A Decomposition Approach for Finding the Setup Number of a Partial Order

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Research and Working Paper Series No. 220
1984

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Abstract

Consider the linear extensions of a partial order. A setup occurs in a linear extension if two consecutive elements are unrelated in the partial order. The setup problem is to find a linear extension of the ordered set which contains the smallest possible number of setups. We present a decomposition approach for this problem. Based on this some new complexity results follow.

1. Introduction

Precedence constrained scheduling problems have been extensively studied in the literature (For a survey see e.g. [10], [11]). The setup problem, we deal with, also belongs to this class: Suppose we are given a finite set V of tasks to be sequenced subject to precedence constraints, which imply that a task $v \in V$ cannot be scheduled unless all of its predecessors ($P(v)$) have been scheduled already. If a task t is scheduled immediately after the task u, then there is a "setup" resulting in a fixed cost if $u \notin P(t)$ and there is no cost (no setup) if $u \in P(t)$. The problem can be considered a special case of the precedence constrained traveling salesman problem. Since the cost of a setup is not dependent on where it occurs the cost of a schedule is completely defined by the structure of the underlying partial order which represents the precedence constraints. Partial orders have been studied from this point of view and an overview of basic results is in [2]. The problem was shown to be NP-hard even in the special case when no task has both successors and predecessors [13]. This explains why attention has concentrated on identifying special classes of precedence constraints for which the problem has a polynomial time solution. Such classes include the series-parallel case ([3], [13]), cycle-free ordered sets [5], N-free ordered sets ([6], [7], [8], [14]) and cycle-series-parallel ordered sets [9]. In this paper we present a decomposition approach for the general case. This leads to identifying new polynomially solvable classes of partial orders and some new complexity results.

2. Preliminary Definitions and Notation

By a <u>partially ordered set</u> (poset) we mean a set P with a binary relation $\leq$ which is reflexive, antisymmetric and transitive. A set $C=\{a_1,a_2,\dots,a_k\}\subseteq P$ is called a <u>chain</u> if $a_i \leq a_{i+1}$ for $i=1,\dots,k-1$. For $a,b \in P$ we say that <u>b covers a</u> ($a \prec b$) if $a< b$ and there is no $c \in P-\{a,b\}$ such that $a \leq c \leq b$. An order preserving bijection (schedule) $f : P \to \{1,2,\ldots,|P|\}$ is called a <u>linear extension of P</u>. We say that <u>f has a setup</u> between $u, v \in P$ (denoted by $f(u) \,||\, f(v)$) if $f(v) = f(u)+1$ and $u \nleq v$ in P. The <u>setup number of f</u> ($s(f)$) is the number of setups in f, and the <u>setup number of P</u> ($s(P)$) is defined by

$$s(P) = \min \{s(f) \mid f \text{ is a linear extension of } P\},$$

and an f for which $s(P) = s(f)$ will be called an <u>optimal linear extension of P</u>.

As is customary, we will represent posets by directed acyclic graphs (dags): A dag $G=(V,E)$ represents a poset P if there is a one-to-one correspondence between the vertices in V and the elements of P and, for any $u,v \in P$ $u \leq v$ if and only if (iff) there is a directed path from u to v in G. The dag G may contain all the transitive edges (directed comparability graph) or no transitive edges (Hasse diagram), and for the remainder of the paper we will not distinguish between a poset P and a dag G representing it.

Consider a transitive dag $G=(V,E)$. A subset $M \subseteq V$ is a <u>module</u> in G iff for every $v \in V-M$ one of the following conditions holds:

i) $(u,v) \in E$ for every $u \in M$

ii) $(v,u) \in E$ for every $u \in M$

iii) $(u,v) \notin E$ and $(v,u) \notin E$ for every $u \in M$.

A module M is <u>nontrivial</u> if $1<|M|<|V|$.

Let $G_0 = (V_0, E_0)$ be a transitive dag on k vertices $V_0=\{v_1,v_2,\ldots,v_k\}$ and let $G_i = (V_i, E_i)$ $(i = 1,\ldots,k)$ be disjoint transitive dags. We define the <u>composition dag</u> $G=(V,E)$ by $V = \bigcup_{i\geq 1} V_i$ and

$$E = \bigcup_{i\geq 1} E_i \cup \{(u,v) \mid u \in V_i,\ v \in V_j \text{ and } (v_i,v_j) \in E_0\}.$$

(Verbally we replace each $v_i \in V_0$ with the dag G_i and make each vertex of G_i precede each vertex of G_j whenever v_i precedes v_j in G_0.) This type of composition is usually referred to as <u>substitution composition</u>. For such a composition we will also use the notation $G = G_0[G_1, G_2, \ldots, G_k]$ and refer to G_0 as the <u>outer factor</u> and $G_1, \ldots G_k$ as the <u>inner factors</u> and we will call v_i the root of G_i.

It is easy to see that each inner factor is a module of G, and G is decomposable (can be written as a composition) iff G contains a nontrivial module. Based on this we say that a transitive dag G is <u>indecomposable</u> iff it contains no nontrivial module.

3. Decomposition Algorithms for the Setup Problem

<u>Lemma 1</u>: Let P be a poset and $M \subseteq P$ a module, then there is always an optimal linear extension which splits M into at most two parts.

<u>Proof</u>: Let f be an optimal linear extension of P and assume that M is split into $\bigcup_{i=1}^{k} M_i$ $(k \geq 3)$, where $\bigcup_{i=1}^{k+1} V_i = P-M$ and

$$f(V_1) \leq f(M_1) \leq f(V_2) \leq f(M_2) \leq \ldots \leq f(V_k) \leq f(M_k) \leq f(V_{k+1}) \tag{1}$$

and $\leq$ holds between two sets if it is true between every pair of elements selected from the two sets.

Let u_i and v_i denote the first and last element of V_i in f, respectively $(i=1, \ldots, k+1)$ and x_i and y_i the first and last element of M_i in f, respectively $(i=1, \ldots, k)$ Because M is a module, any element of P-M separating parts of M in f must be unrelated to any element of M, so

$$f(y_i) \,||\, f(u_{i+1}) \qquad i = 1, \ldots, k$$
$$f(v_i) \,||\, f(x_i) \qquad i = 1, \ldots, k \tag{2}$$

Let g be the linear extension obtained from f by exchanging V_2 and M_2 in f, i.e.

$$g(V_1) \leq g(M_1) \leq g(M_2) \leq g(V_2) \leq g(V_3) \leq g(M_3) \leq \ldots \leq g(V_k) \leq g(M_k) \leq g(V_{k+1}) \tag{3}$$

and leaving the order of the elements within each V_i and M_i unchanged. In this process we eliminated the setups $f(y_1) \,||\, f(u_2)$, $f(v_2) \,||\, f(x_2)$ and $f(y_2) \,||\, f(u_3)$. On the other hand the only places where we may need new setups in g are between y_1 and x_2, y_2 and u_2, v_2 and u_3. This proves that

$s(g) \leq s(f)$ and so g is also optimal.

Let $f:=g$ and repeat the above exchange and argument inductively. After a finite number of exchanges, we obtain the desired linear extension of the lemma.

We note that the above exchange argument is not true for the $k = 2$ case, as it is illustrated in Figure 1 . For the poset shown there an optimal linear extension is $u_1x_1u_2v_2x_2u_3$ and there is no optimal linear extension in which x_2 would be the immediate successor of x_1.

Let us call a module M <u>non-contigous</u> (NC) if there is no optimal linear extension which is contigous on M. The following lemma describes some properties of NC modules (see Figure 2):

<u>Lemma 2</u>: Let $M \subseteq P$ be an NC module and f an optimal linear extension of P splitting M into $M = M_1 \cup M_2$ with $P-M = V_1 \cup V_2 \cup V_3$ and $f(V_1) \leq f(M_1) \leq f(V_2) \leq f(M_2) \leq f(V_3)$. Then

i) $V_1 \neq \emptyset$ and $V_3 \neq \emptyset$

ii) every $u \in V_2$ is incomparable to any element of M

iii) y_1 is incomparable to x_2 (refer to the notation used in Lemma 1)

iv) v_1 precedes every element of M and every element of M precedes u_3 in P

v) if f_M is the linear extension of the subposet M induced by f then f_M is optimal on M

vi) M_1 can be assumed to be a single chain

<u>Proof</u>: ii) follows from M being a module. For i) if we had $V_1 = \emptyset$ or $V_3 = \emptyset$ then we could exchange M_1 and V_2 or V_2 and M_2 respectively, without increasing the number of setups and contradicting M being NC. Similar exchange arguments could be used to show iii) and iv).

To show v) assume that there is a linear extension g_M of M for which $s(g_M) < s(f_M)$. Because of iii) g_M must have at least one setup, say between $x, y \in M$. Define

$$L(x) = \{u \mid g_M(u) \le g_M(x),\ u \in M\} \text{ and } R(y) = \{u \mid g_M(y) \le g_M(u) \quad u \in M\} \tag{4}$$

and extend g_M into a linear extension g of P by (see Figure 3)

$$g(u) = \begin{cases} f(u) & \text{if } u \in V_1 \cup V_3 \\ g_M(u)+|V_1| & \text{if } u \in L(x) \\ f(u)-|M_1|+|L(x)| & \text{if } u \in V_2 \\ g_M(u)+|V_1|+|V_2| & \text{if } u \in R(y) \end{cases} \tag{5}$$

It is clear that $s(g) < s(f)$ contradicting the optimality of f.

For vi) if f breaks up into more than one chain on M_1 then the second, third, etc. chains all could be exchanged with V_2 without increasing the number of setups.

Theorem 3. If g_M is an optimal linear extension of a module M then there is an optimal linear extension g of the whole poset which is consistent with g_M on M, i.e., orders the elements in M the same way as g_M.

Proof: a) If M is not an NC module, let f be an optimal extension, such that f_M is contigous on M. Let g be defined by

$$g(x) = \begin{cases} f(x) \text{ if } x \in M \\ |V'| + g_M(x) \text{ if } x \in M \end{cases} \tag{6}$$

where V' is the set of predecessors of M in f. It is clear that $s(f) = s(g)$.

b) If M is NC, then an optimal f splits M into $M = M_1 \cup M_2$ with

$$f(V_1) \le f(M_1) \le f(V_2) \le f(M_2) \le f(V_3). \tag{7}$$

By Lemma 2 iii) f_M must have a setup between y_1 and x_2, so g_M must have at least one setup too, say between x and y. Define the linear extension g the same way as in (5) and for this g we clearly have again $s(f) = s(g)$.

The concepts of modules and substitution decomposition have wide applicability to many combinatorial optimization problems. (For an extensive bibliography see [1].) This explains why it has been dealt with under different names by many researchers. Buer and Mohring [1] and independently

Steiner [16] have developed polynomial time algorithms to find a (unique) decomposition for a transitive dag (poset). Cunningham [4] has discovered a more general decomposition scheme for digraphs, which contains as a special case the decomposition of posets. For this problem all these methods represent basically equivalent algorithms of the same $O(n^3)$ complexity. Recently Spinrad [15] and Muller and Spinrad [12] improved these by developing implementations of the decomposition algorithm requiring $O(n^2)$ time.

For a transitive dag $G=(V,E)$ the decomposition algorithms identify one of the following three mutually exclusive cases:

i) G has connected components $G_1,\dots, G_k$ $(k>1)$ and $G=G_0[G_1,\dots,G_k]$ where G_0 is the empty graph (no arcs) on k vertices.

ii) The complementary graph of the undirected version of G, $\bar{G}^C$ has connected components $H_1, H_2, \dots, H_k$ $(k>1)$ with vertex sets $V_1, V_2,\dots,V_k$ and $G=G_0[G_1,\dots, G_k]$ where G_0 is a complete dag on k vertices and G_i $(i=1,\dots,k)$ are the induced subgraphs of G by V_i.

iii) Both G and $\bar{G}^C$ are connected, the maximal modules different from V partition V into $V = \bigcup_{i=1}^{k} V_i$ $(k\geq 4)$, and $G=G_0[G_1,\dots, G_k]$ where $G_i=(V_i,E)$ $(i\geq 1)$ and G_0 is the outer factor obtained from G by replacing in it each G_i by a single vertex v_i.

When G is series-parallel then either case i) or ii) applies and iii) can never occur. When G is indecomposable then case iii) applies with $|V_i|=1$ $(i\geq 1)$ and G_0 is isomorphic to G.

The decomposition procedure works in an iterative fashion: It identifies for the original dag G which of the above three cases applies and this is repeated for each factor (module) identified earlier. The process ends when each factor has been decomposed or proved to be indecomposable. At the end, as its output, we obtain a sequence of factors $G_0,G_1,\dots,G_m$ where each G_i is either a factor of the original dag G or a factor of a factor identified earlier. It can be shown, that apart from some sequential differences, these factors define a unique canonical decomposition of a transitive dag [1].

The canonical decomposition of G can be represented by a decomposition tree: The root of the tree is labeled by the vertex set of G and is assigned a type indicator (P for parallel, S for series and N for neighbourhood) depending on which of the above respective cases applies to G. The sons of G in the tree are the inner factors G_i of G, each node labeled by the vertex set (module) V_i. Each internal node V_i gets the type indicator P,S or N depending on which of the three cases applies to the factor $G_i = (V_i, E_i)$. The sons of the node V_i are the inner factors of G_i, etc... This is continued until each factor (module) is decomposed into single vertices corresponding to the leaves of the decomposition tree. An example for this is shown in Figure 4.

The decomposition theory and Theorem 3 enable us to use the following algorithm to solve the setup problem for a poset P represented by the transitive dag G = (V,E).

<u>Algorithm</u> <u>DECOMPOSE</u>

1. Find a module $M \subseteq V$ in G = (V,E) such that $|M| > 1$.
2. Find an optimal linear extension g_M for the poset represented by the induced subgraph (M,E).
3. For each $i,j \in M$ such that $g_M(i) || g_M(j)$ add the setup arc (i,j) to G.
4. If G contains an arc between every pair of vertices STOP, the transitive reduction of G defines the optimal linear extension of P, otherwise G has a module not completely sequenced yet. Apply step 2 to this module.

Since the decomposition theory for posets provides an efficient way of finding the modules and the series or parallel type modules can be efficiently sequenced ([3], [13]), the difficulty in the implementation of DECOMPOSE must lie in executing step 2 for the neighbourhood type modules. In fact this must be NP-hard since the general problem is. On the other hand if we can identify classes of posets, for which step 2 can be executed in polynomial time, then the setup number can be found efficiently for these classes.

The following theorem shows that there is an even stronger relationship between the setup number of a poset and the setup numbers of its modules.

Theorem 4: Let $M \subset P$ be a nontrivial module defining the decomposition $P = P_o[M]$ and let $v_o \in P_o$ be the root of M. Then

i) M is NC iff $s(P) = s(P_o) + s(M) - 1$

ii) M is not NC iff $s(P) = s(P_o) + s(M)$

Proof: Refer to the notation used in lemmas 1 and 2.

First assume that M is NC and let f be any optimal linear extension of P splitting M into $M = M_1 \cup M_2$ as in Lemma 2. Let f_o be the linear extension of P_o defined by

$$f_o(u) = \begin{cases} f(u) & \text{if } u \in V_1 \\ |V_1| + 1 & \text{if } u = v_o \\ f(u) - |M_1| + 1 & \text{if } u \in V_2 \\ f(u) - |M| + 1 & \text{if } u \in V_3 \end{cases} \tag{8}$$

Since V_2 splits M at a setup and the induced f_M must be optimal on M, with the contraction of M into v_o we removed $s(M) + 1$ setups and added two: $f_o(v_o)||f_o(u_2)$ and $f_o(v_2)||f_o(u_3)$. Thus we have

$$s(f_o) = s(f) - s(M) + 1 = s(P) - s(M) + 1 \tag{9}$$

We claim that f_o is optimal for P_o, i.e., $s(P_o) = s(f_o)$: Otherwise there would be a linear extension h_o of P_o such that $s(P_o) = s(h_o) < s(f_o)$. Define h by replacing v_o in h_o by M (ordered by f_M). This h is clearly a linear extension of P and $s(h) = s(P_o) + s(M) < s(f_o) + s(M)$. Substituting $s(f_o)$ by (9) we get $s(h) < s(P) + 1$, which means that h is optimal for P, contradicting the fact that there is no optimal linear extension of P in which M is contigous. In summary we have shown that if M is NC then

$$s(P_o) + s(M) - 1 = s(P). \tag{10}$$

For the second part assume now that M is not NC and let f be an optimal linear extension of P which is contigous on M. It is clear that we must have

$s(M) = s(f_M)$, otherwise we could order the elements in M with less than $s(f_M)$ setups thereby improving on $s(f)$ too. Let f_o be the linear extension of P_o defined by contracting M into v_o and leaving all other elements intact in f. We have $s(f_o) = s(f) - s(M) = s(P) - s(M)$. Furthermore f_o must be optimal for P_o, otherwise there would be a linear extension h_o of P_o with $s(P_o) = s(h_o) < s(f_o)$ and replacing v_o in h_o by M (ordered by f_M) would result in a linear extension h of P such that $s(P) \leq s(h) = s(h_o) + s(M) < s(P)$, a contradiction. Thus we have proved that if M is not NC then

$$s(P_o) + s(M) = s(P) \tag{11}$$

Combining (10) and (11) proves the theorem.

As it has been discussed earlier the canonical decomposition of a poset P into modules can be found in polynomial time, therefore Theorem 4 reduces the setup problem for P to answering the following two questions for each module M_i in the decomposition of P:

1. What is $s(M_i)$?
2. Is M_i NC or not?

Answering question 1 efficiently in the case of general P-s seems to be difficult. Finding an answer for question 2 may be somewhat easier, although whether a module M is NC or not depends not only on M but also on the poset P in which it is embedded. Issues related to these questions are discussed in the remainder of the paper.

Let f be a linear extension of the poset P with setup number $s(f) = t$. It is evident that f splits P into the linear sum of t+1 chains, denoted by $P = C_1 + C_2 + \dots + C_{t+1}$, i.e., $f(C_1) \leq f(C_2) \leq \dots \leq f(C_{t+1})$ and f has its setups between the last element of C_i and the first element of C_{i+1} $(i=1,2,\dots,t)$. Now, fix some i and let $C_{i+1} = \{a = a_1 \prec a_2 \prec \dots \prec a_j\}$ and $C_i = \{b = b_1 \succ b_2 \succ \dots \succ b_k\}$.

Let

$$A = \{a = a_1 \prec a_2 \prec \dots \prec a_{k'}\} \subseteq C_{i+1} \qquad (12)$$

be maximal with respect to $a_{k'} \not\geq b$, and let

$$B = \{b = b_1 \succ b_2 \succ \dots \succ b_{j'}\} \subseteq C_i \qquad (13)$$

be maximal with respect to $b_{j'} \not\leq a$. Now, define $f(a/b)$: $P \dashrightarrow \{1, \dots, |P|\}$ by interchanging the subchains A and B, i.e., $f(a/b)$ orders P into the linear sum $P = C_1 + C_2 + \dots + C_{i-1} + (C_i - B) + A + B + (C_{i+1} - A) + C_{i+2} + \dots + C_{t+1}$. If $f(a/b)$ is a linear extension of P then we call it a <u>chain interchange</u> of f. This concept was introduced by I. Rival [14] and he proved the following:

<u>N-Lemma</u>: Let f be a linear extension of the finite poset P and let $a, b \in P$ be such that $f(b) \,||\, f(a)$. Then one of the following two cases must hold:

i) $f(a/b)$ is a linear extension of P and $s(f(a/b)) \leq s(f)$

ii) $f(a/b)$ is not a linear extension of P and there exists a subset $N = \{a', c, d, b'\}$ such that $a' \prec c$, $d \prec c$, $d \prec b'$, $a' \nprec b'$, $b' \nprec c$, $d \nprec a'$ (see Figure 5).

A poset is called <u>N-free</u> if it contains no cover-preserving subset isomorphic to the poset N of Figure 5. The class of N-free posets ($\overline{N}$) properly contains the class of series-parallel posets (SP), since P is N-free iff the Hasse diagram of P contains no subgraph isomorphic to N while P is series-parallel iff the directed comparability graph of P contains no subgraph isomorphic to N [18]. N-free posets can be recognized in polynomial time [17]. Using the N-Lemma Rival has shown that s(P) can be found for every N-free poset P by a simple "greedy" algorithm [14]. The N-Lemma also can be used to prove additional properties for NC modules:

<u>Theorem 5</u>: Let $M \subset P$ be an NC module and f an optimal linear extension splitting M as in Lemma 2. Then the Hasse diagram of the posets $P_1 = V_1 \cup M_1 \cup V_2$ and $P_2 = V_2 \cup M_2 \cup V_3$ both must contain a cover-preserving subset isomorphic to the poset N shown in Figure 5.

Proof: f splits into chains on P: Let the last one of these chains on V_1 be C_1 and let $V_2 = C_{21} + C_{22} + \ldots + C_{2k}$ $(k \geq 1)$. If y_1 and u_2 are as defined in Lemma 2, then $f(u_2/y_1)$ is either a linear extension of P with $s(f(u_2/y_1)) = s(f)$ or $C_1 \cup M_1 \cup C_{21}$ contains an $N = \{a',c,d,b'\}$ by the N-Lemma, where $a',c \in V_2$, $d \in V_1$ and $b' \in M_1 \cup V_1$. Applying this argument inductively we cannot remove every C_{2i} from between M_1 and M_2 since this would contradict that M is NC, so we must find an N in P_1. The proof is similar for P_2.

Corollary 6: Let $M \subset P$ be a module. If any one of the following conditions is satisfied then M is not NC:

i) M contains an initial element of P.

ii) M contains a terminal element of P.

iii) P is series-parallel

iv) P is N-free

Proof: i) and ii) follow from Lemma 2 i); iii) and iv) follow from Theorem 5 since neither series-parallel nor N-free posets can contain an N.

We note that iii) in Corollary 6 implies that every NC module of a poset P must be contained in a neighbourhood module of P. The following theorem applies to this situation.

Theorem 7: Let M be a son of the neighbourhood module $M' \subseteq P$ in the decomposition tree of P. If M is NC then M' must have other sons U_1, U_2', U_2'', U_3 such that

a) every $u \in U_1$ precedes every $v \in M$

b) no $u \in U_1' \cup U_2''$ is related to any $v \in M$

c) every $u \in U_3$ succeeds every $v \in M$

Proof: Let f be an optimal linear extension of P splitting M as in Lemma 2. For the sets V_1, V_2 and V_3 defined in Lemma 2, there exist $z_1 \in V_1$ and $z_3 \in V_3$ such that z_1 precedes every $v \in M$ and z_3 succeeds every $v \in M$ in P. By Theorem 5 $P_1 = V_1 \cup M_1 \cup V_2$ must contain an $N = \{a',c,d,b'\}$ with $a',c \in V_2$.

It is easy to see that z_1, a', c and z_3 all must be contained in different sons of M'. Let these be U_1, U_2', U_2'' and U_3 respectively, i.e., $z_1 \varepsilon U_1$, $a' \varepsilon U_2'$, $c \varepsilon U_2''$ and $z_3 \varepsilon U_3$. These modules satisfy the statement of the theorem.

Let $\mathcal{B}$ be the class of bipartite posets, i.e., $P \varepsilon \mathcal{B}$ iff no element of P has both predecessor and successor. By Corollary 6 if $P \varepsilon \mathcal{B}$ then P cannot have an NC module. Based on Theorem 7 we can define the class of posets $\overline{NC}$ by $P \varepsilon \overline{NC}$ iff no neighbourhood module $M' \subseteq P$ has sons M, U_1, U_2', U_2'' and U_3 satisfying the conditions of Theorem 7. We note that based on the canonical decomposition of posets we can recognize the elements of the class $\overline{NC}$ in polynomial time. It is evident that $\overline{NC}$ properly contains each of the classes $\mathcal{B}$, $\mathcal{SP}$ and $\overline{N}$. By Theorem 4 for posets in $\overline{NC}$ the setup problem fully reduces to finding the setup number of the factors in the decomposition of the poset:

<u>Corollary 8</u>: Let $P \varepsilon \overline{NC}$ and let $M_1, M_2, \ldots, M_k$ be the sons of P in the decomposition tree of P with outer factor P_0, i.e., $P = P_0[M_1, M_2, \ldots, M_k]$ then

$$s(P) = s(P_0) + \sum_{i=1}^{k} s(M_i). \tag{14}$$

Unfortunately Corollary 8 also implies that finding these $s(M_i)$ values must be NP-hard in general, because $\mathcal{B} \subset \overline{NC}$ and the setup problem is known to be NP-hard on $\mathcal{B}$ [13]. On the other hand we can limit our attention to classes of posets, where we know in advance the $s(M_i)$ values occuring: Consider a set $\psi = \{H_i\}$ of disjoint indecomposable posets; we say that a poset P is <u>ψ-decomposable</u> if in the canonical decomposition of P each factor is either N-free or isomorphic to a member in ψ. We denote this class by $\mathcal{D}(\psi)$. Naturally when $\psi = \emptyset$ then $\mathcal{D}(\psi) = \overline{N}$ and for any other ψ, $\mathcal{D}(\psi) \supset \overline{N}$. For a given <u>finite</u> ψ, we can recognize whether a poset P is in $\mathcal{D}(\psi)$ or not in polynomial time (by looking at the canonical decomposition of P). Futhermore, if we know $s(H_i)$ for each $H_i \varepsilon \psi$ and a poset P is in $\mathcal{D}(\psi) \cap \overline{NC}$ then we can find s(P) in polynomial time, by the repetitive application of Corollary 8 for the factors of P. This way it is possible to define classes of posets for which both the recognition problem and the setup problem are efficiently solvable. Of course this approach can lead to meaningful classes $\mathcal{D}(\psi)$ only if ψ contains only a small number of posets. As an example we present one such class: Let N denote the class of those posets where $P \in N$ iff each neighbourhood factor in the canonical decomposition of P is either N-free or it contains exactly four sons. It is easy to see that $N \equiv \mathcal{D}(\psi_o)$, where ψ_o consists of the single poset N shown in Figure 5. By Theorem 7 $N \subset \overline{NC}$ and so for every $P \in N$ s(P) is the sum of the setup numbers of its factors. But each factor M_i of such a P is either N-free and so $s(M_i)$ can be easily found or M_i is isomorphic to N which means that $s(M_i) = s(N) = 1$. This procedure is demonstrated for the poset P in Figure 4. The poset P has five nontrivial indecomposable modules $M_1, M_2, \ldots, M_5$. M_i is isomorphic to N and we have $s(M_i) = 1$ for $i = 1,2,3$; M_4 is N-free so by a "greedy" algorithm we find $s(M_4) = 2$; since the outer factor of the poset $\{1,2,\ldots,17\}$ is isomorphic to N we get $s(\{1,2,\ldots,17\}) = 1 + s(M_1) + s(M_2) + s(M_3) + s(M_4) = 6$; $s(M_5) = 1$ and $s(\{20\}) = 0$ and because the root node in the decomposition tree represents a chain outer factor, we get $s(P) = s(\{1,2,\ldots,17\}) + s(M_5) = 7$. The optimal linear extension could be constructed similarly "piecewise".

Finally we note that N is not contained in any previously known classes of posets over which the setup problem is polynomially solvable, actually properly contains the class $\overline{N}$ which was dealt with in [14].

Acknowledgement

The author gratefully acknowledges Professor I. Rival for his helpful comments. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, under Grant No. A1798.

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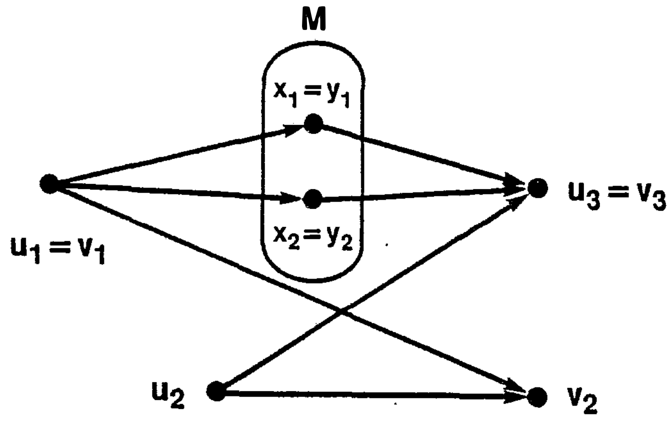


Figure 1

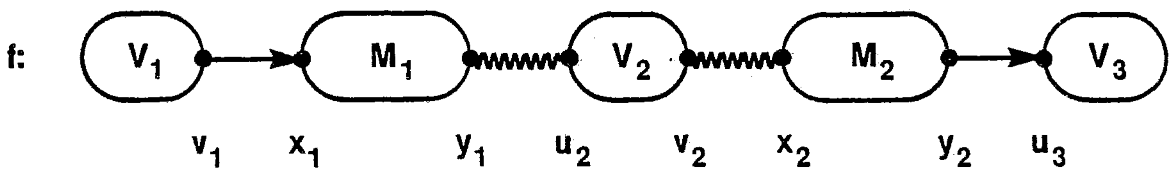


Figure 2

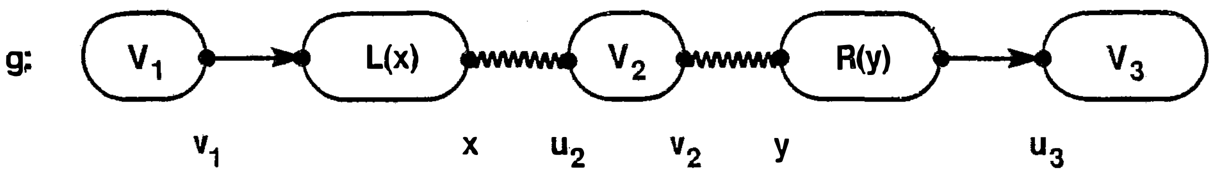
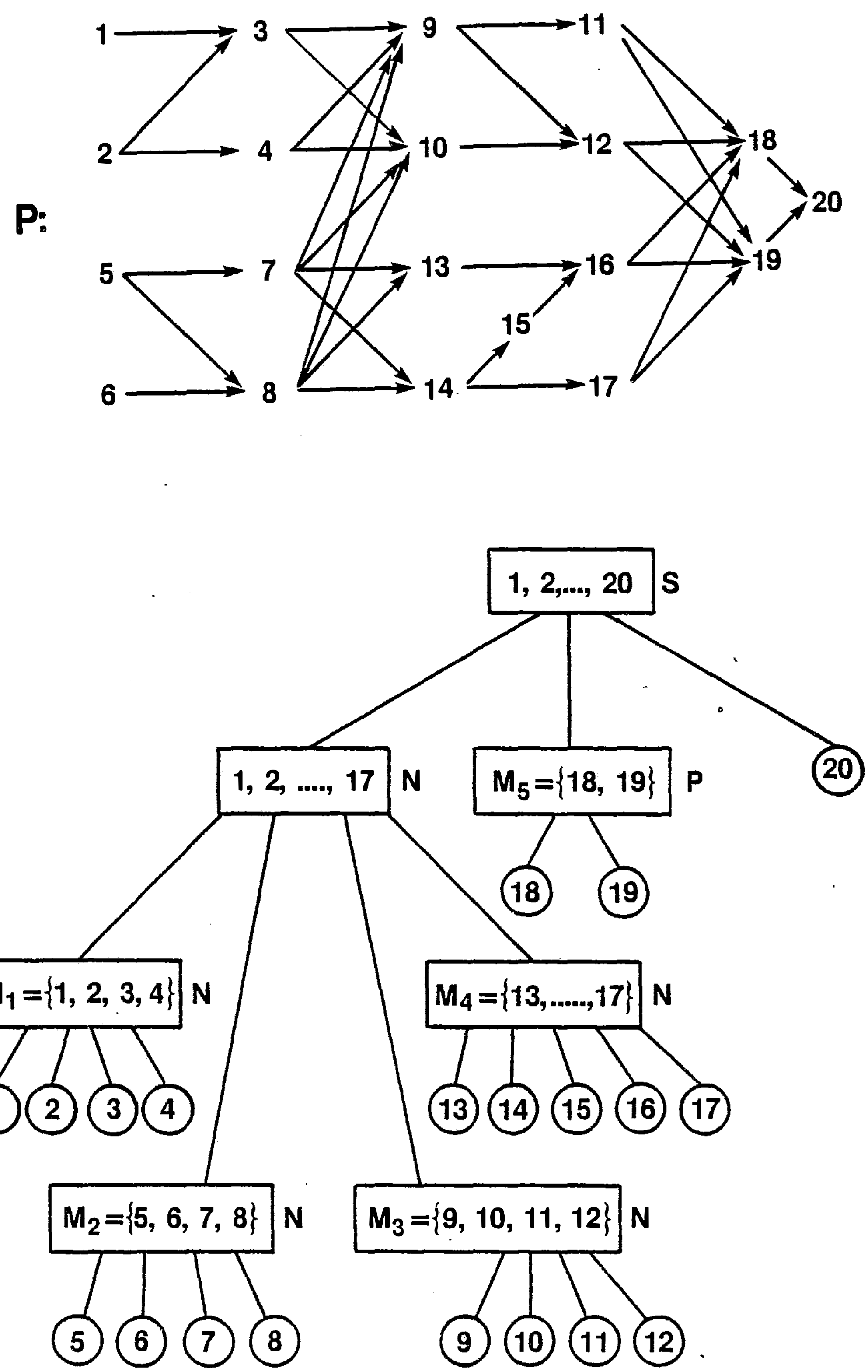


Figure 3



A poset P and its decomposition tree with the type indicators beside each internal node.

Figure 4

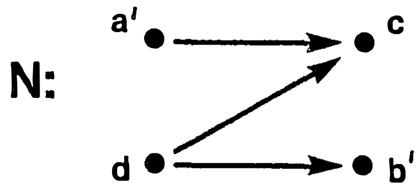


Figure 5

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