

CONTRIBUTIONS TO THE MODEL
THEORY OF HIGHER-ORDER LOGIC

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HIGHER-ORDER LOGIC

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A Thesis Submitted to the School of Graduate Studies in Partial
Fulfilment of the Requirements for the Degree of Master of Science

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McMaster University

MASTER OF SCIENCE (2025)

Hamilton, Ontario, Canada (Dept. of Computing and Software)

TITLE : Contributions to the Model Theory of
Higher-Order Logic

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NUMBER OF PAGES : vii, 53

Abstract

In this thesis, we develop the model theory of higher-order logic by working in Alonzo, a classical higher-order logic based on Church’s formulation of simple type theory that extends first-order logic and that admits undefined expressions. In particular, we sharpen the Löwenheim-Skolem theorem (Theorem 9.39 in William M. Farmer’s *Simple Type Theory*) such that there exists a structural relationship between the starting and produced models, we develop model-theoretic types and prove a corresponding higher-order version of the omitting types theorem, and we give syntactic and semantic characterizations of how first-order theories are embedded in Alonzo.

Acknowledgements

My utmost and sincerest thanks go to my supervisor, William M. Farmer. When I was an undergraduate student, Bill began directing and nourishing my blossoming enthusiasm for mathematical logic by introducing me to simple type theory through his graduate course on the topic. Since then, Bill has been gifting me with an enriching and educational environment through our weekly meetings. It is the exception for our meetings to be short – our mutual interests exclusively permit extensive and quality-dense discussions. In his duties as a supervisor, Bill’s guidance, patience, support, careful eye, and wealth of knowledge made my academic journey rewarding and easy to navigate. When presenting Bill with various drafts of my work, he gave his comments with speed, punctuality, and sharpness. Beyond our scholarly connection, I am grateful to Bill for our interdisciplinary discussions involving philosophy, history, literature, and art; over the gradual course of time, references from the works of S. Ambrose, the Bard, E. Curtis, A. Dürer, J. W. von Goethe, K. Jarrett, St. Jerome, I. Lakatos, P. O’Brien, M. O’Connor, N. Rockwell, R. Smullyan, A. Solzhenitsyn, M. Spivak, T. H. White, and F. L. Wright (to name a few) have found a comfortable place in my lexicon.

Thank you to my supervisory committee – comprised of William M. Farmer, Bradd Hart, and Ridha Khedri – for guiding the direction of my thesis, for your helpful suggestions, and for your timeliness and reliability. Thank you to the Chair of my defence, Jake Doliskani, for handling the administrative and technological aspects of my defence.

Finally, I would like to thank many family members and friends for their warmth, encouragement, and support throughout my thesis writing period.

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Declaration of Academic Achievement

All of the definitions related to the logic Alonzo in Chapter 2 are not my own and have been taken verbatim from [7] with explicit permission from the author. The example at the end of Chapter 2 is inspired by the discussion in Section 13.4 of [7]. The definitions, propositions, lemmas, and theorems in Chapters 3 and 4 are my own work, though they are inspired by corresponding first-order analogues found in [12]. The work in Chapter 5 is entirely my own. The content in the Appendix is taken verbatim from [7] with explicit permission from the author. Unless stated explicitly otherwise, all other work is my own.

Chapter 1

Introduction

Appendix B in Chang and Keisler’s *Model Theory* [2] contains a famous list of open problems in classical model theory. We shall tackle the last open problem:

Develop the model theory of second- and higher-order logic.

In particular, we take “higher-order logic” to mean a version of simple type theory. The nature of this identification is semantic and historical, and so we motivate the task of developing the model theory of higher-order logic by defining and briefly elucidating the history of simple type theory below. A more complete account of the development of simple type theory can be found in [5].

1.1 History

In 1908, Bertrand Russell presented a logic for mathematics that he called *the theory of types* [16]. A recursively defined syntactic type hierarchy of objects was set up such that the *first logical type* consists of a fixed collection of individual terms which act as subjects in propositions with no quantifiers; the *second logical type* consists of propositions, called *first-order propositions*, in which members of the first logical type are quantified over; the *third logical type* consists of *second-order propositions* in which members of the second logical type are quantified over; and so on. Notice that second-order propositions quantify over (first-order) propositions and that a proposition containing a particular quantified variable of a given type α must itself be of type higher than α . The first departure from first-order logic occurs with second-order logic since quantifying over (first-order) propositions is permitted. More generally, n^{th} -order logic is just a logic that has a type hierarchy up to the $(n + 1)^{\text{th}}$ logical type. We thus have a stratification of propositions by type.

Now in 1902, Russell discovered that unrestricted set comprehension leads to the following paradox: if $X = \{x \mid x \notin x\}$ is a set, then

$$X \in X \text{ if and only if } X \notin X.$$

That is, the set X of all sets that do not contain themselves cannot itself be a set since X contains itself if and only if it does not. In particular, Russell noticed that the paradox arose because such a definition is *impredicative*; that is, the definition of X contains the circular property $x \notin x$. So, Russell proposed a *ramified theory of types* in which there is a second hierarchical constraint

based on predicativity: an n^{th} -order *predicative function* is one such that its value is a third-order proposition which has an $(n - 1)^{\text{th}}$ -order predicative function as its largest argument with respect to the hierarchy of logical types.¹

However, in the same 1908 publication mentioned above, Russell saw that the constraint due to predicativity was too restrictive for a logic for mathematics. Consider the definition of the infimum $\inf(X)$ of a partial order $(X, \leq)$. We say that $y = \inf(X)$ if and only if

1. For all $x \in X$, $y \leq x$; that is, y is a *lower bound* of X .
2. For all $z \in X$, z is a lower bound of X implies that $z \leq y$.

The definition of y is impredicative since we quantify over all X , which includes y . Russell’s solution to the restrictiveness of his type theory was to introduce the axiom of reducibility, which states that every function is extensionally equivalent to some predicative function of the same argument. However, the axiom of reducibility negates exactly that which Russell wished to accomplish with the ramified theory of types, as impredicative functions come to exist through their predicative counterparts.

Now Leon Chwistek [4] and Frank Ramsey [14] noticed that the resulting “simplified” type theory, now called *simple type theory* – with the hierarchical constraint of predicativity removed – was adequate for mathematics. Indeed, the hierarchy of logical types prevents set theoretic paradoxes, and the ability to have impredicative definitions – such as the infimum of a partial order – is a virtue rather than a pitfall of non-ramified type theory.

In 1940, Alonzo Church introduced a version of simple type theory called *Church’s type theory* which incorporated features of λ -calculus into simple type theory [3]. The hierarchy of logical types in Church’s type theory is defined inductively by the following formation rules: o and ι are types; and if α, β are types, then $(\alpha\beta)$ is a type. In particular, o denotes the type of booleans, ι denotes the type of individuals, and $(\alpha\beta)$ denotes the type of functions from β to α . Proper symbols are contained in the infinite list

$$N_{oo}, A_{(oo)o}, \Pi_{o(o\alpha)}, \iota_{\alpha(o\alpha)}, a_{\alpha}, b_{\alpha}, \dots$$

and well-formed formulas (of type indicated by the subscript) are defined inductively by the following formation rules: a single proper symbol is a well-formed formula; if $\mathbf{x}_{\beta}$ is a variable and $\mathbf{M}_{\alpha}$ is a well-formed formula, then $(\lambda \mathbf{x}_{\beta} \mathbf{M}_{\alpha})$ is a well-formed formula; if $\mathbf{F}_{\alpha\beta}$ and $\mathbf{A}_{\beta}$ are well-formed formulas, then $(\mathbf{F}_{\alpha\beta} \mathbf{A}_{\beta})$ is a well-formed formula. Notice how the formation rules for well-formed formulas correspond to the variable, λ -abstraction, and function application term formation rules in the λ -calculus. Perhaps most importantly, this addition of λ -terms paired with the hierarchy of logical types induced a built-in theory of functions in type theory, making it a logic well-suited for computing.

1.2 Remarks

Church’s presentation of simple type theory with λ -calculus in his 1940 paper was of a syntactic flavour, in the sense that the logic consisted of a formal notion of syntax along with a proof system. In 1950, Leon Henkin introduced two kinds of semantics for Church’s type theory: the

¹For first-order predicative functions, we alter the definition such that individuals act as (the highest) arguments in place of 0^{th} order predicative functions.

standard semantics considers models whose typed domains are all full, while the *general semantics* considers models where typed function domains contain some (but not necessarily all) functions of corresponding type [9]. These semantics are wildly different: since Peano arithmetic can be developed within the formal system of Church’s type theory [3], then by Gödel’s first incompleteness theorem, there is a well-formed formula which is valid with respect to the standard semantics but is not a theorem of the formal system. However, in [9], Henkin showed that with respect to the general semantics, there is a complete proof system for Church’s type theory. Broadly speaking, then, simple type theory can be viewed in two ways: firstly, with respect to the general semantics, as a *first-order set theory* that is equivalent (with respect to equi-consistency) to bounded Zermelo set theory [13]; and secondly, with respect to the standard semantics, as an ω -order logic.

Recalling the last open problem in Chang and Keisler, in this thesis we will develop the model theory of Alonzo, a classical higher-order logic based on Church’s type theory that extends first-order logic and that admits *undefined* expressions. By undefinedness, we mean that there are expressions of Alonzo that do not denote anything. Unlike the presentation of Church’s type theory above, undefinedness arises in Alonzo since partial functions can populate typed domains. Alonzo employs the *traditional approach to undefinedness*, which is based on three principles [6]:

1. Atomic expressions (i.e., variables and constants) are always defined.
2. Compound expressions may be undefined. A function application $f(x)$ is undefined if f is undefined, x is undefined, or f is undefined at x . A definite description $(\text{I } x \in S. E)$ is undefined if there is no $s \in S$ or more than one $s \in S$ such that $E[x \mapsto s]$ is true, where $E[x \mapsto s]$ denotes the expression resulting from replacing all free occurrences of x in E with s .
3. Formulas are always true or false and hence always defined. This implies that a predicate application $p(x)$ is false, rather than undefined, if p is undefined, x is undefined, or p is undefined at x .

The principles extend to n -ary functions and predicates in the obvious way. Furthermore, undefinedness naturally induces two notions of equality. We say a is *equal* to b , written $a = b$, if a and b are both defined and have the same value. We say that a is *quasi-equal* to b , written $a \simeq b$, if $a = b$ or a and b are both undefined.

Undefinedness is useful for a several reasons. Examples include:

1. We can have meaningful statements, such as $\forall x \in \mathbb{R}. x \geq 0 \Rightarrow e^{\ln(x)} = x$, even though, say, $\ln(-5)$ is undefined.
2. We can define function applications extensionally, such as $e^x \simeq \text{I } y \in \mathbb{R}. y \geq 0 \wedge \ln(y) = x$.

Note that a more extensive discussion on undefinedness, its benefits, and its use in Alonzo can be found in [7].

Finally, we answer the question regarding why we chose to work in Alonzo rather than another version of Church’s type theory. Alonzo is a *highly developed* and *practice-oriented* version of Church’s type theory. It is highly developed in the sense that there exists a graduate-level textbook [7] covering the logic and it is practice-oriented in the sense that the syntax of Alonzo is close to mathematical practice. A possible alternative to Alonzo would be Peter Andrews’ $\mathcal{Q}_0$, which does not admit language families (see Section 55 of [1]), nor does it have the benefits of undefinedness mentioned above.

1.3 Scope

The overarching goal of this thesis is to develop higher-order analogues of various model theoretic definitions, constructions, and techniques from first-order logic. Most of the results are done with respect to the general models semantics.

The structure of this thesis is as follows. In Chapter 2, we introduce the syntax and semantics of Alonzo. In Chapter 3, we develop the corresponding higher-order notions of embedding, inclusion, submodel, elementary submodel, elementary diagram, elementary chain, and the upward and downward Löwenheim-Skolem theorems from first order logic. In Chapter 4, we define model theoretic types for Alonzo, and show how the omitting types theorem and constructions involving partial elementary maps lift to higher-order logic. In Chapter 5, we show how first-order theories embed into Alonzo from syntactic and semantic points of view.

Chapter 2

Syntax and Semantics

We introduce the syntax and semantics of Alonzo, modelling the presentation given in [7]. Most of the text in this chapter is taken verbatim from [7]. For a full treatment that includes (equivalent) formal notation, beta-reduction, alpha-conversion, quasitypes, many auxillary notational definitions, and more, see Chapters 4 – 7 of [7].

2.1 Syntax

In this section, we will define types, which denote nonempty sets of values; and expressions, which either denote values (when they are defined) or do not denote anything (when they are undefined). We start with an overview of the different symbols that make up the syntax of Alonzo.

Let $\mathcal{S}_{\text{bt}}$, $\mathcal{S}_{\text{var}}$, be fixed countably infinite sets of symbols and $\mathcal{S}_{\text{con}}$ be a fixed (possibly uncountably) infinite set of symbols that will serve as names of base types, variables, and constants, respectively. We assume that $\mathcal{S}_{\text{bt}}$ contains the symbols $A, B, C, \dots, X, Y, Z$, etc., $\mathcal{S}_{\text{var}}$ contains the symbols $a, b, c, \dots, x, y, z$, etc., and $\mathcal{S}_{\text{con}}$ contains the symbols $A, B, C, \dots, X, Y, Z$, etc., numeric symbols, nonalphanumeric symbols, and words in lowercase sans serif font.¹ We will employ the following syntactic variables for these symbols and the syntactic entities defined later in this chapter:

1. **a, b**, etc. range over $\mathcal{S}_{\text{bt}}$.
2. **f, g, h, i, j, k, m, n, u, v, w, x, y, z**, etc. range over $\mathcal{S}_{\text{var}}$.
3. **c, d**, etc. range over $\mathcal{S}_{\text{con}}$.
4. $\alpha, \beta, \gamma, \delta$, etc. range over types.
5. **A_α, B_α, C_α, ..., X_α, Y_α, Z_α**, etc. range over expressions of type α .

Definition 2.1 (Type). A *type* of Alonzo is a string of symbols defined inductively by the following formation rules:

1. *Type of truth values*: o is a type.

¹An expression like “ u, v, w , etc.” means, here and elsewhere, the set of symbols that includes u , v , and w , and all possible annotated forms of u , v , and w such as u' , v_1 , and $\tilde{w}$.

2. *Base types*: $\mathbf{a} \in \mathcal{S}_{\text{bt}}$ is a type.
3. *Function type*: Given α, β are types, $(\alpha \rightarrow \beta)$ is a type.
4. *Product type*: Given α, β are types, $(\alpha \times \beta)$ is a type.

Let $\mathcal{T}$ denote the set of types of Alonzo.

Definition 2.2 (Expression). An *expression of type α* of Alonzo is a string of symbols defined inductively by the following formation rules:

1. *Variable*: $(\mathbf{x} : \alpha)$ is an expression of type α .
2. *Constant*: $\mathbf{c}_\alpha$ is an expression of type α .
3. *Equality*: $(\mathbf{A}_\alpha = \mathbf{B}_\alpha)$ is an expression of type o .
4. *Function application*: $(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ is an expression of type β .
5. *Function abstraction*: $(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)$ is an expression of type $(\alpha \rightarrow \beta)$.
6. *Definite description*: $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$ is an expression of type α where $\alpha \neq o$.
7. *Ordered pair*: $(\mathbf{A}_\alpha, \mathbf{B}_\beta)$ is an expression of type $(\alpha \times \beta)$.

Let $\mathcal{E}$ denote the set of expressions of Alonzo. A *formula* is an expression of type o . Notice how the treatment of formulas as special kinds of expressions – in other words, terms – differs from treating formulas as separate from terms in first-order logic. We write $\mathbf{A}_\alpha \equiv \mathbf{B}_\alpha$ when the expressions denoted by $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha$ are the same.

Definition 2.3. We can say the following about occurrences of a variable in an expression:

1. An occurrence of a variable $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$ is *bound* [*free*] if it is [is not] within a subexpression of $\mathbf{B}_\beta$ of either the form $\lambda \mathbf{x} : \alpha . \mathbf{C}_\gamma$ or the form $\mathbf{I} \mathbf{x} : \alpha . \mathbf{C}_o$.
2. A variable $(\mathbf{x} : \alpha)$ is *bound* [*free*] *in* $\mathbf{B}_\beta$ if there is a bound [free] occurrence of $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$.
3. An expression is *closed* if it contains no free variables.
4. A *sentence* is a closed formula.
5. $\mathbf{A}_\alpha$ is *free for* $(\mathbf{x} : \alpha)$ *in* $\mathbf{B}_\beta$ if no free occurrence of $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$ is within a subexpression of $\mathbf{B}_\beta$ of either the form $\lambda \mathbf{y} : \gamma . \mathbf{C}_\delta$ or the form $\mathbf{I} \mathbf{y} : \gamma . \mathbf{C}_o$ where $(\mathbf{y} : \gamma)$ is free in $\mathbf{A}_\alpha$.

Definition 2.4 (Substitution). The *substitution of $\mathbf{A}_\alpha$ for $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$* , written $\mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha]$, is the result of replacing each free occurrence of $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$ with $\mathbf{A}_\alpha$. Notice that $\mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \in \mathcal{E}$ since the free occurrences of $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$ are replaced with $\mathbf{A}_\alpha$, an expression of the same type as the type of $(\mathbf{x} : \alpha)$. This operation on expressions is defined using recursion and pattern matching by the following identities:

1. $(\mathbf{x} : \alpha)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv \mathbf{A}_\alpha$.
2. $(\mathbf{y} : \beta)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\mathbf{y} : \beta)$
where $(\mathbf{x} : \alpha)$ and $(\mathbf{y} : \beta)$ are distinct.

3. $\mathbf{c}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv \mathbf{c}_\beta$.
4. $(\mathbf{B}_\beta = \mathbf{C}_\beta)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] = \mathbf{C}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha])$.
5. $(\mathbf{F}_{\beta \rightarrow \gamma} \mathbf{B}_\beta)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv$
 $(\mathbf{F}_{\beta \rightarrow \gamma}[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha])$.
6. $(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)$.
7. $(\lambda \mathbf{y} : \gamma . \mathbf{B}_\beta)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\lambda \mathbf{y} : \gamma . \mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha])$
where $(\mathbf{x} : \alpha)$ and $(\mathbf{y} : \gamma)$ are distinct.
8. $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$.
9. $(\mathbf{I} \mathbf{y} : \gamma . \mathbf{B}_o)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\mathbf{I} \mathbf{y} : \gamma . \mathbf{B}_o[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha])$
where $(\mathbf{x} : \alpha)$ and $(\mathbf{y} : \gamma)$ are distinct.
10. $(\mathbf{B}_\beta, \mathbf{C}_\gamma)[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha] \equiv (\mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha], \mathbf{C}_\gamma[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha])$.

Definition 2.5 (Language). A *language* of Alonzo is a pair $L = (\mathcal{B}, \mathcal{C})$ where $\mathcal{B}$ is a finite set of base types and $\mathcal{C}$ is a set of constants $\mathbf{c}_\alpha$ where each base type occurring in α is a member of $\mathcal{B}$.

A type α is a *type of L* if all the base types occurring in α are members of $\mathcal{B}$, and an expression $\mathbf{A}_\alpha$ is an *expression of L* if all the base types occurring in $\mathbf{A}_\alpha$ are members of $\mathcal{B}$ and all the constants occurring in $\mathbf{A}_\alpha$ are members of $\mathcal{C}$. Let $\mathcal{T}(L) \subseteq \mathcal{T}$ denote the set of types of L and $\mathcal{E}(L) \subseteq \mathcal{E}$ denote the set of expressions of L . Notice that $\mathcal{B}$ and $\mathcal{C}$ may be empty, but $\mathcal{T}(L)$ and $\mathcal{E}(L)$ are always nonempty since $o \in \mathcal{T}(L)$. The *minimum language* is the language $L_{\min} = (\emptyset, \emptyset)$.

The base types and constants of a language are used to represent, respectively, the base domains and distinguished values of a structure. So it is sometimes convenient, when the set of constants is finite, to write a language

$$L = (\{\mathbf{a}_1, \dots, \mathbf{a}_m\}, \{\mathbf{c}_{\alpha_1}^1, \dots, \mathbf{c}_{\alpha_n}^n\})$$

as the tuple

$$(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{c}_{\alpha_1}^1, \dots, \mathbf{c}_{\alpha_n}^n)$$

in the same way a structure can be written as a tuple.

Definition 2.6. Let $L_i = (\mathcal{B}_i, \mathcal{C}_i)$ be a language for $i \in \{1, 2\}$. L_2 is an *extension* of L_1 (or L_1 is a *sublanguage* of L_2), written $L_1 \leq L_2$, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Notice that $L_{\min} \leq L$ for every language L .

Recall that the *cardinality* of a set S , denoted $|S|$, is the cardinal number κ such that there is a bijection $f : \kappa \rightarrow S$. The *power* of a language $L = (\mathcal{B}, \mathcal{C})$, written $\|L\|$, is $|\mathcal{E}(L)|$. In the usual case, when $\mathcal{C}$ is countable (i.e., finite or countably infinite), $\|L\| = \omega$. When $\mathcal{C}$ is uncountable, $\|L\| = |\mathcal{C}|$.

We end off the section by introducing theories and a proof system for Alonzo.

Definition 2.7 (Theory). A *theory* of Alonzo is a pair $T = (L, \Gamma)$ where L is a language and Γ is a set of sentences of L . We say that L is the *language* of T and Γ are the *axioms* of T .

Definition 2.8 (Proof System). A *proof system* $\mathfrak{P}$ (P in fraktur font) for Alonzo consists of a decidable set of *axioms* and *rules of inference*. Each axiom is a formula of Alonzo and each rule of inference has the following form: From the formulas $\mathbf{A}_o^1, \dots, \mathbf{A}_o^n$, infer the formula $\mathbf{B}_o$ (possibly subject to certain constraints).

Proofs in a proof system $\mathfrak{P}$ are defined as follows:

Definition 2.9 (Proof). A *proof of $\mathbf{A}_o$ in $\mathfrak{P}$* is a finite sequence Π of formulas of Alonzo ending with $\mathbf{A}_o$ such that every formula in Π is an axiom of $\mathfrak{P}$ or inferred from previous formulas in Π by one of the rules of inference of $\mathfrak{P}$. Now let Γ be a set of formulas of Alonzo. A *proof of $\mathbf{A}_o$ from Γ in $\mathfrak{P}$* is a pair (Π_1, Π_2) of finite sequences of formulas of Alonzo ending with $\mathbf{A}_o$ such that Π_1 is a proof in $\mathfrak{P}$, Π_2 ends with $\mathbf{A}_o$, and every formula in Π_2 is a member of Γ , a member of Π_1 (and thus a theorem of $\mathfrak{P}$), or inferred from previous formulas in Π_2 by one of the rules of inference of $\mathfrak{P}$ modified, if necessary, so that the free variables in members of Γ are treated as constants instead of as universally quantified variables as they are in axioms.

In particular, we are interested in the proof system $\mathfrak{A}$, whose axioms and rules of inference are presented in [7, Section 8.2]. $\mathfrak{A}$ is noteworthy for being sound and complete with respect to the general semantics [7, Corollary 8.13].

Definition 2.10. Now let $T = (L, \Gamma)$ be a theory, $\mathbf{A}_o \in \mathcal{E}(L)$, and $\mathfrak{P}$ be a proof system for Alonzo.

1. A *theorem of $\mathfrak{P}$* is a formula that has a proof in $\mathfrak{P}$, written $\vdash_{\mathfrak{P}} \mathbf{A}_o$.
2. Let Γ be a set of formulas. We write $\Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o$ to assert that there is a proof of $\mathbf{A}_o$ from Γ in $\mathfrak{P}$.
3. Let Γ be a set of formulas. Γ is *consistent in $\mathfrak{P}$* if not $\Gamma \vdash_{\mathfrak{P}} F_o$.
4. $\mathbf{A}_o$ is *provable from T in $\mathfrak{P}$* , written $T \vdash_{\mathfrak{P}} \mathbf{A}_o$, if $\Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o$.
5. T is *consistent in $\mathfrak{P}$* if Γ is consistent in $\mathfrak{P}$.

Definition 2.11. Let $L_i = (\mathcal{B}_i, \mathcal{C}_i)$ be a language and $T_i = (L_i, \Gamma_i)$ be a theory for $i \in \{1, 2\}$. T_2 is an *extension* of T_1 if $L_1 \leq L_2$ and $\Gamma_1 \subseteq \Gamma_2$.

2.2 Semantics

We define the general and standard semantics for Alonzo. We start with the corresponding notion of a universe from first-order logic.

Definition 2.12 (Frame). A *frame* for L is a collection $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ of nonempty domains (sets) of values such that:

1. *Domain of truth values:* $D_o = \mathbb{B} = \{\mathbf{F}, \mathbf{T}\}$.
2. *Predicate domain:* $D_{\alpha \rightarrow o}$ is a set of *some* total functions from D_α to D_o for $\alpha \in \mathcal{T}(L)$.
3. *Function domain:* $D_{\alpha \rightarrow \beta}$ is a set of *some* partial and total functions from D_α to D_β for $\alpha, \beta \in \mathcal{T}(L)$ with $\beta \neq o$.
4. *Product domain:* $D_{\alpha \times \beta} = D_\alpha \times D_\beta$ for $\alpha, \beta \in \mathcal{T}(L)$.

A predicate domain $D_{\alpha \rightarrow o}$ is *full* if it is the set of *all* total functions from D_α to D_o , and a function domain $D_{\alpha \rightarrow \beta}$ with $\beta \neq o$ is *full* if it is the set of *all* partial and total functions from D_α to D_β . The frame is *full* if $D_{\alpha \rightarrow \beta}$ is full for all $\alpha, \beta \in \mathcal{T}(L)$. Notice that the only restriction on a *base domain*, i.e., $D_{\mathbf{a}}$ for some $\mathbf{a} \in \mathcal{B}$, is that it is nonempty and that the frame is completely determined by its base domains when the frame is full.

An *interpretation* of L is a pair $M = (\mathcal{D}, I)$ where $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ is a frame for L and I is an *interpretation function* that maps each constant in $\mathcal{C}$ of type α to an element of D_α .

Definition 2.13 (Assignment). Let $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be a frame for L . An *assignment into* $\mathcal{D}$ is a function φ whose domain is the set of variables of L such that $\varphi((\mathbf{x} : \alpha)) \in D_\alpha$ for each variable $(\mathbf{x} : \alpha)$ of L . Given an assignment φ , a variable $(\mathbf{x} : \alpha)$ of L , and $d \in D_\alpha$, let $\varphi[(\mathbf{x} : \alpha) \mapsto d]$ be the assignment ψ in $\mathcal{D}$ such that $\psi((\mathbf{x} : \alpha)) = d$ and $\psi((\mathbf{y} : \beta)) = \varphi((\mathbf{y} : \beta))$ for all variables $(\mathbf{y} : \beta)$ of L distinct from $(\mathbf{x} : \alpha)$. Given an interpretation M of L , let $\text{assign}(M)$ be the set of assignments into the frame of M .

Definition 2.14 (General Model). Let $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be a frame for L and $M = (\mathcal{D}, I)$ be an interpretation of L . M is a *general model* of L if there is a partial binary *valuation function* V^M such that, for all assignments $\varphi \in \text{assign}(M)$ and expressions $\mathbf{C}_\gamma$ of L , (1) either $V_\varphi^M(\mathbf{C}_\gamma) \in D_\gamma$ or $V_\varphi^M(\mathbf{C}_\gamma)$ is undefined² and (2) each of the following conditions is satisfied:

- V1. $V_\varphi^M((\mathbf{x} : \alpha)) = \varphi((\mathbf{x} : \alpha))$.
- V2. $V_\varphi^M(\mathbf{c}_\alpha) = I(\mathbf{c}_\alpha)$.
- V3. $V_\varphi^M(\mathbf{A}_\alpha = \mathbf{B}_\alpha) = \text{T}$ if $V_\varphi^M(\mathbf{A}_\alpha)$ is defined, $V_\varphi^M(\mathbf{B}_\alpha)$ is defined, and $V_\varphi^M(\mathbf{A}_\alpha) = V_\varphi^M(\mathbf{B}_\alpha)$. Otherwise, $V_\varphi^M(\mathbf{A}_\alpha = \mathbf{B}_\alpha) = \text{F}$.
- V4. $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) = V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta})(V_\varphi^M(\mathbf{A}_\alpha))$ — i.e., the application of the function $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta})$ to the argument $V_\varphi^M(\mathbf{A}_\alpha)$ — if $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta})$ is defined, $V_\varphi^M(\mathbf{A}_\alpha)$ is defined, and $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta})$ is defined at $V_\varphi^M(\mathbf{A}_\alpha)$. Otherwise, $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) = \text{F}$ if $\beta = o$ and $V_\varphi^M(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ is undefined if $\beta \neq o$.
- V5. $V_\varphi^M(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)$ is the (partial or total) function $f \in D_{\alpha \rightarrow \beta}$ such that, for each $d \in D_\alpha$, $f(d) = V_{\varphi[(\mathbf{x} : \alpha) \mapsto d]}^M(\mathbf{B}_\beta)$ if $V_{\varphi[(\mathbf{x} : \alpha) \mapsto d]}^M(\mathbf{B}_\beta)$ is defined and $f(d)$ is undefined if $V_{\varphi[(\mathbf{x} : \alpha) \mapsto d]}^M(\mathbf{B}_\beta)$ is undefined.
- V6. $V_\varphi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$ is the $d \in D_\alpha$ such that $V_{\varphi[(\mathbf{x} : \alpha) \mapsto d]}^M(\mathbf{A}_o) = \text{T}$ if there is exactly one such d . Otherwise, $V_\varphi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$ is undefined.
- V7. $V_\varphi^M((\mathbf{A}_\alpha, \mathbf{B}_\beta)) = (V_\varphi^M(\mathbf{A}_\alpha), V_\varphi^M(\mathbf{B}_\beta))$ if $V_\varphi^M(\mathbf{A}_\alpha)$ and $V_\varphi^M(\mathbf{B}_\beta)$ are defined. Otherwise, $V_\varphi^M((\mathbf{A}_\alpha, \mathbf{B}_\beta))$ is undefined.

It follows that V^M is unique when it exists. $V_\varphi^M(\mathbf{C}_\gamma)$ is called the *value of* $\mathbf{C}_\gamma$ *in* M *with respect to* φ when $V_\varphi^M(\mathbf{C}_\gamma)$ is defined. $\mathbf{C}_\gamma$ is said to have no value in M with respect to φ when $V_\varphi^M(\mathbf{C}_\gamma)$ is undefined.

Now the syntax and semantics of Alonzo allow us to define familiar boolean operators and quantifiers such that they are interpreted as expected.

²We write $V_\varphi^M(\mathbf{C}_\gamma)$ instead of $V^M(\varphi, \mathbf{C}_\gamma)$.

T_o	stands for	$(\lambda x : o . x) = (\lambda x : o . x).$
F_o	stands for	$(\lambda x : o . T_o) = (\lambda x : o . x).$
$\wedge_{o \rightarrow o \rightarrow o}$	stands for	$\lambda x : o . \lambda y : o .$ $(\lambda g : o \rightarrow o \rightarrow o . g T_o T_o) =$ $(\lambda g : o \rightarrow o \rightarrow o . g x y).$
$(\mathbf{A}_o \wedge \mathbf{B}_o)$	stands for	$\wedge_{o \rightarrow o \rightarrow o} \mathbf{A}_o \mathbf{B}_o.$
$\Rightarrow_{o \rightarrow o \rightarrow o}$	stands for	$\lambda x : o . \lambda y : o . x = (x \wedge y).$
$(\mathbf{A}_o \Rightarrow \mathbf{B}_o)$	stands for	$\Rightarrow_{o \rightarrow o \rightarrow o} \mathbf{A}_o \mathbf{B}_o.$
$\neg_{o \rightarrow o}$	stands for	$\lambda x : o . x = F_o.$
$(\neg \mathbf{A}_o)$	stands for	$\neg_{o \rightarrow o} \mathbf{A}_o.$
$\vee_{o \rightarrow o \rightarrow o}$	stands for	$\lambda x : o . \lambda y : o . \neg(\neg x \wedge \neg y).$
$(\mathbf{A}_o \vee \mathbf{B}_o)$	stands for	$\vee_{o \rightarrow o \rightarrow o} \mathbf{A}_o \mathbf{B}_o.$
$(\forall \mathbf{x} : \alpha . \mathbf{A}_o)$	stands for	$(\lambda x : \alpha . T_o) = (\lambda \mathbf{x} : \alpha . \mathbf{A}_o).$
$(\exists \mathbf{x} : \alpha . \mathbf{A}_o)$	stands for	$\neg(\forall \mathbf{x} : \alpha . \neg \mathbf{A}_o).$

Table 2.1: Notational Definitions for Boolean Operators and Quantifiers

For example, if M is a general model of L and $\varphi \in \text{assign}(M)$,

$$\begin{aligned} V_\varphi^M(\forall \mathbf{x} : \alpha . \mathbf{A}_o) = \top & \quad \text{iff} \quad V_{\varphi[(\mathbf{x}:\alpha) \mapsto d]}^M(\mathbf{A}_o) = \top \text{ for all } d \in D_\alpha^M, \\ V_\varphi^M(\mathbf{A}_o \wedge \mathbf{B}_o) = \top & \quad \text{iff} \quad V_\varphi^M(\mathbf{A}_o) = \top \text{ and } V_\varphi^M(\mathbf{B}_o) = \top, \end{aligned}$$

and so on. See Exercises 1 and 3 in Chapter 6 of [7].

Definition 2.15.

1. The *size* of M , written $|M|$, is the cardinality of $\bigcup_{\mathbf{a} \in \mathcal{B}} D_{\mathbf{a}}^M$. M is *finite* if its size is finite and *infinite* otherwise.
2. The *power* of M , written $\|M\|$, is the least cardinal κ such that $|D_\alpha^M| \leq \kappa$ for all $\alpha \in \mathcal{T}(L)$.

The power of a model need not exist; whether it exists can depend on the underlying set-theoretic assumptions that one makes. For instance, the power of a model of countably infinite size with a full frame exists if a strongly inaccessible cardinal exists.

Now the following are two useful classifications of general models:

Definition 2.16. Let $M = (\mathcal{D}, I)$ be a general model of L .

1. M is a *standard model* of L if $\mathcal{D}$ is full. Note that it is sufficient for M to be just an interpretation of L by [7, Proposition 5.7].
2. M is a *frugal general model* of L if $\|M\| \leq \|L\|$.

Now we have two kinds of semantics; namely, general models semantics and standard models semantics:

Definition 2.17. Let M be a general model L and $\mathbf{A}_o \in \mathcal{E}(L)$.

1. φ *satisfies* $\mathbf{A}_o$ in M , written $M \models_\varphi \mathbf{A}_o$, if $V_\varphi^M(\mathbf{A}_o) = \top$ for $\varphi \in \text{assign}(M)$.

2. $\mathbf{A}_o$ is *satisfiable in M* if $M \models_\varphi \mathbf{A}_o$ for some $\varphi \in \text{assign}(M)$.
3. $\mathbf{A}_o$ is *satisfiable* if $M \models_\varphi \mathbf{A}_o$ for some general model M and some $\varphi \in \text{assign}(M)$.
4. $\mathbf{A}_o$ is *valid in M* (or M is a *model of $\mathbf{A}_o$*), written $M \models \mathbf{A}_o$, if $M \models_\varphi \mathbf{A}_o$ for all $\varphi \in \text{assign}(M)$.
5. If $\mathbf{A}_o$ is a sentence, then $\mathbf{A}_o$ is *true [false] in M* if $V_\varphi^M(\mathbf{A}_o) = \text{T} [\text{F}]$ (for all $\varphi \in \text{assign}(M)$).
6. $\mathbf{A}_o$ is *valid (in the general sense)*, written $\models \mathbf{A}_o$, if $M \models \mathbf{A}_o$ for all general models M that interpret $\mathbf{A}_o$.
7. $\mathbf{A}_o$ is *valid in the standard sense*, written $\models^s \mathbf{A}_o$, if $M \models \mathbf{A}_o$ for all standard models M that interpret $\mathbf{A}_o$.
8. $\mathbf{A}_o$ is *logically equivalent* to $\mathbf{B}_o$ if $V_\varphi^M(\mathbf{A}_o) = V_\varphi^M(\mathbf{B}_o)$ for all general models M that interpret $\mathbf{A}_o$ and $\mathbf{B}_o$ and all $\varphi \in \text{assign}(M)$.

Now let M be a general model that interprets a set Γ of formulas.

1. φ *satisfies Γ in M* , written $M \models_\varphi \Gamma$, if $M \models_\varphi \mathbf{A}_o$ for all $\mathbf{A}_o \in \Gamma$.
2. Γ is *satisfiable in M* if $M \models_\varphi \Gamma$ for some $\varphi \in \text{assign}(M)$.
3. Γ is *satisfiable* if $M \models_\varphi \Gamma$ for some general model M of L and some $\varphi \in \text{assign}(M)$.
4. M is a *model of Γ* , written $M \models \Gamma$, if $M \models \mathbf{A}_o$ for all $\mathbf{A}_o \in \Gamma$.
5. $\mathbf{A}_o$ is a *semantic consequence of Γ (in the general sense)*, written $\Gamma \models \mathbf{A}_o$, if $M \models_\varphi \Gamma$ implies $M \models_\varphi \mathbf{A}_o$ for all general models M that interpret $\mathbf{A}_o$ and Γ and all $\varphi \in \text{assign}(M)$.
6. $\mathbf{A}_o$ is a *semantic consequence of Γ in the standard sense*, written $\Gamma \models^s \mathbf{A}_o$, if $M \models_\varphi \Gamma$ implies $M \models_\varphi \mathbf{A}_o$ for all standard models M that interpret $\mathbf{A}_o$ and Γ and all $\varphi \in \text{assign}(M)$.

Now let $T = (L, \Gamma)$ be a theory and $\mathbf{A}_o \in \mathcal{E}(L)$.

1. $\mathbf{A}_o$ is *valid in T (in the general sense)*, written $T \models \mathbf{A}$ if $\Gamma \models \mathbf{A}_o$.
2. $\mathbf{A}_o$ is *valid in T in the standard sense*, written $T \models^s \mathbf{A}$ if $\Gamma \models^s \mathbf{A}_o$.
3. A *theorem* of T is a sentence that is valid in T .
4. M is a *model of T* , written $M \models T$, if $M \models \Gamma$.
5. T is *satisfiable* if Γ is satisfiable.

Notice how there are two notions of satisfiability: one that considers all general models, and one that considers just standard models. The following example illustrates the difference:

Example 2.18 (Skolem's paradox). Let $T = (L, \Gamma)$ be the theory of complete ordered fields as specified in [7, Theory Definition 13.4]. The syntax of Alonzo is strong enough to express that the reals are uncountable; namely, we have that

$$(\star) \quad T \models \neg(\exists f : R \rightarrow R . \text{BIJ-ON}(f, N_{\{R\}}, U_{\{R\}})).^3$$

³See Chapter 6 of [7] for a primer on quasitypes and the notational definition of BIJ-ON.

So, if $M \models T$, there is no bijection $f_{R \rightarrow R}$ from the naturals (treated as a subset of the reals) to the reals in $D_{R \rightarrow R}^M$. T is obviously satisfiable by the reals, but the Löwenheim-Skolem Theorem [7, Theorem 9.39] implies the existence of a countable model N of T . How could this be?

The reason is that the Henkin construction for Alonzo (Lemma A.1 and Theorem A.2) produces a frugal general model for a theory consistent in $\mathfrak{A}$. Since $\|L\| = \omega$, the frugality of N implies that we have that $D_{R \rightarrow R}^N \leq \omega$. So, $N \models T$ implies that N is *missing* a bijection from $V^N(N_{\{R\}})$ – the naturals – to $V^N(U_{\{R\}})$ – the reals. Now let S be a standard model such that $S \models T$. The scenario changed: because $D_{R \rightarrow R}^S$ is *full*, $(\star)$ implies that $D_R^S = V^S(U_{\{R\}})$ is uncountable.

Like in first-order logic, the resolution to Skolem’s paradox is found at the level of how a model interprets a sentence akin to

$$\neg(\exists f : R \rightarrow R . \text{BIJ-ON}(f, N_{\{R\}}, U_{\{R\}})).$$

Since Alonzo has a built in theory of functions – vis-à-vis typed function domains in a frame – the notion of “missing” functions is not a metatheoretic one, as it is in first-order logic. Additionally, unlike in first-order logic, we do not have to axiomatize a version of set theory in order for the seeming paradox to arise.

We end off the section by introducing expansions of a general model. Let $f : X \rightarrow Y$ and $X \subseteq X$. Then $f|_A$ (the *strong restriction of f to A*) denotes the function $g : A \rightarrow Y$ such that $g(a) = f(a)$ for all $a \in A$.

Definition 2.19 (Expansion of a model). Let M_i be a general model of L_i for $i \in \{1, 2\}$. Assume $L_1 \leq L_2$. M_2 is an *expansion of M_1 to L_2* (or M_1 is a *reduct of M_2 to L_1*), written $M_1 \leq M_2$, if $\mathcal{D}_1 \subseteq \mathcal{D}_2$ and $I_1 \sqsubseteq I_2$.

If $L_1 \neq L_2$, then M_1 has many possible expansions to L_2 , one for each way of assigning domains to the types in $\mathcal{T}(L_2) \setminus \mathcal{T}(L_1)$ and values to the constants in $\mathcal{C}_2 \setminus \mathcal{C}_1$. However, M_2 has only one reduct to L_1 , namely, the general model M_1 where $\mathcal{D}_1 = \{D_\alpha^2 \in \mathcal{D}_2 \mid \alpha \in \mathcal{T}(L_1)\}$ and $I_1 = I_2|_{\mathcal{C}_1}$. It is noteworthy that expansions and reducts are not only defined similarly but also behave similarly to their counterparts in first-order logic; this is illustrated by the following (Lemma 5.14 in [7]):

Proposition 2.20. *Let M_i be a general model of L_i for $i \in \{1, 2\}$ such that $L_1 \leq L_2$ and M_2 is an expansion of M_1 to L_2 . If $\mathbf{A}_o$ is a formula of L_1 , then $M_1 \models \mathbf{A}_o$ iff $M_2 \models \mathbf{A}_o$.*

In the subsequent chapter, will see that capturing the behaviour of (elementary) extensions from first-order logic requires much more work, particularly because adding members to base domains impacts function and product domains.

Chapter 3

Sharpened Löwenheim- Skolem Theorems

3.1 Preliminaries

Recall Theorem 9.39 in [7]:

Theorem 3.1 (Löwenheim-Skolem Theorem). *Let T be a theory. If T has an infinite general model, then T has a general model of size and power κ for every cardinal $\kappa \geq \|L\|$.*

Given an infinite model of some theory T , Theorem 3.1 allows you to construct smaller (going down) and larger (going up) infinite models of T . However, there is no structural relationship between the model M that T has and a model N produced by Theorem 3.1. To this end, we will develop the analogues of the “sharpened” first-order Löwenheim-Skolem theorems in Alonzo, such that the models M and N will be related by a strong¹ embedding. First, we develop analogues of various definitions and constructions from the first-order world. We model our development after [12, Theorem 2.3.4, Proposition 2.3.5, Theorem 2.3.7].

Definition 3.2 (Embedding). Let $M_i = (\mathcal{D}^i, I^i)$ be interpretations of L for $i \in \{1, 2\}$. An *embedding* from M_1 to M_2 is a set $\mathfrak{E} = \{\varepsilon_\alpha \mid \alpha \in \mathcal{T}(L)\}$ of mappings² such that:

1. ε_α is an injection from D_α^1 to D_α^2 for all $\alpha \in \mathcal{T}(L)$;
2. $\varepsilon_o(\mathbf{F}) = \mathbf{F}$ and $\varepsilon_o(\mathbf{T}) = \mathbf{T}$;
3. $\varepsilon_\beta(f(a)) \simeq \varepsilon_{\alpha \rightarrow \beta}(f)(\varepsilon_\alpha(a))$ for all $\alpha, \beta \in \mathcal{T}(L)$, $f \in D_{\alpha \rightarrow \beta}^1$, and $a \in D_\alpha^1$;
4. $\varepsilon_{\alpha \times \beta}((a, b)) = (\varepsilon_\alpha(a), \varepsilon_\beta(b))$ for all $\alpha, \beta \in \mathcal{T}(L)$, $a \in D_\alpha^1$, and $b \in D_\beta^1$;
5. $\varepsilon_\alpha(I^1(\mathbf{c}_\alpha)) = I^2(\mathbf{c}_\alpha)$ for all $\mathbf{c}_\alpha \in \mathcal{C}$.

¹“Strong” is the analogue of “elementary” from first-order logic.

²We allow the omission of the type of $\varepsilon_\alpha \in \mathcal{E}$ when it can be easily inferred.

Notice how the definition of an embedding is slightly weaker than the definition of an isomorphism in [7, Chapter 5.6]. The single difference is that mappings $\varepsilon_\alpha \in \mathfrak{E}$ only ought to be injective (rather than bijective).

Definition 3.3 (Inclusion). Let $M_i = (\mathcal{D}^i, I^i)$ be interpretations of L for $i \in \{1, 2\}$. We say $\mathfrak{I} = \{\iota_\alpha \mid \alpha \in \mathcal{T}(L)\}$ an embedding from M_1 to M_2 is an *inclusion* if, in addition to the conditions in Definition 3.2, we have that:

$$\iota_{\mathbf{a}}(d) = d \text{ for all } \mathbf{a} \in \mathcal{B} \text{ and } d \in D_{\mathbf{a}}^1.$$

We say M_1 is a *substructure* of M_2 (or conversely, M_2 an *extension* of M_1).

The definition of an inclusion necessitates that for all $\alpha \in \mathcal{T}(L)$, $\iota(d_\alpha^M)$ is an “expanded” version of d_α^M :

Lemma 3.4. Let $M = (\mathcal{D}^M, I^M)$, $N = (\mathcal{D}^N, I^N)$ be interpretations of L and $\mathfrak{I} = \{\iota_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be an inclusion from M to N . Let $Z = \{\zeta_\alpha : \text{ran}(\iota_\alpha) \rightarrow D_\alpha^M \mid \alpha \in \mathcal{T}(L)\}$ be a set of mappings such that:

$$\begin{aligned} \zeta_o(\iota(d_o^M)) &= d_o^M \text{ for all } d_o^M \in D_o^M; \\ \zeta_{\mathbf{a}}(\iota(d_{\mathbf{a}}^M)) &= d_{\mathbf{a}}^M \text{ for all } \mathbf{a} \in \mathcal{T}(L) \text{ and } d_{\mathbf{a}}^M \in D_{\mathbf{a}}^M; \\ \zeta_{\alpha \rightarrow \beta}(\iota(d_{\alpha \rightarrow \beta}^M)) &= \text{the } g_{\alpha \rightarrow \beta} \text{ such that for all } \alpha, \beta \in \mathcal{T}(L), d_\alpha^M \in D_\alpha^M, \text{ and} \\ &\quad d_{\alpha \rightarrow \beta}^M \in D_{\alpha \rightarrow \beta}^M, g_{\alpha \rightarrow \beta}(\zeta_\alpha(\iota(d_\alpha^M))) = \zeta_\beta(\iota(d_{\alpha \rightarrow \beta}^M(d_\alpha^M))); \\ \zeta_{\alpha \times \beta}(\iota((d_\alpha^M, d_\beta^M))) &= (\zeta_\alpha(\iota(d_\alpha^M)), \zeta_\beta(\iota(d_\beta^M))) \text{ for all } \alpha, \beta \in \mathcal{T}(L), \\ &\quad d_\alpha^M \in D_\alpha^M, \text{ and } d_\beta^M \in D_\beta^M. \end{aligned}$$

Then ζ_α acts as the inverse to ι_α . That is,

$$(\star) \text{ for all } \alpha \in \mathcal{T}(L) \text{ and } d_\alpha^M \in D_\alpha^M, \zeta(\iota(d_\alpha^M)) = d_\alpha^M.^3$$

Proof We prove $(\star)$ by structural induction on the complexity of types.

Base: $\alpha = o$ or $\alpha \in \mathcal{B}$. Then $\zeta(\iota(d_\alpha^M)) = d_\alpha^M$ follows directly from the definition of ζ .

Step:

Case 1. $\alpha = \beta \rightarrow \gamma$. Pick any $d_\beta^M \in D_\beta^M$. Then

$$\begin{aligned} &\zeta(\iota(d_{\beta \rightarrow \gamma}^M))(d_\beta^M) \\ &\simeq \zeta(\iota(d_{\beta \rightarrow \gamma}^M))\zeta(\iota((d_\beta^M))) && \text{(induction hypothesis)} \\ &\simeq \zeta(\iota(d_{\beta \rightarrow \gamma}^M(d_\beta^M))) && \text{(definition of } \zeta) \\ &\simeq d_{\beta \rightarrow \gamma}^M(d_\beta^M). && \text{(induction hypothesis)} \end{aligned}$$

Since d_β^M was arbitrary, we conclude

$$\zeta(\iota(d_{\beta \rightarrow \gamma}^M(d_\beta^M))) = d_{\beta \rightarrow \gamma}^M(d_\beta^M).$$

³Here, and below, we allow the omission of the type of $\zeta_\alpha \in Z$ when it can be easily inferred.

Case 2. $\alpha = \beta \times \gamma$. Then

$$\begin{aligned} & \zeta(\iota((d_\beta^M, d_\gamma^M))) \\ &= (\zeta(\iota(d_\beta^M)), \zeta(\iota(d_\gamma^M))) && \text{(definition of } \zeta) \\ &= (d_\beta^M, d_\gamma^M). && \text{(induction hypothesis)} \end{aligned}$$

Thus, we have demonstrated $(\star)$. $\square$

Definition 3.5 (Strong Embedding). Let M_i be an interpretation of L for $i = \{1, 2\}$. Let $\mathfrak{E} = \{\varepsilon_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be an embedding from M_1 to M_2 . Let $\varphi \in \text{assign}(M_1)$ and define $\psi \in \text{assign}(M_2)$ to be $\psi((\mathbf{x} : \alpha)) = \varepsilon_\alpha(\varphi((\mathbf{x} : \alpha)))$ for all variables $(\mathbf{x} : \alpha)$. Call $\mathfrak{E}$ a *strong embedding* if

$$M_1 \models_\varphi \mathbf{A}_o \text{ iff } M_2 \models_\psi \mathbf{A}_o$$

for all $\mathbf{A}_o \in \mathcal{E}(L)$. Furthermore, call M_1 a *strong substructure* of M_2 (or conversely, M_2 a *strong extension* of M_1) if $\mathfrak{E}$ is an inclusion. In this case, we write $M_1 \preceq M_2$.

We can use Definition 3.5 to define a limit structure with respect to the strong embedding relation.

Definition 3.6 (Strong Chain). Let $(\mathcal{I}, <)$ be a well-order of order type $\sigma \leq \omega$. Let M_i be an interpretation of L for $i \in \mathcal{I}$. Call $(M_i \mid i \in \mathcal{I})$ a *strong chain* if $M_i \preceq M_j$ for all $i < j$ where $i, j \in \mathcal{I}$. In particular, denote the inclusion from M_i to M_{i+1} as $\mathfrak{I}^i = \{\iota_\alpha^i \mid \alpha \in \mathcal{T}(L)\}$. Given any $d_\alpha^i \in D_\alpha^i$, let $\iota^{x,i}$ denote the composition⁴ of inclusions $\iota^x \circ \iota^{x-1} \circ \dots \circ \iota^{i+1} \circ \iota^i$, such that $\iota^{x,i}(d_\alpha^i) \in D_\alpha^x$.

Definition 3.7. Let $(\mathcal{I}, <)$ be a well-order of order type $\sigma \leq \omega$ and $(M_i \mid i \in \mathcal{I})$ be a strong chain. Define the corresponding *union general model* of the strong chain to be $M = (\mathcal{D}, I)$, where

$$\begin{aligned} D_o^M &= \{T, F\}; \\ D_{\mathbf{a}}^M &= \bigcup_{i \in \mathcal{I}} D_{\mathbf{a}}^{M_i} \text{ for all } \mathbf{a} \in \mathcal{T}(L); \\ D_{\alpha \rightarrow \beta}^M &= \bigcup_{i \in \mathcal{I}} D_{\alpha \rightarrow \beta}^{M_i^*} \text{ for all } \alpha, \beta \in \mathcal{T}(L); \\ D_{\alpha \times \beta}^M &= D_\alpha^M \times D_\beta^M; \end{aligned}$$

and where $D_{\alpha \rightarrow \beta}^{M_i^*} = \{\lim_{x \rightarrow \sigma} \iota^{x,i}(f_{\alpha \rightarrow \beta}^i) \mid f_{\alpha \rightarrow \beta}^i \in D_{\alpha \rightarrow \beta}^{M_i}\}$.

Immediately, our definition gives us $I^M(\mathbf{c}_\alpha) = \lim_{x \rightarrow \sigma} \iota^{x,i}(I^i(\mathbf{c}_\alpha))$ for all $i \in \mathcal{I}$ and $\mathbf{c}_\alpha \in \mathcal{C}$. Now $D_{\mathbf{a}}^M = \bigcup_{i \in \mathcal{I}} D_{\mathbf{a}}^{M_i}$ is well-defined for all $\mathbf{a} \in \mathcal{B}$, and at this stage, the construction mirrors the one in first-order logic (e.g., see [12, Proposition 2.3.11]). Recall that function and product domains are determined by the base domains. For example, $f \in D_{\mathbf{a} \rightarrow \mathbf{a}}^M$ has domain and range $D_{\mathbf{a}}^M$ and $(a, b) \in D_{\mathbf{a} \times \mathbf{a}}^M$ is a member of $D_{\mathbf{a}}^M \times D_{\mathbf{a}}^M$. It is thus straightforward to show that limits of the form $\lim_{x \rightarrow \sigma} \iota^{x,i}(f_{\alpha \rightarrow \beta}^i)$ and $\lim_{x \rightarrow \sigma} \iota^{x,i}(a_\alpha^i, b_\beta^i)$ are well-defined by induction on the complexity of types.

⁴Note that we define function composition differently than in [7]. There, $(f \circ g)(x)$ denotes $g(f(x))$.

Proposition 3.8. *Let $(\mathcal{I}, <)$ be a well-order of order type $\sigma \leq \omega$, $(M_i \mid i \in \mathcal{I})$ be a strong chain, and M be the corresponding union general model. Given $d \in D_\alpha^{M_i}$, denote the corresponding element⁵ in D_α^M as $\iota_\alpha^\sigma(d)$. This naturally induces the inclusion $\mathfrak{I}^\sigma = \{\iota_\alpha^\sigma \mid \alpha \in \mathcal{T}(L)\}$.⁶ Then it follows that for all $i \in \mathcal{I}$, $\mathbf{A}_\alpha$, and $\varphi \in \text{assign}(M_i)$ we have $(\star)$*

$$V_\varphi^{M_i}(\mathbf{A}_\alpha) \simeq \zeta^\sigma(V_\psi^M(\mathbf{A}_\alpha)),$$

where $\psi = \iota^\sigma(\varphi((\mathbf{x} : \alpha)))$ for all $(\mathbf{x} : \alpha)$, and the collection of mappings $Z^\sigma = \{\zeta_\alpha^\sigma \mid \alpha \in \mathcal{T}(L)\}$ is defined similarly to Z in Lemma 3.4.

Proof We prove $(\star)$ by structural induction on the complexity of expressions.

E1. $\mathbf{A}_\alpha$ is $(\mathbf{x} : \alpha)$. Then $\zeta^\sigma(V_\psi^M((\mathbf{x} : \alpha))) = \zeta^\sigma(\psi((\mathbf{x} : \alpha))) = \zeta^\sigma(\iota^\sigma(\varphi((\mathbf{x} : \alpha)))) = \varphi((\mathbf{x} : \alpha)) = V_\varphi^{M_i}((\mathbf{x} : \alpha))$ follows from Condition V1 of a general model and Lemma 3.4.⁷

E2. $\mathbf{A}_\alpha$ is $\mathbf{c}_\alpha$. Then $\zeta^\sigma(V_\psi^M(\mathbf{c}_\alpha)) = \zeta^\sigma(I^M(\mathbf{c}_\alpha)) = \zeta^\sigma\left(\lim_{x \rightarrow \sigma} \iota^{x,i}(I^i(\mathbf{c}_\alpha))\right) = I^i(\mathbf{c}_\alpha) = V_\varphi^{M_i}(\mathbf{c}_\alpha)$ follows from Condition V2 of a general model and Lemma 3.4.

E3. $\mathbf{A}_\alpha$ is $(\mathbf{B}_\beta = \mathbf{C}_\beta)$.

Case 1. $V_\varphi^{M_i}(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{T}$ or $V_{\varphi'}^M(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{T}$. Then $\zeta^\sigma(V_\psi^M(\mathbf{B}_\beta)) = V_\varphi^{M_i}(\mathbf{B}_\beta) = V_\varphi^{M_i}(\mathbf{C}_\beta) = \zeta^\sigma(V_\psi^M(\mathbf{C}_\beta))$ follows from Condition V3 of a general model and by the induction hypothesis. The definition of ζ^σ implies that $V_\psi^M(\mathbf{B}_\beta = \mathbf{C}_\beta) = V_\varphi^{M_i}(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{T}$.⁸

Case 2. $V_\varphi^{M_i}(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{F}$ or $V_\psi^M(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{F}$. One of $V_\varphi^{M_i}(\mathbf{B}_\beta)$ or $V_\varphi^{M_i}(\mathbf{C}_\beta)$ is undefined iff one of $V_\psi^M(\mathbf{B}_\beta)$ or $V_\psi^M(\mathbf{C}_\beta)$ is undefined by the induction hypothesis; and so, $V_\varphi^M(\mathbf{B}_\beta = \mathbf{C}_\beta) = V_\varphi^{M_i}(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{F}$ from Condition V3 of a general model. $V_\varphi^{M_i}(\mathbf{B}_\beta) \neq V_\varphi^{M_i}(\mathbf{C}_\beta)$ or $V_\varphi^M(\mathbf{B}_\beta) \neq V_\varphi^M(\mathbf{C}_\beta)$ implies that $V_\varphi^M(\mathbf{B}_\beta = \mathbf{C}_\beta) = \text{F}$ by a similar argument to Case 1.

E4. $\mathbf{A}_\alpha$ is $(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$.

Case 1. $V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined. We have that $\zeta^\sigma(\iota^\sigma(V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha}))) = \zeta^\sigma(V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha}))$ and $\zeta^\sigma(\iota^\sigma(V_\varphi^{M_i}(\mathbf{B}_\beta))) = \zeta^\sigma(V_\psi^M(\mathbf{B}_\beta))$ by the induction hypothesis and Lemma 3.4. Since ζ^σ is defined only on images under the injective map ι^σ , we have that $\iota^\sigma(V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha})) = V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha})$ and $\iota^\sigma(V_\varphi^{M_i}(\mathbf{B}_\beta)) = V_\psi^M(\mathbf{B}_\beta)$. The fact that $V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined and $\mathfrak{I}^\sigma$ is an inclusion implies that $\iota^\sigma(V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)) = V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$. Lemma 3.4 implies that

$$V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta) = \zeta^\sigma(V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)).$$

⁵Built via Definition 3.7.

⁶Below, we allow the omission of the type of ι_α^σ when it can be easily inferred.

⁷Throughout the proof, we appeal to this lemma since the definition of ζ^σ is similar to ζ and it is easy to see that a similar result holds.

⁸For example, $\zeta^\sigma(V_\psi^M(\mathbf{B}_\beta))$ is defined implies $V_\psi^M(\mathbf{B}_\beta)$ is in the range of the injection ι^σ .

Case 2. $V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined. By the construction of M , the induction hypothesis, and Lemma 3.4, there is $j \geq i$ such that $\zeta^\sigma(\iota^\sigma(V_\varphi^{M_j}(\mathbf{F}_{\beta \rightarrow \alpha}))) = \zeta^\sigma(V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha}))$ and $\zeta^\sigma(\iota^\sigma(V_\varphi^{M_j}(\mathbf{B}_\beta))) = \zeta^\sigma(V_\psi^M(\mathbf{B}_\beta))$.⁹ Now let

$$\mathbf{C}_o \equiv (\mathbf{y} : \beta \rightarrow \alpha) (\mathbf{x} : \beta) = \mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta \wedge \mathbf{x} = \mathbf{B}_\beta \wedge \mathbf{y} = \mathbf{F}_{\beta \rightarrow \alpha}.$$

Since $M_i \preceq M_j$ by assumption, we have that

$$M_i \models_\varphi \exists \mathbf{y} : \beta \rightarrow \alpha, \mathbf{x} : \beta . \mathbf{C}_o \text{ iff } M_j \models_{\varphi''} \exists \mathbf{y} : \beta \rightarrow \alpha, \mathbf{x} : \beta . \mathbf{C}_o.$$

A similar argument to Case 1 that uses the fact that $\mathcal{J}^{j,i}$ is an inclusion and the notational definition for the existential quantifier allows us to conclude that

$$V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta) = \zeta^\sigma(V_\psi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)).$$

Case 3. $V_\varphi^{M_i}(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is undefined. Holds similarly to Case 2.

Case 4. $V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is undefined. Holds similarly to Case 1.

E5. $\mathbf{A}_\alpha$ is $(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma)$. Pick any $d_\beta^{M_i} \in D_\beta^{M_i}$. Then $V_\varphi^{M_i}(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma)(d_\beta^{M_i}) \simeq V_{\varphi[(\mathbf{x}:\beta) \mapsto d_\beta^{M_i}]}^{M_i}(\mathbf{B}_\gamma) \simeq \zeta^\sigma(V_{\psi[(\mathbf{x}:\beta) \mapsto \iota^\sigma(d_\beta^{M_i})]}^M(\mathbf{B}_\gamma)) \simeq \zeta^\sigma(V_\psi^M(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma)(\iota^\sigma(d_\beta^{M_i}))) \simeq \zeta^\sigma(V_\psi^M(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma))(d_\beta^{M_i})$. Since our choice of $d_\beta^{M_i}$ was arbitrary, we conclude that $V_\varphi^{M_i}(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma) = \zeta^\sigma(V_\psi^M(\lambda \mathbf{x} : \beta . \mathbf{B}_\gamma))$.

E6. $\mathbf{A}_\alpha$ is $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$.

Case 1. $V_\varphi^{M_i}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ is defined. Then for exactly one $d_\alpha^{M_i} \in D_\alpha^{M_i}$, $V_{\varphi[(\mathbf{x}:\alpha) \mapsto d_\alpha^{M_i}]}^{M_i}(\mathbf{B}_o) = \top$ by Condition V6 of a general model. By the induction hypothesis, $V_{\psi[(\mathbf{x}:\alpha) \mapsto \iota^\sigma(d_\alpha^{M_i})]}^M(\mathbf{B}_o) = \top$ for said $d_\alpha^{M_i}$. Suppose that there was another $d_\alpha^M \in D_\alpha^M$ such that $V_{\psi[(\mathbf{x}:\alpha) \mapsto d_\alpha^M]}^M(\mathbf{B}_o) = \top$. Then by construction and the induction hypothesis, there is $j \geq i$ with $\iota^{j,i}(d_\alpha^{M_i}) = d_\alpha^1 \in D_\alpha^{M_j}$ and $(\iota^\sigma)^{-1}(d_\alpha^1) = d_\alpha^2 \in D_\alpha^{M_j}$ such that $V_{\varphi''[(\mathbf{x}:\alpha) \mapsto d_\alpha^1]}^{M_j}(\mathbf{B}_o) = \top$ and $V_{\varphi''[(\mathbf{x}:\alpha) \mapsto d_\alpha^2]}^{M_j}(\mathbf{B}_o) = \top$;⁹ contradicting the assumption that $M_i \preceq M_j$. Therefore, $V_\varphi^{M_i}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o) = \zeta^\sigma(V_\psi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o))$.

Case 2. $V_\psi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ is defined. Then for exactly one $d_\alpha^M \in D_\alpha^M$, $V_{\psi[(\mathbf{x}:\alpha) \mapsto d_\alpha^M]}^M(\mathbf{B}_o) = \top$ by Condition V6 of a general model. By construction and the induction hypothesis, there is $j \geq i$ such that for only the same d_α^M , $V_{\varphi''[(\mathbf{x}:\alpha) \mapsto (\iota^\sigma)^{-1}(d_\alpha^M)]}^{M_j}(\mathbf{B}_o) = \top$. Since $M_i \preceq M_j$ by assumption, $V_\varphi^{M_i}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o) = \zeta^\sigma(V_\psi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o))$.

Case 3. $V_\varphi^{M_i}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ is undefined. Follows similarly to Case 2.

Case 4. $V_\psi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ is undefined. Follows similarly to Case 1.

E7. $\mathbf{A}_\alpha$ is $(\mathbf{B}_\beta, \mathbf{C}_\gamma)$.

⁹ Really, the assignment φ ought to change to reflect the transition from M_i to M_j . We alter φ to φ'' such that the map $(\mathbf{x} : \alpha) \mapsto d_\alpha \in D_\alpha^{M_i}$ is instead $(\mathbf{x} : \alpha) \mapsto \iota^{j,i+1}(d_\alpha) \in D_\alpha^{M_j}$.

- Case 1. One of $V_\varphi^{M_i}((\mathbf{B}_\beta, \mathbf{C}_\gamma))$ or $V_\psi^M((\mathbf{B}_\beta, \mathbf{C}_\gamma))$ is defined. Then $V_\varphi^{M_i}((\mathbf{B}_\beta, \mathbf{C}_\gamma)) = \zeta^\sigma(V_\psi^M((\mathbf{B}_\beta, \mathbf{C}_\gamma)))$ follows from Condition V7 of a general model, the induction hypothesis, and the definition of ζ^σ .⁸
- Case 2. One of $V_\varphi^{M_i}((\mathbf{B}_\beta, \mathbf{C}_\gamma))$ or $V_\psi^M((\mathbf{B}_\beta, \mathbf{C}_\gamma))$ is undefined. By the induction hypothesis and Condition V7 of a general model, $V_\varphi^{M_i}((\mathbf{B}_\beta, \mathbf{C}_\gamma))$, $\zeta^\sigma(V_\psi^M((\mathbf{B}_\beta, \mathbf{C}_\gamma)))$ are both undefined.

□

Corollary 3.9. *Let $(\mathcal{I}, <)$ be a well-order of order type $\sigma \leq \omega$ and $(M_i \mid i \in \mathcal{I})$ be a strong chain. Then the union general model M is a strong extension of M_i for all $i \in \mathcal{I}$.*

Proof Let $\mathcal{E}_o(L)$ be the set of formulas of L . We must show that for all $i \in \mathcal{I}$, $\mathbf{A}_o \in \mathcal{E}_o(L)$, $\varphi \in \text{assign}(M_i)$, and $d_{\alpha_1}^1 \in D_{\alpha_1}^{M_i}, \dots, d_{\alpha_n}^n \in D_{\alpha_n}^{M_i}$, we have

$$\begin{aligned} M_i &\models_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}^1] \dots [(\mathbf{x}_n:\alpha_n) \mapsto d_{\alpha_n}^n]} \mathbf{A}_o \\ \text{iff } M &\models_{\psi[(\mathbf{x}_1:\alpha_1) \mapsto \iota^\sigma(d_{\alpha_1}^1)] \dots [(\mathbf{x}_n:\alpha_n) \mapsto \iota^\sigma(d_{\alpha_n}^n)]} \mathbf{A}_o, \end{aligned}$$

where $\psi \in \text{assign}(M)$ is the assignment defined as $\psi((\mathbf{x} : \alpha)) = \iota^\sigma(\varphi((\mathbf{x} : \alpha)))$. This follows immediately from Proposition 3.8 as $\mathcal{E}_o(L) \subset \mathcal{E}(L)$. □

Definition 3.10 (Strong Diagram). Let $L = (\mathcal{B}, \mathcal{C})$ be a language and let $\mathcal{D} = \{D_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be a frame for L . We want to add a constant that corresponds to every $d \in D_\alpha$ for all $\alpha \in \mathcal{T}(L)$. So, for each α , well-order D_α such that $D_\alpha = \{d_\alpha^1, d_\alpha^2, \dots\}$ and let $\mathcal{C}_\alpha = \{\mathbf{c}_\alpha^{d_1}, \mathbf{c}_\alpha^{d_2}, \dots\}$ be a set of constants such that $\mathcal{C}_\alpha \cap \mathcal{C} = \emptyset$ and $|\mathcal{C}_\alpha| = |D_\alpha|$. Let $\mathcal{C}' = \bigcup_{\alpha \in \mathcal{T}(L)} \mathcal{C}_\alpha$. Let

$$L_{\mathcal{D}} = (\mathcal{B}, \mathcal{C} \cup \mathcal{C}').$$

If $M = (\mathcal{D}, I)$ is an interpretation of L , then L_M denotes $L_{\mathcal{D}}$. Now define $f : \mathcal{E}_o \times \text{assign}(M) \rightarrow \mathcal{C}'$ to be the function that, given some formula $\mathbf{A}_o$ and $\varphi \in \text{assign}(M)$ for a general model M , substitutes the corresponding constants in $\mathcal{C}'$ for the free variables in $\mathbf{A}_o$. More precisely, if $\varphi \in \text{assign}(M)$ and $(\mathbf{x}_1 : \alpha_1), \dots, (\mathbf{x}_n : \alpha_n)$ are free in $\mathbf{A}_o$ for some n , then $f(\mathbf{A}_o, \varphi) = \mathbf{A}_o[(\mathbf{x}_1 : \alpha_1) \mapsto \mathbf{c}^{\varphi(\mathbf{x}_1:\alpha_1)}] \dots [(\mathbf{x}_n : \alpha_n) \mapsto \mathbf{c}^{\varphi(\mathbf{x}_n:\alpha_n)}]$. Then the *strong diagram* of M , denoted $\text{Diag}_{\text{st}}(M)$, is¹⁰

$$\{f(\mathbf{A}_o, \varphi) \mid M \models_\varphi \mathbf{A}_o\}.$$

For convenience, we introduce a notational definition for the full frame defined by some collection of base domains:

Definition 3.11 (Generated Full Frame). Let $L = (\mathcal{B}, \mathcal{C})$ and $B = \{D_{\mathbf{a}}^B \mid \mathbf{a} \in \mathcal{B}\}$. Denote the full frame generated by B as

$$\mathcal{D}^{F,B} = \{D_\alpha^{F,B} \mid \alpha \in \mathcal{T}(L)\},$$

where $D_{\mathbf{a}}^{F,B} = D_{\mathbf{a}}^B$ for all $\mathbf{a} \in \mathcal{B}$ and for all other $\alpha \in \mathcal{T}(L)$, define $D_\alpha^{F,B}$ appropriately such that $\mathcal{D}^{F,B}$ is full. Alternatively, if M is a general model of L , then $\mathcal{D}^{F,M}$ denotes the full frame generated by the base domains of M .

¹⁰We do not specify the type of an expression when it is easily inferred.

3.2 Sharpened Upward Löwenheim-Skolem Theorem

Theorem 3.1 allows us to build an arbitrarily large model N of some theory T with an infinite model M . With an eye toward the sharpened upward Löwenheim Skolem theorem, we want to come up with a theory such that M strongly embeds into N . Conceptually, if $N \models \text{Diag}_{\text{st}}(M)$, then it satisfies the (higher-order) sentences that capture the structure of M . With this motivation in mind, we have the following:

Lemma 3.12. *Let M be a general model of L . Suppose $N = (\mathcal{D}^N, I^N)$ is a general model of $L_M = (\mathcal{B}, \mathcal{C} \cup \mathcal{C}')$ such that $N \models \text{Diag}_{\text{st}}(M)$. Then there exists a strong embedding of M into N .*

Proof Let $\mathfrak{E} = \{\varepsilon_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be the set of mappings such that for all $d \in D_\alpha^M$, $\varepsilon_\alpha(d) = I^N(\mathbf{c}^d)$. That is, $\varepsilon_\alpha(d)$ is the interpretation of the constant $\mathbf{c}^d \in \mathcal{C}'$ that corresponds to $d \in D_\alpha^M$. We will now show that $\mathfrak{E}$ is a strong embedding by showing that (i) $\mathfrak{E}$ is an embedding and that (ii) it is strong. We now demonstrate that criteria 1 – 5 for an embedding as per Definition 3.2 are satisfied.

1. For any $\alpha \in \mathcal{T}(L)$, suppose that $d, e \in D_\alpha$ are distinct. Then $\mathbf{c}^d \neq \mathbf{c}^e \in \text{Diag}_{\text{st}}(M)$, and so $\varepsilon_\alpha(d) = I^N(\mathbf{c}^d) \neq I^N(\mathbf{c}^e) = \varepsilon_\alpha(e)$. Thus for all $\alpha \in \mathcal{T}(L)$, ε_α is an injection.
2. $\varepsilon_o(\top) = I^N(\mathbf{c}^\top) = \top$ by the definition of ε_o and assumption; $\varepsilon_o(\text{F}) = \text{F}$ by similar reasoning.
3. Let $\alpha, \beta \in \mathcal{T}(L)$. Let $d \in D_\alpha^M$ and $f \in D_{\alpha \rightarrow \beta}^M$. Suppose $f(d)$ is defined. Then

$$\mathbf{c}^{f(d)} = \mathbf{c}^f \mathbf{c}^d \in \text{Diag}_{\text{st}}(M),$$

and so

$$\begin{aligned} \varepsilon_\beta(f(d)) &= I^N(\mathbf{c}^{f(d)}) && \text{(definition of } \varepsilon_\beta) \\ &= I^N(\mathbf{c}^f)(I^N(\mathbf{c}^d)) && (N \models \text{Diag}_{\text{st}}(M)) \\ &= \varepsilon_{\alpha \rightarrow \beta}(f)(\varepsilon_\alpha(d)) && \text{(definition of } \varepsilon_{\alpha \rightarrow \beta} \text{ and } \varepsilon_\alpha(d)) \end{aligned}$$

Now suppose that $f(d)$ is undefined. Then

$$\mathbf{c}^f \mathbf{c}^d \uparrow \in \text{Diag}_{\text{st}}(M),$$

and so $\varepsilon_{\alpha \rightarrow \beta}(f)\varepsilon_\alpha(d)$ is undefined by similar reasoning to the above and we can conclude that $\varepsilon_\beta(f(d))$ is undefined.

4. Let $a \in D_\alpha$, $b \in D_\beta$. By the definition of a frame for L , $(a, b) \in D_{\alpha \times \beta}$ is defined [7]. We have that

$$\mathbf{c}^{(a,b)} = (\mathbf{c}^a, \mathbf{c}^b) \in \text{Diag}_{\text{st}}(M),$$

and so

$$\begin{aligned} \varepsilon_{\alpha \times \beta}(a, b) &= I^N(\mathbf{c}^{(a,b)}) && \text{(definition of } \varepsilon_{\alpha \times \beta}) \\ &= (I^N(\mathbf{c}^a), I^N(\mathbf{c}^b)) && (N \models \text{Diag}_{\text{st}}(M)) \\ &= (\varepsilon_\alpha(a), \varepsilon_\beta(b)). && \text{(definition of } \varepsilon_\alpha \text{ and } \varepsilon_\beta) \end{aligned}$$

5. Let $d = I^M(\mathbf{c}_\alpha) \in D_\alpha$. Then

$$\{\mathbf{c}_\alpha = \mathbf{c}^d\} \in \text{Diag}_{\text{st}}(M),$$

and so

$$\begin{aligned} \varepsilon_\alpha(I^M(\mathbf{c}_\alpha)) &= I^N(\mathbf{c}^d) && (\text{definition of } \varepsilon_\alpha) \\ &= I^N(\mathbf{c}_\alpha). && (N \models \text{Diag}_{\text{st}}(M)) \end{aligned}$$

This completes the proof for (i). Let $\varphi \in \text{assign}(M)$. Define $\psi \in \text{assign}(N)$ to be $\psi((\mathbf{x} : \alpha)) = \varepsilon_\alpha(\varphi((\mathbf{x} : \alpha)))$ for all $\alpha \in \mathcal{T}(L)$ and $(\mathbf{x} : \alpha)$.

To demonstrate that $\mathfrak{E}$ is strong, we must show that $M \models_\varphi \mathbf{A}_o$ if and only if $N \models_\psi \mathbf{A}_o$. Recall the substitution function f from Definition 3.10. We have

$$\begin{aligned} M \models_\varphi \mathbf{A}_o & \text{iff } f(\mathbf{A}_o, \varphi) \in \text{Diag}_{\text{st}}(M) && (\text{definition of } \text{Diag}_{\text{st}}(M)) \\ & \text{iff } V^N(f(\mathbf{A}_o, \varphi)) = \top && (N \models \text{Diag}_{\text{st}}(M)) \\ & \text{iff } N \models_\psi \mathbf{A}_o. && (\text{definition of } f, \mathbf{A}_o, \psi, \text{ and } \mathfrak{E}) \end{aligned}$$

This completes the proof for (ii) and thus the lemma. $\square$

Now we can go one step further: we can refactor the resulting model N from the previous lemma and obtain a new model N' such that there is an inclusion from M into N' and $N' \equiv N$.

Lemma 3.13. *Let N be a general model of L_M such that $N \models \text{Diag}_{\text{st}}(M)$. Then there exists a general model N' such that $M \preceq N'$ and $N' \equiv N$.*

Proof By Lemma 3.12, we have a strong embedding $\mathfrak{E} = \{\varepsilon_\alpha \mid \alpha \in \mathcal{T}(L)\}$ from M to N . Let $\mathcal{D}^{F,M,N}$ denote the full frame determined by the base domains $D_{\mathbf{a}}^{M,N} = D_{\mathbf{a}}^M \cup (D_{\mathbf{a}}^N \setminus \text{ran}(\varepsilon_{\mathbf{a}}))$. Define a collection of mappings $\Theta = \{\theta_\alpha : D_\alpha^N \rightarrow D_\alpha^{F,M,N}\}$ as follows:¹¹

$$\begin{aligned} \theta_o(d_o) &= d_o; \\ \theta_{\mathbf{a}}(d_{\mathbf{a}}) &= \begin{cases} \varepsilon_{\mathbf{a}}^{-1}(d_{\mathbf{a}}); & \text{if } d_{\mathbf{a}} \in \text{ran}(\varepsilon_{\mathbf{a}}) \\ d_{\mathbf{a}}; & \text{otherwise} \end{cases} \\ \theta_{\alpha \rightarrow \beta}(d_{\alpha \rightarrow \beta}) &= \text{the } f_{\alpha \rightarrow \beta} \text{ such that for all } d_\alpha \in D_\alpha^N, \\ & \quad f_{\alpha \rightarrow \beta}(\theta_\alpha(d_\alpha)) \simeq \theta_\beta(d_{\alpha \rightarrow \beta}(d_\alpha)); \\ \theta_{\alpha \times \beta}((d_\alpha, d_\beta)) &= (\theta_\alpha(d_\alpha), \theta_\beta(d_\beta)). \end{aligned}$$

Now denote $D_\alpha^{N'} = \{\theta(d_\alpha) \mid d_\alpha \in D_\alpha^N\}$ and define the frame for N' to be

$$\mathcal{D}^{N'} = \{D_\alpha^{N'} \mid \alpha \in \mathcal{T}\}.$$

Since θ_α is an injection for all $\alpha \in \mathcal{T}(L)$, D_α^N and $D_\alpha^{N'}$ are in bijective correspondence under θ_α for all $\alpha \in \mathcal{T}(L)$. Now given $\psi \in \text{assign}(N)$, define $\psi' \in \text{assign}(\mathcal{D}^{N'})$ such that for all $\alpha \in \mathcal{T}(L)$ and $(\mathbf{x} : \alpha)$,

$$\psi'((\mathbf{x} : \alpha)) = \theta(\psi((\mathbf{x} : \alpha))).$$

¹¹We allow for the omission of the type of θ_α for some $\alpha \in \mathcal{T}(L)$ when the type is easily inferred.

Now define a partial evaluation function $V^{N'}$ on L_M such that given some $\psi' \in \text{assign}(\mathcal{D}^{N'})$,

$$V_{\psi'}^{N'}(\mathbf{A}_\alpha) \simeq \theta(V_\psi^N(\mathbf{A}_\alpha)).$$

Let $I^{N'}(\mathbf{c}_\alpha) = V^{N'}(\mathbf{c}_\alpha)$ for all $\mathbf{c}_\alpha \in \mathcal{C} \cup \mathcal{C}'$. It is easy to verify that $N' = (\mathcal{D}^{N'}, I^{N'})$, with the partial evaluation function $V^{N'}$, satisfies conditions V1–V7, so N' is a general model of L_M .¹² It is also easy to verify that there is an inclusion from M into N' and that $N' \models \text{Diag}_{\text{st}}(M)$, and so $M \preceq N'$.

It is easy to verify that Θ is an isomorphism from N to N' by its construction. Thus, $N' \equiv N$.
□

We are now ready to prove the sharpened upward Löwenheim-Skolem theorem.

Theorem 3.14 (Sharpened upward Löwenheim-Skolem theorem). *Let $T = (L, \Gamma)$ be a theory. If T has an infinite general model M of L , then for all cardinals κ such that $\kappa \geq \|M\| + \|L\|$,¹³ T has a general model N of L of size and power κ such that $M \preceq N$.*

Proof We can expand M to a general model M' of L_M such that for all $\mathbf{c}^d \in \mathcal{C}'$, $I^{M'}(\mathbf{c}^d) = d$; it follows that $M' \models \text{Diag}_{\text{st}}(M)$. By Theorem 3.1, $\text{Diag}_{\text{st}}(M)$ has a general model N of size and power κ . By Lemma 3.12, there is a strong embedding from M into N . By Lemma 3.13, there is a model N' such that $M \preceq N'$ and $N' \equiv N$. It is easy to verify that the reduct of N' to L is a model of T of size and power κ .
□

3.3 Sharpened Downward Löwenheim-Skolem Theorem

With an eye toward the sharpened downward Löwenheim-Skolem theorem, we prove the analogue of the Tarski-Vaught test from first-order logic which gives us useful criteria for finding strong submodels. In particular, if there is an inclusion from M into N , always being able to find an existential witness from M inside of N (up to the inclusion) is sufficient and necessary for $M \preceq N$.

Proposition 3.15. *Let M, N be general models of L . Suppose that there is an inclusion $\mathfrak{I} = \{\iota_\alpha \mid \alpha \in \mathcal{T}(L)\}$ from M to N . Define $Z = \{\zeta_\alpha \mid \alpha \in \mathcal{T}(L)\}$ to be the collection of mappings that, like in Lemma 3.4, satisfy the property that $\zeta_\alpha(\iota_\alpha(d_\alpha^M)) = d_\alpha^M$ for all $\alpha \in \mathcal{T}(L)$ and $d_\alpha^M \in D_\alpha^M$. Then for all $\mathbf{A}_\alpha \in \mathcal{E}(L)$, and $\varphi \in \text{assign}(M)$, we have*

$$(\star) \quad V_\varphi^M(\mathbf{A}_\alpha) \simeq \zeta(V_\psi^N(\mathbf{A}_\alpha))$$

iff for any $\mathbf{A}_o \in \mathcal{E}(L)$, if there is $d_{\alpha_1}^N \in D_{\alpha_1}^N$ such that

$$N \models_{\psi[(\mathbf{x}_1 : \alpha_1) \mapsto d_{\alpha_1}^N]} \mathbf{A}_o,$$

then there is $d_{\alpha_1}^M \in D_{\alpha_1}^M$ such that

$$N \models_{\psi[(\mathbf{x}_1 : \alpha_1) \mapsto \iota(d_{\alpha_1}^M)]} \mathbf{A}_o,$$

where $\psi = \iota(\varphi((\mathbf{x} : \alpha)))$ for all $(\mathbf{x} : \alpha)$.

¹²As M, N are general models.

¹³Assuming that $\|M\|$ exists.

Proof The forward direction follows easily from [7, Lemma 6.4] and $M \preceq N$. We prove $(\star)$ by structural induction on the complexity of expressions. Cases E1, E2, E3, E5, and E7 follow similarly to the same cases in the proof of Proposition 3.8. The remaining cases are E4 and E6.

(E4). $\mathbf{A}_\alpha$ is $(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$.

Case 1. $V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined. We have that $\zeta(\iota(V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha}))) = \zeta(V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha}))$ and $\zeta(\iota(V_\varphi^M(\mathbf{B}_\beta))) = \zeta(V_\psi^N(\mathbf{B}_\beta))$ by the induction hypothesis and a similar argument to the one in Lemma 3.4. Since ζ is defined only on images under the injective map ι , we have that $\iota(V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha})) = V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha})$ and $\iota(V_\varphi^M(\mathbf{B}_\beta)) = V_\psi^N(\mathbf{B}_\beta)$. The fact that $V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined and $\mathfrak{I}$ is an inclusion implies that $\iota(V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)) = V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$. Lemma 3.4 implies that

$$V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta) = \zeta(V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)).$$

Case 2. $V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined. Let

$$\mathbf{C}_o \equiv (\mathbf{y} : \beta \rightarrow \alpha) (\mathbf{x} : \beta) = \mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta \wedge \mathbf{x} = \mathbf{B}_\beta \wedge \mathbf{y} = \mathbf{F}_{\beta \rightarrow \alpha}.$$

Since $V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is defined, we have that for some $d_\beta^N \in D_\beta^N$ and $d_{\beta \rightarrow \alpha}^N \in D_{\beta \rightarrow \alpha}^N$, $N \models_{\psi[(\mathbf{x}:\beta) \mapsto d_\beta^N][(\mathbf{y}:\beta \rightarrow \alpha) \mapsto d_{\beta \rightarrow \alpha}^N]} \mathbf{C}_o$. By assumption, (a) $N \models_{\psi[(\mathbf{x}:\beta) \mapsto \iota(d_\beta^M)][(\mathbf{y}:\beta \rightarrow \alpha) \mapsto \iota(d_{\beta \rightarrow \alpha}^M)]} \mathbf{C}_o$ for some $d_\beta^M \in D_\beta^M$ and $d_{\beta \rightarrow \alpha}^M \in D_{\beta \rightarrow \alpha}^M$. Hence,

$$\begin{aligned} & \zeta(V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)) \\ & \simeq \zeta(V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha})(V_\psi^N(\mathbf{B}_\beta))) && (V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta) \text{ is defined}) \\ & = \zeta(V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha}))(\zeta(V_\psi^N(\mathbf{B}_\beta))) && ((a) \text{ and Lemma 3.4}) \\ & = V_{\varphi'}^M(\mathbf{F}_{\beta \rightarrow \alpha})(V_{\varphi'}^M(\mathbf{B}_\beta)) && (\text{induction hypothesis}) \\ & = V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta). && (\text{Condition V4 of a general model}) \end{aligned}$$

Case 3. $V_\varphi^M(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is undefined. Holds similarly to Case 2.

Case 4. $V_\psi^N(\mathbf{F}_{\beta \rightarrow \alpha} \mathbf{B}_\beta)$ is undefined. Holds similarly to Case 1.

(E6). $\mathbf{A}_\alpha$ is $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$. W.l.o.g, let $(\mathbf{y} : \alpha)$ be free for $(\mathbf{x} : \alpha)$ in $\mathbf{B}_o$. Let

$$\mathbf{C}_o \equiv \exists \mathbf{x} : \alpha, \mathbf{y} : \alpha . (\mathbf{x} \neq \mathbf{y}) \wedge \mathbf{B}_o \wedge \mathbf{B}_o[(\mathbf{x} : \alpha) \mapsto (\mathbf{y} : \alpha)].$$

Case 1. $V_\varphi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ is defined. Then there is (exactly one) $d_\alpha^M \in D_\alpha^M$ such that $M \models_{\varphi[(\mathbf{x}:\alpha) \mapsto d_\alpha^M]} \mathbf{B}_o$. By the induction hypothesis, $N \models_{\psi[(\mathbf{x}:\alpha) \mapsto \iota(d_\alpha^M)]} \mathbf{B}_o$. Now suppose that there was some $d_\alpha^N \notin \text{ran}(\iota_\alpha)$ such that $N \models_{\psi[(\mathbf{x}:\alpha) \mapsto d_\alpha^N]} \mathbf{B}_o$. Then $N \models_\psi \mathbf{C}_o$. By assumption and the notational definition for the existential quantifier, there are two distinct members of D_α^M that satisfy $\mathbf{B}_o$, a contradiction. Now suppose that there was some $d_\alpha^N \in \text{ran}(\iota_\alpha)$ such that $N \models_{\psi[(\mathbf{x}:\alpha) \mapsto d_\alpha^N]} \mathbf{B}_o$. Then $d_\alpha^N = \iota(d_\alpha^M)$; for otherwise, there would be two distinct witnesses in M that satisfy $\mathbf{B}_o$, contradicting our induction hypothesis. Hence $\iota(V_\varphi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)) = V_\psi^N(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o)$ by Condition V6 of a general model; and so by Lemma 3.4, $V_\varphi^M(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o) = \zeta(V_\psi^N(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o))$.

Case 2. $V_\psi^N(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)$ is defined. Then there is (exactly one) $d_\alpha^N \in D_\alpha^N$ such that $N \models_{\psi[(\mathbf{x}:\alpha) \mapsto d_\alpha^N]} \mathbf{B}_o$. By assumption, there is exactly one $d_\alpha^M \in D_\alpha^M$ such that $M \models_{\varphi[(\mathbf{x}:\alpha) \mapsto d_\alpha^M]} \mathbf{B}_o$.¹⁴ By the induction hypothesis, $N \models_{\varphi[(\mathbf{x}:\alpha) \mapsto \iota(d_\alpha^M)]} \mathbf{B}_o$. Since $V_\psi^N(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)$ is defined, we have $\iota(V_\varphi^M(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)) = V_\psi^N(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)$ by Condition V6 of a general model; and so by Lemma 3.4, $V_\varphi^M(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o) = \zeta(V_\psi^N(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o))$.

Case 3. $V_\varphi^M(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)$ is undefined. Similar to Case 2.

Case 4. $V_\psi^N(\mathbf{I}\mathbf{x} : \alpha . \mathbf{B}_o)$ is undefined. Similar to Case 1.

□

Corollary 3.16 (Higher-order Tarski-Vaught test). *Let M, N be general models of L . Suppose that there is an inclusion $\mathfrak{J}$ from M to N . Then $M \preceq N$ iff for any $\mathbf{A}_o$ and $\psi \in \text{assign}(N)$, if there is $d_{\alpha_1}^N \in D_{\alpha_1}^N$ such that*

$$N \models_{\psi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}^N]} \mathbf{A}_o,$$

then there is $d_{\alpha_1}^M \in D_{\alpha_1}^M$ such that

$$N \models_{\psi[(\mathbf{x}_1:\alpha_1) \mapsto \iota(d_{\alpha_1}^M)]} \mathbf{A}_o.$$

Proof The forward direction follows easily from [7, Lemma 6.4] and $M \preceq N$. The backward direction follows from Proposition 3.15 as $\mathcal{E}_o(L) \subset \mathcal{E}(L)$. □

We are now ready to prove the sharpened downward Löwenheim-Skolem Theorem.

Theorem 3.17 (Sharpened downward Löwenheim-Skolem theorem). *Let $L = (\mathcal{B}, \mathcal{C})$ be a language and $T = (L, \Gamma)$ be a theory. If T has an infinite general model N of L , then for every infinite cardinal κ such that $\|L\| \leq \kappa \leq \|N\|$, T has a general model M of L of size and power κ such that $M \preceq N$.*

Proof Let κ be an infinite cardinal such that $\|L\| \leq \kappa \leq \|N\|$. Choose a collection of typed domains $\mathcal{D}^0 = \{D_\alpha^0 \mid \alpha \in \mathcal{T}(L)\}$ such that $D_\alpha^0 \subseteq D_\alpha^N$ for all $\alpha \in \mathcal{T}(L)$, $|\bigcup_{\mathbf{a} \in \mathcal{B}} D_{\mathbf{a}}^0| = \kappa$, and $|D_\alpha^0| \leq \kappa$ for all $\alpha \in \mathcal{T}(L)$.¹⁵ Notice how the only restriction we have on our choice of domains is in terms of cardinality. Given $\mathcal{D}^i$, define the collection of typed domains

$$\mathcal{D}^{i+1} = \{D_\alpha^i \cup D_\alpha^{w,i+1} \mid \alpha \in \mathcal{T}(L)\},$$

where

$$\begin{aligned} D_\alpha^{w,i+1} &= \{d_\alpha \mid j \in \mathbb{N}, \varphi \in \text{assign}(N) \text{ and } d_\alpha \text{ witnesses} \\ &\quad V_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto (d_{\alpha_1}^i)] \dots [(\mathbf{x}_j:\alpha_j) \mapsto (d_{\alpha_j}^i)]}^N(\exists \mathbf{x} : \alpha . \mathbf{A}_o), \text{ where} \\ &\quad (\mathbf{x}_k : \alpha_k) \text{ are free in } \mathbf{A}_o \text{ and } d_{\alpha_k}^i \in D_{\alpha_k}^i \text{ for } 1 \leq k \leq j\}. \end{aligned}$$

Define $\mathcal{D}^\infty = \{\bigcup_i D_\alpha^i \mid \alpha \in \mathcal{T}(L)\}$.

¹⁴As $\exists! \mathbf{x} : \alpha . \mathbf{A}_o \equiv \exists y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o) = (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)$, provided $(y : \alpha)$ does not occur in $(\lambda \mathbf{x} : \alpha . \mathbf{A}_o)$.

¹⁵Determine members of the product domains by the choice of members in non-product domains.

Now define a reduction to be the collection of mappings $\{\rho_\alpha : D_\alpha^N \rightarrow D_\alpha^{F,\infty} \mid \alpha \in \mathcal{T}(L)\}$ such that¹⁶

$$\begin{aligned} \rho_o(d_o) &= d_o; \\ \rho_o(d_{\mathbf{a}}) &\simeq \begin{cases} d_{\mathbf{a}}; & \text{if } d_{\mathbf{a}} \in D_{\mathbf{a}}^\infty \\ \text{undefined}; & \text{otherwise} \end{cases} \\ \rho_{\alpha \rightarrow \beta}(d_{\alpha \rightarrow \beta}) &= \text{the } f_{\alpha \rightarrow \beta} \text{ such that for all } d_\alpha \in D_\alpha^N, \text{ if } \rho_\alpha(d_\alpha) \text{ is defined, then} \\ &\quad f(\rho_\alpha(d_\alpha)) \simeq \rho_\beta(d_{\alpha \rightarrow \beta}(d_\alpha)); \\ \rho((d_\alpha, d_\beta)) &\simeq (\rho(d_\alpha), \rho(d_\beta)). \end{aligned}$$

Define the frame for L

$$\mathcal{D}^M = \{\{\rho(d_\alpha) \mid d_\alpha \in D_\alpha^\infty\} \mid \alpha \in \mathcal{T}(L)\}.$$
¹⁷

By construction, $|\bigcup_{\mathbf{a} \in \mathcal{B}} D_{\mathbf{a}}^M| = \kappa$ and $|D_\alpha^M| \leq \kappa$ for each $\alpha \in \mathcal{T}(L)$.

Define $I^M(\mathbf{c}_\alpha) = \rho(I(\mathbf{c}_\alpha))$,¹⁸ $M = (\mathcal{D}^M, I^M)$ an interpretation of L , and a partial valuation function V^M such that given $\varphi \in \text{assign}(M)$,

$$V_\varphi^M(\mathbf{A}_\alpha) \simeq \rho(V_\psi^N(\mathbf{A}_\alpha)),$$

where $\psi \in \text{assign}(N)$ such that if $\varphi((\mathbf{x} : \alpha)) = \rho(d_\alpha)$, $\psi((\mathbf{x} : \alpha)) = d_\alpha$. It is easy to verify that M is a general model of L . Define a set of injective mappings

$$\mathfrak{I} = \{\iota_\alpha : D_\alpha^M \rightarrow D_\alpha^N \mid \alpha \in \mathcal{T}(L)\}$$

where $\iota_\alpha(\rho(d_\alpha)) = d_\alpha$ for all $\alpha \in \mathcal{T}(L)$.

We now show that $M \preceq N$ under $\mathfrak{I}$. Conditions 1, 2, 5, and 6 for an inclusion are satisfied by the definition of ι and ρ . Now take any $\rho(d_\alpha) \in D_\alpha^M$ and $\rho(f_{\alpha \rightarrow \beta}) \in D_{\alpha \rightarrow \beta}^M$. We have that

$$\begin{aligned} &\iota(\rho(f)\rho(d)) \\ &\simeq \iota(\rho(f(d))) && \text{(definition of } \rho) \\ &\simeq f(d) && \text{(definition of } \iota) \\ &\simeq \iota(\rho(f))(\iota(\rho(d))). && \text{(definition of } \iota, \rho) \end{aligned}$$

Hence, Condition 3 is satisfied. Take any $d_{\alpha \times \beta} = \rho(a, b) \in D_{\alpha \times \beta}^M$. Observe that

$$\begin{aligned} &= \iota(\rho((a, b))) \\ &= \iota((\rho(a), \rho(b))) && \text{(definition of } \rho) \\ &= (\iota(\rho(a)), \iota(\rho(b))). && \text{(definition of } \iota) \end{aligned}$$

Hence, Condition 4 is satisfied. By the construction of M and Corollary 3.16, $M \preceq N$. $\square$

¹⁶We allow for the omission of the type of ρ_α for some α when the type is easily inferred.

¹⁷If there are $d_\alpha^1, \dots, d_\alpha^n$ such that $\rho(d_\alpha^1) = \dots = \rho(d_\alpha^n)$, pick one to keep and remove the rest. Construct the product domains appropriately.

¹⁸This is valid since $N \models \exists \mathbf{x} : \alpha. \mathbf{c}_\alpha = (\mathbf{x} : \alpha)$ for any $\mathbf{c}_\alpha \in \mathcal{C}$.

Chapter 4

Model-Theoretic Types

4.1 Preliminaries

We introduce model-theoretic types for Alonzo, following the development in chapter 4 of [12]. Where the meaning is obvious, we omit the prefix and refer to them as just “types”. First, we state the compactness theorem – Corollary 8.16 in [7] – for Alonzo.

Theorem 4.1 (Compactness theorem). *Let Γ be a set of sentences. Then Γ is satisfiable iff every finite subset of Γ is satisfiable.*

Definition 4.2. Let $\mathcal{D}^i$ be frames for L for $i \in \{1, 2\}$. Call $\mathcal{D}^1$ a *subframe* of $\mathcal{D}^2$ if $D_\alpha^1 \subseteq D_\alpha^2$ for all $\alpha \in \mathcal{T}(L)$. We write $\mathcal{D}^1 \leq \mathcal{D}^2$ in this case.

Let $M = (\mathcal{D}, I)$ be a general model of L and $\mathcal{A} \leq \mathcal{D}$. Denote the set of all $L_{\mathcal{A}}$ -sentences true in M by $\text{Th}_{\mathcal{A}}(M)$.

Definition 4.3. Let $M = (\mathcal{D}, I)$ be a general model of L and $\mathcal{A} \leq \mathcal{D}$. Let $L_{\mathcal{A}}^n$ denote the set of $L_{\mathcal{A}}$ -formulas with n free variables. Let $p \subseteq L_{\mathcal{A}}^n$. Call p an *n-type* (over $\mathcal{A}$) if $p \cup \text{Th}_{\mathcal{A}}(M)$ is satisfiable (in the general sense). Call an n -type p *complete* if $\mathbf{A}_o \in p$ or $\neg \mathbf{A}_o \in p$ for all $L_{\mathcal{A}}$ -formulas $\mathbf{A}_o$.

We adopt the convention that if the free variables of $\mathbf{A}_o \in p$ for p an n -type are not mentioned explicitly, they are $(\mathbf{x}_i : \alpha_i)$ for $1 \leq i \leq n$. The set of all complete n -types is denoted as $C_n^M(\mathcal{A})$. Let $d_{\alpha_i} \in D_{\alpha_i}^M$ for $1 \leq i \leq n$. Then $\text{tp}_n^M(\bar{d}/\mathcal{A})$ denotes the complete type

$$\{\mathbf{A}_o \in L_{\mathcal{A}}^n \mid M \models_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}] \dots [(\mathbf{x}_n:\alpha_n) \mapsto d_{\alpha_n}]} \mathbf{A}_o \text{ for any } \varphi \in \text{assign}(M)\}.$$

Definition 4.4. Let p be an n -type over $\mathcal{A}$. We say that $d_{\alpha_i} \in D_{\alpha_i}^M$ for $1 \leq i \leq n$ *realize* p if $M \models_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}] \dots [(\mathbf{x}_n:\alpha_n) \mapsto d_{\alpha_n}]} \mathbf{A}_o$ for any $\varphi \in \text{assign}(M)$ and $\mathbf{A}_o$ in p . If p is not realized, we say M *omits* p .

We can always realize a given n -type in some general model.

Proposition 4.5. *Let $M = (\mathcal{D}, I)$ be a general model of L , $\mathcal{A} \leq \mathcal{D}$, and p be an n -type over $\mathcal{A}$. Then there is a strong extension N of M such that p is realized in N .*

Proof Let $\Gamma = p \cup \text{Diag}_{\text{st}}(M)$. Let $\Delta \subseteq \Gamma$ be a finite subset. W.l.o.g., Δ is equivalent to $\mathbf{A}_o \wedge \mathbf{B}_o$ where $\mathbf{A}_o$ is an $L_{\mathcal{A}}^n$ -formula with $(\mathbf{y}_i : \beta_i)$ free for $1 \leq i \leq n$ – corresponding to a finite subset of p – and $\mathbf{B}_o$ is an L_M formula with parameters in $\mathcal{A}$ and $\mathcal{X} = \{D_{\alpha}^M \setminus D_{\alpha}^A \mid \alpha \in \mathcal{T}(L)\}$ such that

$$\mathbf{B}_o \equiv \mathbf{B}'_o[(\mathbf{x}_1 : \alpha_1) \mapsto \mathbf{c}^{d_1^X}] \dots [(\mathbf{x}_k : \alpha_k) \mapsto \mathbf{c}^{d_k^X}],$$

where $\mathbf{B}'_o$ is the appropriate $L_{\mathcal{A}}$ -formula with $(\mathbf{x}_i : \alpha_i)$ free for $1 \leq i \leq k$ – corresponding to a finite subset of $\text{Diag}_{\text{st}}(M)$. By Definition 4.3, there is a general model N_0 such that $N_0 \models p \cup \text{Th}_{\mathcal{A}}(M)$. Since $\mathbf{B}_o \in \text{Diag}_{\text{st}}(M)$, we have $\exists \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_k : \alpha_k . \mathbf{B}'_o \in \text{Th}_{\mathcal{A}}(M)$, and so

$$N_0 \models \mathbf{A}_o \wedge \exists \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_k : \alpha_k . \mathbf{B}'_o.$$

So, $N_0 \models \Delta$ by interpreting $\mathbf{c}^{d_i^X}$ for $1 \leq i \leq k$ as the appropriate witnesses. By the Compactness Theorem [7, Corollary 8.16], Γ is satisfiable; so let $N' \models \Gamma$. By Lemma 3.12, there is a strong embedding of M into N' . By Lemma 3.13, there is a strong extension N of M such that $N \models \Gamma$. Thus p is realized by the interpretations of $(\mathbf{y}_i : \beta_i)$. $\square$

Corollary 4.6. *Let $M = (\mathcal{D}, I)$ be a general model of L and $\mathcal{A} \leq \mathcal{D}$. Then $p \in C_n^M(\mathcal{A})$ iff there is a strong extension N of M and $d_{\alpha_i} \in D_{\alpha_i}^N$ for $1 \leq i \leq n$ such that $p = \text{tp}_n^N(\bar{d}/\mathcal{A})$.*

Proof ($\Rightarrow$) Let $p \in C_n^M(\mathcal{A})$. By Proposition 4.5, there is a strong extension N of M that realizes p , say with $d_{\alpha_i} \in D_{\alpha_i}^N$ for $1 \leq i \leq n$. Let $\mathbf{A}_o$ be any $L_{\mathcal{A}}^n$ -formula with $(\mathbf{x}_i : \alpha_i)$ free for $1 \leq i \leq n$. Since p is a complete type, exactly one of $\mathbf{A}_o$ or $\neg \mathbf{A}_o$ is in p ; hence $p = \text{tp}_n^N(\bar{d}/\mathcal{A})$.

($\Leftarrow$) Let N be a strong extension of M and $d_{\alpha_i} \in D_{\alpha_i}^N$ for $1 \leq i \leq n$ such that $p = \text{tp}_n^N(\bar{d}/\mathcal{A})$. We have that $p \in C_n^N(\mathcal{A})$; furthermore, since $M \preceq N$, we have that $C_n^N(\mathcal{A}) = C_n^M(\mathcal{A})$.¹ Hence $p \in C_n^M(\mathcal{A})$. $\square$

4.2 Constructions Through Partially Strong Extensions

Definition 4.7. Let $M = (\mathcal{D}, I)$ and N be general models of L and $\mathcal{X} \leq \mathcal{D}$. Let $F = \{f_{\alpha} \mid \alpha \in \mathcal{T}(L)\}$ be a collection of mappings, where given $\alpha \in \mathcal{T}(L)$, f_{α} is a total map from D_{α}^X to D_{α}^N . We say that F is a *partially strong* set of mappings if

$$M \models_{\varphi} \mathbf{A}_o \text{ iff } N \models_{\psi} \mathbf{A}_o$$

for all $\mathbf{A}_o \in \mathcal{E}(L)$ where if φ is an assignment into $\mathcal{X}$, then ψ is the assignment into $\mathcal{D}^N$ defined by $\psi((\mathbf{x} : \alpha)) = f_{\alpha}(\varphi((\mathbf{x} : \alpha)))$.

We proceed to show that we can always grow a partially strong set of mappings by considering strong extensions.

Lemma 4.8. *Let $M = (\mathcal{D}, I)$ and N be general models of L and $\mathcal{X} \leq \mathcal{D}$. Let $F = \{f_{\alpha} \mid \alpha \in \mathcal{T}(L)\}$ be a partially strong set of mappings from $\mathcal{X}$ into N . Then given any $d_{\alpha}^M \in D_{\alpha}^M$, there is a strong extension N' of N and $F' = \{f'_{\beta} \mid \beta \in \mathcal{T}(L)\}$ a partially strong set of mappings extending f , up to the inclusion from N into N' .*

¹The definition of $M \preceq N$ entails that M and N satisfy the same $L_{\mathcal{A}}$ -sentences.

Proof Below, assume $\mathbf{A}_o \in L_M$ has $(\mathbf{x}_i : \alpha_i)$ free for $1 \leq i \leq k$ and $\varphi((\mathbf{x}_1 : \alpha_1)) = d_\alpha^M$.

$$\Gamma = \{ \mathbf{A}_o[(\mathbf{x}_2 : \alpha_2) \mapsto \mathbf{c}^{f(d_{\alpha_2}^X)}] \dots [(\mathbf{x}_k : \alpha_k) \mapsto \mathbf{c}^{f(d_{\alpha_k}^X)}] \mid M \models_\varphi \mathbf{A}_o \text{ and } d_{\alpha_i}^X \in D_{\alpha_i}^X \text{ for } 2 \leq i \leq k \} \cup \text{Diag}_{\text{st}}(N).$$

We will show that Γ is satisfiable. Let $\Delta \subseteq \Gamma$ be a finite subset. The diagram component of Δ is satisfiable by $N_e = (\mathcal{D}^e, I^e)$ an expansion of N where $I^e(\mathbf{c}^{d_\alpha}) = d_\alpha$ for all $d_\alpha \in D_\alpha^e$. Now well-order the non-diagram component $X = \{\mathbf{A}_o^1, \dots, \mathbf{A}_o^n\}$. By definition, there is $\varphi_i \in \text{assign}(M)$ such that $M \models_{\varphi_i} \mathbf{A}_o^i$ for $1 \leq i \leq n$; and across each assignment, we have that $\varphi_i((\mathbf{x}_1 : \alpha_1)) = d_\alpha^M$ remains fixed. Hence we just need to show that there is $\psi \in \text{assign}(N_e)$ such that $N_e \models_\psi \exists \mathbf{x}_1 : \alpha_1 . \mathbf{A}_o$ for any $\mathbf{A}_o \in X$, giving us a witness. Let $\mathbf{A}_o \equiv \mathbf{A}_o^i$ for some $1 \leq i \leq n$. So, we have

$$\begin{aligned} M &\models_{\varphi_i} \mathbf{A}_o \\ \text{implies } M &\models_{\varphi_i} \exists \mathbf{x}_1 : \alpha_1 . \mathbf{A}_o && \text{(by definition)} \\ \text{implies } N_e &\models_\psi \exists \mathbf{x}_1 : \alpha_1 . \mathbf{A}_o, && (f \text{ is partially strong}) \end{aligned}$$

where $\psi \in \text{assign}(N_e)$ is defined as $\psi((\mathbf{x} : \alpha)) = f(\varphi_i((\mathbf{x} : \alpha)))$ for all $(\mathbf{x} : \alpha)$. By the Compactness Theorem [7, Corollary 8.16], Γ is satisfiable; so let $N_s \models \Gamma$. Now take the reduct of N_s to L . By Lemma 3.12 and Lemma 3.13, there is a strong extension N' of N such that $N' \equiv N_s$, with corresponding inclusion $\mathfrak{J} = \{\iota_\alpha : D_\alpha^N \rightarrow D_\alpha^{N'} \mid \alpha \in \mathcal{T}(L)\}$. Let $d_\alpha^n \in D_\alpha^N$ be the witness that satisfies the existential statements in Γ . We construct $F' = \{f'_\beta \mid \beta \in \mathcal{T}(L)\}$ such that $f'_\beta(d) = \iota_\beta(f_\beta(d))$ for all $d \in D_\beta^X$ and $f'_\alpha(d_\alpha^M) = \iota_\alpha(d_\alpha^N)$. $\square$

Corollary 4.9. *Let $M = (\mathcal{D}, I)$ and N be interpretations of L and $\mathcal{X} \leq \mathcal{D}$. Let $F = \{f_\alpha \mid \alpha \in \mathcal{T}(L)\}$ be partially strong from $\mathcal{X}$ into N . Then there is $N' \succeq N$ and $\mathfrak{E} = \{\varepsilon_\alpha \mid \alpha \in \mathcal{T}(L)\}$ a strong embedding from M into N' .*

Proof Well-order $\mathcal{T}(L)$ and denote the j^{th} type in this order as α_j . Let $\kappa_j = |D_{\alpha_j}^M|$. Using induction, we will build a strong chain $(N_j \mid j < \omega)$ and a partially strong set of mappings $F^j = \{f_\alpha^j : X_\alpha^j \rightarrow D_\alpha^{N,j} \mid \alpha \in \mathcal{T}(L)\}$ such that for all $\alpha \in \mathcal{T}(L)$ and $a < b$, f_α^a can be extended to f_α^b .

Base: Let $\{d_{\alpha_0}^i \mid i < \kappa_0\}$ be a well-order of $D_{\alpha_0}^M$. Let $X_{\alpha_0}^{0,k} = X_{\alpha_0} \cup \{d_{\alpha_0}^i \mid i < k\}$. Using transfinite induction, we will build a strong chain $(N_{0,k} \mid k < \kappa_0)$ and a partially strong set of mappings $F^{0,k} = \{f_\alpha^{0,k} : X_\alpha^{0,k} \rightarrow D_\alpha^{N,0,k} \mid \alpha \in \mathcal{T}(L)\}$ such that for all $\alpha \in \mathcal{T}(L)$ and $a < b$, $f_\alpha^{0,a}$ can be extended to $f_\alpha^{0,b}$.

k is 0: Let $F^{0,0} = F$ and $N_{0,0} = N$.

k is a successor ordinal $\xi + 1$: Using Lemma 4.8, we can find a strong extension $N_{0,\xi+1}$ of $N_{0,\xi}$ and a partially strong set of mappings $F^{0,\xi+1}$ where $f_{\alpha_0}^{0,\xi+1}$ extends $f_{\alpha_0}^{0,\xi}$ with $d_{\alpha_0}^\xi$.

k is a limit ordinal: Let $N_{0,k}$ be the union general model of the strong chain $(N_{0,h} \mid h < k)$. Given $f_\alpha^{0,h}$ for some h , let $\lim_{h \rightarrow k} f_\alpha^{0,h}$ denote the function g_α where $g_\alpha(d_\alpha) = \lim_{x \rightarrow k} \iota_\alpha^{x,h}(f_\alpha^{0,h}(d_\alpha))$ for all $d_\alpha \in \text{dom}(f_\alpha^{0,h})$. Let $F^{0,k} = \{ \bigcup_{h < k} \lim_{h \rightarrow k} f_\alpha^{0,h} \mid \alpha \in \mathcal{T}(L) \}$. By

Corollary 3.9, $N_{0,k}$ is a strong extension of $N_{0,h}$ for all $h < k$ and by construction, $F^{0,k}$ is a partially strong set of mappings.

Now let N_0 be the union general model of the strong chain $(N_{0,k} \mid k < \kappa_0)$ and $F^0 = \{\bigcup_{k < \kappa_0} \lim_{k \rightarrow \kappa_0} f_\alpha^{0,k} \mid \alpha \in \mathcal{T}(L)\}$. By Corollary 3.9, N_0 is a strong extension of $N_{0,k}$ for all $k < \kappa_0$ and by construction, F^0 is a partially strong set of mappings. Notice that $\text{dom}(f_{\alpha_0}) = D_{\alpha_0}^M$.

Step: Suppose the claim holds for j . Let $\{d_{\alpha_{j+1}}^i \mid i < \kappa_0\}$ be a well-order of $D_{\alpha_{j+1}}^M$. Let $X_{\alpha_{j+1}}^{j+1,k} = X_{\alpha_{j+1}} \cup \{d_{\alpha_{j+1}}^i \mid i < k\}$. Using transfinite induction, we will build a strong chain $(N_{j+1,k} \mid k < \kappa_{j+1})$ and a partially strong set of mappings $F^{j+1,k} = \{f_\alpha^{j+1,k} : X_\alpha^{j+1,k} \rightarrow D_\alpha^{N,j+1,k} \mid \alpha \in \mathcal{T}(L)\}$ such that for all $\alpha \in \mathcal{T}(L)$ and $a < b$, $f_\alpha^{j+1,a}$ can be extended to $f_\alpha^{j+1,b}$.

The construction is similar to the one in the base case. For the $k = 0$ case, we let $F^{j+1,0} = F^j$ and $N_{j+1,0} = N_j$. The successor and limit ordinal cases follow almost identically. We are left with a similar result: by Corollary 3.9, N_{j+1} is a strong extension of $N_{j+1,k}$ for all $k < \kappa_{j+1}$ and by construction, F^{j+1} is a partially strong set of mappings. Also, $\text{dom}(f_{\alpha_{j+1}}) = D_{\alpha_{j+1}}^M$.

Finally, let N' be the union general model of the strong chain $(N_j \mid j < \omega)$ and $\mathfrak{E} = \{\bigcup_{j < \omega} \lim_{j \rightarrow \omega} f_\alpha^j \mid \alpha \in \mathcal{T}(L)\}$. By Corollary 3.9, N' is a strong extension of N_j for all $j < \omega$ and by construction, $\mathfrak{E}$ is a strong embedding since $\text{dom}(f_\alpha) = D_\alpha^M$ for all $\alpha \in \mathcal{T}(L)$. $\square$

We now show that like in the first-order case, two distinct tuples (in some general model M) realizing the same n -type implies the existence of an automorphism of a strong extension N of M , fixing the appropriate elements and mapping one tuple to the other.

Proposition 4.10. *Let $M = (\mathcal{D}, I)$ be an interpretation of L and $\mathcal{X} \leq \mathcal{D}$. Suppose there are two sequences $\bar{d}^1$ and $\bar{d}^2$ with $d_{\alpha_i}^1, d_{\alpha_i}^2 \in D_{\alpha_i}^M$ for $1 \leq i \leq n$ such that $\text{tp}_n^M(\bar{d}^1/\mathcal{X}) = \text{tp}_n^M(\bar{d}^2/\mathcal{X})$. Then there is a strong extension $N \succeq M$ with $\mathfrak{I} = \{\iota_\alpha : D_\alpha^M \rightarrow D_\alpha^N \mid \alpha \in \mathcal{T}(L)\}$ the corresponding inclusion and $\Theta = \{\theta_\alpha \mid \alpha \in \mathcal{T}(L)\}$ an automorphism of N fixing all elements of $\mathcal{X}$ (up to $\mathfrak{I}$) and $\theta_{\alpha_i}(\iota_{\alpha_i}(d_{\alpha_i}^1)) = \iota_{\alpha_i}(d_{\alpha_i}^2)$ for all $1 \leq i \leq n$.*

Proof Let

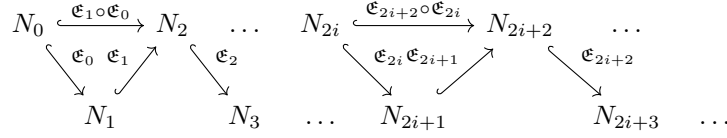
$$F = \{f_\alpha : D_\alpha^X \cup \{d_{\alpha_1}^1, \dots, d_{\alpha_n}^1\} \rightarrow D_\alpha^X \cup \{d_{\alpha_1}^2, \dots, d_{\alpha_n}^2\} \mid \alpha \in \mathcal{T}(L)\}$$

be the collection of mappings where

$$f_\alpha(d) = \begin{cases} d; & \text{if } d \in D_\alpha^X \\ d_{\alpha_i}^2; & \text{if } d \text{ is } d_{\alpha_i}^1 \text{ for some } i \end{cases}$$

Using the fact that $\text{tp}^M(\bar{d}^1/\mathcal{X}) = \text{tp}^M(\bar{d}^2/\mathcal{X})$ and by Corollary 4.9, we can find a strong extension $N_0 \succeq M$ with $\mathfrak{E}_M$ a strong embedding from M into N_0 that extends F . We can recursively construct a strong chain as follows: given $N_{2i} \preceq N_{2i+1}$ with corresponding $\mathfrak{E}_{2i}$, we show how to construct N_{2i+2} , N_{2i+3} such that $N_{2i+1} \preceq N_{2i+2} \preceq N_{2i+3}$ and we have corresponding strong embeddings $\mathfrak{E}_{2i+1}$, $\mathfrak{E}_{2i+2}$. Given $\mathfrak{E}_{2i}$, we can view $F_{2i+1} = \{f_\alpha^{2i+1} : \text{ran}(\varepsilon_\alpha^{2i}) \rightarrow D_\alpha^{N_{2i}} \mid \alpha \in \mathcal{T}(L)\}$ as a partially strong set of mappings. By Corollary 4.9, we can find $N_{2i+2} \succeq N_{2i+1}$ and extend F_{2i+1} to a strong embedding $\mathfrak{E}_{2i+1} = \{\varepsilon_\alpha^{2i+1} : D_\alpha^{N_{2i+1}} \rightarrow D_\alpha^{N_{2i+2}} \mid \alpha \in \mathcal{T}(L)\}$. With $\mathfrak{E}_{2i+1}$ in mind, we can similarly view $F_{2i+2} = \{f_\alpha^{2i+2} : \text{ran}(\varepsilon_\alpha^{2i+1}) \rightarrow D_\alpha^{N_{2i+2}} \mid \alpha \in \mathcal{T}(L)\}$ as a partially strong set of mappings. By Corollary 4.9, we can find $N_{2i+3} \succeq N_{2i+2}$ and extend F_{2i+2} to a strong embedding

$\mathfrak{E}_{2i+2} = \{\varepsilon_\alpha^{2i+2} : D_\alpha^{2i+2} \rightarrow D_\alpha^{2i+3} \mid \alpha \in \mathcal{T}(L)\}$. Pictorially, we have



where strong extensions are represented by $\hookrightarrow$ and labelled with their corresponding strong embeddings. Let $\mathfrak{I}_i = \{\iota_\alpha^i \mid \alpha \in \mathcal{T}(L)\}$ denote the inclusion from N_i into N_{i+1} for all i . Notice that by construction, $\iota_\alpha^{2i+1, 2i} \circ \varepsilon^{2i} \subseteq \varepsilon^{2i+2}$ for all i and $\alpha \in \mathcal{T}(L)$. Now let N be the union general model of the strong chain $(N_{2i} \mid i < \omega)$ and $\mathfrak{I} = \{\iota_\alpha : D_\alpha^M \rightarrow D_\alpha^N \mid \alpha \in \mathcal{T}(L)\}$ be the corresponding inclusion. Define $\Theta = \{\theta_\alpha = \lim_{i \rightarrow \omega} \varepsilon_\alpha^{0, 2i} \mid \alpha \in \mathcal{T}(L)\}$.² By construction, Θ is an automorphism; furthermore, the construction ensures that $\theta_{\alpha_i}(\iota_{\alpha_i}(d_{\alpha_i}^1)) = \iota_{\alpha_i}(d_{\alpha_i}^2)$ for all $1 \leq i \leq n$. $\square$

4.3 Stone Spaces and Omitting Types

As in first-order logic, the *Stone topology* on $C_n^M(\mathcal{A})$ has (basic) open sets

$$[\mathbf{A}_o] = \{p \in C_n^M(\mathcal{A}) \mid \mathbf{A}_o \in p\}.$$

Now

$$\begin{array}{lll}
 \bigvee_{i \in \mathcal{I}} \mathbf{A}_o^i & \text{stands for} & \mathbf{A}_o^1 \vee \dots \vee \mathbf{A}_o^n \\
 \bigwedge_{i \in \mathcal{I}} \mathbf{A}_o^i & \text{stands for} & \mathbf{A}_o^1 \wedge \dots \wedge \mathbf{A}_o^n
 \end{array}$$

where $\mathcal{I} = \{1, \dots, n\}$. Notice that $[\bigvee_{i \in \mathcal{I}} \mathbf{A}_o^i] = \bigcup_{i \in \mathcal{I}} [\mathbf{A}_o^i]$ and $[\bigwedge_{i \in \mathcal{I}} \mathbf{A}_o^i] = \bigcap_{i \in \mathcal{I}} [\mathbf{A}_o^i]$ since open sets are complete types. Furthermore, $[\mathbf{A}_o]$ is closed since $[\mathbf{A}_o] = C_n^M(\mathcal{A}) \setminus [\neg \mathbf{A}_o]$. We now show that $C_n^M(\mathcal{A})$ is indeed a Stone space; that is, it is compact, Hausdorff, and totally disconnected.

Proposition 4.11. $C_n^M(\mathcal{A})$ is compact.

Proof We show that every cover of $C_n^M(\mathcal{A})$ has a finite subcover. A.f.s.o.c.³ that not. Let $X = \{[\mathbf{A}_o^i] \mid i \in \mathcal{I}\}$ be a cover of $C_n^M(\mathcal{A})$. Let $p = \{\neg \mathbf{A}_o^i \mid i \in \mathcal{I}\}$. We will show that $p \cup \text{Th}_{\mathcal{A}}(M)$ is satisfiable. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite subset. By assumption, we have an n -type q such that $q \notin \bigcup_{i \in \mathcal{I}_0} [\mathbf{A}_o^i]$.

By Proposition 4.5 and De Morgan's law, there is $N_0 \succeq M$ with $d_{\alpha_1}^{N_0} \in D_{\alpha_1}^{N_0}, \dots, d_{\alpha_n}^{N_0} \in D_{\alpha_n}^{N_0}$ such that for any $\varphi \in \text{assign}(N_0)$,

$$N_0 \models_{\varphi[(\mathbf{x}_1 : \alpha_1) \mapsto d_{\alpha_1}^{N_0}] \dots [(\mathbf{x}_n : \alpha_n) \mapsto d_{\alpha_n}^{N_0}]} \text{Th}_{\mathcal{A}}(M) \cup \bigwedge_{i \in \mathcal{I}_0} \neg \mathbf{A}_o^i.$$

By the Compactness Theorem [7, Corollary 8.16], $p \cup \text{Th}_{\mathcal{A}}(M)$ is satisfiable, and so p is an n -type. By Proposition 4.5, there is $N \succeq M$ with $d_{\alpha_1}^N \in D_{\alpha_1}^N, \dots, d_{\alpha_n}^N \in D_{\alpha_n}^N$ such that $N \models p \cup \text{Th}_{\mathcal{A}}(M)$.

²See the proof of Corollary 4.9 for the meaning of $\lim_{i \rightarrow \omega} \varepsilon_\alpha^{0, 2i}$.

³Assume for the sake of contradiction.

By definition, $\text{tp}^N(d_{\alpha_i}^N/\mathcal{A}) \in C_n^M(\mathcal{A})$ but since $N \models_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}^N] \dots [(\mathbf{x}_n:\alpha_n) \mapsto d_{\alpha_n}^N]} \neg \mathbf{A}_o^i$ for all $i \in \mathcal{I}$, then $\text{tp}^N(d_{\alpha_i}^N/\mathcal{A}) \notin \bigcup X = C_n^M(\mathcal{A})$ a contradiction. $\square$

Proposition 4.12. $C_n^M(\mathcal{A})$ is totally disconnected.

Proof We will show that for distinct $p, q \in C_n^M(\mathcal{A})$ there is $\mathbf{A}_o$ such that $p \in [\mathbf{A}_o]$ and $q \notin [\mathbf{A}_o]$. Since $p \neq q$ and p, q are complete n -types, there is $\mathbf{A}_o$ such that $\mathbf{A}_o \in p$ and $\neg \mathbf{A}_o \in q$. Hence $p \in [\mathbf{A}_o]$ and $q \notin [\mathbf{A}_o]$. $\square$

Proposition 4.13. $C_n^M(\mathcal{A})$ is Hausdorff.

Proof We will show that distinct $p, q \in C_n^M(\mathcal{A})$ can be separated by open sets. Since $p \neq q$ and p, q are complete n -types, there is $\mathbf{A}_o$ such that $p \in [\mathbf{A}_o]$ and $q \in [\neg \mathbf{A}_o]$. A.f.s.o.c. that $[\mathbf{A}_o] \cap [\neg \mathbf{A}_o] \neq \emptyset$. Then there is $r \in C_n^M(\mathcal{A})$ such that $(\mathbf{A}_o \wedge \neg \mathbf{A}_o) \in r$, a contradiction. Therefore, the claim holds. $\square$

Definition 4.14. An n -type $p \in C_n^M(\mathcal{A})$ is *isolated* if $\{p\}$ is an open set.

Proposition 4.15. Let $p \in C_n^M(\mathcal{A})$. Then the following are equivalent:

1. p is isolated.
2. There exists an $L_{\mathcal{A}}$ formula $\mathbf{A}_o$ such that $\{p\} = [\mathbf{A}_o]$.
3. There exists an $L_{\mathcal{A}}$ formula $\mathbf{A}_o \in p$ such that for all $\mathbf{B}_o \in \mathcal{E}(L_{\mathcal{A}})$ with $(\mathbf{x}_1 : \alpha_1), \dots, (\mathbf{x}_n : \alpha_n)$ free, $\mathbf{B}_o \in p$ iff $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$.

Proof

(1 $\Rightarrow$ 2): Suppose p is isolated. By definition, $\{p\}$ is an open set. So for some well-order $\mathcal{I}$, $\{p\} = \bigcup_{i \in \mathcal{I}} [\mathbf{A}_o^i]$. Hence for some $j \in \mathcal{I}$, we have $\{p\} = [\mathbf{A}_o^j]$.

(2 $\Rightarrow$ 3): Suppose $\{p\} = [\mathbf{A}_o]$ and let $\mathbf{B}_o \in \mathcal{E}(L_{\mathcal{A}})$. We will show that $\mathbf{B}_o \in p$ iff $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$. Suppose $\mathbf{B}_o \in p$. A.f.s.o.c. that $\text{Th}_{\mathcal{A}}(M) \not\models \mathbf{A}_o \Rightarrow \mathbf{B}_o$. Then there is a general model $N \models \text{Th}_{\mathcal{A}}(M)$ and $d_{\alpha_1}^N \in D_{\alpha_1}^N, \dots, d_{\alpha_n}^N \in D_{\alpha_n}^N$ such that for all $\varphi \in \text{assign}(N)$, $N \models_{\varphi[(\mathbf{x}_1:\alpha_1) \mapsto d_{\alpha_1}^N] \dots [(\mathbf{x}_n:\alpha_n) \mapsto d_{\alpha_n}^N]} (\mathbf{A}_o \wedge \neg \mathbf{B}_o)$. It follows that $\mathbf{A}_o, \neg \mathbf{B}_o \in \text{tp}^N(d_{\alpha_i}^N/\mathcal{A})$. Since $\{p\} = [\mathbf{A}_o]$, $p = \text{tp}^N(d_{\alpha_i}^N/\mathcal{A}) \in C_n^M(\mathcal{A})$. But by assumption, $\mathbf{B}_o \in p$, a contradiction. Now suppose $\mathbf{B}_o \notin p$. Because p is complete, $\neg \mathbf{B}_o \in p$. By the same argument, $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \neg \mathbf{B}_o$. Since $\mathbf{A}_o \in p$ by assumption, $\text{Th}_{\mathcal{A}}(M) \cup \{\mathbf{A}_o\}$ is satisfiable. Hence $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$.

(3 $\Rightarrow$ 1): Suppose there exists an $L_{\mathcal{A}}$ formula $\mathbf{A}_o \in p$ such that for all $\mathbf{B}_o \in \mathcal{E}(L_{\mathcal{A}})$, $\mathbf{B}_o \in p$ iff $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$. We will show that $\mathbf{A}_o$ isolates p ; that is, $\{p\} = [\mathbf{A}_o]$. By assumption, $p \in [\mathbf{A}_o]$. Suppose $q \in [\mathbf{A}_o]$ and let $\mathbf{B}_o \in \mathcal{E}(L_{\mathcal{A}})$. If $\mathbf{B}_o \in p$, then by assumption $\text{Th}_{\mathcal{A}}(M) \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$. A.f.s.o.c. that $\mathbf{B}_o \notin q$. Since q is a complete n -type, $\neg \mathbf{B}_o \in q$. But then $\text{Th}_{\mathcal{A}}(M) \cup q$ is unsatisfiable, a contradiction. Hence $\mathbf{B}_o \in q$. If on the other hand $\mathbf{B}_o \notin p$, then p is a complete n -type implies $\neg \mathbf{B}_o \in p$. By the same argument, $\mathbf{B}_o \notin q$. Thus, since $p = q$ for arbitrary $q \in [\mathbf{A}_o]$, $\{p\} = [\mathbf{A}_o]$ is an open set. $\square$

Let $T = (L, \Gamma)$. Then $C_n(T)$ denotes the set of complete n -types (over T) where $p \in C_n(T)$ implies $p \cup T$ is satisfiable. Let basic open sets of $C_n(T)$ be $[\mathbf{A}_o] = \{p \mid \mathbf{A}_o \in p\}$.

Proposition 4.16. $C_n(T)$ is a Stone space.

Proof Follows similarly to the proofs of Propositions 4.11, 4.12, and 4.13. $\square$

Similar to the $C_n^M(\mathcal{A})$ case, p is *isolated* in $C_n(T)$ if $\{p\} = [\mathbf{A}_o]$ for some $\mathbf{A}_o \in \mathcal{E}(L)$.

Definition 4.17. Let $\mathbf{A}_o \in \mathcal{E}(L)$ such that $T \cup \{\mathbf{A}_o\}$ is satisfiable and p be an n -type over T . Then $[\mathbf{A}_o]$ *isolates* p if for all $\mathbf{B}_o \in p$,

$$T \models \forall \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_n : \alpha_n . \mathbf{A}_o \Rightarrow \mathbf{B}_o.$$

Proposition 4.18. If p is a complete n -type and $\mathbf{A}_o$ isolates p , then for all $\mathbf{B}_o \in \mathcal{E}(L)$ with $(\mathbf{x}_1 : \alpha_1), \dots, (\mathbf{x}_n : \alpha_n)$ free, $\mathbf{B}_o \in p$ iff $T \models \mathbf{A}_o \Rightarrow \mathbf{B}_o$.

Proof Follows similarly to the proof of Proposition 4.15. $\square$

The notion of isolation with respect to $C_n^M(\mathcal{A})$ extends naturally to $C_n(T)$.

Proposition 4.19. Suppose $p \in S_n(T)$ is isolated by $\mathbf{A}_o$. Then p is realized in any model of $T \cup \{\exists \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_n : \alpha_n . \mathbf{A}_o\}$.

Proof Suppose $M \models T \cup \{\exists \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_n : \alpha_n . \mathbf{A}_o\}$. Then there are $d_{\alpha_1}^M \in D_{\alpha_1}^M, \dots, d_{\alpha_n}^M \in D_{\alpha_n}^M$ such that for all $\varphi \in \text{assign}(M)$,

$$M \models_{\varphi[(\mathbf{x}_1 : \alpha_1) \mapsto d_{\alpha_1}^M] \dots [(\mathbf{x}_n : \alpha_n) \mapsto d_{\alpha_n}^M]} \mathbf{A}_o.$$

But then by assumption, $d_{\alpha_1}^M, \dots, d_{\alpha_n}^M$ realize p . $\square$

Proposition 4.20. If T is complete, then every isolated type p can be realized.

Proof Let $\{p\} = [\mathbf{A}_o]$. Since p is isolated, $T \cup \{\mathbf{A}_o\}$ is satisfiable. Let $M \models T$. Since T is complete, we must have $T \models \exists \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_n : \alpha_n . \mathbf{A}_o$. But then p is isolated implies we can realize p in M . $\square$

We now aim to prove the omitting types theorem. We first show that given a theory T in a countable language, we can extend T to a theory T' (also in a countable language) such that T' is satisfiable by a frugal general model and the axioms of T' ensure that a given n -type is omitted. Note that there exists a sound and complete proof system for Alonzo called $\mathfrak{A}$ [7, Corollary 8.13]. We include the following definitions from [7, Appendix C].

Definition 4.21. Let $T = (L, \Gamma)$ and $\mathfrak{P}$ be a proof system for Alonzo. T is *syntactically complete* in $\mathfrak{P}$ if either $T \vdash_{\mathfrak{P}} \mathbf{A}_o$ or $T \vdash_{\mathfrak{P}} \neg \mathbf{A}_o$ holds for all $\mathbf{A}_o \in \mathcal{E}(L)$.

Definition 4.22. Let $T = (L, \Gamma)$ and $\mathfrak{P}$ be a proof system for Alonzo. T is *extensionally complete* in $\mathfrak{P}$ if for all $\mathbf{A}_o$ of the form $\mathbf{F}_{\alpha \rightarrow \beta} = \mathbf{G}_{\alpha \rightarrow \beta}$, there is a closed expression $\mathbf{C}_\alpha$ such that:

1. $T \vdash_{\mathfrak{P}} \mathbf{C}_\alpha \downarrow$.
2. $T \vdash_{\mathfrak{P}} (\mathbf{F}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{G}_{\alpha \rightarrow \beta} \downarrow) \Rightarrow (\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \simeq \mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \Rightarrow \mathbf{F}_{\alpha \rightarrow \beta} = \mathbf{G}_{\alpha \rightarrow \beta})$.

Lemma 4.23. Let $L = (\mathcal{B}, \mathcal{C})$ with $\|L\| = \omega$, $T = (L, \Gamma)$, and p be a non-isolated n -type. For each $\alpha \in \mathcal{T}(L)$, let $\mathcal{C}_\alpha$ be a well-ordered set of constants such that $\mathcal{C}_\alpha \cap \mathcal{C} = \emptyset$ and $|\mathcal{C}_\alpha| = \|L\|$. Let $\mathcal{C}_0 = \bigcup_{\alpha \in \mathcal{T}(L)} \mathcal{C}_\alpha$, $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_0$, and $L' = (\mathcal{B}, \mathcal{C}')$. If T is consistent in $\mathfrak{A}$, then there is a theory

$T' = (L', \Gamma')$ such that:

1. $T \leq T'$.

2. T' is consistent in $\mathfrak{A}$.
3. T' is syntactically complete in $\mathfrak{A}$.
4. T' is extensionally complete in $\mathfrak{A}$.
5. $\|L\| = \|L'\| = \omega$.
6. For all $\mathbf{c}_{\alpha_1}, \dots, \mathbf{c}_{\alpha_n} \in \mathcal{C}_0$, there is $\mathbf{A}_o \in p$ such that

$$T' \vdash_{\mathfrak{A}} \neg \mathbf{A}_o[(\mathbf{x}_1 : \alpha_1) \mapsto \mathbf{c}_{\alpha_1}] \dots [(\mathbf{x}_n : \alpha_n) \mapsto \mathbf{c}_{\alpha_n}].$$

Proof We provide a modified version of the construction in Lemma A.1. By construction, $\|L'\| = \|L\|$ and so condition 5 is satisfied.

Now well-order the sentences of L' and n -tuples $\mathbf{c}_{\alpha_1}, \dots, \mathbf{c}_{\alpha_n} \in \mathcal{C}_0$. For each ordinal $\xi < \|L'\|$, denote the ξ^{th} sentence in the well-order as $\mathbf{A}_o^\xi$. For each $\xi \leq \|L'\|$, we will define a set of sentences Γ_ξ of L' by induction such that

$$\zeta < \xi \text{ implies } \Gamma_\zeta \subseteq \Gamma_\xi \quad (\star)$$

Base: $\xi = 0$. Then $\Gamma_0 = \Gamma$.

Step: Let $\zeta = 2i$ for $i \geq 0$.

Case 1. $\xi = 2i + 1$. We consider three subcases:

Subcase a. $\Gamma_\zeta \cup \{\mathbf{A}_o^i\}$ is consistent in $\mathfrak{A}$. Then $\Gamma_\xi = \Gamma_\zeta \cup \{\mathbf{A}_o^i\}$.

Subcase b. $\Gamma_\zeta \cup \{\mathbf{A}_o^i\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{A}_o^i$ does not have the form $\mathbf{B}_{\alpha \rightarrow \beta} = \mathbf{C}_{\alpha \rightarrow \beta}$. Then $\Gamma_\xi = \Gamma_\zeta$.

Subcase c. $\Gamma_\zeta \cup \{\mathbf{A}_o^i\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{A}_o^i$ has the form $\mathbf{B}_{\alpha \rightarrow \beta} = \mathbf{C}_{\alpha \rightarrow \beta}$. Then

$$\Gamma_\xi = \Gamma_\zeta \cup \{\neg(\mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{C}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{C}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha)\}$$

where $\mathbf{c}_\alpha$ is the first constant in $\mathcal{C}_\alpha$ that does not occur in Γ_ζ or $\mathbf{A}_o^i$.

Case 2. $\xi = 2i + 2$. Let $\bar{\mathbf{c}}^i = \mathbf{c}_{\alpha_1}, \dots, \mathbf{c}_{\alpha_n}$ be the i^{th} n -tuple in the well-order of n -tuples from $\mathcal{C}_0$. W.l.o.g. let $\mathbf{d}_{\alpha_1}, \dots, \mathbf{d}_{\alpha_n}$ be the constants in $\bar{\mathbf{c}}^i_\alpha$ that are not in $\mathbf{A}_o^i$. Define⁴

$$\mathbf{B}_o \equiv \mathbf{A}_o^i \wedge (\mathbf{d}_{\alpha_1} = \mathbf{d}_{\alpha_1} \wedge \dots \wedge \mathbf{d}_{\alpha_n} = \mathbf{d}_{\alpha_n}).$$

Let $\mathbf{e}_{\beta_1}, \dots, \mathbf{e}_{\beta_m}$ be the constants in $\mathbf{B}_o$ that are in $\mathcal{C}_0 \setminus \bar{\mathbf{c}}^i_\alpha$. Define

$$\begin{aligned} \mathbf{C}_o \equiv & \exists \mathbf{y}_1 : \beta_1, \dots, \mathbf{y}_m : \beta_m . \mathbf{B}_o[\mathbf{c}_{\alpha_1} \mapsto (\mathbf{x}_1 : \alpha_1)] \\ & \dots [\mathbf{c}_{\alpha_n} \mapsto (\mathbf{x}_n : \alpha_n)] [\mathbf{e}_{\beta_1} \mapsto (\mathbf{y}_1 : \beta_1)] \\ & \dots [\mathbf{e}_{\beta_m} \mapsto (\mathbf{y}_m : \beta_m)]. \end{aligned}$$

That is, we remove any constant from $\mathcal{C}_0$ in $\mathbf{B}_o$ by either existentially quantifying over it or replacing it with a (free) variable.

Because p is non-isolated, there is a formula $\mathbf{D}_o \in p$ such that

$$T \not\models \forall \mathbf{x}_1 : \alpha_1, \dots, \mathbf{x}_n : \alpha_n . \mathbf{C}_o \Rightarrow \mathbf{D}_o. \quad (\dagger)$$

Let $\Gamma_\xi = \Gamma_{\zeta+1} \cup \{\neg \mathbf{D}_o[(x_1 : \alpha_1) \mapsto \mathbf{c}_{\alpha_1}] \dots [(x_n : \alpha_n) \mapsto \mathbf{c}_{\alpha_n}]\}$.

⁴In contrast to the presentations of this construction in, say, [12, Theorem 4.2.3] and [11, 7.2.1], we want to ensure that we have n free variables in $\mathbf{C}_o$ to be in accordance with the definition of p being non-isolated.

It is easy to verify by induction that $(\star)$ holds. Let $\Gamma' = \bigcup_{\xi < \omega} \Gamma_\xi$ and $T' = (L', \Gamma')$. Since $T \leq T'$, condition 1 holds. Conditions 3 and 4 follow by a similar argument to the one in Lemma A.1. Case 2 of the inductive construction ensures that condition 6 holds. Lemma A.1 handles the fact that additions to Γ' from case 1 of the inductive construction preserve consistency. Thus, it only remains to show that case 2 preserves consistency.

We will show that for all $\xi = 2i + 2 < \|L'\|$, Γ_ξ is satisfiable. Let $\mathbf{C}_o$ and $\mathbf{D}_o$ be from step $\xi = 2i + 2$. $(\dagger)$ implies that there is a general model $M \models T$ and $d_{\alpha_1}^M \in D_{\alpha_1}^M, \dots, d_{\alpha_n}^M \in D_{\alpha_n}^M$ such that for all $\varphi \in \text{assign}(M)$,

$$M \models_{\varphi} [(\mathbf{x}_1 : \alpha_1) \mapsto d_{\alpha_1}^M] \dots [(\mathbf{x}_n : \alpha_n) \mapsto d_{\alpha_n}^M] \mathbf{C}_o \wedge \neg \mathbf{D}_o.$$

Therefore we can turn $M = (\mathcal{D}, \mathcal{I})$ into a model of

$$\neg \mathbf{D}_o[(x_1 : \alpha_1) \mapsto \mathbf{c}_{\alpha_1}] \dots [(x_n : \alpha_n) \mapsto \mathbf{c}_{\alpha_n}]$$

by expanding I such that $I(\mathbf{c}_{\alpha_i}) = d_{\alpha_i}^M$ for $1 \leq i \leq n$. By [7, Corollary 8.15], Γ_ξ is consistent in $\mathfrak{A}$. Therefore, condition 2 holds, completing the proof. $\square$

Proposition 4.24. *Let $T = (L, \Gamma)$ be consistent in $\mathfrak{A}$ where $L = (\mathcal{B}, \mathcal{C})$. Let $L' = (\mathcal{B}, \mathcal{C} \cup \mathcal{C}_0)$ and $T' = (L', \Gamma')$ be the language and theory obtained from Lemma 4.23. Let $M = (\mathcal{D}, I)$ be the frugal general model obtained from Theorem A.2 such that $M \models T'$. Then for all $d_\alpha^M \in D_\alpha^M$, there exists $\mathbf{c}_\alpha \in \mathcal{C}_0$ such that $(\star)$ $I(\mathbf{c}_\alpha) = d_\alpha^M$.*

Proof We will prove $(\star)$ by structural induction on types. Recall from Theorem A.2 that for any $\alpha \in \mathcal{T}(L)$, $\overline{\mathcal{E}}_\alpha = \{\mathbf{A}_\alpha \mid \mathbf{A}_\alpha \in \mathcal{E}(L') \text{ that is closed}\}$ and for all $\alpha \in \mathcal{T}(L')$,

- (1 $^\alpha$) $D_\alpha = \{V(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \in \overline{\mathcal{E}}_\alpha \text{ and } V(\mathbf{A}_\alpha) \text{ is defined}\};$
- (2 $^\alpha$) $V(\mathbf{A}_\alpha)$ is defined iff $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha \downarrow$ for all $\mathbf{A}_\alpha \in \overline{\mathcal{E}}_\alpha$;
- (3 $^\alpha$) $V(\mathbf{A}_\alpha) = V(\mathbf{B}_\alpha)$ iff $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha = \mathbf{B}_\alpha$ for all $\mathbf{A}_\alpha, \mathbf{B}_\alpha$;

and $I = V|_{\mathcal{C}'}$. Let $d_\alpha^M \in D_\alpha^M$. Then $d_\alpha^M = V(\mathbf{A}_\alpha)$ for some $\mathbf{A}_\alpha \in \overline{\mathcal{E}}_\alpha$ such that $V(\mathbf{A}_\alpha)$ is defined by (1 $^\alpha$). Since $M \models \exists \mathbf{x} : \alpha. (\mathbf{A}_\alpha = \mathbf{x})$ and T' is syntactically complete, then by [7, Theorem 8.12], $\Gamma' \vdash_{\mathfrak{A}} \exists \mathbf{x} : \alpha. (\mathbf{A}_\alpha = \mathbf{x})$. Since $\neg \exists \mathbf{x} : \alpha. (\mathbf{A}_\alpha = \mathbf{x})$ is inconsistent with T' , then for some $\xi < \|L'\|$, we have that at step $\xi = 2i + 1$ of Lemma 4.23,

$$\Gamma_\xi = \Gamma_\zeta \cup \{\neg(\mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{C}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{C}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha)\}$$

for some $\mathbf{c}_\alpha$ where $\mathbf{B}_{\alpha \rightarrow o} \equiv \lambda x : \alpha. T_o$ and $\mathbf{C}_{\alpha \rightarrow o} \equiv \lambda \mathbf{x} : \alpha. \mathbf{x} \neq \mathbf{A}_\alpha$.⁵ Since function abstractions are always defined and T' is syntactically complete and consistent, we have that $\Gamma' \vdash_{\mathfrak{A}} \mathbf{c}_\alpha = \mathbf{A}_\alpha$. Since $I(\mathbf{c}_\alpha) = V(\mathbf{A}_\alpha) = d_\alpha^M$ by (3 $^\alpha$), the claim holds. $\square$

Theorem 4.25 (Omitting types theorem). *Let $L = (\mathcal{B}, \mathcal{C})$ such that $\|L\| = \omega$, $T = (L, \Gamma)$ be consistent in $\mathfrak{A}$, and p be a non-isolated n -type of T . Then there exists a frugal general model $M \models T$ that omits p .*

⁵ $\neg \exists \mathbf{x} : \alpha. \mathbf{A}_\alpha = \mathbf{x}$ stands for $(\lambda x : \alpha. T_o) \neq (\lambda \mathbf{x} : \alpha. \mathbf{A}_\alpha \neq \mathbf{x})$

Proof Let $T' \geq T$ be obtained from Lemma 4.23 and let $M' = (\mathcal{D}, I') \models T'$ be the frugal general model obtained from Theorem A.2 applied to T' . Suppose that $d_{\alpha_1}^M \in D_{\alpha_1}^{M'}, \dots, d_{\alpha_n}^M \in D_{\alpha_n}^{M'}$. Proposition 4.24 implies that there are constants $\bar{\mathbf{c}}_{\alpha}^j$ such that for all $1 \leq j \leq n$, $I(\mathbf{c}_{\alpha_j}) = d_{\alpha_j}^M$. Let this n -tuple of constants be the i^{th} element in the enumeration of n -tuples from $\mathcal{C}_0$ in Lemma 4.23. At stage $\xi = 2i + 2$, we ensure that $\neg \mathbf{D}_o[(x_1 : \alpha_1) \mapsto \mathbf{c}_{\alpha_1}] \dots [(x_n : \alpha_n) \mapsto \mathbf{c}_{\alpha_n}] \in \Gamma'$ for some $\mathbf{D}_o \in p$. Thus for all $\varphi \in \text{assign}(M')$,

$$M' \models_{\varphi[(x_1:\alpha_1) \mapsto d_{\alpha_1}^M] \dots [(x_n:\alpha_n) \mapsto d_{\alpha_n}^M]} \neg \mathbf{D}_o.$$

Since $d_{\alpha_1}^M, \dots, d_{\alpha_n}^M$ were arbitrary, M' omits p ; and so, the reduct M of M' to L omits p and $M \models T$. $\square$

We can extend the result to omit a countable number of non-isolated types.

Corollary 4.26. *Let $L = (\mathcal{B}, \mathcal{C})$ such that $\|L\| = \omega$, $T = (L, \Gamma)$ be consistent in $\mathfrak{A}$, and P be a countable set of non-isolated types of T . Then there exists a frugal general model $M \models T$ that omits all $p \in P$.*

Proof (Sketch) We will show how to augment Lemma 4.23 such that condition 6 becomes the following: for all finite sequences of constants $\mathbf{c}_{\alpha_1}, \dots, \mathbf{c}_{\alpha_n}$ from $\mathcal{C}_0$ and $p \in P$, if p is an n -type, then there is $\mathbf{A}_o \in p$ such that

$$T' \vdash_{\mathfrak{A}} \neg \mathbf{A}_o[(\mathbf{x}_1 : \alpha_1) \mapsto \mathbf{c}_{\alpha_1}] \dots [(\mathbf{x}_n : \alpha_n) \mapsto \mathbf{c}_{\alpha_n}].$$

To this end, let $\{p_1, p_2, \dots\}$ be a well-order of P and $\{\bar{\mathbf{c}}^1, \bar{\mathbf{c}}^2, \dots\}$ be a well-order of all finite sequences from $\mathcal{C}_0$. Now fix a bijection $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.

We change the case where $\xi = 2i + 2$ in the inductive construction in Lemma 4.23 in the following manner. We have $i = f(m, n)$ for some m, n . Now if p_m is a $|\bar{\mathbf{c}}^n|$ -type, then proceed in the same way as in the original lemma. Otherwise, let $\Gamma_{\xi} = \Gamma_{\xi+1}$.

Now let $T' \geq T$ be obtained from Lemma 4.23 with the refinements mentioned above and M' be the frugal general model obtained from Theorem A.2 applied to T' . Then the reduct M of M' to L omits all $p \in P$ and $M \models T$. $\square$

Chapter 5

First-Order Logic

The notions of language, term, formula, and theory from first-order logic are easily expressible in Alonzo. We will see how to capture first-order theories either syntactically or semantically.

5.1 Syntactic First-Order Theories

In this section we will formalize first-order theories (and languages) from a *syntactic* point of view. That is, a first-order theory (and language) can be captured explicitly using the syntax of Alonzo.

Definition 5.1. A language $L = (\mathcal{B}, \mathcal{C})$ is called *first-order* if $\mathcal{B} = \{\mathbf{a}\}$ and members of $\mathcal{C}$ have the form $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow o}$ or $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow \mathbf{a}}$.

Definition 5.2. Let $L = (\mathcal{B}, \mathcal{C})$ be a first-order language. The set of *first-order L -terms* $\mathcal{T}_{\omega, \omega}$ is the smallest set satisfying:

1. $\mathbf{c}_{\mathbf{a}} \in \mathcal{T}_{\omega, \omega}$ for all $\mathbf{c}_{\mathbf{a}} \in \mathcal{C}$,
2. $(\mathbf{x} : \mathbf{a}) \in \mathcal{T}_{\omega, \omega}$ for all variables $(\mathbf{x} : \mathbf{a})$,
3. $\mathbf{A}_{\mathbf{a}}^1, \dots, \mathbf{A}_{\mathbf{a}}^n \in \mathcal{T}_{\omega, \omega}$ and $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow \mathbf{a}} \in \mathcal{C}$ with arity n implies $\mathbf{c}(\mathbf{A}_{\mathbf{a}}^1, \dots, \mathbf{A}_{\mathbf{a}}^n) \in \mathcal{T}_{\omega, \omega}$.

Definition 5.3. Let $L = (\mathcal{B}, \mathcal{C})$ be a first-order language. The set of *atomic first-order L -formulas* $\Phi_{\omega, \omega}^a$ is the smallest set satisfying:

1. $\mathbf{A}_{\mathbf{a}}^1, \mathbf{A}_{\mathbf{a}}^2 \in \mathcal{T}_{\omega, \omega}$ implies $\mathbf{A}_{\mathbf{a}}^1 = \mathbf{A}_{\mathbf{a}}^2 \in \Phi_{\omega, \omega}^a$,
2. $\mathbf{A}_{\mathbf{a}}^1, \dots, \mathbf{A}_{\mathbf{a}}^n \in \mathcal{T}_{\omega, \omega}$ and $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow o} \in \mathcal{C}$ with arity n implies $\mathbf{c}(\mathbf{A}_{\mathbf{a}}^1, \dots, \mathbf{A}_{\mathbf{a}}^n) \in \Phi_{\omega, \omega}^a$.

Definition 5.4. Let $L = (\mathcal{B}, \mathcal{C})$ be a first-order language. The set of *first-order L -formulas* $\Phi_{\omega, \omega}$ is the smallest set satisfying:

1. $\mathbf{A}_o \in \Phi_{\omega, \omega}$ for all $\mathbf{A}_o \in \Phi_{\omega, \omega}^a$,
2. $\mathbf{A}_o^1, \mathbf{A}_o^2 \in \Phi_{\omega, \omega}$ implies $(\neg \mathbf{A}_o^1), (\mathbf{A}_o^1 \vee \mathbf{A}_o^2), (\mathbf{A}_o^1 \wedge \mathbf{A}_o^2), (\mathbf{A}_o^1 \Rightarrow \mathbf{A}_o^2), (\mathbf{A}_o^1 \Leftrightarrow \mathbf{A}_o^2)$ are in $\Phi_{\omega, \omega}$,

3. $\mathbf{A}_o \in \Phi_{\omega, \omega}$ implies $(\exists \mathbf{x} : \mathbf{a} . \mathbf{A}_o), (\forall \mathbf{x} : \mathbf{a} . \mathbf{A}_o)$ are in $\Phi_{\omega, \omega}$.

Definition 5.5. Let $T = (L, \Gamma)$ be a theory. Call T a *syntactic first-order theory* if $\Gamma \subseteq \Phi_{\omega, \omega}$.

The specific base type $\mathbf{a} \in \mathcal{B}$ is arbitrary; as in first-order logic the universe is made up entirely of individuals. It is easy to modify Definitions 5.1 – 5.5 to accomodate many-sorted first-order logic through a correspondence between base types and sorts.

First-order notions that depend on the relationship between models carry over as well, though we need a way of dealing with higher-order domains. A naïve approach would be to treat those domains as containing only the relevant interpretations of $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow o}$ and $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow \mathbf{a}}$, but this is incorrect due to the definability of various interpretations of Alonzo expressions. For example, the identity function $(\lambda \mathbf{x} : \alpha \rightarrow \alpha . \mathbf{x})$ is defined for all $\alpha \in \mathcal{T}(L)$ [7, Lemma 5.4]. We resolve the problem by adapting the following convention:

Definition 5.6. Given an $\mathcal{L}$ -structure $\mathcal{M} = (M, \mathcal{F}, \mathcal{R})$ from first-order logic, identify it with the corresponding *first-order full interpretation* $N = (\mathcal{D}, I)$ of $L = (\{\mathbf{a}\}, \mathcal{C})$ where

1. $D_{\mathbf{a}} = M$;
2. $\mathcal{D}$ is the full frame generated by the singular base domain $D_{\mathbf{a}}$;
3. For each $f : M^n \rightarrow M \in \mathcal{F}$, we have $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow \mathbf{a}} \in \mathcal{C}$ where for all $m_1, \dots, m_n \in D_{\mathbf{a}}$,

$$I(\mathbf{c})(m_1, \dots, m_n) = f^{\mathcal{M}}(m_1, \dots, m_n);$$

4. For each $R : M^n \rightarrow \{\mathbf{T}, \mathbf{F}\} \in \mathcal{F}$, we have $\mathbf{c}_{(\mathbf{a} \times \dots \times \mathbf{a}) \rightarrow o} \in \mathcal{C}$ where for all $m_1, \dots, m_n \in D_{\mathbf{a}}$,

$$I(\mathbf{c})(m_1, \dots, m_n) = R^{\mathcal{M}}(m_1, \dots, m_n).$$

This embedding preserves isomorphisms across logics:

Proposition 5.7. Let T be an $\mathcal{L}$ -theory from first-order logic. Suppose M, N are (first-order) isomorphic models of T . Let $L' = (\mathcal{B}, \mathcal{C})$ be the corresponding first-order language in Alonzo, $T' = (L', \Gamma)$ be the corresponding first-order theory in Alonzo, and $M' = (\mathcal{D}^M, I^M)$, $N' = (\mathcal{D}^N, I^N)$ be the corresponding first-order full interpretations of T' . Then M', N' are isomorphic (higher-order) standard models of T' .

Proof By assumption, we have an isomorphism $j : M \rightarrow N$. Now let $\Theta = \{\theta_{\alpha} \mid \alpha \in \mathcal{T}(L)\}$ be a set of mappings from D_{α}^M to D_{α}^N defined by:

1. θ_o is the identity function on D_o^M ;
2. $\theta_{\mathbf{a}}(d) = j(d)$ for all $d \in D_{\mathbf{a}}^M$;
3. For all $\alpha, \beta \in \mathcal{T}(L)$ and $f_{\alpha \rightarrow \beta} \in D_{\alpha \rightarrow \beta}^M$, $\theta_{\alpha \rightarrow \beta}(f)$ is the unique function $g_{\alpha \rightarrow \beta} \in D_{\alpha \rightarrow \beta}^N$ such that for all $d \in D_{\alpha}^M$, $g(\theta_{\alpha}(d)) \simeq \theta_{\beta}(f(d))$;¹
4. $\theta_{\alpha \times \beta}(a, b) = (\theta_{\alpha}(a), \theta_{\beta}(b))$ for all $\alpha, \beta \in \mathcal{T}(L)$ and $a \in D_{\alpha}^M, b \in D_{\beta}^M$.

¹This definition is well-defined since there is a bijection from $D_{\mathbf{a}}^M$ to $D_{\mathbf{a}}^N$ and $\mathcal{D}^M$ and $\mathcal{D}^N$ are full.

It is easy to see that conditions (1)–(4) of an isomorphism [7, Ch. 5.6] are satisfied, and condition (5) follows from (2) and (3) above. Since $\mathcal{D}^M$ and $\mathcal{D}^N$ are full, they are isomorphic standard models of L' . $\square$

While our choice of embedding models of first-order logic by way of a corresponding standard model is sufficient, it is not necessary; for example, we could have instead added in all of the higher-order elements necessary to satisfy definability criteria.

5.2 Semantic First-Order Theories

In this section we will formalize first-order theories (and languages) from a *semantic* point of view. That is, we will show how a first-order theory (and language) can be captured using the semantics of Alonzo.

The development in the previous section suggests that we can extend any general model of a first-order theory to a standard model. Indeed, just extend a given (non-full) interpretation of L to the corresponding first-order full interpretation of L . We can define a first-order theory semantically, as follows:

Definition 5.8. Let $L = (\mathcal{B}, \mathcal{C})$ be a language and $T = (L, \Gamma)$ be a theory. T is a *semantic first-order theory* if every model M of T extends to a standard model N of T where $D_{\mathbf{a}}^N = D_{\mathbf{a}}^M$ for all $\mathbf{a} \in \mathcal{B}$.

The advantages of Definition 5.8 over Definition 5.5 are the following: (1) it captures the notion of being agnostic towards higher-order domains: a first-order theory T is one such that for any model M of T , you can lift M to a standard model of T by making each domain full; and (2) it also captures syntactic first-order theories as they are a proper subset.

Chapter 6

Conclusion

6.1 Summary and Insights

In Chapter 3, we developed the corresponding sharpened upward and downward Löwenheim-Skolem theorems for Alonzo. In order to do this, we needed a suitable notion of an inclusion in the context of the submodel and strong submodel relations. Recall from Definition 3.3 that $\mathcal{I} = \{\iota_\alpha \mid \alpha \in \mathcal{T}(L)\}$ is an inclusion if it is an embedding and $\iota_{\mathbf{a}}(d) = d$ for all $\mathbf{a} \in \mathcal{B}$. Unlike in first-order logic, not all maps ι_α act as the identity, as this would be problematic: Suppose we had an inclusion from M into N and that $D_{\mathbf{a}}^M \subset D_{\mathbf{a}}^N$ for some $\mathbf{a} \in \mathcal{B}$. Then $D_{\mathbf{a} \rightarrow \mathbf{a}}^M \cap D_{\mathbf{a} \rightarrow \mathbf{a}}^N = \emptyset$ since $D_{\mathbf{a}}^M \neq D_{\mathbf{a}}^N$. Lemma 3.4 justifies our definition by providing a way to exactly recover the corresponding function $f_{\alpha \rightarrow \beta}^M \in D_{\alpha \rightarrow \beta}^M$ from the extensional behaviour of $f_{\alpha \rightarrow \beta}^N = \iota(f_{\alpha \rightarrow \beta}^M) \in \text{ran}(\iota_\alpha)$. However, the fact that it is possible that $f_{\alpha \rightarrow \beta}^M \neq \iota(f_{\alpha \rightarrow \beta}^M)$ means we have to keep track of inclusions when considering strong chains. Unlike in first-order logic, members $f_{\alpha \rightarrow \beta}^i \in D_{\alpha \rightarrow \beta}^{M_i}$ for some strong chain member M_i embed into the union of the strong chain (of order type σ) as limits $\lim_{x \rightarrow \sigma} \iota^{x,i}(f_{\alpha \rightarrow \beta}^i)$ in which we take repeated compositions of inclusions.

Now there are two main points of interest related to the proofs of Corollary 3.9, i.e., the union general model of a strong chain is a strong extension of all the chain members; and Corollary 3.16, i.e., the higher-order Tarski-Vaught test. First, notice that both are corollaries of more general theorems, which are Proposition 3.8 and Proposition 3.15 respectively. The generality comes from the fact that in Alonzo, formulas are special kinds of expressions/terms of type o , while in first-order logic, formulas and terms are entirely different species. Second, the fact that Alonzo admits undefined expressions required us to have, in some instances, four cases when proving statements by induction on the complexity of expressions. For example, consider (E4) in Proposition 3.15. Notice that Case 1 does not use the assumption of the proposition, while Case 2 does. If we translated the results to a logic that is a version of Church's type theory that *does not* admit undefined expressions, e.g., Peter Andrews' $\mathcal{Q}_0$ [1], we would alter the argument to consider Cases 1 – 4 concurrently, and thus would have to appeal to the assumption and the induction hypothesis simultaneously in a single step of the proof.

In Chapter 4, we defined model-theoretic types for Alonzo and proved some theorems related to them. Notably, we showed that if two sequences in some interpretation M satisfy the same set of $L_{\mathcal{A}}^n$ formulas, then we can construct an automorphism of a strong extension of M that sends one

sequence to the other and fixes all elements otherwise, up to an inclusion $\mathfrak{J}$. We ended off Chapter 4 by showing that the Stone topology with $[\mathbf{A}_o]$ as basic open sets functions similarly as in first-order logic and we also proved a higher-order version of the omitting types theorem. Since the Henkin construction for Church’s type theory is more general than the one in first-order logic, the modified Henkin construction for omitting a particular type is more intricate (see Proposition 4.24) relative to the one in first-order logic.

In Chapter 5, we showed that theories from first-order logic can be captured either syntactically, through a recursive method for capturing the language of first-order logic; or semantically, by considering those theories whose models extend to standard models.

We argue that the results of this thesis are easily translatable to different versions of Church’s type theory (without undefinedness). Translating the work in Chapter 3 related to the Löwenheim-Skolem theorems can be done with little effort: since versions of Church’s type theory share a common syntax, proofs by induction on the complexity of types (e.g., the higher-order Tarski-Vaught test) can be done similarly. The fact that in the proofs of Propositions 3.8 and 3.15 undefinedness forced us to split the cases of the structural induction into subcases – which have corresponding first-order analogues – is a virtue. Many constructions, like the ones involving model-theoretic types in Chapter 4, hinge on the Compactness Theorem [7, Corollary 8.16]; and so, there is no difficulty in extending such results to other versions of Church’s type theory, which have a corresponding theorem due to the Henkin construction. To use $\mathcal{Q}_0$ as an example, [9, Theorems 5501 and 5503] correspond to Henkin’s Theorem (Theorem A.2) and the Compactness Theorem [7, Corollary 8.16] respectively. In addition to the caveats related to inclusions mentioned above, the added complexity in, say, Corollary 4.9 is with regards to the type hierarchy of Alonzo, such that the proof required nested induction. Since versions of Church’s type theory have similar hierarchies of types, this poses no issue in extrapolability.

Finally, this thesis demonstrates the difference between higher-order and first-order model theory. The definitions, lemmas, propositions, and theorems in Chapters 3 and 4 demonstrate the added complexity of having a built-in theory of functions in the logic. As these results utilize the general semantics, they can be thought of as pertaining (more specifically) to a many-sorted first-order theory of functions. In contrast, results related to the standard semantics pertain (more generally) to an ω -order logic.

6.2 Future and Related Work

We list some open problems of interest.

1. The *Hanf number* of a logic $\mathcal{L}$ is the least infinite cardinal κ such that every L -sentence of $\mathcal{L}$ that has a model of size κ has arbitrarily large models. Let $L = (\mathcal{B}, \mathcal{C})$ be a language of Alonzo. The Hanf number of Alonzo with respect to the general semantics is $\omega + \|L\|$, by Theorem 3.14. What is the Hanf number of Alonzo with respect to the standard semantics? We can reduce the problem to proving an equisatisfiability result between sentences of Alonzo and sentences of Hintikka’s version of simple type theory in [10]. Theorems III and IV in [10] imply that sentences of Hintikka’s formulation of simple type theory are equisatisfiable with sentences of second-order logic. We thus conjecture that the Hanf number of Alonzo with the standard semantics is the same as that of second-order logic, which is, by Corollary 5.7 in [18],

$$\sup\{\alpha \mid \alpha \text{ is a } \Sigma_2 \text{ definable ordinal}\}.$$

2. Second-order logic has received a fair amount of attention in the last few years [8, 17, 19]. How translatable are these results to versions of Church's type theory?
3. Continue the development of analogues from first-order logic. Examples include the ultraproduct construction, saturated models, quantifier elimination, and Ehrenfeucht-Fraïssé games.
4. Demonstrate that first-order logic is as strong as Alonzo with the general semantics as per Lindström's theorem.

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Appendix A

Henkin's Theorem for $\mathfrak{A}$

The following is Lemma C.2 taken verbatim from [7].

Lemma A.1 (Extension Lemma). *Let $T = (L, \Gamma)$ be a theory of Alonzo. If T is consistent in $\mathfrak{A}$, then there is a theory $T' = (L', \Gamma')$ such that:*

1. $T \leq T'$.
2. T' is consistent in $\mathfrak{A}$.
3. T' is syntactically complete in $\mathfrak{A}$.
4. T' is extensionally complete in $\mathfrak{A}$.
5. $\|L'\| = \|L\|$.

Proof Let $L = (\mathcal{B}, \mathcal{C})$ and $\kappa = \|L\|$. For each $\alpha \in \mathcal{T}(L)$, let $\mathcal{C}_\alpha$ be a well-ordered set of new constants of type α such that $|\mathcal{C}_\alpha| = \kappa$. Define $L' = (\mathcal{B}, \mathcal{C} \cup \mathcal{C}')$ where

$$\mathcal{C}' = \bigcup_{\alpha \in \mathcal{T}(L)} \mathcal{C}_\alpha.$$

Clearly, $|\mathcal{C}'| = \kappa$, so $\|L'\| = \kappa$, and so $\|L'\| = \|L\|$. Therefore, condition 5 is satisfied.

Well-order the sentences in $\mathcal{E}(L')$ and, for each ordinal $\xi < \kappa$, let $\mathbf{S}_o^\xi$ be the ξ -th sentence of L' in this well-order.

For each ordinal $\xi \leq \kappa$, we will define a set Γ_ξ of sentences of L' by transfinite recursion so that (A) $\zeta \leq \xi$ implies $\Gamma_\zeta \subseteq \Gamma_\xi$ and (B) the cardinality of the set of constants in $\mathcal{C}'$ occurring in the sentences of Γ_ξ is finite if ξ is finite and is less than or equal to the cardinality of ξ if ξ is infinite.

Case 1: $\xi = 0$. Then $\Gamma_0 = \Gamma$.

Case 2: ξ is a successor ordinal $\zeta + 1$. There are three subcases:

Subcase 2.a: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is consistent in $\mathfrak{A}$. Then $\Gamma_{\zeta+1} = \Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$.

Subcase 2.b: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{S}_o^\zeta$ does not have the form $\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. Then $\Gamma_{\zeta+1} = \Gamma_\zeta$.

Subcase 2.c: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{S}_o^\zeta$ has the form $\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. Then

$$\Gamma_{\zeta+1} = \Gamma_\zeta \cup \{\neg(\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha)\}$$

where $\mathbf{c}_\alpha$ is the first constant in $\mathcal{C}_\alpha$ that does not occur in Γ_ζ or $\mathbf{S}_o^\zeta$.

Case 3: ξ is a limit ordinal. Then

$$\Gamma_\xi = \bigcup_{\zeta < \xi} \Gamma_\zeta.$$

It is easy to verify by transfinite induction that conditions (A) and (B) above are satisfied for all ordinals $\xi \leq \kappa$.

We will now prove by transfinite induction that Γ_ξ is consistent in $\mathfrak{A}$ for all ordinals $\xi \leq \kappa$. We have the same three subcases as above:

Case 1: $\xi = 0$. $T = (L, \Gamma)$ is consistent in $\mathfrak{A}$ by assumption. Hence $\Gamma_0 = \Gamma$ must be consistent in $\mathfrak{A}$.

Case 2: ξ is a successor ordinal $\zeta + 1$. We have the same three subcases as above:

Subcase 2.a: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is consistent in $\mathfrak{A}$. Hence $\Gamma_{\zeta+1} = \Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is trivially consistent in $\mathfrak{A}$.

Subcase 2.b: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{S}_o^\zeta$ does not have the form $\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. Hence $\Gamma_{\zeta+1} = \Gamma_\zeta$ is consistent in $\mathfrak{A}$ by the induction hypothesis.

Subcase 2.c: $\Gamma_\zeta \cup \{\mathbf{S}_o^\zeta\}$ is inconsistent in $\mathfrak{A}$ and $\mathbf{S}_o^\zeta$ has the form $\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. Suppose

$$\Gamma_{\zeta+1} = \Gamma_\zeta \cup \{\neg(\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha)\}$$

is inconsistent in $\mathfrak{A}$. Then

$$\Gamma_\zeta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha$$

by the Deduction Theorem [7, Theorem A.50], the notational definition of $\neg$, and the Tautology Rule [7, Corollary A.46]. Let P be a proof of

$$\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha$$

from a finite subset Δ of Γ_ζ , $(\mathbf{x} : \alpha)$ be a variable that does not occur in P or Δ , and P' be the result of replacing each occurrence of $\mathbf{c}_\alpha$ in P with $(\mathbf{x} : \alpha)$. P' is a proof of

$$\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha) \simeq \mathbf{B}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha)$$

from Δ since $\mathbf{c}_\alpha$ does not occur in Γ_ζ , $\mathbf{A}_{\alpha \rightarrow \beta}$, or $\mathbf{B}_{\alpha \rightarrow \beta}$.

$$\Delta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha) \simeq \mathbf{B}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha). \quad (1)$$

$$\Delta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} \downarrow. \quad (2)$$

$$\Delta \vdash_{\mathfrak{A}} \mathbf{B}_{\alpha \rightarrow \beta} \downarrow. \quad (3)$$

$$\Delta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha) \simeq \mathbf{B}_{\alpha \rightarrow \beta} (\mathbf{x} : \alpha). \quad (4)$$

$$\Delta \vdash_{\mathfrak{A}} \forall \mathbf{x} : \alpha. \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{x} \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{x}. \quad (5)$$

$$\Delta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta} \Leftrightarrow \forall \mathbf{x} : \alpha. \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{x} \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{x}. \quad (6)$$

$$\Delta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}. \quad (7)$$

(1) is given; (2), (3), and (4) follow from (1) by the Tautology Rule [7, Corollary A.46]; (5) follows from (4) by Universal Generalization [7, Theorem A.30] since $(\mathbf{x} : \alpha)$ does not occur in Δ ; (6) follows from (2), (3), and Axiom A3 by the Substitution Rule [7, Theorem A.31] and Alpha-Conversion [7, Theorem A.18]; and (7) follows from (6) and (5) by Rule R2' [7, Lemma A.2].

Hence (a) $\Gamma_\zeta \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. However, $\Gamma_\zeta \cup \{\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}\}$ is inconsistent in $\mathfrak{A}$ in Subcase 2.c, and so (b) $\Gamma_\zeta \cup \{\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}\} \vdash_{\mathfrak{A}} F_o$. (a) and (b) imply Γ_ζ is inconsistent in $\mathfrak{A}$, which contradicts the induction hypothesis. Therefore, $\Gamma_{\zeta+1}$ must be consistent in $\mathfrak{A}$.

Case 3: ξ is a limit ordinal. Γ_ζ is consistent in $\mathfrak{A}$ for all $\zeta < \xi$ by the induction hypothesis. Then each finite subset of Γ_ξ is a subset of some Γ_ζ with $\zeta < \xi$, and so Γ_ξ must be consistent in $\mathfrak{A}$.

Define $\Gamma' = \Gamma_\kappa$ and $T' = (L', \Gamma')$. Then $T \leq T'$ and T' is consistent in $\mathfrak{A}$. Therefore, conditions 1 and 2 are satisfied.

Now all we have left to show is that T' is syntactically and extensionally complete in $\mathfrak{A}$. Let $\mathbf{S}_o$ be any sentence of L' . Then $\mathbf{S}_o = \mathbf{S}_o^\zeta$ for some $\zeta < \kappa$.

If $\Gamma_\zeta \cup \{\mathbf{S}_o\}$ is consistent in $\mathfrak{A}$, then $\mathbf{S}_o \in \Gamma_{\zeta+1} \subseteq \Gamma'$ by Subcase 2.a, and so $T' \vdash_{\mathfrak{A}} \mathbf{S}_o$. Otherwise $\Gamma_\zeta \cup \{\mathbf{S}_o\} \vdash_{\mathfrak{A}} F_o$, so $\Gamma_\zeta \vdash_{\mathfrak{A}} \neg \mathbf{S}_o$ by the Deduction Theorem [7, Theorem A.50] and the notational definition of $\neg$, and so $T' \vdash_{\mathfrak{A}} \neg \mathbf{S}_o$. Hence T' is syntactically complete in $\mathfrak{A}$. Therefore, condition 3 is satisfied.

Assume that $\mathbf{S}_o$ has the form $\mathbf{A}_{\alpha \rightarrow \beta} = \mathbf{B}_{\alpha \rightarrow \beta}$. If $\Gamma_\zeta \cup \{\mathbf{S}_o\}$ is consistent in $\mathfrak{A}$, then again $T' \vdash_{\mathfrak{A}} \mathbf{S}_o$, and so

$$T' \vdash_{\mathfrak{A}} (\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow) \Rightarrow (\mathbf{A}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \Rightarrow \mathbf{S}_o)$$

for all expressions $\mathbf{C}_\alpha \in \mathcal{E}(L')$ by the Tautology Rule [7, Corollary A.46]. Notice that there is some expression $\mathbf{C}_\alpha$ that is closed with $T' \vdash_{\mathfrak{A}} \mathbf{C}_\alpha \downarrow$. If $\Gamma_\zeta \cup \{\mathbf{S}_o\}$ is inconsistent in $\mathfrak{A}$, then

$$T' \vdash_{\mathfrak{A}} \neg(\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha)$$

for some $\mathbf{c}_\alpha \in \mathcal{C}'$ by Subcase 2.c, and so

$$T' \vdash_{\mathfrak{A}} (\mathbf{A}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{B}_{\alpha \rightarrow \beta} \downarrow) \Rightarrow (\mathbf{A}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \simeq \mathbf{B}_{\alpha \rightarrow \beta} \mathbf{c}_\alpha \Rightarrow \mathbf{S}_o)$$

by the Tautology Rule [7, Corollary A.46]. Notice that $\mathbf{c}_\alpha$ is closed and $T' \vdash_{\mathfrak{A}} \mathbf{c}_\alpha \downarrow$ by Axiom A5.2. Hence T' is extensionally complete in $\mathfrak{A}$. Therefore, condition 4 is satisfied.

This completes the proof of the Extension Lemma. $\square$

The following is Theorem C.3 taken verbatim from [7].

Theorem A.2 (Henkin's Theorem). *Every theory of Alonzo that is consistent in $\mathfrak{A}$ has a frugal general model.*

Proof Let $T = (L, \Gamma)$ be a theory that is consistent in $\mathfrak{A}$, and let $T' = (L', \Gamma')$ be an extension of T as described in the Extension Lemma. For $\gamma \in \mathcal{T}(L')$, define $\overline{\mathcal{E}_\gamma} = \{\mathbf{A}_\gamma \mid \mathbf{A}_\gamma \in \mathcal{E}(L') \text{ that is closed}\}$.

We will simultaneously define, by recursion on the syntactic structure of the types in $\mathcal{T}(L')$, a frame $\mathcal{D} = \{D_\gamma \mid \gamma \in \mathcal{T}(L')\}$ and a partial function V on the closed expressions in $\mathcal{E}(L')$ so that the following conditions hold for all $\gamma \in \mathcal{T}(L')$:

(1^γ) $D_\gamma = \{V(\mathbf{A}_\gamma) \mid \mathbf{A}_\gamma \in \overline{\mathcal{E}_\gamma} \text{ and } V(\mathbf{A}_\gamma) \text{ is defined}\}.$

(2^γ) $V(\mathbf{A}_\gamma)$ is defined iff $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\gamma \downarrow$ for all $\mathbf{A}_\gamma \in \overline{\mathcal{E}_\gamma}.$

(3^γ) $V(\mathbf{A}_\gamma) = V(\mathbf{B}_\gamma)$ iff $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\gamma = \mathbf{B}_\gamma$ for all $\mathbf{A}_\gamma, \mathbf{B}_\gamma \in \overline{\mathcal{E}_\gamma}.$

Case 1: $\gamma = o$. Define $D_o = \{\mathbf{F}, \mathbf{T}\}$. For each $\mathbf{A}_o \in \overline{\mathcal{E}_o}$, define $V(\mathbf{A}_o) = \mathbf{T}$ if $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_o$ and $V(\mathbf{A}_o) = \mathbf{F}$ if $\Gamma' \vdash_{\mathfrak{A}} \neg \mathbf{A}_o$. By the syntactic completeness of T' in $\mathfrak{A}$, exactly one of $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_o$ and $\Gamma' \vdash_{\mathfrak{A}} \neg \mathbf{A}_o$ holds. Then the definition of V on $\overline{\mathcal{E}_o}$ is well-defined, (a) V is total on $\overline{\mathcal{E}_o}$, and (1^o) is satisfied. (b) $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_o \downarrow$ for all $\mathbf{A}_o \in \overline{\mathcal{E}_o}$ by Proposition [7, Proposition A.8]. (a) and (b) imply (2^o) is satisfied. (3^o) is satisfied by the definition of V on $\overline{\mathcal{E}_o}$ and the Tautology Rule [7, Corollary A.46]. Therefore, (1^o), (2^o), and (3^o) are satisfied.

Case 2: $\gamma = \mathbf{a}$. For each $\mathbf{A}_\mathbf{a} \in \overline{\mathcal{E}_\mathbf{a}}$, define

$$V(\mathbf{A}_\mathbf{a}) = \{\mathbf{B}_\mathbf{a} \mid \mathbf{B}_\mathbf{a} \in \overline{\mathcal{E}_\mathbf{a}} \text{ and } \Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\mathbf{a} = \mathbf{B}_\mathbf{a}\}$$

if $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\mathbf{a} \downarrow$, and otherwise define $V(\mathbf{A}_\mathbf{a})$ to be undefined. Define

$$D_\mathbf{a} = \{V(\mathbf{A}_\mathbf{a}) \mid \mathbf{A}_\mathbf{a} \in \overline{\mathcal{E}_\mathbf{a}} \text{ and } V(\mathbf{A}_\mathbf{a}) \text{ is defined}\}.$$

The definitions of $D_\mathbf{a}$ and V on $\overline{\mathcal{E}_\mathbf{a}}$ obviously satisfy (1^a) and (2^a). They also satisfy (3^a) since $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\mathbf{a} = \mathbf{B}_\mathbf{a}$ is an equivalence relation over $\overline{\mathcal{E}_\mathbf{a}}$. Notice that $D_\mathbf{a}$ is nonempty by (2^a) and the extensional completeness of T' in $\mathfrak{A}$.

Case 3: $\gamma = \alpha \rightarrow \beta$. For each $\mathbf{F}_{\alpha \rightarrow \beta} \in \overline{\mathcal{E}_{\alpha \rightarrow \beta}}$, define $V(\mathbf{F}_{\alpha \rightarrow \beta})$ to be undefined if $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} \uparrow$ and otherwise define $V(\mathbf{F}_{\alpha \rightarrow \beta})$ to be the (partial or total) function from D_α to D_β whose value at an argument $V(\mathbf{A}_\alpha) \in D_\alpha$ is $V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ if $V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ is defined and is undefined if $V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha)$ is undefined. We must show that this definition does not depend on the choice of the particular closed $\mathbf{A}_\alpha$ to represent the argument. If $V(\mathbf{A}_\alpha) = V(\mathbf{B}_\alpha)$, then $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha = \mathbf{B}_\alpha$ by (3^α), and so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha \simeq \mathbf{F}_{\alpha \rightarrow \beta} \mathbf{B}_\alpha$ by part 5 of the Equality Rules [7, Lemma A.13], and so $V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) \simeq V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{B}_\alpha)$ by (2^β), (3^β), and the definition of $\simeq$. Finally, define

$$D_{\alpha \rightarrow \beta} = \{V(\mathbf{A}_{\alpha \rightarrow \beta}) \mid \mathbf{A}_{\alpha \rightarrow \beta} \in \overline{\mathcal{E}_{\alpha \rightarrow \beta}} \text{ and } V(\mathbf{A}_{\alpha \rightarrow \beta}) \text{ is defined}\}.$$

(1^{α→β}) and (2^{α→β}) are satisfied by the definitions of $D_{\alpha \rightarrow \beta}$ and V on $\overline{\mathcal{E}_{\alpha \rightarrow \beta}}$. We will now show that (3^{α→β}) is satisfied. Suppose $V(\mathbf{F}_{\alpha \rightarrow \beta}) = V(\mathbf{G}_{\alpha \rightarrow \beta})$. Then $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} \downarrow$ and $\Gamma' \vdash_{\mathfrak{A}} \mathbf{G}_{\alpha \rightarrow \beta} \downarrow$ by (2^{α→β}). Since T' is extensionally complete in $\mathfrak{A}$, there is some $\mathbf{C}_\alpha \in \overline{\mathcal{E}_\alpha}$ such that $\Gamma' \vdash_{\mathfrak{A}} \mathbf{C}_\alpha \downarrow$ and

$$\Gamma' \vdash_{\mathfrak{A}} (\mathbf{F}_{\alpha \rightarrow \beta} \downarrow \wedge \mathbf{G}_{\alpha \rightarrow \beta} \downarrow) \Rightarrow (\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \simeq \mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \Rightarrow \mathbf{F}_{\alpha \rightarrow \beta} = \mathbf{G}_{\alpha \rightarrow \beta}).$$

Then

$$V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha) \simeq V(\mathbf{F}_{\alpha \rightarrow \beta})(V(\mathbf{C}_\alpha)) \simeq V(\mathbf{G}_{\alpha \rightarrow \beta})(V(\mathbf{C}_\alpha)) \simeq V(\mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha);$$

so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \simeq \mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha$ by (2^β), (3^β), and the definition of $\simeq$; and so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} = \mathbf{G}_{\alpha \rightarrow \beta}$ by the Tautology Rule [7, Corollary A.46]. Now suppose $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} = \mathbf{G}_{\alpha \rightarrow \beta}$. Then, for all $\mathbf{C}_\alpha \in \overline{\mathcal{E}_\alpha}$,

$$\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha \simeq \mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha$$

by part 4 of the Equality Rules [7, Lemma A.13]; so $V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha) \simeq V(\mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha)$ by (2^β), (3^β), and the definition of $\simeq$; and so

$$V(\mathbf{F}_{\alpha \rightarrow \beta})(V(\mathbf{C}_\alpha)) \simeq V(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha) \simeq V(\mathbf{G}_{\alpha \rightarrow \beta} \mathbf{C}_\alpha) \simeq V(\mathbf{G}_{\alpha \rightarrow \beta})(V(\mathbf{C}_\alpha)).$$

Hence $V(\mathbf{F}_{\alpha \rightarrow \beta}) = V(\mathbf{G}_{\alpha \rightarrow \beta})$ by the definition of V on $\overline{\mathcal{E}_{\alpha \rightarrow \beta}}$. Therefore, $(3^{\alpha \rightarrow \beta})$ is satisfied, and this is the end of Case 3.

Case 4: $\gamma = \alpha \times \beta$. Define $D_{\alpha \times \beta} = D_\alpha \times D_\beta$. For each $\mathbf{A}_{\alpha \times \beta} \in \overline{\mathcal{E}_{\alpha \times \beta}}$, define

$$V(\mathbf{A}_{\alpha \times \beta}) = (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta}), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta}))$$

if (a) $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \times \beta} \downarrow$, and otherwise define $V(\mathbf{A}_{\alpha \times \beta})$ to be undefined. (a) implies $\Gamma' \vdash_{\mathfrak{A}} (\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta}) \downarrow$ and $\Gamma' \vdash_{\mathfrak{A}} (\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta}) \downarrow$ by Axioms A5.5, A7.2, A7.3, and A7.4 and Universal Instantiation [7, Theorem A.14], so $V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta})$ and $V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta})$ are defined by (2^α) and (2^β) . Hence the definition of V on $\overline{\mathcal{E}_{\alpha \times \beta}}$ is well-defined.

$(2^{\alpha \times \beta})$ is obviously satisfied by the definitions of $D_{\alpha \times \beta}$ and V on $\overline{\mathcal{E}_{\alpha \times \beta}}$.

Let $D = \{V(\mathbf{A}_\gamma) \mid \mathbf{A}_\gamma \in \overline{\mathcal{E}_{\alpha \times \beta}} \text{ and } V(\mathbf{A}_\gamma) \text{ is defined}\}$. We must show that $D_{\alpha \times \beta} = D$ to show that $(1^{\alpha \times \beta})$ is satisfied. Let $p \in D_{\alpha \times \beta}$. Then $p = (a, b)$ for some $a \in D_\alpha$ and $b \in D_\beta$, and so $a = V(\mathbf{A}_\alpha)$ and $b = V(\mathbf{B}_\beta)$ for some $\mathbf{A}_\alpha \in \overline{\mathcal{E}_\alpha}$ and $\mathbf{B}_\beta \in \overline{\mathcal{E}_\beta}$ by (1^α) and (1^β) , respectively. Then

$$\begin{aligned} & (V(\mathbf{A}_\alpha), V(\mathbf{B}_\beta)) \\ &= (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} (\mathbf{A}_\alpha, \mathbf{B}_\beta)), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} (\mathbf{A}_\alpha, \mathbf{B}_\beta))) \end{aligned} \tag{1}$$

$$= V((\mathbf{A}_\alpha, \mathbf{B}_\beta)). \tag{2}$$

(1) is by (3^α) , (3^β) , and Lemma [7, Lemma A.54]; and (2) is by the definition of V on $\overline{\mathcal{E}_{\alpha \times \beta}}$ since $V((\mathbf{A}_\alpha, \mathbf{B}_\beta))$ is defined by (2^α) , (2^β) , $(2^{\alpha \times \beta})$, and Axiom A7.1. Hence $p \in D$, and so $D_{\alpha \times \beta} \subseteq D$. Now let $p \in D$. Then $p = V(\mathbf{A}_{\alpha \times \beta})$ for some $\mathbf{A}_{\alpha \times \beta} \in \overline{\mathcal{E}_{\alpha \times \beta}}$, and so

$$p = (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta}), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta})).$$

Hence $p \in D_{\alpha \times \beta}$ by (1^α) and (1^β) , and so $D \subseteq D_{\alpha \times \beta}$. Therefore, $(1^{\alpha \times \beta})$ is satisfied.

Now we will show that $(3^{\alpha \times \beta})$ is satisfied.

$$\begin{aligned} & V(\mathbf{A}_{\alpha \times \beta}) = V(\mathbf{B}_{\alpha \times \beta}) \\ \text{iff } & (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta}), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta})) = \\ & (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{B}_{\alpha \times \beta}), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{B}_{\alpha \times \beta})) \end{aligned} \tag{1}$$

$$\begin{aligned} \text{iff } & \Gamma' \vdash_{\mathfrak{A}} \text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta} = \text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{B}_{\alpha \times \beta} \text{ and} \\ & \Gamma' \vdash_{\mathfrak{A}} \text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta} = \text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{B}_{\alpha \times \beta} \end{aligned} \tag{2}$$

$$\begin{aligned} \text{iff } & \Gamma' \vdash_{\mathfrak{A}} (\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \times \beta}, \text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{A}_{\alpha \times \beta}) = \\ & (\text{fst}_{(\alpha \times \beta) \rightarrow \alpha} \mathbf{B}_{\alpha \times \beta}, \text{snd}_{(\alpha \times \beta) \rightarrow \beta} \mathbf{B}_{\alpha \times \beta}) \end{aligned} \tag{3}$$

$$\text{iff } \Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_{\alpha \times \beta} = \mathbf{B}_{\alpha \times \beta}. \tag{4}$$

(1) is by the definition of V on $\overline{\mathcal{E}_{\alpha \times \beta}}$; (2) is by (3^α) and (3^β) ; (3) is by Axioms A5.4, A5.5, A7.2, A7.3, and A7.5, the Equality Rules [7, Lemma A.13], Universal Instantiation [7, Theorem A.14], and the Tautology Rule [7, Corollary A.46]; (4) is by $(2^{\alpha \times \beta})$, Axiom A7.4, Universal Instantiation [7, Theorem A.14], and the Equality Rules [7, Lemma A.13]. Therefore, $(3^{\alpha \times \beta})$ is satisfied, and this is the end of Case 4.

Therefore, we have shown that $\mathcal{D}$ is a frame and conditions (1^γ) , (2^γ) , and (3^γ) are satisfied for all $\gamma \in \mathcal{T}(L')$. V clearly maps each constant of L' of type γ to a member of D_γ . Hence $M' = (\mathcal{D}, I)$ is an interpretation of L' where $I = V \upharpoonright_{\mathcal{C}'}$ and $\mathcal{C}'$ is the set of constants of L' .

We will next show that M' is a general model of L' . Choose a well-order of $\mathcal{E}(L')$ and, for each $\varphi \in \text{assign}(M')$ and variable $(\mathbf{x} : \alpha) \in \mathcal{E}(L')$, let $\theta_\varphi((\mathbf{x} : \alpha))$ be the first closed expression $\mathbf{E}_\alpha$ in this well-order such that $\varphi((\mathbf{x} : \alpha)) = V(\mathbf{E}_\alpha)$. For each $\varphi \in \text{assign}(M')$ and $\mathbf{C}_\gamma \in \mathcal{E}(L')$, let $(\mathbf{C}_\gamma)^\varphi = \mathbf{C}_\gamma^\varphi$ be the expression obtained by simultaneously replacing $(\mathbf{x}_i : \alpha_i)$ with $\theta_\varphi((\mathbf{x}_i : \alpha_i))$ for all i with $1 \leq i \leq n$ where $(\mathbf{x}_1 : \alpha_1), \dots, (\mathbf{x}_n : \alpha_n)$ are the free variables in $\mathbf{C}_\gamma$. Define $V_\varphi^{M'}(\mathbf{C}_\gamma) = V(\mathbf{C}_\gamma^\varphi)$ if $V(\mathbf{C}_\gamma^\varphi)$ is defined, and otherwise define $V_\varphi^{M'}(\mathbf{C}_\gamma)$ to be undefined. Clearly, $\mathbf{C}_\gamma^\varphi \in \overline{\mathcal{E}_\gamma}$, and so $V_\varphi^{M'}(\mathbf{C}_\gamma) \in D_\gamma$ if $V_\varphi^{M'}(\mathbf{C}_\gamma)$ is defined. We are now ready to show that each of the seven conditions of the definition of a general model is satisfied.

1. Let $\mathbf{C}_\gamma$ be a variable $(\mathbf{x} : \alpha)$. Then

$$V_\varphi^{M'}(\mathbf{C}_\gamma) = V_\varphi^{M'}((\mathbf{x} : \alpha)) = V((\mathbf{x} : \alpha)^\varphi) = V(\theta_\varphi((\mathbf{x} : \alpha))) = \varphi((\mathbf{x} : \alpha)).$$

2. Let $\mathbf{C}_\gamma$ be a constant $\mathbf{c}_\alpha$. Then $\mathbf{c}_\alpha \in \overline{\mathcal{E}_\alpha}$, and so

$$V_\varphi^{M'}(\mathbf{C}_\gamma) = V_\varphi^{M'}(\mathbf{c}_\alpha) = V((\mathbf{c}_\alpha)^\varphi) = V(\mathbf{c}_\alpha) = I(\mathbf{c}_\alpha).$$

3. Let $\mathbf{C}_\gamma$ be an equality $\mathbf{A}_\alpha = \mathbf{B}_\alpha$. Then

$$V_\varphi^{M'}(\mathbf{C}_\gamma) = V_\varphi^{M'}(\mathbf{A}_\alpha = \mathbf{B}_\alpha) = V(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi).$$

There are three cases to consider:

- a. $V_\varphi^{M'}(\mathbf{A}_\alpha)$ and $V_\varphi^{M'}(\mathbf{B}_\alpha)$ are defined with $V_\varphi^{M'}(\mathbf{A}_\alpha) = V_\varphi^{M'}(\mathbf{B}_\alpha)$. Then $V(\mathbf{A}_\alpha^\varphi) = V(\mathbf{B}_\alpha^\varphi)$, so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi$ by (3 $^\alpha$), and so $V(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi) = \text{T}$ by the definition of V on $\overline{\mathcal{E}_o}$. Hence $V_\varphi^{M'}(\mathbf{C}_\gamma) = \text{T}$.
 - b. $V_\varphi^{M'}(\mathbf{A}_\alpha)$ and $V_\varphi^{M'}(\mathbf{B}_\alpha)$ are defined with $V_\varphi^{M'}(\mathbf{A}_\alpha) \neq V_\varphi^{M'}(\mathbf{B}_\alpha)$. Then $V(\mathbf{A}_\alpha^\varphi) \neq V(\mathbf{B}_\alpha^\varphi)$, so $\Gamma' \vdash_{\mathfrak{A}} \neg(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi)$ by (3 $^\alpha$), and so $V(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi) = \text{F}$ by the definition of V on $\overline{\mathcal{E}_o}$. Hence $V_\varphi^{M'}(\mathbf{C}_\gamma) = \text{F}$.
 - c. $V_\varphi^{M'}(\mathbf{A}_\alpha)$ or $V_\varphi^{M'}(\mathbf{B}_\alpha)$ is undefined. Then $V(\mathbf{A}_\alpha^\varphi)$ or $V(\mathbf{B}_\alpha^\varphi)$ is undefined, so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha \uparrow$ or $\Gamma' \vdash_{\mathfrak{A}} \mathbf{B}_\alpha \uparrow$ by (2 $^\alpha$), so $\Gamma' \vdash_{\mathfrak{A}} \neg(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi)$ by Axioms A5.4 or A5.5 and the Tautology Rule [7, Corollary A.46], and so $V(\mathbf{A}_\alpha^\varphi = \mathbf{B}_\alpha^\varphi) = \text{F}$ by the definition of V on $\overline{\mathcal{E}_o}$. Hence $V_\varphi^{M'}(\mathbf{C}_\gamma) = \text{F}$.
4. Let $\mathbf{C}_\gamma$ be a function application $\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha$. There are two cases to consider:

- a. $V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta})$ and $V_\varphi^{M'}(\mathbf{A}_\alpha)$ are defined. Then $V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi)$ and $V(\mathbf{A}_\alpha^\varphi)$ are defined, and so

$$\begin{aligned} V_\varphi^{M'}(\mathbf{C}_\gamma) &\simeq V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) \simeq V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi \mathbf{A}_\alpha^\varphi) \simeq \\ &V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi)(V(\mathbf{A}_\alpha^\varphi)) \simeq V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta})(V_\varphi^{M'}(\mathbf{A}_\alpha)). \end{aligned}$$

Hence, if $V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta})$ is defined at $V_\varphi^{M'}(\mathbf{A}_\alpha)$, then $V_\varphi^{M'}(\mathbf{C}_\gamma) = V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta})(V_\varphi^{M'}(\mathbf{A}_\alpha))$, and otherwise $V_\varphi^{M'}(\mathbf{C}_\gamma)$ is undefined.

- b. $V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta})$ or $V_\varphi^{M'}(\mathbf{A}_\alpha)$ is undefined. Then $V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi)$ or $V(\mathbf{A}_\alpha^\varphi)$ is undefined, and so (a) $\Gamma' \vdash_{\mathfrak{A}} \mathbf{F}_{\alpha \rightarrow \beta}^\varphi \uparrow$ by $(2^{\alpha \rightarrow \beta})$ or (b) $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha^\varphi \uparrow$ by (2^α) . If $\beta = o$, then $\Gamma' \vdash_{\mathfrak{A}} \neg(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi \mathbf{A}_\alpha^\varphi)$ follows from (a) or (b) by Axioms A5.7 and A5.8 and the Tautology Rule [7, Corollary A.46], and so

$$V_\varphi^{M'}(\mathbf{C}_\gamma) = V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) = V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi \mathbf{A}_\alpha^\varphi) = \mathbf{F}$$

by the definition of V on $\overline{\mathcal{E}_o}$. If $\beta \neq o$, then $\Gamma' \vdash_{\mathfrak{A}} (\mathbf{F}_{\alpha \rightarrow \beta}^\varphi \mathbf{A}_\alpha^\varphi) \uparrow$ follows from (a) or (b) by Axioms A5.9 and A5.10 and the Tautology Rule [7, Corollary A.46], and so

$$V_\varphi^{M'}(\mathbf{C}_\gamma) \simeq V_\varphi^{M'}(\mathbf{F}_{\alpha \rightarrow \beta} \mathbf{A}_\alpha) \simeq V(\mathbf{F}_{\alpha \rightarrow \beta}^\varphi \mathbf{A}_\alpha^\varphi)$$

is undefined by (2^β) .

5. Let $\mathbf{C}_\gamma$ be a function abstraction $(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)$. Let $V(\mathbf{A}_\alpha) \in D_\alpha$ and $\psi = \varphi[(\mathbf{x} : \alpha) \mapsto V(\mathbf{A}_\alpha)]$. Notice that (a) $\mathbf{A}_\alpha$ is closed and (b) $V(\mathbf{A}_\alpha)$ is defined. Then

$$\begin{aligned} V_\varphi^{M'}(\mathbf{C}_\gamma)(V(\mathbf{A}_\alpha)) \\ \simeq V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta)(V(\mathbf{A}_\alpha)) \end{aligned} \tag{1}$$

$$\simeq V(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta^\varphi)(V(\mathbf{A}_\alpha)) \tag{2}$$

$$\simeq V((\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta^\varphi) \mathbf{A}_\alpha) \tag{3}$$

$$\simeq V(\mathbf{B}_\beta^\psi) \tag{4}$$

$$\simeq V_\psi^{M'}(\mathbf{B}_\beta). \tag{5}$$

(1) is given; (2) and (5) are by the definition of $V_\varphi^{M'}$; (3) follows from (a), (b), Axiom A5.11, and the definition of V on $\overline{\mathcal{E}_{\alpha \rightarrow \beta}}$; and (4) follows from (a), (b), Axiom A4, (2^α) , (2^β) , (3^β) , and the definition of $\simeq$. Hence the condition is satisfied.

6. Let $\mathbf{C}_\gamma$ be a definite description $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o)$. Without loss of generality, we may assume that $(y : \alpha)$ is distinct from $(\mathbf{x} : \alpha)$ and does not occur in $\mathbf{A}_o$. There are two cases to consider:

- a. $V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{A}_o) = V_{\varphi[(y:\alpha) \mapsto d]}^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{x} = (y : \alpha))$ for some $d \in D_\alpha$. Let $\mathbf{B}_\alpha$ be the first member of $\overline{\mathcal{E}_\alpha}$ in the well-order of $\mathcal{E}(L')$ such that $d = V(\mathbf{B}_\alpha)$. Notice that (a) $\mathbf{B}_\alpha$

is closed and (b) $V(\mathbf{B}_\alpha)$ is defined.

$$V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{A}_o) = V_{\varphi[(y:\alpha) \mapsto V(\mathbf{B}_\alpha)]}^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{x} = (y : \alpha)). \quad (1)$$

$$V(\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = V(\lambda \mathbf{x} : \alpha . \mathbf{x} = \mathbf{B}_\alpha). \quad (2)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = (\lambda \mathbf{x} : \alpha . \mathbf{x} = \mathbf{B}_\alpha). \quad (3)$$

$$\Gamma' \vdash_{\mathfrak{A}} \exists y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = (\lambda \mathbf{x} : \alpha . \mathbf{x} = y). \quad (4)$$

$$\Gamma' \vdash_{\mathfrak{A}} \exists! \mathbf{x} : \alpha . \mathbf{A}_o^\varphi. \quad (5)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) (\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi). \quad (6)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \downarrow. \quad (7)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\lambda \mathbf{x} : \alpha . \mathbf{x} = \mathbf{B}_\alpha) (\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi). \quad (8)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = \mathbf{B}_\alpha. \quad (9)$$

$$V(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = V(\mathbf{B}_\alpha). \quad (10)$$

$$V_\varphi^{M'}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o) = V(\mathbf{B}_\alpha). \quad (11)$$

(1) is given; (2) follows from (1) by the definition of $V_\varphi^{M'}$; (3) follows from (2) by $(3^{\alpha \rightarrow o})$; (4) follows from (a), (b), and (3) by Existential Generalization [7, Theorem A.51] and (2^α) ; (5) follows from (4) by the notational definition of $\exists!$; (6) follows from (5) and Axiom A6.1 by Rule R1' [7, Lemma A.1] and the definition of $\in$; (7) follows from (6) and Axiom A5.8 by Rule R1' [7, Lemma A.1]; (8) follows from (3) and (6) by Rule R2' [7, Lemma A.2]; (9) follows from (7) and (8) by Beta-Reduction Rule 1 [7, Lemma A.6]; (10) follows from (9) by (3^α) ; and (11) follows from (10) by the definition of $V_\varphi^{M'}$. Hence the condition is satisfied for this case.

- b. $V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{A}_o) \neq V_{\varphi[(y:\alpha) \mapsto d]}^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{x} = (y : \alpha))$ for all $d \in D_\alpha$. Let $V(\mathbf{B}_\alpha) \in D_\alpha$.

Notice that (a) $\mathbf{B}_\alpha$ is closed and (b) $V(\mathbf{B}_\alpha)$ is defined.

$$V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{A}_o) \neq V_\varphi^{M'}(\lambda \mathbf{x} : \alpha . \mathbf{x} = (y : \alpha)). \quad (1)$$

$$V(\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq V(\lambda \mathbf{x} : \alpha . \mathbf{x} = \mathbf{B}_\alpha). \quad (2)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = \mathbf{B}_\alpha). \quad (3)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\lambda y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)) \mathbf{B}_\alpha. \quad (4)$$

$$V((\lambda y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)) \mathbf{B}_\alpha) = \mathbf{T}. \quad (5)$$

$$V(\lambda y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y))(V(\mathbf{B}_\alpha)) = \mathbf{T}. \quad (6)$$

$$\begin{aligned} V(\lambda x : \alpha . T_o) = \\ V(\lambda y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)). \end{aligned} \quad (7)$$

$$\begin{aligned} \Gamma' \vdash_{\mathfrak{A}} (\lambda x : \alpha . T_o) = \\ (\lambda y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)). \end{aligned} \quad (8)$$

$$\Gamma' \vdash_{\mathfrak{A}} \forall y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \neq (\lambda \mathbf{x} : \alpha . \mathbf{x} = y). \quad (9)$$

$$\Gamma' \vdash_{\mathfrak{A}} \neg(\exists y : \alpha . (\lambda \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) = (\lambda \mathbf{x} : \alpha . \mathbf{x} = y)). \quad (10)$$

$$\Gamma' \vdash_{\mathfrak{A}} \neg \exists! \mathbf{x} : \alpha . \mathbf{A}_o^\varphi. \quad (11)$$

$$\Gamma' \vdash_{\mathfrak{A}} (\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \uparrow. \quad (12)$$

$$V(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o^\varphi) \text{ is undefined.} \quad (13)$$

$$V_\varphi^{M'}(\mathbf{I} \mathbf{x} : \alpha . \mathbf{A}_o) \text{ is undefined.} \quad (14)$$

(1) is given; (2) follows from (1) by the definition of $V_\varphi^{M'}$; (3) follows from (2) by $(3^{\alpha \rightarrow o})$; (4) follows from (a), (b), (3), and axiom A4 by Rule R2' [7, Lemma A.2]; (5) follows from (4) by the definition of V on $\overline{\mathcal{E}_{\alpha \rightarrow o}}$; (6) follows from (5) by the definition of V on $\overline{\mathcal{E}_{\alpha \rightarrow o}}$; (7) follows from (6) since $V(\mathbf{B}_\alpha)$ has been arbitrarily chosen; (8) follows from (7) by $(3^{\alpha \rightarrow o})$; (9) follows from (8) by the notational definition of $\forall$; (10) follows from (9) by the Tautology Rule [7, Corollary A.46] and the notational definition of $\exists$; (11) follows from (10) by the notational definition of $\exists!$; (12) follows from (11) and Axiom A6.2 by Rule R1' [7, Lemma A.1]; (13) follows from (12) by (2^α) ; and (14) follows from (13) by the definition of $V_\varphi^{M'}$. Hence the condition is satisfied for this case.

7. Let $\mathbf{C}_\gamma$ be an ordered pair $(\mathbf{A}_\alpha, \mathbf{B}_\beta)$. Then

$$V_\varphi^{M'}(\mathbf{C}_\gamma) \simeq V_\varphi^{M'}((\mathbf{A}_\alpha, \mathbf{B}_\beta)) \simeq V((\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi)).$$

There are two cases to consider:

- a. $V_\varphi^{M'}(\mathbf{A}_\alpha)$ and $V_\varphi^{M'}(\mathbf{B}_\beta)$ are defined. Then $V(\mathbf{A}_\alpha^\varphi)$ and $V(\mathbf{B}_\beta^\varphi)$ are defined, so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha^\varphi \downarrow$ and $\Gamma' \vdash_{\mathfrak{A}} \mathbf{B}_\beta^\varphi \downarrow$ by (2^α) and (2^β) , and so $\Gamma' \vdash_{\mathfrak{A}} (\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi) \downarrow$ is defined by Axiom A7.1 and Universal Instantiation [7, Theorem A.14].

$$\begin{aligned} & V((\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi)) \\ &= (V(\text{fst}_{(\alpha \times \beta) \rightarrow \alpha}(\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi)), V(\text{snd}_{(\alpha \times \beta) \rightarrow \beta}(\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi))) \end{aligned} \quad (1)$$

$$= (V(\mathbf{A}_\alpha^\varphi), V(\mathbf{B}_\beta^\varphi)) \quad (2)$$

$$= (V_\varphi^{M'}(\mathbf{A}_\alpha), V_\varphi^{M'}(\mathbf{B}_\beta)). \quad (3)$$

(1) is by the definition of V on $\overline{\mathcal{E}_{\alpha \times \beta}}$; (2) is by (3^α) , (3^β) , and Lemma [7, Lemma A.54]; and (3) is by the definition of $V_\varphi^{M'}$. Hence $V_\varphi^{M'}(\mathbf{C}_\gamma) = (V_\varphi^{M'}(\mathbf{A}_\alpha), V_\varphi^{M'}(\mathbf{B}_\beta))$.

- b. $V_\varphi^{M'}(\mathbf{A}_\alpha)$ or $V_\varphi^{M'}(\mathbf{B}_\beta)$ is undefined. Then $V(\mathbf{A}_\alpha^\varphi)$ or $V(\mathbf{B}_\beta^\varphi)$ is undefined, so $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_\alpha^\varphi \uparrow$ by (2^α) or $\Gamma' \vdash_{\mathfrak{A}} \mathbf{B}_\beta^\varphi \uparrow$ by (2^β) , so $\Gamma' \vdash_{\mathfrak{A}} (\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi) \uparrow$ is undefined by Axioms A7.2 or A7.3 and the Tautology Rule [7, Corollary A.46], and so $V((\mathbf{A}_\alpha^\varphi, \mathbf{B}_\beta^\varphi))$ is undefined by $(2^{\alpha \times \beta})$. Hence $V_\varphi^{M'}(\mathbf{C}_\gamma)$ is undefined.

Therefore, M' is a general model of L' .

Let $\mathbf{A}_o \in \Gamma'$. Then $\mathbf{A}_o \in \overline{\mathcal{E}_o}$ and $\Gamma' \vdash_{\mathfrak{A}} \mathbf{A}_o$, and so $V(\mathbf{A}_o) = \top$. Hence $V_\varphi^{M'}(\mathbf{A}_o) = V(\mathbf{A}_o^\varphi) = V(\mathbf{A}_o) = \top$ for all $\varphi \in \text{assign}(M')$, and so $M' \models \mathbf{A}_o$. Therefore, M' is a model of T' , and so the reduct M of M' to L is a model of T .

For all $\gamma \in \mathcal{T}(L')$, $|D_\gamma| \leq |\overline{\mathcal{E}_\gamma}| \leq |\mathcal{E}(L')| = \|L'\|$ since V maps $\overline{\mathcal{E}_\gamma}$ onto D_γ , and so $\|M'\| \leq \|L'\|$. However, $\|L'\| = \|L\|$ by assumption, and so $\|M\| = \|M'\| \leq \|L\|$. Therefore, M is frugal.

At last, this completes the proof of Henkin's Theorem for $\mathfrak{A}$! □