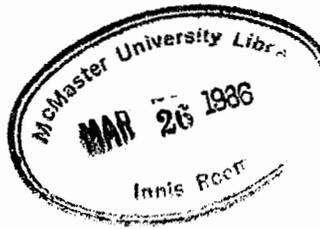




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Cordination of Pricing,
Advertising, Production
and Work-Force Level
Decisions in a Functionally
Decentralized Firm

by

Dr. P.L. Abad

Working Paper No. 242

February, 1986

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ABSTRACT

An optimal control model of a marketing-production system is formulated. The Marketing subsystem in the model is represented by the Nerlove-Arrow model of advertising and the production subsystem by the HMMS model of production planning. Using the proposed model as a reference point, a decentralized procedure is designed for coordinating pricing, advertising, production and work-force level related decisions in a firm. Attention is especially devoted to the case when the demand for the product is highly seasonal. An example is presented to illustrate the procedure.

Introduction

In recent years, there has been significant interest in applying mathematical modeling procedures to the problem of marketing-production planning. Several papers have appeared on the subject matter. Although most of the earlier research assumes that the two policies are planned in a centralized fashion [Bergstram and Smith 1969, Thomas 1970, Leitch 1974, Damon and Schramm 1972, Abad 1982a], recently efforts have been made to consider the problem as a decentralized planning problem [Welam 1977b, Freeland 1980, Abad 1982b, 1982c].

Among the papers that consider the problem in a centralized fashion, the model proposed by Damon and Schramm is the most comprehensive. It includes not only the marketing and production functions, but also the finance function. In this model, demand is assumed to be inversely related to price. The model appears to suffer, moreover, from computational difficulties [Welam 1977b].

One of the first papers to consider the problem in a decentralized fashion is the paper by [Welam 1977a]. In this paper, a sequential procedure for planning marketing-production policies is designed. The procedure assumes the transfer price to be constant during the planning horizon. In [Freeland 1980], only a static version of the problem is considered. In [Abad 1982b], an optimal control model based upon the Vidale-Wolfe model of advertising is used and a suboptimal procedure assuming constant transfer price is designed. The model assumes that price is constant during the planning horizon. Also costs associated with the varying work-force level are ignored. In [Abad 1982c] an optimal procedure assuming a time-varying transfer price is designed for the same model.

In this paper¹, a new optimal control model of marketing-production system is formulated. The marketing subsystem in the model is represented by the Nerlove-Arrow model [Nerlove and Arrow, 1962] and the production subsystem by the HMMS model [Holt et. al 1969]. The model differs in the following significant ways from the previously proposed models of decentralized marketing-production planning [Abad 1982b, 1982c]:

- 1) Price is assumed to be a policy variable,
- 2) Costs associated with the varying work-force level are not ignored,
- 3) The effects of exogenous variables (e.g. seasonality) on demand can be incorporated,
- 4) the ideal level of inventory is assumed to be a square root function of the sales rate, and
- 5) the discount interest rate is incorporated in the model.

Using the model as a reference point, a decentralized procedure for coordinating pricing, advertising, production and work-force level decisions is designed. An example is presented illustrating how marketing as well as production policies can be employed in smoothing the effects of seasonal variations on demand.

1. The Interdependent Model

The problem of marketing-production planning can be modeled as:

$$\text{Max}_{p,u,v,m} \quad J = \int_0^T e^{-rt} (pS - u$$

1. This research is supported by Natural Sciences and Engineering Research Council of Canada Grant A1226.

$$\begin{aligned}
 & -[c_I(I - a_0 - a_1 \sqrt{S})^2 + c_r(a_0 + a_1 \sqrt{S}) \\
 & + c_v v + c_w W \\
 & + c_p(v - k_1 W)^2 - k_2 v W + k_3 v - k_4 W + k_5 m^2] \} dt \\
 & + b_1 G(T) + b_2 I(T) + b_3 W(T)
 \end{aligned} \tag{1}$$

s. t.

$$\dot{G} = u - \delta G \tag{2}$$

$$\dot{I} = v - S \tag{3}$$

$$\dot{W} = m \tag{4}$$

$$S = f(p, G, Z) \tag{5}$$

$$0 \leq u \leq u_{\max} \tag{6}$$

$$G(0) = G_0, I(0) = W(0) = W_0$$

where

G = Level of goodwill at time (t)

u = rate of advertising expenditure at time (t)

p = price at time (t)

Z = Exogenous variables affecting the sales rate

$S = f(p, G, Z)$ = sales rate at time t

We assume

$$\frac{\partial f}{\partial p} < 0, \frac{\partial^2 f}{\partial p^2} - \frac{\partial^2 f}{\partial p^2} \cdot f / \frac{\partial f}{\partial p} \geq 0 \text{ and } \frac{\partial^2 f}{\partial G^2} \leq 0$$

I = level of inventory at time t (units)

v = production rate at time t (unit/day)

W = work-force level at time t (no. of workers)

m = rate of change of work-force level (hire and fire rate) at time t
(workers/day)

δ = the rate of depreciation of the goodwill

r = the discount rate

u_{\max} = the rate of advertising that the firm can effectively maintain

c_v = per unit cost of raw material

c_W = wage rate

$k_1 W$ = the desirable production rate at time t

$c_p (v - k_1 W)^2 + k_2 v W + k_3 v - k_4 W$ = Overtime-undertime costs

$a_0 + a_1 \sqrt{S}$ = desirable level of inventory at time t

$c (I - a - a \sqrt{S}) + c (a + a \sqrt{S})$ = rate of inventory costs
associated with the inventory level I .

$k_5 m^2$ = hiring and layoff costs

b_1 = value, in current dollars, of a unit of goodwill at time T

v_2 = value, in current dollars, of a unit of inventory at time T

b_3 = value, in current dollars, of a worker at time T

G_0 = Initial level of goodwill

I_0 = Initial level of inventory

W_0 = Initial work-force level

Equation (2) above is the Nerlove-Arrow model of advertising. The model was proposed in 1962. The model views advertising as an investment leading to building of goodwill which depreciates at a constant rate, δ . Equation (3) is the inventory identity. The objective function is profit

plus the value of ending goodwill, ending inventory and ending work-force level in current dollars.

The production and work-force level costs in the model are approximated using the HMMS [Holt et. al. 1960] framework. The model does not include the constraints $I(t) \geq 0$, $v(t) \geq 0$ and $W(t) \geq 0$ because it is assumed that the presence of the quadratic terms in the objective function will preclude the possibility of $I(t) < 0$, $v(t) < 0$ and $W(t) < 0$.

The above formulation is a centralized version of the marketing-production planning problem. Marketing-production policies are formulated in a decentralized fashion in most firms. In what follows, the problem described above will be reformulated as a decentralized planning problem.

2. A Decentralized Procedure

Examination of the interdependent model formulated in the previous section reveals that the interdependencies between marketing and production policies are embodied in S (which represents the demand for the firm's product). The system furthermore is serial in nature given that S , which is an output of the marketing subsystem, is an input to the production subsystem. Letting d represent the interconnecting variable, i.e. letting $d = S$. equation (3) can be rewritten as

$$\dot{I} = v - d \quad (8)$$

s.t.

$$d - S = 0 \quad (9)$$

Similarly, the rate of inventory costs can be expressed as

$$c_I(I - a_0 - a_1 \sqrt{d})^2 + c_r (a_0 + a_1 \sqrt{d}) \quad (10)$$

Let λ_1 , λ_2 and λ_3 be the current value adjoint variables corresponding to state equations (2), (8) and (4) respectively. Also let θ be the current-value lagrange multiplier associated with interaction constraint (9). The current value Lagrangian then can be stated as:

$$\begin{aligned}
 &L(p, G, u, I, v, W, m, d, \theta) \\
 &= PS - u - [c_I(I - a_0 - a_1 \sqrt{d})^2 \\
 &\quad + c_r(a_0 + a_1 \sqrt{d}) + c_v v + c_w W \\
 &\quad + c_p(v - k_1 W)^2 + k_2 v W + k_3 v - k_4 W + k_5 m^2] \\
 &\quad + \lambda_1(u - \delta G) + \lambda_2(v - d) + \lambda_3 m \\
 &\quad + \theta(d - S)
 \end{aligned} \tag{11}$$

Thus if Pontryagin's maximum principle is applied, besides the adjoint equations, the optimality conditions associated with u and m and the transversality conditions, the necessary conditions are:

$$\frac{\partial L}{\partial v} = -2c_p(v - k_1 W) - k_2 W - c_v - k_3 + \lambda_2 = 0 \tag{12}$$

$$\frac{\partial L}{\partial \theta} = d - S = 0 \tag{13}$$

and

$$\begin{aligned}
 \frac{\partial L}{\partial d} &= -2c_I(I - a_0 - a_1 \sqrt{d}) (-1/2 a_1 / \sqrt{d}) \\
 &\quad - c_r a_1 / (2\sqrt{d}) - \lambda_2 + \theta = 0
 \end{aligned} \tag{14}$$

From (12), (13) and (14)

$$\begin{aligned}
 \theta &= 2c_p(v - k_1 W) + k_2 W + c_v + k_3 \\
 &\quad - a_1 c_I (I - a_0 - a_1 \sqrt{d}) / \sqrt{d} + c_r a_1 / (2\sqrt{d})
 \end{aligned} \tag{15}$$

2.1 A Two-Level Formulation

As stated before, the marketing-production system is serial in nature. Given this structural characteristic, examination of the Lagrangian suggests the following two-level decomposition of the overall problem (note in Level I, θ is an input to problem (MARK) and d is an input to problem (PROD)):

Level I

$$\text{Max}_{p,u} \quad J_1 = \int_0^T e^{-rt} [(p - \theta) S - u] dt \quad + b_1 G(T)$$

(MARK)

s. t.

$$\dot{G} = u - \delta G$$

$$S = f(p, G, Z)$$

$$0 \leq u \leq u_{\max}$$

$$G(0) = G_0$$

and

$$\text{Min}_{y,m} \quad J_2 = \int_0^T e^{-rt} [c_I (I - a_0 - a_1 \sqrt{d})^2 + c_R (a_0 + a_1 \sqrt{d})$$

$$+ c_V v + c_W W +$$

$$\text{(PROD)} \quad + c_P (v - k_1 W)^2 + k_2 vW + k_3 v$$

$$- k_4 W + k_5 m^2] dt - b_2 I(T) - b_3 W(T)$$

s. t.

$$\dot{I} = v - d$$

$$\dot{W} = m$$

$$I(0) = I_0, W(0) = W_0$$

Level II

Update θ in a suitable fashion.

An algorithm based upon the above formulation is presented in the next sub-section. Before that we make the following important observations:

1. The role of θ in the above decomposition is that of a transfer price. This result is similar to one shown in [Abad 1982c]. θ can be viewed as the current value transfer price charged to the marketing department for every unit it sells at time t .
2. Equation (15) suggests that θ is equal to the marginal value of production plus a correction associated with under or over utilization of capacity and the present level of inventory.
3. Given that d is an input, problem (PROD) is a standard linear quadratic problem. The two point boundary value problem associated with (PROD) can be solved, for example, by IMSL subroutin DVCPR.
4. Solution to problem (MARK) is described in the Appendix. As shown there, if η is price elasticity, the optimal price for (MARK) is given by

$$p = p^* = \frac{\eta\theta}{\eta - 1} \text{ for } t \in [0, T] \quad (16)$$

This result is consistent with the classical economic theory.

2.2 An algorithmic scheme

Given that the system under consideration is a serially connected system, the approach presented in [Abad 1983] is applicable to the problem under consideration. Specifically it is suggested that condition (13) be

enforced in Level I and condition (15) be used to update θ in level II. An algorithm based upon the above approach is:

Step 1. Let $i = 0$ and $\theta^i = \theta^{(0)}$

Step 2. Pass θ^i to the marketing department and have it solve problem (MARK). Let $d = S$. Pass d to the production department and have it solve problem (PROD). Obtain the schedule of marginal cost

$$\theta^i = c_v + 2c_p (v - k_1 W) + k_2 W + k_3 \\ - a_1 c_I (I - a_0 - a_1 \sqrt{d})/\sqrt{d} + c_r a_1/(2\sqrt{d})$$

from the production department

Step 3. Compute

$$e = \int_0^T |\theta^i - \hat{\theta}^i| dt. \text{ If } e < \epsilon \text{ stop. Otherwise go to step 4.}$$

Step 4. Let $\theta^{i+1} = \alpha \theta^i + (1 - \alpha) \hat{\theta}^i$ where $0 \leq \alpha < 1$

Let $i = i + 1$ and go to step 2.

3. The Special Multiplicative Demand Case

One of the special case often considered is the multiplicative demand case. In this case:

$$f(p, G, Z) = k_0 p^{-\eta} G^\beta f_e(Z) \quad (17)$$

where

η = elasticity of demand with respect to price, assumed to be > 1 .

β = elasticity of demand with respect to goodwill, assumed to be < 1 .

$f_e(Z)$ = function of exogenous variables, Z . It could, for example,

reflect the effect of seasonal variations on demand.

It can be shown that in the above special case, the singular goodwill level (described in A-5 in the appendix) is given by [Nerlove and Arrow 1962]:

$$G^S(t) = \left[\frac{(\eta-1)\eta^{-1}}{\eta \theta^{\eta-1}} \frac{k_0\beta}{(r+\delta)} f_e(Z) \right]^{\frac{1}{1-\beta}} \quad (18)$$

Similarly from (16), (A-3) and (17)

$$\pi(G, Z) = \frac{k_0\beta}{\eta-1} \left[\frac{n\theta}{\eta-1} \right]^{-\eta} G^{\beta} f_e(Z) \quad (19)$$

Thus in the stage III boundary value problem described in the appendix,

$$\frac{\partial \pi}{\partial G} = \frac{k_0\beta}{\eta-1} \left[\frac{n\theta}{\eta-1} \right]^{-\eta} G^{\beta-1} f_e(Z) \quad (20)$$

An example illustrating the application of the decentralized procedure to the above special case is presented below.

Test Example

Consider a problem situation where:

T = 365 days	$k_3 = 8$
$\delta = .02$	$k_4 = 50$
r = .000333	$k_5 = 4000$
$\eta = 2.8$	$u_{\max} = 25000 \text{ \$/day}$
$\beta = .56$	$a_0 = 10000$
$c_I = .0000015$	$a_1 = 600$
$c_p = .03$	$b_1 = .95$
$c_v = \$10/\text{unit}$	$b_2 = 25$
$c_w = \$100/\text{worker/day}$	$b_3 = 400$
$k_1 = 8$	$G_0 = 150,000$
$k_2 = 0$	$I_0 = 25000$
$f_e[Z(t)] = 1250 + 400 \left[\sin\left(\pi + \frac{2\pi t}{T}\right) \right]$	$W_0 = 120$

$$c_r = .04$$

Note that in this case $f_e[Z(t)]$ is a cyclical function of time; i.e. demand is assumed to be subject to seasonal variations. Time plots of sales rate, transfer price and price in the optimal plan are shown in Fig. 1. Fig. 2 presents the time plots of level of goodwill, rate of advertising expenditure, inventory level, production rate, work-force level and the rate of change of work-force level. It is seen that in the optimal plan, transfer price is not constant; rather, it is time-varying. In the low demand season, transfer price is low whereas in the peak demand season, it is high. This is also reflected in time plot of price since price is simply a multiple of transfer price. The time-plot of inventory shows that there is considerable building of inventory during the low demand seasons. The plot of work-force level shows that there is some firing of workers during the low demand seasons. Finally, the variation in price in the optimal plan suggests that in this case there is considerable smoothing of demand by marketing policies.

4. Conclusions

A comprehensive model of marketing-production planning is presented and a procedure, is designed for planning pricing, advertising, production and work-force level decisions in a decentralized (yet optimal) fashion. The model provides a mechanism for incorporating exogenous variables (such as seasonality or the effect of business cycle) in the analysis. The model in fact can be used to derive optimal marketing-production policies in a variety of situations.

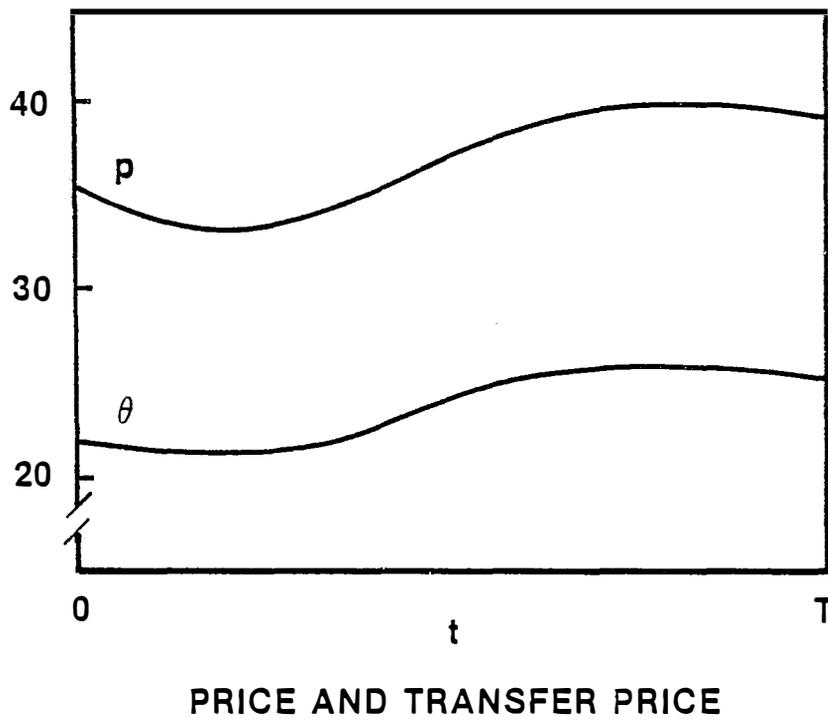
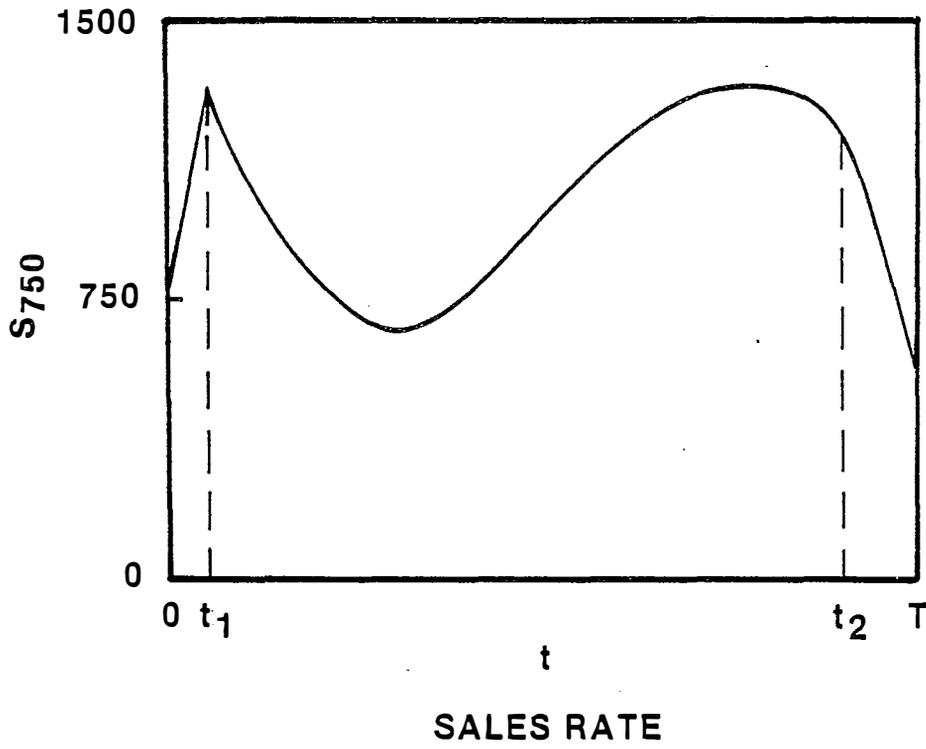


Fig. 1

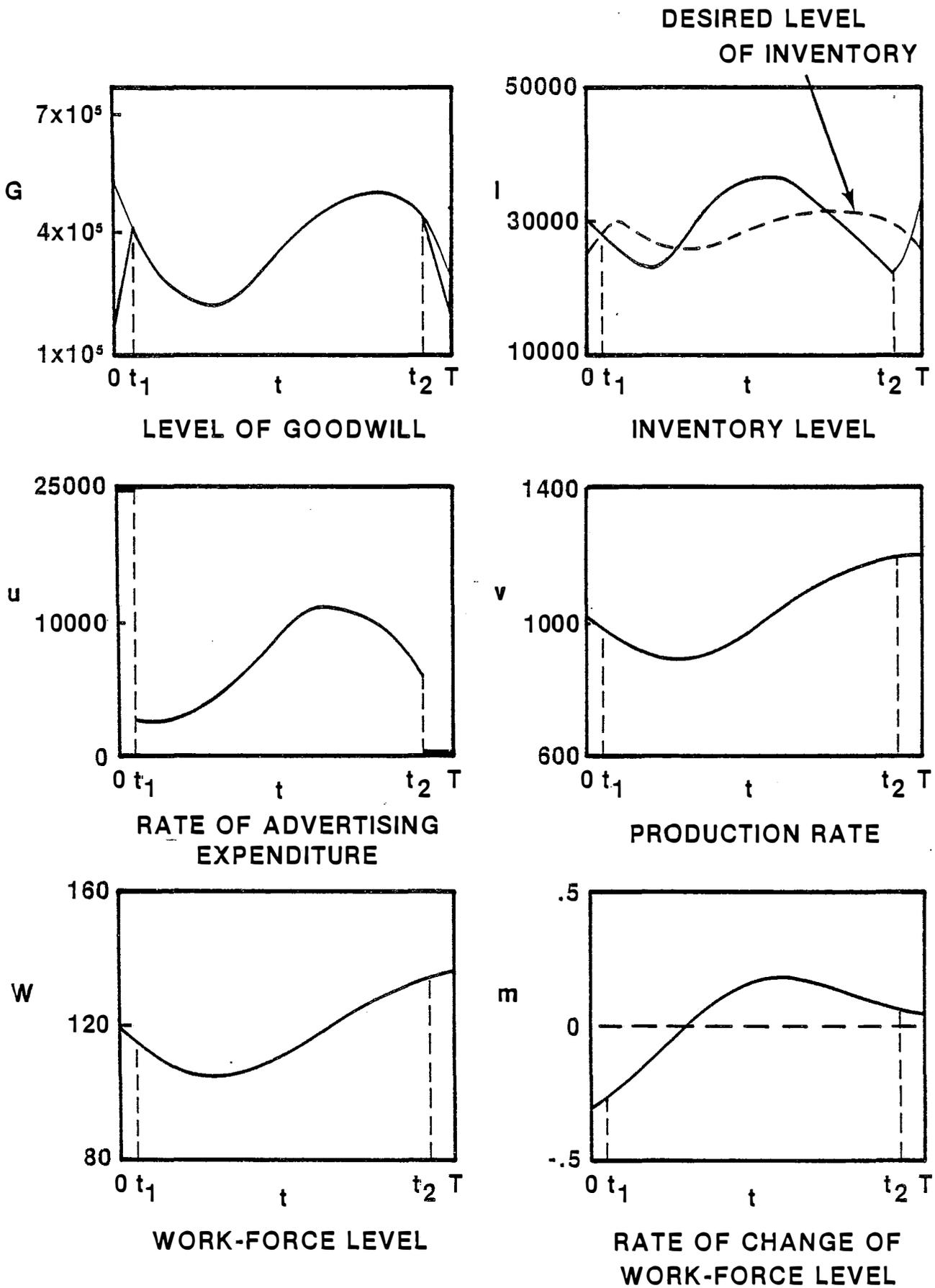


Fig. 2

Appendix

With $S = f(p, G, Z)$, the current value hamiltonian for problem (MARK) is

$$(p - \theta) f(p, G, Z) - u + \lambda_1 (u - \delta G)$$

Where λ_1 is the adjoint variable associated with constraint (2). Since p appears only in the integrand, we can maximize J_1 by first maximizing the integrand with respect to p assuming u fixed and then maximizing the result with respect to u . Thus

$$\frac{\partial}{\partial p} \{ (p - \theta) f(p, G, Z) \} = 0$$

or

$$f(p, G, Z) + (p - \theta) \frac{\partial f}{\partial p} = 0$$

If we let $\eta = - (p/f) \frac{\partial f}{\partial p}$ be the elasticity of demand with (A-1)

respect to price, (A-3) reduces to

$$p = p^* = \frac{\eta \theta}{(\eta - 1)} \tag{A-2}$$

Let

$$\pi(G, Z) = [p^* - \theta] f(p^*, G, Z) \tag{A-3}$$

Problem (MARK) then can be restated as

$$\text{Max}_u \int_0^T e^{-rt} [\pi(G, Z) - u] dt + b_1 G(T)$$

(MARK1) s.t.

$$\dot{G} = u - \delta G$$

$$G(0) = G_0$$

The current-value hamiltonian with problem (MARK1) is

$$\pi(G, Z) - u + \lambda_1 (u - \delta G) \tag{A-4}$$

The above hamiltonian is linear with respect to u ; i.e. problem (MARK1) is singular with respect to u . The solution to problem (MARK1) is provided in [Nerlove and Arrow 1962, Sethi and Thompson 1981]. It is shown that the singular advertising rate is given by

$$G = G^S(t) = \frac{p\beta S}{(r+\delta)\eta} \quad (\text{A-5})$$

$$\lambda_1 = \lambda_1^S(t) = 1 \quad (\text{a-6})$$

where

$$u = u^S(t) = \delta G^S(t) + \dot{G}^S(t) \quad (\text{A-7})$$

The solution for the case $u^S(t) > 0$, $t \in [0, T]$ is shown in Fig. 3. It is easy to show in this case t_1 and the stage I trajectories are determined by the boundary value problem:

$$\dot{G} = u_{ns} - \delta G$$

(BVPI)

$$\begin{aligned} G(0) &= G_0 \\ G(t_1) &= G^S(t_1) \end{aligned} \quad u_{ns} = \begin{cases} 0 \\ u_{\max} \end{cases} \quad \text{if } \begin{cases} G_0 > G^S(0) \\ G_0 < G^S(0) \end{cases}$$

t_1 is free

Similarly t_2 and the stage III trajectories are determined by [Sethi and Thompson 1981]:

$$\dot{G} = -\delta G$$

$$\dot{\lambda}_1 = -\frac{\delta\pi}{\delta G} + (r + \delta)\lambda_1$$

(BVPIII)

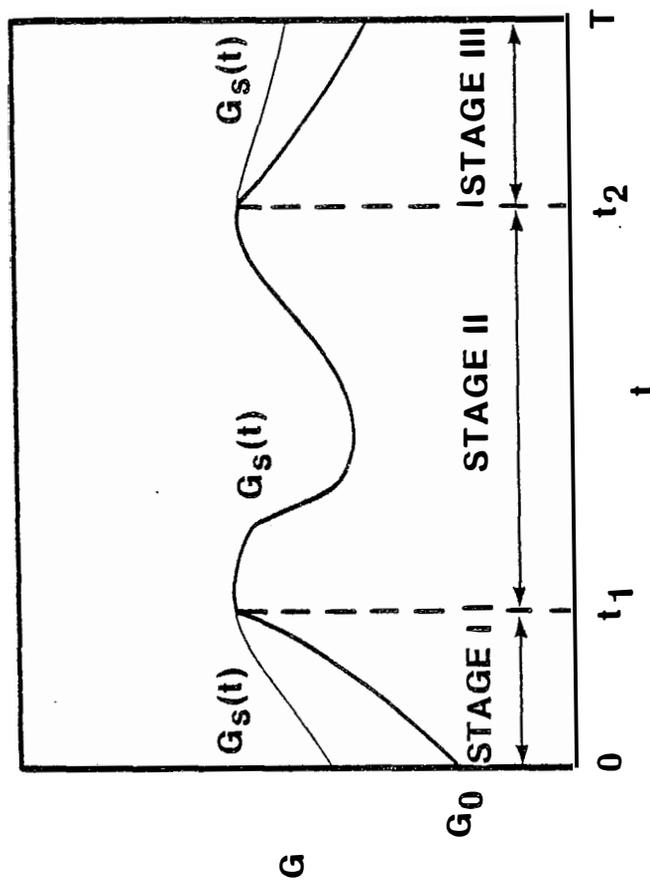


Fig. 3

THE OPTIMAL LEVEL OF GOODWILL

$$G(t_2) = G^S(t_2)$$

$$\lambda_1(t_2) = 1$$

$$\lambda_1(T) = b_1$$

t_2 is free

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