THE TRANSIENT BEHAVIOUR OF TRANSFER LINES WITH BUFFER INVENTORIES

By

G.J. MILTENBURG
Assistant Professor of Production and Management Science

FACULTY OF BUSINESS

McMASTER UNIVERSITY

HAMILTON, ONTARIO

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Abstract

The transfer line models in the literature are planning models rather than operational models. That is, they are very useful for planning or designing the transfer line, but are less useful for controlling daily operations of the line. The performance measure, used in these models is the expected efficiency of the line. In this paper a method is presented for calculating the variance of the efficiency of the line. These two performance measures can be used to construct a confidence interval for the expected production during a specified time interval (say, a shift). This confidence interval is an operational guide for the production manager.

1. Introduction

The transfer line is an important production system. It consists of a number of stations connected so that all work-in-process goes through the same sequence of stations. Figure 1 depicts a K-station transfer line with K-1 buffer inventories. The problem of transfer lines and buffer inventories in transfer lines has had a long history. A good review of the literature is found in Buzacott and Hanifan [1978] and Gershwin and Berman [1981]. The models in the literature are planning models rather than operational models. That is, they are very useful for planning or designing the transfer line, but are of limited use in controlling the daily operation of the transfer line. In most models, the performance measure is the expected efficiency of the line E(A). Gershwin and Schick [1983] interpret the efficiency, A, as the probability that a product emerges from the line during a cycle. Equivalently, A can be interpreted as the ratio of what the system actually produces over some period to what it could have produced in the same period had there been no lost production (Buzacott [1971]). The expected efficiency E(A) is a long-run measure and may be considerably different from the actual efficiency (or actual production) over a production shift. The manager needs a confidence interval estimate for actual production he can expect, so that he can schedule material handling, shipping and overtime.

This paper shows how to calculate V(A) the variance of the efficiency of a transfer line. Confidence intervals for the expected production over specified planning intervals can then be calculated. These confidence intervals are an operational guide for the production manager. (It is interesting to note that Hatcher [1969] first identified the need to calculate the variability of the efficiency of a transfer line.)

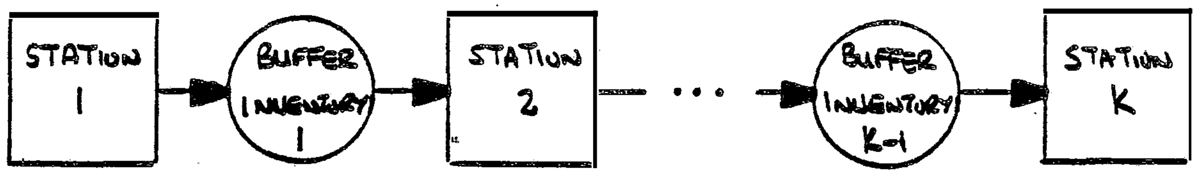


Figure 1 K - Station Transfer Line with K-1 Buffer Inventories

In what follows; Section 2 describes the structure of the transfer line models. Section 3 shows how E(A) and V(A) are calculated. Section 4 gives two illustrative examples using well-known transfer line models. Section 5 outlines some computational considerations. Section 6 discusses extensions to this research.

2. Markov Chain Models

Most of the transfer line models in the literature are Markov chain models. Each of the K stations in the transfer line can be described by up to three variables. They are the processing time q_i, station failure time l_i and station repair time b_i $(i=1,2,\dots,K)$. The buffer inventory is described by specifying its maximum size s_i $(i=1,2,\dots,K-1)$. Each station can be either up (that is, working) or down (that is, under repair). Let Q, L, B and S be vectors whose elements are q_i, l_i, b_i $(i=1,2,\dots,K)$ and s_j $(j=1,2,\dots,K-1)$ respectively. For finite size buffer inventories the states of a K station transfer line constitute a Markov chain. Each station j can be up or down and each buffer inventory can have content $0,1,2,\dots,s_j$. The total number of states (TNS) is

(1) $\quad TNS = 2^K \prod_{j=1}^{K-1}(s_j + 1)$

Transitions between states are functions of the four vectors Q, L, B and S. The corresponding transition probability matrix P, with the elements p_{ij}, can be specified in terms of the elements in Q, L, B and S, and the steady state probabilities of the states can, in principle, be determined. If π is the vector of steady state transition probabilities with elements π_i, $(i=1,2,\dots,TNS)$ then π can be determined from the well known result $\pi=P\pi$, or equivalently,

$$\pi_j = \sum_{i=1}^{TNS} p_{ij}\pi_i \qquad j=1,2,\dots,TNS-1$$

or $\pi_j(p_{jj}-1) + \sum_{\substack{i=1 \\ i \neq j}}^{TNS} p_{ij}\pi_i = 0 \qquad j=1,2,\ldots,TNS-1$ (2)

and $\sum_{j=1}^{TNS} \pi_j = 1$.

There are many computer procedures available for solving this large set of simultaneous equations. (See section 5.) The expected efficiency E(A), is the sum of the π_j, $j \in U$, where U is the set of states that result in a finished unit being produced by the line.

3. Calculation of E(A) and V(A)

As mentioned

(3) $E(A) = \sum_{j \in U} \pi_j$

Let,

(4) $V(A) = \sum_{j \in U} V_j$

where V_j is the variance of the steady state transition probability of state j. In order to develop expressions for the calculation of V_j define the following Markov chain variables.

p_{ij} = element of the transition probability matrix P,
= probability that the process will occupy state j at the next transition given that it currently occupies state i.

$\emptyset_{ij}(n)$ = n-step transition probability from state i to state j,
= probability that the process will occupy state j at time n given it occupied state i at time 0.

$v_{ij}(n)$ = state occupancy random variable,
= the number of times state j is entered through time n given that the system started in state i at time 0.

$\bar{v}_{ij}(n)$ = mean of the state occupancy random variable.

$\hat{v}_{ij}(n)$ = variance of the state occupancy random variable.

θ_{ij} = first passage time random variable,

= the number of transitions to reach state j for the first time if the system was in state i at time 0.

$\bar{\theta}_{ij}$ = mean of the first passage time random variable.

$\hat{\theta}_{ij}$ = variance of the first passage time random variable.

$f_{ij}(n)$ = the probability that $\theta_{ij} = n$.

The random variables that are of interest to us are $\lim_{n\to\infty} \frac{\bar{v}_{ij}(n)}{n}$ and $\lim_{n\to\infty} \frac{\hat{v}_{ij}(n)}{n}$. The first expression represents the probability that a state j is entered given that the process started in state i at time zero. The second expression represents the variability of the state occupancy random variable per transition. We will proceed to show that

$$\lim_{n\to\infty} \frac{\bar{v}_{ij}(n)}{n} = \pi_j = \frac{1}{\bar{\theta}_{jj}} .$$

This equation shows that the probability of entering state j is independent of the starting state i. As well

$$\lim_{n\to\infty} \frac{\hat{v}_{ij}(n)}{n} = \frac{\hat{\theta}_{jj}}{\bar{\theta}_{jj}^{\,3}} = V_j .$$

Similarily the variance is also independent of the starting state i and so is denoted V_j.

From the definition of a Markov chain

(5) $\Phi(n) = P^n \qquad n = 0,1,2,\ldots$

Let

(6) $\Phi(n) = P^n = \Phi + T(n) \qquad n = 0,1,2,\ldots$

where Φ is the limiting steady state transition probability matrix, with elements ϕ_{ij}, where $\phi_{ij} = \phi_{ij}(\infty)$, and T(n) is the matrix of transient terms (which disappear when n is large) with elements $t_{ij}(n)$. (Notice that Φ is a matrix where each row is the vector π.)

When working with Markov chain equations it is often useful to transform the equations and solve the equations in the transformed domain. Taking the inverse of the transform solution gives the solution to the original problem. (The great advantage is that working in the transformed domain permits analysis of the Markov process before it reaches steady state.) For discrete time Markov chains, the geometric transform is used. That is, for a discrete function $f(n) > 0$ $(n=0,1,2,...)$, $f(n) = 0$ $(n < 0)$; the geometric transform $f^g(z)$ is defined

(7) $f^g(z) = \sum_{n=0}^{\infty} f(n)z^n$.

$f^g(z)$ exists if the series converges. (See Appendix 1, for a brief review of geometric transforms in Markov chains).

Equation 6 in transformed form is

$$\Phi^g(z) = \frac{1}{1-z}\,\Phi + T^g(z)$$

or

$$\phi^g_{ij}(z) = \frac{1}{1-z}\,\phi_{ij} + t^g_{ij}(z)\,.$$

But ϕ_{ij} is the limiting steady state probability of state j, and so is independent of the starting state i. That is $\phi_{ij} = \pi_j$, and so

(8) $\phi^g_{ij}(z) = \frac{1}{1-z}\,\pi_j + t^g_{ij}(z)$.

Consider now $v_{ij}(n)$. Clearly,

(9) $\bar{v}_{ij}(n) = \sum_{m=0}^{n} \phi_{ij}(m)$

or, in transformed form

(10) $\bar{v}^g_{ij}(z) = \frac{1}{1-z}\,\phi^g_{ij}(z)$.

It can be shown that the second moment of the mean occupancy is (see pp. 240-270 of the Howard [1971]),

$$\overline{v^2_{ij}}(n) = 2\sum_{m=0}^{n}\sum_{r=m}^{n} \phi_{ij}(m)\quad \phi_{ij}(r-m) - \sum_{m=0}^{n} \phi_{ij}(m) \qquad n=0,1,2,...$$

which simplifies to

(11) $\overline{v^2_{ij}}(n) = 2\sum_{m=0}^{n} \phi_{ij}(m)\bar{v}_{jj}(n-m) - \bar{v}_{ij}(n) \qquad n=0,1,2,...$

In the transformed domain equation 11 becomes

$$\overline{v^2_{ij}}^g(z) = 2\,\phi^g_{ij}(z)\bar{v}^g_{jj}(z) - \bar{v}^g_{ij}(z) \qquad n=0,1,2,\ldots$$

which simplifies to

$$(12)\quad \overline{v^2_{ij}}^g(z) = \frac{1}{1-z}\ \phi^g_{ij}(z)\ [2\,\phi^g_{ij}(z) - 1]\ .$$

$\hat{v}_{ij}(n)$ can be calculated by using the definition of a variance. That is

$$(13)\quad \hat{v}_{ij}(n) = \overline{v^2_{ij}}(n) - [\bar{v}_{ij}(n)]^2$$

where $\bar{v}_{ij}(n)$ is given by equation 9 and $\overline{v^2_{ij}}(n)$ is given by equation 11.

(Alternatively equations 10 and 12 in the transformed domain could be used.)

It turns out the equations for $\bar{v}_{ij}(n)$ and $\overline{v^2_{ij}}(n)$ have a number of terms which become insignificant for large n. Substituting equation 8 into equation 10 gives

$$(14)\quad \bar{v}^g_{ij}(z) = \frac{1}{1-z}\left(\frac{1}{1-z}\ \pi_j + t^g_{ij}(z)\right)$$

$$= \frac{\pi_{ij}}{(1-z)^2} + \frac{t^g_{ij}(z)}{1-z}$$

For large n, the inverse of this transform equation is

$$(15)\quad \bar{v}_{ij}(n) = (n+1)\ \pi_j + t^g_{ij}(1)\ .$$

We can calculate a similar expresssion for $\overline{v^2_{ij}}(n)$ and then use equation 13 to calculate $\hat{v}_{ij}(n)$. The result would be

$$(16)\quad \hat{v}_{ij}(n) = \pi_j[\pi_j+2t^g_{jj}(1)-1]n + \pi_j^{\ 2} + \pi_j[2t^g_{jj}(1)-1]$$
$$- 2\pi_j[t^{g'}_{ij}(1)+t^{g'}_{jj}(1)] + t^g_{ij}(1)[2t^g_{jj}(1)-1-t^g_{ij}(1)] \text{ for large n,}$$

where

$$t^{g'}_{ij}(1) = \left.\frac{dt^g_{ij}(z)}{dz}\right|_{z=1}$$

Finally consider θ_{ij} . Recall that

$$(17)\quad f_{ij}(n) = \text{Prob}\ (\theta_{ij}=n),$$

and so

$$(18)\quad \phi_{ij}(n) = \delta_{ij}\delta(n) + \sum_{m=0}^{n} f_{ij}(m)\phi_{jj}(n-m) \qquad n=0,1,2\ldots$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise; $\delta(n) = 1$ if $n = 0$ and zero otherwise. In transformed form equation 18 is

$$(19)\quad \phi^{g}_{ij}(z) = \delta_{ij} + f^{g}_{ij}(z)\, \phi^{g}_{jj}(z)$$

or

$$(20)\quad f^{g}_{ij}(z) = [\phi^{g}_{ij}(z) - \delta_{ij}] \,/\, \phi^{g}_{jj}(z)\,.$$

Substituting equation 8 into 20 gives

$$(21)\quad f^{g}_{ij}(z) = \begin{cases} \dfrac{\pi_j + (1-z)t^{g}_{ij}(z)}{\pi_j + (1-z)t^{g}_{jj}(z)} & i \neq j \\[2ex] 1 - \dfrac{1-z}{\pi_j + (1-z)t^{g}_{jj}(z)} & i = j\,. \end{cases}$$

Since $f_{ij}(n)$ is a probability distribution function, $f^{g}_{ij}(1)=1$ from equation 7, and the moments of θ_{ij} can be obtained by differentiating $f^{g}_{ij}(z)$, and evaluating the results at $z=1$. In particular

$$(22)\quad \bar{\theta}_{ij} = f^{g'}_{ij}(1) = \begin{cases} [t^{g}_{jj}(1) - t^{g}_{ij}(1)]/\pi_j & i \neq j \\ 1/\pi_j & i = j\,. \end{cases}$$

In a more compact form

$$(23)\quad \bar{\theta}_{ij} = [\delta_{ij} + t^{g}_{jj}(1) - t^{g}_{ij}(1)]/\pi_j\,.$$

Similarly the second moment of the mean first passage time is

$$(24)\quad \overline{\theta^{2}_{ij}} = f^{g''}_{ij}(1) + f^{g'}_{ij}(1)$$

which, after some simplication, is

$$(25)\quad \overline{\theta^{2}_{ij}} = \theta_{ij}\left[\frac{2t^{g}_{jj}(1)}{j} + 1\right] + 2[\,t^{g'}_{jj}(1) - t^{g'}_{ij}(1)]/\pi_j\,.$$

Again, the variance of the first passage time random variable, is calculated from

$$(26)\quad \hat{\theta}_{ij} = \overline{\theta^{2}_{ij}} - (\bar{\theta}_{ij})^{2}$$

using equations 23 and 25. If we are only interested in calculating the recurrence time moments (that is $i=j$) then equations 23 and 26 simplify to

(27) $\bar{\theta}_{jj} = \frac{1}{\pi_j}$

(28) $\hat{\theta}_{jj} = \frac{1}{\pi_j^2} [2t_{jj}^g(1) + \pi_j - 1]$

Using equations 15 and 27 gives

(29) $\lim_{n\to\infty} \frac{\bar{v}_{ij}(n)}{n} = \pi_j = \frac{1}{\bar{\theta}_{jj}}$

while equation 16 gives

(30) $\lim_{n\to\infty} \frac{\hat{v}_{ij}(n)}{n} = \pi_j[\pi_j + 2t_{jj}^g(1) - 1]$

or, after substituting equations 27 and 28;

(31) $\lim_{n\to\infty} \frac{\hat{v}_{ij}(n)}{n} = \frac{\hat{\theta}_{jj}}{\bar{\theta}_{jj}^3}$

In Appendix 2 we show that the matrix $T^g(1)$ with elements $t_{ij}^g(1)$ can be calculated from

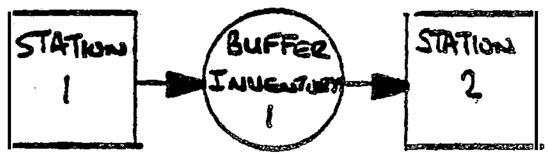
(32) $T^g(1) = [I - P + \Phi]^{-1} - \Phi$

where I is the identity matrix.

4. Illustrative Examples

Example 1 Consider the two station-one buffer inventory transfer line of Figure 2. Suppose an appropriate model for this line is the following model from Buzacott [1971].

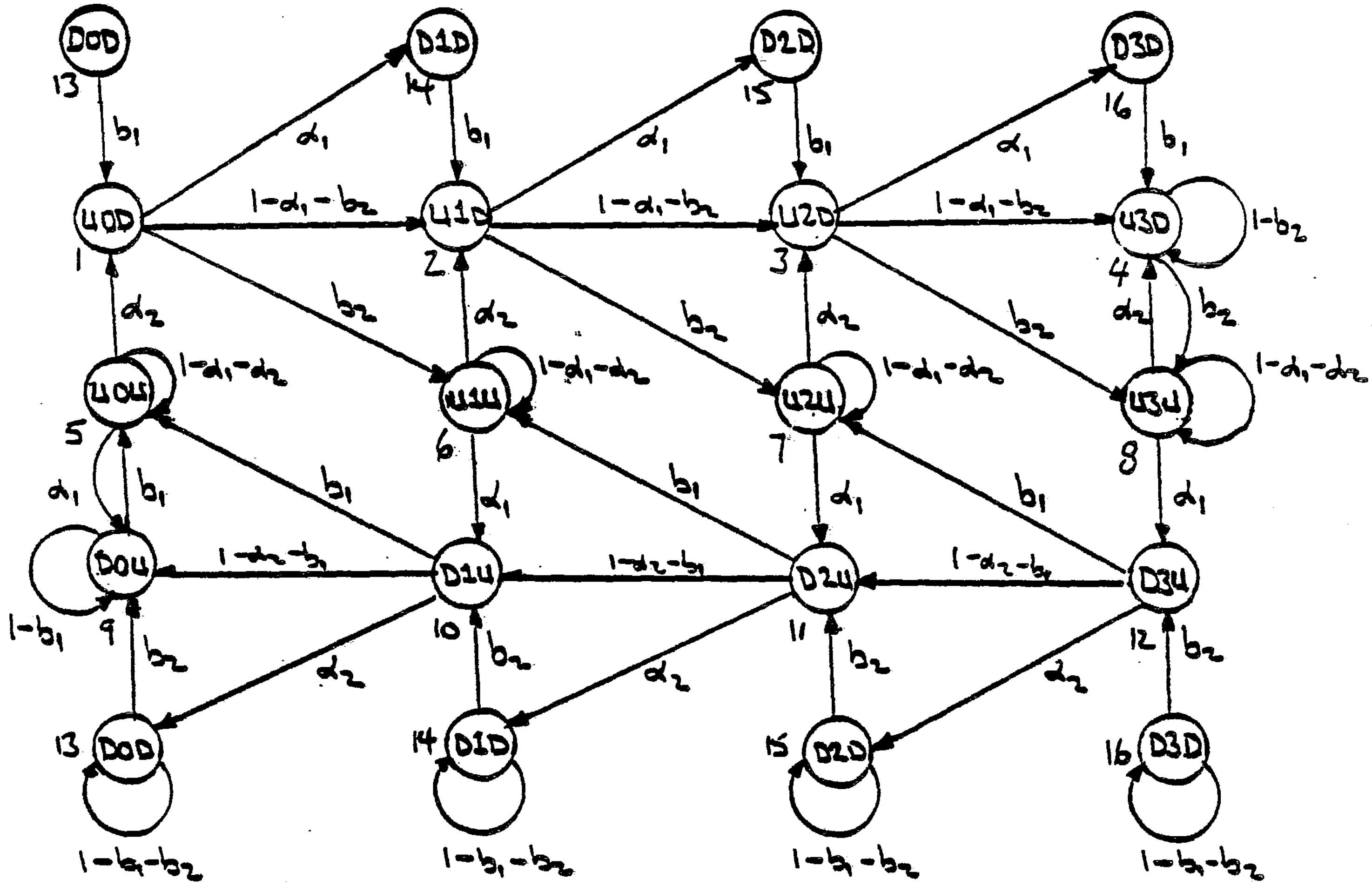
Let α_i be the probability that station i breaks down during a cycle and let b_i be the probability that a repair to a broken down station is completed during a cycle. Suppose that $\alpha_1 = \alpha_2 = 0.20$ and $b_1 = b_2 = 0.30$; and that the buffer inventory between the stations has a capacity of 3 units. (That is, s=3). Buzacott uses a Markov chain formulation (see Figure 3) and derives the following closed form expression for the expected efficiency of a two station transfer line.



$\alpha_1 = 0.20$ $s = 3$ $\alpha_2 = 0.20$

$b_1 = 0.30$ $b_2 = 0.30$

Figure 2 Two Station - One Buffer Transfer Line



Note: D2U represents the state where station 1 is down (undergoing repair), the buffer inventory contains 2 pieces, and station 2 is up. The states are numbered 1 through 16 beginning with state 1 - U0D. (Each state corresponds to a row in equation 34.)

Figure 3 Markov Chain For Model in Example 1

$$(33)\quad E(A) = \begin{cases} \dfrac{1-tC^{s}}{1 + X_1 - (1 + X_2)tC^{s}} & t/1 \\[2ex] \dfrac{1+r - b_2(1+X) + s\, b_2(1+x)}{(1+2X)(1+r-b_2(1+X))+sb_2(1+X)}^{2} & t=1 \end{cases}$$

where

$$C = \frac{(\alpha_1+\alpha_2)(b_1+b_2) - \alpha_1 b_2(\alpha_1+\alpha_2+b_1 + b_2)}{(\alpha_1+\alpha_2)(b_1+b_2) - \alpha_2 b_1(\alpha_1+\alpha_2+b_1+b_2)}$$

$$t = \alpha_2 b_1 / \alpha_1 b_2$$

$$x_1 = \alpha_1/b_1 \qquad x_2 = \alpha_2/b_2$$

and when $t = 1$

$$x_1 = x_2 = x$$

$$r = \alpha_2/\alpha_1 = b_2/b_1$$

For this example C=1, t=1, x=0.6667, r=1 and so equation 33 gives E(A) = 0.50.

The transition probability matrix P for the Markov chain of Figure 3 is

$$(34)\quad P = \begin{bmatrix}
0 & .5 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 \\
0 & 0 & .5 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 & 0 \\
0 & 0 & 0 & .5 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2 \\
0 & 0 & 0 & .7 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.2 & 0 & 0 & 0 & .6 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .2 & 0 & 0 & 0 & .6 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .2 & 0 & 0 & 0 & .6 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .2 & 0 & 0 & 0 & .6 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .5 & 0 & 0 & 0 & .2 & 0 \\
.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .4 & 0 & 0 & 0 \\
0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .4 & 0 & 0 \\
0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .4 & 0 \\
0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 & 0 & .4
\end{bmatrix}$$

Using equation 2 gives

$$(35)\quad \pi = [0.033\ 0.033\ 0.033\ 0.166\ 0.150\ 0.050\ 0.050\ 0.150\ 0.166\ 0.033\ 0.033\ 0.033\ 0.011\ 0.022\ 0.022\ 0.011]\ .$$

Define a vector consisting of the W diagonal elements of a W x W matrix A as A_D. That is A_D has elements a_{ii} i=1,2,...,W. $T^g(1)$ is calculated from equation 32, from which

(36) $T_D^g(1)$ = [0.7592 0.7128 0.7592 3.6750 2.6806 2.2081 2.2081 2.6806
3.6750 0.7592 0.7128 0.7592 1.6049 1.5740 1.5740 1.6049].

Define $\bar{\Theta}_D$ as a vector with elements $\bar{\theta}_{ii}$. Using equation 27 gives

(37) $\bar{\Theta}_D$ = [30.0 30.0 30.0 6.0 6.6 20.0 20.0 6.6 6.0
30.0 30.0 30.0 90.0 45.0 45.0 90.0].

Similarly define $\hat{\Theta}_D$ as a vector with elements $\hat{\theta}_{ii}$. Using equation 28 gives

(38) $\hat{\Theta}_D$ = [496.5 413.1 496.5 234.6 200.5 1386.5 1386.5 200.5 234.6
496.5 413.1 496.5 17989.4 4394.7 4394.7 17989.5].

Substituting equations 37 and 38 into 30 gives the variances of the steady state probabilities

(39) V_D = [0.01839 0.01530 0.01839 1.08625 0.67653 0.17332 0.17332 0.67653
1.08625 0.01839 0.01530 0.01839 0.02468 0.04823 0.04823 0.02468]

where V_D is a vector with elements V_j. For this model the "productive" states are states 5,6,7,8,10,11 and 12. From equations 35 and 39

π_5 = 0.1500	V_5 = 0.6765
π_6 = 0.0500	V_6 = 0.1733
π_7 = 0.0500	V_7 = 0.1733
π_8 = 0.1500	V_8 = 0.6765
π_{10} = 0.0333	V_{10} = 0.0184
π_{11} = 0.0333	V_{11} = 0.0153
π_{12} = 0.0333	V_{12} = 0.0184

Hence

$E(A) = \pi_5 + \pi_6 + \pi_7 + \pi_8 + \pi_{10} + \pi_{11} + \pi_{12} = 0.50$

(40) $V(A) = 1.7517$

The expected efficiency, of course, agrees with our previous results.

Suppose, for example, that the transfer line runs for 800 cycles during a production shift. On average $800E(A) = 400$ of these cycles will be productive and that on $800(1-E(A)) = 400$ cycles no units will be produced. The variance of the expected production is $800V(A) = 1401.4$ and so the standard deviation is $\sqrt{1401.4} = 37.43$. Interestingly, the central limit theorem applies to this large sum of dependent trials. (See for example, p. 275 of Howard [1971] or a copy of the proof by A.A. Markov on pp. 552-576 of Howard [1971]). Consequently, the 95% confidence interval for the expected production during the shift is

$$0.50*800 \pm 1.96\sqrt{1.7517*800} = (327, 473) \text{ units.}$$

This production range is very useful to a manager because it estimates how high and how low the actual production could be. The point estimate 0.50*800 for the expected production on the shift does not give this information.

To illustrate the flexibility of this approach consider the following situation. Suppose the above model is appropriate for the transfer line except that there is a constraint on repairman. In fact, if <u>both</u> stations are down then the probability of repairing a station changes from 0.30 to 0.15. That is

$$(41)\quad b_1 = b_2 = \begin{cases} 0.15 & \text{states 13(DOD), 14(D1D), 15(D2D), 16(D3D)} \\ 0.30 & \text{all other states} \end{cases}$$

The transition probability matrix is the same as before (equation 34) except that rows 13 to 16 are changed to

(42) row 13 ... 0.15 0 0 0 0 0 0 0 0.15 0 0 0 0.7 0 0 0

row 14 ... 0 0.15 0 0 0 0 0 0 0 0.15 0 0 0 0.7 0 0

row 15 ... 0 0 0.15 0 0 0 0 0 0 0 0.15 0 0 0 0.7 0

row 16 ... 0 0 0 0.15 0 0 0 0 0 0 0 0.15 0 0 0 0.7

Again, using equations 2, 27, 28, 29 and 31, gives

$\pi_5 = 0.140625$ $\quad V_5 = 0.65197$

$\pi_6 = 0.046875$ $\quad V_6 = 0.16342$

$\pi_7 = 0.046875$ $\quad V_7 = 0.16342$

$\pi_8 = 0.140625$ $\quad V_8 = 0.65197$

$\pi_{10} = 0.031250$ $\quad V_{10} = 0.01675$

$\pi_{11} = 0.031250$ $\quad V_{11} = 0.01361$

$\pi_{12} = 0.031250$ $\quad V_{12} = 0.01675$

From which

$E(A) = 0.4688$ and $V(A) = 1.6779$

The constraint on repairmen decreases the expected efficiency from 0.50 to 0.4688. Suppose again that the line runs for 800 cycles during a production shift. The 95% confidence interval for the expected production is (303, 447) units. To justify additional repairmen, the expected additional cost must not exceed the profit associated with producing an additional 800 (0.50-0.4688) = 25 units.

Unfortunately if slight adjustments, such as this are made, then the closed form solution of equation 33 cannot be used.

Example 2 In addition to the two station variables (up time and down time) considered in example 1, consider now a model (by Gershwin and Berman [1981]) which adds a third variable - processing time. Using their notation define

U_i = mean uptime in cycles for station i,

D_i = mean repair time in cycles for station i,

R_i = mean processing time in cycles for station i,

$T = \Sigma\ U_i + D_i + R_i$ for all stations.

and let

$\alpha_i = U_i/T, \quad b_i = D_i/T, \quad r_i = R_i/T$

One of their examples has

Station, i	U_i	α_i	D_i	b_i	R_i	r_i
1	1	0.0476	5	0.2381	3	0.1429
2	2	0.0952	6	0.2857	4	0.1905

with s=3. Gershwin and Berman assume that the limiting steady state probabilities are of the form

(43) $\pi(a_1,n,a_2) = c\, X^n\, Y_1^{a_1}\, Y_2^{a_2}$

where $a_i = 0$ if station i is down, $a_i = 1$ if station i is up and $n=0,1,\ldots,s$. An algorithm is given to calculate C, X, Y_1 and Y_2 and hence the limiting steady state probabilities.

The Markov chain for this example is shown in Figure 4. The corresponding transition probability matrix is

$$
(43)\ P = \begin{bmatrix}
.5238 & .1429 & 0 & 0 & .2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .0476 & 0 & 0 & 0 \\
0 & .5238 & .1429 & 0 & 0 & .2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .0476 & 0 & 0 \\
0 & 0 & .5238 & .1429 & 0 & 0 & .2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .0476 & 0 \\
0 & 0 & 0 & .7143 & 0 & 0 & 0 & .2857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.0952 & 0 & 0 & 0 & .7143 & .1429 & 0 & 0 & .0476 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .0952 & 0 & 0 & .1905 & .5238 & .1429 & 0 & 0 & .0476 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .0952 & 0 & 0 & .1905 & .5238 & .1429 & 0 & 0 & .0476 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .0952 & 0 & 0 & .1905 & .6667 & 0 & 0 & 0 & .0476 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & .2381 & 0 & 0 & 0 & .7619 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .2381 & 0 & 0 & .1905 & .4762 & 0 & 0 & 0 & .0952 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .2381 & 0 & 0 & .1905 & .4762 & 0 & 0 & 0 & .0952 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & .2381 & 0 & 0 & .1905 & .4762 & 0 & 0 & 0 & .0952 \\
.2381 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2857 & 0 & 0 & 0 & .4762 & 0 & 0 & 0 \\
0 & .2381 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2857 & 0 & 0 & 0 & .4762 & 0 & 0 \\
0 & 0 & .2381 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2857 & 0 & 0 & 0 & .4762 & 0 \\
0 & 0 & 0 & .2381 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .2857 & 0 & 0 & 0 & .4762
\end{bmatrix}
$$

According to Gershwin and Berman the expected system efficiency is

$E(A) = \sum_{n=1}^{3}[\pi(1,n,1,) + \pi(0,n,1)]$

or $E(A) = \pi_6 + \pi_7 + \pi_8 + \pi_{10} + \pi_{11} + \pi_{12}$

Solving equations 2,27,28,29 and 31 gives

$\pi_6 = 0.1624 \qquad V(\pi_6) = 0.5164$

$\pi_7 = 0.1380 \qquad V(\pi_7) = 0.5385$

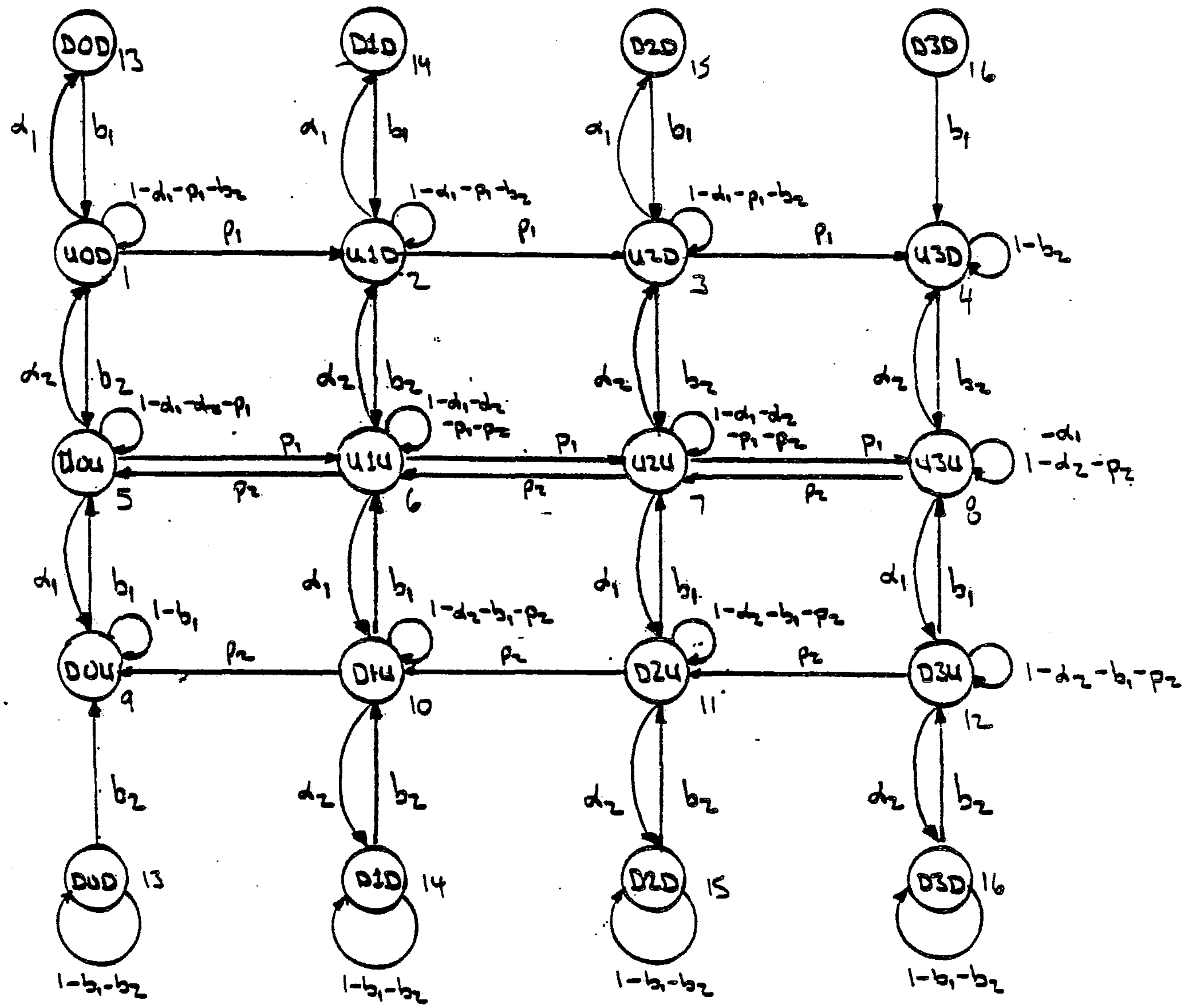


Figure 4 Markov Chain For Model in Example 2

$\pi_8 = 0.1260$ $\qquad$ $V(\pi_8) = 1.0527$

$\pi_{10} = 0.0279$ $\qquad$ $V(\pi_{10}) = 0.0839$

$\pi_{11} = 0.0216$ $\qquad$ $V(\pi_{11}) = 0.0687$

$\pi_{12} = 0.0127$ $\qquad$ $V(\pi_{12}) = 0.0461$

and so

(44) $E(A) = 0.4886$, and $V(A) = 2.3063$.

Notice that the expected efficiency for this model is similar to the expected efficiency for the model of example 1 (equation 40). However, the variance is 32% higher in this model. This is because of the additional station variable. Adding variables to the model increases the variability.

Suppose again that the line runs for 800 cycles during a shift. We expect to be in a productive state 48.86% (equation 44) of the time. When in a productive state the average processing time is $P_2=4$ cycles per unit. Hence the expected production rate E(Prod), is

$$E(\text{Prod}) = E(A)/P_2 = 0.1222 \text{ units/cycle.}$$

and

$$V(\text{Prod}) = V(A)/P_2^2 = 0.1441 \text{ units}^2/\text{cycle}^2$$

The 95% confidence interval for the expected production on the shift is

$$0.1222*800 \pm 1.96 \sqrt{0.1441*800} = (77,\ 119) \text{ units.}$$

5. Computational Considerations

Recall that Buzacott [1971] was able to develop a closed form expression for the expected efficiency E(A) of a two station - one buffer model. Unfortunately, it has not been possible to develop an accompanying closed form expression for the variance V(A). Neither has it been possible to develop closed form expressions for E(A) and V(A) for larger transfer lines (more than two stations). Techniques such as Gershwin and Berman's

[1981] have been used on two station - one buffer models. (The maximum buffer capacity considered is s=20). A similar technique is used by Gershwin and Schick [1983] to calculate E(A) for a three station - two buffer model. The largest problem they solve has s_1=15 and s_2=15. They report computation times of approximately $0.0007(s_1 + s_2)^3$ cpu seconds on a Honeywell 6880 Multics system.

In this paper, we chose to model the transfer line with a Markov chain, formulate the corresponding transition probability matrix and use equations 2 and 3 to obtain E(A) and equations 30, 31 and 32 to obtain V(A). These equations involve large matrices. A FORTRAN program was written to use these equations and, the IMSL [1982] subroutines LEQIF (to calculate E(A)) and LINV1F (to calculate V(A)), to analyze a number of two and three station transfer lines.

Table 1 summarizes the computational requirements for some typical two and three station transfer lines. The storage requirements and computational times are very reasonable for two station transfer lines. For s=15, we require approximately 64 x 64 = 4096 words of storage and 5 cpu seconds to calculate E(A) and V(A) on a VAX 11-780 or 3.56 cpu seconds on the faster CYBER 170-815 computer. For three-station transfer lines, and larger transfer lines, the state space grows rapidly. One of our problems had s_1 = 7, s_2 = 7 and required approximately 512x512 = 262,144 words of storage and 354 cpu seconds (5 min. 54 sec.) to calculate E(A). Gershwin and Schick [1983] estimate that their technique would have required approximately $0.007(7+7)^3$ = 19.2 cpu seconds on a Honeywell 6880 Multics system.

Table 1 - Computational Results on VAX 11/780

Type of Transfer Line	Buffer Capacity	Number of States (Equation 1)	Order of Matrices	CPU Seconds to Calculate * E(A)	V(A)
Two Station	s=15	64	64 x 64	0.82 sec (0.64 sec **)	4.25 sec (2.92 sec **)
Three Station	$s_1=4$, $s_2=4$	200	200 x 200	16.63 sec (9.3 sec)	2 min 27 sec (1 min 20 sec)
Three Station	$s_1=7$, $s_2=7$	512	512 x 512	5 min 54 sec	52 min 43 sec
Three Station	$s_1=8$, $s_2=8$	648	648 x 648	14 min 16 sec	--------
Three Station	$s_1=9$, $s_2=9$	800	800 x 800	25 min 49 sec	--------

* Using IMSL subroutines LEQIF for E(A) and LINV1F for V(A)
** Times for E(A) and V(A) on a CYBER 170-815 computer

6. Summary and Extensions

We began by describing a deficiency in the transfer line models in the literature - namely that they do not accurately model the daily operation of the transfer line. If both the expected value and the variance of the efficiency of the transfer line are calculated then a confidence interval for the expected production over a specified planning horizon (say a production shift) can be calculated. Such a confidence interval allows the manager to plan for shifts where output exceeds the expected output and for shifts where output is less than expected. It also gives him an indication of the probability that overtime will be required.

E(A) and V(A) can be calculated for most of the models in the literature (where lines are modelled as Markov chains). The calculations can be done using standard IMSL subroutines for almost all sizes of transfer lines considered in the literature - namely, all two station - one buffer transfer lines and all three station-two buffer lines where $s_1 + s_2 < 10$.

Clearly the most important extension to this research is to find a more efficient technique for calculating the variance V(A). Such a technique might be similar to Gershwin and Berman's [1981] technique for calculating E(A).

Acknowledgement

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Appendix 1 - Using the Geometric Transform with Markov Processes

(Based on Huggins [1957] and Howard [1971]).

Consider a simple Markov process with transition probability matrix

(A1) $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$

The corresponding transition diagram appears in Figure 5. A mathematician customarily describes the Markov process by writing down a difference equation for each state, which indicates how the probability of being in a state at time n depends on the probabilities of being in adjacent states at time n-1. If $p_1(n-1)$ and $p_2(n-1)$ are the probabilities of being in states 1 and 2, respectively, at time n-1; then one transition step later, the probabilities are

(A2) $p_1(n) = 0.8p_1(n-1) + 0.3p_2(n-1)$

$p_2(n) = 0.2p_1(n-1) + 0.7p_2(n-1)$

This is a set of linear difference equations with constant coefficients. Notice that $p_1(n)$ and $p_2(n)$ are linear combinations of themselves after a unit delay. Taking account of this we can redraw Figure 5 as the linear flow graph in Figure 6. This is written in the standard notation of flow graphs representing linear electrical systems. The signal at each mode is the probability of finding the original process of Figure 5 in that state. The z operators on the branches together with the transition probability gains indicate that the signal which passes down each branch is to be delayed by a unit time and multiplied by a constant. We will find that working with linear flow graphs such the one in Figure 6 will provide insights into the underlying Markov process. For example, $p_1(3)$ the probability of being in state 1 after the third transition could be calculated from the linear flow graph of Figure 6 by calculating the signal

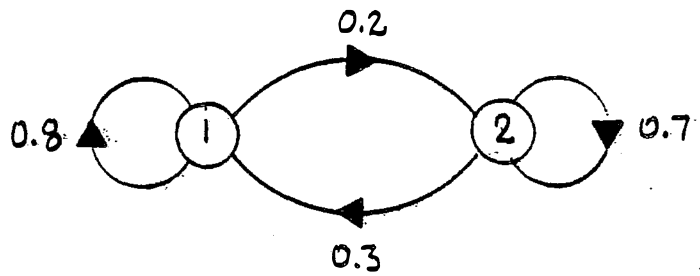

Figure 5 - Transition Diagram for a Simple Markov Process

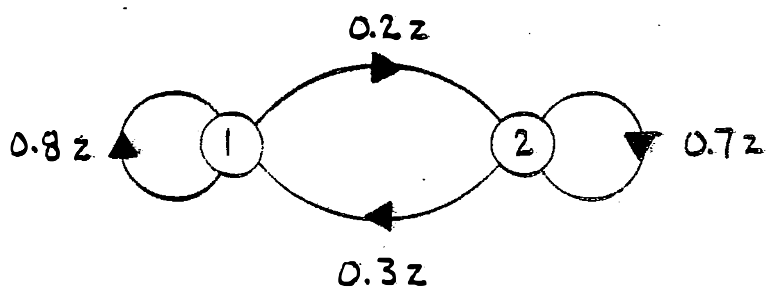

Figure 6 - Linear Flow Graph for a Simple Markov Process

at node 1 corresponding to a delay of z^3. This is done in Figure 7 where, for the sake of convenience, we assume the process began in state 1. Under this assumption, $p_1(0)=1$. As well $p_1(1)=0.8$, while $p_1(2)=0.7$ and $p_2(3)=0.65$. It appears that these probabilities are quickly converging to a steady state probability. It is a simple matter to calculate the steady state probabilities for this Markov process using traditional techniques. The great advantage of the flow graph technique is that it permits analysis of the Markov process before the process reaches steady state.

Before continuing with flow graph techniques let us define $\phi_{ij}(n)$ as the probability that the process will occupy state j at time n, given that it occupied state i at time 0. The quantity $\phi_{ij}(n)$ is called the n-step transition probability of the Markov process from state i to state j. Also define $\Phi(n)$ as the n-step transition probability matrix with elements $\phi_{ij}(n)$. By definition, $\Phi(0) = I$, the identity matrix. If P is the transition probability matrix of the Markov process then

$$\Phi(0)=I$$

$$\Phi(1)=\Phi(1)P=IP=P$$

$$\Phi(2)=\Phi(2)P=P^2$$

$$\Phi(3)=\Phi(3)P=P^3$$

and in general

(A3) $\Phi(n)=P^n \qquad n=0,1,2,3,\ldots$

The behaviour of $\phi_{ij}(n)$ for all values of i,j and n is the most important derived characteristic of the Markov process.

Consider a discrete function f(n), as shown in Figure 8. This function can take on any real value, positive or negative, at any non-negative integer n=0,1,2,... We shall find it convenient to define f(n)=0 for n<0. The geometric transform is then defined by

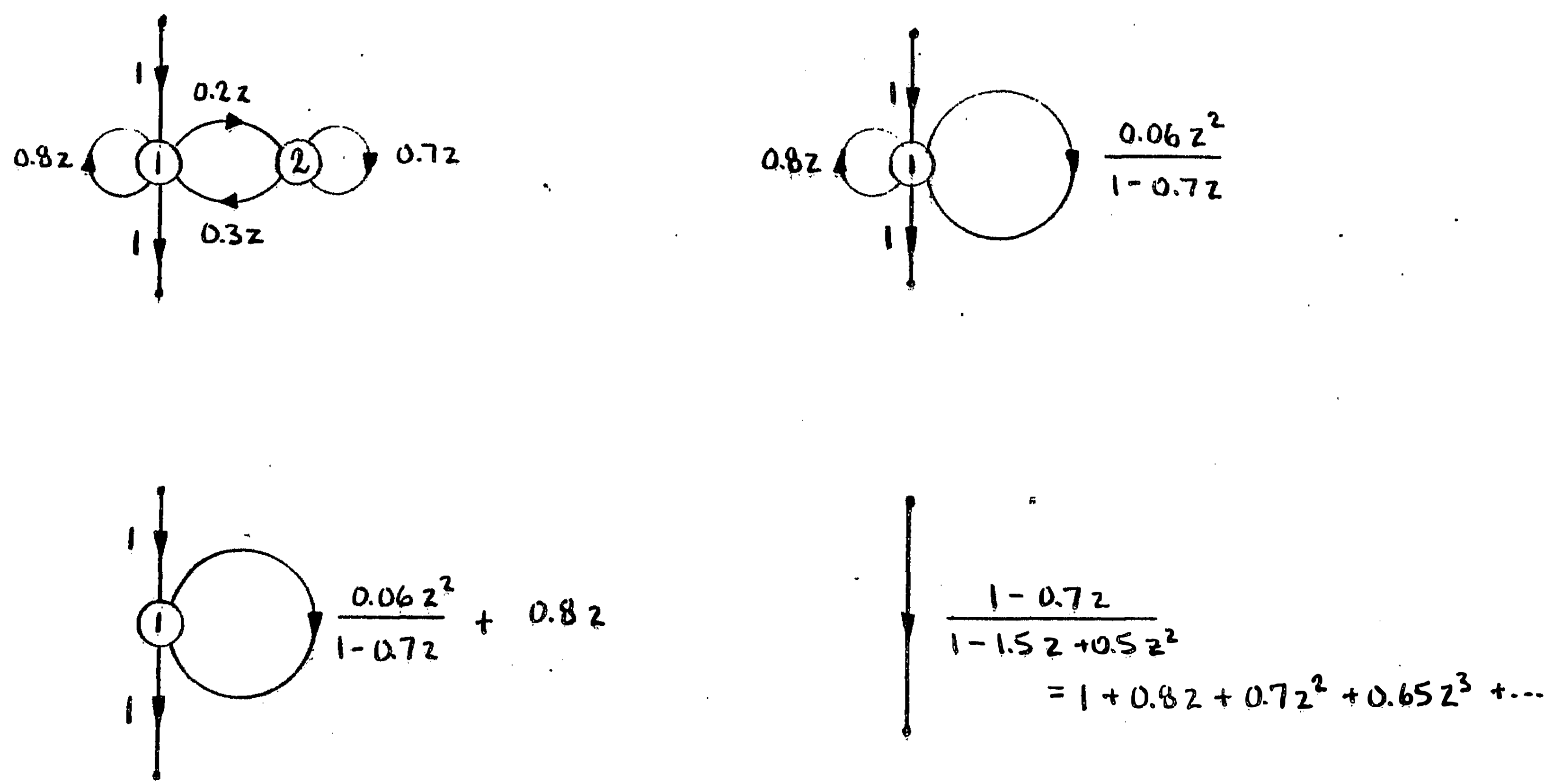

Figure 7 Calculating Signal Transmission

(A4) $f^g(z) = f(0) + f(1)z + f(2)z^2 + \ldots = \sum_{n=0}^{\infty} f(n)z^n$

if the series converges. We speak of the discrete function corresponding to a geometric transform as the inverse of the transform. The process of finding it is called transform inversion. If $f^g(z)$ is given in closed form, then we could expand it in a Taylor series about z=0 and write the inverse discrete time function as the coefficients of the successive powers of z. In some cases we can actually carry out this procedure by division when $f^g(z)$ is expressed as the ratio of two polynomials in z. (This was done in Figure 7). Alternatively we could differentiate equation A4,

(A5) $f(n) = \frac{1}{n!} \frac{d^n}{dz^n}[f^g(z)]\Big|_{z=0}$

Since the series expansion $f^g(z)$ is unique, so is the relationship between the discrete function and its geometric transform. (These transforms are related to Laplace transforms. Tables of geometric transforms are widely published. See, for example, pp. 516-520 of CRC Standard Mathematical Tables [1975] or pp. 43-81 of Howard [1981]).

Consider now, equation A3.

$\Phi(n) = P^n \qquad n=0,1,2,\ldots$

Let us take the geometric transform of this equation.

(A6) $\Phi^g(z) = \sum_{n=0}^{\infty} P^n z^n = I + Pz + P^2z^2 + \ldots = [I-Pz]^{-1}$

To calculate $\Phi^g(z)$ we must calculate the inverse of the [I-Pz] matrix. To illustrate the technique let us consider again our simple Markov process.

$$I-Pz = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - z \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 1-0.8z & -0.2z \\ -0.3z & 1-0.72 \end{bmatrix}$$

The determinant of I-Pz is

$$|I-Pz| = (1-0.8z)(1-0.7z) - (-0.2z)(-0.3z)$$

$$= 1-1.5z+0.5z^2$$

$$= (1-z)(1-0.5z)$$

Then

(A7) $\Phi^{g}(z)=[I-Pz]^{-1} = \frac{1}{(1-z)(1-0.5z)}\begin{bmatrix} 1-0.72 & 0.2z \\ 0.3z & 1-0.8z \end{bmatrix}$

Equation A7 can be simplified using a partial fractional expansion. This gives

(A8) $\Phi^{g}(z) = \frac{1}{1-z}\begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} + \frac{1}{(1-0.5z)}\begin{bmatrix} 0.4 & -0.4 \\ -0.6 & 0.6 \end{bmatrix}$

Taking the inverse transform of $\Phi^{g}(z)$ gives

(A9) $\Phi(n) = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} + 0.5^{n}\begin{bmatrix} 0.4 & -0.4 \\ -0.6 & 0.6 \end{bmatrix}$ $n=0,1,2...$

From equation A9 we see that $\emptyset_{11}(n)=0.6 + 0.5^{n}(0.4)$. Then $\emptyset_{11}(0)=1$, $\emptyset_{11}(1)=0.8$, $\emptyset_{12}(2)=0.7$, $\emptyset_{11}(3)=0.65$. etc. which agrees with our results from Figure 7. The limiting multistep transition probabilities are

(A10) $\pi=[\pi_1 \ \ \pi_2] = [0.6 \ \ 0.4]$

π_1 is the probability of being in state 1 after an infinite number of transitions regardless of the initial starting state. Similarly π_2 is the steady state probability of being in state 2 regardless of the initial starting state. We can summarize the behaviour of $\Phi(n)$ for an N state monodesmic (that is, a single chain process - a process that can make a transition from any state to any other state) by the equation

(A11) $\Phi(n) = p^{n} = \Phi + T(n)$ $n=0,1,2,...$

or in transformed form

(A12) $\Phi^{g}(z) = \frac{1}{1-z}\Phi + T^{g}(z)$

$\Phi(n)$, from equation A11 will consist of N terms. One will be a constant term Φ - the limiting multistep transition probability matrix for the process. The other N-1 terms are combined into a matrix T(n). They are transient terms, whose effect will disappear when n is large.

Appendix 2 Calculating the Transient Sum Matrix $T^g(z)$

Recall equation 6

$$\Phi(n) = P^n = \Phi + T(n) \qquad n=0,1,2,\ldots$$

where P = transition probability matrix for the process,

Φ = limiting steady state transition probability matrix with elements $\emptyset_{ij}$,

$T(n)$ = matrix of transient terms with elements $t_{ij}(n)$,

$\Phi(n)$ = n-step transition probability matrix with elements to $\emptyset_{ij}(n)$.

Rewriting equation 6 gives

$$T(n) = P^n - \Phi$$

The geometric transform of T(n) is (from equation 7)

$$\begin{aligned} T^g(z) &= \sum_{n=0}^{\infty} T(n)z^n \\ &= \sum_{n=0}^{\infty} (P^n - \Phi)z^n \\ &= I - \Phi + \sum_{n=1}^{\infty} (P^n - \Phi)z^n \end{aligned}$$

However

$$P^n - \Phi = (P - \Phi)^n \qquad n = 1,2,\ldots$$

and so

$$\begin{aligned} T^g(z) &= I - \Phi + \sum_{n=1}^{\infty} (P-\Phi)^n z^n \\ &= \sum_{n=0}^{\infty} (P-\Phi)^n z^n - \Phi \\ &= [I - zP + z\,\Phi]^{-1} - \Phi \end{aligned}$$

Hence

$$T^g(1) = [I - P + \Phi]^{-1} - \Phi$$

References

Buzacott, J.A., "Methods of Reliability Analysis of Production Systems Subject to Breakdowns", pp. 211-232 of Operations Research and Reliability, edited by D. Grouchko, Gordon and Breach (1971).

Buzacott, J.A. and L.E. Hanifan, "Models of Automatic Transfer Lines with Inventory Banks - a Review and Comparison", AIIE Transactions, 10,2, 197-201 (1978).

CRC Standard Mathematical Tables, 23rd Edition (1975).

Gershwin, S.B. and O. Berman, "Analysis of Transfer Lines Consisting of Two Unreliable Machines with Random Processing Times and Finite Storage Buffers", AIIE TRANSACTIONS, 13, 1, 2-11 (1981).

Gershwin, S.B. and I.C. Schick, "Modelling and Analysis of Three-Stage Transfer Lines with Unreliable Machines and Finite Buffers", Operations Research, 31,2, 354-380 (1983).

Hatcher, J.M., "The Effect of Internal Storage on the Production Rate of a Series of Stages Having Exponential Service Times", AIIE Transactions, 1,2, 150-156 (1969).

Howard, R.A., Dynamic Probabilistic Systems, Volume 1: Markov Models, Wiley (1971).

Huggins, W.H., "Signal-Flow Graphs and Random Signals", Proceedings of the IRE, Vol. 45, 74-86 (1957).

International Mathematical and Statistical Library, Edition 9, Houston, Texas (June 1982).

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