RESIDUALLY DOMINATED GROUPS

RESIDUALLY DOMINATED GROUPS IN HENSELIAN VALUED FIELDS OF EQUICHARACTERISTIC ZERO

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Abstract

We study the model theory of henselian valued fields of equicharacteristic zero by generalizing results from the complete theory of algebraically closed valued fields (ACVF) in the literature. Haskell, Hrushovski and Macpherson introduced the notion of *stable domination* which provides a tameness condition to study the model theory of ACVF. It was later generalized to *residual domination* in the work by Haskell, Ealy and Simon, and independently by Vicaria for pure henselian valued fields of equicharacteristic zero.

In this thesis, we study residual domination itself, giving characterizations that are similar to those for stably dominated types in ACVF. We then introduce *residually dominated groups*, which are the analogue of the *stably dominated groups* introduced and studied extensively by Rideau-Kikuchi and Hrushovski. We show that the connected components of residually dominated groups are subgroups of stably dominated groups that are definable in the algebraic closure of the given henselian valued field. This allows us to extend results from ACVF to the henselian setting.

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Chapter 1

Introduction

Model theory is a branch of mathematical logic that studies mathematical structures through their properties expressible in a first-order language. Shelah's *classification theory* ([She78]) arranges complete theories along several *dividing lines* according to the complexity of their first-order behaviour, separating the *tame* from the *wild*. Stable theories, for example the theories of vector spaces and algebraically closed fields, lie in the tame end, while the complete theory of integers, as shown by Gödel's Incompleteness Theorems, exhibits extremely wild behaviour. Between these extremes, there are several dividing lines in the tame side, such as simplicity, NIP and NTP_2.

Henselian valued fields provide natural examples for many classes of theories on the tame side. Hence, it is tempting for model theorists to study them to shed light on the complexities of a class of theories. Examples include algebraically closed valued fields, real closed valued fields and, more generally, Hahn fields.

A striking fact about henselian fields is that, although their theories may be complicated, many of their model-theoretic properties are controlled by two simpler components: the residue field and the value group. This was proved by Ax and Kochen in [AK65], and independently by Ershov in [Es65], in terms of first-order properties. This insight is generally referred to in the literature as the AKE-principle.

One research direction in model theory is to generalize results known for stable theories to unstable tame theories. Inspired by the theory of algebraically closed valued fields, which is NIP, Haskell, Hrushovski, and Macpherson introduced the notion of stably dominated types, which are the types that are controlled by their traces on sorts whose induced theories are stable. In algebraically closed valued fields, stably dominated types are characterized by their orthogonality to the value group, meaning that any definable interaction with the value group is trivial. The stable sorts are those internal to the residue field; that is, they lie in the image of a definable map whose domain is the residue field.

This idea was later generalized to henselian valued fields as *residual domination* (also referred to as *residue field domination*), first for real closed valued fields in [EHM19], and later for pure henselian valued fields of equicharacteristic zero in [EHS23] and [Vic22] (see Definition 2.1.20).

One part to understand the complexity of a theory is to analyze definable groups in it. In stable theories definable groups have been studied extensively using the well-behaved notions of generic types, connected components, stabilizers of types, etc. Outside of the stable setting, several generalizations of these tools have been studied. For example, In [HRK19], Hrushovski and Rideau-Kikuchi introduce *stably dominated groups* which are definable groups that are controlled by a stably dominated type, and they have similar properties as groups in a stable theory have. Moreover they show an AKE-type result, where they show that a definable abelian group can be decomposed into a *limit stably dominated group* which is a direct union of stably dominated groups uniformly parametrized by elements in the value group Γ and into a group that is internal to Γ.

In this thesis, we aim to generalize results on stably dominated types and groups from the setting of algebraically closed valued fields to the more general henselian setting. Given a type in the henselian field K, one can also view it as a type in the algebraically closed valued field that contains the henselian field K. This naturally raises the question of whether residually dominated types are stably dominated from the viewpoint of the algebraically closed valued field. In Chapter 2, we will show that they are equivalent over acl-closed bases for types that concentrate on the valued field sort.

We write T for a complete theory of a henselian valued field, and let $\mathcal{U}$ be a universal model of T. We denote by T_0 the complete theory of algebraically closed valued fields of equicharacteristic zero, and write $\widetilde{\mathcal{U}}$ for the field-theoretic algebraic closure of $\mathcal{U}$. Similarly, $\mathrm{tp}_0(a/A)$ denotes the complete type of a over A in T_0.

Theorem 2.2.11. *Let a be a tuple in $\mathcal{U}$ that lies in the valued field sort and $C \subset \mathcal{U}$ be a subfield that is* acl*-closed in the valued field. Then the following are equivalent:*

1. $\mathrm{tp}(a/C)$ *is residually dominated in T,*

2. $\mathrm{tp}_0(a/C)$ *is stably dominated in T_0.*

Using Theorem 2.2.11, we also characterize residually dominated types as those that are orthogonal to Γ, similar to stably dominated types.

Theorem 2.3.5. *Let C be a valued field such that C is* acl-*closed in the valued field sort and assume that p is a C-invariant type. If $p|C$ is residually dominated, then for any model $M \supseteq C$ and $a \models p|M$, we have $\Gamma(Ma) = \Gamma(M)$. Conversely if for all $|C|^+$-saturated model $M \supseteq C$ and $a \models p|M$, $\Gamma(Ma) = \Gamma(M)$, then $p|C$ is residually dominated.*

Regarding groups, we extend the notion of stably dominated groups by introducing residually dominated groups. Given a definable group G, a global type p concentrating on G is called *strongly f-generic* if there exists a small model M such that every translate of p by an element of G does not fork over M. We say that G is *residually dominated* if there exists a global residually dominated type concentrating on G. Another related notion is the *connected component* of the group. For a set A, we define the connected component G^{00}_A to be the intersection of all A-type-definable subgroups H of G such that the index $|G/H|$ is bounded.

We assume that the theory of the henselian field is NTP_2. Using results of Montenegro, Onshuus, and Simon in [MOS20], we can embed the connected component of such groups into an algebraic group. This allows us to view residually dominated groups as subgroups of stably dominated groups, hence enabling the application of known results for stably dominated groups.

Theorem 3.2.11. *Assume T is* NTP_2. *Let $(G, \cdot)$ be a definable residually dominated group in the valued field sort of $\mathcal{U}$, with a strongly f-generic type over M. Then, there exist an* ACF-*definable group $\mathfrak{g}$ over M in the residue field and a pro-M-definable group homomorphism $f : G^{00}_M \to \mathfrak{g}$ such that the generics of G^{00}_M are dominated via f. Namely, for each strongly f-generic p of G^{00}_M and for tuples a, b in $\mathcal{U}$ with $a \models p|M$, we have* $\mathrm{tp}(b/Mf(a)) \vdash \mathrm{tp}(b/Ma)$ *whenever $f(a) \mathop{\smile\!\!\!\!\!\!|}^{alg}_{\mathrm{k}(M)} \mathrm{k}(Mb)$.*

The organization of the paper is as follows. In the first chapter, we provide preliminary definitions and results needed.

In the second chapter, we begin with an expository survey on stable and residual domination, unifying similar statements on residual domination from [EHS23] and [Vic22]. We then show the equivalence of residual and stable domination for certain types. We conclude with a generalization of many properties of stable domination to the context of residual domination.

In the third chapter, we introduce residually dominated groups and present generalizations of results from [HRK19].

Finally, in the last chapter, we offer concluding remarks and discuss directions for future research.

1.1 Pure Model Theory

In this section, we review the relevant notions from pure model theory that will be used throughout this paper. We assume the reader is familiar with the basics of model theory, such as formulas, elementary equivalences and extensions, saturation, quantifier elimination, etc.

Throughout this section, we fix a language $\mathcal{L}$ together with the theory T in the language $\mathcal{L}$. We also fix a sufficiently saturated and homogeneous model of T which we denote by $\mathcal{U}$, and refer to as a *universal model*. A subset $A \subseteq \mathcal{U}$ is said to be *small* if its cardinality is less than the cardinality of $\mathcal{U}$.

We denote the model-theoretic algebraic closure and definable closure of a set A by $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$, respectively. We use $x, y, a, b, \cdots$ to denote tuples. For a tuple a in $\mathcal{U}$, the set of all formulas with parameters from A that are realized by a is called the *type of a over A*, denoted by $\mathrm{tp}(a/A)$. The space of all types over A is denoted by $S(A)$, while $S_n(A)$ denotes the space of types of n-tuples over A. A *global type* is a type in $S(\mathcal{U})$. If Δ is a finite set of $\mathcal{L}$-formulas, then we write $S_\Delta(A)$ for the set of types that consists of the boolean combinations of formulas $\varphi(x;b)$ with $\varphi(x;y) \in \Delta$ and $b \in A^{|y|}$.

A definable set is assumed to be definable over parameters. If φ is an $\mathcal{L}$-formula that has parameters from a set A, we write $\varphi \in \mathcal{L}(A)$. If S is a collection of sorts in $\mathcal{L}$, $\mathcal{L}|_S$ denotes the restriction of the language to those sorts in S. Given a sort $\mathcal{S}$ and a set A, we write $\mathcal{S}(A)$ for $\mathcal{S} \cap \mathrm{dcl}(A)$.

Quantifier elimination

In valued fields, quantifier elimination results are generally given relatively. We begin by discussing the definition of quantifier elimination relative to a collection of sorts. The following definition is quoted from [Rid17, Appendix A].

Definition 1.1.1. Let Σ be a collection of sorts of $\mathcal{L}$.

1. Let $\mathcal{L}' \supseteq \mathcal{L}$ be an expansion of $\mathcal{L}$, and let Σ' be the set of new sorts. We say $\mathcal{L}'$ is a Σ-*enrichment* of $\mathcal{L}$ if $\mathcal{L}' \setminus \mathcal{L}'|_{\Sigma\cup\Sigma'} \subseteq \mathcal{L}$, meaning that the enrichment does not affect the sorts of $\mathcal{L}$ outside of Σ.

2. We define the *Morleyization of $\mathcal{L}$ on the sorts in* Σ to be the language obtained by adding predicates $P_\varphi(x)$ for every $\mathcal{L}{\restriction}_\Sigma$-formula $\varphi(x)$. We write $T^{\Sigma\text{-Mor}}$ for the corresponding expansion of T to the language $\mathcal{L}^{\Sigma\text{-Mor}}$.

3. We say that T *eliminates quantifiers relative to* Σ if $T^{\Sigma\text{-Mor}}$ eliminates quantifiers.

4. We say that T *eliminates quantifiers resplendently relative to* Σ if, for any Σ-enrichment $\mathcal{L}'$ of $\mathcal{L}$ and any $\mathcal{L}'$-theory $T' \supseteq T$, the theory T' eliminates quantifiers relative to $\Sigma \cup \Sigma'$.

Definition 1.1.2. Let $M \models T$, and A be a set with $A \subseteq M$.

1. An $\mathcal{L}(A)$-definable set $X \subseteq M$ is called *stably embedded in* M if for any $n \in \mathbb{Z}^{>0}$, all definable subsets of X^n are definable with parameters from $X \cup A$.

2. A definable set $X \subseteq M$ is called *stably embedded in* M *with control of parameters*, if X is stably embedded, and in addition, for any $\mathcal{L}$-formula $\phi(x; y)$ and a tuple $u \in M^{|y|}$ such that $\phi(x; u)$ defines a subset of X^n for some $n \in \mathbb{Z}^{>0}$, there exists an $\mathcal{L}$-formula $\psi(x, d)$ with $d \in \mathrm{dcl}(u)$ that defines the same subset of X^n.

3. The definable sets X and Y are called *orthogonal* if for any $\mathcal{L}(\mathcal{U})$-formula $\varphi(x, y, a)$ defining a subset of $X^{|x|} \times Y^{|y|}$, there exist formulas

$$\theta_1(x, b_1), \cdots, \theta_n(x, b_n) \text{ and } \psi_1(y, c_1), \cdots, \psi_n(y, c_n)$$

such that φ defines the same set as the formula

$$\bigcup_{i=1}^{n} \theta_i(x, b_i) \wedge \psi_i(y, c_i).$$

4. A sort $\mathcal{S}$ is *stably embedded* (or *stably embedded with control of parameters*) if the set defined by $x \in S$ is stably embedded (or stably embedded with control of parameters) in every model $M \models T$. We say $\mathcal{S}$ is a *purely stably embedded* $\mathcal{L}'$*-structure* where $\mathcal{L}'$ is the restriction of $\mathcal{L}$ to the sort $\mathcal{S}$, if any definable subset X of $\mathcal{S}^n$ is definable by an $\mathcal{L}'$-formula with parameters from $\mathcal{S}$.

5. Two sorts $\mathcal{S}$ and $\mathcal{T}$ are orthogonal to each other if $\mathcal{S}$ and $\mathcal{T}$ are orthogonal as definable sets.

Definition 1.1.3. Let $\mathcal{T}$ be a sort. A definable set X is $\mathcal{T}$*-internal* if there exists a finite set F of parameters such that $X \subseteq \mathrm{dcl}^{eq}(\mathcal{T} \cup F)$. Equivalently, by compactness, there exists a map $f : \mathcal{T}^n \to X$ for some $n \in \mathbb{Z}^{>0}$, definable with parameters from F in T^{eq}.

Notation 1.1.4. *For a sort* $\mathcal{S}$ *and a set* A*, we write* $\mathcal{S}(A) := \mathcal{S} \cap \mathrm{dcl}(A)$.

The stably embedded sets are extensively studied in [CH99]. The following two facts are well-known. For the sake of completeness, we include their proofs.

Fact 1.1.5. *Let $\mathcal{S}$ be a sort in $\mathcal{L}$ that is stably embedded with control of parameters. Let $B \supseteq C$ be subsets of $\mathcal{U}$. Then for any subset $A \subseteq \mathcal{U}$, we have* $\mathrm{tp}(B/C\mathcal{S}(B)) \vdash \mathrm{tp}(B/C\mathcal{S}(B)\mathcal{S}(A))$.

Proof. Let $B \equiv_{C\mathcal{S}(B)} B'$ and σ be an automorphism of $\mathcal{U}$ that fixes $C\mathcal{S}(B)$ pointwise and sends B to B'. We will show that $B \equiv_{C\mathcal{S}(B)\mathcal{S}(A)} B'$. For this, let $\phi(x, y, z)$ be an $\mathcal{L}$-formula, b be a tuple in $\mathcal{S}(B)$, and let a be a tuple in $\mathcal{S}(A)$ such that $\models \phi(B, a, b)$. Then, the set $\{d \in \mathcal{U} : \models \phi(B, d, b)\}$ is definable in the sort $\mathcal{S}$. By assumption, it is definable over $\mathcal{S}(B) = \mathcal{S}(B')$. After applying σ, we have $\{d \in \mathcal{U} : \models \phi(B, d, b)\} = \{d \in \mathcal{U} : \models \phi(B', d, b)\}$. Thus, $\models \phi(B', a, b)$, as desired. □

Fact 1.1.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be sorts of $\mathcal{U}$ that are orthogonal to each other. Then for any sets $A \subseteq \mathcal{S}$ and $B \subseteq \mathcal{T}$, we have $\mathcal{S}(A \cup B) = \mathcal{S}(A)$.*

Proof. Let $c \in \mathcal{S}(A \cup B)$, and let a be a tuple in A, b be a tuple in B, and $\phi(x, y, z)$ be an $\mathcal{L}(\emptyset)$-formula such that $\phi(x, a, b)$ witnesses $c \in \mathrm{dcl}(a, b)$. By the orthogonality of $\mathcal{S}$ and $\mathcal{T}$, the formula $\phi(x, a, z)$ is equivalent to a finite disjunction of formulas of the form $\theta_i(x, d) \wedge \psi_i(z)$, where $i \leq n$, and for each $i \leq n$, θ_i is an $\mathcal{L}|_{\mathcal{S}}$-formula and ψ_i is an $\mathcal{L}|_{\mathcal{T}}$-formula. It follows that c is the unique solution of $\bigvee_{i=1}^{n} \theta_i(x, d)$, which is also an A-definable set. Therefore $c \in \mathcal{S}(A)$. The other direction is trivial. □

Elimination of Imaginaries

Let $X \subseteq \mathcal{U}^n$ be a definable set, and let $E(x, y)$ be a definable equivalence relation on X. For each $c \in X$, the equivalence class c/E is called an *imaginary element*. It is desirable when the imaginary elements can be identified with a tuple in the model. We can expand the language to ensure that these elements are *coded.*

Definition 1.1.7. Let X be a definable set. A finite tuple c is called a *code* for X if for every automorphism σ of $\mathcal{U}$, σ fixes c pointwise if and only if it fixes X setwise. Similarly, a possibly infinite tuple c is a *code* for a type p if for every automorphism σ of $\mathcal{U}$, σ fixes c pointwise if and only if σ leaves p invariant.

Definition 1.1.8. The theory T *eliminates imaginaries* if every $\emptyset$-definable equivalence class c/E has a code within $\mathcal{U}$.

Let $(E_i)_{i\in\mathcal{I}}$ be an enumeration of all $\emptyset$-definable equivalence relations in T. Fix a model $M \models T$. For each $i \in \mathcal{I}$, we expand the language $\mathcal{L}$ by adding new sorts corresponding to the sets X_i/E_i, where E_i is an equivalence relation defined on X_i. We also add functions $\pi_i : X_i \to X_i/E_i$, defined by $\pi_i(x) = x/E_i$ for each $x \in X_i$. The resulting expanded language is denoted by $\mathcal{L}^{eq}$. For a model $M \models T$, we write M^{eq} to denote its expansion in the language $\mathcal{L}^{eq}$. Then, the $\mathcal{L}^{eq}$-theory of M^{eq} is axiomatized by the following conditions:

- $\forall x \exists y_i, (\pi_i(x) = y_i)$, where x is a variable in the home sort and y_i is a variable in the sort X_i/E_i.
- $\forall x_1 x_2, (\pi_i(x_1) = \pi_i(x_2) \longleftrightarrow E_i(x_1, x_2))$.

This expanded theory is denoted by T^{eq}. If T eliminates imaginaries, then for every $\emptyset$-definable equivalence relation E, each class c/E has a code in some model $M \models T$.

We write dcl^{eq} and acl^{eq} to denote definable and algebraic closures taken in T^{eq}, respectively.

Pro-definable Sets

Most of the time, we work with infinite tuples of variables in formulas or types. Pro-definable sets are one of the suitable frameworks for this purpose.

A partial order $\mathcal{I}$ is *filtered* if every pair of elements in $\mathcal{I}$ has an upper bound within $\mathcal{I}$.

Definition 1.1.9. A *pro-definable set* is a family of definable sets $(X_i)_{i\in\mathcal{I}}$ indexed by a filtered order $\mathcal{I}$, together with definable *transition* maps $f_{j,i}\colon X_j \to X_i$ for each $i < j$. We identify it with its projective limit $X := \varprojlim X_i$.

The pro-definable set X is C-definable if each X_i is C-definable and all transition maps $f_{j,i}$ are C-definable.

After fixing an enumeration of the sets X_i and introducing variables x_i corresponding to each X_i, the pro-definable set X can be represented as a partial type in the variables $x = (x_i)_{i\in\mathcal{I}}$. If all transition maps in the system are injective, X can be identified as a subset of each X_i. In this case, X is the intersection of all X_i, making it ∞-definable.

A pro-definable map $f\colon \varprojlim X_i \to Y$, where Y is a definable set, is simply a definable map from some X_j to Y. If Y is also a projective limit, then a pro-definable map $f\colon \varprojlim X_i \to \varprojlim Y_i$ consists of a compatible family of pro-definable maps $f_j\colon \varprojlim X_i \to Y_j$.

Invariant and definable types

Definition 1.1.10. Let A be a small set. A global type $p \in S(\mathcal{U})$ is A-*invariant* if for each $\sigma \in Aut(\mathcal{U}/A)$, each $\mathcal{L}$-formula $\varphi(x;y)$ and each tuple b in $\mathcal{U}$, we have

$$\varphi(x;b) \in p \Longleftrightarrow \varphi(x;\sigma(b)) \in p.$$

When $M \models T$, we will show that any $p \in S(M)$ extends to global M-invariant type. Consider the set

$$\Sigma(x) = \{\phi(x,b) : \phi(x,y) \text{ is an } \mathcal{L}\text{-formula such that } \neg\phi(M,b) \text{ is empty}\}.$$

Then, $p \cup \Sigma$ is consistent. Since otherwise, there would exist some $\varphi(x) \in p$ and $\theta_1, \ldots, \theta_n \in \Sigma$ with $\varphi \wedge \bigwedge_{i=1}^{n} \theta_i$ inconsistent. It follows that $\varphi \vdash \bigvee_{i=1}^{n} \neg\theta_i$. Since M is a model, there exists some $a \in M^{|x|}$ with $\models \varphi(a)$. Hence, $\models \neg\theta_i(a)$ for some i, a contradiction. Let q be a global extension of $p \cup \Sigma$. Notice that for any $\theta \in q$, there exists $\varphi \in p$ with $\varphi \vdash \theta$. Let $\sigma \in Aut(\mathcal{U}/M)$, then for any $a \in M^{|x|}$ and tuple b in $\mathcal{U}$, $\models \theta(a,b) \leftrightarrow \theta(a,\sigma(b))$. It follows that q is M-invariant.

Another important class of invariant types is *definable types*.

Definition 1.1.11. A type $p \in S(A)$ is *definable* if for each $\mathcal{L}$-formula $\varphi(x;y)$, there is an $\mathcal{L}(A)$-formula $(d_p x)\varphi(x,y)$ such that for every tuple $b \in A^{|y|}$, we have

$$\varphi(x;b) \in p \Longleftrightarrow \mathcal{U} \models (d_p x)\varphi(x,b).$$

The collection of formulas $\big((d_p x)\varphi(x,y)\big)_{\varphi}$ is called a *defining scheme of* p. For $B \subseteq A$, p is said to be B-definable if for any $\mathcal{L}$-formula $\varphi(x,y)$, the defining formula $(d_p x)\varphi(x,y)$ is $\mathcal{L}(B)$-definable.

One of the main advantages of working with invariant and definable types is that there is a canonical way to extend them to global types.

Let $p \in S(M)$ be an A-definable type and $N \succeq M$. Let $\{(d_p x)\varphi(x,y)\}_{\varphi}$ be a defining scheme of p over A. Then, p extends to a complete A-definable type $q \in S(N)$, given by

$$q := \{\varphi(x,b) : \varphi(x,y) \text{ is an } \mathcal{L}\text{-formula, } b \in N^{|y|} \text{ and } \mathcal{U} \models (d_p x)\varphi(x,b).\}$$

In this case, q is the unique A-definable extension of p.

Similarly, in the case of invariant types, we also have a canonical extension. Indeed, let $p \in S(M)$ be an A-invariant type where M is $|A|^+$-saturated. For each $\mathcal{L}$-formula $\varphi(x;y)$, define $D_{p,\varphi}$ to be the set of all types $q \in S(A)$ for which there exists some $b \in M^{|y|}$ with $b \models q$ and $\varphi(x,b) \in p$. Then, if $N \succeq M$, p extends to a complete type $r \in S(N)$, defined by

$$r := \{\varphi(x;b) : \varphi(x,y) \text{ is an } \mathcal{L}\text{-formula, } b \in N^{|y|} \text{ and } \exists q \in D_{p,\varphi} \text{ such that } b \models q.\}$$

As in the definable case, r is the unique A-invariant extension of p. We will denote it by $p|_N$. If $N \supseteq B \supseteq M$, then $p|_B$ is the restriction $r|B$ of the type r to B. Moreover, it depends only on p, not on A: if p is B-invariant for some other set B, then the construction yields the same type.

Tensor product of invariant types, Morley sequences

Let $p, q \in S(M)$ be invariant types. Define $p(x) \otimes q(y)$ as follows:

$$(a,b) \models p(x) \otimes q(y) \text{ if and only if } b \models q \text{ and } a \models p|_{Mb}.$$

This type is called the *tensor product of p and q*, and it is well-defined by the invariance of p.

Remark 1.1.12. *The product $\otimes$ on the space of invariant types $S^{inv}(\mathcal{U})$ is associative but need not be commutative.*

Now let p be an A-invariant type. For any positive integer $n \in \mathbb{Z}^{>0}$, we define $p^{(n)}$ inductively as follows: $p^{(1)}(x_1) := p$, and for $n \geq 2$, $p^{(n+1)}(x_1 \ldots, x_{n+1}) := p(x_{n+1}) \otimes p^{(n)}(x_1, \ldots, x_n)$. We write $p^{\omega} = \bigcup p^{(n)}$. Note that if $(a_1, a_2, \cdots) \models p^{(\omega)}$, then for each i, $a_i \models p|_{a_1, \cdots a_{i-1}}$.

For $B \supseteq A$, a *Morley sequence of p over B* is a sequence $b := (b_i : i < \omega)$ that realizes $p^{(\omega)}|_B$.

Classification theory

Classification theory was initiated by Shelah [She78], with the goal of classifying first-order theories using *dividing lines*. In this thesis, we introduce the class of theories we will focus on. The definitions and results are from [Pil96],[Sim15] and [Che14].

Definition 1.1.13. 1. An $\mathcal{L}(\mathcal{U})$-formula $\varphi(x, y)$ is called *stable* if there are no sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ in $\mathcal{U}$ such that

$$\varphi(a_i, b_j) \text{ holds if and only if } i < j \text{ for all } i, j < \omega.$$

2. The theory T is stable if every formula $\varphi(x, y)$ is stable.

In the definition, one can say that a stable formula does not witness the existence of a *linear order* on any infinite tuple.

The following equivalent characterizations of stability show that stability can be checked locally.

Fact 1.1.14. *The following are equivalent.*

1. *T is stable,*
2. *For all $A \subseteq \mathcal{U}$ with $|A| \leq \kappa$, $|S(A)| < \kappa + |\mathcal{L}|$.*
3. *For all $A \subset \mathcal{U}$ and every finite set of formulas Δ, we have $|S_\Delta(A)| < |A|$,*
4. *For all A, and every $p \in S(A)$, p is definable.*
5. *For all A and every 1-type $p \in S_1(A)$, p is definable.*

Example 1.1.15. 1. The following theories are stable: algebraically closed fields, $\mathbb{Z}$-modules, vector spaces, differentially closed fields of equicharacteristic zero.

2. The following theories are unstable, since they admit a definable linear order: dense linear orders, real closed fields, ordered abelian groups, non-trivially valued algebraically closed valued fields.

Definition 1.1.16. 1. Let $\varphi(x; y)$ be an $\mathcal{L}(\emptyset)$-formula, and let A be set of $|x|$-tuples. We say that $\varphi(x; y)$, shatters A if there exists a family $(b_I : I \subseteq A)$ of $|y|$-tuples such that, for all $a \in A$

$$\mathcal{U} \models \varphi(a; b_I) \text{ if and only if } a \in I.$$

2. An $\mathcal{L}(\emptyset)$-formula $\varphi(x; y)$ is IP (the *independence property*) if it shatters an infinite set. We say it is NIP, if it is not IP.

3. The theory is NIP (or *dependent*) if all formulas are NIP.

Example 1.1.17. 1. Let T be the theory of $(\mathbb{Q}, <)$. The formula $x < y$ does not shatter any set A with $|A| \geq 2$. In fact, if $a_1, a_2 \in A$ with $a_1 < a_2$, there is no $b_{\{2\}}$ such that $\neg(a_1 < b_{\{2\}})$ and $a_2 < b_{\{2\}}$ both hold.

2. Any stable formula fails to shatter an infinite set, otherwise it would induce a linear order, contradicting stability.

Example 1.1.18. 1. The following theories are NIP: the theory of algebraically closed valued fields, dense linear orders without endpoints, real closed fields.

2. The following theories are not NIP: random graphs, algebraically closed fields with an automorphism.

Definition 1.1.19. A formula $\varphi(x; y)$ has *the tree property of the second kind* (TP_2) if there exists a family of tuples $(a_{ij})_{i,j\in\mathbb{N}}$ and $k \in \mathbb{N}$ such that:

- $\varphi(x, a_{ij})_{j\in\mathbb{N}}$ is k-inconsistent for every $i \in \mathbb{N}$,
- $\varphi(x, a_{i\sigma(i)})_{i\in\mathbb{N}}$ is consistent for every permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$.

A formula is NTP_2 if it is not TP_2. A theory is NTP_2 if every formula in the theory is NTP_2.

Example 1.1.20. The class of NTP_2-theories generalizes both NIP and simple theories, so any NIP or simple theory is NTP_2. An example of an NTP_2-theory that is neither NIP nor simple is the ultraproduct $\prod_{p\in P} \mathbb{Q}_p/\mathcal{U}$ of p-adically closed fields $\mathbb{Q}_p$, with respect to a non-principal ultrafilter $\mathcal{U}$ on the set P of prime numbers.

Forking and dividing

Notation 1.1.21. *For tuples a and b and a set A, we write $a \equiv_M b$ if* $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$.

Definition 1.1.22. Let κ be a cardinal. A sequence $(a_i)_{i<\kappa}$ is an *A-indiscernible sequence*, if for all $i_1 < \cdots i_n$ and $j_1 < \cdots j_n$, $a_{i_1} \cdots a_{i_n} \equiv_A a_{j_1} \cdots a_{j_n}$.

Definition 1.1.23. Let $\varphi(x; y)$ be an $\mathcal{L}(\emptyset)$-formula, a a tuple of length $|y|$ and A a set. Then,

1. We say $\varphi(x,a)$ *divides over* A, if there exists an A-indiscernible sequence $(a_i)_{i<\omega}$ with $a_1 := a$ and such that the set $\{\varphi(x;a_i){:}i<\omega\}$ is inconsistent. (Equivalently, it is k-inconsistent for some $k \in \mathbb{Z}^{>0}$, meaning that every subset of size k is inconsistent.)

2. We say $\varphi(x,a)$ *forks over* A if it lies in a finite disjunction of dividing formulas. That is, there exist $\mathcal{L}(\mathcal{U})$-formulas $\theta_1(x),\cdots,\theta_n(x)$, where for each i, θ_i divides over A, and $\varphi(x) \vdash \bigvee_{i=1}^{n} \theta_i(x)$. A set of formulas *forks over* A, if it implies a formula that forks over A.

3. If b is a tuple, we say a does not fork from b over A, denoted by $a \mathop{\smile\!\!\!\!\!\!|}_A b$ if $\mathrm{tp}(a/Ab)$ does not fork over A.

It is straightforward that if a formula does not divide over A, then it does not fork over A. However, the converse does not always hold in general.

Nevertheless, in a large class of theories, forking and dividing coincide over models:

Fact 1.1.24. *([CK12, Theorem 1.1]) Assume that T is* NTP_2 *and $M \models T$. Then, for any formula $\varphi(x) \in \mathcal{L}(\mathcal{U})$, $\varphi(x)$ divides over M if and only if it forks over M.*

In stable theories, every type over an acl-closed set admits a unique non-forking extension.

Fact 1.1.25. *Assume that T is stable. Let C be a subset of $\mathcal{U}$ such that $C = \mathrm{acl}(C)$. Then for any tuple a in $\mathcal{U}$, $\mathrm{tp}(a/C)$ is stationary, i.e. for any $M \models T$ with $M \supseteq C$, there exists a unique type $p \in S(M)$ that does not fork over C.*

Germs of Functions on Types

In this subsection, we will recall the notion of germs of definable functions on a definable type. This is analogous to germs of functions on topological spaces, where a germ is an equivalence class of functions which appear identical locally. When the theory is stable, the *strong germs* always exists, which are particularly useful in group configuration arguments, namely, constructing definable groups from data on definable types. The definitions are given below.

Definition 1.1.26. Let $p(x)$ be a global A-definable type, where A is a small set. Let f_c and $g_{c'}$ be definable functions with parameters c and c', respectively. We say that $f_c(x)$ and $g_{c'}(x)$ *have the same p-germ* if $p \vdash f_c(x) = g_{c'}(x)$, or equivalently, $\models (d_p x) f_c(x) = g_{c'}(x)$.

Given an $\mathcal{L}(c)$-definable function f_c, we define a relation E as follows:

$$c_1 E c_2 \text{ if and only if } f_{c_1}(a) = f_{c_2}(a) \text{ for all } a \models p. \tag{1.1.1}$$

By definability of p, the relation E is definable, and it is easy to check that E is an equivalence relation. The equivalence class c/E of c under E is called the *p-germ of f_c on p*.

The p-germ of f_c is said to be *strong over A* if, moreover, for all $a \models p|Ac$, we have $f_c(a) \in \mathrm{dcl}(e, a, A)$, where e is a code for the definable set c/E.

Model theory and groups

Throughout this section, we fix a definable group G in a theory T over a model M - that is, there exist $\mathcal{L}(M)$-formulas that define both the underlying set of G and its multiplication operation. Each definition and fact presented here also holds when G is a type-definable group; that is, when G itself is type definable and the graph of its group operation is given by the restriction of a definable set to G^3.

For a subgroup $H \leq G$, we say H has *bounded index* in G if $[G : H] < \kappa$, where κ is smaller than the cardinality of the universal model $\mathcal{U}$.

Definition 1.1.27. Let A be a small set. The connected components of an $\mathcal{L}(A)$-definable group G over A are defined as follows:

1. G^0_A is the intersection of all A-definable subgroups of G of finite index.
2. G^{00}_A is the intersection of all subgroups of G that are type-definable over A and have bounded index.
3. G^{∞}_A is the intersection of all subgroups of G that are $\mathrm{Aut}(\mathcal{U}/A)$-invariant and have bounded index.

For any A, we note that $G^{\infty}_A \leq G^{00}_A \leq G^0_A$. However, the converse might not hold in general.

Fact 1.1.28. *([Sim15, Theorem Theorem 8.3, Theorem 8.4,8.7]) Assume that T is NIP. Then for any small set A, the connected components G^0_A, G^{00}_A, and G^{∞}_A do not depend on A.*

Therefore, in NIP theories, we simply write G^0, G^{00}, and G^{∞}, omitting the parameter set, for the connected components of G.

The elements of $G(\mathcal{U})$ act on definable subsets of G as follows: for any $\mathcal{L}(\mathcal{U})$-formula $\phi(x)$ defining a subset of G and any $g \in G(\mathcal{U})$, we define $g \cdot \phi(x) := \phi(g^{-1} \cdot x)$. This induces an action of $G(\mathcal{U})$ on $S_G(\mathcal{U})$, the set of global types concentrating on G, where for each $g \in G(\mathcal{U})$ and $p \in S_G(\mathcal{U})$:

$$g \cdot p := \{\varphi(x) : \text{there exists } a \models p \text{ such that } \mathcal{U} \models \varphi(g \cdot a)\}.$$

Hence $g \cdot a \models g \cdot p$ whenever $a \models p$. The *stabilizer of* p under this action is:

$$Stab_G(p) = \{h \in G(\mathcal{U}) : hp = p\}.$$

The following definitions first appeared in [HP11] and were refined in [CS18] in the NIP context. The same definitions were also used in [MOS20] in the NTP_2 context.

Definition 1.1.29.

1. An $\mathcal{L}(\mathcal{U})$-formula is *f-generic* if for all $g \in G(\mathcal{U})$, there exists a small model M such that $g \cdot \varphi$ does not fork over M.
2. A partial type p is f-generic if it contains only f-generic formulas.
3. A global type $p \in S_G(\mathcal{U})$ is *strongly f-generic over a small model M* if for every $g \in G(\mathcal{U})$, $g \cdot p$ does not fork over M. In this case, we say G has a strongly f-generic type.

As discussed in [CS18] and [MOS20], these notions of genericity behave similarly to generics in stable theories *when* there exists a strongly f-generic of the group.

1.2 Valued Fields

In this section, we review both the algebraic and model-theoretic aspects of valued fields.

Definition 1.2.1. Let K be a field. A *valuation* on K is a group homomorphism $v : K \to \Gamma \cup \{\infty\}$ where Γ is an ordered abelian group such that for all $x, y \in K$:

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

In this case, the pair (K, v) is called a *valued field*, and Γ is called the *value group*.

Here, $\Gamma \cup \infty$ is a monoid, where we assume that $\gamma < \infty$ for all $\gamma \in \Gamma$, and that $\infty + \gamma = \gamma + \infty = \infty$.

For a valued field (K, v), its *valuation ring* is the set $\mathcal{O} = \{a \in K \mid v(a) \geq 0\}$. This ring has a unique maximal ideal $\mathfrak{m} := \{a \in \mathcal{O} \mid v(a) > 0\}$. The quotient field $\mathcal{O}/\mathfrak{m}$ is called the *residue field*, and is denoted by k. We also denote the natural projection by res : $\mathcal{O} \to$ k.

Given a valued field K, in terms of characteristic the possibilities are either $char(K) = char(\mathrm{k}) = p$ where $p \geq 0$, or $char(K) = 0$ and $char(\mathrm{k}) = p >$ with $p > 0$. When $char(K) = char(\mathrm{k}) = 0$, we say K is *of equicharacteristic zero*; when when $char(K) \neq char(\mathrm{k})$, we say *$K$ is of mixed characteristic.*

Example 1.2.2. 1. On $\mathbb{Q}$, for a prime p, we can define the valuation $v_p : \mathbb{Q} \to \mathbb{Z}$ as follows: for $a, b \in \mathbb{Q}$ with $b \neq 0$, $v_p(\frac{a}{b}) = k$ where $\frac{a}{b} = p^k mn$ and p does not divide both m and n. Here, v_p is called the *p-adic valuation*.

2. Let k be a field and G be an ordered abelian group. Consider the set of formal sums $\sum_{\gamma \in G} a_\gamma t^\gamma$ where $a_\gamma \in k$ for each $\gamma \in G$. For a formal sum $f = \sum_{\gamma \in G} a_\gamma t^\gamma$, we define its support *supp*(f) to be the set $\{\gamma : a_\gamma \neq 0\}$. Let

$$k((t^G)) := \{f = \sum_{\gamma \in G} a_\gamma t^\gamma : a_\gamma \in k \text{ and } supp(f) \text{ is well-ordered.}\}.$$

Then $k((t^G))$ has a valued field structure, where

- $\sum_{\gamma \in G} a_\gamma t^\gamma + \sum_{\gamma \in G} b_\gamma t^\gamma = \sum_{\gamma \in G} (a_\gamma + b_\gamma t^\gamma)$,

- $\big(\sum_{\gamma\in G} a_\gamma t^\gamma\big)\big(\sum_{\gamma\in G} b_\gamma t^\gamma\big) = \sum_{\gamma\in G}\sum_{\delta+\alpha=\gamma} a_\delta b_\alpha t^\gamma$.
- For $f \in k((t^\Gamma))$, $v(f) = \min(supp(f))$.

The valued field $k((t^G))$ is called a *Hahn field.* Its residue field is k and its value group is G.

Lemma 1.2.3. *Let (K, v) be a valued field. Then,*

1. $v(1) = 0$,
2. *For $x, y \in K$, $v(x + y) = \min\{v(a), v(b)\}$ if $v(x) \neq v(y)$.*

Proof. This is easily verified using definitions. □

Another important quotient for a valued field (K, v) is the set $\mathrm{RV}^\times := K^\times/(1+\mathfrak{m})$. We write $rv : K^\times \to \mathrm{RV}^\times$ for the natural projection map. For $\mathbf{a} = a(1+\mathfrak{m})$ in $\mathrm{RV}^\times$, define $v_{rv}(\mathbf{a}) = v(a)$. This is well-defined: if $\mathbf{a} = \mathbf{a}'$, then $\frac{a}{a'} \in 1+\mathfrak{m}$. In particular, $a = a'(1+u)$ for some $u \in \mathfrak{m}$. By Lemma 1.2.3, $v(a) = v(a') + v(1+u) = v(a')$. Moreover, there is a natural embedding $i : \mathrm{k} \hookrightarrow \mathrm{RV}$ given by sending $a(1+\mathfrak{m}) \in \mathrm{k}$ to $a(1+\mathfrak{m}) \in \mathrm{RV}^\times$. We set $i(0) := 0_{rv}$ and we write $\mathrm{RV} = \mathrm{RV}^\times \cup \{0_{rv}\}$. The maps i and v_{rv} give a short exact sequence of abelian groups:

$$1 \to \mathrm{k}^\times \hookrightarrow \mathrm{RV} \twoheadrightarrow \Gamma \to 0,$$

and this extends to a short exact sequence of monoids by setting $v_{rv}(0_{rv}) = \infty$.

$$1 \to \mathrm{k} \to \mathrm{RV} \to \Gamma_\infty \to 0.$$

A valuation also induces a natural topology on K, generated by *open balls*, where an open ball centered at a point $a \in K$ and a radius $\gamma \in \Gamma$ is the set $B_{>\gamma}(a) = \{b \in K : v(b-a) > \gamma\}$. Then, a closed ball is a set of the form $B_{\geq\gamma}(a) = \{b \in K : v(b-a) \geq \gamma\}$. Notice that $\mathfrak{m}$ is the open ball $B_{>0}(0)$, and $\mathcal{O}$ is the closed ball $B_{\geq 0}(0)$.

For $\gamma \in \Gamma$, the *fiber* of γ under v_{rv} is the set $\mathrm{RV}_\gamma = \{a(1+\mathfrak{m}) : v(a) = \gamma\}$ of γ. An element $a(1+\mathfrak{m})$ can be identified with the open ball $B_{>v(a)}(a)$ and this identification does not depend on the representative of $a(1+\mathfrak{m})$. For $a(1+\mathfrak{m}) = b(1+\mathfrak{m})$, then $v(a) = v(b)$, and thus if $a - b = au_1 - bu_2$ for some $u_1, u_2 \in \mathfrak{m}$, then $v(a-b) > v(a) = v(b)$.

Remark 1.2.4. *Given $\gamma \in \Gamma$, fix a non-zero $d_\gamma \in \mathrm{RV}_\gamma$. Let $f : \mathrm{k} \to \mathrm{RV}_\gamma$ be defined by $f(a) = ad_\gamma$, for all $a \in \mathrm{k}$. One can show that f is well-defined. Morever, f is a definable bijection, and hence RV_γ is* k*-internal. This also induces a multiplication on RV_γ as follows: for $x, y \in \mathrm{RV}_\gamma$, define $xy = abd_\gamma$ where $x = f(a)$ and $y = f(b)$ for some $a, b \in \mathrm{k}$. There is also an induced addition on the fiber $\mathrm{RV}_\gamma \cup \infty$, interpreted as $x + y = f(a + b)$, where $f(a) = x$ and $f(b) = y$ for $a, b \in \mathrm{k}$. Note that this addition does not depend on the choice of d_γ.*

Therefore, $\mathrm{RV}_\gamma \cup \{0_{\mathrm{RV}}\}$ has a 1-dimensional k*-vector space structure.*

One can define the partial addition on the RV-sort as follows $rv(a + b) = rv(c)$ if and only if there exist $a', b', c' \in K$ such that $rv(a) = rv(a')$, $rv(b) = rv(b')$, $rv(c) = rv(c')$ and $a' + b' = c'$. Otherwise, we set $\mathrm{RV}(a + b) = 0_{\mathrm{RV}}$ However, this is not always well-defined. The following fact provides the necessary condition for well-definedness.

Fact 1.2.5. *([Fle11, Proposition 2.5]) Let (K, v) be a valued field and $a_1 \dots , a_n \in K$ such that $v(a_1 + \dots , a_n) = \min\{v(a_1), \dots , v(a_n)\}$. Then, we have $rv(a_1 + \dots + a_n) = rv(a_1) + \dots + rv(a_n)$.*

Valued Field Extensions

For a valued field K, let $\mathrm{k}_K = \{\mathrm{res}(a){:}a \in K\}$ and $\Gamma_K = \{v(a){:}a \in K\}$. For another valued field L, we write $L \geq K$ if $L \supseteq K$ and the valuation of L resticted to K gives the valuation of K. In this case, we say L is a *valued field extension of K*.

When L has a finite transcendence degree over K, this reflects to the value group and the residue field as in the following fundamental inequality known as *Zariski-Abhyankar Inequality.* Here, for fields $A \subseteq B$, $tr.deg(A/B)$ refers to the transcendence degree of A over B.

Fact 1.2.6. *([vdD14, Theorem 3.25]) Let $L \geq K$ be an extension of valued fields and suppose that $tr.deg(L/K)$ is finite. Then both $tr.deg(\mathrm{k}_L/\mathrm{k}_K)$ and the dimension of the $\mathbb{Q}$-vector space Γ_L/Γ_K are finite. Moreover,*

$$tr.deg(L/K) \geq tr.deg(\mathrm{k}_L/\mathrm{k}_K) + dim_{\mathbb{Q}}(\Gamma_L/\Gamma_K),$$

where $dim_{\mathbb{Q}}(\Gamma_L/\Gamma_K)$ is the dimension of Γ_L/Γ_K as a $\mathbb{Q}$-vector space.

Definition 1.2.7. A valued field extension $L \geq K$ is called

- *unramified* if $\Gamma_K = \Gamma_L$
- *immediate* if $\Gamma_K = \Gamma_L$ and $\mathrm{k}_K = \mathrm{k}_L$.

Definition 1.2.8. A valued field (K, v) is *maximally complete*(or *maximally valued*) if it has no proper immediate extensions.

Fact 1.2.9. *([Hil18, Proposition 5.6]) A Hahn field $k((t^G))$ is maximally complete.*

Henselian Fields

An important class of valued fields are *henselian* valued fields. They are the valued fields that satisfy the *Hensel's Lemma*, given in the definition, below. For a polynomial $P(x) \in \mathcal{O}[X]$, we write $\mathrm{res}(P)(X) \in \mathrm{k}[X]$ to denote the polynomial obtained by applying the map res to the coefficients of $P(x)$.

Definition 1.2.10. A valued field (K, v) is called *henselian* if for every polynomial $P \in \mathcal{O}[X]$ and every $a \in \mathcal{O}$ satisfying $\mathrm{res}(P)(\mathrm{res}(a)) = 0$ and $\mathrm{res}(P)'(\mathrm{res}(a)) \neq 0$, there exists a unique $b \in \mathcal{O}$ such that $v(b - a) > 0$ and $P(b) = 0$.

Example 1.2.11. Hahn fields, algebraically closed valued fields and p-adically closed valued fields.

A valued field (K, v) is called *finitely ramified* if for any $n \in \mathbb{Z}^{>0}$, the interval $[0, v(n)]$ is finite.

Fact 1.2.12. *(Theorem [EP05, Lemma 4.1.1, Theorem 4.1.10]) Let (K, v) be a valued field of equicharacteristic zero, or finitely ramified of mixed characteristic. Then, the following are equivalent:*

1. *(K, v) is henselian.*
2. *(K, v) is algebraically maximal, i.e. any any proper algebraic extension is immediate.*
3. *The valuation v extends uniquely to K^{alg}: if w is a valuation on K^{alg} extending v, then there is a valued-field isomorphism between (K, v) and (K, w).*

By Fact 1.2.12, whenever (K, v) is henselian it embeds into its algebraic closure (K^{alg}, w) in a particularly nice way: any valued field automorphism of (K, v) extends to a valued field automorphism of (K^{alg}, w)).

Definition 1.2.13. Let (K, v) be a valued field of equicharacteristic zero. Then, there exists a henselian valued field extension (K^h, w) of (K, v) such that if (K', v') is a henselian valued extension of (K, v) there exists a valued field embedding $\iota(K^h, w) \to (K', v')$ such that $\iota|_K$ is an identity map. In this case, (K^h, w) is called the *henselization of* (K, v).

Definition 1.2.14. Let K be a field of characteristic zero, possibly with extra structure. We say K is *algebraically bounded* if for all tuple a in K, $\mathrm{acl}(a)$ is contained in $\mathrm{alg}(a)$, where $\mathrm{alg}(a)$ is the field-theoretic algebraic closure of $\mathbb{Q}(a)$.

Fact 1.2.15. *([vdD89]) Let (K, v) be an henselian field. Then K is algebraically bounded.*

Quantifier elimination

There are many different choices of language when working with valued fields. The classical one $\mathcal{L}_{\mathrm{div}}$ was introduced by Robinson. This language extends the language of rings by a single binary relation symbol div, interpreted as follows: $a \operatorname{div} b$ if and only if $v(a) \leq v(b)$. Robinson showed that the theory of algebraically closed valued fields admits quantifier elimination in $\mathcal{L}_{\mathrm{div}}$. However, this fails in general for the theories of henselian fields. Simply adding sorts for value groups and the residue field to the language might not even be sufficient; one might need the *angular component*, a section of the valuation map.

We define the language $\mathcal{L}_{val} = (\mathrm{VF}, \mathrm{k}, \Gamma)$ of valued fields as follows: The sorts VF and k are equipped with the language of rings and the sort Γ is equipped with the language of ordered abelian groups expanded by a symbol ∞. In addition, there are maps $v : \mathrm{VF} \to \Gamma$ and $Res : \mathrm{VF} \times \mathrm{VF}^{\times} \to \mathrm{k}$, where v is interpreted as the valuation map and for all $a \in \mathrm{VF}$ and $b \in \mathrm{VF}^{\times}$, $Res(a, b)$ equals $\mathrm{res}(\frac{a}{b})$ if $v(a) \geq v(b)$ and 0, otherwise.

The following result is a well-known.

Fact 1.2.16. *A complete theory of algebraically closed valued fields eliminates quantifiers in $\mathcal{L}_{val}$.*

We define the language $\mathcal{L}_{val,ac}$ to be the expansion of $\mathcal{L}_{val}$ by adding a map $ac : \mathrm{VF} \to \mathrm{k}$. For each $a \in \mathrm{VF}$, $ac(a) = \mathrm{res}(a)$ if $a \in \mathcal{O}$ and $ac(a) = 0$ otherwise.

Fact 1.2.17. *([Pas90]) The theory of henselian valued fields of equicharacteristic zero with angular components eliminates quantifiers resplendently relatively to Γ and k in the language $\mathcal{L}_{val,ac}$.*

Next, we define the language $\mathcal{L}_{\mathrm{VF,RV}}$ of valued fields, consisting of two sorts: VF and RV. Each sort is equipped with the language of rings. The addition on RV is defined as in Fact 1.2.5: for $x, y \in K$,

$$rv(x) + rv(y) = \begin{cases} rv(x+y), & \text{if } v(x+y) = \min\{v(x), v(y)\}, \\ 0_{\mathrm{RV}}, & \text{otherwise.} \end{cases}$$

We also include the map $rv : \mathrm{VF} \to \mathrm{RV}$ given by the natural projection on $\mathrm{VF}^\times$, with $rv(0) := 0_{rv}$.

The following fact was originally proved in [Bas91], we cite it from a more recent source.

Fact 1.2.18. *([Rid17, Theorem 1.4]) The complete theory of henselian valued fields of equicharacteristic zero eliminates quantifiers resplendently and relatively to* RV *in the language* $\mathcal{L}_{\mathrm{VF,RV}}$

As an immediate consequence of the previous two facts, we have the following corollary.

Corollary 1.2.19.
1. *Let $T_{val,ac}$ be a complete theory of henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{val,ac}$. Then the sorts* k *and* Γ *are purely stably embedded and orthogonal to each other.*
2. *Let T_{rv} be a complete theory of henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{\mathrm{RV}}$. Then the sort* RV *is purely stably embedded.*

Theory of henselian valued fields also eliminates quantifiers if we choose a different expansion of the language $\mathcal{L}_{val}$. We can add the multiplicative sorts $\mathrm{k}^\times/(\mathrm{k}^\times)^n$. This approach is analogous to the quantifier elimination for p-adically closed fields, where predicates for nth roots are added to the language (see [Mac76]). In the henselian case, introducing these additional sorts ensures that the short exact sequence $1 \to \mathrm{k}^\times \to \mathrm{RV}^\times \to \Gamma \to 0$ splits. The language was introduced in [ACGZ22] and as explained in [Vic22], the quantifier elimination result of the field quantifiers was given.

Definition 1.2.20.
1. We define $\mathcal{L}_{val,\mathcal{A}}$ to be the expansion of $\mathcal{L}_{val}$ by the *power residue sorts* $\mathcal{A}_n = \mathrm{k}^\times/(\mathrm{k}^\times)^n \cup \{\mathbf{0}_n\}$. For each $n \in \mathbb{N}$, we add the maps $\pi_n : \mathrm{k} \to \mathcal{A}_n$ where π_n is the projection map on $\mathrm{k}^\times$ and $\pi_n(0) := 0_n$. We also add the maps $\mathrm{res}_n : \mathrm{VF} \to \mathcal{A}_n$ defined as follows: for all $a \in \mathrm{VF}$, if $v(a) \in n\Gamma$, then $\mathrm{res}_n(a) = \pi_n(\mathrm{res}(\frac{a}{b}))$ for any b with $v(a) = nv(b)$. We note that this is well-defined and does not depend on the choice of b. We set $\mathrm{res}_n(a) := 0_n$ otherwise.

2. We define $\mathcal{L}_{\mathrm{RV},\mathcal{A}} = \big(\mathrm{RV}, \mathrm{k}, \Gamma, \mathcal{A}, v_{rv}, \iota, (\rho_n)_{n\in\mathbb{Z}}, (\pi_n)_{n\in\mathbb{N}}\big)$ as the language of short exact sequences of monoids. Here, the sort Γ is in the language of ordered groups expanded by a symbol ∞. The sorts RV and k have the language of monoids $\{\cdot, 1, 0\}$, with the sort k further expanded by the sorts $\mathcal{A}$ as described in part (1). The map $\iota : \mathrm{k} \to \mathrm{RV}$ is the natural embedding, and $v_{rv} : \mathrm{RV} \to \Gamma$ is the induced valuation map on RV. For $a \in \mathrm{RV}$, the function ρ_n is defined as follows: if $a \in v_{rv}^{-1}(n\Gamma)$, then $\rho_n(a) = \rho_n(ab^{-1})$ for some $b \in \mathrm{RV}$ with $nv_{rv}(b) = v_{rv}(a)$; otherwise, $\rho_n(a) := 0$. This is a well-defined map and does not depend on the choice of b.

Fact 1.2.21. *([ACGZ22, Theorem 5.15]) Every $\mathcal{L}_{val}$ formula $\varphi(x, y, z)$ where x, y and z are variables belonging to* VF, Γ, k, *respectively, is equivalent to a $\mathcal{L}_{val,\mathcal{A}}$-formula $\psi(t_1(x), \cdots, t_n(x), y, z)$ where t_i's are one of the following forms: $v(p(x))$, $\mathrm{res}(\frac{p(x)}{q(x})$ or $\mathrm{res}^n(p(x))$ where p and q are polynomials with integer coefficients. In particular, if D is a subset of $\mathrm{k}_{\mathcal{A}}$ (respectively Γ) over a parameter set C in the valued field sort, then there exists an $\mathrm{k}_{\mathcal{A}}$-formula (respectively, a Γ-formula) and some $u \in \mathrm{dcl}(C)$ such that $\psi(x, u)$ defines D. Hence, the sorts $\mathrm{k}_{\mathcal{A}}$ and Γ are stably embedded with the control of parameters.*

The following, then is a direct consequence of Fact 1.2.21.

Fact 1.2.22. *Let T be a theory of henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{val,\mathcal{A}}$. Then T eliminates field quantifiers. Moreover, the sorts $\mathrm{k}_{\mathcal{A}}$ and Γ are purely stably embedded and orthogonal to each other.*

A suitable language for the elimination of quantifiers for the theories of the short exact sequence of abelian groups is given in [ACGZ22]. Here, we state this result in the context of valued field structures.

Fact 1.2.23. *([ACGZ22, Corollary 4.8]) The theory of $(\mathrm{RV}, \Gamma, \mathrm{k})$ in the language $\mathcal{L}_{\mathrm{RV},\mathcal{A}}$ eliminates the* RV*-quantifiers. Moreover, the sorts $\mathrm{k}_{\mathcal{A}}$ and Γ are purely stably embedded and orthogonal to each other.*

The Ax-Kochen-Ershov Principle

For henselian valued fields of equicharacteristic zero or unramified mixed characteristic with a perfect residue field, the theory is completely determined by the value group and residue field. This was proven by Ax and Kochen in [AK65] and, independently, by Ershov in [Es65].

Fact 1.2.24. *Let $(K, \Gamma_K, \mathrm{k}_K)$ and $(M, \Gamma_M, \mathrm{k}_M)$ be henselian fields of equicharacteristic or unramified henselian valued field of mixed characteristic and with perfect residue field. Then*

$(K, \Gamma_K, \mathrm{k}_K) \equiv (L, \Gamma_L, \mathrm{k}_L)$ *if and only if* $\Gamma_K \equiv \Gamma_L$ *as ordered abelian groups and* $\mathrm{k}_K \equiv \mathrm{k}_L$ *as fields.*

In the light of Fact 1.2.24, one can consider which model theoretical properties can be lifted from the residue and the value group to the theory of the valued field. This approach is known as the *AKE*-principle. For example, some tameness properties of the theory are transferred from the residue field and the value group. Such a result first appeared in [Del81] in which it is proved that the theory of henselian valued fields is NIP if and only if the theory of the residue field is NIP (the theory of the value group is always NIP). A further instance has been proved more recently.

Fact 1.2.25. *([Che14, Theorem 7.6]) A henselian valued field of equicharacteristic zero is* NTP_2 *if and only if the theory of its residue field is* NTP_2*,*

The following is a folklore fact and it can be deduced from Theorem 1.2.24.

Fact 1.2.26. *Let K be a henselian valued field which is either equicharacteristic zero or is unramified of mixed characteristic with perfect residue field. Then K has an elementary extension M, which is a maximally complete model.*

Algebraically Closed Valued Fields

In this section, we survey model-theoretic properties of algebraically closed valued fields that we will need in this thesis. We have seen that a complete theory of algebraically closed valued fields eliminates quantifiers in 1-sorted language $\mathcal{L}_{div}$. This leads to a characterization of definable subsets in the field sort.

A *swiss cheese* is a set of the form $B \setminus (D_1 \cup \ldots \cup D_n)$, where $B, D_1, \ldots, D_n$ are balls and $D_1, \cdots, D_n$ are pairwise disjoint and are proper subballs of B.

Fact 1.2.27. *([Hol95]) Let K be an algebraically closed valued field. If $X \subseteq K$, then X is a union of finitely many swiss-cheeses.*

Next, we recall the characterization of definable and algebraic closure in the valued field sort.

Notation 1.2.28. *Let K be a field and $C \subseteq K$. Then C^{alg} denotes the field-theoretic algebraic closure of C. If a is a tuple of elements (possibly infinite) or a set of elements, then $C(a)$ denotes the field generated by C and a.*

Fact 1.2.29. *([Hil18, Corollary 4.13]) Let K be an algebraically closed valued field, seen as an $\mathcal{L}_{div}$-structure. Then, for $A \subseteq K$,*

1. $\operatorname{acl}(A) = \mathbb{F}(A)^{alg}$*, where F is $\mathbb{Q}$ if $\operatorname{char} K = 0$ and is $\mathbb{F}_p$ otherwise.*
2. $\operatorname{dcl}(A) = \mathbb{F}(A)^h$*, the henselization of the field $\mathbb{F}(A)$ (see Definition 1.2.13).*

The following is another well-known fact that follows from the stable embeddedness with control of parameters of the sorts k and Γ.

Fact 1.2.30. *Let $A \subseteq K$. Then,* $\operatorname{acl}(\mathrm{k}_A) = \mathrm{k}_{A^{alg}}$, $\operatorname{acl}(\Gamma_A) = \operatorname{dcl}(\Gamma_A) = \mathbb{Q} \otimes \Gamma_A$.

The Language of Geometric Sorts

In this section, we recall the the language of geometric sorts for which the theory of algebraically closed valued fields eliminate imaginaries.

For $n \geq 1$, let S_n be the set of codes for free $\mathcal{O}$-submodules of K^n. We can identify each free $\mathcal{O}$-submodule λ of K^n with an element in $GL_n(K)/GL_n(\mathcal{O})$. We define $T_n = \bigcup_{s \in S_n} \Lambda(s)/\mathfrak{m}\Lambda(s)$, where $\Lambda(s)$ is the $\mathcal{O}$-submodule coded by s. Let $\tau_n : S_n \to T_n$ be the natural projection map. Furthermore, for each $n \geq 1$ we define the natural projection maps $\lambda_n : GL_n(K) \to S_n$ and $\sigma_n : GL_n(K) \to T_n$.

Definition 1.2.31. Let $\mathcal{L}_0$ denote the expansion of the language $\mathcal{L}_{val}$ of valued fields obtained by adding sorts $\mathcal{S} = \bigcup_n S_n$ and $\mathcal{T} = \bigcup_n T_n$ along with the maps $(\tau_i, \lambda_n, \sigma_n)_{i>0}$ as described above.

Example 1.2.32.

1. Let $\gamma\mathcal{O} = \{a \in K : v(a) \geq \gamma\}$. It is easy to check that $\gamma\mathcal{O}$ is an $\mathcal{O}$-module, hence lies in S_1. Conversely, using Fact 1.2.27, one can show that any $\mathcal{O}$-submodule of K is of this form.
2. If $\Lambda(s) = \gamma\mathcal{O}$, then the quotient element $a + \mathfrak{m}\Lambda(s)$ is the open ball $B_{>\gamma}(a)$. Hence, the open balls are coded in T_1.

Fact 1.2.33. *([HHM06]) A complete theory of algebraically closed valued fields eliminates imaginaries in $\mathcal{L}_0$.*

We close this section by stating another key property of algebraically closed valued fields.

Fact 1.2.34. *([HHM08, Corollary 8.16]) Let $\widetilde{\mathcal{U}}$ be a universal model of algebraically closed valued fields in the language $\mathcal{L}_0$. Let $C = \mathrm{acl}(C)$ be a subset and a be a finite sequence in $\widetilde{\mathcal{U}}$. Let M be a model containing C. Then,* $\mathrm{tp}(a/C)$ *has a C-invariant extension to some type $p \in S(M)$.*

Chapter 2

Residual Domination

The notion of stable domination was introduced by Haskell, Hrushovski and Macpherson in [HHM08] for algebraically closed valued fields. Later, it was generalized to *residual domination* (also referred to as *residue field domination*) first by Ealy, Haskell and Marikova in [EHM19] for real closed valued fields and then to a broader class of henselian valued fields by Ealy, Haskell and Simon in [EHS23], by Vicaria in [Vic22] and recently by Kovacsics, Rideau-Kikuchi and Vicaria in [KRKV24]. In this chapter, we will provide an algebraic characterization of residual domination in pure henselian valued fields of equicharacteristic zero, analogous to the result given for algebraically closed valued fields in [EHS23]. Our proofs are very similar to those in [EHS23]. Using this result, we relate the residual domination in the henselian field to the stable domination in the algebraically closed valued field that it embeds into.

For algebraically closed valued fields, it is well known that the stable sorts are precisely those internal to the residue field. In the setting of henselian valued fields, it was shown in [Vic22] and [EHS23] that certain types are dominated by a collection of sorts internal to the residue field over the value group. We will further show that these types are stably dominated over the value group in the algebraic closure of the henselian field. Conversely, we will show that a type that is stably dominated over the value group is dominated in the henselian field by the same collection of sorts internal to the residue field.

The organization of this chapter is as follows: we start by giving a survey on domination in valued fields, which covers the facts on stable domination in algebraically closed valued fields and residual domination in henselian valued fields. In the subsequent sections, we will present our main results.

2.1 A Survey on Domination in Valued Fields

In this section, we recall definitions and results on domination in valued fields. An abstract notion of domination was introduced in [EHM19]. Let $\mathcal{L}$ be an arbitrary language, and let T be an $\mathcal{L}$-theory with a universal model $\mathcal{U}$. Suppose $\mathcal{S}$ and $\mathcal{T}$ are sorts (or collections of sorts) that are stably embedded in $\mathcal{U}$. Additionally, assume that there exists a ternary relation, denoted by $\downarrow^{\mathcal{S}}$ on $\mathcal{S}$, which provides a notion of independence.

For a set A, we write $\mathcal{S}(A)$ to denote the set $\mathcal{S} \cap \operatorname{dcl}(A)$.

Definition 2.1.1. Let a be a tuple, and let C be a set of parameters in $\mathcal{U}$. We say that $\operatorname{tp}(a/C)$ is dominated by $\mathcal{S}$ if for every tuple b in $\mathcal{U}$ such that $\mathcal{S}(Ca) \downarrow^{\mathcal{S}}_C \mathcal{S}(Cb)$, we have

$$\operatorname{tp}(a/C\mathcal{S}(Cb)) \vdash \operatorname{tp}(a/Cb).$$

We say $\operatorname{tp}(a/C)$ is *dominated by $\mathcal{S}$ over $\mathcal{T}$* if, $\operatorname{tp}(a/C\mathcal{T}(Ca))$ is dominated by $\mathcal{S}$.

In the context of henselian valued fields, the stably embedded sorts that we will work with will be the residue field k and the value group Γ. We will use the field theoretic algebraic independence in the sort k. As stated in [EHS23], the existence of *good separated basis* is crucial to achieve the desired type implication.

Separated Extensions

Let $(C, v) \subseteq (L, v)$ be valued fields, let V be a finite dimensional C-vector subspace of L. We say V is *separated over C* if there exists a C-basis $\vec{b} = (b_1, \ldots, b_n)$ of V such that for all $c_1, \ldots, c_n \in C$,

$$v(\sum_i c_i b_i) = \min\{v(c_i) + v(b_i)\}_i.$$

In this case, $\vec{b}$ is called a *separated basis*. A separated basis is *good* if in addition, for all $1 \leq i, j \leq n$, we have $v(b_i) = v(b_j)$ or $v(b_i) + \Gamma(C) \neq v(b_j) + \Gamma(C)$.

Finally, we say L has *the separated basis property over C* (or equivalently, the field extension *L/C is separated*) if every finite-dimensional C-vector subspace of L has a separated basis over C. We say L has *the good separated basis property over C*, if every finite-dimensional C-vector subspace of L has a good separated basis over C

In fact, separated basis property always implies the good separated basis property.

Remark 2.1.2. *Let $(C, v) \subseteq (L, v)$ be valued fields, and suppose that L has the separated basis property over C. Then L also has the good separated basis property over C.*

Proof. We will prove by induction on the dimension of C-vector subspace of L. Let U be a finite-dimensional C-vector space of dimension n, with a good separated basis $b_1, \ldots, b_n$. Let $b \in L \setminus U$. By assumption, we may assume that $(b_1, \ldots, b_n, b)$ is separated over C. First, suppose that $v(b) \neq v(u)$ for any $u \in U$. Then, in particular, $v(b) \neq v(c) + v(b_i)$ for any $c \in C$ and $i \leq n$. Hence, $(b_1, \ldots, b_n, b)$ remains a good basis.

Now suppose that $v(b) = v(u)$ for some $u \in U$. Then there exists $i_0 \leq n$ and $c \in C$ such that $v(b) = v(c) + v(b_{i_0})$. After replacing b with $c^{-1}b$, we may assume that $v(b) = v(b_{i_0})$. Therefore, $(b_1, \ldots, b_n, b)$ is again a good basis. □

Fact 2.1.3. *([Del88, Corollary 7]) Any algebraic extension of a henselian field is separated.*

The following fact follows from the definition.

Fact 2.1.4. *([Del88, Lemma 5]) Let $(C, v) \subseteq (L, v)$ be valued fields. Assume that U and V are finite-dimensional C-vector subspaces of L with $U \subseteq V$ and V is separated over K. Then U is separated over C.*

As mentioned in [Del88], the transitivity of having the separated basis property can be derived using Fact 2.1.4:

Proposition 2.1.5. *Let $(C, v) \subseteq (L, v) \subseteq (M, v)$ be valued fields such that L/C and M/L are separated. Then M/C is separated.*

Proof. Let $(f_i)_{i \in I}$ be a separated basis of M over L and $(e_j)_{j \in J}$ a separated basis of L over C. For any $x \in M$, we can write $x = \sum l_i f_i = \sum \left(\sum c_{i,j} e_j\right) f_i$. It is easy to see that the products $e_i f_j$ are distinct and the set $\{e_j f_i \mid j \in J, i \in I\}$ is C-linearly independent. Moreover, it remains separated over C since the elements f_i's are separated over L and $ce_j \in L$ for any $c \in C$ and $j \in J$. Thus, $\{e_j f_i \mid j \in J, i \in I\}$ is a separated basis of M over C.

If U is a finite-dimensional C-vector subspace of M, then U lies in the C-span of finitely many $e_j f_i$'s. By Fact 2.1.4, this implies U is separated over C. □

The following is a well-known example of separated extensions. This field extension is called the *Gauss extension.*

Example 2.1.6. Let C be a valued field and let a be a singleton with $v(a) = 0$ and $\mathrm{res}(a)$ is transcendental over C. Let L be the field generated by C and a. Then L is separated over C.

Proof. We will show that for every $n \in \mathbb{Z}^{>0}$ and $c_1, \ldots, c_n \in C$, $v(\sum_{i=1}^{n} c_i a^i) = \min\{v(c_i)\}$. Suppose not. Let $\gamma = \min\{v(c_i)\}_i$ and $I = \{j : v(c_j) = \gamma\}$. We may assume that $1 \in I$. Then $v(\sum_{j \in I} c_j c_1^{-1} a^j) > 0$. It follows that $\sum_{j \in I} \mathrm{res}(c_j c^{-1}) \mathrm{res}(a)^j = 0$, which contradicts with the transcendentality of $\mathrm{res}(a)$ over k_C. The separatedness of L over C then follows immediately. □

Recall that a valued field is maximally complete if it does not have a proper immediate extension. The following is a well-known fact. Recall that by Remark 2.2.8, the separated basis property implies the good separated basis property.

Fact 2.1.7. *[HHM08, Lemma 12.2] Let C be maximally complete. Then every valued field extension of C has the separated basis property over C.*

Stably Dominated Types

Throughout this subsection, we fix a language $\mathcal{L}$ and its theory T. We will work in a universal model $\mathcal{U}$ of T.

Let C be a set. A C-definable set D is said to be *stable* if for any C-definable formula $\varphi(x; y)$, the formula $\varphi(x, y) \wedge \psi(x)$, where $\psi(x)$ defines D, is stable. If D is stable, then it is also stably embedded. However, in the literature, such sets are referred to as stable and stably embedded, and we will stick to this convention.

Let St_C denote the collection of all C-definable, stable and stably embedded sets in $\mathcal{U}$. We will consider St_C as a multi-sorted structure whose sorts are the stable and stably embedded C-definable sets, equipped with their induced structure. Additionally, for each relation on finitely many C-definable stable and stably embedded sets $D_1, \ldots, D_n$ in $\mathcal{U}$, we include the corresponding relations in St_C.

Fact 2.1.8. *([HHM08, Lemma 3.2]) The structure St_C is stable.*

We informally refer to the sort St_C as the *stable sorts*. Then a type is stably dominated if it is dominated by the stable sorts, where we use the forking independence, denoted by $\downfree$. The formal definition is given below.

Definition 2.1.9. Let C be a parameter set and a be a tuple. Then the type $\mathrm{tp}(a/C)$ is *stably dominated* if for every tuple $b \in \mathcal{U}$,

$$\mathrm{tp}(a/C\mathrm{St}_C(Cb)) \vdash \mathrm{tp}(a/Cb),$$

whenever $\mathrm{St}_C(Ca) \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{St}_C(Cb)$.

Let M be a model. We say that a type $p \in S(M)$ is *stably dominated* if there exists $C \subseteq M$ such that the restriction $p|C$ is stably dominated.

Example 2.1.10. 1. In any theory, if a type p lies in the stable sorts, then p is stably dominated.

2. Let T be a complete theory of algebraically closed valued fields in the language $\mathcal{L}_0$ of geometric sorts. For any ball B, define

$$p^B = x \in B \cup x \notin B' : B' \subset B \text{ is a proper subball of } B.$$

By Fact 1.2.27, every definable set in a model M is a boolean combination of M-definable balls. Hence, p^B is a *complete* type, called the *generic type of* B.

When $B = \mathcal{O}$, we have that $a \models p^{\mathcal{O}}$ implies $a \notin \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal, i.e., the open ball $B_0^{op}(0)$. Therefore, $v(a) = 0$, and so $\mathrm{res}(a) \notin 0$.

For a set C, we have $a \models p|C$ if and only if $\mathrm{res}(a)$ does not lie in any $\mathrm{acl}(C)$-definable subset of k. Indeed, suppose $a \models p^{\mathcal{O}}|C$. If $\mathrm{res}(a) = \alpha$ for some $\alpha \in \mathrm{acl}(C)$, then $a \in \mathrm{res}^{-1}(\alpha)$, which is a proper $\mathrm{acl}(C)$-definable subball of $\mathcal{O}$, a contradiction. Conversely, suppose $\mathrm{res}(a) \notin \mathrm{acl}(C)$, and assume there exists a proper closed subball $B_d^{\mathrm{cl}}(\delta) \subset \mathcal{O}$ with $d, \delta \in \mathrm{acl}(C)$ such that $a \in B_d^{\mathrm{cl}}(\delta)$. Then $v(a - d) \geq \delta > 0$. By Lemma 1.2.3, it follows that $v(a) = v(d) = 0$. Applying res, we get $\mathrm{res}(a) = \mathrm{res}(d)$, but $\mathrm{res}(d) \in \mathrm{acl}(C)$, a contradiction.

Since the residue field k is an algebraically closed field, it is stable. We will show that $p^{\mathcal{O}}|C$ is stably dominated via the map res for any set C.

Let b be any tuple such that $\mathrm{res}(a) \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{St}(Cb)$. Suppose $b \equiv_{C\mathrm{res}(a)} b'$, and let $\sigma \in Aut(\mathcal{U}/C\mathrm{res}(a))$ with $\sigma(b) = b'$ and $\sigma(a) = a'$. We will show that $b \equiv_{Ca} b'$. The forking independence in particular implies that $\mathrm{res}(a) \notin \mathrm{acl}(Cb)$, hence $a \models p^{\mathcal{O}}|Cb$. As $b \equiv_{C\mathrm{res}(a)} b'$, it follows that $\mathrm{res}(a) \notin \mathrm{acl}(Cb')$, hence $a \models p^{\mathcal{O}}|Cb'$ as well.

Now let $\varphi(x, y)$ be any $\mathcal{L}(C)$-formula. Then $\models \varphi(a, b)$ if and only if $\models \varphi(a', b')$. Since $a, a' \models p^{\mathcal{O}}|Cb'$, we have $a \equiv_{Cb'} a'$, which implies $\models \varphi(a, b')$. As φ was arbitrary, it follows that $\mathrm{tp}(b/Ca) = \mathrm{tp}(b'/Ca)$, hence $b \equiv_{Ca} b'$, as desired.

With a similar argument one can show that the generic type of any closed ball is stably dominated.

3. The generic type $p^{\mathfrak{m}}$ of the maximal ideal $\mathfrak{m}$, which is the open ball $B_0^{op}(0)$ is not stably dominated. Suppose on the contrary that $p^{\mathfrak{m}}|C$ is stably dominated over some set C. If $a \models p^{\mathfrak{m}}|C$, then we must have $0 < v(a) < \delta$ for all positive $\delta \in \mathrm{dcl}(C)$, in particular $v(a) \notin \Gamma(C)$. It follows that for $c_1, \dots, c_n \in \mathrm{VF}(C)$, we have $v(\sum_{i=1}^{n} c_i a^i) = \min\{v(c_i) + v(a^i)\}$, since for distinct i, j, we have $v(c_i) + v(a_i) \neq v(c_j) + v(a_j)$. In particular this means for all $u \in \mathrm{VF}(Ca) \setminus \mathrm{VF}(C)$, $v(u) \notin \Gamma(C)$. Moreover, since $a \notin \mathrm{acl}(C)$, the Zariski–Abhyankar inequality (Fact 1.2.6) implies that $\mathrm{k}(Ca) = \mathrm{k}(C)$. By Fact 2.1.18, we can identify St_C with $\mathrm{k} \cap \mathrm{acl}(C)$, hence $\mathrm{St}_C(Ca) = \mathrm{St}_C(C)$. It follows trivially that $\mathrm{St}_C(Ca) \perp_C \mathrm{St}_C(Ca)$. Therefore, we must have $\mathrm{tp}(a/C\mathrm{St}_C(Ca)) \vdash \mathrm{tp}(a/Ca)$. However, $\mathrm{tp}(a/C\mathrm{St}_C(Ca)) = \mathrm{tp}(a/C)$ is not a realized type, while $\mathrm{tp}(a/Ca)$ is realized by a — a contradiction. Thus, $p^{\mathfrak{m}}|C$ is not stably dominated.

Stably dominated types behave similarly to types in stable theories. In the following facts, we outline their main properties.

Fact 2.1.11. *([Proposition 3.13, Corollary 3.31 (iii), Corollary 6.12][HHM08], [HRK19, Proposition 2.10 (1)]) For all tuples a and C,*

(i) Suppose $C = \mathrm{acl}(C)$. If $\mathrm{tp}(a/C)$ is stably dominated then it has a unique C-definable extension p that satisfies the following: for all $B \supseteq C$, $a \models p|B$ if and only if $\mathrm{St}_C(Ca) \perp_C \mathrm{St}_C(B)$. Moreover, p is symmetric*: $(a, b) \models p \otimes p$ if and only if $(b, a) \models p \otimes p$*

(ii) $\mathrm{tp}(a/C)$ is stably dominated if and only if $\mathrm{tp}(a/\mathrm{acl}(C))$ is.

(iii) If $\mathrm{tp}(a/C)$ is stably dominated and $b \in \mathrm{acl}(Ca)$, then $\mathrm{tp}(b/C)$ is stably dominated.

Fact 2.1.12. *([HHM08, Theorem 4.9, Corollary 6.12]) Let a be a tuple and C be a set.*

(i) Suppose that p is an $\mathrm{acl}(C)$-invariant global type and $p|C$ is stably dominated. Then for every $B \supseteq C$, $p|B$ is stably dominated.

(ii) If p is a global C-invariant type, stably dominated over $B \supseteq C$, and if $\mathrm{tp}(B/C)$ has a global C-invariant extension, then p is stably dominated over C.

Another key property of stably dominated types is the existence of strong codes for germs (see Definition 1.1.26), which is another similarity to types in stable theories.

Fact 2.1.13. *([HHM08, Theorem 6.3]) Let $C = \mathrm{acl}(C)$ and p be a global C-invariant type, stably dominated over C. Let f be a definable function which is defined on the set of realizations of p. Then the p-germ of f is strong over C. Moreover, if $f(a) \in \mathrm{St}_C$ where $a \models p$, then the code of the germ of f lies in St_C.*

In the case of algebraically closed valued fields, the stably dominated types are characterized as types that are *orthogonal* to the value group in the following sense.

Fact 2.1.14. *([HHM08, Theorem 10.7]) Let p be a global C-invariant type in an algebraically closed valued field. Then p is stably dominated if and only if for any model $M \supseteq C$ and $a \models p|M$, we have $\Gamma(Ma) = \Gamma(M)$.*

Domination in valued fields can be expressed purely as an algebraic statement, in which case, the type implication can be rephrased as finding an extension of valued field isomorphisms. In [HHM08], algebraic domination statements are given under the assumption that the base of the type is a maximally complete field, which guarantees that any of its extensions has the separated basis property. In [EHS23], it was emphasized that it is the separated basis property that is crucial, rather than the maximality of the base field. The following result combines [HHM08, Proposition 12.11] and [EHS23, Proposition 3.1] to make it explicit. We include the proof for completeness.

Notation 2.1.15. *Let A be a subset of a valued field K. Then we write $\Gamma_A := \{v(a) : a \in A\}$ and $\mathrm{k}_A := \{\mathrm{res}(a) : a \in A\}$.*

Fact 2.1.16. *Let C, L and M be valued fields such that $C \subseteq L \cap M$. Assume that $\Gamma_M \cap \Gamma_L = \Gamma_C$, and k_M and k_L are linearly disjoint over C. Further, suppose that L (respectively, M) has the separated basis property over C. Then the following are satisfied:*

(i) L has the separated basis property over M (respectively, M has it over L),

(ii) If N is the field LM generated by L and M, then the value group Γ_N is generated by Γ_L and Γ_M as groups, and k_N is generated by k_L and k_M as fields.

(iii) If $\sigma : L \to L'$ is a valued field isomorphsim fixing C, Γ_L and k_L, then there exists a valued field isomorphism $\tau : LM \to L'M$ which is the identity on M and extends σ.

Proof. We begin by proving (i). Let $\vec{u} = (u_1, \cdots, u_n)$ be a C-linearly independent tuple in L. We will first show that $\vec{u}$ is M-linearly independent and its M-span has a good separated basis. By assumption, we may assume that $\vec{u}$ is already a good separated basis over C. If $v(u_1) = v(u_2) = \ldots = v(u_n)$, then one can observe that the tuple $1, \mathrm{res}(\frac{u_2}{u_1}), \ldots, \mathrm{res}(\frac{u_n}{u_1})$ is k_C-linearly independent.

Now fix $x = \sum_{i=1}^{n} m_i u_i$, where $m_1, \cdots, m_n \in M$. Let $J := \{j \ : \ v(m_j u_j) = \gamma\}$, where $\gamma = \min\{v(l_i m_i) \ : \ i \leq n\}$. Without loss of generality, we can assume that $1 \in J$. Suppose on the contrary that $v(\sum_{i \in J} m_i u_i) > v(m_1 u_1)$. For all distinct $i, j \in J$, we have $v(\frac{u_i}{u_j}) = v(\frac{m_i}{m_j})$, and it follows that $v(\frac{u_i}{u_j}) \in \Gamma_L \cap \Gamma_M = \Gamma_C$. As $\vec{u}$ is a good basis over C and $v(u_i)$, and $v(u_j)$ lie in the same coset in Γ_L/Γ_C, we must have $v(u_i) = v(u_j)$. It follows that $v(m_i) = v(m_j)$ for all $i, j \in J$. Now, we have $v(1 + \sum_{i \in J} \frac{u_i m_i}{u_1 m_1}) > 0$. After applying the residue map, we obtain $1 + \sum_{i \in J} \mathrm{res}(\frac{u_i}{u_1})\mathrm{res}(\frac{m_i}{m_1}) = 0$. By the assumption that k_M and k_L are linearly disjoint over k_C, we can conclude that the tuple $1, \mathrm{res}(\frac{u_2}{u_1}), \ldots, \mathrm{res}(\frac{u_k}{u_1})$ is k_C-linearly dependent. But, this contradicts with the choice of $\vec{u}$. Hence, $\vec{u}$ is separated over C.

For (ii), it is enough to check $x \in N$ of the form $x = \sum_{i \leq n} l_i m_i$ where $l_1, \ldots, l_n \in L$ and $m_1, \ldots, m_n \in M$. By part (i), we can assume that $l_1, \ldots, l_n$ is a good separated basis over M. We may assume that $v(x) = v(l_1) + v(m_1)$. It then follows immediately that Γ_N is generated by Γ_L and Γ_M. Now, further assume that $x \in \mathcal{O}_N^{\times}$, so that $v(x) = 0$. As in the proof of (i), let $J = \{i \leq n : \ v(l_i m_i) = 0\}$. We can assume that $1 \in J$. Then

$$\mathrm{res}(x) = \mathrm{res}(l_1 m_1)(1 + \sum_{i \in J \setminus \{1\}} \frac{l_i m_i}{l_1 m_1}).$$

Since $v(l_1 m_1) = 0$, it follows that $v(l_1) = -v(m_1) \in \Gamma_C$. Let $c \in C$, with $v(l_1) = v(c)$ and define $l = l_1 c^{-1}$ and $m = m_1 c$. Then we have $\mathrm{res}(l_1 m_1) = \mathrm{res}(l)\mathrm{res}(m)$. Moreover, as shown in part (i), we also have $v(\frac{l_i}{l_1}) = v(\frac{m_i}{m_1})$ for all $i \in J \setminus \{1\}$. It follows that

$$\mathrm{res}(x) = \mathrm{res}(l)\mathrm{res}(m)(1 + \sum_{i \in J \setminus \{1\}} \mathrm{res}(\frac{l_i}{l_1})\mathrm{res}(\frac{m_i}{m_1}).$$

Hence, $\mathrm{res}(x) \in \mathrm{k}_L \mathrm{k}_M$, as desired.

For (iii), since L and M are linearly disjoint over C, there exists a field isomorphism $\tau : LM \to L'M$ fixing M and extending σ. We will show that τ is a valued field isomorpshism.

By part (i), for any $x \in LM$, we can write $v(x) = v(l) + v(m)$ for some $l \in L$ and $m \in M$. Since σ fixes Γ_L and τ fixes M, it follows that $v(x) = v(\tau(x))$. As ACVF, eliminates field quantifiers, we conclude that τ is a valued field isomorphism, as desired. □

An algebraic characterization of stably dominated types in algebraically closed valued fields is given in [EHS23].

Fact 2.1.17. *([EHS23, Theorem 3.6]) Assume that $\mathcal{U}$ is an algebraically closed valued field in the language of geometric sorts $\mathcal{L}_0$. Let $C \subseteq \mathcal{U}$ be a subfield and a be a tuple of valued field elements from $\mathcal{U}$. Let $L = \mathrm{dcl}(Ca)$. Assume that L is a regular extension of C. Then the following are equivalent.*

1. *$\mathrm{tp}(a/C)$ is stably dominated.*
2. *L is an unramified extension of C and has the good separated basis property over C.*

We close this section, by recalling a characterization of stable sorts in algebraically closed valued fields.

Fact 2.1.18. *([HHM06, Proposition 3.4.11]) Let C be an algebraically closed valued field and D be a C-definable set in a complete theory of algebraically closed valued fields. Then the following are equivalent:*

(i) D is stable and stably embedded, i.e. $D \subseteq \mathrm{St}_C$,

(ii) D is k-internal.

Residually Dominated Types

In this subsection, we work in the language $\mathcal{L}_{val,\mathcal{A}}$ (see Definition 1.2.20), which we denote by $\mathcal{L}$ for simplicity. We fix a complete theory, T, of a henselian valued field of equicharacteristic zero. Let $\mathcal{U}$ be its universal model.

Notation 2.1.19. *Let L, M and C be fields with $C \subseteq L \cap M$. We write $L \mathop{\smile\!\!\!\!\!\!|}_C^{alg} M$ if L is field-theoretically algebraically independent from M over C. Namely, for every finite tuple $a_1, \ldots, a_n \in L$, we have $tr.deg(a_1, \cdots, a_n/C) = tr.deg(a_1, \cdots, a_n/M)$.*

The following definition is given in [Vic22]. In our notation, we recall that $\mathrm{k}_{\mathcal{A}}$ denotes $\mathrm{k} \cup \mathcal{A}$.

Definition 2.1.20. Let a be a tuple in $\mathcal{U}$ and C be a subset of $\mathcal{U}$. We say $\operatorname{tp}(a/C)$ is *residually dominated* if for every tuple b in $\mathcal{U}$, we have

$$\operatorname{tp}\big(b/C\mathrm{k}_{\mathcal{A}}(Ca)\big) \vdash \operatorname{tp}(b/Ca)$$

whenever $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\!\mid}_C^{alg} \mathrm{k}(Cb)$.

Let f be a C-definable map into the sort $\mathrm{k}_{\mathcal{A}}$ whose domain contains realizations of $\operatorname{tp}(a/C)$. Then we say $\operatorname{tp}(a/C)$ is *residually dominated via* f if for every tuple b in $\mathcal{U}$, $\operatorname{tp}(b/Cf(a)) \vdash \operatorname{tp}(b/Ca)$ holds whenever $f(a) \mathop{\smile\!\!\!\!\!\!\mid}_C^{alg} \mathrm{k}(Cb)$.

The following result is essentially [Vic22, Theorem 3.8]. In the original statement, the base set is assumed to be a maximal model of T_{Hen}; here, we weaken this assumption assuming the existence of the separated basis property. For simplicity, we use a stronger assumption on the value group. Aside from these modifications, the proof remains essentially unchanged, but we include it here for the sake of completeness.

Fact 2.1.21. *([Vic22, Theorem 3.8]) Let L and M be valued fields and C be a common subfield. Assume that*

1. *L or M has the separated basis property over C,*
2. *$\Gamma(L) = \Gamma(C)$,*
3. *$\mathrm{k}(M)$ and $\mathrm{k}(L)$ are linearly disjoint over $\mathrm{k}(C)$.*

Then $\operatorname{tp}(L/C\mathrm{k}_{\mathcal{A}}(M)\Gamma(M)) \vdash \operatorname{tp}(L/M)$.

Proof. Let $L' \models \operatorname{tp}(C\mathrm{k}_{\mathcal{A}}(M)\Gamma(M))$ and let $\sigma : LM \to L'M'$ be an $\mathcal{L}$-isomorphism, sending L to L', $M \to M'$ and fixing $C \cup \mathrm{k}_{\mathcal{A}}(M) \cup \Gamma(M)$. Consider $\sigma^{-1}|M : M' \to M$, the restriction of σ^{-1} on M'. By Fact 2.1.16, we know that there exists a valued field isomorphism $\tau : L'M' \to L'M$, fixing L' and extending $\sigma^{-1}|_{M'}$. Then $h = \tau \circ \sigma$ is a valued field isomorphism between LM and $L'M$ which fixes M and sends L to L'. It remains to show that h is an $\mathcal{L}$-isomorphism.

For this, we will show that for $a, b \in LM$, if $\operatorname{res}^n(a) = \operatorname{res}^n(b)$, then we have $\operatorname{res}^n(h(a)) = \operatorname{res}^n(h(b))$. We know that $v(a) = v(l) + v(m)$ for some $l \in L$ and $m \in M$. By assumption, there is $c \in \operatorname{dcl}(C)$ with $v(c) = v(l)$. Let $l' = lc^{-1}$ and $m' = mc$, then both $v(l')$ and $v(m')$ lie in $n\Gamma$. Let $x = \frac{a}{l'm'}$, then we have

$$\operatorname{res}^n(a) = \pi_n(\operatorname{res}(x))\operatorname{res}^n(l')\operatorname{res}^n(m').$$

Similarly, we can find $y \in \mathcal{O}_{LM}^{\times}$ and $l'' \in L$ and $m'' \in M$ such that

$$\mathrm{res}^n(b) = \pi_n(\mathrm{res}(y))\mathrm{res}^n(l'')\mathrm{res}^n(m'').$$

By Fact 2.1.16 (*ii*), $\mathrm{res}(x)$ and $\mathrm{res}(y)$ lie in $\mathrm{k}_L\mathrm{k}_M$. Moreover, $\mathrm{res}^n(m')$ and $\mathrm{res}^n(m'')$ are elements of $\mathcal{A}(M)$, so we can conclude that $\mathrm{res}^n(a) = \mathrm{res}^n(b)$ can be represented by a formula in $\mathrm{tp}(L/C\mathrm{k}_{\mathcal{A}}(M)\Gamma(M))$. Since h extends an $\mathcal{L}$-elementary map σ that is the identity map on $C \cup \mathrm{k}_{\mathcal{A}}(M) \cup \Gamma(M)$, we conclude that $\mathrm{res}^n(h(a)) = \mathrm{res}^n(h(b))$. Hence, we can extend h to an $\mathcal{L}$-definable map by sending $\mathrm{res}^n(x)$ to $\mathrm{res}^n(h(x))$. By Fact 1.2.22, h is elementary and can be extended to an $\mathcal{L}$-isomorphism of $\mathcal{U}$.

□

Example 2.1.22.

1. Any type that lies in the residue field is residually dominated. Indeed, if $a \in \mathrm{k}_{\mathcal{A}}$, then for every tuple b in $\mathcal{U}$, the type implication $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(b/Ca)$ holds, since the sort $\mathrm{k}_{\mathcal{A}}$ is stably embedded with control of parameters (see Fact 1.2.21).

2. Analogously to Example 2.1.10(2), we can define a *generic type* of a ball within a henselian field of equicharacteristic zero. Consider the set

$$p^{\mathcal{O}} = \{x \notin B : B \subset \mathcal{O} : \text{is a proper subball of } \mathcal{O}\}.$$

Unlike in the case of ACVF, this set is not a complete type, since it does not contain any information on the sort $\mathcal{A}$. Let p be a complete type that contains $p^{\mathcal{O}}$. We will show that p is residually dominated.

In this case, we first show that for any set C, if $a \models p|C$, then we have $\mathcal{A}(Ca) \subseteq \mathcal{A}(C\mathrm{res}(a))$. In fact, by Example 2.1.6 (2), we know that for every $c_1, \dots, c_n \in C$ and $v(\sum_{i=1}^{n} c_i a^i) = \min\{v(c_i) + v(a)\}$. Then for $x = \sum_{i=1}^{n} c_i a^i$ with $v(x) \in n\Gamma$, we have

$$\mathrm{res}^n(\sum_{i=1}^{n} c_i a^i) = \pi_n\left(\mathrm{res}\Big(\frac{\sum_{i=1}^{n} c_i a^i}{c_{i_0} a^{i_0}}\Big)\right)\mathrm{res}^n(c_{i_0})\mathrm{res}^n(a^{i_0})$$

where $v(x) = v(c_{i_0}) + v(a)$. Since $\mathrm{res}(a) = 0$, we have $\mathrm{res}\Big(\frac{\sum_{i=1}^{n} c_i a^i}{c_{i_0} a^{i_0}}\Big) = \sum_{i=1}^{n} \mathrm{res}(\frac{c_i}{c_{i_0}})\mathrm{res}(a)^{i-i_0} \in$

$\mathrm{dcl}(C\mathrm{res}(a))$. We conclude $\mathrm{res}^n(x) \in \mathrm{dcl}(C\mathrm{res}(a))$.

Let b be a tuple in $\mathcal{U}$ such that $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\!\vert}_C^{\mathrm{alg}} \mathrm{k}(Cb)$. We may assume that b is a tuple in the field sort, since we can take b to be an enumeration of a model $M \supseteq Cb$ with $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\!\vert}_C^{\mathrm{alg}} \mathrm{k}(M)$. Our goal is to show that $\mathrm{tp}(b/C\mathrm{res}(a)) \vdash \mathrm{tp}(b/Ca)$. Let $L := \mathrm{dcl}(Ca)$ and $M := \mathrm{dcl}(Cb)$.

By Example 2.1.6, we know that L is separated over C, and as shown in its proof, $\Gamma_L = \Gamma_C$. Since C is acl-closed, it follows that k_M and k_L are linearly disjoint over C. Thus, by Fact 2.1.21, we have

$$\mathrm{tp}(L/C\mathrm{k}_{\mathcal{A}}(M)\Gamma(M)) \vdash \mathrm{tp}(L/M).$$

By the stable embeddedness of the sorts $\mathrm{k}_{\mathcal{A}}$ and Γ, this is equivalent to $\mathrm{tp}(M/C\mathrm{k}_{\mathcal{A}}(L)) \vdash \mathrm{tp}(M/L)$. (Note that $\Gamma(L) = \Gamma(C)$). We conclude $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(b/Ca)$.

By the discussion above, we have in fact shown that the type $p|C$ is residually dominated via the map res, since $\mathrm{k}_{\mathcal{A}}(L) \subseteq \mathrm{dcl}(C\mathrm{res}(a))$.

3. Any completion of a generic type of an open ball is not residually dominated. The proof is similar to the ACVF case and we refer to Example 2.1.10 (3).

We continue by quoting the following fact from [Vic22] on forking.

Fact 2.1.23. *([Vic22, Theorem 4.4]) Let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ where the residue field is equipped with the imaginary expansion k^{eq}, likewise the value group is equipped with its imaginary expansion Γ^{eq}. Let M be a maximally complete model and a, b be tuples. Then $a \mathop{\smile\!\!\!\!\!\!\vert}_M b$ if and only if $\mathrm{k}_{\mathcal{A}}(Ma)\Gamma(Ma) \mathop{\smile\!\!\!\!\!\!\vert}_M \mathrm{k}^{eq}(Ma)\Gamma^{eq}(Ma)$.*

Notice that Fact 2.1.21 does not state that $\mathrm{tp}(L/C)$ is residually dominated, we also need to consider the parameters outside of the valued field sort. We will show that it is indeed enough to consider the parameters inside the valued field sort.

Lemma 2.1.24. *Let $a \in \mathcal{U}$ be a tuple and C be a parameter set in the valued field sort. Then for any tuple b in $\mathcal{A}$, $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(a/Cb)$.*

Proof. If $\varphi(x, b, c) \in \mathrm{tp}(a/Cb)$ where c is a tuple in C, then by Fact 1.2.21, $\varphi(x, b, c)$ is equivalent to a formula $\psi(t(x), b)$ where t is a tuple of terms with parameters from C and a variable x such that $t(x)$ lies in $\mathrm{k}_{\mathcal{A}}$. Since ψ is $C\mathrm{k}_{\mathcal{A}}(Cb)$-definable, the type implication follows immediately. □

Theorem 2.1.25. *Let a be a tuple in $\mathcal{U}$ and C be a parameter set in the valued field sort. Then the following are equivalent:*

(i) For any finite tuple $b \in \mathcal{U}$, if $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{alg}_{\mathrm{k}(C)} \mathrm{k}(Cb)$, then $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(a/Cb)$,

(ii) For any finite tuple $b \in \mathrm{VF}(\mathcal{U})$, if $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{alg}_{\mathrm{k}(C)} \mathrm{k}(Cb)$, then $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(a/Cb)$.

Proof. The direction $(i) \Rightarrow (ii)$ is obvious. For the converse direction, take a tuple $b \in \mathcal{U}$ with $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{alg}}_{\mathrm{k}(C)} \mathrm{k}(Cb)$. Note that for all b in $\mathcal{A}$, by Lemma 2.1.24, we have $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(a/Cb)$. Thus, we may assume that the tuple b does not lie entirely in $\mathcal{A}$. There exists a model M containing Cb such that $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{alg}_{C\mathrm{k}(Cb)} \mathrm{k}(M)$. By transitivity of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{alg}}$, it follows that $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{alg}}_{C} \mathrm{k}(M)$. By (ii), we then have $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(M)) \vdash \mathrm{tp}(a/M)$, which is equivalent to $\mathrm{tp}(M/\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(M/Ca)$. It immediately follows that $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(b/Ca)$, which is equivalent to $\mathrm{tp}(a/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(a/Cb)$. □

Finally, using Fact 2.1.21 and Theorem 2.1.25, we obtain the following corollary.

Corollary 2.1.26. *Let a be a tuple in the valued field sort, and let C be a subset of the valued field such that C is algebraically closed in T. Define $L := \mathrm{dcl}(Ca)$. Assume that L has the separated basis property over C, $\Gamma(L) = \Gamma(C)$, and $\mathrm{k}(L)$ is a regular extension of $\mathrm{k}(C)$. Then $\mathrm{tp}(a/C)$ is residually dominated.*

Types Dominated by k-Internal Sorts

Another instance of domination observed in henselian valued fields is domination by certain collections of k-internal sorts. For this, we will work in the language $\mathcal{L}_{\mathrm{RV},\mathcal{A}}$ (see Definition 1.2.20), and assume that T is a complete theory of henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{\mathrm{RV},\mathcal{A}}$. For $\Delta \subseteq \Gamma$ and a subset M in the valued field, we define

$$\mathrm{kInt}^{M}_{\Delta} := \mathrm{k}(M) \cup \{\mathrm{RV}_{\gamma}(M)\}_{\gamma \in \Delta},$$

where RV_{γ} is the fiber above γ under the map $v_{rv} : \mathrm{RV} \to \Gamma$. Recall that each fiber RV_{γ} can be regarded as a 1-dimensional k-vector space after adding the constant 0_{RV}, and is definably isomorphic to k via the map $f_{\gamma}(a) = \frac{a}{d_{\gamma}}$ for some fixed $d_{\gamma} \in \mathrm{RV}_{\gamma}$. Therefore, the collection of kInt-sorts is k-internal. Furthermore, when L is a model, $\mathrm{kInt}^{L}_{\Gamma_L}$ coincides with the residue field of L, since $\mathrm{RV}_{\gamma}(L)$ is non-empty.

We refer such sorts as kInt-sorts in short, but note that it does not cover all k-internal definable sets. As shown in [HHM08], these sorts serve as a suitable analogue for stable sorts in henselian valued fields.

Fact 2.1.27. *([HHM08, Lemma 12.9]) Let C and L be algebraically closed valued fields with $C \subseteq L$. Let $a_1, \dots, a_n \in L$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and $d_1, \dots, d_n \in \mathrm{RV}$ such that for each $1 \leq i \leq n$, $v(a_i) = \gamma_i$ and $d_i = rv(a_i)$. Define $C' := \mathrm{acl}_0(C\gamma_1 \dots \gamma_n)$ and $C'' := \mathrm{acl}(Cd_1 \dots d_n)$. Then $\mathrm{St}_{C'}(L) = \mathrm{acl}_0(C''\mathrm{k}(L)) \cap \mathrm{St}_{C'}$.*

Notation 2.1.28. *Let C, L, and M be subfields of $\mathcal{U}$ such that $C \subseteq L \cap M$, $L = \mathrm{dcl}(L)$ and $\Gamma_L \subseteq \Gamma_M$. We write $\mathrm{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!|}^0_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}$ if and only if in the stable structure $\mathrm{St}_{C\Gamma_L}$, $\mathrm{kInt}^L_{\Gamma_L}$ and $\mathrm{kInt}^M_{\Gamma_L}$ are independent over $C\Gamma_L$.*

Definition 2.1.29. Let $C \subseteq L$ be substructures of $\mathcal{U}$ with Γ_L/Γ_C torsion-free. Then the type $\mathrm{tp}(L/C\Gamma_L)$ is *dominated by* kInt-*sorts* if, for every M containing $C\Gamma_L$, we have

$$\mathrm{tp}(M/C\Gamma_L\mathrm{kInt}^L_{\Gamma_L}) \vdash \mathrm{tp}(M/L)$$

whenever $\mathrm{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!|}^0_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}$.

The characterization for the independence in $\mathrm{St}_{C\Gamma_L}$ is given in [HHM08].

Fact 2.1.30. *([HHM08, Lemma 12.10]) Let C, L and M be substructures of $\mathcal{U}$. Assume that $\Gamma_L \subseteq \Gamma_M$. Let $a_1, a_2, \dots, a_r$ and $b_1, \dots, b_s$ be tuples in L (where r and s are possibly infinite) such that $v(a_1), \dots, v(a_r)$ forms a $\mathbb{Q}$-basis of Γ_L over Γ_C, and $\mathrm{res}(b_1), \dots, \mathrm{res}(b_s)$ form a transcendence basis of k_L over k_C. Assume that $e_1, \dots, e_r \in M$ such that for each $1 \leq i \leq r$, $v(e_i) = v(a_i)$. Then the following statements are equivalent:*

1. *The elements*
$$\mathrm{res}(\frac{a_1}{e_1}), \dots, \mathrm{res}(\frac{a_r}{e_s}), \mathrm{res}(b_1), \dots, \mathrm{res}(b_s)$$
are algebraically independent over k_M.

2. $\mathrm{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!|}^0_{C\Gamma_L} \mathrm{kInt}^M_{C\Gamma_L}$.

The algebraic conditions for domination by kInt-sorts are given in both [EHS23, Theorem 3.9] and [Vic22, Theorem 5.12]. In the former, the base set C is not assumed to be a maximal model, whereas in the latter it is assumed to be a model. Again, the existence of the separated basis property is crucial here. Since the languages are different in these two settings, we will include the proof for the sake of completeness. Before proceeding, we quote a fact concerning the composition of places.

Fact 2.1.31. *([HHM08, Lemma 12.16]) Let (L, v) be a valued field with value group Γ, and let $p : L \to \mathrm{k}_L \cup \infty$ be the corresponding place. Fix a subfield $F \subseteq \mathrm{k}_L$, and assume that $p' : \mathrm{k}_L \to F \cup \infty$ is a place that restricts to the identity on F.*

Define the composed place $p^* := p' \circ p : L \to F \cup \infty$, and let $v : L \to \Gamma^*$ be the valuation corresponding to p^*. Suppose $a \in L$ satisfies $p(a) \in \mathrm{k}_L \setminus 0$ and $p(a) = 0$. Then:

(i) If $b \in L$ with $v(b) \geq 0$, then

$$0 < v^*(a) \ll v^*(b),$$

where $v^*(a) \ll v^*(b)$ means that for all $n \in \mathbb{Z}^{>0}$, $nv^*(a) < v^*(b)$.

(ii) There exists a convex subgroup $\Delta \subseteq \Gamma^*$ defined by

$$\Delta := \{\pm v^*(x) \mid x \in L,\ p(x) \notin \{0, \infty\},\ p^*(x) = 0\} \cup \{0_{\Gamma^*}\},$$

such that there is an isomorphism $g : \Gamma^*/\Delta \to \Gamma$ satisfying $g \circ v^* = v$.

Fact 2.1.32. *Let $C, L,$ and M be subfields of $\mathcal{U}$ such that L has a finite transcendence degree over C, $C \subseteq L \cap M$, and $M \supseteq C \cup \Gamma_L$. Assume that the following are satisfied:*

1. *L has the separated basis property over C,*
2. *k_L is a regular extension of k_C,*
3. *$\mathrm{kInt}^L_{\Gamma_L} \mathop{\smile\!\!\!\!\!\!\,|}^{0}_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}$*

Then $\mathrm{tp}(L/C\Gamma_L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L}) \vdash \mathrm{tp}(L/M)$.

Proof. Let $L' \models \mathrm{tp}(L/C\Gamma_L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L})$, and let σ be an automorphism of $\mathcal{U}$ witnessing this. We will show that there exists an $\mathcal{L}$-isomorphism $\tau : LM \to L'M$ that is the identity on M and extends $\sigma|_L$.

***Claim* 1.** *We may assume that σ is identity on Γ_M.*

Proof. We have $\mathrm{tp}(L/C\Gamma_L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L}) \vdash \mathrm{tp}(L/C\Gamma(L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L}))$. Since Γ is stably embedded with control of parameters in $\mathcal{L}_{\mathrm{RV},\mathcal{A}}$, by Fact 1.1.5, we know that the right-hand of this type implication implies $\mathrm{tp}(L/C\Gamma(L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L})\Gamma(M))$. If $\Gamma(L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L}) = \Gamma(L)$, we're done. As $M \supseteq \Gamma_L$, for each $\gamma \in \Gamma_L$, $\mathrm{RV}_\gamma(M)$ is non-empty, hence they are isomorphic to $\mathrm{k}(M)$ via a definable map with parameters from $\mathrm{RV}_\gamma(M)$. We know that the sorts Γ and $\mathrm{k}_\mathcal{A}$ are orthogonal to each other in the structure RV, it follows that $\mathrm{kInt}^M_{\Gamma_L}$ and Γ are orthogonal to each other. By Fact 1.1.6, Then it follows that $\Gamma(L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L}) = \Gamma(L)$. $\square$

Now, we outline the rest of the proof. We will construct a finer valuation v' on LM by perturbing v. We will show that $\Gamma_{(L,v')} \cap \Gamma_{(M,v')} = \Gamma_{(C,v')}$ and $\mathrm{k}_{(L,v')}$ and $\mathrm{k}_{(M,v')}$ are linearly disjoint over $\mathrm{k}_{(C,v')}$. Then we will apply Fact 2.1.16 to deduce there is a valued field isomorphism $\tau : LM \to L'M$ which is the identity map on M and extends $\sigma|_L$. As a final step, using Theorem 2.2.12 below, we will show that τ can be assumed to be an $\mathcal{L}_{\mathrm{RV},\mathcal{A}}$-isomorphism.

We will construct a finer valuation on the field LM using the compositions of places. Let

- $a_1, \ldots, a_r \in L$ such that $(v(a_i))_{i \leq r}$ is a $\mathbb{Q}$-basis of Γ_L over Γ_C.
- $e_1, \ldots, e_n \in M$ such that for each $i \leq r$, $v(a_i) = v(e_i)$,
- $b_1, \ldots, b_s \in L$ such that $(\mathrm{res}(b_i))_{i \leq s}$ forms a transcendence basis of k_L over k_C.

As $\mathrm{kInt}^L_{\Gamma_L} \downarrow_{C\Gamma_L} \mathrm{kInt}^M_{C\Gamma_L}$, by Fact 2.1.30, the elements

$$\mathrm{res}(\frac{a_1}{e_1}), \ldots, \mathrm{res}(\frac{a_r}{e_r}), \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)$$

are algebraically independent over k_M.

For $0 \leq j \leq r$, we define

$$R^{(j)} := \mathrm{acl}(\mathrm{k}_M, \mathrm{k}_L, \mathrm{res}(\frac{a_1}{e_1}), \ldots, \frac{a_r}{e_r}) \cap \mathrm{k}_{LM}.$$

Note that $R^{(0)} = \mathrm{acl}(\mathrm{k}_M), \mathrm{k}_L) \cap \mathrm{k}_{LM}$.

Next, for each $0 \leq j \leq r-1$, we choose a place

$$p^{(j)} : R^{(j+1)} \to R^{(j)}$$

that fixes $R^{(j)}$. We can also assume that $\mathrm{res}(\frac{a_{j+1}}{e_{j+1}})$ is sent to 0, since, it does not lie in the algebraically closed set R^j.

Let $p_v : LM \to \mathrm{k}_{LM}$ be the corresponding place to the valuation v, and let $p^* : \mathrm{k}_{LM} \to R^{(r)}$ be a place fixing R^r. Let $p_{v'} : LM \to R^{(0)}$ be the place given by the composition

$$p_{v'} = p^{(0)} \circ \cdots \circ p^{(r-1)} \circ p^* \circ p_v.$$

Let v' be the corresponding valuation on LM.

We observe that $p_{v'}$ and p_v coincide on k_M, since all $p^{(j)}$ and p^* are the identity map on k_M. Thus, we will identify (M, v) and (M, v'). Similarly, they coincide on k_L, thus (Γ, v) and (Γ, v') are isomorphic. However, we cannot simultaneously identify (Γ, v) with (Γ, v') and (M, v) with (M, v').

Let $x \in M$ with $v(x) > 0$. Then after applying Fact 2.1.31 (i) repeatedly, we obtain

$$0 < v^*(\frac{a_1}{e_1}) \ll \cdots \ll v^*(\frac{a_r}{e_r}) \ll v^*(x).$$

Let

$$\Delta = \{v^*(x), -v^* \ : \ x \in LM,\ p_v(x \notin \{0, \infty\}) \text{ and } p'(x) = 0\} \cup \{0_{\Gamma_{(LM,v)}}\}.$$

Then by Fact 2.1.31, Δ is a convex subgroup of $\Gamma_{(LM,v')}$. Moreover, it contains $v'(\frac{a_i}{e_i})$ for all $i \leq r$, by construction.

For each $1 \leq i \leq r$, we write $v'(\frac{a_i}{e_i}) = \delta_i$, and $v'(e_i) = v(e_i) = \epsilon_i$.

***Claim* 2.** $\Gamma_{(L,v')} \cap \Gamma_{(M,v')} = \Gamma_{(C,v')}$.

Proof. Let $m \in M$ and $l \in L$ with $v'(l) = v'(m)$. By the choice of $\{v(a_i)\}_i$, we can find $p_1, \ldots, p_n \in \mathbb{Q}$, and $\gamma \in \Gamma_C$ such that

$$v(l) = \sum_{i=1}^{r} p_i v(a_i) + \gamma$$

As $\Gamma_{(L,v')}$ and $\Gamma_{(L,v)}$ are isomorphic over Γ_C, we further have

$$\begin{aligned} v'(l) &= \sum_{i=1}^{r} p_i v'(a_i) + \gamma \\ &= \sum_{i=1}^{r} p_i \delta_i - \sum_{i=1}^{r} p_i \epsilon_i + \gamma. \end{aligned}$$

By assumption, $\epsilon_1, \ldots, \epsilon_r$ are $\mathbb{Q}$-independent elements of $\Gamma_{(M,v')}$ over Γ_C, so we complete it to a basis $\{\epsilon_i\}_{i \leq r} \cup \{\mu_{\alpha_j}\}_{j \in J}$. Then we can find $p'_1 \ldots, p'_r, q_1, \ldots q_t \in \mathbb{Q}$ and $\gamma' \in \Gamma_C$ such that

$$v'(m) = \sum_{i=1}^{r} p'_i \epsilon_i + \sum_{j=1}^{n} q_i \mu_{\alpha_i} + \gamma'$$

Because $v'(l) = v'(m)$, we observe

$$\sum_{i=1}^{r} p_i\delta_i + \sum_{i=1}^{r}(p_i' - p_i)\epsilon_i - \sum_{j=1}^{n} q_i\mu_{\alpha_i} + \gamma - \gamma_i = 0.$$

Note that $\sum_{i=1}^{r} p_i\delta_i \in \Delta$, where we know that the elements of Δ are infinitisemally small with respect to the elements of $\Gamma_{(M,v')}$. Thus, we must have $p_i = 0$ for all $1 \leq i \leq r$. Moreover, the elements $\{\epsilon_i\}_{i\leq r} \cup \{\mu_{\alpha_j}\}_{j\in J}$ forms a $\mathbb{Q}$-basis of $\Gamma_{(M,v')}$ over Γ_C. Thus, we must have $p_i' = 0$ for all i and $q_j = 0$ for all j. As a result,

$$v'(l) = \gamma = \gamma' = v'(m),$$

where $\gamma = \gamma' \in \Gamma_C$, as desired. □

***Claim* 3.** $\mathrm{k}_{(L,v')}$ *and* $\mathrm{k}_{(M,v')}$ *are linearly disjoint over* $\mathrm{k}_{(C,v')}$

Proof. By construction of p', we have $\mathrm{k}_{(L,v')} = \mathrm{k}_{(L,v)}$ and $\mathrm{k}_{(M,v')} = \mathrm{k}_{(M,v)}$. As k_L is a regular extension of k_C, the result follows from Fact 2.2.6. □

Note that (L, v') has the separated basis property over (C, v'), since (L, v) and (L, v') are isomorphic over $\Gamma_{(C,v)} = \Gamma_{(C,v')}$. Together with Claims 2 and 3, this ensures that the hypothesis of Fact 2.1.16 is satisfied. Therefore, there exists a valued field isomorphism $\tau : LM \to LM'$ that is the identity on M and extends $\sigma|_L$. Since v' is finer than v, τ is also a valued field isomorphism with respect to v.

We have shown that there exists a valued field isomorphism $LM \to LM'$ extending an $\mathcal{L}$-isomorphism $\sigma : L \to L'$ over the set $C\Gamma_L\mathcal{A}(M)\mathrm{kInt}^M_{\Gamma_L}$, where $\mathrm{kInt}^L_{\Gamma_L} \downarrow_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}$. Finally, by Theorem 2.2.12 below, this is equivalent to the existence of an $\mathcal{L}$-isomorphism $LM \to LM'$ that fixes L. □

Resolutions

We close this section by showing that finite tuples in the geometric sorts are *resolved* in $\mathcal{U}$. This was proved in [EHS23], but we also show that the result holds in the language with power residue sorts.

Definition 2.1.33. Let $C \subset \mathrm{VF}(\mathcal{U})$ and e be a finite tuple of imaginaries in $\mathcal{U}$. A set $B \subseteq \mathrm{VF}(\mathcal{U})$ is a *resolution* of Ce if it is acl-closed in the valued field sort and $Ce \subseteq \mathrm{dcl}(B)$. Moreover, the resolution is *prime* if it embeds over Ce into every resolution of Ce.

The following fact is essentially Theorem 4.2 in [EHS23]. However, we restate it in a slightly different form, as presented in the proof of Theorem 4.2.

Fact 2.1.34. *([EHS23, Theorem 4.2]) Let $C \subseteq \mathrm{VF}(\mathcal{U})$ be a subfield, e be a tuple that lies in the geometric sorts of $\mathcal{U}$. Then there exists a prime resolution $B \subseteq \mathcal{U}$ of Ce. Moreover there exists $B_0 \subseteq B$ with $\mathrm{acl}(B_0) = \mathrm{acl}(B)$, $Ce \in \mathrm{dcl}(Ce)$, $\mathrm{k} \cap \mathrm{dcl}_0(B_0) = \mathrm{k} \cap \mathrm{dcl}_0(Ce)$ and $\Gamma \cap \mathrm{dcl}_0(B_0) = \Gamma \cap \mathrm{dcl}_0(Ce)$.*

Theorem 2.1.35. *Let e be a finite tuple that does not contain any element from the sort $\mathcal{A}$, or more generally, lies in the geometric sorts. Then there exists a prime resolution of B with $\mathrm{k}_{\mathcal{A}}(B) = \mathrm{k}_{\mathcal{A}}(\mathrm{acl}(Ce))$ and $\Gamma(B) = \Gamma(Ce)$.*

Proof. Let B and B_0 be as in the conclusion of Fact 2.1.34. Using Fact 1.2.21 we can conclude that in $\mathcal{U}$ we have $\mathrm{k}(B_0) = \mathrm{k}(Ce)$ and $\Gamma(B_0) = \Gamma(Ce)$. Moreover, $\mathcal{A}(B_0) \subseteq \mathcal{A}(Ce)$, since $\mathrm{RV}(B_0) = \mathrm{RV}(Ce)$. Thus $\mathcal{A}(B_0) = \mathcal{A}(Ce)$. It follows that $\mathrm{k}_{\mathcal{A}}(B) = \mathrm{k}_{\mathcal{A}}(\mathrm{acl}(Ce))$. By Remark 1.2.21, we can similarly show that $\Gamma(B) = \Gamma(Ce)$ in $\mathcal{U}$. $\square$

2.2 Equivalence of Residual Domination and Stable Domination

In this section, we will provide our main results on residually dominated types. Throughout this section, we fix a complete theory of henselian valued fields of equicharacteristic zero in the language $\mathcal{L}_{val,\mathcal{A}}$, which we will denote by $\mathcal{L}$, for simplicity.

Sometimes, we will want to work within an algebraically closed valued field. In this case, we consider $\mathcal{U}$ as a substructure of an algebraically closed valued field $\widetilde{\mathcal{U}}$, where the valuation on $\mathcal{U}$ extends uniquely to the valuation on $\widetilde{\mathcal{U}}$, as stated in Fact 1.2.12 (3). Thus, we will regard the valued fields inside $\mathcal{U}$ as substructures of the algebraically closed field.

Before stating our proofs, we will need the following results from [EHS23].

Fact 2.2.1. *([EHS23, Lemma 1.13]) In any theory, let C, F and L be subsets of a universal model with $C \subseteq L \cap F$. Then, $\operatorname{tp}(L/C) \vdash \operatorname{tp}(L/F)$ is equivalent to $\operatorname{tp}(F/C) \vdash \operatorname{tp}(F/L)$.*

Notation 2.2.2. *Let C be a subset of a field. We write C^{alg} for the field-theoretic algebraic closure of C. If $a_1, \ldots, a_n$ is a tuple of field elements, then $\operatorname{alg}(a_1, \ldots, a_n)$ denotes K^{alg}, where K is the field generated by $\mathbb{Q}$ and $a_1, \ldots, a_n$.*

Fact 2.2.3. *([EHS23, Lemma 1.19]) Let $C \subseteq L$ be valued fields such that L is a regular extension of C and L is henselian. Then, $\operatorname{tp}_0(L/C) \vdash \operatorname{tp}_0(L/C^{alg})$.*

Fact 2.2.4. *([Lan02, VIII, 4.12]) Let L and C be fields such that $C \subseteq L$ and L is a regular extension of C. Then, for any field M, $L \mathop{\smile\!\!\!\!\!\!\;|}_C^{alg} M$ implies that L and M are linearly disjoint over C.*

Notation 2.2.5. *For a subset A of a field K, we define $\operatorname{alg}(A) := \mathbb{Q}(A)^{\mathrm{alg}}$, where $\mathbb{Q}(A)$ is the field generated by A, and $\mathbb{Q}(A)^{\mathrm{alg}}$ is its field-theoretic algebraic closure.*

Lemma 2.2.6. *Let C, F and L be valued fields in $\mathcal{U}$ such that $C \subseteq F \cap L$. Assume that L is a regular extension of C and $\operatorname{tp}(L/C) \vdash \operatorname{tp}(L/F)$. Then, L and F are linearly disjoint over C.*

Proof. Suppose on the contrary that L and F are not linearly disjoint over C. By Fact 2.2.4, there exists a finite tuple $l_1, \cdots, l_n \in L$ which is algebraically independent over C but algebraically dependent over F. We may assume $l_1 \in \operatorname{alg}(Fl_1, \cdots, l_n)$, hence $l_1 \in \operatorname{acl}(Fl_1, \cdots, l_n)$. Let X be a set of all conjugates of l_1 over $Fl_1, \cdots, l_n$ in $\mathcal{U}$. By assumption, there exists an automorpshim σ of U fixing $Cl_2, \cdots, l_n$ and sending l_1 to some $m \notin X$.

Then, by the type implication, there exists an automorphism fixing $Fl_2, \cdots, l_n$ and sending l_1 to $m \notin X$, a contradiction. $\square$

We now show that if a type is residually dominated, then its realization does not add any new information to the value group.

Lemma 2.2.7. *Let a be a tuple in $\mathrm{VF}(\mathcal{U})$ and C be a subfield such that $\mathrm{tp}(a/C)$ is residually dominated. Then $\Gamma(Ca) = \Gamma(C)$.*

Proof. Suppose, on the contrary, that there exists $\gamma \in \Gamma(Ca) \setminus \Gamma(C)$. First observe that $\mathrm{k}(C\gamma) = \mathrm{k}(C)$. Indeed, let $\alpha \in \mathrm{k}(C\gamma)$ and let $\varphi(x, \gamma, d)$ be a formula witnessing this, where d is a tuple in $\mathrm{dcl}(C)$. By the orthogonality of $\mathrm{k}_{\mathcal{A}}$ and Remark 1.2.21, there is a formula $\psi(x, d')$ in the sort $\mathrm{k}_{\mathcal{A}}$ with $d' \in \mathrm{k}_{\mathcal{A}}(C)$ such that α is the unique realization. Hence, $\alpha \in \mathrm{k}(C)$. Then the independence $\mathrm{k}(Ca) \downarrow^{alg}_{\mathrm{k}(C)} \mathrm{k}(C\gamma)$ holds trivially since since $\mathrm{k}(C) = \mathrm{k}(C\gamma)$. By domination, we obtain $\mathrm{tp}(\gamma/C\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(\gamma/Ca)$, leading to a contradiction. In fact, by orthogonality again, if $\gamma \in \mathrm{dcl}(C\mathrm{k}_{\mathcal{A}}(Ca))$, then $\gamma \in \Gamma(C)$, contradicting the assumption. $\square$

Lemma 2.2.8. *Let $C \subseteq L$ be subfields of $\mathcal{U}$. Suppose L is a regular extension of C and $\mathrm{tp}(L/C)$ is residually dominated. Then, for any maximal immediate extension F of C in $\mathcal{U}$, we have $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/F)$ in $\widetilde{\mathcal{U}}$.*

Proof. Fix a maximal immediate extension F of C. Notice that $\mathrm{k}(L) \downarrow^{alg}_{\mathrm{k}(C)} \mathrm{k}(F)$ holds trivially, since $\mathrm{k}(F) = \mathrm{k}(C)$. Then, by domination, $\mathrm{tp}(L/C\mathrm{k}_{\mathcal{A}}(F)) \vdash \mathrm{tp}(L/F)$ holds. Since, $\mathrm{k}(F) = \mathrm{k}(C)$, in particular, $\mathcal{A}(C) = \mathcal{A}(F)$, it follows that $\mathrm{tp}(L/C) \vdash \mathrm{tp}(L/F)$.

We will show that every valued field isomorphism $\sigma : L \to L'$ fixing C extends to a valued field isomorphism $\tau : LF \to L'F$, fixing F. By Lemma 2.2.6, L and F are linearly disjoint over C. Hence, there is a ring isomorphism $\tau : LF \to L'F$ that sends L to L' and fixes F. Moreover, by [HHM08, Proposition 12.1], any F-span of a finite tuple from L has a separated basis over F. So, for every $d \in LF$, there exists a separated F-linearly independent tuple $l_1, \cdots, l_n \in L$ and coefficients $f_1, \cdots, f_n \in F$ such that $d = \sum f_i l_i$, and $v(d) = \min\{v(f_i) + v(l_i)\}$. Since $\Gamma(L) = \Gamma(C) = \Gamma(F)$, it follows that $v(\tau(d)) = \tau(v(d))$. Then, τ is a valued field isomorphism and thus $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/F)$. $\square$

Theorem 2.2.9. *Let C be a field and a be a tuple from the field sort. Assume that $\mathrm{tp}(a/C)$ is residually dominated and $L := \mathrm{dcl}(Ca)$ is a regular extension of C. Then, L has the separated basis property over C.*

Proof. First, note two easy cases. If $a \in \mathrm{dcl}(C)$, then $a \in C^{\mathrm{alg}}$. Since L is a regular extension of C, it follows that $a \in C$, and the result follows immediately. If $\Gamma(C)$ is trivial, then by Lemma 2.2.7, $\Gamma(L)$ is also trivial, and the result follows trivially. Hence, assume a does not lie in $\mathrm{dcl}(C)$ and C is not trivially valued.

Let $C' = \mathrm{dcl}(C)$ and F be a maximal immediate extension of C inside $\mathcal{U}$. Note that C' is henselian since it is dcl_0-closed. Moreover, we may assume that F is an elementary extension of C' in the theory $T' := Th(C')$. Note that T and T' implies the same quantifier free sentences with parameters in C'.

As the next step, we will show that L and F are linearly disjoint over C. Since L is a regular extension of C, by Fact 2.2.4, it suffices to prove $L \mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{alg}}_{C} F$. Suppose, for contradiction, that there are tuples $l_1, \ldots, l_n$ which are algebraically independent over C but algebraically dependent over F. We may assume that $l_1 \in \mathrm{alg}(Fl_2, \ldots, l_n)$.

Notice that by Lemma 2.2.8, we have $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/F)$. Let $\theta \in \mathrm{tp}_0(l_1, \ldots, l_n/F)$ be a formula witnessing that $l_1 \in \mathrm{alg}(Fl_2, \ldots, l_n)$. Then there exists $\varphi \in \mathrm{tp}_0(l_1, \ldots, l_n/C)$ that implies θ. Hence, φ witnesses that $l_1 \in \mathrm{acl}(Cl_2, \ldots, l_n)$, and thus $l_1 \in \mathrm{alg}(Cl_2, \ldots, l_n)$, a contradiction.

Now, we will prove the claim by induction on the dimension n of C-vector subspaces of L. For $n = 1$, the claim is obvious. Assume that $\vec{l} = (l_1, \ldots, l_n) \in L^n$ is C-linearly independent and separated over C. Let $v \in L$ be such that v does not lie in the C-span $C \cdot \vec{l}$ of $\vec{l}$. Since L and F are linearly disjoint over C, the tuple $(l_1, \ldots, l_n, v)$ is F-linearly independent. By [HHM08, Proposition 12.1], there exists $u \in F \cdot \vec{l}$ such that $v(u) = \max\{v(w - v) : w \in C \cdot \vec{l}.\}$

Since F is maximally complete, by Fact 2.1.7, LF is a separated extension of F. It follows that Γ_{LF} is generated by $\Gamma_L = \Gamma(L)$ and $\Gamma_F = \Gamma(F)$. By Lemma 2.2.7, we have $\Gamma(L) = \Gamma(C) = \Gamma(F)$. Hence $v(u) \in \Gamma(C)$. Set $\gamma := v(u)$.

Define $\theta(x_1, \cdots, x_n)$ to be the formula $v(\sum_{i=1}^{n} x_i l_i) = \gamma$. Clearly, $\theta \in \mathrm{tp}_0(F/L)$. By Lemma 2.2.8 we have $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/F)$, and by Fact 2.2.1, $\mathrm{tp}_0(F/C) \vdash \mathrm{tp}_0(F/L)$. Thus, there is $\rho(x_1, \cdots, x_n) \in \mathrm{tp}_0(F/C)$ that implies θ (in the theory T). As F is an elementary extension of C' in the theory T', there is a tuple $d_1, \cdots, d_n \in C'$ with $T' \models \rho(d_1, \cdots, d_n)$. Since ρ is quantifier-free, it follows that $T \models \rho(d_1, \cdots, d_n)$. Hence $T \models \rho(d_1, \cdots, d_n)$.

Now let $u = \sum_{i=1}^{n} d_i l_{n+1} \in C' \cdot \vec{l}$. We need to show that there is $w \in C \cdot \vec{l}$ with $v(w - l) = \gamma$. We will use the same proof as in the second claim of the proof of [EHS23, Proposition 2.3]. By Fact 2.2.3, $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/C^{alg})$ and by Fact 2.2.1, we have $\mathrm{tp}_0(C^{alg}/C) \vdash \mathrm{tp}_0(C^{alg}/L)$. Recall that $\mathrm{dcl}(C) \subseteq C^{alg}$, so let $u_1, \cdots, u_m$ be the orbit of u under the action of $Aut(C^{alg})$.

The type implication ensures that $v(u_i - v) = \gamma$ for each i. Then for $w = \frac{1}{m}\sum u_i$, we have $v(w - v) = v(\frac{1}{m}\sum(u_i - v)) = \gamma$.

Let $l_{n+1} = v - w$. Then, $l_1, \dots, l_n, l_{n+1}$ is a separated basis over C. In fact, for $c_1, \dots, c_n, c_{n+1} \in C$, if on the contrary, $v(\sum_{i=1}^{n+1} c_i l_i) > \min\{v(c_{n+1}l_{n+1}), v(\sum_{i=1}^{n} c_i l_i)\}$. It follows that

$$\begin{aligned} v(l_{n+1} + \sum_{i=1}^{n} c_i l_i) &> v(l_{n+1}), \\ v(v - w + \sum_{i=1}^{n} c_i l_i) &> v(v - w), \\ v(v - (w - \sum_{i=1}^{n} c_i l_i)) &> \gamma \end{aligned}$$

where $\gamma > \max\{v(v - w') : w' \in C \cdot \vec{l}\}$. Hence, we have a contradiction. □

The following is an analog of Fact 2.1.17 for henselian fields of equicharacteristic zero.

Corollary 2.2.10. *Let C be a subfield and a be a tuple in the valued field sort of $\mathcal{U}$. Assume that $L := \operatorname{dcl}(Ca)$ is a regular extension of C. Then the following are equivalent:*

1. $\operatorname{tp}(a/C)$ *is residually dominated.*
2. *L is an unramified extension of C and L has the separated basis property over C.*

Proof. By Lemma 2.2.7 and Theorem 2.2.9, 1. implies 2. Conversely, since L is henselian and the extension L is a regular extension of C, it follows that $\mathrm{k}(L)$ is a regular extension of $\mathrm{k}(C)$. Applying Corollary 2.1.26, we conclude that $\operatorname{tp}(a/C)$ is residually dominated. □

Finally, we combine all the results to relate residual domination to stable domination.

Theorem 2.2.11. *Let a be a tuple in the valued field sort and C be subfield of $\mathrm{VF}(\mathcal{U})$. Assume that C is* acl*-closed in the valued field sort. Then the following are equivalent:*

1. $\operatorname{tp}(a/C)$ *is residually dominated in the $\mathcal{L}$-structure $\mathcal{U}$,*
2. $\operatorname{tp}_0(a/C)$ *is stably dominated in the $\mathcal{L}_0$-structure $\widetilde{\mathcal{U}}$.*

Proof. Let $L := \mathrm{dcl}_0(Ca)$ and $M := \mathrm{dcl}(Ca)$ First assume that $\mathrm{tp}(a/C)$ is residually dominated. Since C is acl-closed in $\mathcal{U}$ and $L, M \subseteq \mathcal{U}$, both L and M are regular extensions of C. By Corollary 2.2.10, M has the separated basis property over C and $\Gamma(M) = \Gamma(C)$. As $M \supseteq L$, it follows that $\Gamma_0(L) = \Gamma_0(C)$ and L has the separated basis property over C. By Fact 2.1.17 then, $\mathrm{tp}_0(a/C)$ is stably dominated.

Conversely, assume that $\mathrm{tp}_0(a/C)$ is stably dominated. By Fact 2.1.17, $\Gamma_0(L) = \Gamma_0(C)$ in $\widetilde{\mathcal{U}}$ and L has the separated basis property over C. Since $M = \mathrm{dcl}(Ca)$ is an algebraic extension of L and L is henselian, it follows from Fact 2.1.3 that M is separated over L. Then, by Proposition 2.1.5, M is separated over C. Then, by Corollary 2.2.10, we conclude that $\mathrm{tp}(a/C)$ is residually dominated. □

A similar equivalence holds when considering domination by kInt-sorts.

Theorem 2.2.12. *Let C be a substructure of $\mathcal{U}$ and a be a tuple of field elements. Let $L := \mathrm{dcl}(Ca)$. Then, the following are equivalent:*

(i) $\mathrm{tp}(L/C\Gamma_L)$ *is dominated by the* kInt*-sorts and $\mathcal{A}$ in $\mathcal{U}$.*

(ii) $\mathrm{tp}_0(a/C\Gamma_L)$ *is stably dominated in $\widetilde{\mathcal{U}}$.*

Proof. We will prove that $\mathrm{tp}_0(a/\mathrm{acl}_0(C\Gamma_L))$ is stably dominated, which is equivalent to *(ii)*. Let B be a resolution of $C\Gamma_L$ such that $\Gamma \cap \mathrm{dcl}_0(B) = \Gamma \cap \mathrm{dcl}_0(C\Gamma_L)$ and $\mathrm{k} \cap \mathrm{dcl}_0(B) = \mathrm{k} \cap \mathrm{acl}_0(C\Gamma_L)$. We may assume that B is maximally complete. Then, $\Gamma \cap \mathrm{dcl}_0(Ba) = \Gamma \cap \mathrm{dcl}_0(B) = \Gamma \cap \mathrm{dcl}_0(Ca)$, so $\mathrm{dcl}_0(Ba)$ is an unramified extension of $\mathrm{dcl}_0(B)$. Since B is maximally complete, this extension is in fact separated. Note that B is acl_0-closed, so any field extension of B is regular. Therefore, by Fact 2.1.17, $\mathrm{tp}_0(a/B)$ is stably dominated.

Let p be a B-invariant extension of $\mathrm{tp}_0(a/B)$ in $\widetilde{\mathcal{U}}$. By the choice of B, we have $\mathrm{tp}_0(a/\mathrm{acl}_0(C\Gamma_L)) \vdash \mathrm{tp}_0(a/B)$. In fact, for any $a' \models \mathrm{tp}_0(a/\mathrm{acl}_0(C\Gamma_L))$, we have $a \equiv_{\Gamma(B)} a'$ and $a \equiv_{\mathrm{k}(B)} a'$. Then, by relative quantifier elimination in ACVF, we conclude that $a \equiv_B a'$. Thus, p is $\mathrm{acl}_0(C\Gamma_L)$-invariant. Finally, by Theorem 2.1.12, it follows that $\mathrm{tp}_0(a/\mathrm{acl}_0(C\Gamma_L))$ is stably dominated, as desired.

Conversely, assume that *(ii)* holds. Let $M \subseteq \mathcal{U}$ be a substructure with $M \supseteq C\Gamma_L$ such that

$$\mathrm{kInt}^L_{\Gamma_L} \mathop{\downarrow}^0_{C\Gamma_L} \mathrm{kInt}^M_{\Gamma_L}. \tag{2.2.1}$$

Let $\sigma \in Aut(\mathcal{U}/C\Gamma_L\mathcal{A}(M)\mathrm{kInt}^M_{\Gamma_L})$. Let $L' = \sigma(L)$. By Fact 2.1.30, the independence in 2.2.1 is equivalent to

$$\mathrm{St}_{C\Gamma_L}(L) \mathop{\downarrow}_{C\Gamma_L} \mathrm{St}_{C\Gamma_L}(M). \tag{2.2.2}$$

***Claim* 1.** *In $\widetilde{\mathcal{U}}$, we have $L \equiv_{C\Gamma_L St_{C\Gamma_L}(M)} L'$.*

Proof. Since σ extends to an automorphism of $\widetilde{\mathcal{U}}$, we already have $L \equiv_{C\Gamma_L \mathrm{kInt}^M_{\Gamma_L}} L'$ in $\widetilde{\mathcal{U}}$. Then, by stable embeddedness of the stable sorts, it is enough to show $\mathrm{St}_{C\Gamma_L}(L) \equiv_{\mathrm{St}_{C\Gamma_L}(M)} \mathrm{St}_{C\Gamma_L}(L')$. By the choice of L', we have $\mathrm{St}_{C\Gamma_L}(L') \downarrow_{\mathrm{St}_{C\Gamma_L}} \mathrm{St}_{C\Gamma_L}(M)$. Together with the independence relation 2.2.2, we can find an isomorphism fixing $\mathrm{St}_{C\Gamma_L}$ and sending $\mathrm{St}_{C\Gamma_L}(L)$ to $\mathrm{St}_{C\Gamma_L}(L')$, as desired. □

Hence, by stable domination, we can find an $\mathcal{L}_0$-isomorphism $\tau : LM \to L'M$ which is the identity map on M. We will show that τ is an $\mathcal{L}$-isomorphism. For this, it suffices to show that τ induces an isomorphism on the sorts $\mathcal{A}_n$ and commutes with res^n for each n.

Fix $a_1 \ldots, a_r, b_1, \ldots, b_s \in L$ and $e_1 \ldots, e_r \in M$ such that $\{v(a_i)\}_i$ is a $\mathbb{Q}$-basis of Γ_L over Γ_C, $v(a_i) = v(e_i)$ for each $i \leq r$, and $\{\mathrm{res}(b_i)\}_{i \leq s}$ is a transcendence basis of k_L over k_C. Recall that, by Fact 2.1.30, the independence relation 2.2.1 implies that the elements

$$\mathrm{res}(b_1), \ldots, \mathrm{res}(b_s), \mathrm{res}(\frac{a_1}{e_1}), \ldots \mathrm{res}(\frac{a_r}{e_r})$$

are algebraically independent over k_M.

We will use this fact to show that for each $x \in LM$, $rv(x)$ can be expressed in terms of the elements from $\mathrm{kInt}^M_{\Gamma_L}$ and $rv(L)$. First, we need the following subsequent claims.

***Claim* 2.** *Let $a = lm$ for $l \in L$ and $m \in M$. Assume that $v(lm) = 0$. Then, $\mathrm{res}(a)$ lies in the field generated by the field k_M and the elements $\mathrm{res}(b_1), \ldots, \mathrm{res}(b_s), \mathrm{res}(\frac{a_1}{e_1})^{p_1}, \ldots \mathrm{res}(\frac{a_r}{e_r})^{p_r}$ for some $p_i \in \mathbb{Q}$ for each $i \leq s$.*

Proof. There are $p_1, \ldots, p_r \in \mathbb{Q}$ and $c \in C$ such that $v(l) = \big(\sum_{i \leq r} p_i v(a_i)\big) + v(c)$. It follows that there exists some $\hat{l} \in L$ with $v(\hat{l}) = 0$ such that $l = \big(\Pi_{i \leq r} a_i^{p_i}\big) c\hat{l}$. Since $v(l) = -v(m)$, we can similarly write $m = \frac{\hat{m}}{(\Pi e_i^{p_i}) c}$, where $\hat{m} \in M$ with $v(\hat{m}) = 0$. Then, we have,

$$lm = \frac{\big(\Pi_{i \leq r} a_i^{p_i}\big) c\hat{l}\hat{m}}{\big(\Pi_{i \leq r} e_i^{p_i}\big) c} = \big(\Pi_{i \leq r} (\frac{a_i}{e_i})^{p_i}\big) \hat{l}\hat{m},$$

Note that each factor in the above product has valuation zero. After applying the residue map, we obtain

$$\mathrm{res}(lm) = \big(\Pi_{i\leq r}\mathrm{res}(\frac{a_i}{e_i})^{p_i}\big)\mathrm{res}(\hat{l})\mathrm{res}(\hat{m}).$$

After replacing $\mathrm{res}(\hat{l})$ with $\sum\limits_{i\leq s}\mathrm{res}(c_i)\mathrm{res}(b_i)$ for $c_1,\dots,c_s \in C$, the result follows. □

***Claim* 3.** *L and M are linearly disjoint over C.*

Proof. Let $l_1,\dots,l_n$ be a C-linearly independent tuple. We will first show that for any $m_1,\dots,m_n \in$, we have $v(\sum\limits_{i=1}^{n} m_i l_i)) = \min\{v(m_i)+v(l_i)\}_{i\leq n}$. In particular, this will prove that $\Gamma_{LM} = \Gamma_M$. Suppose on the contrary that for some $m_1,\dots,m_n \in M$, not all equal to zero, we have $v(\sum\limits_{i\leq n} m_i l_i) > \min\{v(m_i)+v(l_i)\}_{i\leq n} = \gamma$. Let $J = \{i \leq n : v(l_i m_i) = \gamma\}$. We may assume that $1 \in J$. Then,

$$v\big(1+\sum_{i\in J}\frac{l_i m_i}{l_1 m_1}\big) > 0.$$

After applying the residue map,

$$1+\sum_{i\in J}\mathrm{res}(\frac{m_i l_i}{m_1 l_1}) = 0. \tag{2.2.3}$$

By Claim 2, each term $\mathrm{res}(\frac{m_i l_i}{m_1 l_1})$ lies in the algebraic closure of k_M and the variables $\mathrm{res}(\frac{a_i}{e_i})_{i\leq r}$ and $\mathrm{res}(b_i)_{i\leq s}$. Hence the equation 2.2.3 witnesses the algebraic dependence of the set $\{\mathrm{res}(b_1),\dots,\mathrm{res}(b_s),\mathrm{res}(\frac{a_1}{e_1}),\dots\mathrm{res}(\frac{a_r}{e_r})\}$ over k_M, which is a contradiction.

To show linear disjointness, suppose on the contrary that there exists a C-linearly independent tuple $l_1,\dots,l_n \in L$ such that for some $m_1,\dots,m_n \in M$, we have $\sum\limits_{i\leq n} m_i l_i = 0$. Then $v(\sum\limits_{i\leq n} m_i l_i) = \min\{v(l_i m_i)\}$, hence cannot be ∞, a contradiction. □

***Claim* 4.** *Let $a \in LM$. Then there are $a' \in LM$ and $m' \in M$ such that $a = a'm'$, $v(a') = 0$ and $rv(a') \in \mathrm{dcl}(L \cup \mathrm{kInt}_L^M)$.*

Proof. First assume that a lies in the ring generated by L and M. We may assume that $a = \sum\limits_{i\leq n} m_i l_i$ where $l_1 \dots, l_n$ are M-linearly independent and separated over M, and $m_1 \dots, m_n \in M$. Without loss of generality, assume further that $v(a) = v(l_1) + v(m_1)$ and let $J =$

$\{i : v(m_i l_i) = v(m_1 l_1)\}$. Let $m' \in M$ such that $v(m') = v(m_1) + v(l_1)$. Note that such m' exists since $\Gamma_M \supseteq \Gamma_L$. Let $a = \frac{a}{m'}m'$ and notice that $v(\frac{a}{m'}) = 0$. By Fact 1.2.5, we have $rv(\frac{a}{m'}) = \sum_{i \in J} rv(\frac{l_i m_i}{m'})$. Thus,

$$rv(a) = rv(\frac{a}{m'})rv(m') = \big(\sum_{i \in J} rv(l_i)rv(\frac{m_i}{m'})\big)rv(m'). \tag{2.2.4}$$

For each $i \in J$, we have $v(\frac{m_i}{m'}) \in \Gamma_L$. Thus, $rv(\frac{m_i}{m'}) \in \mathrm{kInt}^M_{\Gamma_L}$ for each $i \in J$. This shows $rv(\frac{a}{m'}) \in \mathrm{dcl}(L \cup \mathrm{kInt}^M_{\Gamma_L})$.

If $a \in LM$, then we can write $a = \frac{a'_1 m'_1}{a'_2 m'_2}$, where $v(a'_1) = v(a'_2) = 0$, $m'_1, m'_2 \in M$ and $rv(a'_1), rv(a'_2) \in \mathrm{dcl}(L \cup \mathrm{kInt}^M_{\Gamma_L})$. Then, $rv(\frac{a'_1}{a'_2}) = \frac{rv(a'_1)}{rv(a'_2)}$ lies in $\mathrm{dcl}(L \cup \mathrm{kInt}^M_{\Gamma_L})$. Hence, the result follows for $a \in LM$, as well. □

Now let $a, b \in LM$ with $v(a), v(b) \in n\Gamma$, and assume that $\mathrm{res}^n(a) = \mathrm{res}^n(b)$. Then $\rho_n(rv(a)) = \rho_n(rv(b))$. By Claim 4, we can find $a', b' \in LM$ and $m_1, m_2 \in M$ such that $a = a'm_1$ and $b = b'm_2$, where $v(a') = v(b') = 0$, and a', b' lie in $\mathrm{dcl}(L \cup \mathrm{kInt}^M_{\Gamma_L})$. Then

$$\rho_n(rv(a'))\rho_n(m_1) = \rho_n(rv(b'))\rho_n(m_2)$$

where $\rho_n(m_1), \rho_n(m_2) \in \mathcal{A}_M$ and $\rho_n(rv(a'))$, $\rho_n(rv(b')) \in \mathrm{dcl}(L \cup \mathrm{kInt}^M_{\Gamma_L})$. Therefore, the equality $\mathrm{res}^n(a) = \mathrm{res}^n(b)$ is expressible in the type $\mathrm{tp}(L/C\Gamma_L\mathcal{A}_M\mathrm{kInt}^M_{\Gamma_L})$.

As τ fixes $C\Gamma_L\mathrm{kInt}^M_{\Gamma_L}$ and $\tau(L) = L'$ we can conclude $\rho_n\big(rv(\tau(a))\big) = \rho_n\big(rv_n(\tau(b))\big)$. This shows that τ naturally induces an isomorphism between $(\mathcal{A}_n)_{LM}$ and $(\mathcal{A}_n)_{L'M}$ for each $n \in \mathbb{Z}^{>0}$, by sending $\mathrm{res}^n(a) = \rho_n(rv(a))$ to $\mathrm{res}^n(\tau(a)) = \rho_n(rv(a))$. Hence τ is an $\mathcal{L}_{\mathcal{A},\mathrm{RV}}$-isomorphism, as desired. □

2.3 Properties of Residual Domination

In this section, we use Theorem 2.2.11 to show that residually dominated types share similar properties with stably dominated types in ACVF. Assuming the existence of a non-forking global type, we will show that residually dominated types are orthogonal to Γ, and invariant under base change and definable pushforwards.

To work with these results, we first need a characterization of forking over models. Such a characterization was given in [EHM19] for maximally complete models. The maximality assumption guarantees the good separated-basis property, but the proof goes through unchanged if we instead assume the good separated basis property over C.

Proposition 2.3.1. *([EHM19, Theorem 3.4]) Let C be a valued field, and let a and b be tuples from the field sort. Assume that $L := \mathrm{dcl}_0(Ca)$ has the good separated basis property over C, k_L is a regular extension of k_C and Γ_L/Γ_C is torsion free (or $\Gamma_L \cap \Gamma_M = \Gamma_C$, where $M = \mathrm{dcl}_0(Mb)$). Then in the structure $\widetilde{\mathcal{U}}$, we have $a \mathop{\smile\!\!\!\!\!\!\mid}\,_C b$ if and only if $\mathrm{k}(Ca)\Gamma(Ca) \mathop{\smile\!\!\!\!\!\!\mid}\,_C \mathrm{k}(Cb)\Gamma(Cb)$.*

When the type is residually dominated, the independence in Γ comes for free, and independence in the residue field of $\mathcal{U}$ implies independence in the stable sorts.

Lemma 2.3.2. *Let C be a valued field, with $C = \mathrm{acl}(C)$ in the field sort. Let a and b be tuples of field elements. Suppose that $\mathrm{tp}(a/C)$ is residually dominated and $\mathrm{k}(Ca) \mathop{\smile\!\!\!\!\!\!\mid}\,^{\mathrm{alg}}_{\mathrm{k}(C)} \mathrm{k}(Cb)$. Then, $a \mathop{\smile\!\!\!\!\!\!\mid}\,_C b$ in $\widetilde{\mathcal{U}}$. In particular, $\mathrm{St}_C(Ca) \mathop{\smile\!\!\!\!\!\!\mid}\,_C \mathrm{St}_C(Cb)$ in the structure $\widetilde{\mathcal{U}}$.*

Proof. Let $L = \mathrm{dcl}_0(Ca)$ and $M = \mathrm{dcl}_0(Cb)$. The theory of the residue field in $\widetilde{\mathcal{U}}$ is ACF. Thus, the algebraic independence in the residue field implies $\mathrm{k}_L \mathop{\smile\!\!\!\!\!\!\mid}\,_C \mathrm{k}_M$ in $\widetilde{\mathcal{U}}$. Moreover, since $\mathrm{tp}(a/C)$ is residually dominated, by Corollary 2.2.10, we have $\Gamma(Ca) = \Gamma(C)$, which implies $\Gamma_0(C) = \Gamma_0(Ca)$. Thus, it follows trivially that $\Gamma_L \mathop{\smile\!\!\!\!\!\!\mid}\,_C \Gamma_M$. Since C is acl-closed, k_L is a regular extension of k_C. Then, applying Proposition 2.3.1, we conclude $a \mathop{\smile\!\!\!\!\!\!\mid}\,_C b$. In particular, $\mathrm{St}_C(Ca) \mathop{\smile\!\!\!\!\!\!\mid}\,_C \mathrm{St}_C(Cb)$. □

Below we show that residual domination is invariant under base changes. It suffices to assume the existence of a nonforking global extension rather than requiring invariant extensions. The following fact about type implications will be useful.

Fact 2.3.3. *([EHS23, Lemma 1.19]) Let $C \leq L$ be a valued field extension such that L/C is regular and L is henselian. Then, $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/\mathrm{acl}_0(C))$. In particular, if a is a tuple in the field sort such that $\mathrm{dcl}_0(Ca)$ is a regular extension of C, then $\mathrm{tp}_0(a/C) \vdash \mathrm{tp}_0(a/\mathrm{acl}_0(C))$.*

Theorem 2.3.4. *Let $C \subseteq B$ be subfields that are* acl*-closed in the valued field sort. Let p be a global type that does not fork over C.*

(i) If $p|C$ is residually dominated, then $p|B$ is also residually dominated.

(ii) Suppose further that p is C-invariant and $p|B$ is residually dominated. Then $p|C$ is also residually dominated.

Proof. We start with (i). Let $a \models p|C$. By Theorem 2.2.11, $\mathrm{tp}_0(a/C)$ is stably dominated and by Fact 2.1.11 (ii), $\mathrm{tp}_0(a/\mathrm{acl}_0(C))$ is stably dominated. Let q be the unique $\mathrm{acl}_0(C)$-definable extension of $\mathrm{tp}_0(a/\mathrm{acl}_0(C))$ as in Fact 2.1.11. Our aim is to show that q and p_0 coincide on $\mathcal{U}$, where p_0 is the ACVF-type in $S(\widetilde{\mathcal{U}})$ with a realization $b \models p$. Then, using Theorem 2.2.11, it will enable us to lift Fact 2.1.12 to the henselian setting, and the results will follow.

First, we show that q does not fork over C. By Fact 2.3.3, we have $\mathrm{tp}_0(L/C) \vdash \mathrm{tp}_0(L/\mathrm{acl}_0(C))$, where $L = \mathrm{dcl}_0(Ca)$ is henselian. It follows that $q|\mathrm{acl}_0(C)$ does not fork over C, therefore q also does not fork over C.

Now, let $M \preceq \mathcal{U}$ with $C \subseteq M$, and take $b \models p|M$. Since p does not fork over C, we have, $\mathrm{k}(Cb) \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{alg}}_{\mathrm{k}(C)} \mathrm{k}(M)$. By Lemma 2.3.2, this gives $\mathrm{St}_C(Cb) \mathop{\smile\!\!\!\!\!\mid}_C \mathrm{St}_C(M)$. Then, by Fact 2.1.11 (i), $b \models q|M$. Since M is arbitrary, we conclude $p_0|\mathcal{U} = q|\mathcal{U}$.

Let $B \supseteq C$ and assume that $B = \mathrm{acl}(B)$. Then $p_0|B = q|B$ is stably dominated by Fact 2.1.12. Applying Theorem 2.2.11, we conclude that $p|B$ is residually dominated.

For (ii), let $a \models p|M$, where $M \supseteq B$ is a model. Since p does not fork over B, by part (i), we in particular have that $p|M$ is residually dominated. Write $M_0 := \mathrm{acl}_0(M)$. By Fact 2.3.3, we have $\mathrm{tp}_0(a/M) \vdash \mathrm{tp}_0(a/M_0)$, and by Fact 2.1.11 (ii), $\mathrm{tp}_0(a/M_0)$ is stably dominated. We will show that $\mathrm{tp}_0(a/C) \vdash \mathrm{tp}_0(a/M)$ in $\widetilde{\mathcal{U}}$, which implies that $\mathrm{tp}_0(a/M_0)$ is C-invariant in $\widetilde{\mathcal{U}}$. So let $a' \models \mathrm{tp}_0(a/C)$. We will show that $a' \models \mathrm{tp}_0(a/M)$. Since C and M are acl-closed and $p|M$ is not a realized type, a and a' are algebraically independent over C. Hence, they remain algebraically independent over M, so there is a ring isomorphism τ between $M(a)$ and $M(a')$.

By Corollary 2.2.10, we have $\Gamma(Ma) = \Gamma(M)$. By invariance of p over C, we also have $\Gamma(Ma) = \Gamma(C)$. Therefore, $\Gamma_0(Ca) = \Gamma_0(Ca') = \Gamma_0(C)$, and it follows that τ is in fact a valued field isomorphism.

Now let q be a C-invariant extension of $\mathrm{tp}_0(a/\mathrm{acl}_0(M))$. Then q is also $\mathrm{acl}_0(C)$-invariant, and by Fact 2.1.12, it follows that $q|\mathrm{acl}_0(C)$ is stably dominated; equivalently, $q|C$ is stably dominated. By construction, $a \models q|C$, so $\mathrm{tp}_0(a/C)$ is stably dominated. By Theorem 2.2.11, it follows that $\mathrm{tp}_0(a/C)$ is residually dominated, as required.

□

Using this theorem, we show that residually dominated types are orthogonal to Γ, assuming the existence of a global invariant extension.

Theorem 2.3.5. *Let C be a valued field such that C is* acl*-closed in the valued field sort and assume that p is a C-invariant type. If $p|C$ is residually dominated, then for any model $M \supseteq C$ and $a \models p|M$, we have $\Gamma(Ma) = \Gamma(M)$. Conversely if for all $|C|^+$-saturated model $M \supseteq C$ and $a \models p|M$, $\Gamma(Ma) = \Gamma(M)$, then $p|C$ is residually dominated.*

Proof. First, assume that $p|C$ is residually dominated. Let M be a model containing C. Since M is acl-closed, it follows by Theorem 2.3.4 that $p|M$ is residually dominated. Consequently, by Corollary 2.2.10, for $a \models p|M$, we have $\Gamma(M) = \Gamma(Ma)$.

For the converse, fix a maximally complete, model $F \succeq M$ where M is $|C|^+$-saturated. Let $a \models p|F$. By assumption, $\Gamma(Fa) = \Gamma(F)$. By Fact 2.1.7, $L = \mathrm{dcl}(Fa)$ has the separated basis property over F. Moreover, since F is a acl-closed, L is a regular extension of F. Applying Corollary 2.2.10, we conclude that $\mathrm{tp}(a/F)$ is residually dominated. Since p is C-invariant with $C = \mathrm{acl}(C)$ and F is acl-closed, we can apply Theorem 2.3.4 to conclude that $p|C$ is residually dominated.

□

We finish this section by showing that a pushforward of the residually dominated type under a definable map remains residually dominated. We will need the following facts.

Fact 2.3.6. *([HHM08, Lemma 10.14]) Let C be a valued field with $C = \mathrm{acl}_0(C)$ in the valued field sort. Assume that $\mathrm{tp}_0(a/C)$ is stably dominated. Then, there exists a resolution A of Ca such that $\mathrm{tp}(A/C)$ is stably dominated.*

The following is essentially [EHS23, Lemma 4.3]; although the language is different, the proof is identical.

Fact 2.3.7. *Let C be a parameter set, and let a be a tuple in $\mathcal{U}$. Let B be a resolution of Ca such that $\mathrm{k}_{\mathcal{A}}(B) = \mathrm{k}_{\mathcal{A}}(\mathrm{acl}(Ca))$. Suppose that $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(B)) \vdash \mathrm{tp}(a/B)$. Then, $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(Ca)) \vdash \mathrm{tp}(b/Ca)$.*

Proof. Let $\varphi(x,a) \in \mathrm{tp}(b/Ca)$. By assumption, there exists $\psi(x,d) \in \mathrm{tp}(a/Ck_{\mathcal{A}}(B))$ implying $\varphi(x,a)$. Let X be the set of the conjugates of d over Ca in $\mathcal{U}$. By Remark 1.2.21, X is $k_{\mathcal{A}}(Ca)$-definable. Then, $\bigvee_{d_i \in X} \psi(x,d_i)$ is $k_{\mathcal{A}}(Ca)$-definable and implies $\varphi(x,a)$. □

Theorem 2.3.8. *Let C be a valued field that is* acl*-closed in the valued field and let a be a tuple in the field sort with* $\mathrm{tp}(a/C)$ *is residually dominated. Let f be an $\mathcal{L}(C)$-definable map whose domain contains the set of realizations of* $\mathrm{tp}(a/C)$*. Then* $\mathrm{tp}(f(a)/C)$ *is residually dominated.*

Proof. First note that if $f(a) \in \mathcal{A}$, then, by Remark 1.2.21, we have $\mathrm{tp}(f(a)/C\mathrm{k}_{\mathcal{A}}(Cb)) \vdash \mathrm{tp}(f(a)/Cb)$ for any b. Thus, we may assume that f is an $\mathcal{L}$-definable map whose image is lies in an ACVF-definable sort.

By Theorem 2.2.11, $\mathrm{tp}_0(a/C)$ is stably dominated. Moreover, since $f(a) \in \mathrm{dcl}(Ca)$, we have $f(a) \in \mathrm{acl}_0(Ca)$. Then Fact 2.1.11 (*iii*) implies that $\mathrm{tp}_0(f(a)/C)$ is stably dominated and therefore $\mathrm{tp}_0(f(a)/\mathrm{acl}_0(C))$ is stably dominated. By Fact 2.3.6, there exists a resolution A of $\mathrm{acl}_0(C)f(a)$ such that $\mathrm{tp}_0(A/\mathrm{acl}_0(C))$ is stably dominated. Note that A is also a resolution of $Cf(a)$.

Let B be a prime resolution of $Cf(a)$, as described in Theorem 2.1.35. The set B embeds inside A over C, hence $\mathrm{tp}_0(B/C)$ is stably dominated. Since $C = \mathrm{acl}(C)$, applying Theorem 2.2.11, we obtain $\mathrm{tp}(B/C)$ is residually dominated. Also recall that we have $\mathrm{k}_{\mathcal{A}}(B) = \mathrm{k}_{\mathcal{A}}(\mathrm{acl}(Cf(a)))$.

Now, assume that b is a tuple in $\mathcal{U}$ with $\mathrm{k}(Cf(a)) \mathop{\smile\!\!\!\!\!\!\;|}^{alg}_{\mathrm{k}(C)} \mathrm{k}(Cb)$. Then, $\mathrm{k}(B) \mathop{\smile\!\!\!\!\!\!\;|}^{alg}_{\mathrm{k}(C)} \mathrm{k}(Cb)$, since $\mathrm{k}(Cf(a))^{alg} = \mathrm{k}(B)^{alg}$. By residual domination of $\mathrm{tp}(B/C)$, $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(B)) \vdash \mathrm{tp}(b/B)$. Finally, by Fact 2.3.7, $\mathrm{tp}(b/C\mathrm{k}_{\mathcal{A}}(Cf(a)) \vdash \mathrm{tp}(b/Cf(a))$. □

Chapter 3

Residually Dominated Groups

In this section, we introduce *residually dominated groups*, defined analogously to the notion of *stably dominated groups* presented in [HRK19]. For a broader setting, the notion of groups admitting a *strongly f-generic type*, introduced in [CS18] provides a useful generalization of generic types of definable groups in stable theories. In henselian valued fields, we define a group to be *residually dominated* if it admits a residually dominated strongly f-generic type.

As in the ACVF case, we will first establish the existence of a surjective definable group homomorphism from a residually dominated group onto a pro-definable group in the residue field. Moreover, we observe a similar phenomenon regarding the occurrence of residually dominated types: every definable non-trivial *abelian* group contains a directed union of residually dominated groups parameterized by realizations of a type in the value group Γ. Moreover, any definable abelian group can be decomposed into such a union and a Γ-internal group.

3.1 Stabilizer Theorems, Stably Dominated Groups

In stable theories, the notion of generics and stabilizers are primary tools in analyzing groups. It was generalized to unstable theories by various authors. For example, recent generalizations are provided for NIP and later for NTP_2 theories in [CS18] and [MOS20], respectively. In this section, we will present the results from mainly [MOS20].

Throughout this section, we fix a theory T, with its universal model $\mathcal{U}$. We assume that T eliminates imaginaries. By *definable*, we always mean definable with parameters.

We will work in a definable group G, meaning both the set G and its group operation are definable. For elements $x, y \in G$, we denote their group product by $x \cdot y$, and we use xy to denote the tuple obtained by concatenating x and y.

We will identify definable sets with their defining formulas.

Definition 3.1.1. A collection μ of definable sets is called an *ideal* if it satisfies the following:

1. $\emptyset \in \mu$,
2. If $A \subseteq B$ and $B \in \mu$, then $A \in \mu$,
3. If $A \in \mu$ and $B \in \mu$, then $A \cup B \in \mu$.

Two important classes of ideals of our interest are given below.

Definition 3.1.2. Let μ be an ideal of definable subsets of G and let A be a set. Let $M \models T$ and A be any set.

1. We say that μ is *A-invariant* if for all $\phi(x, d) \in \mu$ and $\sigma \in Aut(\mathcal{U}/A)$, we have $\phi(x, \sigma(d)) \in \mu$.
2. Let μ be an M-invariant ideal. We say that μ *has the $S1$-property* if, for any formula $\varphi(x, y)$ and any M-indiscernible sequence $(a_i)_{i<\omega}$, if $\varphi(x, a_i) \wedge \varphi(x, a_j) \in \mu$ for some (or equivalently, any) $i, j < \omega$, then there exists some (or equivalently, any) $i < \omega$ such that $\varphi(x, a_i) \in \mu$.
3. We say that μ *has the $S1$-property on an M-definable set A* if $A \notin \mu$ and the condition above is satisfied for formulas $\varphi(x, a_i)$ that define subsets of A.

Example 3.1.3. Let A be a set. Let $\mathcal{D}(A)$ be the set of formulas that divides over A, and $\mathcal{F}(A)$ be the set of formulas that forks over A. Then $\mathcal{F}(A)$ is an A-invariant ideal. Moreover, by definition of forking, we have $\mathcal{D}(A) \subseteq \mathcal{F}(A)$. However, $\mathcal{D}(A)$ is an ideal only when $\mathcal{D}(A) = \mathcal{F}(A)$

The following fact says that the forking ideal is contained in any ideal with the $S1$-property.

Fact 3.1.4. *([Hru12, Lemma 2.9]) Let $M \models T$ and μ be an M-invariant ideal which has the $S1$-property on some definable set X. Then for any type $p(x)$ that concentrates on X, if p does not belong to μ, then p does not fork over M.*

Fact 3.1.5. *([MOS20, Lemma 3.14]) Let $M \models T$ and G be an M-definable group with a strongly f-generic over M. Let μ be an ideal of formulas that do not extend to a strongly f-generic type over M. Then μ is M-invariant, M and has the $S1$-property.*

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Definition 3.1.6. Let M be a small model of a theory T, and let $(G, \cdot)$ be an M-type-definable group. Let μ be the ideal of formulas that are not contained in any strongly f-generic global type in G. For a type p, we say p is *μ-wide* if p does not concentrate on any $D \in \mu$.

1. For $p, q \in S_G(M)$, we define $Stab_G(p) := \{g \in G(\mathcal{U}) \mid g \cdot p = p\}$, and $Stab_G(p, q) := \{g \in G(\mathcal{U}) \mid g \cdot p = q\}$.
2. If $p \in S_G(M)$ is an f-generic type, we define

$$St_\mu(p) := \{g \in G(\mathcal{U}) \mid g \cdot p \cap p \text{ is wide.}\}.$$

   We write $Stab_\mu(p)$ for the group generated by $St_\mu(p)$.

Fact 3.1.7. *Let G be a type definable group with a strongly f-generic over M in a theory T. Let μ be an ideal of definable subsets of G that do not extend to a strongly f-generic over M.*

(i) ([CS18, Proposition 3.8]) If T is NIP*, then a global type p is an f-generic of G if and only if it has a bounded orbit under the translation by elements of G, which happens if and only if $G^{00} = Stab_G(p)$.*

(ii) ([MOS20, Theorem 3.18]) If T is NTP_2 *and $p \in S_G(M)$ is an f-generic, then $G^{00}_M = Stab_\mu(p) = (p \cdot p^{-1})^2$. Moreover, $G^{00}_M \setminus St_\mu(p)$ is contained in a union of M-definable sets that belong to μ.*

A definable group G is called *definably amenable* if there exists an additive probability measure on the definable subsets of G, which is invariant under the left translations by the elements of G. The examples include definable abelian groups and solvable groups.

Fact 3.1.8. *([MOS20, Proposition 3.20])Assume that G is a definably amenable group in an* NTP_2*-theory, then G admits a strongly f-generic type.*

The following fact, in particular, applies to definable groups in the pure henselian valued fields whose theory is NTP_2.

Fact 3.1.9. *([MOS20, Theorem 2.19]) Let T be an* NTP_2 *theory which extends the theory of fields and is algebraically bounded. Assume that any model of T is definably closed in the theory of its field-theoretic algebraic closure. Let G be a group definable in an ω-saturated $M \models T$, and assume that G has a strongly f-generic type over M. Then there exist an M-definable algebraic group H and an M-definable group homomorphism $\iota : G_M^{00} \to H$ with a finite kernel.*

Groups with definable f-generics

In this section, our aim is to present the equivalence of categories between groups with definable f-generics and pro-definable group chunks.

Let f be a definable function and p be a definable type over some set C. We denote by $\mathfrak{F}(p)$, the set of definable functions that are defined on the set of realizations of p. We denote $\mathfrak{G}(A)$ for the set of p germs of elements of $\mathfrak{F}(p)$. Then $\mathfrak{G}(p)$ has a group structure: if $f, g \in \mathfrak{F}(p)$ with the p-germs $\bar{f}$ and $\bar{g}$, respectively, then $\bar{f} \cdot \bar{g}$ is as the p-germ of the composition $f \circ g$. We note that it is well-defined.

Definition 3.1.10. An *abstract group chunk* is a pair (p, F) where p is a C-definable type and F is a definable map defined on the realizations of $p^{\otimes 2}$ that satisfies the following:

(i) For each $a \models p|C$, $(F_a)_* p = p$, where $F_a(x) = F(a, x)$.

(ii) For each $a, b \models p|C$, we have $a \in \mathrm{dcl}(b, F(a, b))$ and $b \in \mathrm{dcl}(a, F(a, b))$.

(iii) For all $(a, b, c) \models p^{\otimes 3}|C$, we have $F(F(a, b), c) = F(a, F(b, c))$.

In the following fact, a group G is pro-definable if G is a pro-definable set and its group multiplication is a pro-definable map from $G \times G$ to G.

Fact 3.1.11. *([HRK19, Proposition 3.15]) Let (p, F) be an abstract group chunk where p is definable over C. Then there exists a pro-C-definable group $(G, \cdot)$ and a pro-C-definable injective map $f : p \to G$ such that $p^{\otimes 2} \vdash f(F(x, y)) = f(x) \cdot g(x)$ and $G = Stab\big((f)_*p\big)$, (hence $(f)_*p$ is a definable f-generic of G.)*

The following fact states that the choice of G is unique.

Fact 3.1.12. *([HRK19, Proposition 3.16]) Let G_1 and G_2 be a pro-C-definable groups and p be a C-definable f-generic of G_1. Let $f : p \to G_2$ be a pro-C-definable map such that $p^{\otimes 2}(x, y) \vdash f(x \cdot y) = f(x) \cdot f(y)$. Then there exists a unique pro-C-definable homomorphism $g : G_1 \to G_2$ with $p(x) \vdash f(x) = g(x)$. Furthermore, if f is injective, then so is g.*

Fact 3.1.13. *([HRK19, Proposition 3.4]) Let G be a pro-C-definable group with a definable f-generic over C. Then G is pro-C-definably isomorphic to pro-limit of C-definable groups.*

Stably Dominated Groups

In this subsection, we will survey the definitions and the results we need from [HRK19] on *stably dominated groups.* Although the results in [HRK19] apply to more general theories, our focus will be on algebraically closed fields.

By a pro-C-definable group, we mean a pro-C-definable set G equipped with a group operation that is given by a pro-C-definable map.

Definition 3.1.14. Let G be a pro-C-definable group. We say G is *stably dominated* if it has a stably dominated definable f-generic.

In [HRK19], several examples are provided. Here, we include some of them explicitly.

Example 3.1.15. Let $K \models$ ACVF.

1. Let G be the multiplicative group of the valuation ring $\mathcal{O}$. Note that $G = \mathcal{O}^\times = \{u \in \mathcal{O} : v(u) = 0\}$. We will show that G is a stably dominated, connected group with $G = G^{00} = G^0$. Let p be the generic type of $\mathcal{O}$ given in Example 2.1.10(2), which is stably dominated. We will show that for every $g \in G$, the translate $g \cdot p$ equals p. Let φ be an $\mathcal{L}$-formula. By Fact 1.2.27, we may assume that φ defines a formula of the form $x \in B$ where B is a C-definable ball. When B is a C-definable proper subball of $\mathcal{O}$, the definable set $g \cdot x \in g \cdot B$ implies $x \in B'$ where B' is a Cg-definable proper subball of $\mathcal{O}$. Then, by definition of p, for all $a \models p|C$, $a \notin B'$. Thus, a realization of

$g \cdot p$ does not lie in any proper subball B' of $\mathcal{O}$. It follows that $g \cdot p = p$ and we have $Stab_G(p) = G$. Hence, G is a connected group with the stably dominated generic type p, as desired.

2. Let $G = SL_n(\mathcal{O})$ be the multiplicative group of $n \times n$ matrices with entries from $\mathcal{O}$ and determinant 1. Then G is a connected stably dominated group. To illustrate, consider the case $n = 2$, let $q(x_1, x_2, x_3, x_4)$ be the global completion of the type $p^{\otimes 3} \cup \{x_1x_3 - x_2x_4 = 1\}$, where p is as in part (1) the generic type of $\mathcal{O}$. As p is stably dominated, the tensor product $p^{\otimes 3}$ is stably dominated. Moreover, for any $a, b, c, d \models q$, we have the equality $\mathrm{St}_C(Cabcd) = \mathrm{St}_C(abc)$. It follows that q is stably dominated. We will show that $G = Stab(q)$. If

$$\begin{bmatrix} g & h \\ k & l \end{bmatrix} \in G,$$

then for $(a, b, c, d) \models q$, we have

$$\begin{bmatrix} g & h \\ k & l \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ga + hc & gb + hd \\ ka + lc & kb + ld \end{bmatrix}.$$

Since $a \models p|_c$, we have $ga + hc \models p$. Similarly, $gb + hd \models p$ and $ka + lc \models p$. This shows that $G = Stab(q)$ is a connected stably dominated group.

We write T_0 for the theory of algebraically closed valued fields in the language $\mathcal{L}_0$ of geometric sorts.

Fact 3.1.16. *(Proposition 4.6 and Lemma 4.9, [HRK19]) Let G be a pro-C-definable, stably dominated group over C. Then:*

1. *There exists a pro-C-definable group $\mathfrak{g}$ in St_C and a pro-C-definable group homomorphism $\theta : G \to \mathfrak{g}$ such that every generic of G is dominated by θ. Moreover, the pair $(\mathfrak{g}, \theta)$ is universal, i.e. if $\mathfrak{g}'$ is a pro-C-definable group in St_C and $\theta' : G \to \mathfrak{g}'$ is a pro-C-definable map over C then there exist a unique homomorphism $\tau : \mathfrak{g} \to \mathfrak{g}'$ with $\tau \circ \theta = \theta'$.*

2. *If $\mathfrak{g}$ is pro-C-definable in St_C, and $\theta : G \to \mathfrak{g}$ is a surjective pro-C-definable group homomorphism which dominates G, Then any type $p \in S_G(C)$ is generic in G if and only if its image $\theta_*(p)$ is generic in $\mathfrak{g}$.*

3.2 Residually Dominated Groups

In this section, we will introduce *residually dominated groups* which generalizes stably dominated groups for henselian valued fields. Our main result will be the generalization of Fact 3.1.16.

Throughout this section, we will write T to be a complete theory of henselian valued fields of equicharacteristic zero. We denote its universal structure by $\mathcal{U}$. We will also see $\mathcal{U}$ as a substructure of the algebraically closed valued field $\widetilde{\mathcal{U}}$, which is obtained by taking field-theoretic-algebraic closure of $\mathcal{U}$. We will see $\widetilde{\mathcal{U}}$ as an $\mathcal{L}_0$-structure where $\mathcal{L}_0$ is the language of geometric sorts, and we denote its theory by T_0.

Definition 3.2.1. Let G be a definable (or more generally pro-definable) group in G with a strongly f-generic type over M. We say G is *residually dominated* if there exists a residually dominated type $p \in S_G(M)$ that extends to a global strongly f-generic type of G.

Below, we will see that the $\mathcal{L}_0$-definable groups given in Example 3.1.15 are again residually dominated. However, in the henselian setting the groups might not have a unique generic.

Example 3.2.2.

1. Since any type that lies in $\mathrm{k}_{\mathcal{A}}$ is residually dominated, any definable group that lies in $\mathrm{k}_{\mathcal{A}}$ with a strongly f-generic type is residually dominated.

2. Let $G = \mathcal{O}^\times = \{u \in \mathcal{O} : v(u) = 0\}$ be the multiplicative group of the valuation ring, and assume that G has a strongly f-generic type over a model M. We will show that G is residually dominated; we will find a global type $p \in S_G(\mathcal{U})$ where no translation of p forks over M.

   The residue map $\mathrm{res} : G \to \mathrm{k}^\times$ is a group homomorphism. We may assume that $\mathrm{k}^\times$ has a strongly f-generic type q over M. Choose $p \in S_G(\mathcal{U})$ extending $p^{\mathcal{O}}$, the *generic type* of $\mathcal{O}$ defined in Example 2.1.10 (2), such that for every $a \models p$, $\mathrm{res}(a) \models q$. In Example 2.1.22 (2), we showed any completion of $p^{\mathcal{O}}$ is residually dominated, so p is residually dominated as well.

   Now, fix $g \in G(\mathcal{U})$, and $M \models T$ where G has a strong f-generic type over M. Replacing M by a maximally complete elementary extension if necessary, we may assume that M itself is maximally complete. First notice that p does not fork over M. In fact, let $N \succeq M$ be a small model and take $a \models p|N$. Then, $\mathrm{res}(a) \mathop{\smile\!\!\!\!\!|}_M \mathrm{k}(N)$ since q does not fork over M in $\mathrm{k}^\times$. Since $\mathrm{k}(Ma)$ is the field generated by $\mathrm{k}(M)$ and $\mathrm{res}(a)$, it follows that $\mathrm{k}_{\mathcal{A}}(Ma) \downarrow_M \mathrm{k}(N)$. Moreover, since $p|M$ is residually dominated and $a \models p|M$ by Lemma 2.2.7, $\Gamma(Ma) = \Gamma(M)$, and hence $\Gamma(Ma) \downarrow_M \Gamma(N)$ holds immediately. Thus $\mathrm{k}_{\mathcal{A}}(Ma) \downarrow_M \mathrm{k}^{eq}(Mb)\mathcal{A}^{eq}(Mb)$ and $\Gamma(Ma) \downarrow_M \Gamma^{eq}(Mb)$ hold. By Fact 2.1.23, $a \downarrow_M N$ holds, hence $p|N$ does not fork over M as required.

The type $g \cdot p$ is residually dominated since its corresponding $\mathcal{L}_0$-type is stably dominated. Moreover, its pushforward $\mathrm{res}_*(g \cdot p)$ is $\mathrm{res}(g) \cdot q$, which does not fork over M by the choice of q. Repeating the previous arguments we can show that $g \cdot p$ also does not fork over M. Thus, p is a strongly f-generic type of G, and therefore G is residually dominated.

3. Let $G = SL_n(\mathcal{O})$. Applying the residue map componentwise to each element $g \in G$ gives a group homomorphism $G \to SL_n(\mathrm{k})$. Using Example 3.1.15 (3), together with arguments analogous to those in part (2), we conclude that G is residually dominated. In this case, if p is a residually dominated strongly f-generic type of G, then its corresponding $\mathcal{L}_0$-type is the stably dominated $(p^{\mathcal{O}})^{\otimes 3} \cup \{x_1x_4 - x_2x_3 = 1\}$.

4. Let T be the theory of real closed valued fields, and let G denote the multiplicative group of positive elements of the valuation ring $\mathcal{O}$. In this case, G has exactly two strongly f-generic types, namely q_{0^+} and q_∞, where
   - An element $a \models q_{0^+}$ if and only if $a \models p^{\mathcal{O}}$ and a is infinitesimally close to 0 from the right.
   - An element $a \models q_\infty$ if and only if $a \models p^{\mathcal{O}}$ and a is larger than every element G.

Remark 3.2.3. *When T is* NTP$_2$ *and G is a definable group with strongly f-generic type over M. Then G is residually dominated if and only there exists a residually dominated $p \in S_G(M)$, which is an f-generic type.*

If $p \in S_G(M)$ is f-generic and residually dominated, then by [MOS20, Proposition 3.10], p extends to a strongly f-generic type over M. Here, we also note that any model is an extension base in an NTP$_2$*-theory, as shown in [CK12].*

Domination witnessed by a group homomorphism

In this section, we generalize Fact 3.1.16 to henselian valued fields of equicharacteristic zero for groups definable in the valued field sort. Throughout, we assume that the theory of the henselian field is NTP$_2$.

In any theory, a global type p is called *generically stable over A* if it is A-invariant and for any Morley sequence $(a_i)_{i\leq\alpha}$ of p over A and any formula $\varphi(x,y)$ with parameters from the universal model $\mathcal{U}\models T$, the set $\{i<\alpha: \models\varphi(a_i)\}$ is finite or cofinite. Generically stable types coincide with stably dominated types in ACVF.

Fact 3.2.4. *([Hru14, Theorem 3.3]) In the theory of non-trivially algebraically closed valued fields, for an* acl_0*-closed set C, a global type is stably dominated over C if and only if it is generically stable over C.*

In the proof of the lemma below, we will use results from [PT11] regarding generically stable types in arbitrary theories.

Fact 3.2.5. *([PT11, Proposition 1]) Let p be a global A-invariant type in an arbitrary theory T. Assume that p is generically stable over A. Then p is the unique non-forking extension of $p|A$.*

Lemma 3.2.6. *Let a be a tuple in the valued field sort of $\mathcal{U}$ and M be a model. Assume that* $\mathrm{tp}_0(a/M)$ *is residually dominated. Then,* $\mathrm{tp}_0(a/M)$ *is stationary in the structure $\widetilde{\mathcal{U}}$.*

Proof. By Theorem 2.2.11, $\mathrm{tp}_0(a/M)$ is stably dominated, and equivalently, $\mathrm{tp}_0(a/\mathrm{acl}_0(M))$ is stably dominated. Since M is a model, $\mathrm{dcl}_0(Ma)$ is a regular extension of M, and by Fact 2.2.3, we have $\mathrm{tp}_0(a/M)\vdash\mathrm{tp}_0(a/\mathrm{acl}_0(M))$ in $\widetilde{\mathcal{U}}$.

Let p be the $\mathrm{acl}_0(M)$-definable extension of $\mathrm{tp}_0(a/\mathrm{acl}_0(M))$. By the type implication above, p is M-invariant. Thus, p is generically stable over M. By Fact 3.2.5, $p|M$ is stationary. □

Using this lemma, we can show that all non-forking extensions of a residually dominated type have the same ACVF-type.

Notation 3.2.7. *Let p be a global type in $\mathcal{U}$. We define p_0 to be a global type in $\widetilde{\mathcal{U}}$ such that some realization $a\models p$ also realizes p_0.*

Lemma 3.2.8. *(i) Let $M\preceq\mathcal{U}$ and a be a tuple in $\mathcal{U}$ such that* $\mathrm{tp}(a/M)$ *is residually dominated. Since M is a model,* $\mathrm{tp}_0(a/M)$ *is finitely satisfiable in M. Let q be the unique non-forking extension of* $\mathrm{tp}_0(a/M)$ *as in Lemma 3.2.6. Then, for any $B\supseteq M$ in $\widetilde{\mathcal{U}}$, $a\models q|B$ if and only if* $\mathrm{St}_M(Ma)\mathop{\smile\!\!\!\!\!\!\mid}_M\mathrm{St}_M(B)$.

(ii) If $p\in S(\mathcal{U})$ is a non-forking extension of the residually dominated type $\mathrm{tp}(a/M)$*, then $p_0|\mathcal{U}=q|\mathcal{U}$, where q is as in (i).*

Proof. For (i), if $\mathrm{St}_M(Ma) \mathop{\smile\!\!\!\!\!\!\mid}_M \mathrm{St}_M(B)$, then by stable domination, $\mathrm{tp}_0(a/M\mathrm{St}_M(B)) \vdash \mathrm{tp}_0(a/B)$. By stable embeddedness of the stable sorts, the independence also implies that $\mathrm{tp}_0(a/M\mathrm{St}_M(B))$ does not fork over M. Thus, $\mathrm{tp}(a/B)$ does not fork over M, and by the uniqueness of q, $a \models q|B$. The converse direction is trivial.

For (ii), let $N \succeq M$ and suppose that $b \models p|N$ (hence, $b \models p_0|N$). Since $p|N$ does not fork over M, it follows that $\mathrm{k}(Mb) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_{\mathrm{k}(M)} \mathrm{k}(N)$. By Lemma 2.3.2, we obtain $\mathrm{St}_M(Mb) \mathop{\smile\!\!\!\!\!\!\mid}_M \mathrm{St}_M(N)$, as $\mathrm{tp}(b/M)$ is residually dominated. By part (i), we conclude that $b \models q|N$. Since N is arbitrary, it follows that $q|\mathcal{U} = p_0|\mathcal{U}$. □

Theorem 3.2.11 below is an analogue to Fact 3.1.16 (i). The construction of the pro-definable group homomorphism θ is nearly identical; however, instead of using the existence of strong germs of stably dominated types, we apply Proposition 3.2.9 below, which allows us to remain within the residue field of $\mathcal{U}$.

Proposition 3.2.9. *Let C be a subfield and a be a tuple of valued field elements. Let $L = \mathrm{dcl}_0(Ca)$ such that L is a regular extension of C. Assume that $\mathrm{tp}_0(a/C)$ is stably dominated. Then for any tuple b from the valued field sort with $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$ or $b \mathop{\smile\!\!\!\!\!\!\mid}_M a$ (or only with $\mathrm{k}\cap\mathrm{dcl}_0(Ca) \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_M \mathrm{k}\cap\mathrm{dcl}_0(Cb)$), we have $\mathrm{k}\cap\mathrm{dcl}_0(Mab) = \mathrm{dcl}_0(\mathrm{k}\cap\mathrm{dcl}_0(Ma), \mathrm{k}\cap\mathrm{dcl}_0(Mb))$.*

Proof. Define $L_1 = \mathrm{dcl}_0(Ca)$ and $L_2 = \mathrm{dcl}_0(Cb)$. Given that either $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ or $b \mathop{\smile\!\!\!\!\!\!\mid}_C a$, we have $\mathrm{k}_{L_1} \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_{\mathrm{k}_C} \mathrm{k}_{L_2}$. Since k_{L_1} is a regular extension of k_C, it follows from Fact 2.2.4 that k_{L_1} and k_{L_2} are linearly disjoint over k_C. Moreover, by Fact 2.1.17, we have $\Gamma_{L_1} = \Gamma_C$ and that L_1 has the good separated basis property over M. Now, let $N = L_1L_2$. By Fact 2.1.16, k_N is generated by k_{L_1} and k_{L_2} as fields. This implies that $\mathrm{k}\cap\mathrm{dcl}_0(Cab) = \mathrm{dcl}_0(\mathrm{k}\cap\mathrm{dcl}_0(Ca), \mathrm{k}\cap\mathrm{dcl}_0(Cb))$. □

The following Fact will be used later.

Fact 3.2.10. *([BMPW11, Lemme 2.1]) Let $\mathcal{L}$ be any language, and let $\mathcal{L}_0$ be a reduct of $\mathcal{L}$. Let T and T_0 be $\mathcal{L}$- and $\mathcal{L}_0$-theories, respectively, with T_0 stable. If C is* acl*-closed in T, and a does not fork with b over C in T, then a does not fork with b over C in T_0.*

Theorem 3.2.11. *Assume T is* NTP$_2$. *Let $(G,\cdot)$ be a definable residually dominated group in the valued field sort of $\mathcal{U}$, with a strongly f-generic type over M. Then, there exist an* ACF-*definable group $\mathfrak{g}$ over M in the residue field and a pro-M-definable group homomorphism $f : G^{00}_M \to \mathfrak{g}$ such that the generics of G^{00}_M are dominated via f. Namely, for each strongly f-generic p of G^{00}_M and for tuples a, b in $\mathcal{U}$ with $a \models p|M$, we have $\mathrm{tp}(b/Mf(a)) \vdash \mathrm{tp}(b/Ma)$ whenever $f(a) \mathop{\smile\!\!\!\!\!\!\mid}^{alg}_{\mathrm{k}(M)} \mathrm{k}(Mb)$.*

Proof. Let H be an M-definable algebraic group and $\iota : G_M^{00} \to H$ be an M-definable group homomorphism with finite kernel, as in Fact 3.1.9. Let $p \in S(\mathcal{U})$ be a strongly f-generic of G_M^{00}, which is residually dominated.

Let p_0 be the corresponding ACVF-type in $\widetilde{\mathcal{U}}$. Note that $\iota_*(p)$ concentrates on H, where H is ACVF-definable, hence $\iota_*(p_0)$ concentrates on H. Moreover, by Theorem 2.2.11, $p_0|M$ is stably dominated. Since M is a model of T and a substructure of M_0, the type $p_0|M$ is finitely satisfiable in M. Thus, by Lemma 3.2.6, it is stationary. Hence, p_0 is its unique non-forking extension.

Let $q = \iota_*(p)$. Note that $q_0|\mathcal{U}$ and $\iota_*(p)$ are consistent in T. Since p is strongly f-generic in G, we know that q is strongly f-generic in $\iota(G) \leq H$. Moreover, since p concentrates on G_M^{00}, we have $q \in \iota(G)_M^{00}$. We will denote the multiplication of H by $\cdot_H$. Note that since H is an algebraic group, $\cdot_H$ is ACVF-definable.

***Claim* 1.** *q_0 concentrates on $G_1 := Stab_H(q_0)$.*

Proof. First, note that $Stab_H(q_0)$ is $\mathcal{L}_0(M)$-type-definable, thus it suffices to find a realization $b \models q_0|M$ that lies in $Stab_H(q_0)$. For this, we will find independent realizations $b \models q_0|M$, $a \models q_0|Mb$ such that $b \cdot_H a \models q_0|Mb$.

Let μ be the ideal of definable subsets of H that do not extend to a generic over M. Then, q, being strongly f-generic, does not concentrate in any $D \in \mu$. In fact, by Fact 3.1.7 (ii), since $\iota(G)_M^{00} \setminus St_\mu(q)$ is contained in a union of M-definable sets in μ, we know that q concentrates on $St_\mu(q)$. So, let b be a realization of $q|M$. Then, since $b \in St_\mu(q)$, the partial type over Mb given by $q|M \cap b \cdot q|M$ is not in μ. Then it extends into a global type $q' \in S(\mathcal{U})$, such that q' is strongly f-generic of $\iota(G_M^{00})$ over M. Let a realize q', in particular $a \models q|M$. Now, by construction, both $\mathrm{tp}(a/M)$ and $\mathrm{tp}(b \cdot_H a/M)$ are equal to $q|M$. Also, by genericity of q', we have $a \mathop{\smile\!\!\!\!\!\!|}_M b$ and $b \cdot_H a \mathop{\smile\!\!\!\!\!\!|}_M b$.

We observe that these independence relations also hold in $\widetilde{\mathcal{U}}$. By Theorem 2.3.8, q is residually dominated since it is a pushforward of the residually dominated type p under an $\mathcal{L}(M)$-definable map. Then Lemma 2.3.2 implies $b \cdot_H a \mathop{\smile\!\!\!\!\!\!|}_M^0 b$ and $a \mathop{\smile\!\!\!\!\!\!|}_M^0 b$ in $\widetilde{\mathcal{U}}$. By the stationarity of $q_0|M$, we conclude $b \cdot_H a \models q_0|Mb$ and $a \models q_0|Mb$, as desired. □

By Fact 3.1.7 (ii), the group G_M^{00} is generated by the set of realizations of $p|M$ and $p^{-1}|M$. Then $\iota(G_M^{00})$ is generated by $q|M$ and $q^{-1}|M$. Since q_0 is the unique generic of G_1, we must have $q_0 = (q_0)^{-1}$. Then, by the above claim, $\iota(G_M^{00}) \leq G_1$.

***Claim* 2.** *Each strongly f-generic of G_M^{00} is residually dominated.*

Proof. Let p, q, q_0 and G_1 be as above. Fix a strongly f-generic type r over M, and let $s := \iota_*(r)$. Let s_0 be the corresponding global $\mathcal{L}_0$-type in $\widetilde{\mathcal{U}}$. Let $\mathfrak{g}$ be a pro-$\mathcal{L}_0$-definable group in the stable sorts, and let $\theta : G_1 \to \mathfrak{g}$ be the surjective $\mathcal{L}_0$-definable group homomorphism from Fact 3.1.16 (2). We wish to show that $s|M$ is residually dominated.

By the previous paragraph, we have $\iota(G_M^{00}) \leq G_1$. In particular, both s and s_0 concentrate on G_1. Choose $b \in \iota(G_M^{00})(\mathcal{U})$ such that $b \models q|M$, and let $a \in \iota(G_M^{00})(\mathcal{U})$ realize $s|Mb$. Then $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$. Since s is a strongly f-generic of $\iota(G_M^{00})$, we also have $b \cdot_H a \mathop{\smile\!\!\!\!\!\!\;|}_M b$.

Note that $\mathfrak{g}$ is also definable in $\mathcal{U}$, since all parameters are in $\mathcal{U}$. Then we have $\theta(b \cdot a) \mathop{\smile\!\!\!\!\!\!\;|}_M \theta(b)$ in $\mathcal{U}$. Because the reduct is stable, by Fact 3.2.10, the non-forking independence also holds in $\widetilde{\mathcal{U}}$, and by symmetry in stable theories, we also have $\theta(b) \mathop{\smile\!\!\!\!\!\!\;|}_M^0 \theta(b \cdot_H a)$, where $\mathrm{tp}_0(\theta(b)/M)$ is stationary and extends to the unique generic of $\mathfrak{g}$. It follows from Fact 3.1.16 (2) that $b \models q_0|M(b \cdot_H a)$, and in particular $b \models q_0|M(b \cdot_H a)^{-1}$. By the genericity of q_0, the element $(a \cdot_H b)^{-1} \cdot_H b = a^{-1}$ realizes $q_0|M$. Since $q_0|M$ is stably dominated and $a^{-1} \in \mathrm{acl}_0(Ma)$, it follows that $\mathrm{tp}_0(a/M) = s_0|M$ is stably dominated. Hence, by Theorem 2.2.11, the type $s|M = \iota_*(r|M)$ is residually dominated. Since ι has finite kernel, the type $r|M$ is residually dominated as well. This concludes the proof. $\square$

Now, we will construct the desired group homomorphism. For a tuple a in the valued field sort, let $\theta(a)$ be an enumeration of $\mathrm{k} \cap \mathrm{dcl}_0(Ma)$, which can be seen as a pro-definable map. Let $a \models q_0|M$, $b \models q_0|Ma$ and $c = a \cdot_H b$. Then, $\theta(c) \in \mathrm{k} \cap \mathrm{dcl}_0(Mab)$. Since $q_0|M$ is stably dominated, by Proposition 3.2.9, $\theta(c) \in \mathrm{dcl}_0(\theta(a), \theta(b))$. Then there exists an ACVF-definable map F such that $F(\theta(a), \theta(b)) = \theta(c)$. Since F is $\emptyset$-definable, F is defined for all such a and b.

Define $\pi = \theta_*(q_0)$. By Claim 1, for such a, b and c as above, $c \models q_0|Ma$. By Proposition 3.2.9 again, we have $\theta(b) \in \mathrm{dcl}_0(\theta(a), \theta(c))$. Similarly, we can show that $\theta(a) \in \mathrm{dcl}_0(\theta(b), \theta(c))$. Then (F, π) is a group chunk and by Fact 3.1.11, one obtains a pro-M-definable group $\mathfrak{g}$ that lies in the residue field, which is the sort of the codomain of F. By Fact 3.1.12, θ extends to a pro-M-definable map from G_1 to $\mathfrak{g}$, which we again denote it by θ.

Let $f = \theta \circ \iota$.

***Claim* 3.** *Each strongly f-generic of G_M^{00} is residually dominated via f.*

Proof. Fix a strongly f-generic r of G_M^{00}, and let b be a tuple such that $f(a) \mathop{\smile\!\!\!\!\!\!\;|}_{\mathrm{k}(M)}^{alg} \mathrm{k}(Mb)$. Assume that $b \equiv_{Mf(a)} b'$. We will show that $b \equiv_{Ma} b'$.

By Theorem 2.1.25, we may assume that b is a tuple in the valued field. First, we note that $b \equiv_{M\mathrm{k}_{\mathcal{A}}(M\iota(a))} b'$. Indeed, by residual domination of $\mathrm{tp}(\iota(a)/M)$, we have $\Gamma(M\iota(a)) = \Gamma(M)$. Then, for any $x \in \mathrm{dcl}(M\iota(a))$ with $v(x) \in n\Gamma$, since M is a model, there exists $b \in M$ such that $nv(b) = v(x)$. Hence,

$$\mathrm{res}^n(x) = \pi_n\left(\frac{x}{b^n}\right) \in \mathrm{k}(M\iota(a)) \subseteq \mathrm{dcl}(\mathrm{k} \cap \mathrm{dcl}_0(M\iota(a))).$$

It follows that $b \equiv_{M\mathrm{k}_{\mathcal{A}}(M\iota(a))} b'$.

As $\mathrm{tp}(\iota(a)/M)$ is residually dominated and $\mathrm{k}(M\iota(a)) \mathop{\smile\!\!\!\!\!\!\perp}^{alg}_{M} \mathrm{k}(Mb')$, it follows that $b \equiv_{M\iota(a)} b'$. Therefore, it remains to show that $b \equiv_{Ma} b'$.

As $Ker(\iota)$ is an M-definable finite set and M is a model, we have $Ker(\iota) \subset M$. Therefore, any a' with $\iota(a) = \iota(a')$ are interdefinable over M, meaning that we have $a' \in \mathrm{dcl}(Ma)$ and $a \in \mathrm{dcl}(Ma')$.

Now let $\phi(x, y)$ be an $\mathcal{L}(M)$-formula with $\phi(x, a) \in \mathrm{tp}(b/Ma)$. Let $X := \iota^{-1}(\iota(a)) = a_1, \ldots, a_n$, where $a_1 = a$. Note that X is $\mathcal{L}(M\iota(a))$-definable. Since the elements of X are interdefinable over M, it follows that $\phi(x, a)$ is $\mathcal{L}(Ma_i)$-definable for each $a_i \in X$.

Let σ be an automorphism fixing $M\iota(a)$ and sending b to b'. Then $\sigma(a) = a_i$ for some $i \leq n$, and hence $\phi(b', a_i)$ holds. By our observation, it follows that $\phi(b', a)$ also holds. As ϕ was arbitrary, this proves that $b \equiv_{Ma} b'$. □

It remains to show that $\mathfrak{g}$ can be assumed to be definable group rather than a pro-definable one. By Fact 3.1.13, $\mathfrak{g}$ is a projective limit of M-definable groups $(\mathfrak{g}_i)_{i\in I}$. Since each $\mathfrak{g}_i$ are definable in a stable structure, we can assume that $\theta_i(G_1)$ is definable for each $i \in \mathcal{I}$.

Let $a \models p$, where p is a strongly f-generic of G^{00}_M. Then, since a is a finite tuple, $f(a) = \mathrm{k} \cap \mathrm{dcl}_0\big(M\theta(\iota(a))\big)$ is the field generated by k_M and by a finite tuple α. Let $i_0 \in \mathcal{I}$ such that $\alpha \in \theta_{i_0}(\iota(a))$, then for all $i \geq i_0$, $\theta_i(\iota(a)) \in \mathrm{dcl}_0\big(\theta_{i_0}(\iota(a))\big)$. It follows $\mathrm{tp}(a/M)$ is residually dominated via $\theta_{i_0} \circ \iota$. So we can assume $\mathfrak{g} = \mathfrak{g}_{i_0}$, as desired.

□

Chapter 4

Final Discussions

In this final section, we discuss future directions and the questions that are not addressed in this thesis around residual domination and definable groups.

For abelian groups, a decomposition theorem is given in [HRK19] as given in the following fact:

Fact 4.0.1. *([HRKW24, Theorem 5.16]) Let H be a pro-limit of definable Abelian groups in T_0. Then, the limit stably dominated subgroup $S = \bigcup_{t \models q} S_t$ of H exists and H/S is almost internal to the sort Γ. Moreover, if H is definable, then S is connected and H/S is internal to Γ.*

Here, a limit stably dominated group refers to the direct union of connected stably dominated groups S_t, each uniformly defined using a realization of a type q in the value group. A natural next step is to extend this result to the setting of *definably amenable* groups. In NIP theories, Chernikov and Simon [CS18] showed that definably amenable groups admit a well-behaved notion of generic types and stabilizers; this was later generalized to NTP_2 theories by Montenegro, Onshuus, and Simon [MOS20]. However, the case of algebraically closed valued fields remains open.

Question 1. Does Fact 4.0.1 extend to definably amenable groups in non-trivially valued algebraically closed valued fields?

Question 2. Can Fact 4.0.1 be generalized to the henselian valued field setting?

A recent study by Cubides Kovacsics, Rideau-Kikuchi and Vicaria [KRKV24] shows the equivalence of residual domination for henselian valued fields of equicharacteristic zero (and for certain henselian valued fields with operators) in languages that eliminate imaginaries. As a continuation of previous questions, we may ask:

Question 3. For any interpretable group G in a henselian valued field of equicharacteristic zero, can G be decomposed as a direct limit of residually dominated groups and a Γ-internal group?

The algebraic statements for the instances of residual domination are provided when the field is equicharacteristic zero. Finally, we turn to the mixed-characteristic setting. To hope for an AKE-principle we restrict to unramified henselian valued fields of mixed characteristic with perfect residue field.

Question 4. Are there instances of residual domination for unramified henselian valued fields of mixed characteristic with perfect residue field?

We hope that with these open questions, one can have a AKE-type decomposition theorem for definable groups in a complete theory of henselian field, hence reduce any problem related to interpretable groups in henselian fields to a problem in a simpler structures of residue field and the value group.

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