Facility Location with Rectilinear Tour Distances

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Abstract

This problem concerns the location of a facility among n points where the points are serviced by "tours" taken from the facility. Tours include m points at a time and each group of m points may become active (may need a tour) with some known probability. Distances are assumed to be rectilinear. An exact solution procedure is provided for $m \leq 3$ and a bounded heuristic algorithm is suggested when some tours have 4 or more points. It is shown that in the latter case the objective function becomes multimodal.

INTRODUCTION

The problem discussed in this paper may be thought of as an extension of the Weber single facility location problem. In the well known Weber problem, the facility is to be located among n points on a plane with the object of minimizing the sum of weighted distances between the facility and the points. A commonly quoted scenario is that a warehouse must be so placed that the sum of delivery costs to n customers is as small as possible. Distances are weighted by constants to represent the appropriate costs incurred when different volumes are demanded by the customers. This problem and its variations is discussed in [2]. An inherent assumption in the model is that a separate trip is required for the service of each customer. It is assumed here, on the contrary, that two or more points may be covered on a single trip; this trip will be called a tour.

A possible scenario for this problem is that a truck from the warehouse may be called upon to deliver to, say, four customers and return, using the shortest route. It is assumed that the probability of those four demands becoming "active" for a tour is known. The overall objective of the problem is to locate the facility where it will minimize the expected delivery cost over all such possible tours in the system. We assume that in each tour distances have the same weight. It would be trivially easy to extend our problem by assigning a different weight for each tour. However, if each component of the tour, a facility to point distance or a point to point distance, has a different weight per unit distance then our formulation does not apply.

This problem has been called the Traveling Salesman Location Problem. Burness and White [1] provided a heuristic solution method similar to many location-allocation heuristics. A location was found by using a derived Weber problem, then the best tours were "allocated" by a Traveling Salesman algorithm; the process was repeated to form iterations. Heuristic solution methods [4], [5] have been presented for versions of the Traveling Salesman Location Problem where the facility must be located on a network.

Our method takes advantage of the special properties of rectilinear distances which are often a good approximation when travel must occur through a grid of streets or a grid of aisles in a plant. We concentrate mainly on problems where m, the number of points on a tour, is small. An optimal solution is given for $m \leq 3$ and the method turns into a good heuristic when $m>4$ is possible, but is unlikely. Although the rectilinear Traveling Salesman Problem has some special properties, it was shown to be NP-complete [3] and only restricted versions of it have been solved efficiently so far [6].

The following section begins with a somewhat general treatment of rectilinear tours.

<u>Optimal Tours</u>

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two points on the plane. The rectilinear distance between a and b, d(a,b), is defined by

$$d(a,b) = |a_1 - b_1| + |a_2 - b_2|.$$

Although d(a,b) is uniquely defined, there are, of course, many different paths between a and b, all having the length d(a,b) as is illustrated in Figure 1.

We define the distance between a set S and a point a by $d(S,a) = \min\{d(b,a) \mid b \in S\}$.

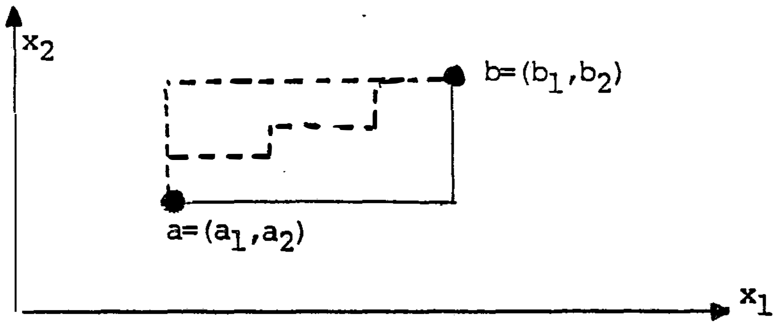

Figure 1 Rectilinear paths

In this section we will study tours which go through a fixed subset of m points. Accordingly we will represent a tour T_m through the m points $\{1,2,\ldots,m\}$ by a "permutation" $p = (p(1),p(2),\ldots,p(m), p(m+1))$ of the numbers $\{1,2,\ldots,m\}$; $p(1) = p(m+1)$ because it is assumed that the sequence in which the points are visited on the tour begins and ends with $p(1)$. The length of such a tour, $\ell(T_m)$ is uniquely defined by $\ell(T_m) = \sum_{i=1}^{m} d(p(i),p(i+1))$. This unique length, however, has infinitely many tour realizations, depending on which path between each pair of points in the sequence is selected from among the infinitely many paths of the same length. Which path between two points is selected, has, of course, no effect on $\ell(T_m)$.

Consider a tour T_m. For any pair of points $p(i)$ and $p(j)$ $(1 \leq i < j \leq m)$ let us define the rectangle $R_{p(i)p(j)}$ which has $p(i)$ and $p(j)$ as its diagonally opposite corner points. It is clear that a point x (for example, the facility) can be included in the tour T_m between the points $p(i)$ and $p(i+1)$ $(1 \leq i \leq m)$ without increasing the length of the tour if and only if $x \varepsilon R_{p(i)p(i+1)}$. Accordingly we define the set of <u>free points for the tour T_m</u> by

$$FR(T_m) = \bigcup_{i=1}^{m} R_{p(i)p(i+1)}$$

and $FR(T_m)$ clearly represents the set of those points in the plane which can be included in the tour T_m without increasing its length. $FR(T_m)$ will be important in our optimization procedure when we seek to extend a tour T_m to include a point x at a minimal cost increase.

In general we are going to denote by R_m the smallest rectangle enclosing the given set of m points, and define the <u>nonfree set for the tour T_m</u> by

$$NFR(T_m) = R_m - FR(T_m).$$

Naturally, the tour T_m can be extended many different ways to include an additional point x, depending on between which pair of adjacent points p(i) and p(i+1) x is visited. In general, we are going to denote by $T_m(x)$ any such extension of T_m for which the length of $T_m(x)$, $\ell(T_m(x))$, is as small as possible; we will call T_m a base tour and $T_m(x)$ its extension. We define the distance of x from the tour T_m, denoted by $d(T_m,x)$, as the shortest distance between x and any point traversed by any realization of T_m. It is clear that $\ell(T_m(x)) = \ell(T_m) + 2d(T_m,x)$ and $d(T_m,x) = 0$ if and only if $x \in FR(T_m)$.

We define the <u>free region</u> for a set of m points by

$$FR = \bigcup_{T_m \in \tau} FR(T_m), \quad \text{where}$$

$$\tau = \{T_m \mid \ell(T_m) \text{ is minimal}\}.$$

The <u>nonfree region</u> is defined by $NFR = R_m - FR$. It is clear that a point x can be included without additional cost in some optimal tour through the m points if and only if $x \in FR$.

If $p = (p(1),\ldots,p(m), p(1))$ represents a tour on m points then any cyclical rearrangement $p' = (p(i),p(i+1),\ldots,p(m),p(1),\ldots,p(i-1),p(i))$ $(1 \le i \le m)$ clearly represents the same tour, therefore without the loss of generality we can always consider only those sequences for which p(1) is fixed, e.g., p(1) = 1. Furthermore since $p = (p(1),p(2),\ldots,p(m),p(1))$ and $p' = (p(1),p(m),p(m-1),\ldots,p(2),p(1))$ represent two tours, where one is the reverse of the other and since our distance function is symmetric, all together it is enough to consider $(m-1)!/2$ sequences to represent all the tours on m points.

<u>Theorem 1</u>: For any set of m points and any tour T_m

$$\ell(T_m) \geq \ell(R_m), \tag{1}$$

where $\ell(R_m)$ is the length of the circumference of R_m.

Proof: Due to the cyclical nature of the sequences used to represent a tour we can start measuring the length of a tour at any point in the sequence representing it. If, in the horizontal direction, we take a leftmost point a as the start of T_m, and b is a rightmost point among the m points then T_m has to go from a to b and eventually get back to a, no matter in what sequence the other m-2 points are covered. Therefore in the horizontal direction $\ell(T_m)$ is at least 2 times the distance between a and b. A similar argument in the vertical direction proves the Theorem.

Corollary 2: Let x be an arbitrary point in the plane and let $\tau_m(x)$ be any tour through m given points and x. If $R_m(x)$ is the smallest rectangle enclosing the m points and x, and R_m is the smallest rectangle enclosing the m points then

$$\ell(\tau_m(x)) \geq \ell(R_m(x)) = \ell(R_m) + 2d(R_m.x). \tag{2}$$

Proof: To prove the inequality apply Theorem 1 to the set of m + 1 points which consists of the m fixed points and x. The equality in (2) is obvious.

Corollary 3: Let $x \varepsilon R_m$ be an arbitrary point. If there exists a tour $T_m^*(x)$ through the m points and x for which $\ell(T_m^*(x)) = \ell(R_m)$ then $T_m^*(x)$ is optimal.

Proof: Obvious by Corollary 2.

Lemma 4: Assume that $x \not\varepsilon R_m$. Let $x' \varepsilon R_m$ be the point where $d(R_m,x)$ is obtained (i.e., x' is the closest point of R_m to x) and let $T_m^*(x')$ be a minimum length tour through the m points and x'. If $T_m^*(x)$ is obtained from $T_m^*(x')$ by replacing in it x' by x (without changing the sequence in which the points are visited) then $T_m^*(x)$ is a minimum length tour through the m points and x and

$$\ell(T_m^*(x)) = \ell(T_m^*(x')) + 2d(x',x) = \ell(T_m^*(x')) + 2d(R_m,x).$$

Proof: Assume that to the contrary there exists a tour $T_m(x)$ through the m points and x for which $\ell(T_m(x)) < \ell(T_m^*(x))$. Obtain $T_m(x')$ from $T_m(x)$ by replacing in it x by x' without changing the sequence. It is clear that $\ell(T_m(x')) = \ell(T_m(x)) - 2d(x',x) < \ell(T_m^*(x)) - 2d(x',x) = \ell(T_m^*(x'))$ contradicting the optimality of $T_m^*(x')$.

Lemma 4 means that when looking for the best tour through m given points and an arbitrary x, it is sufficient to consider only x's which are contained in the smallest rectangle enclosing the m points.

3-point tours

When we are looking for the minimum length tour through 3 given points and an arbitrary x then there are (4-1)!/2 = 3 possible tour sequences. The next two results show that we do not have to consider all of these:

Lemma 5: Assume 3 distinct fixed points and x are all located on a horizontal (vertical) line. Then, there are exactly two optimal tours and their length is 2 times the length of the smallest interval containing all four points.

Proof: Without the loss of generality assume that the three fixed points, 1, 2 and 3 have been numbered so that 2 is between 1 and 3 on the line. Then it is easy to see by inspecting Figure 2 that regardless of the position of x exactly two tour sequences are always optimal and their length is as stated in the Lemma.

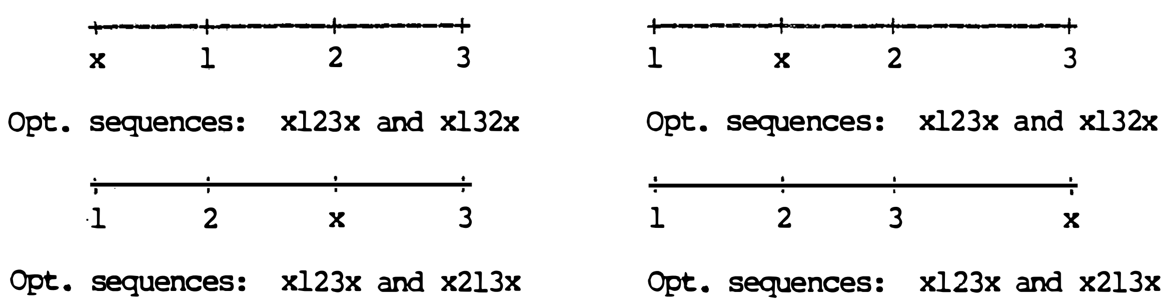

Figure 2 Optimal Sequences

Corollary 6: If $a_1 \le a_2 \le a_3$ are the horizontal coordinates of the points 1, 2 and 3 located on a horizontal line and x_1 is the horizontal coordinate of x then

$$\ell(T_3^*(x)) = \ell(T_3^*(x_1)) = |x_1-a_1|+|x_1-a_3|+|a_1-a_3|$$

Proof: By direct inspection of the smallest interval containing the four points in Lemma 5.

Theorem 7: For 3 arbitrarily located points and x there is an optimal tour sequence that is also optimal in each of the two dimensions separately.

Proof: By Lemma 5 two tour sequences are optimal in the first dimension and two tour sequences are optimal in the second dimension. These four tour sequences cannot be all different since only three tour sequences are possible. Therefore, there must be one which is also optimal in both dimensions. This tour must be of minimum length overall since the length of a tour with rectilinear distances is the sum of its lengths in the two dimensions.

Theorem 7 will enable us to separate the 3 point tour optimization problem into two independent one dimensional optimization problems. It also implies that for m = 3 there is a tour T_3^* for which equality holds in (1).

Corollary 8: Let $T_3^*(x)$ be an optimal tour through 3 given points and an arbitrary x. Then

$$\ell(T_3^*(x)) = \ell(R_3(x)) = \ell(R_3) + 2d(R_3,x)$$

Proof: By Theorem 7 there is an optimal tour which is also optimal in the two dimensions separately. The length of this tour in each dimension, is equal by Lemma 5, to 2 times the length of the interval spanned by the four points in each respective dimension.

4-point tours

Assume now that we want to find a minimum length tour through four given points i,j,k and ℓ. Since Corollary 8 holds for any x, it also means that for an optimal tour T_4^* through i,j,k and ℓ we always have $\ell(T_4^*) = \ell(R_4)$, i.e., equality can be achieved in (1) for m = 4. In the following development we are going to construct such optimal tours and discuss the possibilities which may occur with respect to their extendability to include a fifth point x at minimum cost.

The rectangle R_4 for i, j, k and ℓ may contain on its sides 2, 3 or 4 of these points. Our general rule to construct an optimal tour through them can be formulated as follows: Sequence the points on the sides of R_4 in the counter-clockwise direction and insert the remaining points between these "for free". Arbitrarily we are going to start the tour sequences with the lowest leftmost point on the sides of R_4.

Case a: Exactly two of the four points (say k and ℓ) are on the sides of R_4, i.e., k and ℓ are diagonally opposite corner points of R_4 with (say) k being the left corner point. We further distinguish between two possibilities depending on the relative location of the remaining two points i and j:

a1. $R_{ki} \supseteq R_{kj}$ (Figure 3.a1.)

a2. $R_{ki} \cap R_{kj} \neq \emptyset$ but neither one contains the other, i.e. $R_{ki} \not\subseteq R_{kj}$ and $R_{ki} \not\supseteq R_{kj}$ (Figure 3.a2.)

In case a1., since $R_{k\ell} = R_4$, all three possible tour sequences (k,j,ℓ,i,k), (k,j,i,ℓ,k) and (k,i,ℓ,j,k) are optimal with a length equal to $\ell(R_4)$. We note that of these, for the tour sequence (k,j,i,ℓ,k), $FR(kji\ell k) = R_{\ell k} = R_4$, so $NFR = \emptyset$ i.e., for any $x \in R_4$ the tour can be extended to include x for free.

It can be easily proved in case a2 that (k,i,ℓ,j,k) is the only optimal tour sequence. In this case $FR(ki\ell jk) = R_{ki} \cup R_{i\ell} \cup R_{\ell j} \cup R_{jk} \neq R_4$, so NFR consists of the two corner rectangles, henceforth called "blind" corners, C_1 and C_2 and the interior rectangle I_R, shown as cross hatched areas in Figure 3.a2.

<u>Case b</u>: Exactly three of the four points (say i,j and k) are on the sides of R_4 with i being the lowest of the leftmost of these. Again we distinguish between two possibilities:

b1. Of the rectangles R_{ij}, R_{jk} and R_{ki} exactly one (say R_{jk}) contains the fourth point ℓ. According to our general rule we insert ℓ between j and k for free to get the tour sequence (i,j,ℓ,k,i) with length equal to $\ell(R_4)$. It can be easily seen that in this case this is the only optimal sequence, therefore NFR consists of the corner rectangle ("blind" corner) C_1 and the interior rectangle I_R, shown as cross hatched areas in Figure 3.b1.

b2. Two of the rectangles (say R_{ij} and R_{jk}) contain the fourth point ℓ. In this case ℓ can be inserted for free between both (i,j) and (j,k). So (i, ℓ, j,k,i) and (i,j, ℓ,k,i) are both optimal tour sequences with length $\ell(R_4)$. Furthermore, although each of these two sequences has a blind corner; this is always covered by the other sequence, i.e., $FR(i\ell jki) \cup FR(ij\ell ki) = R_4$ with $NFR = \emptyset$.

<u>Case c</u>: All four points are on the sides of R_4. We sequence them in counter-clockwise direction, starting with the lowest leftmost point and get the optimal sequence say (i,j,k,ℓ,i) with $FR(ijk\ell i) = R_4$, so $NFR = \emptyset$. (See Figure 3.c)

Now consider the problem of extending the base tours through the four given points, to include an x at the smallest possible cost. In all of the above cases where $NFR = \emptyset$, for any $x \in R_4$, we can extend an optimal base tour to include x for free, i.e., there is a tour $T_4^*(x)$ for which $\ell(T_4^*(x)) = \ell(R_4)$ which by Corollary 3 must be an optimal tour. Unfortunately for cases a2. and b1. (when $NFR \neq \emptyset$) this may not be possible; moreover, for certain $x \in NFR$ the extension of a suboptimal base tour may be shorter than the extension of the optimal base tour. If we have the case a2. or b1. NFR consists of an interior rectangle (I_R) and one or two blind corners (C_1,C_2). Let us denote by I_1 and I_2 the length of the horizontal and vertical sides of I_R respectively and let $I = \min\{I_1,I_2\}$. When $I_R = \emptyset$ we let $I = 0$ by convention.

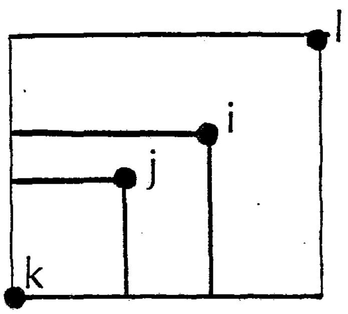

Figure 3.a1.

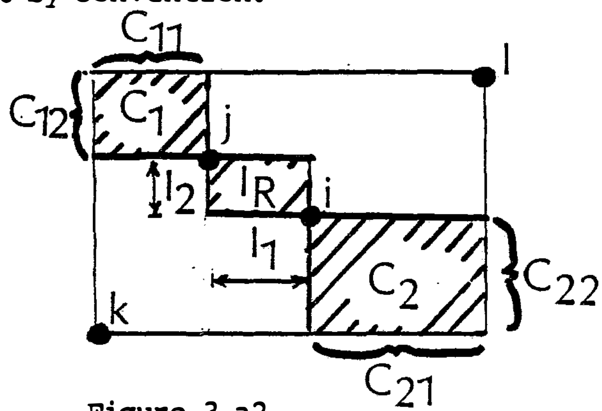

Figure 3.a2.

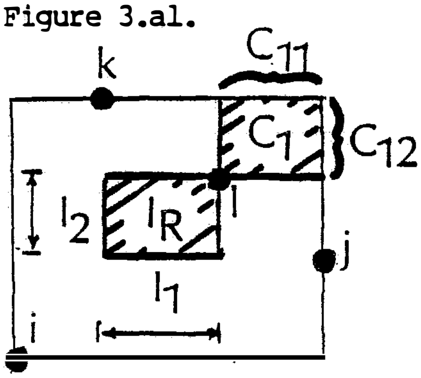

Figure 3.b1.

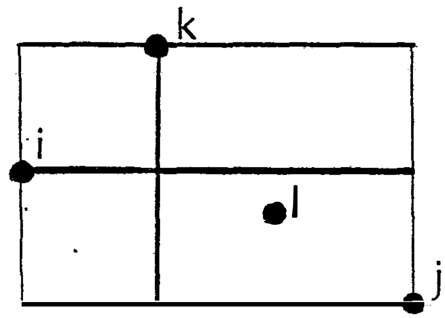

Figure 3.b2.

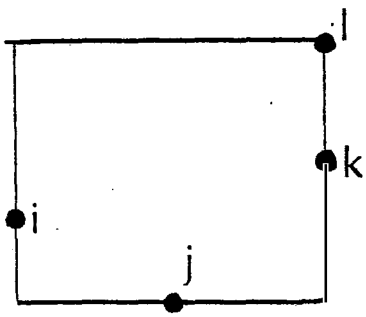

Figure 3.c.

Figure 3: Four point tours

<u>Lemma 9</u>: Assume that the four points belong to case a2. or b1., i.e., $I_R \neq \emptyset$ and let T_4' be the optimal base tour through them. For any $x \in I_R$ the shortest extension of the optimal base tour, $T_4'(x)$ is the optimal tour through the four points and x and

$$\ell(R_4) = \ell(T_4'(x)) = \ell(R_4) + I \leq 1.25\,\ell(R_4) \tag{3}$$

Proof: It is clear that for any $x \varepsilon I_R$, for the distance of x from T_4', $d(T_4',x)$, we have $d(T_4',x) \leq 0.5 \min\{I_1,I_2\} = 0.5I$. Since $I_R \subseteq R_4$ it follows that $I \leq 0.25\,\ell(R_4)$ which clearly means that (3) holds. We have to show that $T_4'(x)$ is optimal: Let T_4^2 and T_4^3 be the other possible (and suboptimal) base tours. We note that the difference between $\ell(T_4')$ and the lengths of the suboptimal base tours $\ell(T_4^2)$ and $\ell(T_4^3)$ is $2I_1$ or $2I_2$ as can easily be seen from Figures 3.a2. and 3.b1. Therefore

$\min\{\ell(T_4^2(x)),\ \ell(T_4^3(x))\} \geq \min\{\ell(T_4^2), \ell(T_4^3)\} \geq \ell(T_4') + 2I > \ell(T_4'(x))$ which proves that $T_4'(x)$ is optimal.

Lemma 10: Assume that the four points belong to case a2. or b1. and let x be a point from the "blind corners" (C_1 or C_2), i.e., $x \varepsilon NFR - I_R$. If $T_4^*(x)$ is the optimal tour through the four points and x then

$$\ell(R_4) \leq \ell(T_4^*(x)) \leq 1.25\,\ell(R_4)$$

Proof: Let C_{11} and C_{12} be the length of the horizontal and vertical sides respectively of the "blind corner" C_1 and similarly let C_{21} and C_{22} be the lengths of the horizontal and vertical sides respectively of C_2, if it exists. Let $T_4'(x)$ be the shortest extension of the optimal base tour T_4'; then if $x \varepsilon C_i$ (i = 1 or 2) we have

$$\ell(T_4'(x)) \leq \ell(T_4') + 2\min\{C_{i1}, C_{i2}\} = \ell(R_4) + 2\min\{C_{i1}, C_{i2}\} \qquad (4)$$

We note that any $x \varepsilon NFR$ can be included for free in either one of the two suboptimal tours, T_4^2 or T_4^3. Therefore, for their extensions $T_4^2(x)$ and $T_4^3(x)$

$$\ell(T_4^2(x)) = \ell(T_4^2) \text{ and } \ell(T_4^3(x)) = \ell(T_4^3) \text{ so}$$

$$\min\{\ell(T_4^2(x)), \ell(T_4^3(x))\} = \min\{\ell(T_4^2), \ell(T_4^3)\} \leq \ell(R_4) + 2\min\{I_1, I_2\} \qquad (5)$$

From (4) and (5) we get

$$\ell(T_4^*(x)) = \min\{\ell(T_4'(x)),\ \ell(T_4^2(x)),\ \ell(T_4^3(x))\}$$

$$\leq \ell(R_4) + 2\min\{I_1,I_2,C_{i1}, C_{i2}\} \text{ for any } x \varepsilon C_i \qquad (6)$$

Let A and B denote the length of the horizontal and vertical sides, respectively, of R_4. Then

$$C_{i1} + I_1 \leq A \quad \text{and} \quad C_{i2} + I_2 \leq B \quad i = 1,2 \tag{7}$$

(see Figures 3.a2. and 3.bl.).

On the other hand

$$\min\{C_{i1}, C_{i2}, I_1, I_2\} \leq 0.5 \ (C_{i1} + I_1)$$
$$\min\{C_{i1}, C_{i2}, I_1, I_2\} \leq 0.5 \ (C_{i2} + I_2) \tag{8}$$

and combining (7) and (8) we get

$$2 \min\{C_{i1}, C_{i2}, I_1, I_2\} \leq 0.5 \ (A+B) = 0.25 \ \ell(R_4)$$

In view of (6), this proves the lemma.

Theorem 11: Consider four given points with smallest enclosing rectangle R_4. Let $T_4^*(x)$ be the optimal tour through the four points and an arbitrary x. Then

$$\ell(R_4) + 2d(R_4,x) \leq \ell(T_4^*(x)) \leq 1.25 \ \ell(R_4) + 2d(R_4,x) \tag{9}$$

Proof: a) If $x \in R_4$ we note that $d(R_4, x) = 0$ for any $x \in R_4$, and (9) is equivalent to

$$\ell(R_4) \leq \ell(T_4^*(x)) \leq 1.25 \, \ell(R_4) \tag{10}$$

If $x \in FR$ (10) clearly holds since $\ell(T_4^*(x)) = \ell(R_4)$.

If $NFR \neq \emptyset$ and $x \in NFR$ then if $x \in I_R$ then Lemma 9 and if $x \in NFR - I_R$ then Lemma 10 proves the Theorem.

b) If $x \notin R_4$ then Lemma 4 reduces this case to case a) and so (9) follows from the fact that (10) holds for $x' \in R_4$, where x' is the closest point of R_4 to x.

Theorem 11 will enable us to approximate the optimal 4-point tours in a heuristic optimization procedure. The following Corollary generalizes the bounds of Theorem 11 for m>4 although its usefulness is limited as the difference between the lower and upper bound increases quickly as m is increased.

Corollary 12: Consider m points and an arbitrary x and let $T_m^*(x)$ be the optimal tour through them. Then

$$\ell(R_m) + 2d(R_{m,x}) \leq \ell(T_m^*(x)) \leq (1.25 + 0.5(m-4))\ \ell(R_m) + 2d(R_{m,x}) \qquad (11)$$

for any $m \geq 4$.

Proof: The lower bound in (11) was proved in Corollary 2. For the upper bound it is sufficient to consider only the case when $x \varepsilon R_4$ (in view of Lemma 4). For m=5 the upper bound in (11) follows from (9): Consider any four of the five points; then for these $\ell(T_4^*(x)) \leq 1.25\ \ell(R_4)$. If the fifth point is inserted at minimal cost in $T_4^*(x)$ then the increase in $\ell(T_4^*(x))$ will be no more than 2 times the length of the shortest side of R_5. From this it follows that $\ell(T_5^*(x)) \leq \ell(T_4^*(x)) + 0.5\ \ell(R_5) \leq 1.25\ \ell(R_4) + 0.5\ \ell(R_5) \leq 1.75\ \ell(R_5)$. A similar induction proves (11) for higher m.

Optimal facility location

For the m-point tour location problem, let us define S(k) as the set of all distinct subsets of the n points that have k members ($k \leq m \leq n$). Let

$S(k) = \{s(k,1),\ s(k,2),\ldots,s(k,r(k))\}$, where $s(k,i)$ is the i'th set of k points and $r(k) = \binom{n}{k}$. Let $T_{ki}^*(x)$ the minimum length tour through the points in $s(k,i)$ and an arbitrary x and let p_{ki} be the probability of the set $s(k,i)$ becoming active. The m point tour location problem then can be formulated as

$$\underset{x}{\text{minimize}}\ F(x) = \sum_{k=1}^{m} \sum_{i=1}^{r(k)} p_{ki}\ \ell(T_{ki}^*(x)) \qquad (12)$$

If m=1 then problem (12) is simply a rectilinear distance Weber problem with each distance multiplied by 2. When tours including two points exist (m=2) the problem is also immediately reduced to a Weber problem because the distance between the two fixed points in each tour is a constant. However, when points can become active three at a time the choice of the optimal tour enters the problem.

Locating the facility with tours through triples

The methods developed in this section can be easily applied when 1 and 2 point tours are also included but in order to simplify the discussion we are going to assume that $p_{ki} = 0$ for $k \neq 3$. The following result is a direct consequence of Theorem 7.

Corollary 13: For k=3 the objective function in (12) is separable into two one-dimensional functions; thus (12) becomes

$$\underset{x_1}{\text{minimize}}\ F_1(x_1) = \sum_{i=1}^{r(3)} p_{3i}\ \ell(T_{3i}^*(x_1)) \tag{13a}$$

and

$$\underset{x_2}{\text{minimize}}\ F_2(x_2) = \sum_{i=1}^{r(3)} p_{3i}\ \ell(T_{3i}^*(x_2)) \tag{13b}$$

where the tours $T_{3i}^*\ (x_j)$ are the optimal tours through s(3,i) and x_j in the j-th dimension (j=1,2), with all distances measured along the corresponding axis and $F(x) = F_1(x_1) + F_2(x_2)$

From now on we restrict our attention to the one dimensional problem (13a) but everything can be easily applied to (13b) too. Let a_{1i}, a_{2i} and a_{3i} be the coordinates in the x_1 dimension of the three points in s(3,i) and without loss of generality assume that $a_{1i} \leq a_{2i} \leq a_{3i}$. Using Corollary 6 we can rewrite the function in (13a) as

$$F_1(x_1) = \sum_{i=1}^{r(3)} p_{3i}(|x_1-a_{1i}|+|x_1-a_{3i}|+|a_{1i}-a_{3i}|)$$

$$= \sum_{i=1}^{r(3)} p_{3i}|a_{1i}-a_{3i}| + \sum_{i=1}^{r(3)} p_{3i}(|x_1-a_{1i}|+|x_1-a_{3i}|) \tag{14}$$

Note that (14) is a sum of convex terms in x_1 and is hence convex. Assume that $a_1 \leq a_2 \leq .... \leq a_n$ are the coordinates in the x_1 dimension of the n fixed points and define subsets of S(3) by

$A_j = \{s(i)\,|\,a_j$ is the coordinate of the leftmost point in s(3,i))$\}$

$B_j = \{s(i)\,|\,a_j$ is the coordinate of the rightmost point in s(3,i)$\}$

for j=1,...,n

Then (14) can be rewritten as

$$F_1(x_1) = \sum_{i=1}^{r(3)} p_{3i}|a_{1i} - a_{3i}| + \sum_{j=1}^{n} \sum_{i \varepsilon A_j \cup B_j} p_{3i}|x_1 - a_j| = \sum_{i=1}^{r(3)} p_{3i}|a_{1i} - a_{3i}|$$

$$+ \sum_{j=1}^{n} w_j|x_1 - a_j|$$

$$\text{where } w_j = \sum_{i \varepsilon A_j \cup B_j} p_{3i} \qquad j=1,\ldots,n \qquad (15)$$

Since the first sum is constant and the second sum forms a Weber problem this proves the following:

Theorem 14: The 3 point tour location problem can be solved by solving a Weber problem in each dimension separately.

Example

There are five points as in Figure 4 and therefore n=5.

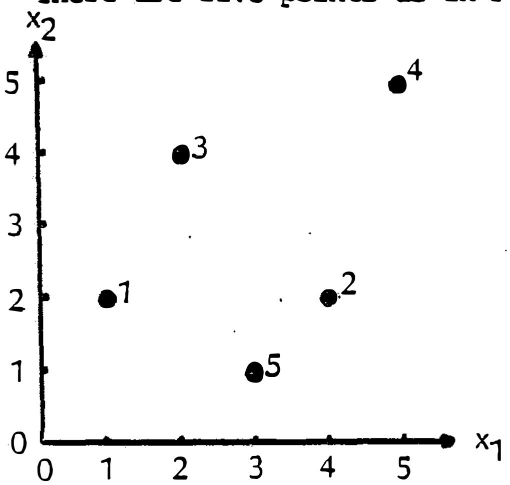

Figure 4: Locations of five points

The possible triplets and their probabilities are listed in Table 1

i	p_{3i}	s(k,i)	a_{1i}	a_{3i}	$\|a_{1i}-a_{3i}\|$
1	0	{1,3,5}	1	3	2
2	.1	{1,3,2}	1	4	3
3	.1	{1,3,4}	1	5	4
4	0	{1,5,2}	1	4	3
5	.2	{1,5,4}	1	5	4
6	.1	{1,2,4}	1	5	4
7	.1	{3,5,2}	2	4	2
8	0	{3,5,4}	2	5	3
9	.4	{3,2,4}	2	5	3
10	0	{5,2,4}	3	5	2

Table 1

We can now find the sets A_j, B_j, and $A_j \cup B_j$

$$\begin{array}{lll} A_1 = \{1,2,3,4,5,6\} & B_1 = \emptyset & A_1 \cup B_1 = \{1,2,3,4,5,6\} \\ A_2 = \emptyset & B_2 = \{2,4,7\} & A_2 \cup B_2 = \{2,4,7\} \\ A_3 = \{7,8,9\} & B_3 = \emptyset & A_3 \cup B_3 = \{7,8,9\} \\ A_4 = \emptyset & B_4 = \{3,5,6,8,9,10\} & A_4 \cup B_4 = \{3,5,6,8,9,10\} \\ A_5 = \{10\} & B_5 = \{1\} & A_5 \cup B_5 = \{1,10\} \end{array}$$

The weights w_j in (15) are now calculated:

$w_1=.5$, $w_2=.2$, $w_3=.5$, $w_4=.8$, $w_5=0$. Therefore

$$F_1(x_1) = 3.3 + .5|x_1-1| + .2\ |x_1-4|\ +.5|x_1-2|+.8|x_1-5|$$

The optimum x_1 is found (see [2]) by finding the median of the numbers 1,2,4 and 5 when they are weighted by .5, .5, .2 and .8 respectively; x_1 is therefore optimal in the closed range [2,4]. A similar procedure would yield the optimal x_2.

Suppose that the example is altered so that all the p_{3i} are equal to 0.1. That is, each triple is equally likely. It is easy to verify that in this case $w_1=.6$, $w_2=.3$, $w_3=.3$, $w_4=.6$ and $w_5=.2$. The optimal x_1 is then 3. Actually it is a consequence of the following corollary that when all triples are equally likely x_1^* and x_2^* are found by determining the median co-ordinates in the horizontal and vertical directions respectively.

<u>Corollary 15</u>: When $p_{3i} = p_o$ for each i then (15) is minimized at

$$x_1 = a_{\ell+1} \quad \text{if } n=2\ell+1 \quad \text{(i.e. n is odd)}$$

and

$$a_\ell \le x_1 \le a_{\ell+1} \text{ if } n = 2\ell \quad \text{(i.e. n is even)}$$

<u>Proof</u>: It can be easily seen that for each j ($3 \le j \le n-2$)

$$|A_j| = \binom{n-j}{2} \text{ and } |B_j| = \binom{j-1}{2} \text{ and for } j=1,2 \quad B_j = \emptyset$$

and $j=n-1,n$ $A_j=\emptyset$. Therefore when all the probabilities are the same

$$w_j = \begin{cases} \binom{n-j}{2} p_o & j=1,2 \\ \left[\binom{n-j}{2} + \binom{j-1}{2}\right] p_o & 3 \leq j \leq n-2 \\ \binom{j-1}{2} p_o & j=n-1,n \end{cases}$$

Because of the obvious symmetry in the w_j's the weighted median is always in the "middle" as defined in the Corollary.

Locating the facility with four point tours

The methods developed in this section can be easily modified to include 1,2 and 3 point tours but in order to simplify the development we will assume that $p_{ki} = 0$ for $k \neq 4$.

Four point tours change the characteristics of F(x) drastically. As an illustration consider the four points in Figure 5. Figure 5b plots F(x) along the line through points 4 and 2. τ_1 = x1243x, τ_2 = x3241x, τ_3=x1234x, τ_4=x4123x, τ_5=x2341x and τ_6=x3412x are the optimal sequences in the designated segments along this line.

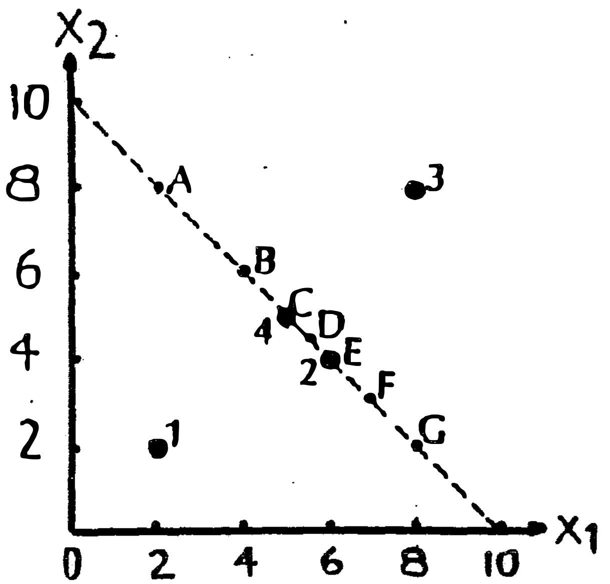

Figure 5a

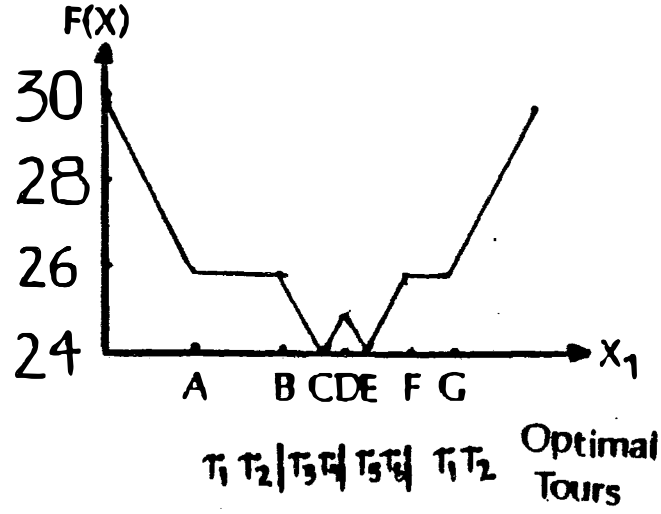

Figure 5b

Figure 5: F(x) along a line (4 pt. tours)

The function $F(x)$ is no longer always convex and can be multimodal as seen in the Figure. Furthermore $F(x)$ is no longer always separable in x_1 and x_2; the minimum in the x_1 direction is affected by x_2 and vice versa.

Therefore, we will not try to find and substitute into (12) the optimal tours, $T^*_{ki}(x)$ for each $s(4,i)$ and x. We develop a heuristic method instead which reduces the problem to a 3 point tour problem, for which the optimal solution is reasonably close to the optimum of the 4 point problem.

Corollary 16: Let R_{4i} be the smallest enclosing rectangle for the four points in $s(4,i)$ $(1 \leq i \leq r(4))$. Define

$$B_1(x) = \sum_{i=1}^{r(4)} p_{4i} \, (\ell(R_{4i}) + 2d(R_{4i},x)) \quad \text{and}$$

$$B_2(x) = \sum_{i=1}^{r(4)} p_{4i} (1.25 \, \ell(R_{4i}) + 2d(R_{4i},x)) \quad \text{then}$$

$$B_1(x) \leq \sum_{i=1}^{r(4)} p_{4i} \, \ell(T^*_{4i}(x)) \leq B_2(x) \quad \text{for any } x.$$

Proof: Substitute the bounds of Theorem 11 for each set $s(4,i)$.

We note that $B_1(x)$ and $B_2(x)$ differ only by a constant, therefore they attain their minimum value at the same x.

Theorem 17: Assume that $B_1(x)$ obtains its minimum at x^*_B and x^* is the optimal location for the 4 point tour problem

Then

$$0 \leq \frac{F(x^*_B) - F(x^*)}{F(x^*)} \leq \frac{1}{4}$$

Proof: Since x^*_B minimizes $B_1(x)$: $B_1(x^*_B) \leq B_1(x^*)$

Since x^* minimizes $F(x)$: $F(x^*) \leq F(x^*_B)$

If follows from these, using Corollary 16 that

$B_1(x^*_B) \leq B_1(x^*) \leq F(x^*) \leq F(x^*_B) \leq B_2(x^*_B)$ and

$$0 \le \frac{F(x_B^*)-F(x^*)}{F(x^*)} \le \frac{B_2(x_B^*)-B_1(x_B^*)}{B_1(x_B^*)} = \frac{1/4 \sum_{i=1}^{r(4)} p_{4i}^* \ \ell(R_{4i})}{\sum_{i=1}^{r(4)} p_{4i}(\ell(R_{4i})+2d(R_{4i},x_B^*))} \le \frac{1}{4}$$

which is what we had to prove.

By Theorem 17 if we minimize $B_1(x)$ instead of $F(x)$ then the solution is at most 25% above the minimum of $F(x)$.

In the following we show that minimizing $B_1(x)$ is equivalent to solving the 3 point tour location problem. For each set $s(4,i)$ the enclosing rectangle R_{4i} can always be defined by three appropriately chosen points on its sides e.g. by three of its corner points. Let the set of these three points be $t(3,i)$ $(1 \le i \le r(4))$ and let us denote by R_{3i} the smallest enclosing rectangle for the points in $t(3,i)$. By definition, $R_{4i}=R_{3i}$ for each i. Substituting these into $B_1(x)$

$$B_1(x) = \sum_{i=1}^{r(4)} p_{4i}(\ell(R_{3i})+2d(R_{3i},x))$$

and by Corollary 8 minimizing this function is equivalent to solving the 3 point tour problem on the sets $t(3,i)$ $(1 \le i \le r(4))$.

Finally we note that the same heuristic procedure could also be applied for m>4; however, Corollary 12 indicates that the bound on the error of the procedure increases very quickly as m increases.

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