempel: "Permutation graphs and transitive graphs", J. Assoc, Comput. Mach. 19 (1972), 400-410.

7. M.C. Golumbic: "Algorithmic Graph Theory and Perfect Graphs", (Academic Press, New York 1980).

8. M. Held, R.M. Karp, R. Shareshian: "Assembly line balancing". Opns. Res. 11 (1967), 442-458.

9. E.L. Lawler, R.E. Tarjan, J. Valdes: "The recognition of series parallel graphs", unpublished manuscript.

10. J.B. Orlin: Private communication.

11. E. Szpilrajn: "Sur l'extension de l'ordre partiel", Fund. Math. 16, (1930), 386-389.

Faculty of Business
McMaster University

WORKING PAPER SERIES

101. Torrance, George W., "A Generalized Cost-effectiveness Model for the Evaluation of Health Programs," November, 1970.

102. Isbester, A. Fraser and Sandra C. Castle, "Teachers and Collective Bargaining in Ontario: A Means to What End?" November, 1971.

103. Thomas, Arthur L., "Transfer Prices of the Multinational Firm: When Will They be Arbitrary?" (Reprinted from: Abacus, Vol. 7, No. 1, June, 1971).

104. Szendrovits, Andrew Z., "An Economic Production Quantity Model with Holding Time and Costs of Work-in-process Inventory," March, 1974.

111. Basu, S., "Investment Performance of Common Stocks in Relation to their Price-earnings Ratios: A Text of the Efficient Market Hypothesis," March, 1975.

112. Truscott, William G., "Some Dynamic Extensions of a Discrete Location-Allocation Problem," March, 1976.

113. Basu, S. and J.R. Hanna, "Accounting for Changes in the General Purchasing Power of Money: The Impact on Financial Statements of Canadian Corporations for the Period 1967-74," April 1976. (Reprinted from Cost and Management, January-February, 1976).

114. Deal, K.R., "Verification of the Theoretical Consistency of a Differential Game in Advertising," March, 1976.

114a. Deal, K.R., "Optimizing Advertising Expenditures in a Dynamic Duopoly," March, 1976.

115. Adams, Roy J., "The Canada-United States Labour Link Under Stress," [1976].

116. Thomas, Arthur L., "The Extended Approach to Joint-Cost Allocation: Relaxation of Simplifying Assumptions," June, 1976.

117. Adams, Roy J. and C.H. Rummel, "Worker's Participation in Management in West Germany: Impact on the Work, the Enterprise and the Trade Unions," September, 1976.

118. Szendrovits, Andrew Z., "A Comment on 'Optimal and System Myopic Policies for Multi-echelon Production/Inventory Assembly Systems'," [1976].

119. Meadows, Ian S.G., "Organic Structure and Innovation in Small Work Groups," October, 1976.

Continued on Page 2...

120. Basu, S., "The Effect of Earnings Yield on Assessments of the Association Between Annual Accounting Income Numbers and Security Prices," October, 1976.

121. Agarwal, Naresh C., "Labour Supply Behaviour of Married Women - A Model with Permanent and Transitory Variables," October, 1976.

122. Meadows, Ian S.G., "Organic Structure, Satisfaction and Personality," October, 1976.

123. Banting, Peter M., "Customer Service in Industrial Marketing: A Comparative Study," October, 1976. (Reprinted from: European Journal of Marketing, Vol. 10, No. 3, Summer, 1976).

124. Aivazian, V., "On the Comparative-Statics of Asset Demand," August, 1976.

125. Aivazian, V., "Contamination by Risk Reconsidered," October, 1976.

126. Szendrovits, Andrew Z. and George O. Wesolowsky, "Variation in Optimizing Serial Multi-State Production/Inventory Systems, March, 1977.

127. Agarwal, Naresh C., "Size-Structure Relationship: A Further Elaboration," March, 1977.

128. Jain, Harish C., "Minority Workers, the Structure of Labour Markets and Anti-Discrimination Legislation," March, 1977.

129. Adams, Roy J., "Employer Solidarity," March, 1977.

130. Gould, Lawrence I. and Stanley N. Laiken, "The Effect of Income Taxation and Investment Priorities: The RRSP," March, 1977.

131. Callen, Jeffrey L., "Financial Cost Allocations: A Game-Theoretic Approach," March, 1977.

132. Jain, Harish C., "Race and Sex Discrimination Legislation in North America and Britain: Some Lessons for Canada," May, 1977.

133. Hayashi, Kichiro. "Corporate Planning Practices in Japanese Multinationals." Accepted for publication in the Academy of Management Journal in 1978.

134. Jain, Harish C., Neil Hood and Steve Young, "Cross-Cultural Aspects of Personnel Policies in Multi-Nationals: A Case Study of Chrysler UK", June, 1977.

135. Aivazian, V. and J.L. Callen, "Investment, Market Structure and the Cost of Capital", July, 1977.

Continued on Page 3...

136. Adams, R.J., "Canadian Industrial Relations and the German Example", October, 1977.

137. Callen, J.L., "Production, Efficiency and Welfare in the U.S. Natural Gas Transmission Industry", October, 1977.

138. Richardson, A.W. and Wesolowsky, G.O., "Cost-Volume-Profit Analysis and the Value of Information", November, 1977.

139. Jain, Harish C., "Labour Market Problems of Native People in Ontario", December, 1977.

140. Gordon, M.J. and L.I. Gould, "The Cost of Equity Capital: A Reconsideration", January, 1978.

141. Gordon, M.J. and L.I. Gould, "The Cost of Equity Capital with Personal Income Taxes and Flotation Costs", January, 1978.

142. Adams, R.J., "Dunlop After Two Decades: Systems Theory as a Framework For Organizing the Field of Industrial Relations", January, 1978.

143. Agarwal, N.C. and Jain, H.C., "Pay Discrimination Against Women in Canada: Issues and Policies", February, 1978.

144. Jain, H.C. and Sloane, P.J., "Race, Sex and Minority Group Discrimination Legislation in North America and Britain", March, 1978.

145. Agarwal, N.C., "A Labour Market Analysis of Executive Earnings", June, 1978.

146. Jain, H.C. and Young, A., "Racial Discrimination in the U.K. Labour Market: Theory and Evidence", June, 1978.

147. Yagil, J., "On Alternative Methods of Treating Risk," September, 1978.

148. Jain, H.C., "Attitudes toward Communication System: A Comparison of Anglophone and Francophone Hospital Employees," September, 1978.

149. Ross, R., "Marketing Through the Japanese Distribution System", November, 1978.

150. Gould, Lawrence I. and Stanley N. Laiken, "Dividends vs. Capital Gains Under Share Redemptions," December, 1978.

151. Gould, Lawrence I. and Stanley N. Laiken, "The Impact of General Averaging on Income Realization Decisions: A Caveat on Tax Deferral," December, 1978.

152. Jain, Harish C., Jacques Normand and Rabindra N. Kanungo, "Job Motivation of Canadian Anglophone and Francophone Hospital Employees, April, 1979.

153. Stidsen, Bent, "Communications Relations", April, 1979.

154. Szendrovits, A.Z. and Drezner, Zvi, "Optimizing N-Stage Production/ Inventory Systems by Transporting Different Numbers of Equal-Sized Batches at Various Stages", April, 1979.

Continued on Page 4...

155. Truscott, W.G., "Allocation Analysis of a Dynamic Distribution Problem", June, 1979.

156. Hanna, J.R., "Measuring Capital and Income", November, 1979.

157. Deal, K.R., "Numerical Solution and Multiple Scenario Investigation of Linear Quadratic Differential Games", November, 1979.

158. Hanna, J.R., "Professional Accounting Education in Canada: Problems and Prospects", November, 1979.

159. Adams, R.J., "Towards a More Competent Labor Force: A Training Levy Scheme for Canada", December, 1979.

160. Jain, H.C., "Management of Human Resources and Productivity", February, 1980.

161. Wensley, A., "The Efficiency of Canadian Foreign Exchange Markets", February, 1980.

162. Tihanyi, E., "The Market Valuation of Deferred Taxes", March, 1980.

163. Meadows, I.S., "Quality of Working Life: Progress, Problems and Prospects", March, 1980.

164. Szendrovits, A.Z., "The Effect of Numbers of Stages on Multi-Stage Production/Inventory Models - An Empirical Study", April, 1980.

165. Laiken, S.N., "Current Action to Lower Future Taxes: General Averaging and Anticipated Income Models", April, 1980.

166. Love, R.F., "Hull Properties in Location Problems", April, 1980.

167. Jain, H.C., "Disadvantaged Groups on the Labour Market", May, 1980.

168. Adams, R.J., "Training in Canadian Industry: Research Theory and Policy Implications", June, 1980.

169. Joyner, R.C., "Application of Process Theories to Teaching Unstructured Managerial Decision Making", August, 1980.

170. Love, R.F., "A Stopping Rule for Facilities Location Algorithms", September, 1980.

171. Abad, Prakash L., "An Optimal Control Approach to Marketing - Production Planning", October, 1980.

172. Abad, Prakash L., "Decentralized Planning With An Interdependent Marketing-Production System", October, 1980.

173. Adams, R.J., "Industrial Relations Systems in Europe and North America", October, 1980.

Continued on Page 5...

174. Gaa, James C., "The Role of Central Rulemaking In Corporate Financial Reporting", February, 1981.

175. Adams, Roy J., "A Theory of Employer Attitudes and Behaviour Towards Trade Unions In Western Europe and North America", February, 1981.

176. Love, Robert F. and Jsun Y. Wong, "A 0-1 Linear Program To Minimize Interaction Cost In Scheduling", May, 1981.

177. Jain, Harish, "Employment and Pay Discrimination in Canada: Theories, Evidence and Policies", June, 1981.

178. Basu, S., "Market Reaction to Accounting Policy Deliberation: The Inflation Accounting Case Revisited", June, 1981.

179. Basu, S., "Risk Information and Financial Lease Disclosures: Some Empirical Evidence", June, 1981.

180. Basu, S., "The Relationship between Earnings' Yield, Market Value and Return for NYSE Common Stocks: Further Evidence", September, 1981

181. Jain, H.C., "Race and Sex Discrimination in Employment in Canada: Theories, evidence and policies", July 1981.

182. Jain, H.C., "Cross Cultural Management of Human Resources and the Multinational Corporations", October 1981.

183. Meadows, Ian, "Work System Characteristics and Employee Responses: An Exploratory Study", October, 1981.

184. Svi Drezner, Szendrovits, Andrew Z., Wesolowsky, George O. "Multi-stage Production with Variable Lot Sizes and Transportation of Partial Lots", January, 1982.

185. Basu, S., "Residual Risk, Firm Size and Returns for NYSE Common Stocks: Some Empirical Evidence", February, 1982.

186. Jain, Harish C. and Muthuchidambram, S. "The Ontario Human Rights Code: An Analysis of the Public Policy Through Selected Cases of Discrimination In Employment", March, 1982.

187. Love Robert F., Dowling, Paul D., "Optimal Weighted ℓ_p Norm Parameters For Facilities Layout Distance Characterizations", April, 1982.