Optimal Weighted l_p Norm Parameters For Facilities Layout Distance Characterizations

Robert F. Love
McMaster University

Paul D. Dowling
McMaster University

FACULTY OF BUSINESS

McMASTER UNIVERSITY

HAMILTON, ONTARIO

Research and Working Paper Series No. 187
May, 1982
Reprinted April, 1983

Optimal Weighted ℓ_p Norm Parameters For Facilities Layout Distance Characterizations

Robert F. Love*
Paul D. Dowling*

The distances used in certain types of industrial, office or street grid layouts are often assumed to be rectangular. Because of the "doubling back" effect caused by finite block and facility sizes, actual distances travelled are often much greater than indicated by the ℓ_1 norm. In this study the weighted ℓ_p norm is fitted to a certain commonly occurring class of layout patterns. The optimal best-fit parameters are reported in addition to other observations which should be useful to users of distance models. The results give strong support for the use of the ℓ_1 norm in determining optimal facility locations when travel distances are rectangular and doubling back occurs. However, the ℓ_1 norm is not appropriate for modelling actual travel distances under the same circumstances.

* McMaster University, Hamilton, Ontario

Optimal Weighted ℓ_p Norm Parameters For Facilities Layout Distance Characterization

Robert F. Love
Paul D. Dowling

1. Introduction

A great deal of research has been carried out dealing with the characterization and solution of floor layout problems [1,7]. One of the most common assumptions made in the development of floor layout models for buildings such as factories, warehouses or offices, is that travel distance between pairs of points can be modelled by the ℓ_1 norm (also commonly referred to as the rectangular or rectilinear distance). The ℓ_1 norm assumption is widely used and applications also include modelling rectangular street patterns and piping and wiring networks where the conduits must follow the orthogonal outlines of building structures. It has been the experience of one of the authors that in practical applications the distances between point pairs are often greater than would be indicated by the ℓ_1 norm. This extra travel distance may be caused by the necessity to "double back". Doubling back may occur in practice due to the existence of finitely-sized blocks or bays into which a floor is divided. Doubling back may also be caused by the necessity to travel between closed departments or areas. By "closed area" we mean, for example, a walled-off office area or production department with an entrance on one side. In order to travel from one closed area to another the traveller may find that he must travel in directions which are different from those which would be travelled if the assumption of perfect rectangular distances prevailed.

Figure 1 illustrates a typical industrial floor layout* with ten of the

----
*This is the current layout of the main manufacturing area of ITW Canada Inc., Toronto.

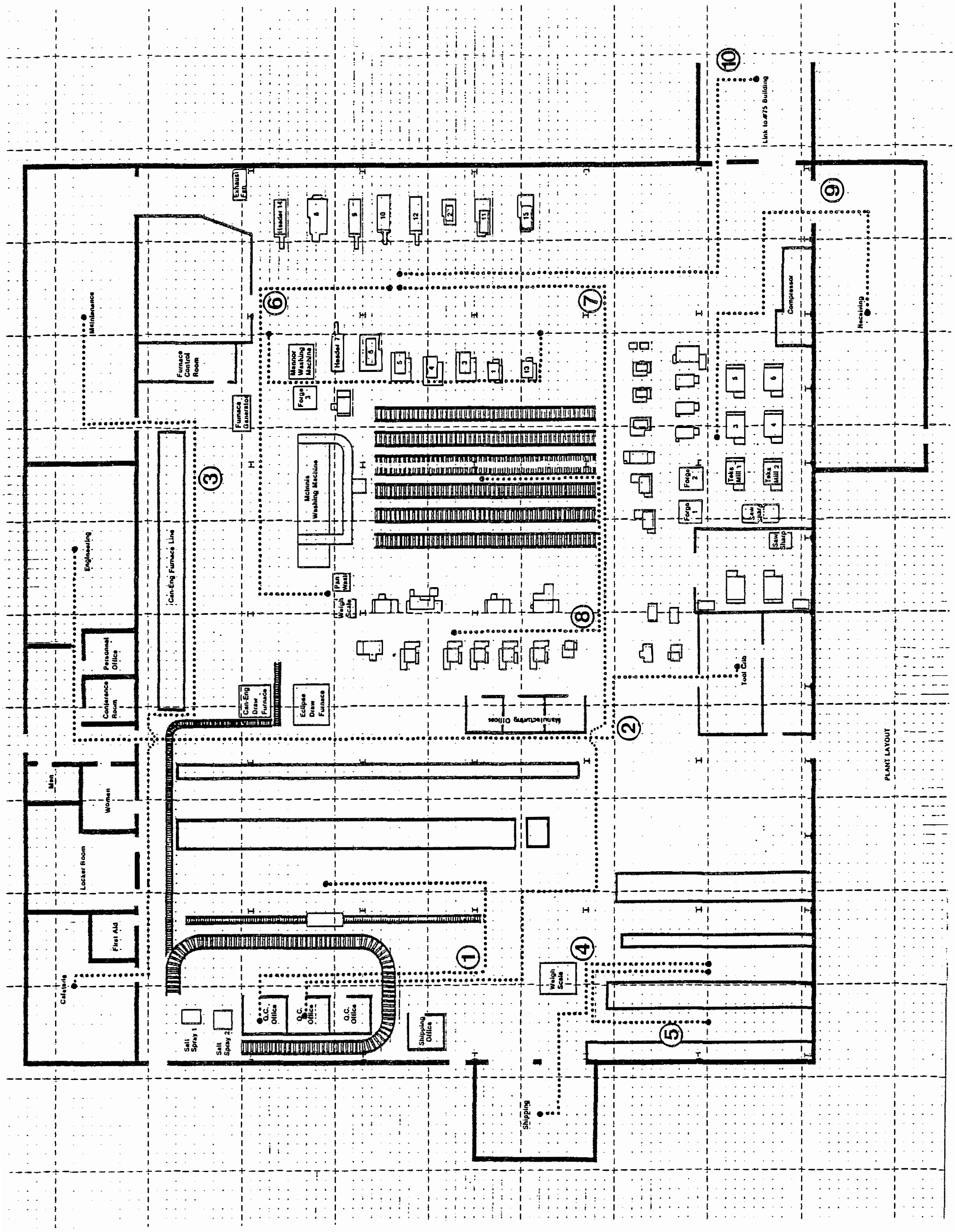

Figure 1 Floor Layout and Inter-Departmental Routes

more frequently travelled inter-departmental routes marked on it. While the flow of materials from machine to machine within a department or machine grouping is handled quite efficiently, it is the movement of parts and people between departments which can create travel distances greater than those specified by the ℓ_1 norm. In this layout, machine groupings can usually be characterized as departments with only one or two access points. This occurs because of the machine locations and the space occupied by completed parts, work in process, and buffer stock located adjacent to the machines. For example, routes 6 and 7 in Figure 1 are from a machine grouping which can be dealt with as if it is a department with two doorways. Table 1 gives the rectangular and actual distances for each route, as well as the excess distance between the actual and rectangular distances that is due to doorway locations and obstacles on the shop floor. In most cases, it is the doubling back which accounts for the increased distances travelled, and thus the extra distance due to doubling back is mainly a function of the lengths of the department sides and not the total travel distance.

Route	Description	Distance Between Midpoints (ft)		Excess Distance Travelled Due To	
		Rectangular	Actual	Doorway Locations	Shop Floor Obstacles
1	Quality Control to Finished Goods Inventory	42	110	68	0
2	Tool Storage to Engineering	168	200	32	0
3	Maintenance to Cafeteria	144	196	28	24
4	Finished Goods Inventory to Shipping	68	68	0	0
5	Finished Goods Inventory Records to Finished Goods Inventory	12	56	44	0
6	Machining Dept. to Weigh/Wash Dept.	82	104	22	0
7	Quality Control to Machining Dept.	176	248	60	12
8	Work in Process Inventory to Trimming	38	86	48	0
9	Receiving Dept. to Milling/Forging	58	102	44	0
10	Raw Material Inventory to Machining Dept.	110	110	0	0
	Total	898	1280	364	36

TABLE 1 Routes and Distances For Floor Layout in Figure 1

The phenomenon of doubling back is not confined to travel distance on floor layouts. It may also occur on street grids due to one-way streets or restrictive traffic rules (for example, no left turns at certain intersections). Doubling back also occurs in piping and wiring circuits since it is often impossible to have conduits follow perfectly rectangular distance paths.

Various distance norms have been evaluated to obtain models with few parameters which accurately describe actual road distances. The ℓ_{kp} norm has proved to be valuable in road distance modelling [3,4]. When doubling back is not present, the ℓ_1 norm provides an ideal fit for travel along rectangular paths. However, in situations which require doubling back, the actual distances are greater than rectangular distances and the ℓ_1 norm may no longer be suitable to model these. The purpose of this study is to investigate the appropriateness of the ℓ_{kp} and ℓ_1 norms when the distances between departments on a plant or office floor layout are equal to or greater than rectilinear distances. The model used here to describe the distance between departments i and j with coordinates $x_i = (x_{i1}, x_{i2})$ and $x_j = (x_{j1}, x_{j2})$ respectively is $\ell_{k,p} = k\,[|x_{i1}-x_{j1}|^p + |x_{i2}-x_{j2}|^p]^{1/p}$ where k and p are the parameters to be fitted. We also refer to this model as the $\ell_{k,p}$ norm or the $\ell_{k,p}$ distance function. The results obtained from using the ℓ_{kp} norm will then be compared with the ℓ_1 norm, which has commonly been used to model floor layouts.

In real situations, office clusters or production areas may have entrances on more than one side. In the layouts studied, (Figure 1 is typical), the majority of departments had a single access. Therefore, it was decided to adopt the convention that a department would have only one entrance, but the entrance to that department would be randomly assigned to one of the four sides with equal probability.

Unlike previous studies [3,4] in which several distance functions were fitted to empirical data, the research reported here concentrates on fitting the ℓ_1 and $\ell_{k,p}$ norms to a set of computer-generated floor layout representations. The weighted ℓ_p norm was chosen for the following reasons.

1. The distances being modelled were chosen from surfaces which were assumed to be flat. In the previous studies of road distances [3,4], city pairs were chosen from fairly large geographical areas. This meant that the existence of natural topographical formations such as mountains, lakes and/or the earth's curvature were factors to be incorporated in the distance function. This suggested the use of distance models such as the spherical distance norm.

2. Previous empirical studies have indicated that, unless more than two parameters are to be determined, the $\ell_{k,p}$ model is as good as or superior to other distance models [3,4].

3. The $\ell_{k,p}$ distance function has desirable convexity properties and is widely used in facility location models [5,6] and other applications (for descriptions of three other applications see reference 3).

2. Design Of The Study

Figure 1 is representative of several floor layouts examined, and in most cases the departments are rectangularly shaped and have a single entrance. Furthermore, in Figure 1, including the machine groupings as departments, the departments cover roughly 55% of the total floor area. These observations, along with the fact that the doubling back effect is mainly a function of departmental side length, form the basis for creating sample floor layouts.

A computer program was written to assign rectangularly shaped departments to a floor layout. The midpoint of each department was randomly generated. The length and width of each department was also randomly generated from a specified minimum and maximum allowable dimension. Various minimum and maximum allowable department sizes were used in the study. In one set of tests carried out, the department sizes were systematically increased in order to examine the effect of this expansion on the parameters of the fitted $\ell_{k,p}$ function. The department areas ranged from 0.002% of the total area (to represent points on a plane) up to approximately 60% of the total floor area.

After each department's dimensions were generated, the location was examined to see if it extended outside the floor layout boundaries. Any protrusion was corrected by a translation to map the department length(s) onto the boundary line(s). Then the department location was tested against all other department locations. If there was any overlap with an existing location, a new location was generated. After a department location passed this overlap check, a doorway was randomly assigned to the midpoint of a side. Figure 2 shows the convention used in assigning the doorway locations.

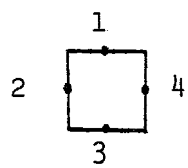

Figure 2. Doorway Location Convention

After the desired number of departments was generated, the inter-departmental travel distances between all pairs of departments were calculated. In the model, when doubling back occurred in a trip from one department to another, the appropriate department lengths were added to the rectangular distance to obtain the actual distance travelled. Thus actual distances were simulated by the model. All travel was assumed to take place parallel to the orthogonal axes since this corresponds to what commonly occurs in industrial plants, warehouses, offices and many city street grids. These distances were then used as the data inputs for the distance function fitting calculations. The parameters (one or both of k and p) were computed using two measures of merit.

The first measure involves minimization of the sum of absolute deviations given by:

$$AD = \sum_{\text{all sample pairs}} |d(q,r)-A(q,r)|,$$

where

A(q,r) = actual distance between departments q and r; and

d(q,r) = estimated distance between departments q and r.

The implication of using this criterion is that a function must estimate greater actual distances relatively more accurately than shorter distances. This can be seen by noting that a 50 per cent error in estimating a 10 unit actual distance is relatively unimportant when compared to a 50 per cent error in estimating a 100 unit actual distance, although the errors are proportionately the same. The reasonableness of such a criterion rests on the assumption that a user may be more concerned with absolute deviations of estimates than with proportional deviations.

The second measure of merit used in the study involves minimizing the sum of squares given by:

$$SD = \sum_{\text{all sample pairs}} \left[\frac{d(q,r)-A(q,r)}{\sqrt{A(q,r)}}\right]^2.$$

This criterion is more sensitive than the first to large values of $|d(q,r)-A(q,r)|$ in relation to A(q,r) [4]. Division by $\sqrt{A(q,r)}$ accomplishes a certain sensitivity in the criterion such that shorter actual distances are to be estimated at least as accurately (in a relative sense) as greater actual distances. Thus the second criterion measures goodness-of-fit in a significantly different way from the first criterion.

The parameters k and p of the distance function were defined as those which best fit the given criterion for the given sample. This allows information to be gained about how the "best" parameters change with changes in criteria and layout patterns. For example, if $\ell_{k,p}$ has optimal parameters p* and k* which minimize AD with respect to p and k, these may not be the parameters which minimize SD. Computer programs were written to perform exhaustive searches for the optimal parameters in the intervals in which they were known to occur. Sample floor layouts with 15 departments were generated with minimum and maximum department lengths and widths of 4 and 12 units respectively. The locations and dimensions for the departments are displayed in Table 2 and the distances between departments are given in Table 3. A random sample of 50 department pairs was selected and the actual distances (Table 2) were compared with the predicted distances to determine the AD and SD best-fit value differences.

Department	Location	Dimensions Length x Width	Doorway Location
A	(25,61)	6 x 8	1
B	(79,99)	7 x 12	4
C	(73,2)	12 x 4	4
D	(5,179)	9 x 8	4
E	(91,172)	11 x 8	3
F	(45,174)	4 x 9	4
G	(29,157)	5 x 6	1
H	(88,171)	9 x 7	1
I	(88,153)	6 x 9	2
J	(23,149)	6 x 11	2
K	(4,177)	4 x 4	2
L	(50,150)	7 x 10	1
M	(38,176)	4 x 6	3
N	(5,116)	10 x 11	1
O	(13,61)	5 x 10	4

Table 2 15 Departments on a 100 x 200 Floor Layout

	B	C	D	E	F	G	H	I	J	K	L	M	N	O
A	99	127	138	177	137	106	180	155	96	141	124	128	86	20
B		110	161	85	116	121	88	63	119	164	97	125	109	111
C			257	188	212	217	191	166	215	260	193	221	205	131
D				101	49	46	91	109	48	16	74	42	72	131
E					56	77	4	28	97	104	63	65	142	189
F						37	46	64	57	52	29	13	102	149
G							80	69	26	49	34	28	71	118
H								31	100	94	66	55	145	192
I									75	112	41	73	120	167
J										51	44	48	51	98
K											77	45	66	134
L												38	89	136
M													93	140
N														79

Table 3 Actual Distances Between Departments

3. Computation Results and Discussion

One and Two Parameter Best-Fit Results

Since one of the purposes of the study was to determine how the unweighted ℓ_p distance model, i.e., the $\ell_{1,p}$ model, was affected by typical floor layout patterns, it was decided to fit the $\ell_{1,p}$ model to sets of test data. In these one parameter fits, the AD and SD values were computed for p in the interval [0.5,1.5] (computational experience indicated that optimum p values would not be found outside this interval). The total search procedure used step sizes (the amount by which p was incremented) of 0.1, 0.03, and 0.02. The best fit values of p according to tests 1 and 2 for the 3 step sizes are shown in Table 4, along with the results for the $\ell_{1,1}$ norm.

p Value Test 1-AD	Test 2-SD	Best Fit Value Differences Test 1-AD	Test 2-SD	Step Size
0.9	0.9	5.058×10^2	6.031×10	0.1
0.92	0.92	4.957×10^2	6.022×10	0.03
0.92	0.92	4.957×10^2	6.022×10	0.02
1.00	1.00	6.290×10^2	8.674×10	

Table 4 One Parameter Fit, k = 1

In the determination of best fits for the $\ell_{k,p}$ model, both parameters were allowed to vary over prescribed intervals using specified step sizes. The results shown in Table 5 are for $k \varepsilon [0.5,1.5]$ and $p \varepsilon [0.5,1.5]$ with step sizes of 0.1, 0.03, and 0.02.

k Value Test 1	Test 2	p Value Test 1	Test 2	Best Fit Value Differences Test 1-AD	Test 2-SD	Step Size
1.1	1.1	1.1	1.1	4.819×10^2	5.534×10	0.1
1.07	1.07	1.04	1.01	4.644×10^2	5.412×10	0.03
1.06	1.08	1.02	1.04	4.641×10^2	5.370×10	0.02

Table 5 Two Parameter Fit

With the layout representations created by the methodology in this study, for a single parameter fit (the best value of p for k=1), the value of p varied between 0.90 and 0.92 as shown in Table 4. It is not surprising that all the p values are less than unity since all sample distances were equal to or greater than their rectangular distance counterparts.

From the results of fitting the $\ell_{k,p}$ distance function (Table 5), it is interesting to note that the best-fit p values remain close to unity whereas the k values tend to be larger than unity. This could indicate that the k values rather than the p values account for the larger-than-rectangular distances which are being modelled. We investigate this possibility in the next section.

Effect of Varying Department Sizes

The "best-fit" values of k and p in the $\ell_{k,p}$ norm must be unity when each department is represented by a single point on the plane. What was not known was the effect on the best-fit values of k and p when department sizes are increased relative to the total floor space. For experimental purposes the layout with 15 departments was chosen and 12 layouts were generated randomly using various minimum and maximum department lengths. The results for one and two parameter best-fits for a step size of 0.02 are shown in Table 6, along with the best fit differences for the $\ell_{1,1}$ norm. For each layout, two samples of 50 randomly selected department location pairs were used to obtain one and two parameter best-fit models labelled (a) and (b) in Table 6.

The data in Table 6 indicate that the optimal p value for (k,p) fits is usually close to unity. The most striking exception to this was the best fit parameter values found for the case involving department lengths between 12 and 36, as shown in Table 6. Under the first goodness-of-fit measure (sum of absolute deviations) the optimum (k,p) fit found was (0.74, 0.58). This fit was 6.28% better than the (k,1) best-fit, 3.04% better than the (1,p) best-fit, and 23.10% better than the (1,1) fit.

For the one parameter (p) fit, it is quite apparent that, as the department sizes shrink, the p values increase to approach p = 1. A similar pattern occurs with the two parameter goodness-of-fit data. For the smallest department size, both k and p values are 1.02 for both goodness-of-fit criteria. It would seem logical to theorize that, for p=1, the best-fit values of k would increase as department sizes increase. The reason for this is that as department sizes increase relative to a fixed total layout space, the "doubling back" effect will increase. Similarly, for k=1, the best fit values of p should decrease as department sizes increase. The decreasing values of p, of course, reflect larger average distances being modelled.

Dept. Lengths Min	Max	Parameter(s) Fitted		k Value Test1	k Value Test2	p Value Test1	p Value Test2	Best Fit Differences Test1-AD	Best Fit Differences Test2-SD
1	2	p (k=1)	(a)	1.00	1.00	0.98	0.98	53.25	3.136
			(b)	1.00	1.00	0.98	0.98	49.17	2.883
		k (p=1)	(a)	1.00	1.02	1.00	1.00	60.00	3.157
			(b)	1.00	1.02	1.00	1.00	58.00	3.065
		k,p	(a)	1.02	1.02	1.02	1.02	51.85	2.845
			(b)	1.02	1.02	1.02	1.02	48.11	2.743
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	60.00	3.157
			(b)	1.00	1.00	1.00	1.00	58.00	3.065
2	4	p (k=1)	(a)	1.00	1.00	0.96	0.96	105.2	8.362
			(b)	1.00	1.00	0.96	0.96	77.65	3.177
		k (p=1)	(a)	1.02	1.02	1.00	1.00	98.54	7.747
			(b)	1.02	1.02	1.00	1.00	88.34	3.333
		k,p	(a)	1.02	1.04	1.00	1.02	98.54	7.512
			(b)	1.00	1.00	0.96	0.96	77.65	3.177
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	128.0	10.62
			(b)	1.00	1.00	1.00	1.00	121.0	5.974
3	6	p (k=1)	(a)	1.00	1.00	0.96	0.94	183.3	10.81
			(b)	1.00	1.00	0.96	0.94	164.4	12.20
		k (p=1)	(a)	1.04	1.04	1.00	1.00	173.6	9.354
			(b)	1.04	1.04	1.00	1.00	157.2	11.22
		k,p	(a)	1.06	1.04	1.06	1.00	163.4	9.354
			(b)	1.04	1.04	1.02	1.00	148.3	11.22
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	208.0	16.26
			(b)	1.00	1.00	1.00	1.00	192.0	17.27
4	8	p (k=1)	(a)	1.00	1.00	0.94	0.94	190.8	20.69
			(b)	1.00	1.00	0.94	0.92	180.9	18.69

Table 6 One and Two Parameter Fits For 15 Departments, step size 0.02

Dept. Lengths Min	Max	Parameter(s) Fitted		k Value Test1	Test2	p Value Test1	Test2	Best Fit Differences Test1-AD	Test2-SD
4	8	k (p=1)	(a)	1.04	1.04	1.00	1.00	182.8	20.81
			(b)	1.04	1.04	1.00	1.00	175.1	16.62
		k,p	(a)	1.02	1.02	0.98	0.98	181.5	20.39
			(b)	1.04	1.06	1.00	1.02	175.1	16.02
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	243.0	26.95
			(b)	1.00	1.00	1.00	1.00	221.0	19.96
4	12	p (k=1)	(a)	1.00	1.00	0.94	0.92	254.9	26.59
			(b)	1.00	1.00	0.92	0.90	264.7	39.71
		k (p=1)	(a)	1.04	1.06	1.00	1.00	236.3	24.08
			(b)	1.06	1.06	1.00	1.00	252.1	37.05
		k,p	(a)	1.06	1.06	1.02	1.02	234.3	24.02
			(b)	1.06	1.08	1.00	1.02	252.1	36.77
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	303.0	38.63
			(b)	1.00	1.00	1.00	1.00	349.0	56.21
6	15	p (k=1)	(a)	1.00	1.00	0.88	0.86	399.8	65.73
			(b)	1.00	1.00	0.82	0.82	405.4	72.69
		k (p=1)	(a)	1.10	1.12	1.00	1.00	365.1	54.79
			(b)	1.12	1.14	1.00	1.00	384.0	69.04
		k,p	(a)	1.18	1.22	1.12	1.16	356.6	51.69
			(b)	1.10	1.12	0.94	0.96	382.2	68.01
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	696.0	140.6
			(b)	1.00	1.00	1.00	1.00	696.0	153.0
7	20	p (k=1)	(a)	1.00	1.00	0.82	0.80	611.5	104.5
			(b)	1.00	1.00	0.76	0.80	568.4	100.0
		k (p=1)	(a)	1.16	1.16	1.00	1.00	525.1	85.12
			(b)	1.18	1.16	1.00	1.00	534.3	85.29

Table 6 One and Two Parameter Fits For 15 Departments, step size 0.02

Dept. Lengths Min	Max	Parameter(s) Fitted		k Value Test1	Test2	p Value Test1	Test2	Best Fit Differences Test1-AD	Test2-SD
7	20	k,p	(a)	1.18	1.20	1.04	1.06	519.2	84.23
			(b)	1.14	1.18	0.94	1.02	532.2	85.26
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	898.0	202.8
			(b)	1.00	1.00	1.00	1.00	837.0	191.3
8	24	p (k=1)	(a)	1.00	1.00	0.90	0.86	501.0	104.9
			(b)	1.00	1.00	0.84	0.82	521.5	99.67
		k (p=1)	(a)	1.06	1.10	1.00	1.00	510.9	97.76
			(b)	1.12	1.14	1.00	1.00	479.1	89.04
		k,p	(a)	0.94	1.14	0.82	1.08	498.8	96.77
			(b)	1.14	1.16	1.02	1.04	477.4	88.74
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	546.0	138.9
			(b)	1.00	1.00	1.00	1.00	760.0	169.5
10	30	p (k=1)	(a)	1.00	1.00	0.84	0.84	766.2	193.5
			(b)	1.00	1.00	0.88	0.82	726.0	149.2
		k (p=1)	(a)	1.10	1.12	1.00	1.00	702.1	180.1
			(b)	1.12	1.14	1.00	1.00	658.7	134.8
		k,p	(a)	1.14	1.16	1.08	1.08	692.2	179.4
			(b)	1.16	1.16	1.10	1.04	639.3	134.6
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	831.0	251.0
			(b)	1.00	1.00	1.00	1.00	876.0	223.7
12	30	p (k=1)	(a)	1.00	1.00	0.76	0.76	826.4	223.3
			(b)	1.00	1.00	0.78	0.76	902.5	285.9
		k (p=1)	(a)	1.18	1.20	1.00	1.00	791.1	218.5
			(b)	1.18	1.20	1.00	1.00	880.7	279.0
		k,p	(a)	1.18	1.14	1.00	0.92	791.1	216.7
			(b)	1.18	1.16	1.00	0.99	880.7	278.0

Table 6 One and Two Parameter Fits For 15 Departments, step size 0.02

Dept. Lengths Min	Max	Parameter(s) Fitted		k Value Test1	Test2	p Value Test1	Test2	Best Fit Differences Test1-AD	Test2-SD
12	30	k=1, p=1	(a)	1.00	1.00	1.00	1.00	1095.	362.5
			(b)	1.00	1.00	1.00	1.00	1089.	414.1
12	36	p (k=1)	(a)	1.00	1.00	0.82	0.82	620.1	125.4
			(b)	1.00	1.00	0.80	0.80	739.2	125.7
		k (p=1)	(a)	1.16	1.14	1.00	1.00	580.1	112.4
			(b)	1.16	1.16	1.00	1.00	764.7	126.2
		k,p	(a)	1.22	1.20	1.14	1.10	567.9	111.3
			(b)	0.74	1.08	0.58	0.90	716.7	124.0
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	732.0	192.1
			(b)	1.00	1.00	1.00	1.00	932.0	231.6
20	36	p (k=1)	(a)	1.00	1.00	0.76	0.74	899.2	264.3
			(b)	1.00	1.00	0.70	0.68	753.2	189.6
		k (p=1)	(a)	1.20	1.24	1.00	1.00	868.8	250.2
			(b)	1.30	1.32	1.00	1.00	698.1	151.8
		k,p	(a)	1.18	1.24	0.96	1.00	867.4	250.2
			(b)	1.30	1.30	1.02	0.98	687.1	151.7
		k=1, p=1	(a)	1.00	1.00	1.00	1.00	1590.	495.5
			(b)	1.00	1.00	1.00	1.00	1132.	420.5

Table 6 One and Two Parameter Fits For 15 Departments, step size 0.02

In order to test these conjectures, statistical tests were used to check for trend effects in the sequences of p and k values generated in Table 6. The following hypothesis was tested:

H_0: independence of p or k value and department size,

versus the alternate hypothesis:

H_1: a trend exists (upward or downward).

One of the two samples was randomly selected for each of the 12 layouts and the p and/or k values for Tests 1 and 2 were recorded as shown in Table 7(a). The corresponding best-fit differences are given in Table 7(b).

Dept. Size Rank	best-fit k value (p=1) Test 1	Test 2	best-fit p value (k=1) Test 1	Test 2	best-fit p value (for k,p best fit) Test 1	Test 2	best-fit k value (for k,p best fit) Test 1	Test 2
1	1.00	1.02	0.98	0.98	1.02	1.02	1.02	1.02
2	1.02	1.02	0.96	0.96	1.00	1.02	1.02	1.04
3	1.04	1.04	0.96	0.94	1.02	1.00	1.06	1.04
4	1.04	1.04	0.94	0.94	1.00	1.02	1.04	1.06
5	1.04	1.06	0.92	0.90	1.02	1.02	1.06	1.08
6	1.12	1.14	0.88	0.86	1.12	1.16	1.10	1.12
7	1.16	1.16	0.76	0.80	1.04	1.06	1.18	1.20
8	1.12	1.14	0.84	0.82	1.02	1.04	1.14	1.16
9	1.12	1.14	0.88	0.82	1.08	1.08	1.14	1.16
10	1.18	1.14	0.76	0.76	1.00	0.92	1.18	1.16
11	1.16	1.16	0.80	0.80	1.14	1.10	0.74	1.08
12	1.30	1.32	0.70	0.68	1.02	0.98	1.30	1.30

Table 7(a) p and k Values For Increasing Department Sizes

Dept. Size Rank	Best-Fit Differences p(k=1) AD	SD	k(p=1) AD	SD	k,p AD	SD
1	49.17	2.883	58.0	3.065	48.11	2.743
2	77.65	3.177	88.34	3.333	77.65	3.177
3	183.3	10.81	173.6	9.354	163.4	9.354
4	190.8	20.69	182.8	20.81	181.5	20.39
5	254.9	26.59	236.3	24.08	234.3	24.02
6	405.4	76.29	384.0	69.04	382.2	68.01
7	611.5	104.5	525.1	85.12	519.2	84.23
8	501.0	104.9	510.9	97.76	498.8	96.77
9	726.0	149.2	658.7	134.8	639.3	134.6
10	826.4	223.3	791.1	218.5	791.1	216.7
11	739.2	125.7	764.7	126.2	716.7	124.0
12	753.2	189.6	698.1	151.8	687.1	151.7

Table 7(b) Best-Fit Differences For Increasing Department sizes

A non-parametric test as described by Lehman [1] was used to test for an upward or downward trend in the values of p and k and the best-fit differences as the department sizes increased. This test is based on the statistic D (or D^* in the case of tied ranks), where D and D^* are given by

$$D = \sum_{i=1}^{N} (T_i - i)^2, \text{ and } D^* = \sum_{i=1}^{N} (T_i^* - i)^2,$$

where N is the number of departments,

i is the rank of a department,

T_i is the rank of the response (p, k or Difference), and

T_i^* is the midrank in the case of ties.

The ranks and $D(D^*)$ values are given in Table 8.

For an upward trend, small values of D lead to rejection of H_o. For a downward trend, large values of D lead to rejection of H_o. For $N \geq 12$, $Pr(D \leq d) = \alpha$ and $Pr(D \geq d) = \alpha$ can be calculated using a normal approximation. Then $Z = \frac{D - E_{H_o}(D)}{\sqrt{V_{H_o}(D)}}$ is approximately N(0,1), where

$$E_{H_o}(D) = \frac{N^3 - N}{6} \text{ and } V_{H_o}(D) = \frac{N^2(N+1)^2(N-1)}{36}.$$

For N=12, $E_{H_o}(D) = 286$ and $V_{H_o}(D) = 7436$. For $\alpha = .05$, $Pr(D \leq d) = Pr\left[Z \leq \frac{D - E_{H_o}(D)}{\sqrt{V_{H_o}(D)}}\right] = .05$. If $Z \leq -1.645$ then reject H_o. For $Pr(D \geq d) = .05$, reject H_o if $Z \geq 1.645$.

Since D takes on even integer values only, a continuity correction should be utilized, so that

$$Pr\left[Z \leq \frac{(D+1) - E_{H_o}(D)}{\sqrt{V_{H_o}(D)}}\right] = .05 \text{ and } Pr\left[Z \geq \frac{(D-1) - E_{H_o}(D)}{\sqrt{V_{H_o}(D)}}\right] = .05.$$

For the best-fit differences we first test the largest D value, D = 12, to see if the resulting Z score leads to the rejection of H_o. If this occurs, then all D values less than 12 will also reject H_o. For D = 12,

Rank of Response (T_i or T_i^*) i	1	2	3	4	5	6	7	8	9	10	11	12	D(D^*)
p (k=1) Test 1	12	10.5	10.5	9	8	6.5	2.5	5	6.5	2.5	4	1	545.5*
p (k=1) Test 2	12	11	9.5	9.5	8	7	3.5	5.5	5.5	2	3.5	1	556.5*
p (p,k) Test 1	6	2	6	2	6	11	9	6	10	2	12	6	174*
p (p,k) Test 2	5.5	5.5	3	5.5	5.5	12	9	8	10	1	11	2	257*
k (p=1) Test 1	1	2	4	4	4	7	9.5	7	7	11	9.5	12	17.5*
k (p=1) Test 2	1.5	1.5	3.5	3.5	5	7.5	10.5	7.5	7.5	7.5	10.5	12	24.5*
k (p,k) Test 1	2.5	2.5	5.5	4	5.5	7	10.5	8.5	8.5	10.5	1	12	123*
k (p,k) Test 2	1	2.5	2.5	4	5.5	7	11	9	9	9	5.5	12	50*
AD Differences p (k=1)	1	2	3	4	5	6	8	7	9	12	10	11	8
SD Differences p (k=1)	1	2	3	4	5	6	7	8	10	12	9	11	12
AD Differences p,k	1	2	3	4	5	6	8	7	9	12	11	10	10
SD Differences p,k	1	2	3	4	5	6	7	8	10	12	9	11	10
AD Differences k (p=1)	1	2	3	4	5	6	8	7	9	12	11	10	10
SD Differences k (p=1)	1	2	3	4	5	6	7	8	10	12	9	11	10

Table 8 Ranks and D Statistic For p,k Values And Best-Fit Differences

$Z = \dfrac{12+1-286}{86.23} = -3.17$. Hence we reject H_0 and accept H_1; there is an upward trend in the best-fit differences as department sizes increase. To calculate the normal approximation for D^*,

$$E_{H_0}(D^*) = \frac{N^3-N}{6} - \frac{1}{12}\sum_{i=1}^{e}(d_i^3-d_i), \text{ and}$$

$$V_{H_0}(D^*) = \frac{N^3(N+1)^2(N-1)}{36}\left[1 - \frac{\sum_{i=1}^{e}(d_i^3-d_i)}{N^3-N}\right],$$

where e is the number of tied groups and $d_1, d_2, \ldots, d_e$ are the number of elements in the first, second,..., e^{th} tied group respectively. The mean and variance and resulting Z value are shown in Table 9 for the eight D^* values.

For the $\ell_{1,p}$ model, reject H_0 and accept the hypothesis that there is a downward trend in the p values as department sizes increase (0.05 significance level). For the $\ell_{k,1}$ model, reject H_0 and accept the hypothesis that there is

	D^*	e	d_1	d_2	d_3	d_4	$E_H\ (D^*)$	$V_H\ (D^*)$	Z
p (k=1) Test 1	545.5	3	2	2	2		284.5	7358	3.04
p (k=1) Test 2	556.5	3	2	2	2		284.5	7358	3.04
p (p,k) Test 1	174	2	3	5			274	6812	-1.21
p (p,k) Test 2	257	1	4				281	7176	-0.28
k (p=1) Test 1	17.5	3	3	3	2		281.5	7202	-3.11
k (p=1) Test 2	24.5	4	2	2	4	2	279.5	7098	-3.02
k (p,k) Test 1	123	4	2	2	2	2	284	7332	-1.88
k (p,k) Test 2	50	3	2	2	3		283	7280	-2.73

Table 9 Z Scores For k,p Values

an upward trend in the k values as department sizes increase (0.05 significance level).

For the $\ell_{k,p}$ model, reject H_0 and accept the hypothesis that there is an upward trend in the k values as department sizes increase. However, for this model we cannot accept H_1; i.e., that there is a downward or upward trend in p values as department sizes increase.

It is of special interest to observe the behaviour of the sequence of p values obtained for the best-fit k and p values as department size increases. In practice when determining an optimal facility location the practitioner often assumes the value p=1 (probably for numerical expediency since location models for p=1 are easily computed). We are now able to comment on whether this common assumption is sound or not. If there is no discernible trend away from p=1 and there is an increasing trend in the k values for increasing department size, it means that the increased average distance caused by doubling back is being accounted for by the increasing k values. We observe that there is no discernible trend away from p=1, and there is an increasing trend in the k values as department size increases. We conclude that the increasing average distance is being accounted for by the increasing k values rather than decreasing p values.

In order to further verify this rather surprising result we decided to test the one parameter model (k,1) to see if it would give fits as good as the two-parameter model. In order to compare the two models, the percentage difference between the best-fit differences were calculated for the 24 samples as:

$$\% \text{ difference} = \frac{\text{Best-fit Difference(k)} - \text{Best Fit Difference(k,p)}}{\text{Best Fit Difference(k)}} \times 100\%$$

The results for Test1-AD and Test2-SD are displayed in Table 10.

The best-fit differences for the two-parameter model were always less than or equal to the best-fit differences for the one parameter model. However, these percentage differences were very small. The medians for the percentage differences were 1.5% and 1.0% for the Test 1 - AD and Test 2 - SD samples respectively.

The hypothesis H_0 was tested against the alternative hypothesis H_1 using the D statistic, where H_0 and H_1 are given by

H_0: independence of department size and percentage differences,

H_1: downward trend for percentage difference as department sizes increase.

Sample	1(a)	1(b)	2(a)	2(b)	3(a)	3(b)	4(a)	4(b)	5(a)	5(b)
% Difference -AD	13.58	17.05	0	12.10	5.88	5.66	0.71	0	7.06	0
% Difference -SD	9.88	10.51	3.03	4.68	0	0	2.02	3.61	0.25	1.03
Sample	6(a)	6(b)	7(a)	7(b)	8(a)	8(b)	9(a)	9(b)	10(a)	10(b)
% Difference -AD	2.33	0.47	1.12	0.39	2.37	0.35	1.41	2.95	0	0
% Difference -SD	5.65	1.49	1.50	0.04	1.01	0.34	0.39	0.15	0.82	0.36
Sample	11(a)	11(b)	12(a)	12(b)						
% Difference -AD	2.10	6.28	0.16	1.58						
% Difference -SD	0.98	1.74	0	0.07						

Table 10 Percentage Difference Between One And Two Parameter Models

One of the two samples for each of the 12 layouts was randomly selected, and the percentage differences were ranked. After the Z value for the normal approximation was calculated, it was decided to repeat this process for another random sample since the Z value obtained was close to the value for rejecting H_o. The results for both samples for AD and SD percentage differences are in Table 11. Since $Pr(Z \geq 1.645) = 0.05$, we cannot reject H_o at the 0.05 significance level. This result tends to verify the previous discussion. The best-fit differences are always slightly less for the two-parameter model and this is not surprising. The interesting result is that these differences are very small and that there is no discernible trend in them as department sizes increase.

Dept. Size Rank	AD Percentage Differences and Rank: Sample 1		AD Sample 2		SD Percentage Difference and Rank: Sample 1		SD Sample 2	
1	13.58	12	17.05	12	10.51	12	10.51	12
2	0	1.5	12.10	11	3.03	10	4.68	10
3	5.88	10	5.88	9	0	1.5	0	1.5
4	0.71	4	0	2	3.61	11	2.02	9
5	7.06	11	0	2	1.03	7	0.25	5
6	2.33	8	2.33	7	1.49	8	5.65	11
7	1.12	5	0.39	5	0.04	3	0.04	3
8	2.37	9	0.35	4	0.34	4	1.01	7
9	1.41	6	2.95	8	0.39	6	0.15	4
10	0	1.5	0	2	0.36	5	0.36	6
11	2.10	7	6.28	10	1.74	9	1.74	8
12	0.16	3	1.58	6	0	1.5	0	1.5
D^*		393.5		374		424.5		414.5
$E_{H_o}(D^*)$		285.5		284		285.5		285.5
$V_{H_o}(D^*)$		7410		7332		7410		7410
Z		1.25		1.05		1.61		1.50

Table 11 Z Scores For Percentage Differences

4. Conclusions

For the ℓ_{kp} model we accept the hypothesis that there is an upward trend in the k values as department sizes increase. Thus, the k values rather than the p values account for the larger-than rectangular distances being modelled. The results also show that, from an applications viewpoint, for situations of this type, practitioners can use p=1 with no cause for concern that they may be using an invalid model.

A further point may be made. If the distance norm is being utilized in a location model and the only result that is important is that of obtaining the optimal location or locations (rather than, for example, values of the total cost function), then it may not be necesary to know the value of k which is correct for the particular situation. To illustrate this, consider the single facility location model,

$$\text{minimize} \quad W(x,y) = \sum_{i=1}^{n} w_i d_i(x,y),$$

where there are n existing facilities, (x,y) is the location to be determined, and $d_i(x,y)$ is the distance from the i^{th} existing facility to the new, unknown location. Assume that for p=1 the correct value of k has been determined. Then the location problem to be solved is

$$\text{minimize } W(x,y) = \sum_{i=1}^{n} w_i k \left[|x-x_i| + |y-y_i| \right],$$

where (x_i,y_i), i=1,2,...,n, are the locations of the fixed points. k is a common multiplicative value in W(x,y) and is not relevant to the optimal value of (x,y). These results lead to an interesting interpretation. Faced with a situation of the type considered here, the practitioner can imagine that a study has been carried out to determine the optimal (k,p) parameters. Knowing that p=1 is optimal (or close to it) and that there is some (unknown) optimal value of k, the user can proceed to use the rectangular location model with k=1 and be assured of having the optimal location for the new facilities. However, if the user is interested in modelling distances rather than obtaining optimal facility locations, the $\ell_{1,1}$ norm may not be appropriate.

From Table 6, we find that the k values for the $\ell_{k,1}$ norm range from 1.00 to 1.32. Thus, any omission of the k factor (which corresponds to using the $\ell_{1,1}$ norm) in estimating distances could result in a serious understatement of the total distance.[1]

[1]This research was supported by a grant from the Natural Sciences and Engineering Research Council, Canada.

The authors would like to thank the three unknown referees for their comments and suggestions for improvements to the original paper.

REFERENCES

1. Gupta, R.M., and Deisenroth, M.P., "Comments On The Complexity Rating Factor For Layout Problems," Management Science, Vol. 27, No. 12 (1981), pp. 1460-1464.

2. Lehman, E.L., "Non Parametrics: Statistical Methods Based on Ranks", Holden-Day, Inc., San Francisco, 1975, pp. 290-297.

3. Love, R.F., and Morris, J.G., "Mathematical Models Of Road Travel Distances", Management Science, Vol. 25, No. 2 (1979), pp. 130-139.

4. ______________________________, "Modelling Inter-city Road Distances by Mathematical Functions", Operational Research Quarterly, Vol 23, No. 1 (1972), pp. 61-71.

5. Love, R.F. and Yeong, W.Y., "A Stopping Rule for Facilities Location Algorithms", A.I.I.E. Transactions, Vol. 13, No. 4 (1981), pp. 357-362.

6. Morris, J.G., "Convergence of the Weiszfeld Algorithm for Weber Problems Using A Generalized Distance Function," Operations Research, Vol. 29, No. 1 (1981), pp. 37-48.

7. Scriabin, M., and Vergin, R.C., "Comparison of Computer Algorithms and Visual Based Methods for Plant Layout," Management Science, Vol. 22, No. 2 (1975), pp. 172-181.