The Demand for Risky Financial Assets by the U.S. Household Sector

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I. Introduction

This paper has two interrelated objectives. First, we shall test empirically the extent to which the U.S. household sector's investments in financial assets can be described by a Mean-Variance portfolio optimization framework. Unlike much of the empirical portfolio literature, we shall not make use of concepts of efficient sets and systematic risk. Instead, we shall revert to a more basic utility function dependent approach. In contrast to the earlier portfolio literature, we will empirically estimate the underlying parameters of the utility function assuming that the household sector makes its investment decisions on the basis of mean-variance portfolio optimization. This in turn will allow us to test the adequacy of the portfolio paradigm at least at the aggregate level. In so doing, we provide some empirical support to Levy and Markowitz's [1979] study which demonstrates that expected utility can be approximated by a judiciously chosen function defined over mean and variance, at least for some utility functions.

The second objective of the study is to estimate own and cross elasticities for the financial assets held by the U.S. household sector. We will estimate three types of elasticities: (i) the impact of a change in the expected return of asset A on the household sector's demand for asset B; (ii) the impact of a change in the riskiness (i.e., variance) of asset A on the demand for asset B; and (iii) the impact of a change in the covariance between assets A and B on the demand for asset C. Estimating elasticities of these types is of potential importance to a number of outstanding issues in economics and finance. Included in these issues are such problems as: What effects do risk reduction regulations in the equity market have on the demand for other securities and, hence, financial intermediaries? Does risk play an important role in the definition of money and money substitutes? Does instability of the stock market have any influence on the impact of federal
Our methodology for studying these issues is based on a synthesis of portfolio theory and the use of flexible functional forms in demand system analysis. Specifically, we assume that the household sector's utility function, defined over mean and standard deviation of end-of-period wealth, can be approximated by a generalized Box-Cox flexible functional form. This latter function takes on the generalized Leontief, generalized square root quadratic, and translog utility function as special or limiting cases. Budget share equations for risky assets are derived from the generalized Box-Cox utility function using a standard portfolio optimization framework. These budget share equations are then estimated from data on the financial asset holdings of the U.S. household sector and the associated market yields. A Chi-square test is used to determine which of the three specific flexible functional forms mentioned earlier best fits the data. Those functions that are not rejected are then checked to see which, if any, yield signs for marginal utilities and comparative static conditions consistent with the theory. In this fashion, we hope to validate at least partially the mean-variance approach although we are unable to specify the true underlying utility function. Finally, having determined that utility function which best fits the theory and the data, it is a straightforward matter to generate estimated mean, variance, and covariance elasticities for financial asset demands.

In what follows, Section II develops the elasticities and budget share equations for risky assets by utilizing a standard optimization procedure and a generalized Box-Cox utility function. Section III estimates the budget share system on U.S. household financial asset holdings and market return data. After using the data to determine the "optimal" form of the utility function, estimates of expected return, variance and covariance elasticities
are obtained and analysed. Section IV briefly concludes the paper. Appendix A develops the budget share equations, while Appendix B describes the data and the data sources.

II. Elasticities and Budget Share Equations for Risky Assets

(i) The Elasticities

To make the theory and empirical work manageable, we adopt the commonly-made assumption of homothetic separability — that the household sector's investment decision in specific financial assets is independent both of the overall consumption-investment decision and the investment in non-financial assets. This means that the total amount of wealth to be invested in financial assets is exogenous to the model and the only issue of consequence is the proportion of wealth to be invested in each financial asset.

The household sector's investment preferences are assumed to be captured by a Lancaster-type utility function defined over portfolio characteristics

\[ U = U(E, V) \]

where \( E \) is the expected end-of-period wealth of the portfolio and \( V \) is its standard deviation. This utility function is assumed to be continuous and twice differentiable with \( U_E > 0 \) and \( U_V < 0 \) where the subscripts denote partial derivatives. In short, the household sector is assumed to be risk averse with indifference curves in \( E-V \) space which are upward sloping and — given additional assumptions to be made further — convex from below.

The personal sector's financial asset choice framework is assumed to be described by the program:

\[
\text{Maximize } U(E, V)
\]

\[ X_1, \ldots, X_m \]
subject to: \[ E = W_0 \left[ 1 + \sum_{i=1}^{m} X_i E_i \right] \] (1b)
\[ V = [W_0^2 \sum_{i=1}^{m} \sum_{j=1}^{m} X_i X_j G_{ij}]^{1/2} \] (1c)
\[ \sum_{i=1}^{m} X_i = 1 \] (1d)

where

\( X_i = \) the proportion of the household sector's wealth invested in financial asset \( i \), \( i=1, 2, \ldots, m \)
\( E_i = \) the expected rate of return on asset \( i \), \( i=1, \ldots, m \)
\( G_{ij} = \) the covariance of returns between assets \( i \) and \( j \), \( i,j=1,2,\ldots,m \)
\( W_0 = \) initial wealth invested in financial assets

Solving the utility maximizing program (1) yields the first-order conditions:

\[ W_0 U E_i + W_0^2 U V^{-1} \sum_{j=1}^{m} X_j G_{ij} = \lambda \quad (i,j=1,2,\ldots,m) \] (2a)
\[ 1 - \sum_{i=1}^{m} X_i = 0 \] (2b)

where \( \lambda \) is the Lagrange multiplier.

The second-order condition for a maximum require the principal minors of the determinant \( D \) — obtained by differentiating (2a) and (2b) with respect to the \( X_i \)'s — to alternate in sign. In particular,

\[ D = \begin{bmatrix}
Z_{11} & \cdots & Z_{1m} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & & \ddots & \vdots \\
Z_{m1} & \cdots & Z_{mm} & 1 \\
1 & \cdots & 1 & 0 \\
\end{bmatrix} \] (3)

where
\[ Z_{ij} = W_0^2 U_{EE} E_i E_j + V^{-1} W_0^3 U_{EV} \left( \sum_{i=1}^{m} E_i \Sigma X_{iG_{ji}} + \sum_{j=1}^{m} E_j \Sigma X_{jG_{ji}} \right) \]

\[ + W_0^4 (U_{VV} V^{-2} - U_{V} V^{-3} \Sigma \Sigma X_{G_{ji}} X_{G_{ij}} + V^{-1} W_0^2 U_{vG_{ij}} \right) \]

The impact of a change in the \( r \)th asset return on the demand for the \( k \)th asset is determined by differentiating the first-order conditions [Equations (2a) and (2a)] with respect to \( E_r \). This procedure yields the matrix equation

\[
\begin{bmatrix}
  Z_{11} & \cdots & Z_{1m} & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  Z_{r1} & \cdots & Z_{rm} & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  Z_{ml} & \cdots & Z_{mm} & 1 \\
  1 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial X_1}{\partial E_r} \\
  \vdots \\
  \frac{\partial X_r}{\partial E_r} \\
  \vdots \\
  \frac{\partial X_m}{\partial E_r}
\end{bmatrix}
= 
\begin{bmatrix}
  T_1 \\
  \vdots \\
  T_r \\
  \vdots \\
  T_m
\end{bmatrix}
\]

where \( T_i = -W_0 U_{EE} \delta_{ir} - X_r [W_0^2 U_{EE} E_i + V^{-1} W_0^3 U_{EV} \Sigma \Sigma X_j G_{ij}] \)

and \( \delta_{ir} = \begin{cases} 
1 & \text{for } i = r \\
0 & \text{for } i \neq r 
\end{cases} \)

Solving for \( \frac{\partial X_k}{\partial E_r} \) yields:

\[
\frac{\partial X_k}{\partial E_r} = -W_0 U_{EE} \frac{D_{rk}}{|D|} - X_r [W_0^2 U_{EE} \Sigma E_i D_{ik} + V^{-1} W_0^3 U_{EV} \Sigma \Sigma X_j G_{ij} D_{ik}] \left( \Sigma \frac{D_{rk}}{|D|} \right) \]

where \( D_{rk} \) is the \( r \)th \( k \)th cofactor of \( D \). The demand elasticity of asset \( k \) with respect to the expected return on asset \( r \) is easily calculated from equation (6) as
\[
\eta(X_k, E_r) = \frac{\partial X_k}{\partial E_r} \cdot \frac{E_r}{X_k}
\]

(7)

In an analogous fashion, one can derive the impact of a change in the covariance between assets \( r \) and \( f \) on the demand for asset \( k \), namely,

\[
\frac{\partial X_k}{\partial G_{rf}} = -V^{-1}U_V \omega_0^2 (X_r \frac{D_{rk}}{|D|} + X_f \frac{D_{fk}}{|D|}) - V^{-1}X_r X_f \omega_0^3 U_{EV} \sum_{i=1}^{m} \frac{E_i}{|D|} \frac{D_{ik}}{|D|}
\]

\[
+ \omega_0 V^{-1}u_V \sum_{i=1}^{m} \sum_{j=1}^{m} X_{ij} \frac{G_{ij}}{|D|} \frac{D_{ik}}{|D|}
\]

(8)

Similarly, the demand elasticity of asset \( k \) with respect to a change in the covariance between assets \( r \) and \( f \) is given by

\[
\eta(X_k, G_{rf}) = \frac{\partial X_k}{\partial G_{rf}} \cdot \frac{G_{rf}}{X_k}
\]

(9)

The variance cross elasticities are obtained from (9) by setting \( r = f \).

Unlike the traditional theory of demand underlying the Slutsky equation where prices appear in the budget equations, in the theory of portfolio choice expected rates of return (or variance--covariance of returns) are not in the budget equations, but in the preference function affecting the ranking of portfolios. Allingham and Morishima (1973) identify effects of changes and distinguish them from the wealth and substitution effects of the traditional Slutsky equations. Thus they identify the first term in (6) [or (8)] as a Relative Want-Pattern effect and the second term as an Absolute Want-Pattern effect. However, equation (6) [or (8)] does not represent effect of taste changes in E-V characteristics space. It simply represents the effects of changes in asset attributes. Thus equation (6) does not involve a change in the investor's preference function defined over E-V characteristics. It simply involves a shift in the E-V efficiency locus due to a change in the "productivity" of \( X_r \) in yielding \( E \) (i.e., \( E_r \)). For this reason we identify such effects as productivity effects.
Equation (6) (and similarly (8)) can be decomposed into two effects. As Aivazian (1976) has shown the last term in (6) is equal to \(-x_r \frac{3x_k}{3T_E}\) where \(T_E\) is a "lump-sum tax" on the portfolio's (expected) return. Since a change in \(T_E\) is equivalent to a change in the average-productivity of each asset in producing \(E\), we identify the last term in (6) as the average-productivity effect of a change in \(E_r\), while the first term on the right is the pure marginal productivity effect of a change in \(E_r\). Notice that for given initial quantities of the assets, a small increase in \(E_r\) produces an increase in \(E\) of \(x_r \Delta E_r\), while \(V\) remains unchanged. Thus the increase in \(E_r\) increases the average-productivity of each asset in producing \(E\) (in proportion to the change in \(E\)). We can adjust the average-productivity of each asset to the original level by "lump-sum taxing" away the above change in \(E\). The effect on \(x_k\) of such a compensation in the average-productivities of the assets is \(x_r \frac{3x_k}{3T_E}\), which is equal to the negative of the last term in (6). Hence the first term on the right in (6) represents the effect of a pure change in the marginal productivity of asset \(r\) in producing \(E\), netting out average productivity changes. Since we are dealing with an expected return rather than a price effect, \(r\) and \(k\) are defined to be net complements (substitutes) if this first term is positive (negative). It can be shown that the "own" expected return effect (i.e., when \(r = k\)) is unambiguously positive. The sign of the average productivity effect is ambiguous even when \(r = k\). It is obvious that equation (8), the equation for risk, can be decomposed similarly into a marginal and average productivity effect in the production of portfolio risk. The "own" pure marginal productivity effect in the case of risk is unambiguously negative.\(^5\)

(ii) Budget Share Equations from Flexible Functional Form Utility Functions

To operationalize the theory developed in the previous section, we assume that the utility function \(U(E, V)\) can be specified as a generalized Box-Cox
function of the form\(^6\)

\[ U(\delta) = \alpha_0 + \alpha_1 E(\lambda) + \alpha_2 V(\lambda) + \frac{1}{2} \alpha_3 [E(\lambda)]^2 + \frac{1}{2} \alpha_4 [V(\lambda)]^2 + \alpha_5 E(\lambda)V(\lambda) \quad (10) \]

where \( U(\delta) \), \( E(\lambda) \) and \( V(\lambda) \) are the Box-Cox transformations

\[
\begin{align*}
U(\delta) &= (U^{2\delta} - 1)/2\delta \\
E(\lambda) &= (E^{-1}/\lambda) \\
V(\lambda) &= (V^{-1}/\lambda)
\end{align*}
\]

As the parameters \( \delta \) and \( \lambda \) take on different values, one obtains the following alternative flexible functional forms.

Case (a): \( \delta, \lambda \rightarrow 0 \): \( U(\delta) = \ln U; E(\lambda) = \ln E; V(\lambda) = \ln V \)

This case yields the translog utility function

\[
\ln U = \alpha_0 + \alpha_1 \ln E + \alpha_2 \ln V + \frac{1}{2} \alpha_3 (\ln E)^2 + \frac{1}{2} \alpha_4 (\ln V)^2 + \alpha_5 (\ln E)(\ln V) \quad (12)
\]

Case (b): \( \delta, \lambda = 1/2 \): \( U(\delta) = U - 1; E(\lambda) = 2(E^{1/2} - 1); V(\lambda) = 2(V^{1/2} - 1) \)

This case gives the generalized Leontief utility function

\[
\begin{align*}
U &= 2\alpha_3 E + 2\alpha_4 V + 4\alpha_5 E^{1/2}V^{1/2} + (2\alpha_1 - 4\alpha_3 - 4\alpha_5)E^{1/2} \\
&\quad + (2\alpha_2 - 4\alpha_4 - 4\alpha_5)V^{1/2} + 2\alpha_3 + 2\alpha_4 + 4\alpha_5 - 2\alpha_1 - 2\alpha_2 + 1
\end{align*}
\]

Case (c): \( \delta, \lambda = 1 \): \( U(\delta) = (U^2 - 1)/2; \)

\[ E(\lambda) = E - 1; V(\lambda) = V - 1 \]

This case results in the square root quadratic utility function

\[
\begin{align*}
U &= (\alpha_3 E^2 + \alpha_4 V^2 + 2\alpha_5 E.V + 2(\alpha_1 - \alpha_3 - \alpha_5)E + 2(\alpha_2 - \alpha_4 - \alpha_5)V \\
&\quad + 2\alpha_5 + \alpha_3 + \alpha_4 - 2\alpha_1 - 2\alpha_2 + 1)^{1/2}
\end{align*}
\]

It is worth noting that the ordinary quadratic can be obtained by setting \( \delta = 1/2 \) and \( \lambda = 1 \). However, the ordinary quadratic yields the same budget share equations as the square root quadratic, so a test of the latter is a test effectively of both functional forms.\(^7\)

The budget share equations for the generalized Box-Cox utility function can be obtained by substituting equation (10) (and its partial derivatives
such as $U_E$, $U_V$, etc.) into the first-order conditions \[\text{Equations (2a)}\] and \[\text{(2b)}\].

It is shown in Appendix A that the resulting budget share (demand) system can be written as

$$X = KG^{-1}E^*$$

(15)

where

$$X = \begin{bmatrix}
  X_1 \\
  \vdots \\
  X_m
\end{bmatrix}, \quad \begin{bmatrix}
  \tilde{G}_{11} \\
  \vdots \\
  \tilde{G}_{mm}
\end{bmatrix} = \begin{bmatrix}
  \tilde{G}_{11} \\
  \vdots \\
  \tilde{G}_{mm}
\end{bmatrix}$$

and

$$G_{rj}^* = G_{1j} - G_{rj} \quad (r = 2, \ldots, m) \quad (j = 1, 2, \ldots, m)$$

By adding to equation (15) a serially uncorrelated multivariate normal disturbance term $v$, we obtain the budget share system to be estimated:

$$X = KG^{-1}E^* + v$$

(15a)

where

$$v = \begin{bmatrix}
  v_1 \\
  \vdots \\
  v_m
\end{bmatrix}$$

The budget share systems corresponding to the translog, square root quadratic (and quadratic), and generalized Leontief utility functions can be obtained from equation (15a) by setting $\lambda$ equal to zero, one, and one-half, respectively.
III. Estimation and Empirical Results

(i) The Data

The data base that we used to estimate the demand system (15a) is comprised of the annual financial asset holdings of the U.S. household sector from 1949 to 1973, and associated monthly yields. We categorized these financial holdings into six asset types: (i) money broadly defined (MY), (ii) short-term fixed income securities (primarily government bonds) (SB), (iii) long-term U.S. government savings bonds (LB), (iv) corporate and foreign bonds (CB), (v) mortgages (MT), and (vi) equities (ST). The data sources and definitions of each asset category with its associated yield are provided in Appendix B.

A rolling sample technique was used to estimate the mean and standard deviation for the financial asset portfolio held by the U.S. household sector. Specifically, the first two years of yield data, 1949 and 1950, (24 data points for each asset category) were employed to calculate sample mean returns and standard deviation for each asset as well as sample covariances between asset yields. These sample estimates were then used to calculate the (sample) expected return and variance for the portfolio for the year 1951. The $X_i$'s for 1951 were the actual proportions of each asset held in 1951. $W_0$ was assumed to be the dollar holdings in financial assets in 1950. Therefore, the calculated 1951 sample mean returns, variances and covariances for the separate assets as well as the portfolio mean and standard deviation represent one data point to be utilized in estimating the demand system. The second data point (for 1952) was calculated by an updating or rolling sample technique. Sample means, variance and covariances were recalculated after hopping the 1949 monthly yield data and substituting the 1951 data. Again,
two years of data, 1950 and 1951, were used to generate the asset portfolio expected return and standard deviation. These new estimates together with the asset proportions held by the U.S. household sector in 1952 (and the 1951 datum for \( W_0 \)) provide another data point. By means of this procedure, a time series of 23 data points (1951-1973) was generated and utilized to estimate the demand system \([\text{Equation (15a)}]\).\(^{11}\)

(ii) The Estimation Procedure and the "Optimal" Utility Function

As is well known, a utility function defined over mean and standard deviation is appropriate provided the function is quadratic or approximately so or if the underlying returns are normally distributed. Since we are in fact testing if the quadratic (or square-root quadratic) is the appropriate approximation for the true underlying utility function, we are forced to assume that the distribution of returns is normal. To see that this assumption is not completely untenable, we performed a two-tailed Kolmogorov-Smirnov test on asset yields. Table 1 provides the Komogorov-Smirnov statistic \((D)\) and various critical values for each asset category. Since the statistic does not exceed the critical values for any of the listed significance levels, we conclude that the normality assumption cannot be rejected.

The demand or budget share system \([\text{Equation (15a)}]\) is non-linear in the parameters (in \( K \)) and was estimated by a maximum likelihood technique.\(^{12}\) Since the expenditure shares sum to unity, the \( m \) components of the disturbance term \( v \) must add up to zero for each annual observation. Thus, the disturbance variance-covariance matrix (assumed to be time independent) is singular and non-diagonal. To eliminate this problem, and yet take into account the disturbance variance-covariance, one share equation was dropped prior to estimating the system. As shown by Barten [1969], it is completely irrelevant which equation is in fact dropped from the system.\(^{13}\) In addition, since the
share equations are homogeneous of degree zero in the $a_i$ parameters, these
parameters were normalized with respect to $a_1$.

In our previous discussion, it was pointed out that one purpose of this
paper is to compare and discriminate among the three specific flexible
functional forms. However, it is impossible to discriminate among the three
forms on pure economic grounds since each of the forms can represent arbitrary
well-behaved preferences in the neighbourhood of a given point with second
order accuracy. A priori, we are also unable to choose among the forms on
econometric grounds. The estimation of each one of the forms involves the
same dependent variable, the same number of free parameters and the
maximization of a similar likelihood function. In order to use traditional
tests, a fourth form is estimated, namely the unrestricted system where $\lambda$ is a
free parameter. Thus, the three "original" forms are nested (i.e., they are
special cases of the unrestricted case). Therefore, four different budget
share models were estimated; the translog ($\lambda=0$), the generalized Leontief
($\lambda=1/2$), the square root quadratic ($\lambda=1$), and the unrestricted system where $\lambda$
is a free parameter. The unrestricted model involves non-linear estimation ofive free parameters, the $a_i$ (normalized) and $\lambda$. In all other versions of
the model only four free (normalized) $a_i$ parameters need to be estimated.

Table 2 summarizes the results for each of the four estimated systems.
The unrestricted system yielded an estimate for $\lambda$ of .91409 which is close to
the value for the square root quadratic. However, this result can be tested
more rigorously. In particular, it can be shown that $-2\ln L$ is asymptotically
distributed $\chi^2(1)$ where $L$ is the ratio of the value of the unrestricted
likelihood function (i.e., when $\lambda$ is a free parameter) to the value of the
restricted likelihood function (where $\lambda$ is constrained to a specific value).14

For our estimated demand systems, the test statistic ($-2\ln L$) takes on the
values
The Translog is clearly acceptable as the premiere utility function at virtually any reasonable significance level. The square root quadratic, as it turns out, is also not problematic and can be rejected at the 1% significance level. The Leontief function, on the other hand, is somewhat more ambiguous since it can be rejected at the 5% level but not at the 1% level.

Given the ambiguities about the Leontief function, we decided to discriminate between it and the Translog on alternative grounds. From the theory of asset demand, we expect our "optimal" utility function to satisfy the following conditions: (i) the sign of $U_{\beta}$ should be positive, (ii) the sign of $U_{\gamma}$ should be negative, (iii) the "own" elasticities with respect to expected return should be positive for all assets,\(^{15}\) (iv) the "own" elasticities with respect to variance should be negative for all assets; (iv) the principal minors of the border Hessian ($Z$) should alternate in sign.

The results for the test were reasonably unambiguous, showing that the Translog satisfied none of the above criteria. Thus, despite the fact that the Translog was not rejected by the chi-squar test, it was rejected as the optimal utility function because it gave signs for marginal utilities and comparative static conditions inconsistent with the theory of demand for risky assets. In contradistinction to the Translog, the generalized Leontief yielded the appropriate signs for $U_{\beta}$ and $U_{\gamma}$. In addition, as required by the theory, the own expected return elasticities were positive and the own variance elasticities negative for all assets. Tables 3 and 4 list the expected return and the variance elasticities, respectively, for the generalized Leontief for the year 1973. The results for other years are very similar. The boxed-in numbers in Tables 3 and 4 show the own expected return

<table>
<thead>
<tr>
<th>Utility Function</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translog</td>
<td>2.2738</td>
</tr>
<tr>
<td>Leontief</td>
<td>6.5258</td>
</tr>
<tr>
<td>Square Root Quadratic</td>
<td>6.8094</td>
</tr>
</tbody>
</table>

\(^{15}\)
and variance elasticities.\textsuperscript{16} We also checked the signs of the principal
minors of the bordered Hessian but these were unfortunately ambiguous. This
latter finding does not contradict the theory.\textsuperscript{17} It just means that the
\textbf{sufficient} condition for a maximum was not obtain.\textsuperscript{17} Therefore, we argue
that, broadly speaking, the generalized Leontief is both consistent with data
and the theoretical requirements of mean-variance portfolio theory. We thus
conclude that the asset holdings of the U.S. household sector are consistent
with mean-variance portfolio analysis.

(iii) The Elasticities and their Implications
Assuming that the generalized Leontief is the appropriate utility
function, we can now evaluate the expected return (Table 3), variance (Table
4) and covariance (Table 5) elasticities to see what they teach us about the
U.S. household sector's investment preferences. Consider first the expected
return elasticities; especially the own elasticities. The latter indicate
that a one per cent change in expected return has a much bigger impact on
bonds and mortgages than on money or especially stocks. This is intuitively
plausible since money is likely to be held for reasons other than expected
return (or variance for that matter) and is therefore less likely to be
affected by changes in expected return.\textsuperscript{18} Also, given the volatility of stock
returns, a one per cent change in expected return is unlikely to impact very
much on the demand for equities by comparison to fixed income securities.

The cross expected return elasticities appear to be somewhat less
plausible. In particular, short-term bonds and money are apparently
complements, whereas corporate bonds and stocks are substitutes for money.\textsuperscript{19}
Intuition would have suggested the reverse. Also of particular interest in
the expected return elasticities is the fact that stocks seem to be
independent of other assets. Thus, the demand for stocks are little affected
by changes in the expected returns of other assets and changes in expected
stock returns have little affect on the demand for other asset categories.

The variance elasticities are, with one notable exception, much smaller in absolute value than the expected return elasticities. Thus, it would appear that changes in risk have a smaller impact on asset demand than do changes in expected return. The one notable exception to this generalization has to do with stocks. While it is clear that changes in the riskiness of other assets have no effect on the demand for stocks (the last row of Table 4), the effect is not symmetrical. Changes in the variance of stock returns appear to have a marked impact on the demand for all other asset categories (the last column of Table 4). Indeed, there is a strong portfolio effect in that increased riskiness of stocks has the household sector moving out of all other asset categories and into money and mortgages. This effect is not unreasonable since as stocks get riskier, one would expect individuals to respond by holding more "staid" assets. What is surprising, however, is both the magnitude of the response and the fact that mortgages are considered more like money than are short-term bonds.

Table 5 lists the cross-covariance elasticities. Again, while most of these elasticities are small, certainly by comparison to the expected return elasticities, there is one exception, namely stocks. Changes in the covariance between stocks (asset category 6) and other assets seem to have a marked impact on the demand for all assets. Why this should be so is not immediately obvious. However, if we accept these results, the policy implications are clear. Changes in the variability of stock returns or changes in the co-variability of stock returns with other assets are to be avoided. Any destabilization of the stock market, for example, via governmental policies could have a potentially strong impact on money management and on the mortgage market and, hence, on the supply and demand for housing. Risk reduction regulations in the equity market will also have a
dramatic impact both in terms of increasing the demand for stocks and for bonds of all types and decreasing the demand for money. Clearly a proper definition of money must take into account the risk structure of the equity market as well as so-called near-money substitutes.

Table 5 also has implications for federal debt management. As argued by Roley [1979], the impact of federal debt management policies is a function of the covariances between Treasury securities (both short and long-term) and private securities. The question that naturally arises is to what extent are federal debt management policies stable, given potential fluctuations in covariances between short and long-term treasury securities and other private securities. The answer from Table 5 is that as long as the covariances do not involve stocks, stability seems to be assured. However, should the covariance between short-term bonds and stocks, for example, change this could have a strong effect on the demand for long-term treasury securities and hence the effectiveness of federal debt management.

IV. Conclusion

This paper has had a twofold purpose. Firstly, we tested to see whether the U.S. household sector's demand for risky financial assets could be described by a mean-variance portfolio optimization framework. We found that of three specific flexible functional forms, the generalized Leontief yielded signs for marginal utilities and comparative static conditions consistent with the underlying data and the theory of mean-variance portfolio optimization. Therefore, we concluded that, broadly speaking, the U.S. household sector's demand for risky financial assets could be described by a mean-variance portfolio framework. Secondly, accepting the generalized Leontief as our premier utility function, we derived estimated expected return, variance and covariance elasticities for different financial asset categories. Besides
determining the degree of substitutability or complementarity among financial assets (both on expected return and risk dimensions), we saw that changes in expected returns tended to have a much larger impact on asset demand than changes in variances or covariances with one exception. The one exception concerned equities. While the demand for stocks is basically independent of other asset returns, the reverse is not the case. Changes in the variance of stock returns or even changes in covariances between stock and other asset returns have a marked impact on the demand for other assets. This was argued to have important implications for monetary, housing and debt management policies. Thus, for example, it was argued that stability of the stock market is an important factor in determining the stability and effectiveness of monetary and debt management policies.
Appendix A: Solving for the Budget Share System

From the first-order conditions, [Equations (2a)] in the text, we get that for assets \( r = 2, \ldots, m \)

\[
U_E (E_l - E_r) + W_0 V^{-1} \left[ \sum_{j=1}^{m} X_j (G_{lj} - G_{rj}) \right] = 0
\]  

(A1)

Rearranging (A1), we get

\[
\sum_{j=1}^{m} X_j (G_{lj} - G_{rj}) = - \frac{U_E (E_l - E_r)}{W_0 V^{-1}} \quad (r = 2, \ldots, m)
\]  

(A2)

Differentiating the Box - Cox Utility function [Equation (10)] yields

\[
U_E = \left[ \alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda) \right] E^{\lambda - 1}
\]  

(A3)

\[
U_V = \left[ \alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda) \right] V^{\lambda - 1}
\]  

(A4)

Define

\[
K = \frac{U_E}{W_0 V^{-1}} = \frac{\left[ \alpha_1 + \alpha_3 E(\lambda) + \alpha_5 V(\lambda) \right] E^{\lambda - 1}}{\left[ \alpha_2 + \alpha_4 V(\lambda) + \alpha_5 E(\lambda) \right] W_0 V^{\lambda - 2}}
\]  

(A5)

from (A3) and (A4). Define

\[
G^{(rj)} = G_{lj} - G_{rj} \quad (r = 2, \ldots, m)
\]  

(A6)

and \( E^* = E_l - E_K \quad (K = 2, \ldots, m) \)  

(A7)

Then, the first-order conditions [Equations (2a) and (2b)] can be rewritten in the matrix form (as claimed in the text)

\[
X = KG^{-1} E^*
\]

where

\[
X = \begin{bmatrix}
X_1 \\
\vdots \\
X_m
\end{bmatrix} \quad E^* = \begin{bmatrix}
E^*_2 \\
\vdots \\
E^*_m
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
G_{21} & G_{22} & \cdots & G_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
G_{m1} & G_{m2} & \cdots & G_{mm}
\end{bmatrix}
\]

\[
1 \quad 1 \quad \ldots \quad 1
\]
Appendix B: Asset Definitions and Data Sources

This appendix defines the six asset categories, their associated yields and data sources. Note that all yields are monthly.

(i) Money (MY) = Currency and Demand Deposits + Commercial Bank Savings Accounts + Savings Institutions Savings Accounts

Yield for MY = Maximum Interest Rates Payable on Time and Savings Deposits at Federally Insured Institutions

(ii) Short-term Bonds (SB) = U.S. Government Short Term Securities + U.S. State and Local Bonds + Commercial Paper

Yield for SB = weighted average yield on the above three sub-categories

(iii) Long-Term U.S. Government Savings Bonds (LB)

Yield for LB = Long-Term U.S. Government Bond Yields

(iv) Corporate and Foreign Bonds (CB)

Yield for CB = Corporate Bond Yields

(v) Mortgages (MT)

Yield for MT = Mortgage Yields

(vi) Stocks (ST)

Yield for ST = Standard and Poor's Index of Stocks

Data Sources

1. Federal Reserve Bulletin, various issues


2. See Roley [1979] for an analysis of the relationship between the equity market and federal debt management.

3. This technique was first developed and used by Khaled [1977], Berndt and Khaled [1979], and Appelbaum [1979] but in a non-portfolio riskless framework.

4. See Roberts [1975].

5. Aivazian [1976] provides a detailed discussion of these effects.

6. A direct rather than an indirect form of the utility function is used in this paper, since in a portfolio framework share equations can be easily obtained from the direct function. Furthermore, given the lack of analogy between our comparative static equations and ordinary Slutsky equations, the application of Shephard's lemma is not straightforward, and remains to be worked out in the literature. On practical grounds, one should expect the number of arguments to appear in the indirect utility function to be larger since they include the individual means and covariances among assets' returns.

7. See Appelbaum (1975).

8. A similar system was derived by Krinsky [1983].

9. This specification ignores the requirement that budget shares must lie between zero and one by giving positive probability to shares outside this range. See Woodland [1979] for justifications for continued use of the normal distribution specification in the estimation of share equations.

10. It is assumed that investor portfolio selection horizons do not exactly correspond to either of the short-term or long-term savings bond maturities, making these assets risky.

11. There is of course potential aggregation bias in estimating a representative consumer utility function from aggregate data. It is therefore important that this study be replicated on panel data. However, most of the flexible functional form literature dealing with utility function estimation is based on aggregate data. See, for example, Christiensen, Jorgenson, and Lau (1975), Christiensen and Manser (1977), Donovan (1978), and Applebaum (1979).

12. The algorithm used in our study is a Quasi-Newton method.

13. The Barten proof relates only to Full Information Maximum Likelihood (FIML) parameter estimates. Independently, Kmenta and Gilbert (1968) showed that iterated OLS converged to FIML using Monte Carlo techniques and Dhrymes (1973) proved this convergence analytically; that is, he proved that iterated Seemingly Unrelated Regression (SUR) is asymptotically equivalent to FIML. This later technique is, in fact, used in this paper.

15. This is not exactly the case. As pointed out earlier, only the own substitution effects need be positive. However, the wealth effects turned out to be of much lower magnitude than the substitution effects, so that in fact the latter determined the signs of the elasticities. A similar statement holds true for the own variances as well. Thus, we decided to present the own elasticities rather than just the own substitution effects.

16. Note that money only appears on the vertical axis in Table 4. Since money is riskless by assumption, the impact of changes in money variance on other asset demands, is not a meaningful concept. However, the impact on the demand for money as the variances of other assets change can be determined (See row 1, Table 4).

17. Since we are using aggregate data, it would be somewhat fortuitous for these sufficient conditions to obtain.

18. A more general model would take such factors into account.

19. Complements and substitutes are well defined here in that wealth effects were dominated by substitution effects — see also footnote 15 — so that net and gross substitutes (and complements) definitions are equivalent. Also, as can be seen by comparing Tables 3 and 4, expected return and variance elasticities are always of opposite sign. Thus, assets A and B are substitutes (complements) if their expected return elasticities are negative (positive) and variance elasticities are positive (negative). It bears repeating that since we are using returns rather than prices, the sign of substitutes and complements are opposite to the norm.

20. The SB category contains other than treasury securities so that statements about this category are tentative. However, the LB category is comprised of only Treasury securities.
**TABLE 1**

The Kolmogorov - Smirnov Test Results*

<table>
<thead>
<tr>
<th>Asset</th>
<th>K-S Statistic</th>
<th>Critical Values (N=23)</th>
<th>Significant Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>MY</td>
<td>.1758</td>
<td></td>
<td>.20 .10 .05 .02 .01</td>
</tr>
<tr>
<td>SB</td>
<td>.1729</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LB</td>
<td>.1235</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CB</td>
<td>.2177</td>
<td></td>
<td>.216 .247 .275 .307 .330</td>
</tr>
<tr>
<td>MT</td>
<td>.1790</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ST</td>
<td>.0968</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Reject the hypothetical distribution F(X) if Dₙ = max|Fₙ(X) - F(X)| exceeds the tabulated (critical) value.
## TABLE 2

### PARAMETER ESTIMATES FOR DIFFERENT FUNCTIONAL FORMS

<table>
<thead>
<tr>
<th>Functional form</th>
<th>Specified Value of $\lambda$</th>
<th>Estimated* parameters</th>
<th>Value of Log Likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unrestricted Maximum</td>
<td>-</td>
<td>$\lambda = .91409$</td>
<td>307.3941</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_2/\alpha_1 = 1800.0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_3/\alpha_1 = .59308 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_4/\alpha_1 = -.72400$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_5/\alpha_1 = -.62544 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>Translog</td>
<td>$\lambda = 0$</td>
<td>$\alpha_2/\alpha_1 = 6207.6$</td>
<td>306.2572</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_3/\alpha_1 = .16138$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_4/\alpha_1 = -925.65$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_5/\alpha_1 = -.34711$</td>
<td></td>
</tr>
<tr>
<td>Generalized Leontief</td>
<td>$\lambda = 1/2$</td>
<td>$\alpha_2/\alpha_1 = -549.6$</td>
<td>304.1312</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_3/\alpha_1 = .51584$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_4/\alpha_1 = -280.78$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_5/\alpha_1 = -.31362 \times 10^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Quadratic</td>
<td>$\lambda = 1$</td>
<td>$\alpha_2/\alpha_1 = 1200.0$</td>
<td>303.9894</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_3/\alpha_1 = -.3097 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_4/\alpha_1 = 9.8209$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha_5/\alpha_1 = .84549 \times 10^{-2}$</td>
<td></td>
</tr>
</tbody>
</table>
## TABLE 3
Expected Retrun Elasticities for 1973

<table>
<thead>
<tr>
<th>Asset Type</th>
<th>MY</th>
<th>SB</th>
<th>LB</th>
<th>CB</th>
<th>MT</th>
<th>ST</th>
</tr>
</thead>
<tbody>
<tr>
<td>MY</td>
<td>14.47</td>
<td>3.25</td>
<td>-13.17</td>
<td>1.33</td>
<td>-12.28</td>
<td>-1.77</td>
</tr>
<tr>
<td>SB</td>
<td>23.69</td>
<td>51.77</td>
<td>-28.95</td>
<td>0.02</td>
<td>-81.47</td>
<td>2.58</td>
</tr>
<tr>
<td>LB</td>
<td>-143.22</td>
<td>-43.19</td>
<td>334.95</td>
<td>-206.64</td>
<td>61.49</td>
<td>0.50</td>
</tr>
<tr>
<td>CB</td>
<td>12.95</td>
<td>0.03</td>
<td>-186.02</td>
<td>503.95</td>
<td>-266.97</td>
<td>1.55</td>
</tr>
<tr>
<td>MT</td>
<td>-94.08</td>
<td>-85.60</td>
<td>43.28</td>
<td>-208.88</td>
<td>436.88</td>
<td>-0.43</td>
</tr>
<tr>
<td>ST</td>
<td>-0.56</td>
<td>0.11</td>
<td>0.01</td>
<td>0.04</td>
<td>-0.03</td>
<td>1.58</td>
</tr>
</tbody>
</table>
### TABLE 4

Variance Elasticities for 1973

<table>
<thead>
<tr>
<th>Asset Type</th>
<th>SB</th>
<th>LB</th>
<th>CB</th>
<th>MT</th>
<th>ST</th>
</tr>
</thead>
<tbody>
<tr>
<td>MY</td>
<td>-0.35</td>
<td>0.14</td>
<td>-0.02</td>
<td>0.17</td>
<td>3932.90</td>
</tr>
<tr>
<td>SB</td>
<td>-5.62</td>
<td>0.31</td>
<td>0.00</td>
<td>1.16</td>
<td>-5691.00</td>
</tr>
<tr>
<td>LB</td>
<td>4.69</td>
<td>-3.64</td>
<td>2.40</td>
<td>-0.88</td>
<td>-1102.50</td>
</tr>
<tr>
<td>CB</td>
<td>0.00</td>
<td>2.02</td>
<td>-5.85</td>
<td>3.80</td>
<td>-3399.00</td>
</tr>
<tr>
<td>MT</td>
<td>9.30</td>
<td>-0.47</td>
<td>2.43</td>
<td>-6.22</td>
<td>959.44</td>
</tr>
<tr>
<td>ST</td>
<td>-0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-3522.70</td>
</tr>
</tbody>
</table>
### TABLE 5
Covariance Elasticities for 1973*

<table>
<thead>
<tr>
<th>Assets</th>
<th>MY</th>
<th>SB</th>
<th>LB</th>
<th>CB</th>
<th>MT</th>
<th>ST</th>
</tr>
</thead>
<tbody>
<tr>
<td>G23</td>
<td>0.04</td>
<td>-0.43</td>
<td>-1.53</td>
<td>1.13</td>
<td>0.74</td>
<td>0.00</td>
</tr>
<tr>
<td>G24</td>
<td>-0.07</td>
<td>-1.05</td>
<td>2.17</td>
<td>-3.15</td>
<td>3.05</td>
<td>0.00</td>
</tr>
<tr>
<td>G25</td>
<td>0.04</td>
<td>-0.42</td>
<td>0.41</td>
<td>2.51</td>
<td>-2.14</td>
<td>0.00</td>
</tr>
<tr>
<td>G26</td>
<td>-0.97</td>
<td>123.97</td>
<td>-91.48</td>
<td>6.99</td>
<td>-187.73</td>
<td>7.43</td>
</tr>
<tr>
<td>G34</td>
<td>0.06</td>
<td>0.13</td>
<td>-0.96</td>
<td>-0.51</td>
<td>0.36</td>
<td>0.00</td>
</tr>
<tr>
<td>G35</td>
<td>0.07</td>
<td>0.30</td>
<td>-1.22</td>
<td>1.26</td>
<td>-1.23</td>
<td>0.00</td>
</tr>
<tr>
<td>G36</td>
<td>-0.80</td>
<td>-0.73</td>
<td>13.23</td>
<td>-7.06</td>
<td>1.63</td>
<td>0.25</td>
</tr>
<tr>
<td>G45</td>
<td>0.06</td>
<td>0.44</td>
<td>0.51</td>
<td>-0.61</td>
<td>-1.52</td>
<td>0.00</td>
</tr>
<tr>
<td>G46</td>
<td>-3.14</td>
<td>5.12</td>
<td>-59.70</td>
<td>151.06</td>
<td>-62.21</td>
<td>3.18</td>
</tr>
<tr>
<td>G56</td>
<td>-6.84</td>
<td>-21.94</td>
<td>20.53</td>
<td>-83.14</td>
<td>139.45</td>
<td>2.59</td>
</tr>
</tbody>
</table>

*2 = SB
3 = LB
4 = CB
5 = MT
6 = ST
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Continued on Page 2...


Continued on Page 3...


154. Szendrovits, A.Z. and Drezner, Zvi, "Optimizing N-Stage Production/Inventory Systems by Transporting Different Numbers of Equal-Sized Batches at Various Stages", April, 1979. Continued on Page 4...


Continued on Page 5


