OPTION PRICING — VALUING DERIVED CLAIMS IN INCOMPLETE SECURITY MARKETS

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ABSTRACT

The purpose of this paper is to investigate the inherent relationship between the theory of the economic role of options and various approaches to option pricing. It is argued that the economic theory underlying the pricing of options should dictate why and how options are valued. The fundamental issue is that the assumptions of existing option pricing models cause options to be valued in a market environment in which options serve no economic purpose and therefore would not neecessarily be issued. Hence a paradox exists.

A new approach to option pricing is suggested. Options should be valued based on the assumption that they exist for the purpose of providing completeness for the capital market. Thus the efficiency of the economic system is enhanced. The approach which is suggested is developed in the context of the standard Capital Asset Pricing Model. This has resulted in a new option pricing formula which contains a market-effect variable - the expected return on the underlying stock. The reason as why such a variable exists is explained.

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1. INTRODUCTION

1.1 Statement of Problem

This paper approaches the question of option valuation by considering the economic role of options. It specifically treats the problem of understanding why options are created in the marketplace and at what prices they should be valued. A fundamental paradox inherent in existing option-valuation models is revealed.

The valuation of options has attracted a great deal of attention in the financial literature in the past ten years or so. This is because option pricing theory has been viewed as a general theory pertinent to one of the most essential problems in finance--the valuation of derived assets. Many financial economists have attempted to apply existing option pricing models to a large variety of corporate financial problems. For example, Galai and Masulis (1976) have applied the Black-Scholes model to a firm's decision in the issuance of corporate debts and equities. Myers (1977) even has suggested that the investment decision of a firm can be analyzed in an option pricing framework.

Although the history of option pricing theory dates back to 1900, the first successful attempt to price options in a general equilibrium context was theBlack-Scholes model in 1973. Since then, the theory has developed further, using their initial methodological approach as a basis from which a number of extensions and generalizations have been generated.

However, recently Kreps (1979) and Harrison-Pliska (1981) have formally contended that the Black-Scholes model contains an important implicit assumption--a complete market. This complete market basis for the Black-Scholes option valuation expression, being consistent with the explicit complete market construction in the Arrow-Debreu sense, creates a fundamental paradox. This is because the Black-Scholes model is shown to be based upon an idealized market structure in which no rationale for the creation of options of any kind can be found.

For some time it has been realized that the potential value of options as financial instruments stems from their capacity to capture a portion of the return distribution of their underlying assets. Consequently, the rationale for their existence stems from the prospect of a more agreeable distribution of returns, in the sense of expected utility maximization. In other words, they serve to make the market more complete by offering opportunities that could not otherwise be created by the set of underlying assets (stocks and bonds) available in the market.

Therefore, it is contended in this paper that options should be valued in a market environment in which the assumption of a complete market is not necessary.

1.2 Definition of an Option

An option is any instrument that provides its holder with the right to purchase or sell an asset for a specified period of time at a specified price. Any financial arrangement which provides such a right is classified as an option. Obviously, in financial markets there are a large number of different types of option transactions. Some options provide an opportunity for the holder to purchase an asset, while others provide an opportunity for the sale of an asset. In some cases the option arrangement is a contract negotiated between two parties. The underlying asset on which the financial transaction is based need not be held by either of the parties to the contract. In fact, in some option arrangements there is no guarantee that delivery would be forthcoming.

Options are widely used in our complex financial environment. There are options that allow the corporation to take certain actions with respect to securities they have issued. Call options on outstanding bonds are an example. There are options as well which are part of another security agreement. Warrants to purchase common stock that may be associated with a bond issued by a corporation, or convertible features of bonds or preferred stock, are such examples. There are also options available in many leasing arrangements. A leasee may have the option to extend the lease period. In general, Black and Scholes (1973) suggest that all corporate liabilities may be viewed as options one way or another.

However, the specific kind of option arrangmenet that will be dealt with in the paper is that which is generally labeled a call option. For the purposes of this discussion, a call option is defined as a financial arrangement allowing the holder of an option to purchase from the issuer of an option a fixed given number of shares of some underlying common stock at a specific single exercise price for a specific period of time. Initially, it will be assumed that the option is of the "European" type. European options provide the opportunity to purchase an underlying asset only at maturity. Such options are derived financial instruments, that is they are simply inventions of the parties to a contract, the value of which is a function of the underlying security. An option of this sort may contain utility values because it provides a means for shifting the payoffs between parties to the option. It provides ex ante benefits to both parties to the transaction by arranging for a distribution of outcomes or a sharing rule that increases the utility of both parties.

1.3 The Purpose of Security Markets

Since the work of Arrow (1963) and Debreu (1959) it has been recognized that securities markets exist to aid in the allocation of risk in an economy. Moreover, in a well functioning securities market the resulting allocation of risk is Pareto optimal, or more precisely, first-best Pareto optimal.

Arrow-Debreu demonstrate that, in order to achieve a Pareto optimal distribution of risk in an economy securities markets must be well functioning or efficient in three critical ways. First, the number of securities in the market must equal the number of states of nature which can occur in an economy. Second, these securities collectively must have payoffs in every state of nature. Third, the payoffs of any security can not be duplicated by forming portfolios of other securities in the market. When these three conditions are satisfied, the market is complete in an Arrow-Debreu sense.

As a result of the Arrow-Debreu thesis, options—like any security—will be created only if they cheaply offer payoffs which cannot be obtained by forming portfolios of existing securities. The rationale for the existence of options lies in their role as market completing securities. For example Ross (1976) points out that the justification for the creation of options in the marketplace is that the market becomes more efficient as a result.

One major implication of a complete market concerns the pricing of securities when the number of securities exceeds the number of states of nature. In such a case, the price of some securities can be written as a linear combination of the prices of other securities in the market place. In such a world, once the number of securities are sufficient to span all the states of nature, any additional securities are redundant. It is the fact that a security is redundant that allows for this simple linear approach to pricing. There is, however, a paradox associated with the "redundant asset" approach to pricing. In the absence of tansaction costs why do redundant securities exist as they serve no purpose in attaining efficiency? To the extent that options are redundant secrutities, it would seem that they perform no usual economic function in market place. Therefore, there would be no reason for active market in options to exist under such conditions.

1.4 The Purpose and Organization of the paper

The main purpose of this paper will be to develop option pricing models under the codition of an incomplete market. Options in this world are derived financial instruments formed to aid in completing the market. Thus, the derivation of option-pricing models within this framework will generalize current results in the literature. This approach is counter to most recent option valuation models that have derived preference-free results based on the assumption of complete markets. The premise of this paper is that generalizations should extend toward valuing options in less than complete markets. conceptually, this paper can be seen as having two parts. The first part contains a complete development of the connection between existing option pricing models and implicit assumptions that are made regarding the completeness of the market assumed within these models. Based on these proofs the paradox of attempting to value options with the complete market assumption and the implicatin for risk neutral valuation models is established. This analysis sets the stage for the second part of the study that prices options in less than complete securities markets: within the framework of the capital asset pricing model.

This paper is organized as follows: Part 2 begins with a somewhat detailed discussion of complete markets, spaning and Pareto optimal allocations. Within this part, the rationale for the existence of derived secutities is discussed with reference to complete and incomplete securities markets. Part 3 reviews existing option pricing literature and suggests the reason behind the preference free pricing results contained in the literature. The applicability of the derived preference free pricing results to incomplete markets is discussed and the basis established for the rationale for the development of incomplete market option pricing equations. Part 4 summarizes the few incomplete market pricing equations that exist and then extends this work to price options within the context of the standard Capital Asset Pricing Model. A closed-form option pricing formula is derived. Part 5 offers a summary.

2. COMPLETE MARKET, PARETO EFFICIENCY, DERIVED-SECURITY VALUATION, AND THE ECONOMIC ROLE OF DERIVED-SECURITIES

2.1 Introduction

The Arrow-Debreu (Arrow 1964, Debreu 1959) approach to general equilibrium in a pure exchange economy has long been recognized as one of the most general and conceptually elegant frameworks for the study of financial problems under uncertainty. It is well known that, within the Arrow-Debreu framework, complete market conditions imply Pareto efficiency, that in terms of expected utility there is no other feasible allocation of securities which would make everyone at least as well off and some better off than before. It is also well known that under complete market conditions, the pricing of any derived security is preference-free. In other words, the price of any derived security can be determined in terms of the prices of primitive securities without invoking the preferences of market investors. Nonetheless, prices of primitive securities do reflect these preferences.

Since under complete market conditions the pricing of any derived security is preference free, it has also been argued that these derived securities serve no economic purpose. The reason is that the trading of these securities does not increase the risk-allocation function of primitive securities in the economy. Therefore, these derived securities are redundant.

The purpose of this part is to give a somewhat detailed discussion of these ideas, and to demonstrate that the Ross (1976) risk-neutral valuation technique can be extended to any derived asset under complete market conditions.

2.2 Definition of a Complete Market

Let there be S possible states of nature which can occur at the end of a given period. These states are mutually exclusive and collectively exhaustive, i.e., exactly one state will occur and it will be identifiable to all investors. By definition, therefore, prices, output, level of utility, etc., may be expressed as functions of the state of nature alone. In a complete market, for every state an investor can purchase for e_s a financial instrument, a "state security" (alternatively, an Arrow-Debreu elementary security) which pays $1 if state s occurs and $0 if any other state occurs. The return, or cash flow, on this security is

$$(1)\quad e(S) = \begin{cases} 1 & S = s \\ 0 & S \neq s \end{cases}$$

Let π_s^k denote the "probability" that state s will occur as judged by investor k.

In this economy there are also N securities whose values next period, i.e., v(s), will depend upon the kind of state occurring. The current value is denoted as p. If two securities have the same expected values in all states of nature over the course of the next period, their current market prices must then be equal or arbitrage would result.

It is important however, to note that state securities are fictitious securities which really do not exist in the marketplace. Nonetheless, these securities can be derived from an appropriate portfolio of real securities. The sufficient and necessary conditions for this derivation are:

1) the number of securities in such a portfolio must equal the number of states of nature which can occur in an economy;

2) the pay-offs of any given security in this portfolio cannot be duplicated by forming portfolios of other securities in the same portfolio.

Such a portfolio is referred to as a "basis" of the market.

Among the N securities assumed in a complete market economy, and set of S securities which satisfies the conditions set out above can be a basis. All securities within a basis are referred to as primitive securities, while all others are derived securities. If N<S, then of course no basis can exist.

The construct of a complete market is in fact derived from the construct of a vector space in linear algebra. The state securities are parallel to unit vectors, while the number of states, the number of securities, and the basis correspond to the dimension, the number of vectors, and the basis, respectively.

The following example explains the derivation of state securities from a basis.

<u>EXAMPLE</u>

For simplicity, assume that there are only two states in the world and that there are three securities in theee financial market. The pay-off matrix of these securities is assumed to be

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

The payoff of the first security, v_1, is \$1 if state 1 occurs, and \$1 if state 2 occurs, etc. Supposing that the first two securities form a basis B, then the pay-offs of state securities can be derived from the following well-known result in linear algebra,

(2) $E = B^{-1} B$

where E is the vector of the pay-offs of state securities, and B^{-1} is the inverse matrix of B. If B^{-1} exists, then the conditions for B as a basis are satisfied. In this example, B^{-1} exists, i.e.,

$$(3) \qquad B^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}}{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

thus

$$(4) \qquad E = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is the pay-off matrix of state securities.

In the portfolio context, B^{-1} is in fact the matrix of portfolio composition. For instance, the first row of the matrix indicates that to form the pay-off of the state security 1, an investor should short one share on the first security and long on share on the second security. Consequently, the price of the state security 1, e_1, can be expressed as a linear combination of the prices of the first and second securities:

$$(5) \qquad e_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -P_1 + P_2$$

and similarly:

$$(6) \qquad e_2 = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 2P_1 - P_2$$

If no basis exists in the financial market, then the market is "incomplete".

2.3 <u>The Economic Role and Pricing of Derived Securities under Complete Market Conditions</u>

2.3.1 <u>Complete Markets and the Economic Role of Derived Securities</u>

Since the work of Arrow (1963) and Debreu (1959), it has been recognized that securities markets develop in order to allocate risk in an economy.

Morever, in a complete securities market the resulting allocation of risk is Paeto optimal, or more precisely first-best Pareto-optimal. This means that there is no other feasible allocation of securities which would make everyone at least as well off and some better off than before.

One of the major implications of a complete market concerns the pricing of derived securities. Under complete market conditions, the pricing of any derived security is preference-free. In other words, the price of any derived security can be determined using a linear combination of the prices of primitive securities. Investors' preferences enter into pricing insofar as they determine the prices of primitive securities.

For the sake of simplicity, let it be assumed that state securities are primitive securities (as demonstrated previously, under complete market conditions prices of state securities can always be derived from prices of primitive securities). Given an arbitrary derived asset which can be constructed as a portfolio of V_i (s) shares of state securities in which s=1,2,.......,S, the value of the portfolio will clearly be e_s if state s occurs. Hence one share of firm i must sell for

$$(7)\quad P_i = \sum_{s=1}^{S} V_i(s)e_s$$

In a case such as this, the price of any derived security can be written using a linear combination of the prices of the primitive securities in the marketplace. In a complete markt, because the set of primitive securities is sufficient to span all states of nature, any additional securities (i.e., derived securities) would be economically redundant. The fact that a security is redundant permits this simple linear approach to pricing. There is, however, a paradox inherent in this "redundant asset" approach to pricing. Redundant securities serve no purpose in attaining efficiency. Therefore, in the absence of transaction costs, the question of whether or not redundant securities should exist invitably arises. To the extent that derived assets are redundant securities, it would seem that they perform no usful economic function in the marketplace. For this reason, there would be no social need for an active market in derived assets to exist.

2.3.2 The Complete Market and the Risk-Neutral Valuation Technique

Cox-Ross (1976) suggest a technique for the pricing of options. Observing that the Black-Scholes option pricing result is preference-free, they suggest that an investor can assume any preference structure which permits equilibrium. Therefore, the investor may choose the structure which proves most tractable mathematically—a risk-neutral world.

In the proofs which follow, it is shown that the Cox-Ross technique can be extended to any derived security under complete market conditions in both the discrete-state case and the continuous-state case.

Lemma 1 The Cox-Ross technique can be applied to the pricing of any derived asset under complete market conditions in which the state distribution is discrete.

(Proof)

The price of a state-security can be derived using the discounted-value approach:

$$(8)\quad e_s = \frac{\pi_s^k}{1 + \rho_s}$$

where ρ_s is the appropriate discounting factor on state-security s.

In the same fashion, the price of one share of security i can be expressed as:

$$(9)\quad P_i = \frac{\sum_{s=1}^{S} [\pi_s^k V_i(s)]}{1 + \rho_i}$$

Substituting (8) into (9),

$$(10) \qquad P_i = \frac{\sum_s [e_s(1+\rho_s)V_i(S)]}{1+\rho_i}$$

$$(11) \qquad = \sum_s [e_s V_i(S) \frac{1+\rho_s}{1+\rho_i}]$$

According to th Cox-Ross risk-neutral valuation technique, i can be priced as if all investors were risk-neutral. Thus,

$$(12) \quad \rho_s = \rho_i = r \qquad \forall s \in S$$

where r is the risk-free interest rate in the market.

Substituting (12) into (11) yields the price of a derived asset in a complete market:

$$(13) \qquad P_i = \sum_s^S e_s V_i(s)$$

Noting that (13) is identical to (7), the Cox-Ross technique is proven to generate the price of a derived asset in a complete market.

——Q.E.D.——

Lemma 2 The Cox-Ross technique can be applied to the pricing of any derived security in a complete market in which the state variable has a continuum of states.

(Proof)

Let ds denote the current price of one dollar to be paid if the terminal state is in the interval (s, s + ds). Then the current value of any security with final value V(s), s=1,2,..,S is by analogy with eq. (17):

$$(14) \qquad P_i = \int_S V_i(s) e_s ds$$

where $e_s ds$ can be derived using the discounted-value approach:

$$(15) \quad e_s ds = \frac{\pi_s^k \, ds}{e^{\rho_s T}}$$

where ρ_s is the instantaneous discounting factor on state security, and is assumed to be a constant, and T is the time interval of a period.

The price of one share of a derived security can be alternatively derived using the discounted-value approach:

$$(16) \qquad P_i = \frac{\int_S V_i(s) \, \pi_s^k \, ds}{e^{\rho_i T}}$$

where ρ_i is the instantaneous discounting factor on i, and is also assumed to be a constant.

Substituting (15) into (16),

$$(17) \quad P_i = \frac{\int_S V_i(s) e^{\rho s T} e_s ds}{e^{\rho_i T}}$$

$$(18) \qquad = \int_S V_i(s) e^{(\rho_s - \rho_i)T} e_s ds$$

According to the Cos-Ross risk-neutral valuation technique, i can be priced as if all investors were risk-neutral. Thus,

$$(19) \quad \rho_s = \rho_i = r \qquad \forall s \in S$$

where r is the instantaneous risk-free interest rate, and is also assumed to be a constant.

Substituting (19) into (18),

$$(20) \quad P_i = \int_S V_i(s) e_s \, ds$$

which is identical to eq. (22). Therefore the Cox-Ross technique is valid.

——Q.E.D.——

2.4 Incomplete Market and Efficiency

In an incomplete market the number of existing securities, either elementary or complex, is not sufficient to span the state space. In other words, combinations of existing securities are not sufficient to create elementary securities corresponding to each and every state. Consequently, there are some states of nature in which "risks" cannot be insured, and in which ultimate consumption is not guaranteed.

In such a market, some form of Pareto efficiency can still be achieved. The marginal rates of substitution for production, consumption and investment create a set of prices in which, given the incompleteness of the market, no one can be made better off without someone else being made worse off. Lipsey and Lancaster (1956-7) have termed this condition "second best" Pareto optimality in the sense that the costless formation of new securities to aid in completing the market would provide the potential for a new "first-best" Pareto optimality.

Thus there are incentives for individuals or corporations to tailor the issuance of securities in a way that aids market completeness. These new securities in a way that aids market completeness. These new securities are not redundant and obviously serve to improve the nature of market efficiency. For instance, options as previously defined, are not costless to create, but they may have a sufficiently low cost to serve as a relatively efficient mechanism for completing the market. Moreover, the nature of the most typical option contracts (calls, puts, straddles, warrants, etc.) with single or multiple exercise prices may provide the best structure for optimal movement toward market completeness and a substantial gain in efficiency.

The following example, similar to the one in Ross (1976), explains the market-completing function of a simple option.

Example

Let it be assumed for the sake of simplicity that there are only three relevant states in the world, and that in the capital market there is only one stock. This stock is assumed to have payoffs in all three states, i.e.,:

$$V_1 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

Supposing that call options are created on this stock with exercise prices 2 and 3, then the payoffs of these options are:

$$C(V_1, 2) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C(V_1, 3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

If these three securities are chosen to form a basis of the market, i.e., B, then the payoff of B is:

$$B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

The inverse matrix of B can be derived using linear algebra:

$$B^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & -3/2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Since B^{-1} exists, the conditions for B as a basis of the market are satisfied. Consequently, the pay-off matrix of state securities (the unit matrix) can be derived using eq. (2), and thereby a complete market is achieved.

The presence of transaction and set-up costs, however, does not permit all states to be spanned when the number of states is sufficiently large. But because the formation of options involves relatively low costs, options will be created up to the point at which gains are outweighed by costs.

3. OPTION PRICING UNDER COMPLETE MARKET CONDITIONS

3.1 Introduction

Serious study of option valuation in financial literature goes back a very long way; at least to the long-neglected thesis by Bachelier (1900). In a study of the French bourse, he gives the first mathematical characterization of the price of a call option under continuous-time conditions. In the 1960's his work was revived, and since then a number of authors have attempted to price options under continuous-time conditions. Merton (1973a) has given a derivation of a "continuous-time Capital Asset Pricing Model" based on Ito integral price processes. Later in the same year, Black and Scholes (1973) derived the first closed-form option pricing formula by employing a "continuous-hedging" strategy. In this study, Black-Scholes also show that an identical result can be derived based upon Merton's model.

One of the two important and often debated features of the Black-Scholes pricing result is that the call price is only a function of the current price of the underlying stock. Investors' preferences enter into the pricing formula only insofar as they determine the current price of the stock. This kind of pricing result is known as a "preference-free" pricing result. As Brennan (1979) pointed out, a central feature of modern option pricing theory has been the derivation of such preference-free results. Since Black-Scholes developed their formula, their model has been considerably generalized along the lines suggested by Brennan.

The other important and often-debated feature of the Black-Scholes result (Kreps 1979) concerns the fact that they price options as redundant securities. There are an infinite number of states of nature, yet the option (a derived security whose value is a nonlinear function of the terminal prices of two securities—the underlying stock and the risk-fre discount bond) can be priced in such a way that it is redundant.

Kreps (1979) and Harrison-Pliska (1981) prove that most of the continuous-time models, including the Black-Scholes model, implicitly assume that the market of the underlying securities is complete. This complete market assumption explains why the Black-Scholes model and most other existing option pricing results are preference-free, and also why options in their models can be priced as redundant securities.

The purpose of this part is to review option-pricing results which implicitly or explicitly assume a complete market. Care is given to explain why the Black-Scholes model implicitly assumes a complete market, and why as a consequence their model cannot be applied under incomplete market conditions.

The organization of this part is as follows. Section 2 re-derives the Black-Scholes result in a more direct manner. Section 3 generalizes this rederivation into a multi-security world. It is emphasized that in this multi-security world an identical formula results, and that the market equilibrium balance equation derived is identical to that of the capital asset pricing model. Section 4 relates the Black-Scholes model to the complete market assumption. It is demonstrated that under incomplete market conditions the Black-Scholes model does not yield option prices which are consistent with the general first-degree stochastic dominance argument.

3.2 Rederivation of the Black-Scholes Option Pricing Result

In 1973, Black and Scholes developed the first closed form equilibrium value for pricing a European call option. This valuation expression is:

$$C = SN(d_1) - Ke^{-rT}N(d_2) \tag{20}$$

where $d_1 = \dfrac{\ln S/_K + (r + 1/2\ \sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 \sigma \sqrt{T}$ and N( ) is

the standard cumulative normal distribution. This expression contains the following exogenously given variables:

(1) S: the current price of the underlying stock

(2) K: the exercise price of the option

(3) r: the instantaneous risk-free interest rate

(4) T: the time to maturity of the option

(5) σ^2: the constant instantaneous variance rate of return;

The model development is based on the following assumptions:

(1) The capital markets are perfect.

(2) No restrictions exist on the free use of proceeds from short selling.

(3) trading takes place continuously.

(4) The pricing behavior of the underlying stock is exogenous and is defined by an Ito differential equation with the instantaneous variance rate of return being constant throughout the life of the option. The option price at any time during its life span is a continuous, twice-differentiable function of the stock price at that time instance and a continuous differentiable function of time.

(5) The risk-free interest rate exists and is a known constant throughout the life of the option.

(6) The option is a European call and, therefore, can be exercised only at the point of maturity of the option agreement.

(7) The underlying stock pays no dividends throughout the life of the option.

Thus, the value of the call option according to Black and Scholes focuses on two terms. In a risk-neutral world*, the first term represents the discounted expected value of the terminal stock price when it exceeds the exercise price times the probability that it is greater than the exercise

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* In more general worlds, the interpretation given here is not correct.

price; the second term represents the discounted exercise price times the probability that the terminal stock price exceeds the exercise price. Essentially the model was developed with insight provided by the assumption that capital markets are perfect and will set prices to eliminate arbitrage profits on options, given that the price of the underlying security is set exogenously and independently of the pricing of this and all other calls. In other words, as the basis for their solution procedure (which culminated in equation (20)), Black and Scholes relied on the assumption that arbitrage profits would be eliminated. An important consequence of the exogeneity of stock prices is that the call price is only a function of the current price of the underlying stocks. Thus, any "market wide effects" (reflected on expected future returns on stocks) enter into the Black-Scholes model only insofar as it determines the price of the stock. In other words, the option is valued as if it were risk free, where no risk-adjusted discount is used in deriving the present value of the option.

Since options are themselves risky securities the result seemed counter intuitive. Yet it has been explained by appealing to the argument that the current stock price reflects the expected rate of return which is appropriate to the risk level of the option. Thus, two securities with the same price may reflect different discount functions, and all that matters is the current stock price (i.e., the current stock price reflects the higher or lower demanded return).

As this point an alternative derivation for the black-Scholes option-pricing formula is provided. This derivation is somewhat simpler and more direct than the original derivation that is provided in Black-Scholes' 1973 work. The formula that is arrived at is the same. However, this method of derivation provides a more intuitive basis for the development of a generalized formula in the next section.

The logic that underlies the derivation of the Black-Scholes formula is a partial defferential equation developed as a result of the Black-Scholes' economic insight that a continuous perfect hedging strategy, which is available with the use of an option and its underlying stock, ought to supply a rate of return equal to the risk-free rate in equilibrium*. Thus, the equilibrium option price is set as a result of both the normal equilibrium condition existing in a security market where perfectly hedged portfolio offers neither more nor less than the risk-free rate (or in cases where a portfolio has no invested wealth returns zero, a zero rate of return), and the terminal condition that at the maturity of the option the option value ought to equal the maximum of the difference betwen the current stock price and the exercise price, or zero.

The essence of the difference between this derivation and that of Black and Scholes involves the method used to derive the differential equation. In both models, this differential equation serves as a key for the ultimate development of an option pricing formula.

Let the market price of the stock and the corresponding call option on that stock at time t be S_t and C_t, respectively. The formation of a portfolio, consisting of N_s shares of stock owned and N_c call options held short against the stock, indicates a net value of the position at any time t as,

(21) $$I_t = N_{st}S_t - N_{ct}C_t$$

The net value of the portfolio per shares of stock held would then be

*A detailed discussion of the original derivation is found in Black-Scholes (1973) and in Smith (1976). The discussion that follows the derivation undertaken here assumes a reasonably thorough understanding of both the optin pricing problem and its original derivation. Reference to Smith, in particular, will reveal a more detailed discussion of the salient points that were part of the original derivation.

indicated as,

(22) $I_t/N_{st} = S_t - C_t/(N_{st}/N_{ct})$

Designating α_t as the ratio of stocks held long to options issued (sold) at any point t, then (22) can be restated as,

(23) $I_t/N_{st} = S_t - C_t/\ \alpha_t$

Ito's lemma* can be used to find the instantaneous change in the net value of the portfolio per share.

(24) $d(I_t/N_{st}) = dS_t - d(C_t/\ \alpha_t)$

$$\text{where } d(C_t/\ \alpha_t) = d(\frac{1}{\alpha_t})C_t + \frac{1}{\alpha_t}\,dC_t + d(\frac{1}{\alpha_t})\ dC_t$$

In principle, since the continuous hedging assumption allows α_t to be continuously <u>determined</u>, α_t is not an endogenous variable, and thus $d(\frac{1}{\alpha_t}) = 0$. Accordingly, (24) simplifies to

(25) $dI_t/N_{st} = dS_t - \frac{1}{\sigma_t}dC_t$

According to the Black-Scholes assumption, the option price is only a function of the price of its underlying stock and time. In other words, both the expected rate of return and the variance rate of return are assumed to be constant. Thus,

$$C_t = C(S_t,t)$$

where the price of the stock S_t follows an Ito equation of the form

(26) $dS_t = \rho_{st}\ dt + \sigma_{st}dZ$

As a consequence dC_t also satisfies an Ito equation and applying Ito's lemma, the instantaneous change in the value of the option would be

*Ito's lemma is used to find the total derivative of a "stochastic variable" (Astrom, 1970)

(27) $$dC_t = \rho_{ct}dt + \sigma_{ct}dZ$$

$$\text{where } \rho_{ct} = \frac{\partial C_t}{\partial S_t}\rho_{st} + \frac{\partial C_t}{\partial_t} + 1/2\,\frac{\partial^2 C_t}{\partial S_t^2}$$

$$\sigma_{ct} = \frac{\partial C_t}{\partial S_t}\sigma_{st}$$

Substituting dS_t and dC_t into (25), the instantaneous value of the portfolio per share of stock held is described by the compound process

(28) $$d(I_t/N_{st}) = (\rho_{st} - (\rho_{ct}/\alpha_t))\,dt + (\sigma_{st} - (\sigma_{ct}/\alpha_t))\,dZ$$

Equation (28) simply describes the behavior of the hedge portfolio on a per share of stock basis. If α_t is now set so that the second term in the bracket on the right hand side of (28) is zero then the intuition underlying the continuous perfect-hedge equilibrium concept is completed.

that is, α_t is set as:

(29) $$\alpha_t = \frac{\sigma_{ct}}{\sigma_{st}}$$

From (27) the pricing behavior of the option σ_{ct} is given as:

(30) $$\sigma_{ct} = \frac{\partial C_t}{\partial S_t}\sigma_{st}$$

Therefore, substituting (30) into (29) yields:

(31a) $$\alpha_t = \frac{\partial C_t}{\partial S_t}(\sigma_{st})/\sigma_{st}$$

(31b) $$= \frac{\partial C_t}{\partial S_t}$$

In words, the hedge ratio α_t which provides zero variability to the portfolio I_t, is the one which is equal to the relative ratio of the change of option price to stock price. Using the form from α_t given in (29), equation (28) can be reformed conditional upon the selection of α_t to eliminate risk as follows:

$$(32a) \qquad d({}^{I}t/N_{st}) = [\ \rho_{st} - [\ \frac{\rho_{ct}}{(\sigma_{ct}/\sigma_{st}}\ ]\ ]\ dt$$

$$(32b) \qquad = [\rho_{st} - \frac{\sigma_{st}}{\sigma_{ct}} \rho_{ct}]\ dt$$

Having no risk, such a portfolio must at any point in time change its value in such a way that only the risk free rate of interest is earned. In other words

$$(33) \qquad [\rho_{st} - \frac{\sigma_{st}}{\sigma_{ct}} \rho_{ct}]\ dt = ({}^{I}t/N_{st})\ rdt$$

where r is the known (by assumption) risk free rate. Cancelling dt terms from both sides of equation (33) yields:

$$(34) \qquad \rho_{st} - \frac{\sigma_{st}}{\sigma_{ct}} \rho_{ct} = r({}^{I}t/N_{st})$$

Since $I_t/N_{st} = S_t - C_t/\alpha_t$ from (23), substituting it into (34) yields:

$$(35a) \qquad \rho_{st} - \frac{\sigma_{st}}{\sigma_{ct}} \rho_{ct} = r(S_t - C_t/\alpha_t)$$

$$(35b) \qquad = r(s_t - \frac{\sigma_{st}}{\sigma_{ct}} C_t)$$

Rearranging (35b) results in a "balance equation" for a capital market equilibrium (which is in the same general form as that arrived at for pricing risky securities with reference to market portfolio, for example the capital asset pricing model; (fama & Miller (1972), Black (1972)):

$$(36a) \qquad \rho_{st} - rS_t = \rho_{ct} \frac{\sigma_{st}}{\sigma_{ct}} - rC_t \frac{\sigma_{st}}{\sigma_{ct}}$$

$$(36b) \qquad \rho_{sr} - rS_t = \frac{\sigma_{st}}{\sigma_{ct}} (\rho_{ct} - rC_t)$$

Thus the excess expected dollar return on the option must be proportional to the excess expected dollar return on the underlying stock that determines the option's value. Rearranging (36) gives,

$$(37) \qquad \frac{\rho_{st} - rS_t}{\sigma_{st}} = \frac{\rho_{ct} - rC_t}{\sigma_{ct}}$$

Obviously the absolute price of the option C_t in equilibrium according to (37) must depend upon the stock price S_t and its instantaneous variance σ_{st}. Given the S_t following an Ito equation, to solve explicitly for C_t requires further steps. These further steps require the formation of a partial differential equation.

substituting (27) into (37) yields:

$$(38) \qquad \frac{\dfrac{\partial C_t}{\partial S_t}\rho_{st} + \dfrac{\partial C_t}{\partial t} + 1/2\,\dfrac{\partial^2 C_t}{\partial S_t^2}\sigma_{st}^2 - rC_t}{\dfrac{\partial C_t}{\partial S_t}\sigma_{st}} = \frac{\rho_{st} - rS_t}{\sigma_{st}}$$

Further rearranging and simplifying yields:

$$(39) \qquad \frac{\partial C_t}{\partial t} + 1/2\,\frac{\partial^2 C_t}{\partial S_t^2}\sigma_{st}^2 - rC_t = -\frac{\partial C_t}{\partial S_t} rS_t$$

Since this expression must hold in equilibrium at all periods, it holds at t = o.

Thus the option price must conform to (39) as well as being subject to ther terminal condition at time t*

$$(40) \qquad C_{t^*} = \max(S_{t^*} - K,\ 0)$$

To solve the partial differential equation (39), the best approach is to use an isomorphic transformation of (39) for which a known solution exists. This is the procedure followed by Black and Scholes using the heat ransfer equation for which a known solution exists (churchill (1963), p. 155). The solution is then expressed as in eq. (20).

3.3 The Generalization of the Black-scholes Option Pricing Formula in a Multi-Security World

3.3.1 Introduction

The purpose of this development is to derive a generalization of the Black-Scholes model in a multi-security world. The point of departure from the work done by Black-Scholes is the contention that it is not possible to determine an investor's most desirable course of action by considering each possible act in isolation from others. Since at any one time he may be confronted with an entire menu of choices, the investor is faced with the necessity of determining a rational strategy for choosing. An investor may buy any number of options at any one time.

Specifically, Black-Scholes explicitly assume in their derivation that the price of a call option is determined in isolation from any other options in the marketplace. In other words, they do not differentiate between the pricing of a single option portfolio and that of an option as one of the individual securities in a portfolio. In this discussion, their world of construction is specifically referred to as a single-security world. The generalization taken here, however, allows investors to price options in relation to their overall portfolio positions. An investor may buy any number of options at any one time. This kind of construction is referred to as a multi-security world.

Recently, Harrison-Pliska (1981) have derived a generalized Black-Scholes formula in a multi-dimensional diffusion model. The model is characterized by a bond and many correlated stocks. This setting is very similar to the multi-security world employed here. However, there are differences in terms of the techniques used to derive the results. They use stockastic calculus, while the derivation here is a generalization of the technique employed in the last section.

One interesting contribution of the results here is that continuous-hedging produces the same market balance equation as that derived in a continuous-time Capital Asset Pricing Model. This finding probably simply verifies the law of one price.

3.3.2 ASSUMPTIONS

All assumptions in the Black-Scholes model are retained here except the one that there is only a single stock in the model, i.e., the single security world assumption. The particular type of multi-security world assumed here contains n correlated stocks. The joint price movement of these stocks is assumed to be a multi-dimensional diffusion process. Further, the determination of this process is assumed to be exogenous.

A multi-dimensional diffusion process looks like

$$(41) \qquad dS_i = \rho_i dt + \sum_{k=1}^{n} \sigma_{ik}\, dZ_k, \quad i = 1,\ldots,n$$

where $S = (S_1, \ldots, S_i, \ldots, S_n)$ represent the set of stocks, ρ_i is the drift factor. $A = [\sigma_{ik}]_{n \times n}$ defines a non-singular covariance matrix (symmetric and positive definite), $Z_1,\ldots,Z_n$ are standard Wiener processes and the correlation between dZ_i and dZ_k is the correlation coefficient μ_{ik}.

In a two-stock world, the process looks like

$$dS_1 = \rho_1 dt + \sigma_{11} dZ_1 + \sigma_{12} dZ_2$$
$$dS_2 = \rho_2 dt + \sigma_{21} dZ_1 + \sigma_{22} dZ_2$$

In other words, the price change of stock comes as a result of not only its noise but also the noise of stock 2 (i.e., the rest of the market).

In order to simplify the derivation process taken here, however, it is further assumed that the price change of a stock can be explained by its own noise and the noise of a common market index. The correlation between the two is assumed to be zero. therefore, (41) becomes:

$$(42) \qquad dS_i = \rho_i dt + \beta_i\, \sigma_m\, dZ_m + \sigma_{ei}\, dZ_i$$

where β_i is a coefficient measuring the effect of the market noise on the price change of stock i, Z_m is a Wiener process describing the noise of the price dynamic of the market index, σ_m denotes the instantaneous standard derivation rate of return on the market index, σ_{ei} denotes the specific standard derivation rate of return on stock i.

3.3.3 Derivation

Suppose an investor engages in continuous-hedging by writing covered call options on stock. According to Ito's Lemma, the pricing dynamic of any call can be stated as

$$(43) \qquad dC_i = \rho_{ci}dt + \beta_{ci}\sigma_m dZ_m + \frac{\partial C_i}{\partial S_i}\sigma_{ei}\, dZ_i$$

where

$$\rho_{ci} = \frac{\partial C_i}{\partial S_i}\rho_i S_i + \frac{\partial C_i}{\partial t} + 1/2\,\frac{\partial^2 C_i}{\partial S_i^2}(\beta_i{}^2\sigma_m{}^2 + \sigma_{ei}{}^2)$$

$$\beta_{ci} = \frac{\partial C_i}{\partial S_i}\beta_i$$

(Proof)

Applying Ito's Lemma on C_i,

$$(44) \qquad dC_i = \frac{\partial C_i}{\partial S_i}dS_i + \frac{\partial C_i}{\partial t} + 1/2\,\frac{\partial^2 C_i}{\partial S_i^2}(dS_i)^2$$

where

$$(45) \qquad (dS_i)^2 = (\rho_i dt + \beta_i\sigma_m\, dZ_m + \sigma_{ei}\, dZ_i)^2$$

$$= \rho_i^2\,(dt)^2 + \beta_i^2\sigma_m^2(dZ_m)^2 + \sigma_{ei}^2\,(dZ_i)^2 + 2\rho_i\beta_i\sigma_m dtdZ_m$$

$$+ 2\rho_i\,\sigma_{ei}\, dtdZ_i + 2\beta_i\sigma_m\sigma_{ei}\, dZ_m dZ_i$$

Since according to the well-known "laws" of stochastic calculus (Astrom (1970), Soong (1973)),

$$dtdZ_m = 0$$

$$(dZ_m)^2 = (dZ_i)^2 = dt$$

$$dtdZ_i = 0$$

and by construction

$$(dZ_m)(dZ_i) = 0$$

equation (45) can be simplified as

$$(46) \qquad (dS_i)^2 = (\beta^2{}_i\sigma_m^2 + \sigma_{ei}^2)\, dt$$

It is interesting to note that the two terms in the right hand bracket of equation (46) are precisely the instantaneous systematic risk and instantaneous unsystematic risk.

Substituting equations (42) and (46) into equation (44) and rearranging,

$$(47) \qquad dC_i = [\frac{\partial C_i}{\partial S_i} \quad i + \frac{\partial C_i}{\partial t} + \frac{1}{2}\frac{\partial^2 C_i}{\partial S_i^2}(\beta_i{}^2\sigma_m{}^2 + \sigma_{ei}{}^2)]dt + \frac{\partial C_i}{\partial S_i}\beta_i\sigma_m dZ_m + \frac{\partial C_i}{\partial S_i}\sigma_{ei}dZ_i$$

Defining ρ_{ci} as the value of the coefficient term of dt, and β_{ci} as $\frac{\partial C_i}{\partial S_i}\beta_i$, equation (47) is equal to

$$dC_i = \rho_{ci}dt + \beta_{ci}\sigma_m\, dZ_m + \frac{\partial C_i}{\partial S_i}\sigma_{ei}\, dZ_i$$

—— Q.E.D. ——

To hedge away the risk of any stock i, this investor may write covered call options on it with a hedging ratio α_i per share of the stock. The value of the hedging portfolio I_i thus is (48) and the total derivative is (49),

$$(48) \qquad I_i = S_i - \alpha_i C_i$$

$$(49) \qquad dI_i = dS_i - d(\alpha_i C_i)$$
$$= dS_i - [(d\alpha_i) C_i + \alpha_i(dC_i) + (d\alpha_i)(dC_i)]$$

In principal, since continuous hedging allows α_i to be continuously determineed, α_i is not an endogenous variable, and thus

$$(50) \qquad d\alpha_i = 0$$

Substituting equation (50) into equation (49),

$$(51) \qquad dI_i = dS_i - \alpha_i dC_i$$

Substituting equations (42) and (43) into equation (51),

$$(52) \qquad dI_i = (\rho_i - \rho_{ci}\alpha_i)\, dt + (\beta_i\sigma_m - \beta_{ci}\sigma_m\alpha_i)\, dZ_m + (\sigma_{ei} - \frac{\partial C_i}{\partial S_i}\sigma_{ei})\, dZ_i$$

There are two risk terms which make up the total risk of this hedging portfolio in equation (52). The first term corresponds to the impact of the market risk and the second term corresponds to the stock's specific risk.

To design a rational hedging strategy, the investor has the choice of buying one or many hedge portfolios. In the extreme case, he may engage in hedging on all stocks simultaneously.

When he buys only one hedging portfolios (the Black-Scholes case), the instantaneous change in the value of the hedging portfolio is described by equation (52). However, according to portfolio theory, if he buys many hedging portfolios simultaneously, the risk contribution of an individual portfolio to his overall position should be less than the total risk of that portfolio. In the extreme case, the risk contribution of an individual portfolio to his overall position should only be the systematic portion of the total risk position of that portfolio, i.e., the term in equation (52) which corresponds to the risk of the market portfolio. A rational investor, in light of this advantage of risk reduction, <u>will buy a portfolio of hedging portfolios at any time</u>. In this portfolio context each security is priced according to its contribution to a well-diversified (market) portfolio risk and return, while each security itself (with the exception of the market security) is not "efficient."

The following procedure supplies an approach to price an option as a non-efficient asset.

To continuously hadge the non-efficient portfolio i, the investor must continuously readjust the hedging ratio α_i in such a way that the coefficient of the market risk term in equation (52) is always zero. To do this,

$$(53) \qquad \beta_i\sigma_m - \beta_{ci}\sigma_m\alpha_i = 0$$

thus:

$$(54) \qquad \alpha_i = \frac{\beta_i}{\beta_{ci}}$$

From the derivation of β_{ci}, $\beta_{ci} = \beta_i \dfrac{\partial C_i}{\partial S_i}$, thus:

$$(55) \qquad \alpha_i = \frac{\partial S_i}{\partial C_i}$$

Substituting equation (54) into equation (52) and assuming total diversification yields:

$$(56) \qquad dI_i = (\rho_i - \rho_{ci} \frac{\beta_i}{\beta_{ci}}) \; dt$$

In order that the value of risk-free hedging portfolio be market-determined at equilibrium, this portfolio must earn a risk-free interest rate over time:

$$(57) \qquad dI_i = rI_i dt$$

where r is the instantaneous risk-free interest rate.

Equating equations (56) and (57),

$$(58) \qquad \rho_i - \rho_{ci} \frac{\beta i}{\beta_{ci}} = rI_i$$

$$= r(S_i - C_i \alpha_i)$$

$$= r(S_i - C_i \frac{\beta_i}{\beta_{ci}})$$

Rearranging equation (58),

$$(59) \qquad \frac{\rho_i - rS_i}{\beta_i} = \frac{\rho_{ci} - rC_i}{\beta_{ci}}$$

This balance equation is, as it turns out, in the same general form as that arrived in a continuous-time CAPM (Merton 1970 and 1973b) between an option and its underlying stock. It is different from the one in the Black-

Scholes world, i.e.,

$$\text{(60)} \qquad \frac{\rho_i - rS_i}{\sigma_i} = \frac{\rho_{ci} - rC_i}{\sigma_{ci}}$$

Apparently this difference eminates from the fact that, in the Black-Scholes world, there is only one stock, thus the market prices this stock in relation to its own variances and consequently supplies a risk-premium that is solely a function of this variance. However, in a multi-security world, the relevant risk measure of securities is β (the systematic risk portion), thus results in (59).

It is not surprising that a sufficient condition (use the continuous hedging strategy) in the derivation of a continuous-time CPAM relationship can be shown here. That is because there should be only <u>one</u> balance equation in a multi-security <u>mean-variance</u> world. Arbitrage-free condition for market equilibrium will not allow simultaneous existences of different market risk-return trade-offs.

To price an option according to the balance equation (59), requires substituting the values for ρ_{ci} and β_{ci} as they are defined in equation (43). To do so results in

$$\text{(61)} \qquad \frac{\rho_i - rS_i}{\beta_i} = \frac{\frac{\partial C_i}{\partial S_i}\rho_i + \frac{\partial C_i}{\partial t} + 1/2\,\frac{\partial^2 C_i}{\partial S_i^2}(\beta_i^2\sigma_m^2 + \sigma_{ei}^2) - rC_i}{\beta_i \frac{\partial C_i}{\partial S_i}}$$

Rearranging and simplifying yields:

$$\text{(62)} \qquad \frac{\partial C_i}{\partial t} + rS_i \frac{\partial C_i}{\partial S_i} + 1/2\,\frac{\partial^2 C_i}{\partial S_i^2}\sigma_i^2 = rC_i$$

where $\qquad \sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{ei}^2$

It is interesting to note that the resulting multi-security option-pricing differential equation is exactly identical to the single-security Black-Scholes equation. It follows that the pricing solution for options should also be the same. In other words it is shown that the value of the option is irrelevant to the diversification position of investors. The original single-security formula can also be applied even if investors buy a portfolio of options. In equation (61), the <u>expcted rate of return</u> terms are cancelled out from both sides, thus the resulting differential equation is preference-free. This means that the value of the option is uniquely determined by the value of its underlying stock. Since the effect of diversification (the market effect) is only reflected in a lower $_i$ because that certain risk is diversified away, the fact that $_i$ is irrelevant means that diversification is also irrelevant.

3.4 <u>The Black-Scholes Model and a Complete Market</u>

Since Black-Scholes developed the first satisfactory option pricing result, their model has been considerably generalized and many of their assumptions have been relaxed. However as Brennan (1979) pointed out, most of these models have derived Black-Scholes type of preference-free results. In here, it is explained why Black-Scholes type of preference-free models assume a complete market.

Cox-Ross-Rubenstein (1979) have demonstrated that the Black-Scholes pricing result can be derived as a limiting form of their model explicitly assumes a complete market. Thus it is logical to expect that the Black-Scholes model also assumes a complete market.

Harrison-Pliska (1981) have proven that the Black-Scholes model implicitly assumes that the market for underlying securities is complete. An intuitive explanation of the Harrison-Pliska proof follows.

In a continuous market, the usual definition of a complete market cannot be applied because the state space is no longer finite. However, a complete market can be alternatively defined in this way: "a market is complete if and only if any derived asset in the market is attainable." The term "attainable" means that the cash flow of the deried asset can be perfectly duplicated by a portfolio of underlying assets. The alternative definition here presented is a logical counterpart to the following statement:

> "A market is incomplete if not all derived assets in the market are attainable."

The unique feature of the Black-Scholes model is the use of "continuous hedging." In other words, the cash flow of the option can be perfectly and continuously duplicated by the cash flow of a portfolio made up of the underlying stock and a risk free discount bond. The hedging ratio of course has to be continuously adjusted. In fact, the possibility of continuous hedging enables the investor to hedge on <u>any</u> contingent claim because the hedging ratio is simply the relative price change between the contingent claim and the underlying asset or assets. Thus, any contingent claim in their world is attainable, and it follows from the definition of a complete market that their model assumes a complete market.

Harrison-Pliska state that the fact that the price of stock follows a continuous process is not a sufficient condition to a complete market. It is also required that the process of the stock price satisfy certain martingale represeentation properties; loosely adaping their terminology, this means that the return process on the stock using the risk-free rate as the discounting factor must be a martingale and must be written as a stochastic integral. Other models which have already been shown to assume a complete market include multi-dimensional Brownian Motion, the Poisson martingale, and others.

So far the idea that the Black-Scholes model implicitly assumes a complete market has been established. As a consequence, it can be logically deduced that their model should not price options in equilibrium in an incomplete market. This contention is more formally examined here. In particular, it will be demonstrated that the Black-Scholes option pricing formula prices options inconsistent with the First-degree Stochastic Dominance argument in a one-period incomplete market. The particular kind of market structure on which this analysis is based is a standard Capital Asset Pricing Model. It is important to notice that what is intent to be shown is that th Black-Scholes result cannot be applied in a larger environment (incomplete market) than the one (complete market) in which their result is derived. Their result is nonetheless a market equilibrium result in their own setting (complete market).

To create a larger environment, an additional assumption is required for this analysis. This is a general assumption that different securities may potentially off different expected equilibrium returns. It does not matter why these securities may offer different expected equilibrium returns, though we choose to follow a set of assumptions in our development that is consistent with the formulation of the capital asset pricing model. Thus, if for no other reason, securities offering different levels of risk (i.e., different beta coefficients) would offer different expected equilibrium returns.

Consider, for example, the existence of two underlying securities. These securities face the same market risk return tradeoff and differ only with respect to the level of their beta coefficient. Call options on these two securities, with the same underlying stock price, total variance of stock return, time to maturity, risk free interest rate, and exercise prices would, according to the Black-Scholes formula, be priced identically. However, it can be shown in the following that an application of stochastic dominance techniques, that if beta coefficients on the two securities are different, the value of the options on the two securities would not be equivalently desired by market participants. Hence, the application of Black-Scholes options pricing formula to these two instruments would not lead to a situation in which there were no arbitrage opportunities between the two options. This is a contradiction to the Black-Scholes result, and, in fact, all of the preference-free option pricing equation results that have been developed in the literature up to this point in time.

Stochastic dominance (Ali 1975, Bawa 1975, 1978, Josef-Williams 1971, Levy-Yoram 1976, and Vickson 1975) is a very powerful and general method for ranking or valuing investments, and any alternative method for ranking or valuing investments that violates stochastic dominance is subject to question. It is a mechanism that allows discrimination between different probability distributions characterizing uncertain events in certain cases. For example, the idea that one security's uncertain return stochastically dominates another allows comparisons of values of securities without the need to specify the form of the utility function that individuals may apply to their choices.

The one theorem which will be applied here is the First Degree Stochastic Dominance (FSD). The theorem states: if investors prefer more to less, and if the cumulative probability of A is never greater than the cumulative probability of B and sometimes less, then A is preferred to B.

Assume equilibrium stock prices are determined according to the CAPM valuation formula

(63) $S_{io} = \overline{S}_{iT}\ (1 + r_f + (\overline{r}_m - r_f)\ \beta_i]^{-1}$

where $\beta_i = \text{Corr}\ (r_i\ r_m)\ \sigma\ (r_i)/\sigma(r_m)$

$\text{Corr}\ (r_i r_m)$ is the correlation coefficient between the return on stock i and the return on the market index, $\sigma(r_i) = \sigma_i/S_{io}$ is the standard derivation of return of stock i with σ_i the standard derivation of the prices of the stock at the end of one period, and finally $\bar{S}_{iT}$ is the expected terminal price of stock i.

In the context of the CAPM, two stocks, i and j can have equilibrium expected returns, $\bar{r}_i$ and $\bar{r}_j$, that differ even though they have the same current price and the same variance for the future price. For example, assume current stock prices are determined according to (63), and for stock i and j, $S_{io} = S_{jo}$, $\sigma_i = \sigma_j$, $\sigma(r_i) = \sigma(r_j)$, but $\bar{S}_{iT} \neq \bar{S}_{jT}$ and $Corr(r_i, r_m) \neq Corr(r_j, r_m)$, and consequently and $\beta_i \neq \beta_j$. Because the systematic risks of the two stocks differ, their expected returns differ, with $\bar{r}_i = r_f + (\bar{r}_m - r_f)\beta_i$

Now consider the calls on these two stocks. Assume each stock has an associated call with the features of the call the same for the two stocks: both calls expire at T; and the exercise prices are the same, $k_i = k_j$. With the Black-Scholes formula the values of these calls would be the same, $C_{10} = C_{20}$, independent of the fact that the stocks have different levels of systematic risk and different expected returns.

To apply the stochastic dominance criterion, the cumulative probability distributions of the call prices at expiration can be shown to be truncated lognormal

$$(64)\quad P_r(C_{iT} \leq a) = P_r[S_{iT} \leq k_i + a]$$

$$(65)\quad = N\left[\frac{\ln(k_i + a)/S_i) - (\rho_i - \sigma_i^2/2)T}{\sigma_i\sqrt{T}}\right]$$

The argument of the distribution function (the term in brackets) is monotonically decreasing in the expected return, ρ_i, and this distribution function is monotonically increasing in its argument. Thus, if $\rho_1 > \rho_2$, the term in brackets for call 1 is smaller than for call 2, and

$$(66)\quad P_r(C_{1T} \leq a) < P_r(C_{2T} \leq a)$$

for any $a \geq 0$.

Note that expression (66) fits the definition of first degree stochastic dominance, so that all investors that prefer more wealth to less will prefer to purchase call 1 at price C_{10}.

Yet, it was assumed that $C_{10} = C_{20}$, which seems to be a disequilibrium. That is, if all investors prefer call 1, it would seem that its price could not continue to be the same as call 2. At the very least, equilibrium should require that some investors are willing to hold call 2.

The simple application of first degree stochastic dominance shows that all investors would prefer to purchase the call for the stock with higher expected returns, when all other parameters (which appear in the Black-Scholes formula) are equal. As it has been shown before, the one crucial assumption which Black-Scholes implicitly make in their model is that the underlying security market is complete. In a complete market, an option can be priced relative to the price of its underlying security as if the world were composed of risk-neutral investors. In other words, the expected returns on underlying stock must all equal the risk-free interest rate <u>in the expression for cumulative probability distribution of option prices</u>, i.e., equation (65) (however, it is important to note that preference free valuation can only be applied to the pricing of a <u>derived security</u> in which the market of the underlying primitive security is complete. The relationship cannot be applied to a primitive security). Thus, the expected rate of return is invariant in a complete market, and the stochastic dominance argument fails to apply. It rounds out that the Black-Scholes formula cannot be shown to be inconsistent with the stochastic dominance argument in a complete market, but it violates the argument in an incomplete market case and their model thus cannot yield equilibrium option prices.

One existing criticism of this application of stochastic dominance is that the calls are in apparent disequilibrium because it is the stocks that are in disequilibrium.

The Black-Scholes formula purports to price the call only in relation to its underlying stock. If the stock is not priced correctly relative to other stocks, then neither will the call be correctly priced relative to other calls and stocks.

This critical view would note that stock #1 stochastically dominates stock #2 just as in the case of the calls, and suggest the stocks are incorrectly priced rather than the calls.

That is with $\rho_1 > \rho_2$, and $\sigma_1 = \sigma_2$. For the stocks,

(67) $P_r(S_{1T} \leq a) \leq P_r(S_{2T} \leq a)$

for all $a \geq 0$, and #1 exhibits first degree stochastic dominance over #2 when $S_{10} = S_{20}$.

The forgoing argument would be valid if there were just the two stocks available. With just stocks #1 and #2 available to investors, investors would prefer #1, and prices would have to change until investors are willing to hold both stocks. However, this argument is incorrect in an incomplete market with many securities wherein investors can eliminate part of their risk through diversification. In such a market individual stocks may exhibit stochastic dominance over other stocks and still be correctly priced because stochastic dominance is irrelevant for individual stocks, but is a valid concept for the portions comprised of these stocks. That is, it is correct to rank portfolios according to stochastic dominance, but the securities should be ranked and valued with the CAPM that measures risk as covariance or beta.

Thus, with portfolios A and B, it can be said that all investors prefer A over B if

$$P_r(r_A \leq a) \leq P_r(r_B \leq a)$$

for all a (with strict inequality for some a), but two stocks 1 and 2 conforming to (67) would be correctly priced with $\rho_1 > \rho_2$ and $\sigma_1 = \sigma_2$ because $\beta_1 > \beta_2$.

4. OPTION PRICING UNDER INCOMPLETE MARKET CONDITIONS

4.1 Introduction

In the Arrow-Debreu type of approach to uncertainty, a complete market has been treated as a bench mark of ultimate capital market efficiency. The attraction of such a market rests upon its unique ability to achieve a Pareto efficient allocation of capital assets among investors under heterogeneous preferences and beliefs. Despsite this decided virtue, in the real world, a complete capital market has not come to existence, because of problems associated with market frictions such as transaction costs, moral hazard, and adverse selection. However, it has been suggested by many authors (pointed out in the working paper by Senbet-Taggart, 1981) that the issuances of corporate securities have the economic function of helping to complete a capital market. In fact, these authors have argued that the economic role of corporate securities lies in helping to complete a capital market.

Options, as a particular kind of corporate security, have attracted some attention in the field of finance lately (Friesen 1974 and Ross 1976) as having efficient market-completing function. That is because the issuance of options is relatively easy and inexpensive and the tranaction costs involed in the trading of options are also relatively low. They have note that options may bring an incomplete market into completeness under the following sufficient conditions:

(i) To every state of nature corresponds a unique set of prices for primitive securities, and

(ii) markets exist for call options written on all portfolios of securities at all dates and at all striking prices. In a two period setting, Ross (1976) shows that it suffices to have call options on a single portfolio of the primary securities (the portfolio chosen is one whose terminal payoff distinguishes the state of nature).

In a more general setting, Arditti-John (1980) shows that options written on almost any portfolioo of assets x can make the market complete with respect to the number of distinct states x can span - "maximum efficiency principal.".

The work in here is to review a few existing option-pricing models which are applicable under incomplete market conditions, and to derive an option-pricing formula within the framework of CAPM. It is emphasized that these existing models are not derived under the objective of correcting other models which assume complete markets. In fact, these models were not addressed to resolving the paradox associated with complete-market-based option-pricing models.

4.2 Existing Option Pricing Models Which Do Not Necessarily Imply Complete Market

4.2.1 Kwon Model (1980)

Farka (1902) developed a lemma which has been proven to be very useful in formulating a certain type of mathematical programming problems. The lemma proves the following: A vector P will satisfy $WP \geq 0$ for all W satisfying $WR \geq 0$ if and only if there exists a $\Pi \geq 0$ such that $P = R\Pi$. In an investment context, the lemma says that if every portfolio in the market must involve a nonnegative amount of investment ($WP \geq 0$) in order to have nonnegative returns ($WR \geq 0$) (a typical arbitrage-free condition), then the current prices of assets (p) must be expressible as nonnegative ($\Pi \geq 0$) linear combination of their future payoffs across all possible states (R).

In this context, Farka' Lemma is very essential because it proves the existence oof Arrow-Debreu type of state securities even in the absence of complete market. Of course, now the prices of those securities are no longer unique and their determination is dependent upon investors' utility functions.

In fact, one can imagine an environment in which investors as a group behave according to a "consensus" utility function and expectation, and consequently, the price of state security equals the relative marginal utility of wealth.

Kwon has extended Farka's Lemma to a continuous-state framework, and then suggests that the price of an option can be determined via the Lemma when a "consensus" utility function can be derived. He does not demonstrate the kinds of utility function that are apropriate. However, he derives a technique to show how options can be priced given a consensus utility function. In fact, he shows that the Black-Scholes result can be derived under the assumption of a constant-proportional-risk-aversion consensus utility function. He further suggests that as there can be different consensus utility functions, there can be different coption pricing formulas.

The distinct feature of Kwon's technique concerns the involvement of investors' preferences—the need of assuming a consensus utility function. This kind of technique has been called "absolute-pricing" technique (Garman, 1978). A problem associated with this kind of technique is that it is practically very hard to determine a consensus utility function.

4.2.2 Lee-Rao-Auchmuty Model (1981)

Lee-Rao-Auchmuty have employed Bawa's Lognormal Capital Asset Pricing Model (1981) to price options. They state:

> "This paper derives a call option valuation equation assuming discrete trading in the securities markets where the underlying asset and the market returns are bivariate lognormally distributed and investors have increasing, concave utility functions exhibiting skewness preference. Since the valuation does not require the continuous time riskfree hedging of Black and Scholes, nor the discrete time riskfre hedging of Cox, Ross and Rubinstein, market effects are introduced into the option valuation relation. The new

option valuation seems to correct for the systematic mispricing of well-in and well-out of the money options by the Black and Scholes option pricing formula."

and

"......the new discrete trading option valuation equation is based on a larger admission set of utility functions than permitted in the Rubinstein-Brennan State preference framework......"

Essentially, Lee-Rao-Auchmuty attempt to price options under discrete-trading conditions, by determining the expected rate of return of an option through a particular kind of CAPM, and then by discounting the expected terminal price of this option using the return factor. Therefore, their formula yields market equilibrium prices. Since the lognormal CAPM does not assume a complete market, their result is applicable under incomplete market conditions. However, it is emphasized here that their derivation is not motivated by the idea of correcting existing complete-market-based results. In fact, they did not discuss this idea at all.

4.2.3 O'Brien-Schwarz Model (1982)

O'Brien-Schwarz have presented a third model which is applicable under incomplete market conditions. They developed an ad hoc argument that an option-pricing formula developed in the standard CAPM may be useful. Though they were not able to derive a closed-form pricing result, they suggested a numerical procedure to price options. Applying the model in the over-the-counter gold market, they suggest that their model outperforms the Black-Scholes model.

Compared to the Lee-Rao-Auchmuty model, they suggest that their model is easier to apply because it contains one less unobservable variable. Their model is applicable under incomplete market conditions becuase the standard CAPM does not make a complete market assumption. It is emphasized that they did not offer any discussion about the complete market idea and about the fact that the Black-Scholes model makes that assumption.

4.3 <u>Option-Pricing in The Standard CAPM Framework</u>

In here the option valuation problem is approached under one form of incomplete market conditions. The particular model employed is the standard CAPM. Within this model, a European call option will be priced in the same way as any other security in the market place. That is the price of a security equals the present value of its future cash-flow. Present value is defined using expected returns from the CAPM. It is emphasized that this model and O'Brien-Schwarz model are developed independently, although both of them are based upon the standard CAPM.

The CAPM is a partial market equilibrium which does not make an assumption of a complete market. In such a model, in general, options are not redundant securities. The cash-flow of an option cannot be totally replicated by linear combination of the cash-flow of other securities in the market place.

A unique feature of the CAPM is the derivation of a market equilibrium risk-return relationship -- the "security market line". This relationship enables investors to determine the expected rate of return of a security according to the risk of this security. The development taken here supplies a way of determining the risk of an option.

The incomplete-market-based approach employed in here is different from the approaches discussed in Section B in the following ways:

1) While previous approaches are not addressed to resolving the paradox associated with complete-market-based option-pricing models, this approach is developed for the purpose of correcting the paradox,
2) it does not relate to investors' utility functions and thus is

fundamentally different from the Kwon approach.

3) it is based on the standard CAPM and thus is different from the Lee-Rao-Auchmuty approach which employes the lognormal CAPM,

4) it results in a simple, closed-from option-pricing formula, and thus improves the Schwarz-O'Brien result.

4.3.1 The Assumptions and Result of Standard CAPM

CAPM can be derived under the following set of assumptions (not a necessary set; many of the assumptions can be relaxed, see Elton and Gruber 1981):

(a) All individuals have a strictly concave Von Neuman-Morganstern utility function and are one-period expected utility maximizers.

(b) Investors have hooomogeneous expectations about the terminal firms asset values and security prices.*

(c) The capital market is perfect: no transaction costs or taxes and all traders have free and costless access to all available inforation. All traders are price takers in the market.

(d) There are no costs of liquidation or bankruptcy

(e) Investors can lend or borrow any amount at the risk free interest rate.

(f) Borrowing and short-selling by all investors and free use of all proceeds is allowed.

(g) All assets are infinitely divisible and marketable.

(h) The returns of any asset and the market portfolio follow a joint normal distribution.

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* Options may exist even under the condition of homogenous expectations if investors have heterogeneous preferences.

The result of the model is a market equilibrium risk-return relationship called the "security market line":

$$\bar{R}_i = R_f + \beta_i(\bar{R}_m - R_f)$$

where, $\bar{R}_i$ is the expected return on security i, R_f is the risk free interest rate, $\bar{R}_m$ is the expected return on the market portfolio and β_i is the amount of risk of security i and is defined as:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

Verbally, the CAPM structures the demanded compensation of the market for time and risk.

4.3.2 The Drivation of the Standard CAPM Option-Pricing Formula

Notations

C: the present value of the option.

C_T: the terminal value of the option which is uncertain as of to date.

T: the time to maturity of the option.

$\bar{C}_T$: the expected terminal value of the option.

$\bar{R}_C$: the expected gross rate of return on the option.

$\bar{R}_S$: the expected gross rate of return on the stock.

K: the exercise price of the option.

S: the current price of the stock.

σ_S: the standard-deviation of the rate of return on the stock.

R_m: the gross rate of return on the market.

β_C, β_S: the systematic risk of the option and of the stock, respectively.

R_f: the risk-free interest rate.

Derivation

The present worth of a European call option equals

$$C = \frac{\bar{C}_T}{\bar{R}_C} \tag{68}$$

The derivations which follow determine the values of $\bar{C}_T$ and $\bar{R}_c$ and hence C.

First of all, according to the definitioon of expected value, the assumption that R_s follows a normal distribution and the terminal condition of the option that $C_T = S_T - K$ if $R_s \geq \frac{K}{S}$ and othewise zero, the expected terminal price of the option can be determined as:

$$(69) \qquad \bar{C}_T = \int_{K/S}^{\infty} (S_T - K) \quad dN(R_s)$$

Standardizing the normal distribution $N(R_s)$ by setting

$U = \dfrac{R_s - \bar{R}_s}{\sigma_s}$, then $S_T = SR_s = S\,\sigma_s U + S\bar{R}_s$, and (69) becomes

$$(70) \qquad \bar{C}_T = \int_{-b}^{\infty} (S\sigma_s U + S\bar{R}_s - K) \; dN(U)$$

Where N(U) is the standard cumulative normal distribution and $b = \dfrac{-K/S + \bar{R}_s}{\sigma_s}$.

Rearranging and simplifying (70) yeild $\bar{C}_T$:

$$(71a) \qquad \bar{C}_T = \sigma_s S \int_{-b}^{\infty} U dN(U) + (\bar{R}_s S - K)\; N\; (b)$$

$$(71b) \qquad = \frac{\sigma_s S}{\sqrt{2\Pi}}\, e^{-1/2b^2} \qquad + \quad (\bar{R}_s S - K)\; N\; (b)$$

Secondly, $\bar{R}_c$ can be determined via the following alternative form of the SML:

$$(72)\ \bar{R}_c = R_f + \beta_c \left(\frac{\bar{R}_s - R_f}{\beta_s}\right)$$

In (72) the only unknown variable is β_c and it can be derived from its definition and the terminal condition of the option:

$$(73) \qquad \beta_c = \frac{\mathrm{Cov}(R_c, R_m)}{\mathrm{Var}(R_m)}$$

$$(74) \qquad R_c = \begin{cases} (R_s - \frac{K}{S}) \frac{S}{C} & \text{if } R_s \geq \frac{K}{S} \\ 0 & \phantom{\text{if }} R_s < \frac{K}{S} \end{cases}$$

The discussion which follows is the derivation of β_c.

Multiplying R_c by $\frac{C}{S}$, then from (74),

$$(75) \qquad R_c \frac{C}{S} = \begin{cases} R_s - \frac{K}{S} & \text{if } R_s \geq \frac{K}{S} \\ 0 & \phantom{\text{if }} R_s < \frac{K}{S} \end{cases}$$

It essentially defines a truncated normal distribution with the trucation taken place at $\frac{K}{S}$. According to the results derived by Lintner (1977, p111) concerning the properties of the truncated normal distribution,

$$(76) \qquad \text{Cov } (R_c \frac{C}{S}, R_m) = \text{Cov } (R_s, R_m) N(b)$$

Thus rearranging (76) yields:

$$(77) \qquad \text{Cov } (R_c, R_m) = \frac{S}{C} N(b) \text{ Cov } (R_s, R_m)$$

Substituting (77) into (73) yields:

$$(78) \qquad \beta_c = \frac{S}{C} N(b) \text{ Cov } \frac{(R_s, R_m)}{\text{Var } (R_m)}$$

$$(78b) \qquad = \frac{S}{C} N(b) \beta_s$$

This result indicates that the risk of the option comes entirely from the risk of its underlying stock.

Substituting (78b) into (72),

$$\text{(79a)} \qquad \bar{R}_c = R_f + \frac{S}{C} N(b)\, \beta_s \left(\frac{\bar{R}_s - R_f}{\beta_s}\right)$$

$$\text{(79b)} \qquad = R_f + \frac{S}{C} N(b) \quad (\bar{R}_s - R_f)$$

Finally, substituting (79b) and (71b) into (68) yields the present worth of the option as:

$$\text{(80)} \qquad C = \frac{\frac{S\sigma_s}{\sqrt{2\pi}} e^{-1/2b^2} + (\bar{R}_s S - K)\, N(b)}{R_f + \frac{S}{C} N(b) \quad (\bar{R}_s - R_f)}$$

Simpllifying and rearranging (80) yields the option-pricing formula:

$$\text{(81a)} \qquad C = \frac{\frac{S\sigma_s}{\sqrt{2\pi}} e^{-1/2b^2} + (\bar{R}_s S - K)\, N(b) - S\, N(b)\, (\bar{R}_s - R_f)}{R_f}$$

$$\text{(81b)} \qquad = \frac{\frac{S\sigma_s}{\sqrt{2\pi}} e^{-1/2b^2} - KN(b) + SN(b) R_f}{R_f}$$

Where $b = \dfrac{\frac{-K}{S} + \bar{R}_s}{\sigma_s}$ and N( ) is the standard cumulative normal distribution.

This formula has "certainty-equivalence" form. In (81a), the first two numerator terms correspond to the expected terminal price of the option while the third numerator term represents the demanded compensation from the risk of option. The formula contains the following five determining variables: S, K, R_f, $\bar{R}_s$ and σ_s. The time to maturity, T, is reflected in the values of R_f, $\bar{R}_s$ and σ_s.

The new formula being derived from the CAPM is more general than the Black-scholes formula, because the CAPM does not make a necessary assumption of a complete market. Consequently, the new formula contains one more determining factor than that of Black-Scholes—the expected rate of return on the underlying asset of the option under consideration. This factor reflects the effect of investors' preferences upon the value of the option.

The new formula is very similar to the Lee-Rao-Auchmuty formula, because both of them are derived in the CAPM context. Because of that, they are both preference-related. The difference between the two formulas comes as a direct result of different distribution assumptions—Lee-Rao-Auchmuty assume that the return of the underlying asset of the option in consideration follows a lognormal distribution instead of a normal distribution.

However, the new formula is much simpler and has a practical advantage over that of Lee-Rao-Auchmuty, which is that its implementation requires the estimation of one less unobservable determining variable. The one variable which is not required is the covariance of the underlying asset with the market index.

The new formula should have been identical to the Schwarz-O'Brien formula, had they been able to derive a closed-form solution. This is because both of these attempts are based upon the same kind of CAPM although they have been developed independently.

5. SUMMARY AND CONCLUSION

This paper approaches the question of option pricing by considering the economic theory of options. This attempt results in the classification of existing option pricing models into two categories, one in which options serve no economic purpose and one in which they do.

It has long been argued that the economic role of options, as well as other securities in the marketplace, is to provide completeness to the existing incomplete capital market. A distinct feature of an incomplete capital market is that marginal rates of substitution between current consumption and all conceivable patterns of future returns are not necessarily driven to equality for all investors. The reasons for this incompleteness are transactioon costs, moral hazard, adverse selection and so on. These problems are not of direct concern in this paper. However, because of them, there is a clear incentive for investors and corporations to tailor their issuances of securities so as to eliminate or more realistically reduce these divergences (in marginal rates of substitution). These actions provide completeness to the market. It is known in economic theory that a complete market has the unique ability to achieve ultimate "efficiency" or Pareto-optimal, which has long been recognized as the ultimate goal of economic development (in the context that social welfare is held to rise with individual happiness). In this "positive" sense, the argument that securities exist to complete the capital market is assumed here to be the economic theory which underlies the development of options.

In their role as market completing securities, options have some particular advantages as opposed to stocks or bonds—the relative cheapness of issuance and the virtual infinity in which they can be written based upon a single underlying asset. These two advantages distinguish options as superior market completing securities.

Recently, however, it has been found that most of the existing option pricing models implicitly or explicitly assume that the market of the underlying security is already complete. This finding leads to the "paradox' raised in this dissertation. This paradox exists because these existing models attempt to price optioons in a complete market environment in which options would not necessarily be issued if they do not serve the economic purpose of completing the market. This category of models includes the seminal Black-Scholes model.

The Black-Scholes option pricing model has long been recognized as the first successful attempt to price options in a general equilibrium context. since then, option-pricing theory has been developed using their initial methodological approach as a basis from which a number of extensions and generalizations have been formulated. Because of the importance of their model, a significant portion of this paper is devoted to reviewing, generalizing, and commenting on their model.

Because the original derivation of their result is not totally satisfactory, a rederivation of their result is provided. The alternative derivation is somewhat clearer. One conclusion that emerges from this derivation has the important implication that the inherent market equilibrium balanced equation derived is in the same general form as that of the Capital Asset Pricing Model. The difference is that the Black-Scholes balance equation implies that the market prices securities in relation to their own variances but not their systematic risks. The reason given is that black-Scholes assume a single stock in their model. As a consequence, the unsystematic portion of the risk of the securities in their model cannot be diversified away.

In order to investigate whether the Black-Scholes formula is valid even in a market in which there are many stocks, a generalization of their model is given. The contention is that it is not possible to determine an investor's most desirable course of action by considering the purchase of one option in isolation from others. The investor can, at any time, undertake a number of different acts, such as the purchase of a single option vs. the purchase of a portfolio of options. He must determine a rational strategy when confronted with an entire menu of choices, and he must determine the value of an option in the context of this rational strategy. The development of this generalization is based upon the construction that the set of stock prices in the model follows a multi-dimensional diffusion process. the technique used to bring about market equilibrium is the seminal "continuous hedging" technique. The result of this generalization is quite surprising at first glance. The resulting formula is identical to that of Black-Scholes in the single security world. The implication is that within the framwork of this generalization, an option is priced indifferently to the portfolio position an investor may take. In other words, diversification is irrelevant. In order that this counter-intuitive result can be understood, it is necessary initially to explain why the Black-Scholes approach assumes a complete market. It is also worthwhile to note another point which emerges in the generalization: that the market equilibrium balance equation derived in this generalization is identical to that of the continuous-time CAPM. In other words, the market prices securities according to their systematic levels of risk. This finding, in my opinion, confirms the "law of one price."

The formal proof that the Black-Scholes model and its generalization in the multi-security world implicitly assume a complete market has been given very recently by the joint efforts of Kreps and Harrison-Pliska. In this dissertation only an intuitive explanation based upon their proof is attempted, that is, the complete market assumption is a direct result of the continuous-hedging technology employed in the Black-Scholes model and its generalization. this technology enables an investor to duplicate the return distribution of any option by a continuously adjusted portfolio of its underlying asset and a risk-free discount bond. Such an act ensures that any option in their world is attainable and thus redundant. This condition in turn implies that the market assumed in their model is complete. Because most existing option-pricing models are extensions of their model, it is also true that these extensions likewise assume complete markets.

One unique feature of the complete market approach to option-pricing is that the resulting formula is preference-free. This means that the price of an option can be determined without invoking the preferences of investors. Consequently, the factor which reflects these preferences, i.e., the expected rate of return, is irrelevant. However, the price of the underlying security as one of the determining variables of the price of the option itself reflects the preferences of investors. In the work here, it is demonstrated that it is the complete market assumption which causes the preference-free pricing result and further that the so-called "risk neutral valuation relationship" technique is applicable to the valuation of any derived security in a complete market.

After understanding that a complete market implies a preference-free pricing result, the counter-intuitive result that diversification is irrelevant in the Black-Scholes approach can be explained. The effect of diversification, being reflected only in the value of the expected rate of return of a security, is irrelevant if the pricing of that security does not involve the expected rate of return as a result of the preference-free property.

After an investigation of the paradox, the discussion shifts to pricing options under incomplete market conditions. A few existing models which do not assume a complete market are presented. It is emphasized that the development of these models is not based upon the rationale of treating options as market-completing securities. Therefore the pricing of options is developed under the standard Capital Asset Pricing Model. The model does not necessarily make a complete market assumption.

The option pricing result developed under the standard CAPM is shown to contain one more factor than the Black-Scholes result. This factor accounts for the fact that different investors have different preferences. In essence, this factor is the expected rate of return on the underlying asset. The theoretical drawback of this model is the assumption that the price of the underlying asset follows an Arithmetic Brownian Motion. As a consequence, the resulting formula implies both positive probability of negative price for the underlying asset and an option price greater than the price of its underlying asset for a sufficiently long time to maturity.

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