Working Pap. McMASTER

On a Decomposition Algorithm for Sequencing Problems with Precedence Constraints

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Research and Working Paper Series No. 213
1983

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ABSTRACT

In this paper we introduce a class of sequencing problems, which includes many widely-studied problems; for example the one-machine total weighted completion time problem, the directed linear ordering problem, the least cost fault detection problem and the total weighted exponential completion time problem. Sidney developed a decomposition algorithm for the one-machine total weighted completion time problem which was later shown to be applicable to all the problems in the class. We discuss how recently developed decomposition theories for directed acyclic graphs enable us to execute efficiently one of the two main steps in Sidney's algorithm. This is followed by complexity results, where it is shown that all of the above sequencing problems remain NP-complete even if we restrict the precedence constraints between the jobs to a special class of precedence graphs. At the end we introduce a class of precedence constraints for which the sequencing problems have polynomial time and space complexity.

ON A DECOMPOSITION ALGORITHM FOR SEQUENCING PROBLEMS WITH PRECEDENCE CONSTRAINTS

Introduction

In this paper we study a class of sequencing problems, which includes many widely-studied problems; for example the one-machine total weighted completion time problem [8, 16, 23, 25] the directed linear ordering problem [16] the least cost fault detection problem [18] and the total weighted exponential completion time problem [9]. Sidney [23] developed a decomposition algorithm for the one-machine total weighted completion time problem which was later shown to be applicable to all the problems in the class. We discuss how recently developed decomposition theories for directed acyclic graphs [4, 5, 26] enable us to execute efficiently one of the two main steps in Sidney's algorithm. This is followed by complexity results, where it is shown that all of the above sequencing problems have the same complexity even if we restrict the precedence constraints between the jobs to a special class of precedence graphs. Many results are known in scheduling theory which show that restricting the parameters defining a sequencing problem to a small set of allowable values does not change the complexity of the problem [14, 15, 16, 17]. Our complexity results show that the precedence graphs also may be restricted to special classes without affecting NP-completeness of the problems involved. At the end we introduce a class of precedence constraints for which the sequencing problems have polynomial time and space complexity. We assume that the reader is familiar with the concepts of algorithmic complexity. [1, 6].

1. Preliminary definitions, notation and results

Consider the set $V = \{1,2,\ldots,n\}$ of n jobs to be sequenced for processing on a single machine. Each job is characterized by certain parameters. (E.g. in the total weighted completion time problem the job i is specified by its processing time $p_i > 0$ and a weighting factor w_i.) A sequence s of k jobs is a function from $\{1,2,\ldots,k\}$ to V and will be represented by $(s(1),s(2),\ldots,s(k))$, where s(i) is the i-th job in the sequence s. A cost function f assigns a real value to each sequence. (E.g. for the total weighted completion time problem ($\sum w_i C_i$) $f(s) = \sum_{i=1}^{n} w_{s(i)} C_i^s$, where $C_i^s = \sum_{j=1}^{i} p_{s(j)}$ is the completion time of the i-th job in the sequence s.)

If we denote a permutation of the jobs in V by π, then a sequencing problem, on a set of jobs V with cost function f, is to find a permutation of V contained in a set of feasible permutations F, which minimizes f, i.e.

$$\underset{\pi \in F}{\text{minimize}}\ f(\pi)$$

The problem is called unconstrained if all permutations are feasible. In many cases however, the set of feasible permutations is restricted by a set of precedence constraints represented by a directed acyclic graph (dag) $G=(V,A)$, where the vertices are identified with the jobs of the sequencing problem and the arc $(i,j) \in A$ if and only if (iff) j must be preceded by i in every feasible sequence of F. A subset $V' \subseteq V$ is called a <u>compound job</u> if the jobs in V' must be sequenced <u>consecutively</u> in every feasible sequence. If, in addition, a compound job must be processed in a fixed sequence of its jobs in every feasible permutation of V, then we refer to it as a <u>string</u>. The precedence constraints could be extended to strings of jobs, denoted by $\vec{s}$, $\vec{t}$, $\vec{u}$, etc.: $\vec{s} \to \vec{t}$ means, that the jobs in both $\vec{s}$ and $\vec{t}$ must be sequenced consecutively in the fixed order of each string and every job in $\vec{s}$ must precede every job in $\vec{t}$.

A common property of the class of sequencing problems discussed in this paper is the <u>adjacent pairwise interchange (API) property</u>: We say that a cost function f satisfies the API property if there is a transitive and complete binary preference relation $\preceq$ defined on the jobs of V s.t. for all $i,j \in V$,

$i \preceq j$ implies that

$f(u,i,j,v) \leq f(u,j,i,v)$ for all sequences u and v.

We say that i is strictly preferable to j ($i \prec j$) if $i \preceq j$ but $j \preceq i$ does not hold.

Smith [25] presented a very simple and powerful solution to any unconstrained sequencing problem satisfying the API property, by showing that if a permutation is consistent with $\prec$, i.e. $i \prec j$ implies that i precedes j in the permutation, then this permutation must be optimal.

The API property can be generalized [16, 21] from jobs to sequences of jobs (job-strings): A function f satisfies the <u>adjacent sequence interchange (ASI) property</u> if there exists a transitive and complete binary preference relation $\preceq$ defined on sequences by the following:

For all sequences s and t, $s \preceq t$ implies that

$f(u,s,t,v,) \leq f(u,t,s,v)$ for all sequences u and v.

If the reverse is also true, i.e. $f(u,s,t,v) \leq f(u,t,s,v)$ implies $s \preceq t$, we say that f satisfies the <u>strong ASI property</u>. $\sum w_i C_i$ is an example for the cost function satisfying the strong ASI property.

We end this section by defining some other sequencing problems studied in this paper and all satisfying the strong ASI property.

In the least cost fault detection problem ($\sum Q_i c_i$) a system consisting of n components is to be inspected by sequentially applying tests to each component until one fails (i.e. the system is "defective") or all components pass their test (i.e. the system passes inspection). Associated with each component j is a testing cost c_j and a probability q_j of passing its test, $0 < q_j < 1$. The tests are assumed to be statistically independent and so for any sequence s, $Q_i^s = q_{s(1)} q_{s(2)} \cdots q_{s(i-1)}$ is the probability that the i-th component in s will be tested (by convention $Q_1^s=1$). The expected testing cost for a sequence s of length k is given by $\sum_{i=1}^{k} Q_i^s c_{s(i)}$. The problem is to find a feasible permutation which minimizes the expected testing cost, i.e.

$$\underset{\pi \varepsilon F}{\text{minimize}} \sum_{i=1}^{n} Q_i c_{\pi(i)}$$

The total weighted exponential completion time problem ($\sum w_i \exp(-rC_i)$) is similar to $\sum w_i C_i$ but cost accumulates exponentially rather than linearly. The cost of a sequence s of length k is $\sum_{i=1}^{k} w_{s(i)} \exp(-rC_i^s)$, where $r>0$ is constant. The problem is to find a feasible permutation which minimizes the total weighted exponential completion time, i.e.

$$\underset{\pi \varepsilon F}{\text{minimize}} \sum_{i=1}^{n} w_{\pi(i)} \exp(-rC_i^{\pi}).$$

$\sum w_i \exp(-rC_i)$ was shown to be equivalent to $\sum Q_i c_i$ [21] and $\sum w_i C_i$ can be considered a special case of $\sum Q_i c_i$ [13].

In the directed linear ordering problem the vertices of a given dag $G = (V,A)$ are to be ordered on the real line by assigning the coordinate x_j to vertex j ($j \varepsilon V$) in such a fashion that i) if $(i,j) \varepsilon A$ then $x_i < x_j$; ii) if j is the leftmost vertex then $x_j = p_j$ where $p_j > 0$ is the given "width" of vertex j; iii) if the vertices i and j are placed at consecutive positions on the line then $x_j = x_i + p_j$; and the objective function $f = \sum_{(i,j) \varepsilon A} w_{ij}(x_j - x_i)$ is to be minimized, where the w_{ij}-s are the given weights of arc $(i,j) \varepsilon A$.

2. <u>Job Modules and The Decomposition of Transitive Dags</u>.

Sidney [23] suggested the following decomposition approach for the solution of $\sum w_i C_i$. It is based on decomposing the job set (and the precedence graph) into subsets called (job) modules: A subset $M \subseteq V$ is a <u>(job) module</u> in $G=(V,A)$ iff for every $k \varepsilon V \setminus M$ one of the following conditions holds:

i) $(k,i) \varepsilon A$ for every $i \varepsilon M$

ii) $(i,k) \varepsilon A$ for every $i \varepsilon M$

iii) $(i,k) \notin A$ and $(k,i) \notin A$ for every $i \varepsilon M$.

Job Module Algorithm

1. Find a job-module $M \subseteq V$ in $G=(V,A)$ such that $|M|>1$.
2. Find an optimal sequence s_1 for the subproblem induced on M with precedence constraints $G_M = (M,A)$.
3. Set $A=A \cup \{(i,j) \mid i \text{ precedes } j \text{ in } s_1\}$. If G contains an arc between every pair of vertices (i.e. G is completely sequenced) STOP, otherwise go to 1.

This algorithm results in an optimal sequence for many sequencing problems including $\sum w_i C_i$ [23] and the equivalent directed linear ordering problem [16], $\sum Q_i c_i$ [11] and the equivalent $\sum w_i \exp(-rC_i)$ [23] and certain stochastic problems described in [9]. Monma [19] identified general sufficient conditions which ensure that a permutation produced by the algorithm is optimal.

The usefulness of this algorithm depends on whether efficient implementation can be found for steps 1 and 2. So far this can be done only in special cases, such as the polynomial time algorithms for series-parallel precedence constrained sequencing problems ([16], [23]) can be considered applications of the Job Module Algorithm.

The main source of difficulty for sequencing problems with precedence constraints is that a conflict may exist between ordering the jobs according to the preference relation and preserving feasibility according to the precedence constraints. The following theorem shows how this conflict can be resolved in certain cases.

Theorem 1. [23] Let f satisfy the ASI property. Consider a job-module $\{\vec{s},\vec{t}\}$ where $\vec{s}$ and $\vec{t}$ are strings of jobs for which $\vec{s} \rightarrow \vec{t}$ and $\vec{t} \preceq \vec{s}$ hold. Then there is an optimal permutation with $\vec{s}$ immediately preceding $\vec{t}$.

The following theorem can be considered the dual of Theorem 1.

Theorem 2: Consider a sequencing problem defined by a sequencing function f, which satisfies the strong ASI property and precedence constraints $G = (V,A)$. Consider a job-module $\{\vec{s},\vec{t}\}$ where $\vec{s} \rightarrow \vec{t}$ and $\vec{s} < \vec{t}$. Then the precedence constraint between $\vec{s}$ and $\vec{t}$ can be eliminated, i e., any optimal sequence in $G_1 = (V, A_1)$ is also optimal in G, where $A_1 = A \setminus \{(\vec{s},\vec{t})\}$.

Proof: Consider a feasible permutation in G_1 with $\vec{t}$ preceding $\vec{s}$ i e., one of the form (u,t,v,s,w). No string in v may be related to $\vec{t}$ or $\vec{s}$ since $\{\vec{s},\vec{t}\}$ is a job-module; hence v may be interchanged with $\vec{t}$ or $\vec{s}$. If $v \leq \vec{t}$ then interchanging v and $\vec{t}$ does not increase the cost. On the other hand, if $v > \vec{t}$, we have $v > \vec{s}$ by transitivity and so interchanging v and $\vec{s}$ does not increase the cost. In either case, after the interchange we have $\vec{t}$ immediately preceding $\vec{s}$ in the permutation. By the strong ASI property interchanging $\vec{s}$ and $\vec{t}$ strictly decreases the cost, so the original permutation could not be optimal in G_1.

We note that a similar result to the above theorem appeared in [20]. We decided to state it and prove if for modules, since the development in the latter part of the paper heavily depends on it.

The concept of modules has a wide applicability outside the field of scheduling, e.g. in graph theory. This explains why it has been defined and dealt with under different names (autonomous sets [4], [12], splits [5] closed sets [7], stable sets [22], clumps [3], simplifiable sub-networks [24], partitive sets [10]) by several researchers. Buer and Mohring [4] and independently Steiner [26] have developed polynomial-time algorithms to decompose a transitive dag or equivalently a sequencing problem into modules. Cunningham [5] has developed a more general decomposition scheme for digraphs which contains as a special case the decomposition of sequencing problems into modules. For this problem all these methods represent basically equivalent decomposition algorithms of the same $O(n^3)$ time complexity, where n is the number of jobs (vertices).

If $G = (V,A)$ is a transitive dag, then it is clear that the sets consisting of a single vertex from V and V itself are modules. These are called the trivial modules and G is indecomposable or prime if it has only trivial modules. We call the dag G uniquely partially orderable (UPO) if its undirected version, $\bar{G}$ has exactly two transitive orientations G and G^{-1}, where in G^{-1} each arc is directed in the opposite direction compared to G. Shevrin and Filippov [22] and independently Trotter, Moore and Sumner [27] proved that a transitive dag G is UPO iff every nontrivial module of G is an independent set (a set of vertices with no arc between them). Since every indecomposable transitive dag satisfies this condition vacuously it follows that every indecomposable dag is UPO.

For a transitive dag $G = (V,A)$ the decomposition algorithms may result in one of the following three mutually exclusive cases [4]:

a) The graph G has more than one connected component and each of these is a module.

b) The complementary graph of the undirected version of G, $\bar{G}^c$ has more than one connected component and each of these defines a module.

c) When both G and $\bar{G}^c$ are connected then the maximal modules different from V partition V.

Comparing these cases to the decomposition of transitive series-parallel dags ([16], [28]), it is clear that for such graphs case a) is equivalent to the graph being the parallel composition of its connected components, case b) is equivalent to the graph being the series composition of the connected components of the complementary graph and case c) can never occur. Accordingly we will refer to the above cases a), b) and c) as of type P (parallel), S (series) or N (non-series-parallel) respectively.

A composition tree, T(G) can be used to represent the decomposition of a general precedence graph $G = (V,A)$: The root of T(G) is V. If a node U of

T(G) with $|U| \geq 2$ is unlabeled, determine the type of the decomposition for the induced subgraph (U,A) and accordingly label the node by P,S or N. The leaves of the tree are the vertices of G and they are the only unlabeled nodes of T(G). For an example see Figure 1. In the following we will refer to a graph G as N-type if the root of its composition tree T(G) is of type N.

3. Complexity Results

The decomposition algorithms for transitive dags, referred to in the previous section provide an efficient implementation for the first step of the Job Module Algorithm. This is an indication that the problem of finding optimal sequences for job modules (i.e., step 2) must be just as difficult as the original sequencing problem. It is known however that series-parallel job-modules can be sequenced very efficiently ([16], [23]). In light of this the following theorem is a direct consequence of the decomposition theory for transitive dags.

Theorem 3: Let Ω be the family of sequencing problems, to which the Job Module Algorithm applies and let $\Omega_1 \subset \Omega$ be the sub-family in which the precedence graph of the problems cannot be decomposed into series or parallel components, i.e., it is N-type. If we have an algorithm α capable to sequence any member of Ω_1 in polynomial amount of time and space (as a function of the size of the sequencing problem) then the Job Module Algorithm in combination with α and the series-parallel algorithm solves any problem in Ω in polynomial amount of time and space.

This result can be further sharpened. In order to do this we are going to use Theorems 1 and 2, which will enable us to replace completely sequenced job modules by a set of composite jobs with no precedence constraints between them. First we give an example how to define composite jobs, which can be used to replace strings of individual jobs without affecting the optimal sequences.

For example for $\sum w_i \exp(-rC_i)$ define a composite job k for the string (i,j) by $p_k = p_i + p_j$ and $w_k = w_i \exp(rp_j) + w_j$. Job k is a valid job, since $p_k > 0$. In order to prove the equivalence of the objective function f on (i,j) and k we note that f can be defined recursively as follows:

$f(m) = w_m \exp(-rp_m)$ for job m

$f(s,t) = f(s) + \exp(-rp(s)) \cdot f(t)$ for sequences s and t,

where for $s = (s(1), s(2),\ldots,s(r))$ we define $p(s) = \sum_{\ell=1}^{r} p(\ell)$.

Using this for arbitrary sequences u and v

$$\begin{aligned} f(u,k,v) &= f(u) + \exp(-rp(u))f(k) + \exp(-r(p(u)+p_k))f(v) = \\ &= f(u) + \exp(-rp(u))[w_k\exp(-rp_k) + \exp(-rp_k)f(v)] = \\ &= f(u) + \exp(-rp(u))[(w_i\exp(rp_j)+w_j)\exp(-r(p_i+p_j)) + \\ &\exp(-r(p_i+p_j))f(v)] = f(u) + \exp(-rp(u))[w_i\exp(-rp_i)+w_j\exp(-r(p_i+p_j)) \\ &+ \exp(-r(p_i+p_j))f(v) = f(u,i,j,v). \end{aligned}$$

Thus the job k and the string (i,j) may be used interchangeably when sequencing a $\sum w_i \exp(-rC_i)$ problem.

For the other problems composite jobs can be defined in a similar fashion.

<u>Theorem 4</u>: Consider a sequencing problem from the family Ω of Theorem 3 with sequencing function f, which satisfies the strong ASI property. Let the precedence graph be $G = (V,A)$ and let $M \subseteq V$ be a job module in G. If we know an optimal sequence for the jobs in M, then these jobs can be replaced by a set of independent composite jobs, so that any optimal sequence for the new problem corresponds to an optimal sequence for the original one.

<u>Proof</u>: Let $s = (s(1), s(2),\ldots, s(k))$ be an optimal sequence in the subgraph (M,A). By the definition of Ω, G can be replaced by the precedence graph $G_1 = (V,A_1)$ where $A_1 = A \cup \{(i,j) \mid i \text{ precedes } j \text{ in } s\}$ and any optimal permutation in G_1 is also optimal for the original problem. It is clear that any contigous subsequence $s_1 = (s(i), s(i+1),\ldots, s(i+j))$ $(1 \leq i \leq i+j \leq k)$

defines a job module in G_1. Let us consider then the strings in s from the beginning, starting with s(1) and s(2). If $s(1) \prec s(2)$ then by Theorem 2 the arc (s(1), s(2)) can be eliminated from A_1 without affecting the optimal sequence. If $s(1) \succeq s(2)$ then by Theorem 1 there is an optimal sequence in which s(1) and s(2) can be replaced by the composite job (string) (s(1), s(2)). Applying this process in a repetitive fashion for subsequent elements of s proves the theorem.

Theorem 4 means that all the job modules for which we can find an optimal sequence in polynomial amount of time (this certainly includes all the series or parallel job modules) can be replaced by sets of independent composite jobs. Consider a sequencing problem with precedence graph G = (V,A) and composition tree T(G). Starting from the bottom up in T(G), i.e., the leaf job modules, we can sequence all the series or parallel job modules, until we reach an N-type node of T(G) with all its sons sequenced. Since by Theorem 4 these sequenced modules can be replaced by independent composite jobs, we can assume that all the sons of the N-type node are independent jobs. This means however, by an earlier referred to result of Shevrin and Filippov [22], that the induced subgraph corresponding to such an N-type node belongs to a special class of precedence graphs, called UPO graphs. Based on Theorem 3 this proves the following:

Theorem 5: Let $\Omega_2 \subset \Omega_1$ be the subfamily of sequencing problems, where the precedence graph is UPO. If there exists an algorithm β, which solves any problem from Ω_2 in polynomial amount of time and space then any problem from Ω can also be solved in polynomial amount of time and space, i.e., the complexity of the problems in Ω_2 is the same as in Ω.

Corollary 6: The total weighted completion time problem ($\sum w_iC_i$) is NP-complete even if we restrict the precedence graphs to the class of UPO graphs.

Proof: Lawler [16] proved that $\sum w_iC_i$ is NP-complete with general precedence constraints. Since this problem belongs to Ω, the subclass of $\sum w_i C_i$ with UPO precedence graphs has the same complexity.

Corollary 7: The directed linear ordering problem is NP-complete even on the class of UPO graphs.

Proof: Lawler [16] has shown the equivalence of the directed linear ordering problem and $\sum w_i C_i$ with general precedence constraints and this equivalence was established using the same dag in both problems.

Corollary 8: The least cost fault detection problem ($\sum Q_i c_i$) is NP - complete on the class of UPO graphs.

Proof: Kelly [13] has shown that $\sum w_i C_i$ reduces to $\sum Q_i c_i$, which by Corollary 6 proves the statement.

Corollary 9: The total weighted exponential completion time problem ($\sum w_i \exp(-rC_i)$) is NP - complete on the class of UPO graphs.

Proof: Monma and Sidney [21] have shown that $\sum w_i \exp(-rC_i)$ is equivalent to $\sum Q_i c_i$.

The question, whether there exists a polynomial time algorithm for the class of NP-complete problems has been for years one of the central unsolved problems in combinatorial optimization [6]. Strong empirical evidence suggests that no such algorithm exists. Assuming that this is true it raises the question what separates the polynominally solvable problems in Ω from the difficult problems in Ω_2. The largest known polynomially solvable subclass of problems from Ω has been the subclass SP with series-parallel precedence constraints. In the following we define an extension of this class which narrows the gap between SP and Ω_2.

4. Generalized Series-Parallel Sequencing

Consider a finite set ψ of indecomposable transitive dags. The class of Transitive Generalized Series-Parallel dags over ψ (TGSP(ψ)) is defined as follows:

i) A dag consisting of a single vertex, e.g. $G=(\{i\},\emptyset)$ is in TGSP (ψ).

ii) Any dag $G_i \in \psi$ is in TGSP (ψ).

iii) If $G_1=(V_1, A_1)$ and $G_2=(V_2, A_2)$ are in TGSP (ψ) and $V_1 \cap V_2=\emptyset$ then either one of the following dags is also in TGSP (ψ).

a) Parallel Composition: $G_p=(V_1 \cup V_2, A_1 \cup A_2)$

b) Series Composition: $G_S=(V_1 \cup V_2,\ A_1 \cup A_2 \cup (O_1 \times I_2))$, where O_1 is the set of sinks in G_1 and I_2 is the set of sources in G_2.

A dag is Generalized Series-Parallel over ψ (GSP(ψ)) iff its transitive closure is in TGSP (ψ). We note that when $\psi = \emptyset$, GSP (ψ) is exactly the class of series-parallel dags.

It is clear that the composition tree T(S) of any $S \in \psi$ is of height one with its leaves corresponding to the single vertices of S, the only modules different from the whole vertex set. The composition tree T(G) of any $G \in$ TGSP(ψ) will contain N-type nodes only on the level directly above these leaves. All higher level nodes of T(G) must be either P- or S-type. In other words the graphs in TGSP(ψ) can be derived by taking the composition tree of a series-parallel graph and replacing in it some leaves by the composition tree of the appropriate member of ψ . How rich the class TGSP(ψ) is clearly depends on what ψ is.

Theorem 10: Consider the subclass of sequencing problems from Ω for which the precedence graph belongs to GSP(ψ) for a fixed, finite ψ . Then any problem from this subclass is solvable in polynomial time and space.

Proof: If all the N-type job-modules were sequenced in a GSP(ψ) precedence graph, then for sequencing the remaining P- or S-type job-modules we can use the Series-Parallel Algorithm, which needs polynomial time and space. For sequencing the N-type job-modules we can make use of the fact, that their number and individual size is fixed, by having fixed ψ , even as the size of the precedence graph varies in GSP(ψ). So even if none of the efficient solution methods (e.g. transformations [20] or suboptimization over p-minimal subsets [9,11,16,23] works for the N-type job-modules we can use branch and bound or dynamic programming methods, which may require exponential time and space, but exponential in the size of the N-type job-modules, which is bounded from above by a constant, determined by ψ.

We note that this whole approach can be meaningful only for such classes GSP(ψ), where the number of graphs is relatively small in ψ and its members can be relatively easily sequenced. As an example we present GSP(ψ_0) where ψ_0 contains one member, the dag Z shown in Figure 2. The dag Z is the forbidden subgraph for transitive series-parallel dags [28] and if we add a source and sink to it, it becomes the well-known Wheatstone bridge. Accordingly we call this class <u>Wheatstone Generalized Series-Parallel (WGSP)</u>.

Monma [19] has shown that from the sequencing problems in Ω it is exactly the subclass SP which always can be solved based only on ordinal information, i.e. using the preference ordering. Consider now a sequencing problem from Ω on four jobs with the precedence graph Z of Figure 2. It is easy to see that an optimal sequence can always be identified without using any cardinal information about the jobs with the exception of the case when the preference ordering on the jobs is $3 \prec 4 \prec 2 \prec 1$. In this case either (2,1,3,4) or (1,4,2,3) is the optimal sequence, depending on the cardinal information for the jobs. Table I shows an example of this for $\sum w_i C_i$. But even in this case, a simple test, we call the Z test, can be used to establish which one of the above two sequences is optimal. As an example we show this for $\sum w_i C_i$ in the following lemma.

<u>Lemma 11</u>: For $\sum w_i C_i$ on four jobs with precedence graph Z $f(1,4,2,3) \leq f(2,1,3,4)$ if and only if

$$w_2 (p_1 + p_4) + p_4 w_3 \leq p_2 (w_1 + w_4) + p_3 w_4$$

<u>Proof</u>: By direct substitution into f.

So for a WGSP precedence constrained problem from Ω , we can first sequence all the N-type job modules using only ordinal information or the Z-test and once these have been dealt with the problem becomes a series-parallel precedence constrained sequencing problem. Sequencing one N-type job module requires O(1) time and space and since their number is clearly not more than n/4, in total sequencing all the N-type job modules requires O(n) time and space. This proves the following corollary of Theorem 10 based on the complexity results for the SP algorithm [16,23].

Corollary 12: A WSGP precedence constrained sequencing problem from Ω can be solved in O(nlogn) time.

As an example consider the $\sum w_i C_i$ problem defined on 16 jobs by the data in Table II and the precedence graph in Figure 1. For the N-type job module $Z_1=\{4,3,5,6\}$ we have $5 \lessdot 6 \lessdot 3 \lessdot 4$. Applying the Z-test we have $w_3(p_4 + p_6)+ p_6w_5 = 11 < 13 = p_3(w_4 + w_6) + p_5w_6$, so Z_1, can be replaced by the optimal chain (4,6,3,5). For $Z_2 = \{8,9,11,10\}$ we have $8 \lessdot 11 \lessdot 10 \lessdot 9$, so based on this 8 must be the first job in any optimal sequence, so we can add the arc (8,9) which replaces Z_2 by a series-parallel dag. For $Z_3 = \{12,13,15,14\}$ we have $15 \lessdot 14 \lessdot 13 \lessdot 12$ and $w_{13} (p_{12} + p_{14}) +p_{14}w_{15} = 15 > 8 = p_{13}(w_{12} + w_{14}) + p_{15}w_{14}$, so Z_3 can be replaced by the optimal chain (13,12,15,14). Having dealt with all the N-type job modules, the problem has been reduced to a series-parallel problem for which the SP algorithm [16] finds the optimal sequence (1,7,8,10,9,11,2,4,6,3,5,13,12,15,14,16).

Acknowledgement

The author gratefully acknowledges C.L. Monma for his helpful comments and useful suggestions for the presentation of this material.

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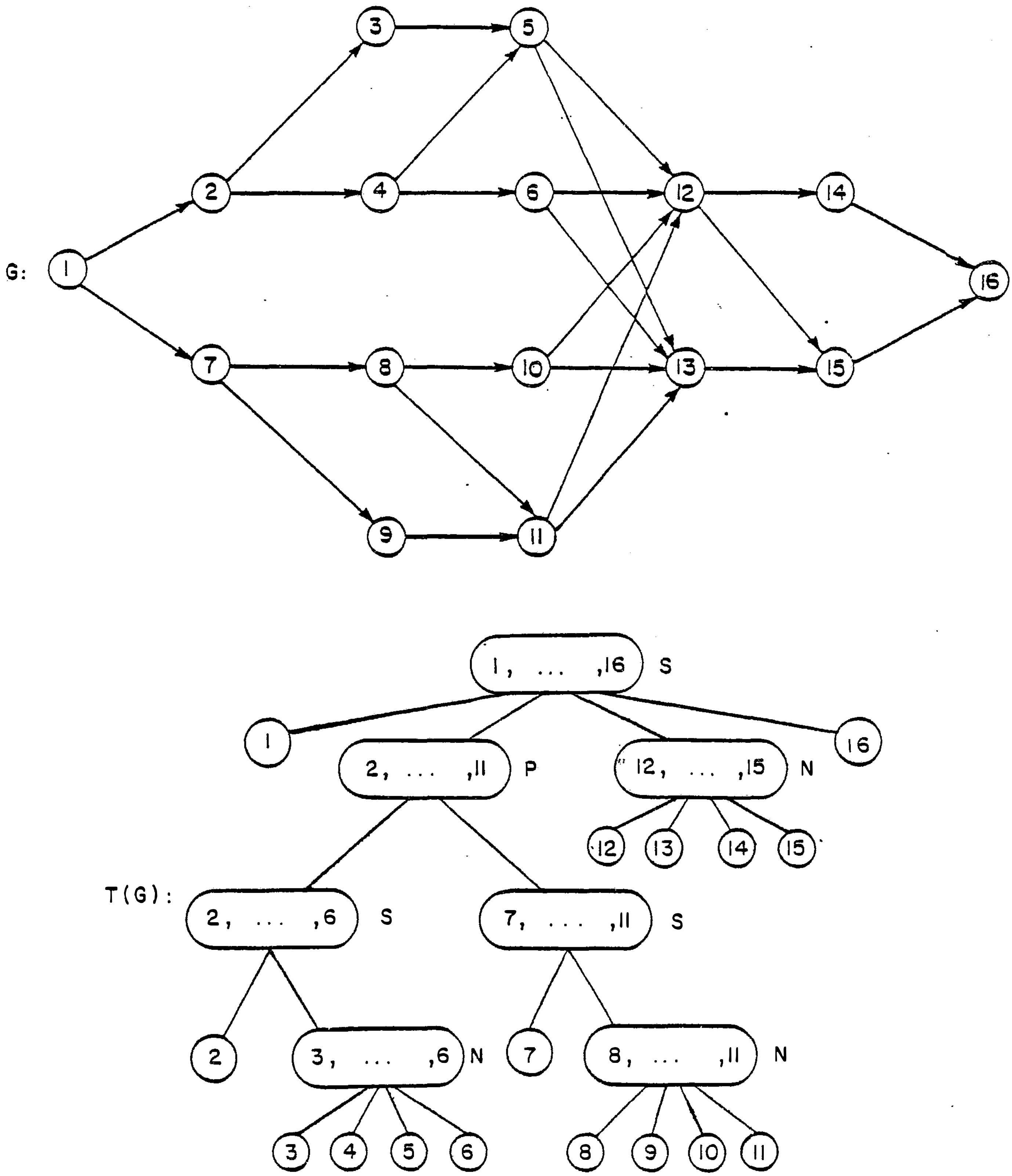

A precedence graph and its composition tree
Figure 1.

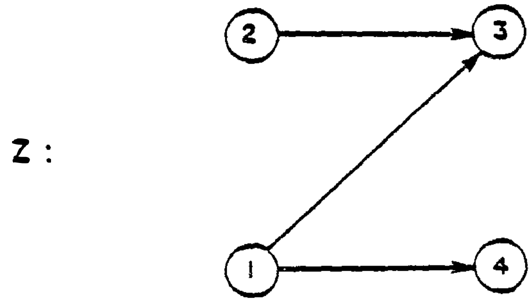

Figure 2

Table I

	p_1	w_1	p_2	w_2	p_3	w_3	p_4	w_4	Optimal sequence
Problem 1	7	1	5	1	3	1	4	1	(2,1,3,4)
Problem 2	7	1	6	1	1	1	2	1	(1,4,2,3)

Table 2

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
p_i	1	4	6	7	1	2	1	5	10	7	8	7	5	4	3	6
w_i	1	1	1	1	1	1	10	3	1	3	4	1	1	1	1	1

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