COMPUTABLE CONTINUOUS MODEL THEORY IN VON NEUMANN ALGEBRAS

ON THE COMPUTABILITY THEORETIC AND CONTINUOUS MODEL THEORETIC STRUCTURE OF GENERAL VON NEUMANN ALGEBRAS

By JANANAN ARULSEELAN, BSc

A Thesis Submitted to the School of Graduate Studies in Partial
Fulfillment of the Requirements for
the Degree Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2025)

Hamilton, Ontario, Canada (Dept. of Mathematics and Statistics)

TITLE:	On the Computability Theoretic and Continuous Model Theoretic Structure of General von Neumann Algebras
AUTHOR:	Jananan Arulseelan BSc (Mathematics and Computer Science, University of Toronto), Toronto, Canada
SUPERVISOR:	Dr. Bradd Hart
NUMBER OF PAGES:	ix, 160

Abstract

In this thesis, we introduce and develop the model theory of general von Neumann algebras with a faithful normal semifinite weight. Our framework admits various computability theoretic properties that align it with recent work on uncomputable universal theories in the tracial von Neumann algebra setting. We study the ultraproduct that our framework suggests and we prove analogues of known theorems about the Ocneanu ultraproduct for this new ultraproduct, ultimately providing 3 new operator algebraic characterizations of our ultraproduct. We show that our framework captures the Connes-Takesaki decomposition which is central to the classification of injective factors. We capture the Connes-Takesaki decomposition via definable groupoids, examining other aspects of definability in continuous logic along the way. Finally, we study the uncomputability of various classes of operator algebras and, as an application, give consequences about ultraproduct embeddings.

Dedications

To Ansh, Aksha and Neil

Acknowledgements

I must first thank my advisor, Bradd Hart. His patience, generosity, and unending mathematical curiosity have inspired me and kept me motivated even in the face of a worldwide pandemic. His wisdom, both practical and mathematical has made this work possible. I would also like to thank Deirdre Haskell and Patrick Speissegger for serving on my committee and for significant academic and emotional support over the past few years. I thank David Sherman for serving as the External Examiner of this thesis; for catching many typos, errors and stylistic issues in an earlier draft; and for very kind and enthusiastic reviews of my work. I thank Isaac Goldbring for collaborating with me since even the early days of my graduate school career; for many long email correspondences; and for listening to my ideas and always pointing out where they failed or needed elaboration. I thank Thomas Sinclair and Christopher Eagle for their collaboration and hospitality in my visits to Purdue and the University of Victoria respectively. I thank everyone who has shown me friendship and companionship throughout my time in grad school, you are too many to list by name but all deeply appreciated. I thank my friend Sasha Shmakov for the years of interesting, enthusiastic and inspiring mathematical conversations and generosity with emotional and practical support, jokes, and music recommendations. I thank my friend Thomas Jakovac for many interesting non-mathematical conversations and for joining me on many, often eventful, walks. I thank Raymond Grinnell for fostering my interests in logic and functional analysis as a second year undergraduate and encouraging me to get in contact with Bradd Hart, thereby beginning this journey. Last, but not least, I thank my family for their love and patience while I spent many years in graduate school. I must especially thank my niece and nephews for always brightening my days with their smiles, their antics and their unconditional love. I have learned more from them than they have from me.

Contents

Declaration of Academic Achievement

Nothing in chapters 2, 3, or 4 is my own work and these chapters are present as background material. Sections 5.2 and 5.3 are also background material and not my own work. The language, axiomatization, proof of equivalence of categories and the discussions of modular automorphisms in chapter 5 are mild alterations of joint work with Goldbring, Hart and Sinclair. Chapter 8 is drawn from joint work with Goldbring and Hart. Everything else, unless stated otherwise, is my own work.

Chapter 1

Introduction

The model theory of tracial von Neumann algebras was first given explicit treatment, complete with an axiomatization in continuous logic, by Farah-Hart-Sherman in [20]. This was prefigured by their work begun earlier (though published later) in [21]. These works facilitated the import of concepts and techniques from logic to the study of tracial von Neumann algebras and their ultraproducts - a practice which is now commonplace (for a small sample, see [22], [7], [28], [36]). These von Neumann algebras are, in turn, central to a vast array of research directions and programs in mathematics and physics.

More recently, the more specialized topic of computable continuous model theory and its applications has gained prominence. This topic began to receive widespread attention near the beginning of the author's graduate school career with the publication of [43] by Ji-Natarajan-Vidick-Wright-Yuen who, using techniques from quantum complexity, resolve the famous Connes Embedding Problem (CEP) in the negative. The negative solution to the CEP means that there are separable II_1 factors which do not embed in any (tracial) ultrapower of the hyperfinite II_1 factor $\mathcal{R}$. Soon after this result was announced, Goldbring-Hart synthesized its proof with a computability and continuous model theoretic perspective in [32] to, among other things, prove a vast strengthening of the negative resolution of the CEP by showing that $\mathcal{R}$ does not have a computable universal theory. This kicked off a flurry of work and proliferation of interesting questions in computable continuous model theory (see [24], [32] or [31] for a survey).

In [4], the author, with Goldbring-Hart, expands the scope of computable continuous model theory to include von Neumann algebras that are equipped with a faithful normal state (so-called W*-probability spaces), and the associated Ocneanu ultraproducts. In [5], the author, with Goldbring-Hart-Sinclair, develops a language of W*-probability spaces that is better suited to computable continuous logical study than the previous Dabrowski language.

Note that not all von Neumann algebras admit a faithful normal state, though they all admit a generalization called a faithful normal semifinite weight. To a von Neumann algebra $\mathcal{M}$ and a choice of faithful normal semifinite weight Φ on it, we can associate a one-parameter group of automorphisms σ_t^Φ of $\mathcal{M}$ called the modular automorphism group or simply the modular group (see Section 4.5 for more details). The modular group is trivial in the tracial von Neumann algebra setting, but it is indispensable to the structure theory of more general W*-probability spaces. In particular, the study of the modular group, also known as Tomita-Takesaki theory, is the key to understanding the so-called hyperfinite type III factors, which admit natural choices of faithful normal states but do not admit faithful normal traces. More generally, Tomita-Takesaki theory is the source of all known structural studies of type III von Neumann algebras. Strikingly, the modular group also plays an important role in quantum field theory (see [62] or [9] for a gentle introduction).

In this thesis, we further expand the scope of continuous model theory to include von Neumann algebras in complete generality by considering full left Hilbert algebras. Previous work in continuous model theory has yielded significant tools and insights in the type I and type II_1 settings; this thesis sets the stage for the extension of these tools and insights to the type III setting, about which less is presently known (though see [34] for several interesting applications of model theory in the type III setting). The language we give sticks closely to that given for tracial von Neumann algebras by Farah-Hart-Sherman, further easing the transition from the tracial paradigm to the completely general setting for model theorists and operator algebraists. The computability of our theory and the various extensions we give allows the recent work in computable continuous model theory to be readily adapted to this setting.

This thesis straddles multiple broad areas of mathematics, and therefore draws from a wide range of prerequisites. In Chapter 2, Chapter 3, and Chapter 4, we aim to acquaint the reader with three of these: continuous logic, operator algebras, and Hilbert algebras (and the accompanying Tomita-Takesaki theory) respectively. Rather than simply listing definitions, lemmas and theorems, or recapitulating standard references, we attempt to tailor the exposition to our unique purposes and to provide the reader with relevant insights.

In Chapter 5, we present a natural language and axiomatization of a theory of full left Hilbert algebras or, equivalently, von Neumann algebras equipped with a choice of faithful normal semifinite weight. We dub the latter "weighted von Neumann algebras". The existence of such an axiomatization has been conjectured by experts in continuous logic, likely even since shortly after the original work of Farah-Hart-Sherman. Dabrowski, in [16], gives an abstract axiomatizability result, using Keisler-Shelah, for the class of (preduals of) von Neumann algebras in his language of so-called "tracial matrix ordered operator spaces". However, this axiomatization (provably) cannot possibly include the modular automorphism group and therefore does not induce the correct morphisms of models (namely inclusions of von Neumann algebras admitting a conditional expectation).

The language presented here required multiple innovations, which may explain post-hoc why this problem remained open for so long. The first such problem is that multiplication is not uniformly continuous on the operator norm unit ball. Dealing with this issue required the introduction of totally K-bounded elements. By taking our sorts S_K to be the totally K-bounded elements, we are able to include multiplication in our language, which is highly desirable since the multiplication is ubiquitous in all applications. The second problem is that in order to ensure that our models have the correct morphisms, we need to be able to express the modular group. This is a difficult task since the classical approaches to Tomita-Takesaki theory rely on the use of unbounded (and therefore discontinuous) operators. We resolve this issue by adapting techniques of Rieffel-Van Daele from [57]. After multiple calculations to prove that the formalism of Rieffel-Van Daele is compatible with totally K-bounded elements, we use their formalism to exhibit a computable definitional expansion of theory which axiomatizes the modular group as a one-parameter group of automorphisms associated to our choice of weight. The last hurdle that we will discuss here is one that arises when showing that the axiomatization we have is correct, namely that there is an equivalence of categories between models of our theory and the category we are axiomatizing. Call the category we want to capture $\mathcal{C}$ and the category of models of our theory $\mathrm{Mod}(T)$. The functors witnessing this equivalence are the Dissection functor taking an object of $\mathcal{C}$ and returning a metric structure in $\mathrm{Mod}(T)$ and the Interpretation functor taking structures in $\mathrm{Mod}(T)$ and returning objects of $\mathcal{C}$. It is often easy to show that if we start with $c \in \mathrm{Ob}(\mathcal{C})$, dissect it and then interpret the resulting metric structure, we get something isomorphic to c. On the other hand, it is often non-trivial to show that if we start with $M \in \mathrm{Mod}(T)$, interpret it and then dissect the result, we have an isomorphic metric structure. The reason for this is that we need to rule out the possibility of extra elements finding their way into the sorts that M did not see. In the tracial setting, the analogous problem was resolved by adding extra axioms to mimic the proof of the Kaplansky Density Theorem (see [20, Section 3]). Since the polynomials involved there do not necessarily preserve totally bounded elements, we must find a new approach. To that end, inspired by results of Kadison in [45], we prove Theorem 5.5.18 characterizing the totally bounded elements of a weighted von Neumann algebra. The proof takes up the majority of section 5.5 and is rather involved and technical. We believe this result is of independent interest to operator algebraists.

In the tracial von Neumann algebra and W*-probability space settings, the model theoretic ultraproduct that arose from the axiomatizations corresponded to the tracial ultraproduct and the Ocneanu ultraproduct respectively (and the latter subsumes the former). These ultraproducts were already well-understood by operator algebraists, having been introduced by Sakai in 1962 and Ocneanu in 1985 respectively. In contrast, the only ultraproduct found in the literature for general von Neumann algebras is the Groh-Raynaud ultraproduct. The Groh-Raynaud ultraproduct is well-known to not behave well with respect to modular automorphisms (see, for example, [3]). Since our axiomatization can define the modular group, our ultraproduct must not be the Groh-Raynaud ultraproduct. In Chapter 6, building on ideas of Ando-Haagerup in [3] and of Masuda-Tomatsu in [49], we provide multiple purely operator algebraic ultraproduct constructions for weighted von Neumann algebras. Ultimately, we show that all of these constructions are equivalent to the model theoretic ultraproduct associated to our theory. These results and the associated ultraproducts can be of interest even to any von Neumann algebraists who are unfamiliar with model theoretic language and techniques.

In Chapter 7, we extend some of the definability machinery of continuous logic. Considering definable groupoids, we show that our theory "remembers" the Connes-Takesaki decomposition. The Connes-Takesaki decomposition plays a crucial role in Connes' work on the classification of injective factors in [13]. This gives a partial answer to a question of Bradd Hart regarding the model-theoretic content of Connes' classification.

In [4], we proved various extensions of the negative solution to CEP as well as the undecidability of the universal theories of various hyperfinite type III von Neumann algebras and associated results about ultraproduct embeddings of von Neumann algebras. In Chapter 8, we reprove many of these results in the language developed in Chapter 5. Some of the proofs we provide here are notably simpler than those given in [4], demonstrating the utility of our formalism. We also prove further results that are made possible by this language, remarkably including the undecidability of the universal theory of the hyperfinite II_∞ factor. This result is the first such undecidability result for a von Neumann algebra equipped with an unbounded weight.

Anticipating more such results, we prove Theorem 8.6.4, which can be used to extend an undecidability result for the centralizer of an algebra to the algebra itself.

Chapter 2

Preliminaries on Continuous Logic

2.1 Overview

Model theory is the branch of mathematics that studies classes of mathematical objects by attaching semantics (called structures or models) to syntax. Classical model theory was formally initiated by Alfred Tarski and his school in 1933 with the development of his semantic theory of truth. Truth, in this setting, follows a bivalent logic, meaning it assumes exactly two truth values (namely True and False). In the near century since then, model theory has become a central topic in mathematics, with deep connections and applications to algebra, algebraic geometry, graph theory, computer science, differential equations and nearly every other field.

For some purposes, such as the study of Banach spaces, two truth values turn out to be insufficient. It becomes more natural to consider logics with continuum many truth values corresponding to real numbers. Multi-valued logics such as those considered by Łukasiewicz and Pavelka have been studied since the 1920s. Various model theories corresponding to such logics were studied by Chang and Keisler in the

1960s. A related model theory for Banach spaces was studied by Henson and Iovino. Continuous model theory in its most recognizable form was initiated by Ben Yaacov and Usvyatsov. Since then, continuous logic has been successfully applied to a wide range of subjects in operator algebras and metric geometry.

Many of the proofs in this chapter are standard and can be found in [30, Chapter 2], among other sources, and follow the classical proofs closely and will thus be omitted. For the fundamentals of classical logic, the reader should see [19]. For a more comprehensive account of classical model theory, we recommend [11]. To learn more about continuous logic, the reader should see [30].

2.2 Continuous Logic

We will begin by recalling the basic objects and theorems of continuous logic. The aim of this section is to establish notation, conventions and terminology for self-containment and the convenience of the reader.

A foundational idea in model theory is that of a signature. It forms the basis for constructing languages and syntax and also provides the framework upon which structures will hang. Recall that a **modulus of uniform continuity** or **continuity modulus** for a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a function $\delta : \mathbb{R} \to \mathbb{R}$ such that $d_Y(f(x), f(y)) \leq \delta(d_X(x, y))$ for all $x, y \in X$.

Definition 2.2.1. A **signature** is a triple $\Sigma := (\mathfrak{S}, \mathfrak{F}, \mathfrak{R})$ such that:

- $\mathfrak{S}$ is a set of **sort symbols** S together with a real number K_S for each S.
- $\mathfrak{F}$ is a set of **function symbols** f together with an **arity** $(S_1, ..., S_n)$ with each $S_i \in \mathfrak{S}$ representing the domain, $S \in \mathfrak{S}$ representing the codomain and a

modulus of uniform continuity δ_F for each F.

- $\mathfrak{R}$ is a set of **relation symbols** R together with an arity as above for the domain of R and a compact interval K_R in $\mathbb{R}$, the codomain of R, and a modulus of uniform continuity δ_R for each R.

The language $\mathcal{L}$ associated to a signature Σ is defined recursively as in classical logic. Here the **connectives** are continuous functions from cartesian powers of compact intervals to the reals. **Quantifiers** will be supremum and infimum (for universal and existential quantification respectively).

Definition 2.2.2. We define the main syntactic notions in continuous logic. We will always assume that, for each sort, we have infinitely many variables available that range over that sort.

- **$\mathcal{L}$-terms** are valid compositions of functions, variables, and constants. They naturally come with continuity moduli by the respective composition of the continuity moduli for the functions.

- **Atomic $\mathcal{L}$-formulae** are given by

  - $R(t_1, ..., t_n)$ where $R \in \mathfrak{R}$ with arity $(S_1, ..., S_n)$ and each t_i is a term of sort S_i;

  - $d_S(t_1, t_2)$ where t_1 and t_2 are terms of sort S. We will interpret d_S as the underlying metric of the sort S.

- **Formulae** are obtained by closing under connectives and quantifiers.

- **Sentences** are formulae with no free variables.

With a language, we can introduce the notion of a structure in that language. This is the starting point for the semantic aspect of logic.

Definition 2.2.3. Given a signature $\Sigma = (\mathfrak{S}, \mathfrak{F}, \mathfrak{R})$ and its associated language $\mathcal{L}$, we define an **$\mathcal{L}$-structure** to be a triple $M := (\mathfrak{S}(M), \mathfrak{F}(M), \mathfrak{R}(M))$ of indexed families where:

- $\mathfrak{S}(M)$ is a family of complete bounded metric spaces S^M indexed by $S \in \mathfrak{S}$ such that S^M has diameter bounded by K_S.
- $\mathfrak{F}(M)$ is a family of functions f^M indexed by $f \in \mathfrak{F}$ with domain, codomain and modulus of uniform continuity as specified by the data associated to f.
- $\mathfrak{R}(M)$ is a family of relations R^M indexed by $R \in \mathfrak{R}$ with domain, codomain and modulus of uniform continuity as specified by the data associated to R and range contained in K_R.

For an $\mathcal{L}$-structure M, and a formula ϕ (resp. a term t), we can recursively define ϕ^M (resp. t^M) in the obvious way. We will call this the **interpretation** of ϕ (resp. t) in M. Note that if σ is a sentence, then for any $\mathcal{L}$-structure M, the interpretation σ^M of σ in M is a real number. We now have the ingredients to start studying theories and the models thereof.

Definition 2.2.4. An **$\mathcal{L}$-theory** is a set of $\mathcal{L}$-sentences. An $\mathcal{L}$-structure M is a model of an $\mathcal{L}$-theory T, denoted $M \vDash T$, if $\sigma^M = 0$ for all $\sigma \in T$. We let $\mathrm{Mod}(T)$ denote the class of all models of T. A class $\mathcal{C}$ of $\mathcal{L}$-structures is called an **elementary class** if there is an $\mathcal{L}$-theory T such that $\mathcal{C} = \mathrm{Mod}(T)$ in which case, we will say that T **axiomatizes** $\mathcal{C}$.

Until further notice, fix a language $\mathcal{L}$ and an $\mathcal{L}$-theory T. We denote the set of sentences in $\mathcal{L}$ by $\mathrm{Sent}(\mathcal{L})$. Let $\overline{x}$ denote a tuple of variables $(x_1, \ldots, x_n)$ and $M^{\overline{x}}$ denote the set of tuples of elements in M with the same arity as $\overline{x}$.

Definition 2.2.5. For an $\mathcal{L}$-structure M, the **evaluation map** is $\mathrm{ev}_M : \mathrm{Sent}(\mathcal{L}) \to \mathbb{R}$ defined by $\mathrm{ev}_M(\phi) = \phi^M$.

The **theory** $\mathrm{Th}(M)$ of a structure M is the set consisting of sentences ϕ such that $\mathrm{ev}_M(\phi) = 0$.

Definition 2.2.6. Two $\mathcal{L}$-structures M and N are **elementarily equivalent**, written $M \equiv N$, if $\mathrm{Th}(M) = \mathrm{Th}(N)$.

Definition 2.2.7. A map $\rho : M \to N$ is a **homomorphism** if

- for all function symbols f of $\mathcal{L}$ and $\overline{a} \in M^{\overline{x}}$, we have $\rho(f^M(\overline{a})) = f^N(\rho(\overline{a}))$.
- for all relation symbols R in $\mathcal{L}$ and $\overline{a} \in M^{\overline{x}}$, we have $R^M(\overline{a}) \geq R^N(\rho(\overline{a}))$.

Definition 2.2.8. ρ is an **embedding** if it is a homomorphism and for all relation symbols R in $\mathcal{L}$ and $\overline{a} \in M^{\overline{x}}$, we have $R^M(\overline{a}) = R^N(\rho(\overline{a}))$.

Remark 2.2.9. If ρ is an embedding, then ρ^S is an isometric embedding for all sorts S of $\mathcal{L}$.

Definition 2.2.10. An embedding $\rho : M \to N$ is an **elementary embedding** if for all formulae ϕ with parameters in M, we have $\phi^M = \phi^N \circ \rho$.

Definition 2.2.11. A set of sentences Γ is **complete** if it is satisfiable, and whenever M and N satisfy Γ , then $M \equiv N$.

Note that if M is an $\mathcal{L}$-structure, then $\mathrm{Th}(M)$ is a complete $\mathcal{L}$-theory.

Definition 2.2.12. If N is a substructure of M, then we say that N is an **elementary substructure** of M, written $N \preceq M$, if whenever $\phi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in N$ is a sequence sorted in the same manner as the variables $\overline{x}$, then $\phi^N(\overline{a}) \equiv \phi^M(\overline{a})$. In this case, we call N an **elementary expansion** of M.

We write $N \prec M$ if $N \preceq M$ and $N \neq M$. If $\rho : M \hookrightarrow N$ is an embedding, then it is an elementary embedding if $\rho(M) \preceq N$.

Let Mod(T) be the category of models of T with morphisms given by elementary embeddings between them. We consider the pseudometric space $(\mathcal{F}^0_{\overline{x}}, d_0)$ with $\mathcal{F}^0_{\overline{x}}$ the space of $\mathcal{L}$-formulae together with the pseudometric

$$d_0(\phi, \psi) = \sup\{|\phi^M(\overline{a}) - \psi^M(\overline{a})| \ : \ M \vDash T, \overline{a} \in M^{\overline{x}}\}.$$

The reason d_0 is only a pseudometric and not a metric is because, as defined, many pairs of distinct formulae ϕ and ψ may satisfy $d_0(\phi, \psi) = 0$. For example, consider any formula ϕ and let $\psi := \phi + 2 - 2$.

We will say that ϕ and ψ are T-**equivalent** if $d(\phi, \psi) = 0$. We denote by $(\mathcal{F}_{\overline{x}}, d)$ the metric space completion of $\mathcal{F}^0_{\overline{x}}$ with respect to d_0 together with the completed metric d.

We call an element of $\mathcal{F}_{\overline{x}}$ a T-**formula**. Notice that T-equivalent formulae are identified in $\mathcal{F}_{\overline{x}}$. This will generally not lead to any issues as it will be clear from context when we want to refer to the syntactic concept of a formula or the concept up to T-equivalence.

Theorem 2.2.13. *Let (ϕ_n) be a sequence of $\mathcal{L}$-formulae representing the T-formula ϕ in $\mathcal{F}_{\overline{x}}$. Then $\phi^M(\overline{a}) = \lim_{n\to\infty} \phi_n^M(\overline{a})$ for any $\overline{a} \in M^{\overline{x}}$. In other words, ϕ^M has a* *well-defined* ***interpretation*** *in any* $M \vDash T$.

Proof. Since (ϕ_n) is a Cauchy sequence, we have for every $\epsilon > 0$ that there exists an N such that for every $n, m > N$

$$d(\phi_n, \phi_m) \leq \epsilon \text{ therefore } T \vDash \sup_{\overline{x}} |\phi_n(\overline{x}) - \phi_m(\overline{x})| \leq \epsilon$$

so this limit is defined. Further, for any other (ψ_n) representing ϕ, we have for every $\epsilon > 0$ that there exists an N such that for all $n > N$

$$d(\phi_n, \psi_n) \leq \epsilon \text{ therefore } T \vDash \sup_{\overline{x}} |\phi_n(\overline{x}) - \psi_n(\overline{x})| \leq \epsilon$$

so that this limit is unique. □

Next, we will study definability. To define definability, we need the notion of a T-functor. Let Met denote the category of bounded metric space with morphisms isometric embeddings between them.

Definition 2.2.14. A functor $X : \mathrm{Mod}(T) \to \mathrm{Met}$ is called a T**-functor over** $\overline{x}$ if

- For every $M \vDash T$, $X(M)$ is a closed subset of $M^{\overline{x}}$.
- $X(f : M \to N)$ is the morphism given by restriction of the map induced on $X(M)$ by f.

The most natural examples of T-functors are given by zerosets.

Definition 2.2.15. The **zeroset** of a T-formula ϕ is given by

$$Z(\phi)(M) = Z(\phi^M) = \{\overline{a} \in M^{\overline{x}} \ : \ \phi(\overline{a}) = 0\}.$$

Definition 2.2.16. $\phi(\overline{x})$ is called an **almost-near formula** for M if, for any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ so that, for any $\overline{a} \in M^{\overline{x}}$, if $|\phi^M(\overline{a})| < \delta(\epsilon)$, then there is $\overline{b} \in M^{\overline{x}}$ such that $\phi^M(\overline{b}) = 0$ and $d(\overline{a}, \overline{b}) \leq \epsilon$. In this case, we refer to the function $\delta(\epsilon)$ as a modulus for ϕ. If ϕ is an almost-near formula for M, then we refer to the zeroset of ϕ^M in $M^{\overline{x}}$, denoted $Z(\phi^M)$, as the definable set corresponding to ϕ.

Definition 2.2.17. A T-functor is called a T-**definable set** if and only if it is the zeroset of an almost near formula.

Proposition 2.2.18. *X is a T-definable set if and only if for every T-formula $\psi(\overline{x}, \overline{y})$, the functions $\sup_{\overline{x} \in X} \psi(\overline{x}, \overline{y})$ and $\inf_{\overline{x} \in X} \psi(\overline{x}, \overline{y})$ are realized by T-formulae.*

Remark 2.2.19. The above proposition confirms that this is in fact the right definition of definability. Not all zerosets satisfy this property. This is the key use of definable sets in classical model theory.

Fix a tuple of variables $\overline{x}$ from the sorts of $\mathcal{L}$.

Definition 2.2.20.

- For $M \vDash T$ and $\overline{a} \in M^{\overline{x}}$, the function $\mathrm{tp}^M(\overline{a}) : \mathcal{F}_{\overline{x}} \to \mathbb{R}$ given by $\mathrm{tp}^M(\overline{a})(\phi) = \phi^M(\overline{a})$ is called the **type** of $\overline{a}$ in M.
- The **space of $\overline{x}$-types** of T, denoted $S_{\overline{x}}(T)$, is the set of functions of the form $\mathrm{tp}^M(\overline{a})(\phi(\overline{x})) = \phi^M(\overline{a})$, where M is a model of T, $\overline{a} \in M^{\overline{x}}$, and ϕ is an $\mathcal{L}$-formula whose free variables are among $\overline{x}$.
- An element of the space of $\overline{x}$-types is called a **complete type**.
- $p \in S_{\overline{x}}(T)$ is **realized** in M if there is an element $a \in M$ such that $\mathrm{tp}^M(\overline{a}) = p$.

Definition 2.2.21. Let Σ be a set of $\mathcal{L}$-formulae and p a complete type. The **partial type** associated to Σ is the restriction of p to Σ.

Definition 2.2.22. The **quantifier free type** $\mathrm{tp}_{qf}(\overline{a})$ of $\overline{a}$ is the restriction of $\mathrm{tp}(\overline{a})$ to the set of all quantifier free formulae.

Definition 2.2.23. Suppose that M is an $\mathcal{L}$-structure and $A \subseteq M$. p is a **(partial) type over** A if p is a (partial) type in $\mathcal{L}_A$ with respect to the theory of M_A.

Just as in classical model theory, the space of $\overline{x}$-types $S_{\overline{x}}(T)$ naturally carries a topology. Unlike in classical model theory, this topology is not necessarily totally disconnected.

Definition 2.2.24. The **logic topology** on $S_{\overline{x}}(T)$ is the topology generated by the basis consisting of the sets:

$$B(\phi, (r, s)) = \{p \in S_{\overline{x}}(T) \ : \ p(\phi) \in (r, s)\}$$

where ϕ and $r, s \in \mathbb{R}$ so that (r, s) represents the open interval.

Remark 2.2.25. The logic topology is the coarsest topology on $S_{\overline{x}}(T)$ such that the function $f_\phi(p) = p(\phi)$ (the value of the type p evaluated at ϕ) is continuous for all T-formulae ϕ.

When T is complete, $S_{\overline{x}}(T)$ carries a second topology, finer than the logic topology, that measures how close realizations of types can be.

Definition 2.2.26. The **metric topology** on $S_{\overline{x}}(T)$, for complete T, is the topology induced by the metric

$$d(p, q) = \inf_{M \models T} \inf_{\overline{a}, \overline{b}} d(\overline{a}, \overline{b})$$

where $\overline{a}$ ranges over realizations of type p and $\overline{b}$ ranges over realizations of type q.

The relationship between the logic topology and the metric topology is a key conceptual tool in stability theory and the study of categoricity in continuous model theory. There is a fruitful analogy between the logic and metric topologies on $S_{\overline{x}}(T)$ and the weak and norm topologies on $B(\mathcal{H})$, the bounded linear operators on a Hilbert space.

Classical model theory often concerns itself with the cardinalities of both languages and structures. The analogue of cardinality in continuous logic is the concept of density character.

Definition 2.2.27. The **density character** $\chi(X)$ of a topological space X is the smallest cardinality of a dense subset of X.

Definition 2.2.28. The **density character** $\chi(T)$ of T is the smallest cardinality of a dense subset of $F_{\overline{x}}$.

Now that we have some of the relevant jargon, we will state some of the main theorems of continuous logic. Many of these theorems are ubiquitous in continuous logic and its applications and will be used frequently in this thesis so it is convenient to record them here. We begin by stating analogues of what are arguably the most important and defining (see Lindström's theorem) theorems of classical first order logic.

Theorem 2.2.29 (Downward Löwenheim-Skolem)**.** *Let $\mathcal{L}$ be a separable first order language and let T be a $\mathcal{L}$-theory with an infinite model M of infinite density character. Then M has a separable elementary submodel N.*

We say that a metric structure M is **compact** if S^M is a compact metric space for every sort S in the language.

Theorem 2.2.30 (Upward Löwenheim-Skolem)**.** *Let $\mathcal{L}$ be a separable language and let T be an $\mathcal{L}$-theory with an infinite non-compact model M with density character κ. Then for every $\kappa' > \kappa$, there is an elementary extension N of M of density character $\chi(N) = \kappa'$.*

Definition 2.2.31. Γ is **satisfiable** if there is an $\mathcal{L}$-structure M such that $M \models \Gamma$.

Definition 2.2.32. Γ is **finitely satisfiable** if every finite subset of Γ is satisfiable.

We will often use the notation $a \dot{-} b$ to mean the function $\max\{a - b, 0\}$. This serves as a convenient way to express $a \leq b$ in terms of continuous functions.

Definition 2.2.33. Γ is **approximately finitely satisfiable** if the set

$$\{|\phi| \dot{-} \epsilon \ : \ \phi \in \Gamma, \epsilon > 0\}$$

is finitely satisfiable.

Theorem 2.2.34 (Compactness)**.** *The following are equivalent:*

1. Γ *is satisfiable.*

2. Γ *is finitely satisfiable.*

3. Γ *is approximately finitely satisfiable.*

Remark 2.2.35. The compactness theorem is equivalent to the fact that $S_{\overline{x}}(T)$ is compact with respect to the logic topology for all T and $\overline{x}$. Compactness also implies that every partial type over A is realized in an elementary extension of A.

Next we state a convenient criterion for elementary substructures.

Theorem 2.2.36 (Tarski-Vaught)**.** *Suppose that $N \subseteq M$. Then the following are equivalent:*

- $N \preceq M$.
- *For all basic $\mathcal{L}$-formulae $\phi(y, \overline{x})$ and all $\overline{a} \in N$*

$$\inf\{\phi^M(b, \overline{a}) \ : \ b \in N\} = \inf\{\phi^M(b, \overline{a}) \ : \ b \in M\}.$$

2.3 Ultraproducts and Saturation

We introduce the concept of an ultraproduct in the continuous setting as well as Łoś' theorem, the so-called "fundamental theorem of ultraproducts". Ultraproducts are an indispensable tool in both classical and continuous model theory and their applications. In particular, we will soon provide a criterion for definability in terms of ultraproducts. We will also use the existence of notions of ultraproduct in a class of objects as both a guide for axiomatizing that class and a way to study that class. First we recall the definition of an ultrafilter.

Definition 2.3.1. A **filter** $\mathcal{U}$ on a set I is a set $\mathcal{U} \subset P(I)$ of subsets of I such that:

- $I \in \mathcal{U}$;
- $A \in \mathcal{U}$ and $A \subseteq B$ implies $B \in \mathcal{U}$; and
- $A \in \mathcal{U}$ and $B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$.

Definition 2.3.2. An **ultrafilter** is a maximal filter.

Definition 2.3.3. An ultrafilter is said to be **non-principal** if it does not contain a singleton.

Remark 2.3.4. Non-principal ultrafilters exist by Zorn's lemma. Throughout the rest of this thesis, all ultrafilters will be taken to be non-principal.

Definition 2.3.5. Let (a_i) be a sequence of real numbers and $\mathcal{U}$ an ultrafilter on I. Suppose there is a real number B such that $\{i \in I \ : \ |a_i| < B\} \in \mathcal{U}$. Then there is a unique real number r such that for every $\epsilon > 0$, we have $\{i \in I \ : \ |a_i - r| < \epsilon\} \subseteq \mathcal{U}$. We call r the **ultralimit** of (a_i) with respect to $\mathcal{U}$.

Let $(M_i)_{i \in I}$ be a family of $\mathcal{L}$-structures and let $\mathcal{U}$ be an ultrafilter on I. For each sort S, define $d_{S,\mathcal{U}}$ as the pseudometric on $\prod_{i \in I} S(M_i)$ defined by

$$d_{S,\mathcal{U}}(\overline{x}, \overline{y}) = \lim_{\mathcal{U}} d_{S(M_i)}(\overline{x_i}, \overline{y_i}).$$

Definition 2.3.6. The **ultraproduct** of (M_i), denoted $\prod_{\mathcal{U}} M_i$ has $\prod S(M_i)/d_{S,\mathcal{U}}$ as sorts and symbols interpreted pointwise. If $M_i = M$ for all i, then we call this the **ultrapower** $M^{\mathcal{U}}$.

Remark 2.3.7. The uniform continuity moduli in our languages guarantee that functions and relations on the ultraproduct are well-defined.

Theorem 2.3.8 (Łos' theorem). *For an arbitrary $\mathcal{L}$-formula $\phi(\overline{x})$ and $\overline{a} = (\overline{a_i}) \in \prod_{\mathcal{U}} M_i$ we have that*

$$\phi^{\prod_{\mathcal{U}} M_i}(\overline{a}) = \lim_{\mathcal{U}} \phi^{M_i}(\overline{a_i})$$

where ϕ ranges over $\mathcal{F}_{\overline{x}}$.

We now give an extremely useful characterization of definable sets in terms of ultraproducts.

Theorem 2.3.9. *A T-functor X is a T-definable set if and only if for all sets I, all families of models $(M_i)_{i\in I}$ of T, and all ultrafilters $\mathcal{U}$ on I, we have $X(\prod_{\mathcal{U}} M_i) = \prod_{\mathcal{U}} X(M_i)$.*

It is often useful in model theory to consider models that realize many types.

Definition 2.3.10. Let κ be an infinite cardinal and a M be an $\mathcal{L}$-structure. Then M is said to be κ-**saturated** if for every $A \subset M$ where $\chi(A) < \kappa$, every type (equivalently, every 1-type) over A is realized in M.

Definition 2.3.11. M is said to be **saturated** if it is $\chi(M)$-saturated.

Proposition 2.3.12. *For every infinite cardinal κ, there is an ultrafilter $\mathcal{U}$ on κ such that if $\mathcal{L}$ has density character at most κ and M_α is an $\mathcal{L}$-structure for all $\alpha < \kappa$, then $\prod_{\mathcal{U}} M_\alpha$ is κ^+-saturated.*

Definition 2.3.13. $\mathcal{U}$ as in the above proposition is called a **good ultrafilter**.

In the case that $\kappa = \aleph_0$, we get a simpler statement.

Proposition 2.3.14. *If $\mathcal{L}$ is a separable language, M_n an $\mathcal{L}$-structure for all $n \in \mathbb{N}$, and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$, $\prod_{\mathcal{U}} M_n$ is $\aleph_1$-saturated.*

Theorem 2.3.15 (Keisler-Shelah)**.** *For $\mathcal{L}$-structures M and N, we have that $M \equiv N$ if and only if there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that $M^{\mathcal{U}} \cong N^{\mathcal{V}}$.*

Definition 2.3.16. Let $M^{\mathcal{U}}$ be an ultrapower of M. Then we call M an **ultraroot** of $M^{\mathcal{U}}$.

Proposition 2.3.17. *Suppose that C is a class of $\mathcal{L}$-structures. The following are equivalent:*

- *C is an elementary class.*
- *C is closed under isomorphisms, ultraproducts, and elementary submodels.*
- *C is closed under isomorphisms, ultraproducts, and ultraroots.*

2.4 Computable Continuous Logic

Throughout this section, fix a language $\mathcal{L}$. An immediate concern in developing computable continuous logic is the fact that our languages are uncountable and thereby too big to be computable in the usual sense. Namely, we have all continuous functions as connectives. This is no serious issue though, as the Stone-Weierstrass theorem guarantees we have countable dense subsets of this set. Moreover, we can demand these subsets be computable. Denote by $C_c(\mathbb{R}^k)$ the set of compactly supported continuous functions to $\mathbb{R}$ on $\mathbb{R}^k$.

We fix a countable collection (u_n) of continuous functions $f : \mathbb{R}^k \to \mathbb{R}$ over all k with compact support satisfying the following two conditions:

- For each k, the set $(u_n) \cap C_c(\mathbb{R}^k)$ is dense in $C_c(\mathbb{R}^k)$.
- There is an algorithm that takes a computable $f \in C_c(\mathbb{R}^k)$ and a rational $\delta > 0$ and returns n such that $u_n \in C_c(\mathbb{R}^k)$ and $\|f - u_n\|_\infty < \delta$. For convenience, we assume that the following functions are among the sequence (u_n):
  - the binary functions $+$ and $\times$;

- for each $\lambda \in \mathbb{Q}$, the unary function $x \mapsto \lambda x$;
- the binary function $\dot{-}$ given by $x \dot{-} y = max(x - y, 0)$; and
- the unary functions $x \mapsto 0$, $x \mapsto 1$, and $x \mapsto \frac{x}{2}$.

We call an $\mathcal{L}$-formula ϕ a **restricted $\mathcal{L}$-formula** if it only uses functions from (u_n) as connectives. We further say ϕ is **computable** if it only uses computable functions as connectives. We fix an enumeration (ϕ_n) of restricted $\mathcal{L}$ formulae.

Proposition 2.4.1. *There is an algorithm such that takes as inputs a computable $\mathcal{L}$-formula $\phi(\overline{x})$ and rational $\delta > 0$ and returns n such that $\phi_n(\overline{x})$ has the same arity as ϕ and $\|\phi - \phi_n\| < \delta$, the distance being the usual logical distance between $\mathcal{L}$-formulae. Moreover, if ϕ is quantifier-free, then so are the ϕ_n.*

Proposition 2.4.2. *There is an algorithm such that takes a computable almost near formula $\phi(\overline{x})$ for M that has a computable modulus and a rational $q > 0$ and returns $n \in \mathbb{N}$ so that, for all $\overline{a} \in M^{\overline{x}}$, we have $|d(\overline{a}, Z(\phi^M)) - \phi_n(\overline{a})^M| < q$. Moreover, if ϕ is quantifier-free, then each ϕ_n is existential.*

Chapter 3

Operator Algebras Preliminaries

3.1 Overview

The study of von Neumann algebras was first initiated in the 1930s, motivated by unitary representations and by mathematical formalisms for quantum mechanics, by von Neumann himself and Murray in [50], [51], [69] and [52]. There, they proved many of the foundational results of the theory. The more general class of C*-algebras was studied by Rickart, Gelfand, Naimark, Krein and others in their abstract form in the early 40s. Segal studied their concrete form represented on a Hilbert space and introduced the term "C*-algebra" in [60]. Work of Gelfand and Naimark in 1943 showed that they can be seen as a noncommutative generalization of topological spaces. This perspective later, in turn, led to important connections to K-theory and geometry via index theory. The classification program for simple nuclear C*-algebras is one of the great mathematical feats of the last century, involving many researchers and many important insights (see [63]). In the almost 100 years since their introduction, the study of von Neumann algebras has advanced significantly.

It has developed connections to dynamical systems, representation theory, logic and mathematical physics. In the 1970s, through work by Sakai, McDuff, Connes and others, ultraproducts became a central tool in the analysis of von Neumann algebras. Also from the 1950s to the 1970s, Tomita and Takesaki further revolutionized the field with the theory of modular automorphism groups. Today, operator algebras continue to extend their reach into broader mathematics through Connes' noncommutative geometry program (see [14]). Operator algebras can be found as noncommutative or "quantum" analogues of various classical mathematical objects. However, in this thesis, we will only consider von Neumann algebras and occasionally more general C^*-algebras.

Due to time and space constraints, we assume the reader is familiar with the essentials of functional analysis including, but not limited to: Banach space and Hilbert space theory, linear operators and functionals, weak convergence and Baire category results. The reader lacking such familiarity should consult one of [48], [55] or [58].

While von Neumann algebras are a special case of C^*-algebras, often the techniques used and results obtained differ greatly between the subjects. For the fundamental concepts used to study C^*-algebras and von Neumann algebras, we recommend [55]. More advanced looks at C^*-algebras are [17] and [44]. For details about the Elliot classification program, the reader should consult [63]. A very thorough account of von Neumann algebra theory can be found in [65]. An encyclopedic reference for both topics is [8].

3.2 Definitions

Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$ together with the operator norm.

Definition 3.2.1. For $a \in B(\mathcal{H})$, denote by a^* the unique operator such that

$$\langle av \, , \, w \rangle = \langle v \, , \, a^* w \rangle$$

for all $v, w \in \mathcal{H}$. We call a^* the **adjoint** of a.

A unital *-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a C***-algebra** if it is closed in the topology induced by the operator norm. We also have the following, more abstract, characterization.

Definition 3.2.2. A C***-algebra** is a (complex) Banach space $\mathcal{A}$ equipped with:

- A multiplication that is submultiplicative: $\|ab\| \leq \|a\|\|b\|$ for all $a \in \mathcal{A}$ ($\mathcal{A}$ is a Banach algebra).
- A conjugate linear involution $^* : \mathcal{A} \to \mathcal{A}$ (which we will call the **adjoint**) satisfying:
  - $(xy)^* = y^* x^*$; and
  - $\|x^* x\| = \|x\|^2$ (the C***-identity**).

$\mathcal{A}$ is called a **unital** C*-algebra if, furthermore, $\mathcal{A}$ admits a multiplicative unit 1.

One can check that the above definitions are equivalent using the GNS construction introduced later in this chapter. We remark that if we remove the C*identity requirement, we get the more general notion of a Banach *-algebra.

Example 3.2.3. Let $n \in \mathbb{N}$ be a natural number at least 1. The algebra $M_n(\mathbb{C})$ of $n \times n$ matrices is a (unital) C*-algebra in the operator norm with * the conjugate transpose.

Example 3.2.4. Let X be a compact Hausdorff topological space. The algebra $C(X)$ of complex-valued continuous functions with pointwise addition and multiplication is a (unital) C*-algebra in the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$ with * defined by $f(x)^* = \overline{f(x)}$ for all $x \in X$.

A unital *-algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is called a **von Neumann algebra** if it is closed with respect to either of the following topologies.

Definition 3.2.5. The **weak operator topology** (often abbreviated **WOT**) on $B(\mathcal{H})$ is the weakest topology such that $T \mapsto \langle Tv\,,\, w\rangle$ is continuous for every $v, w \in \mathcal{H}$.

Definition 3.2.6. The **strong operator topology** (often abbreviated **SOT**) on $B(\mathcal{H})$ is the weakest topology such that $T \mapsto Tv$ is continuous for every $v \in \mathcal{H}$.

Equivalently, the strong operator topology is the topology in which a net of operators (T_α) converges to T if and only if (T_α) converges pointwise to T.

Notice that the strong operator topology is in general stronger than the weak operator topology (hence the names). However, both topologies agree on which linear subspaces (and hence subalgebras) are closed. Note also that both topologies are weaker than the norm topology. Hence every von Neumann algebra is a C*-algebra but not vice-versa. Another topology we will use often is the following.

Definition 3.2.7. The **strong* topology** or SOT* is the topology in which a net of operators (T_α) converges to T if and only if (T_α) strongly converges to T and (T_α^*) strongly converges to T^*.

Definition 3.2.8. Let $S \subseteq B(\mathcal{H})$. The **commutant** S' of S is the set of all elements of $B(\mathcal{H})$ that commute with every element of S.

Von Neumann proved his celebrated double commutant theorem in 1930 in [68]. This gives us another concrete definition of von Neumann algebras as those *-algebras closed equal to their own double commutant. Note that such *-algebras are automatically unital.

Theorem 3.2.9 (Double Commutant Theorem (von Neumann))**.** *Let $\mathcal{M}$ be a subalgebra of $B(\mathcal{H})$ closed under *. The closure of $\mathcal{M}$ in the WOT (equivalently the SOT) is $\mathcal{M}''$, the second commutant of $\mathcal{M}$.*

It is useful to define the following types of elements of a C*-algebra.

Definition 3.2.10. Let $\mathcal{A}$ be a C*-algebra and let $x \in \mathcal{A}$ be an element.

- x is **self adjoint** if $x^* = x$.
- x is **normal** if $x^*x = xx^*$.
- x is a **projection** if $x^* = x$ and $x^2 = x$.
- x is a **unitary** if $xx^* = x^*x = 1$.
- x is a **partial isometry** if $x^*x = p$ and $xx^* = q$. for some projections p and q.
- x is **positive** if $x = y^*y$ for some $y \in \mathcal{A}$.

Notice that in the case of bounded linear operators on $B(\mathcal{H})$, the definitions above agree with their standard usage. Note that one can define an order on self-adjoint operators by $a \leq b$ if $b - a$ is a positive operator. We will use the notation $\mathcal{A}_{\mathrm{sa}}$ to denote the set of all self adjoint elements of $\mathcal{A}$ and $\mathcal{A}_+$ to mean the set of all positive elements of $\mathcal{A}$.

Example 3.2.11. Consider the case of continuous function algebras $C(X)$.

- Self-adjoint elements are those with range in the reals. Unitaries are those with range in the complex unit circle. Positive elements are those with range in the positive (non-negative) reals. Note that the image of a continuous function is the same as its spectrum in $C(X)$.
- In the case of continuous function algebras $C(X)$, projections are those elements with range in $\{0, 1\}$; notice this means that it is an indicator function. Since elements of $C(X)$ are meant to be continuous, these are specifically indicator functions of clopen subsets of X. Notice that a C*-algebra may lack non-trivial projections, as in the case when X is connected.
- All elements of $C(X)$ are normal and all partial isometries are projections. These two notions will only become useful in the non-commutative setting.

We will now state the Kaplansky Density Theorem. This is an incredibly central theorem in von Neumann algebras. It is arguably even more important in the continuous logic of von Neumann algebras. See [55, Theorem 2.2.3] for a proof.

Theorem 3.2.12. *Given a *-subalgebra $\mathcal{A}$ of $B(\mathcal{H})$, denote by $\overline{\mathcal{A}}$ the weak closure of $\mathcal{A}$ in $B(\mathcal{H})$, then the unit ball $\mathcal{A}_1$ of $\mathcal{A}$ is weakly dense in the unit ball $\overline{\mathcal{A}}_1$ of $\overline{\mathcal{A}}$. Further, the self-adjoint part of $\mathcal{A}_1$ is weakly dense in the self-adjoint part of $\overline{\mathcal{A}}_1$.*

3.3 States, Traces and Weights

Definition 3.3.1. Let φ be a linear functional with operator norm $\|\varphi\| = 1$ on a unital C*-algebra $\mathcal{A}$.

- φ is called a **state** if $\varphi(y)$ is a positive real for all positive $y = x^*x$ and $\varphi(1) = 1$. A functional satisfying the first condition is called **positive**.
- A state is called **faithful** if $\varphi(x^*x) = 0$ implies $x = 0$.
- If a state preserves suprema of increasing nets of self adjoint operators, it is called **normal**.
- If a faithful normal state furthermore satisfies $\varphi(xy) = \varphi(yx)$ for all $x, y \in \mathcal{A}$, we say φ is a **trace**.

Example 3.3.2. Consider $\mathcal{A} = M_n(\mathbb{C})$. The normalized trace $\tau(a) = \frac{1}{n}tr(a)$ is a tracial state on $\mathcal{A}$. Furthermore, for every $x \in \mathcal{A}_+$ such that $\tau(a) = 1$, we can form another state $\varphi_x(a) = \tau(xa)$ which is not necessarily tracial. We will see later that this exhausts all states on $\mathcal{A}$.

Proposition 3.3.3. *The space of states on $\mathcal{A}$ forms a closed convex subspace of the space of all continuous linear functionals on $\mathcal{A}$. In the case that $\mathcal{A}$ admits a trace, the space of tracial states on $\mathcal{A}$ also forms a closed convex subspace of the space of all continuous linear functionals.*

Definition 3.3.4. $\mathcal{M}$ is **σ-finite** if it admits a faithful normal state.

Remark 3.3.5. The terminology "σ-finite" refers to the fact that, in such von Neumann algebras, the identity is a SOT-limit of a countable increasing family of finite projections. Compare this with the notion of a σ-finite measure.

Definition 3.3.6. A σ-finite von Neumann algebra together with a choice of faithful normal state is called a W***-probability space**.

Remark 3.3.7. The terminology "W*-probability space" comes from the analogy with probability spaces. The state φ can be thought of as a "noncommutative probability measure" on the "noncommutative measure space" $\mathcal{M}$. This analogy is used heavily in the subject of Free Probability.

Definition 3.3.8. A **weight** on $\mathcal{M}$ is a map $\Phi : \mathcal{M}_+ \to [0, \infty]$ such that:

1. $\Phi(x+y) = \Phi(x)+\Phi(y)$ for all $x, y \in \mathcal{M}_+$, with the convention that $\infty+a = \infty$ for any a; and

2. $\Phi(\lambda x) = \lambda\Phi(x)$ for all $x \in \mathcal{M}_+$ and $\lambda \geq 0$ where we use the convention $0\infty = 0$.

Definition 3.3.9. We will often require our weights satisfy more properties:

- A weight is **semifinite** if $P_\Phi = \{x \in \mathcal{M}_+ | \Phi(x) < \infty\}$ generates $\mathcal{M}$.

- A weight is **faithful** if $\Phi(x) = 0$ for $x \in \mathcal{M}_+$ implies $x = 0$.

- A weight is **normal** if $\Phi(\sup_k x_k) = \sup_k \Phi(x_k)$ for all increasing nets $(x_k) \in \mathcal{M}_+$.

- A weight is a **trace** or **tracial weight** if $\Phi(aa^*) = \Phi(a^*a)$ for all $a \in \mathcal{M}$.

Note that any positive linear functional defines a weight by restricting to the positive elements of $\mathcal{M}$.

Definition 3.3.10. Let φ and Φ be weights on a von Neumann algebra $\mathcal{M}$. We say Φ **majorizes** φ, denoted $\varphi \leq \Phi$ if $\varphi(x) \leq \Phi(x)$ for all $x \in \mathcal{M}_+$.

The following is a key result in the theory of weights due to Haagerup [38], and Pederson-Takesaki [56, Theorem 7.2].

Theorem 3.3.11 (Haagerup Approximation of Weights)**.** *Let Φ be a weight. The following are equivalent.*

- *Φ is normal.*
- *Φ is σ-weakly lower semicontinuous.*
- *$\Phi(x) = \sup\{\varphi(x) \ : \ \varphi$ is a positive functional and $\varphi \leq \Phi\}$ for all $x \in \mathcal{M}_+$.*
- *$\Phi(x) = \sum_{i \in I} \langle xv_i \, , \, v_i \rangle$ for some net (v_i) in $\mathcal{H}$.*

The next theorem is classical. See [65, Theorem 2.7] for a proof. Notice that in that proof, the weight constructed majorizes a state (in fact, many) and so we may assume that any faithful normal semifinite weight we consider has this property.

Theorem 3.3.12. *Every von Neumann algebra admits a faithful normal semifinite weight. Moreover, at least one such faithful normal semifinite weight majorizes a state.*

We discuss various possible morphisms between C*-algebras. Let $f : \mathcal{A} \to \mathcal{B}$ be a linear map between C*-algebras $\mathcal{A}$ and $\mathcal{B}$. The **matrix amplification** of dimension n is the map $f^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by $f^{(n)}(a_{ij}) = f(a_{ij})$ for all $1 \leq i, j \leq n$.

Definition 3.3.13.

- f is said to be a ***-homomorphism** if, $f(1) = 1$, $f(ab) = f(a)f(b)$ and $f(x^*) = f(x)^*$.
- f is said to be **positive** if the image of any positive element is positive.

- f is said to be n-**positive** if all of its matrix amplifications up to dimension n are positive.

- f is said to be **completely positive** or **c.p.** if it is n-positive for all n.

- f is said to be **unital completely positive** or **u.c.p.** if it is completely positive and unital.

Let $\mathcal{A}$ be a *-algebra.

Definition 3.3.14.

- A ***-representation** of $\mathcal{A}$ is a *-homomorphism π from $\mathcal{A}$ to $B(\mathcal{H})$. We say π is nondegenerate if $\{\pi(a)v \ : \ a \in \mathcal{A}, v \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

- Let $v \in \mathcal{H}$ be a vector in $\mathcal{H}$. v is **cyclic** for $\mathcal{A}$ if the image of $\mathcal{A}$ under $a \mapsto \pi(a)v$ is a dense subset of $\mathcal{H}$.

- Let $v \in \mathcal{H}$ be a vector in $\mathcal{H}$. v is **separating** for $\mathcal{A}$ if $\pi(a)v = 0$ implies $a = 0$ for $a \in \mathcal{A}$.

Proposition 3.3.15. *Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be a *-representation. Then v is cyclic for $\mathcal{A}$ with respect to π if and only if v is separating for its commutant $\mathcal{A}' := \pi(\mathcal{A})'$. Similarly, v is separating for $\mathcal{A}$ if and only if v is cyclic for $\mathcal{A}'$.*

Theorem 3.3.16. *Let φ be a state on $\mathcal{A}$. There is a *-representation π of $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ with a unit cyclic vector v such that $\varphi(a) = \langle \pi(a)v, v\rangle$ for every a in $\mathcal{A}$.*

Proof. First, we construct $\mathcal{H}$. Define a pre-inner product on $\mathcal{A}$ by $\langle a \, , \, b\rangle = \varphi(b^*a)$. One can check that $K = \{a \in \mathcal{A} \ : \ \varphi(a^*a) = 0\}$ is a left ideal of $\mathcal{A}$. $\mathcal{H}$ is the result of taking the Hilbert space completion of $\mathcal{A}/K$.

Next we construct π. Define $\pi(a)(b+K) = (ab+K)$. One can check that this is a bounded operator on $\mathcal{A}/K$ and hence extends to one on $\mathcal{H}$. □

The construction of π from φ in the proof of the above theorem is called the **Gelfand-Naimark-Segal construction** or **GNS construction**.

Remark 3.3.17. If φ is faithful, then the step involving K above is unnecessary. If $\mathcal{A}$ is unital, then 1 is a cyclic vector for π.

Remark 3.3.18. States on $\mathcal{A}$ separate points and so by taking the direct sum of GNS constructions over all states yields a faithful representation of $\mathcal{A}$ on the direct sum of the Hilbert spaces.

This implies the following theorem.

Theorem 3.3.19. *Every abstract* C**-algebra can be realized as a concrete* C**-algebra.*

The abstract definition of von Neumann algebras is not quite as simple as the abstract definition of C*-algebras. One was provided by Sakai with the proof of his theorem on preduals.

Recall that the **dual** $\mathcal{V}^*$ of a Banach space $\mathcal{V}$ is the space of continuous linear functionals on $\mathcal{V}$. We note that $\mathcal{V}^*$ is naturally also a Banach space.

Definition 3.3.20. Given Banach spaces $\mathcal{V}$ and $\mathcal{V}_*$, we call $\mathcal{V}_*$ the **predual** of $\mathcal{V}$ if $\mathcal{V}$ is isometrically isomorphic to the dual of $\mathcal{V}_*$.

Theorem 3.3.21 (Sakai)**.** *A* C**-algebra is a von Neumann algebra if and only if it admits a predual as a Banach space.*

The proof of the above theorem builds an explicit such predual. If the C*-algebra is $B(\mathcal{H})$ then we can show that the ideal of trace-class operators $B(\mathcal{H})_1$ is the predual of $B(\mathcal{H})$. In fact, every continuous functional f on $B(\mathcal{H})$ is of the form $f(a) = tr(ba)$ for some $b \in B(\mathcal{H})_1$ For $\mathcal{A}$ an arbitrary C*-algebra, then we can represent $\mathcal{A}$ on $B(\mathcal{H})$. We can check that the image of this representation is weakly closed if and only if the set

$$K = \{b \in B(\mathcal{H})_1 \ : \ tr(ba) = 0 \text{ for all } a \in \mathcal{A}\}$$

is closed. A predual of $\mathcal{A}$ in this case is $B(\mathcal{H})_1/K$.

This does not yet give us an abstract definition of von Neumann algebras as the construction of the predual involves a Hilbert space action. But it turns out that more is true. The predual of a von Neumann algebra is essentially unique and can be presented in terms of positive normal functionals.

Theorem 3.3.22. *[59] Any predual of a von Neumann algebra $\mathcal{M}$ is isometrically isomorphic to the space of positive normal linear functionals on $\mathcal{M}$.*

This gives both the abstract definition of a von Neumann algebra and its equivalence to the concrete definition. When working with W*-probability spaces, we often want to consider a von Neumann algebra without worrying about carrying around an action on a Hilbert space. We will therefore often switch between perspectives.

3.4 Commutative Operator Algebras and Duality

Definition 3.4.1. The **Gelfand Spectrum** of a commutative C*-algebra $\mathcal{A}$ is defined as the set $\mathrm{sp}(\mathcal{A}) = \{f \in \mathcal{A}^* : f \text{ is an algebra homomorphism}\}$ together with its topology as a subset of the dual of $\mathcal{A}$.

Theorem 3.4.2. *Every unital commutative* C*-*algebra $\mathcal{A}$ is isomorphic to the* C*-*algebra of continuous functions on its Gelfand spectrum* $\mathrm{sp}(\mathcal{A})$.

In fact, this construction defines an equivalence of categories.

Let $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_0 = p \in \mathbb{C}[x]$ be a formal polynomial in one variable and let $a \in \mathcal{A}$ be given. It is easy to see that we can "define" $p(a)$ by formally replacing x with a. In other words we take $p(a) = c_n a^n + c_{n-1}a^{n-1} + \ldots + c_0 1$. Since $\mathcal{A}$ is an algebra, it is clear that this actually defines a unique element of $\mathcal{A}$. Using an evaluation map, we can identify the formal expressions with their associated elements. This is the most primitive example of a **functional calculus**. We can do the same thing for non-commuting polynomials $p \in \mathbb{C}\langle x, \overline{x}\rangle$ by defining $\overline{a} = a^*$.

When $a \in \mathcal{A}$ is a normal element, we can say more. Since normal elements commute, we can consider polynomials in two commuting variables $p \in \mathbb{C}[x, \overline{x}]$ and evaluation defines a homomorphism $ev_a : \mathbb{C}[x, \overline{x}] \to \mathcal{A}$ by $ev_a(p) = p(a)$.

If $\mathcal{A}$ is a unital C*-algebra, we can do even better. Consider the commutative subalgebra $\mathcal{A}(a, 1)$ generated by a and 1. This is commutative since a is normal. By Gelfand duality, we can consider the Gelfand spectrum X of $\mathcal{A}(a, 1)$. It turns out that X is deeply connected to the eigenvalues of a.

Definition 3.4.3. The spectrum of an element $a \in \mathcal{A}$ is the set $\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$.

Proposition 3.4.4. $\mathrm{sp}(\mathcal{A}(a, 1))$ *is homeomorphic to* $\mathrm{sp}(a)$ *as a subset of* $\mathbb{C}$.

Moreover, the polynomial functional calculus plays well with the eigenvalues of a.

Proposition 3.4.5. $\mathrm{sp}(p(a)) = p(\mathrm{sp}(a))$.

So the polynomial functional calculus can be seen as applying functions to the spectrum of a. In the case of a continuous function on $\mathbb{C}$ that is continuous in a neighbourhood of $\mathrm{sp}(a)$, we can define a more general functional calculus. By

Weierstrass approximation, we can approximate any such function f by polynomials p_n. For each n, $p_n(a)$ is well-defined and one can check that $p_n(a)$ converges to a unique element $f(a)$ such that $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$. This defines what is called the **continuous functional calculus**.

Proposition 3.4.6. *Continuous functional calculus defines an isometric* **-isomorphism* ev_a *from* $C_0(\mathrm{sp}(a))$ *to* $C_0(\mathcal{A}(a, 1))$.

Where C*-algebras are "noncommutative topological spaces", von Neumann algebras are "noncommutative measure spaces". For X a measure space (a set equipped with a σ-algebra of measurable subsets), define $L^\infty(X)$ to be the algebra of equivalence classes of essentially bounded measurable functions from X to $\mathbb{C}$.

Proposition 3.4.7. *Every commutative von Neumann algebra is isomorphic to* $L^\infty(X)$ *where* X *is some "standard" measure space.*

We will not define a "standard" measure space as it is not important for our purposes. We simply note that not every measure space is standard, but every one we will use is. When restricted to the category of standard measure spaces, the above defines an equivalence of categories.

Similar to the continuous functional calculus, von Neumann algebras are closed under a broader functional calculus called the **Borel functional calculus**.

Proposition 3.4.8. *For every self adjoint element* $m \in \mathcal{M}$ *and Borel function* f *from* $\mathrm{sp}(m)$ *to* $\mathbb{C}$*, there is a well-defined* $f(m) \in \mathcal{M}$ *such that* $\mathrm{sp}(f(m) = f(\mathrm{sp}(m))$.

Proposition 3.4.9. *The construction above induces an isometric* **-homomorphism from* $L^\infty(\mathrm{sp}(m))$ *the von Neumann algebra generated by* m *and* 1.

3.5 Factors and Classification

Definition 3.5.1. A **factor** is a von Neumann algebra whose center consists only of scalar multiples of 1.

Murray and von Neumann defined factors in their study of "factorizations" of quantum mechanical systems into mutually independent sets of quantum mechanical systems. Murray and von Neumann further divided factors into three classes.

- Type I: Those with a minimal projection. These are equivalently those isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- Type II: Those with no minimal projections but admitting a faithful normal semifinite tracial weight. Further subdivided into types II_1 and II_∞ based on if the range of the projection under the tracial weight is $[0, 1]$ or $[0, \infty]$ respectively.
- Type III: Those admitting no faithful normal semifinite tracial weight.

Von Neumann would later, in 1949, show that any separably acting von Neumann algebra could be decomposed as a direct integral of factors. As a result, the study of von Neumann algebras in general can be systematically reduced to the study of factors.

Definition 3.5.2. A von Neumann algebra is called **hyperfinite** if it contains a union of finite dimensional subalgebras as a subset dense in the weak operator topology.

Already in [50], Murray and von Neumann completely classified the type I factors using their concept of "relative dimensionality" of projections (the precursor to what is now known as Murray-von Neumann equivalence). In [50], [51], [69], and [52], they also defined and characterized the hyperfinite II_1 factor and showed that there are at least 2 non-isomorphic II_1 factors. In [69], they showed, using their group-measure space construction, that there exists a factor of type III. However, it was not known that there was a pair of non-isomorphic type III factors until 1956. Identification of new isomorphism classes of type II and III factors remained slow until in 1967, Powers shook the subject by presenting an uncountable family of non-isomorphic factors. We will construct these so-called Powers factors below, but we must first make a brief detour.

Consider a pair of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We will often want to take a "tensor product" of $\mathcal{M}$ and $\mathcal{N}$. Taking the algebraic tensor product $\mathcal{M} \odot \mathcal{N}$ produces a *-algebra but in general, this is not even a C*-algebra. One might wish to just complete the algebraic tensor product with respect to a canonical norm. Such a canonical norm is not always guaranteed to exist. There can be an uncountable family of such norms but, usually, operator algebraists will work with the smallest (giving $\otimes_{min}$) or the largest (giving $\otimes_{max}$) such. While thinking more deeply about such norms leads to a wide number of interesting ideas, we will not need them here. Instead, we will leverage the actions on Hilbert spaces to form a canonical tensor product.

Definition 3.5.3. The **spatial tensor product** $\mathcal{M}\hat{\otimes}\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ acting faithfully on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively is defined as the weak* closure of $\mathcal{M} \cup \mathcal{N} \subseteq B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Here, $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the canonical Hilbert space tensor product and $\mathcal{M} \cup \mathcal{N}$ is an abbreviation for the union in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of the images of $\mathcal{M} \hookrightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ defined by $m \mapsto m \otimes 1$ and $\mathcal{N} \hookrightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ defined by $n \mapsto 1 \otimes n$.

Proposition 3.5.4. *The spatial tensor product up to isomorphism depends only on*

$\mathcal{M}$ and $\mathcal{N}$.

Defining tensor products of finitely many von Neumann algebras analogously, we can now construct the hyperfinite II_1 factor and the Powers factors.

Consider $M_2(\mathbb{C})$ together with the normalized trace

$$\tau\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}a + \frac{1}{2}d$$

Given $\mathcal{A}$ we can embed $\mathcal{A}$ into $\mathcal{A} \otimes M_2(\mathbb{C})$ by $\mathcal{A} \to \mathcal{A} \otimes Id$. Thus we can form a nested union

$$M_2(\mathbb{C}) \hookrightarrow M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \hookrightarrow M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$$

together with their normalized traces, which we will denote by Tr. Consider the union $\mathcal{A} = \bigcup_{n=0}^{\infty} M_{2^n}(\mathbb{C})$ together with the induced trace thereupon. Taking the GNS construction of $\mathcal{A}$ with respect to the trace yields a factor which is known as the **hyperfinite II_1 factor $\mathcal{R}$**.

Let $\lambda \in (0, 1)$ be given. Consider $M_2(\mathbb{C})$ together with the state

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{\lambda + 1}a + \frac{\lambda}{\lambda + 1}d.$$

This gives us a nested union of finite dimensional W*-probability spaces $\mathcal{A}_n = \bigotimes_{i=1}^{n} M_2(\mathbb{C})$ with the state given by tensor products of φ. Taking the GNS construction with respect to the tensor state gives us a factor known as the **Powers factor $\mathcal{R}_\lambda$**. The resulting state is called the Powers state.

The two examples above are special cases of what are called **infinite tensor products of finite type Is** or **ITPFI** factors. These were defined by Araki-Woods, motivated by the work of Powers.

Definition 3.5.5. For any countable family $(\mathcal{A}_i, \varphi_i)$ of finite dimensional matrix algebras $\mathcal{A}_i = M_{n_i}(\mathbb{C})$ with a state φ_i thereupon, we can form the tensor products over i of $\mathcal{A}_i$ and φ_i. The **ITPFI** algebra associated to this family is the GNS construction with respect to the resulting state.

Araki-Woods also gave a way to classify ITPFIs in terms of the spectral theory of the corresponding states. Connes' 1976 paper [13] on the classification of injective factors amounted to a huge improvement on our understanding of type II and type III factors. Generalizing the Araki-Woods classification and using the then-new Tomita-Takesaki theory, Connes further subdivided the type III factors into types III$_\lambda$ for $\lambda \in [0, 1]$. Under this classification, for $\lambda \in (0, 1)$, the Powers factor $\mathcal{R}_\lambda$ is the unique hyperfinite factor of type III$_\lambda$ up to isomorphism. As such, we will refer to $\mathcal{R}_\lambda$ as the hyperfinite type III$_\lambda$ factor. There is also a unique hyperfinite factor of type III$_1$ as seen in the next theorem.

Theorem 3.5.6. *[41] All ITPFI factors of type III$_1$ are isomorphic to $\mathcal{R}_\infty$ defined as*

$$\mathcal{R}_\infty = \bigotimes_{n \in \mathbb{N}} (M_3(\mathbb{C}), \mathrm{Tr}(\rho -))$$

where

$$\rho = \begin{bmatrix} \frac{1}{1+\lambda+\mu} & 0 & 0 \\ 0 & \frac{\lambda}{1+\lambda+\mu} & 0 \\ 0 & 0 & \frac{\mu}{1+\lambda+\mu} \end{bmatrix}$$

where $0 < \lambda, \mu$ *and* $\frac{\log(\lambda)}{\log(\mu)}$ *is irrational.*

There are hyperfinite factors which are not covered by the above examples. These are called type III_0. The type III_0 case is more complicated. The hyperfinite factors of type III_0 with separable predual which are ITPFI was classified [15]. However, the existence of hyperfinite factors of type III_0 with separable predual which are not ITPFI was proved by Connes in [12].

3.6 Ultraproducts of von Neumann Algebras

Let $(\mathcal{M}_i, \tau_i)_{i\in I}$ be a family of tracial von Neumann algebras with trace τ_i and let $\mathcal{U}$ be an ultrafilter on I. Define

$$\ell^\infty(\mathcal{M}_i) = \{(x_n) \in \prod_{i\in I} \mathcal{M}_i \ : \ \sup_{i\in I} \|x_i\| < \infty\}$$

and the ideal

$$\mathcal{I}_\mathcal{U} := \{(m_i) \in \ell^\infty(\mathcal{M}_i, I) \ : \ \lim_{\mathcal{U}} \tau(m_i^* m_i) = 0\}$$

Definition 3.6.1. The **tracial ultraproduct** is defined as the quotient

$$\prod^{\mathcal{U}} \mathcal{M}_i := \ell^\infty(\mathcal{M}_i)/\mathcal{I}_\mathcal{U}.$$

For $(m_i) \in \ell^\infty(\mathcal{M}_i)$, we will write $(m_i)^\bullet$ for its equivalence class in $\prod^{\mathcal{U}} \mathcal{M}_i$.

Proposition 3.6.2. $\prod^{\mathcal{U}} \mathcal{M}_i$ *is a tracial von Neumann algebra with trace* $\tau_i^{\mathcal{U}}$ *defined by* $\tau_i^{\mathcal{U}}((m_i)^\cdot) = \lim_{\mathcal{U}} \tau_i(m_i)$.

Definition 3.6.3. A C*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ has the **weak expectation property**

or **WEP** if there is a u.c.p. map $\Phi : B(\mathcal{H}) \to \mathcal{A}^{**}$ that is the identity on $\mathcal{A}$.

Definition 3.6.4. $\mathcal{A}$ has the **QWEP (quotient of the weak expectation property)** if $\mathcal{A}$ is isomorphic to a quotient of a C*-algebra with the WEP.

The **Connes Embedding Problem** asks whether every tracial von Neumann algebra embed in a tracial ultrapower of $\mathcal{R}$. The name comes from its appearance as an off-hand comment in [13].

Kirchberg's QWEP problem asks if every separable C*-algebra has QWEP. In [46], Kirchberg, while asking this question, also proves it to be equivalent to the Connes Embedding Problem (CEP). Fritz proves that both of these are equivalent to Tsirelson's problem from quantum information theory in [25]. See [54] for more about this early work.

By [43], CEP and therefore also the Kirchberg's QWEP problem are known to be resolved in the negative. Later in this thesis, we will prove various strengthenings of the negation of this problem. See [29] for more about these problems and their relationships with model theory from a modern perspective.

Given a family $\mathcal{M}_i$ of von Neumann algebras, acting on Hilbert spaces $\mathcal{H}_i$, we can define Banach space ultraproducts $\Pi_{B,\mathcal{U}}\mathcal{M}_i$ and $\Pi_{B,\mathcal{U}}\mathcal{H}_i$ of the von Neumann algebras and Hilbert spaces respectively. Moreover, $(x_i)_\mathcal{U} \in \Pi_{B,\mathcal{U}}\mathcal{M}_i$ acts naturally on $(\xi_i)_\mathcal{U} \in \Pi_{B,\mathcal{U}}\mathcal{H}_i$ by the formula

$$(x_i)_\mathcal{U}(\xi_i)_\mathcal{U} = (x_i\xi_i)_\mathcal{U}.$$

Definition 3.6.5. The **abstract ultraproduct** $\Pi_\mathcal{U}(\mathcal{M}_i, \mathcal{H}_i)$ of the family $(\mathcal{M}_i, \mathcal{H}_i)$ is defined to be the strong closure of the action of $\Pi_{B,\mathcal{U}}\mathcal{M}_i$ on $\Pi_{B,\mathcal{U}}\mathcal{H}_i$ given above.

For the definition below, see Section 4.5 for the definition of the standard Hilbert space.

Definition 3.6.6. When $\mathcal{H}_i$ is the standard Hilbert space associated to $\mathcal{M}_i$ for all i, the abstract ultraproduct $\Pi_{\mathcal{U}}(\mathcal{M}_i, \mathcal{H}_i)$ is called the **Groh-Raynaud ultraproduct**.

Theorem 3.6.7. *[3, Theorem 3.22]* $(\prod_{\mathcal{U}} \mathcal{M}_i)' = \prod_{\mathcal{U}} \mathcal{M}_i'$.

For a W*probability space $(\mathcal{M}, \varphi)$, we define the sharp norm

$$\|x\|_{\varphi}^{\#} = \frac{\sqrt{\varphi(x^*x) + \varphi(xx^*)}}{2}.$$

Consider a family of W*-probability spaces $(\mathcal{M}_i, \varphi_i)_{i\in I}$ and an ultrafilter $\mathcal{U}$ on I. Define

$$\ell^\infty(\mathcal{M}_i) = \{(x_n) \in \prod_{i\in I} \mathcal{M}_i \ : \ \sup_{i\in I} \|x_i\| < \infty\}$$

and the subset

$$\mathcal{I}_{\mathcal{U}} := \mathcal{I}_{\mathcal{U}}(\mathcal{M}_i, \varphi_i) := \{(m_i) \in \ell^\infty(\mathcal{M}_i, I) \ : \ \lim_{\mathcal{U}} \|m_i\|_{\varphi_i}^{\#} = 0\}.$$

Unlike in the tracial case, we do not in general have that $\mathcal{I}_{\mathcal{U}}$ is a two-sided ideal of $\ell^\infty(\mathcal{M}_i)$. Consider the two sided normalizer of $\mathcal{I}_{\mathcal{U}}$ given by

$$\mathcal{N}_{\mathcal{U}} := \mathcal{N}_{\mathcal{U}}(\mathcal{M}_i, \varphi_i) := \{(m_i) \in \ell^\infty(\mathcal{M}_i, I) \ : \ (m_i)\mathcal{I}_{\mathcal{U}} + \mathcal{I}_{\mathcal{U}}(m_i) \subset \mathcal{I}_{\mathcal{U}}\}.$$

It is readily apparent that $\mathcal{I}_{\mathcal{U}}$ is a two-sided ideal of $\mathcal{N}_{\mathcal{U}}$.

Definition 3.6.8. [3, Section 3.1] The **Ocneanu ultraproduct** is defined as the quotient $\prod^{\mathcal{U}} \mathcal{M}_i := \mathcal{N}_{\mathcal{U}}/\mathcal{I}_{\mathcal{U}}$. For $(m_i) \in \mathcal{N}_{\mathcal{U}}$, we will write $(m_i)^\bullet$ for its equivalence class in $\prod^{\mathcal{U}} \mathcal{M}_i$.

Proposition 3.6.9. *[3, Proposition 3.2.] $\prod^{\mathcal{U}} \mathcal{M}_i$ is a σ-finite von Neumann algebra with faithful normal state $\varphi_i^{\mathcal{U}}$ defined by $\varphi_i^{\mathcal{U}}((m_i)^{\cdot}) = \lim_{\mathcal{U}} \varphi_i(m_i)$.*

3.7 Model Theory of Tracial von Neumann Algebras

We begin this section by describing the language of tracial von Neumann algebras. Recall the 2-norm, defined by $\|x\|_2 = \sqrt{\tau(x^*x)}$. We will need to reference the following fact in the next definition.

Fact 3.7.1. *[30, Chapter 2, Fact 3.11] For every $n \in \mathbb{N}$, there is a sequence $(\hat{q}_k^n)$ of polynomials in one variable such that for every* C**-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, $\|a\| \leq n$ implies $\|\hat{q}_k^n(a)\| \leq 1$. Moreover, if $\|a\| \leq 1$, then* $\lim_{k\to\infty} \hat{q}_k^n(a) = a$ *where this limit is taken in the operator norm.*

The following is adapted from [30, Chapter 2] and [20].

Definition 3.7.2. We define the language $\mathcal{L}_{\text{tr}}$ of tracial von Neumann algebras to consist of:

- For each $n \in \mathbb{N}$, we have a sort S_n with bound $2n$ representing the n-ball in the operator norm, with a metric symbol d_n representing the 2-norm, defined as: $d_n(x, y) = \|x - y\|_2 = \sqrt{\tau((x - y)^*(x - y))}$.
- Binary function symbols $+_n$ and $-_n$ with domain S_n^2 and range S_{2n}. The intended interpretation is simply addition and subtraction restricted to the operator norm ball of radius n. The modulus of continuity for each is $\delta(\epsilon) = \epsilon$.

- a binary function $\cdot_n$ with domain S_n^2 and range S_{n^2}. The intended interpretation is the restriction of multiplication to the operator norm ball of radius n. The modulus of continuity is $\delta(\epsilon) = \frac{\epsilon}{n}$.

- two constant symbols 0_n and 1_n which lie in the sort S_n. The intended interpretation of these symbols are the elements 0 and 1.

- for every $\lambda \in \mathbb{C}$, there is a unary function symbol λ_n whose domain is S_n and range is S_{mn}, where $m = \lceil |\lambda| \rceil$ if $\lambda \neq 0$ or $m = 1$ if $\lambda = 0$. The intended interpretation is scalar multiplication by λ restricted to the operator norm ball of radius n. The modulus of continuity is $\delta(\epsilon) = \frac{\epsilon}{|\lambda|}$ when $\lambda \neq 0$ and $\delta(\epsilon) = 1$ when $\lambda = 0$.

- a unary function symbol $^*{}_n$ with domain and range S_n. The intended interpretation is the restriction of the adjoint to the operator norm ball of radius n. The modulus of continuity is $\delta(\epsilon) = \epsilon$.

- For every $m > n$, we include unary function symbols $i_{n,m}$ with domain S_n and range S_m. The intended interpretation is the inclusion map between the balls of the given radii. The modulus of continuity is $\delta(\epsilon) = \epsilon$.

- For each $n \in \mathbb{N}$, there is one relation symbol τ_n whose intended interpretation is the restriction of the trace to the operator norm unit ball of radius n. Its domain is S_n and the range is D_n, the complex ball of radius n. Technically we have separate symbols for the real and imaginary parts but this is of no great importance.

- For $n, k \in \mathbb{N}$, we include function symbols q_k^n with domain S_n and range S_1.

These will be interpreted as the corresponding polynomials asserted to exist in Fact 3.7.1 and their moduli of continuity will be the same as said polynomials.

Viewing a von Neumann algebra $\mathcal{M}$ as being divided into the above sorts with all the intended interpretations, we have its **dissection** $D(\mathcal{M})$. We will now present the axioms of tracial von Neumann algebras. A dissection will come from a dissection as above if it satisfies the following axioms. They will be stated informally, but it is easy to translate them into formal sentences in our language.

Definition 3.7.3. The set T_{tr} of sentences in the language $\mathcal{L}_{\text{tr}}$ consists of the following:

1. Axioms on the function symbols saying they define a complex *-algebra.
2. Axioms that say that τ defines a tracial state on that algebra.
3. axioms saying that for $m > n$, $i_{m,n}$ preserves addition, multiplication, the adjoint and the trace.
4. $d_n(x, y) = \|x-y\|_2$ requiring that the distance on each sort is given by the norm coming from the trace.
5. $\sup_{x\in S_n} \sup_{y\in S_1}(\|xy\|_2 \dot{-} n\|y\|_2)$. This says that any element of the sort S_n has operator norm at most n in the GNS representation with respect to τ.
6. Axioms saying that q_k^n is interpreted as in Fact 3.7.1. Since q_k^n is a polynomial, we can express this simply in terms of the scalar multiplication, multiplication and addition operations already defined. Let S_m be the naive range of q_k^n we obtain from composing such operations. We also add, for each n, k, the axiom $\sup_{x\in S_n} i_{1,m}(q_k^n(x)) = \hat{q}_k^n(x)$.

Given a model A of T_{tr}, we define the **interpretation** $M(A)$ to be the tracial von Neumann algebra given by taking the nested union of the sorts of A and putting together the operations and trace in the evident manner.

We will refer to the final item above and analogous constructions as "the Kaplansky density trick". We call it such because it mimics the cutdown polynomials used to prove the Kaplansky density theorem. It is a key piece of the proof (omitted here) of the following.

Proposition 3.7.4. *The category* $\text{Mod}(T_{\text{tr}})$ *of models of* T_{tr} *is equivalent to the category of tracial von Neumann algebras. Moreover, this equivalence is witnessed by the functors* D *and* M*.*

Remark 3.7.5. The ultraproducts in T_{tr} are exactly the tracial ultraproducts we defined earlier. More precisely, the equivalence of categories asserted in the previous proposition can be upgraded to an ultra-equivalence when the category of tracial von Neumann algebras is equipped with the tracial ultraproduct.

Proposition 3.7.6. *The following sets are definable:*

- *The set* U *of unitaries.*
- *The set* P *of projections.*
- *The set* P_2 *of pairs of projections of the same trace.*

Consider the sentences given by

$$\sigma_{factor} = \sup_{(p,q)\in P_2} \inf_{u\in U} d(upu^*, q)$$

and

$$\sigma_{II_1} = \inf_{p \in P} \|\tau(p) - \frac{1}{\pi}\|.$$

Proposition 3.7.7. $T_{\mathrm{tr}-factor} = T_{\mathrm{tr}} \cup \{\sigma_{factor} = 0\}$ *axiomatizes the class of tracial factors.*

Theorem 3.7.8. $T_{II_1-factor} = T_{\mathrm{tr}-factor} \cup \{\sigma_{II_1} = 0\}$ *axiomatizes II_1 factors.*

Chapter 4

Hilbert Algebras Preliminaries

4.1 Overview

The term "Hilbert algebras" was first defined to describe the special case of unimodular Hilbert algebras by Hidegorô Nakano, motivated by group algebras of locally compact groups in [53]. Warren Ambrose, in [2], had previously treated the compact group case using what he called H^*-algebras. Nakano generalized this by considering those functions on G with finite integral. Nakano only considered unimodular groups (those whose left and right Haar measures coincide). Irving Segal [61] and Roger Godement [26] then independently identified how the left and right representations that arise in the unimodular case are related. Jacques Dixmier, in [18], later introduced "quasi-unitary algebras" to treat the non-unimodular case. Here, Dixmier even identifies and uses the modular operator. However, it was Minoru Tomita who first realized the importance of the modular operator, describing how it arises from the polar decomposition of the adjoint in 1959 and proving the tensor product commutation theorem in 1967. Tomita's work also included "modular Hilbert algebras", known today as Tomita algebras. This thesis will not treat Tomita algebras, so we will end our discussion of them here. Following the latter paper, Masamichi Takesaki, in [64], reformulated and corrected some errors in Tomita's work, bringing it to mainstream relevance. Here, Takesaki also clarifies the connection between Hilbert algebras and statistical mechanics via the KMS condition. Where previous work on Hilbert algebras was inextricably linked to harmonic analysis and representation theory, Takesaki's work was the beginning of Hilbert algebras being studied for their own sake. Christopher Lance gave a treatment of direct integrals of continuous fields of Hilbert algebras in [47]. In [67], Alfons Van Daele provided a simplified treatment of the material in [64], including a new proof of Tomita's theorem. Providing new proofs of Tomita's theorem is now a long-standing tradition, with the most recent example as of the writing of this thesis being due to Jonathan Sorce in [62]. In [57, Section 5], Rieffel and Van Daele gave a new treatment of left (and right) Hilbert algebras using only bounded operators. Hilbert algebras soon became indispensable to the study of crossed products (see [39] and [40] for example). As such, they played a central role in Alain Connes' classification work in [12], [13] and [15].

4.2 Unbounded Operators

The domain of an unbounded operator will generally only be given on a dense subspace. A reason for this restriction is explained after the following definition.

Definition 4.2.1. An **unbounded operator** T on a Hilbert space $\mathcal{H}$ is a linear operator defined on a subspace $D(T)$ of $\mathcal{H}$. One usually assumes $D(T)$ is dense in $\mathcal{H}$.

Note that any bounded operator $T : \mathcal{H} \to \mathcal{H}$ is an unbounded operator with

$D(T) = \mathcal{H}$. The existence of everywhere-defined unbounded operators that are not bounded is non-constructive and depends on the Hahn-Banach theorem. The unbounded operators that will appear in this thesis (namely the modular operator, introduced later in this chapter) will generally not be everywhere-defined. However they will be defined on a subset of $\mathcal{H}$.

Definition 4.2.2. T is said to be **densely defined** if $D(T)$ is dense in $\mathcal{H}$.

Definition 4.2.3. Two unbounded operators T and S are equal if and only if $D(T) = D(S)$ and for all $x \in D(T) = D(S)$, one has $Tx = Sx$.

Definition 4.2.4. We say that S **extends** T, denoted $T \subseteq S$, if $D(T) \subseteq D(S)$ and $Sx = Tx$ for all $x \in D(T)$.

Suppose T is densely defined. Let $\mathcal{K}$ be the set of all η such that $\xi \mapsto \langle T\xi\,,\,\eta\rangle$ extends to a bounded linear functional on $\mathcal{H}$. Then $\mathcal{K}$ is a subspace of $\mathcal{H}$. By Riesz representation, there is a unique $\zeta \in \mathcal{H}$ such that $\langle T\xi\,,\,\eta\rangle = \langle \xi\,,\,\zeta\rangle$ for all $\xi \in D(T)$. Define the adjoint of T, T^* to be the operator such that $T^*\eta = \zeta$. Note that $D(T^*) = \mathcal{K}$. Unlike in the bounded setting, it is necessary to distinguish between Hermitian and self-adjoint unbounded operators.

Definition 4.2.5. An unbounded operator T is **symmetric** or **Hermitian** if $T = T^* \mid_{D(T)}$.

Definition 4.2.6. T is **self-adjoint** if it is symmetric and $D(T) = D(T^*)$.

Definition 4.2.7. A symmetric operator T is **essentially self-adjoint** if its closure is self-adjoint.

Theorem 4.2.8 (Hellinger-Toeplitz)**.** *An everywhere defined symmetric linear operator on $\mathcal{H}$ is bounded.*

A common technique for working with unbounded operators is by using its graph.

Definition 4.2.9. The **graph** of T is the set $\mathrm{gr}(T) = \{(x, Tx) \mid x \in D(T)\}$ in $\mathcal{H} \oplus \mathcal{H}$.

Lemma 4.2.10. *A linear subspace $\mathcal{K} \subseteq \mathcal{H} \oplus \mathcal{H}$ is the graph of a linear operator if and only if $\mathcal{K} \cap 0 \oplus \mathcal{H} = \{0 \oplus 0\}$. Moreover, the linear operator is uniquely defined by $\mathcal{K}$.*

Corollary 4.2.11. *$T \subseteq S$ if and only if $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$.*

Definition 4.2.12. T is said to be **closed** if its graph $\mathrm{gr}(T)$ is a closed set in $\mathcal{H} \oplus \mathcal{H}$ in the product topology.

Theorem 4.2.13 (Alternative Characterization of Closedness)**.** *T is closed if and only if $D(T)$ is complete with respect to the graph norm $\|x\|_T = \sqrt{\|x\|^2 + \|Tx\|^2}$.*

Note that $\mathcal{H} \oplus \mathcal{H}$ is a Hilbert space with respect to the inner product $\langle x \oplus y \, , \, x' \oplus y' \rangle = \langle x \, , \, x' \rangle + \langle y \, , \, y' \rangle$.

Definition 4.2.14. T is said to be **closeable** if it extends to a closed linear operator.

Proposition 4.2.15 (Closure of Graph is Graph of Closure)**.** *If an operator T is closeable, then the closure of its graph is the graph of an operator and that operator is the closure of T.*

Remark 4.2.16. If T is closed, densely defined and continuous on its domain, then its domain is all of $\mathcal{H}$.

We now describe the functional calculus of self-adjoint unbounded operators and spectral projections.

Definition 4.2.17. Let T be an unbounded operator on $\mathcal{H}$. The **resolvent set** of T, denoted $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ such that there exists $S \in B(\mathcal{H})$ such that

$$S(T - \lambda I) \subseteq (T - \lambda I)S = I.$$

In other words, S is an inverse for $(T - \lambda I)$.

Similarly to the bounded case, we define the spectrum.

Definition 4.2.18. Let T be an unbounded operator on $\mathcal{H}$. The **spectrum** of T, denoted $\sigma(T)$ is the complement of the resolvent set of T.

Definition 4.2.19. Let Σ be a σ-algebra on a set Ω and let $\mathcal{H}$ be a Hilbert space. A **resolution of the identity** or **spectral measure** on Σ is a function

$$E : \Sigma \to B(\mathcal{H})$$

satisfying the following properties:

1. $E(U)$ is a projection for every $U \in \Sigma$;
2. $E() = 0, E(\Omega) = I$;
3. $E(U \cap V) = E(U)E(V)$ for every $U, V \in \Sigma$;
4. If $U, V \in \Sigma$ are disjoint, then $E(U \cup V) = E(U) + E(V)$; and
5. The function $E_{x,y} : \Sigma \to \mathbb{R}$ defined by $E_{x,y}(U) := \langle E(U)x\,,\, y\rangle$ is a complex measure for every $x, y \in \mathcal{H}$.

In this thesis, we only consider situations where $\Omega = \mathbb{R}$ and Σ is the Borel σ-algebra on $\mathbb{R}$. Our final theorem and definition will be crucial to our proofs in Chapters 5 and 6. They provide the framework for functional calculus on unbounded operators.

Theorem 4.2.20. *Let T be an unbounded operator on $\mathcal{H}$. There is a unique resolution of the identity E such that*

$$\langle Tx\,,\,y\rangle = \int_{\mathbb{R}} t dE_{x,y}(t) \qquad x \in D(T), y \in \mathcal{H}.$$

Moreover, E is concentrated on $\sigma(T)$. In other words, $E(\sigma(T)) = I$.

Definition 4.2.21. E as in the theorem above is called the **spectral decomposition** of T. If U is a Borel subset of $\mathbb{R}$, then $E(U)$ is called the **spectral projection** of T associated to U.

4.3 Hilbert Algebras and Semicyclic Representations

Here we will exposit the basics of Hilbert algebras.

Let $\mathfrak{A}$ be an algebra over $\mathbb{C}$ with inner product $\langle \cdot\,,\,\cdot\rangle$ and involution $^\#$ and denote by $\mathcal{H}$ the Hilbert space completion of $\mathfrak{A}$ with respect to the inner product.

Definition 4.3.1. We call $\mathfrak{A}$ a **left Hilbert algebra** if it satisfies the following conditions.

1. Left multiplication is continuous. In other words, for any $a \in \mathfrak{A}$, the operator $\pi(a) : b \mapsto ab$ extends to a bounded operator, also denoted $\pi(a)$ on $\mathcal{H}$.

2. We have $\langle ab\,,\,c\rangle = \left\langle b\,,\,a^\# c\right\rangle$ for $a, b, c \in \mathfrak{A}$ so that the representation π of $\mathfrak{A}$ on $\mathcal{H}$ is a *-representation.

3. The subalgebra $\mathfrak{A}^2$ spanned by elements of the form ab for $a, b \in \mathfrak{A}$ is dense in $\mathfrak{A}$. This implies π is nondegenerate.

4. $^{\#}$ is closable as an unbounded operator on $\mathcal{H}$.

We will now give a bounded operator reformulation of axiom (4) in line with [57]. Denote by $\mathcal{K}$ the real subspace of $\mathcal{H}$ given by the closure of the set $\{a^{\#}a \ : \ a \in \mathfrak{A}\}$.

Theorem 4.3.2. *[57, Definition 4.5 and the preceding remarks] Let $\mathfrak{A}$ be an involutive algebra with inner product $\langle \cdot \, , \, \cdot \rangle$ satisfying* (1) – (3) *above. Then the following are equivalent.*

1. $^{\#}$ is closable as an unbounded operator on $\mathcal{H}$.

2. $\mathcal{K} \cap i\mathcal{K} = \{0\}$.

We will denote by $\mathcal{R}_l(\mathfrak{A})$ the von Neumann algebra $\pi(\mathfrak{A})''$ in $B(\mathcal{H})$ and refer to it as the left von Neumann algebra.

Definition 4.3.3. We say a vector $a \in \mathcal{H}$ is **right bounded** if the operator $\pi'(a)$ defined by $\pi'(a)b = \pi(b)a$ when $b \in \mathfrak{A}$ and extended to all of $\mathcal{H}$ is bounded. We denote the set of right bounded elements as $\mathfrak{A}'$.

Next, we of course consider the double commutant.

Definition 4.3.4. We say a vector $a \in \mathcal{H}$ is **left bounded** if the operator $\pi(a)$ defined by $\pi(a)b' = \pi'(b')a$ when $b' \in \mathfrak{A}'$ and extended to all $\mathcal{H}$ is bounded. We define the set of left bounded elements as $\mathfrak{A}''$.

Definition 4.3.5. We define a left Hilbert algebra $\mathfrak{A}$ to be **full** or **achieved** if $\mathfrak{A} = \mathfrak{A}''$.

Notice the previous definition is reminiscent of the double commutant theorem. This is shown to be an apt comparison in the next section.

Recall that we call a pair $(\mathcal{M}, \Phi)$ a **weighted von Neumann algebra** if $\mathcal{M}$ is a von Neumann algebra and Φ is a faithful normal semifinite weight on $\mathcal{M}$. The next result is fundamental to the ideas used in the rest of this chapter. It says that we can switch perspectives between weighted von Neumann algebras and full left Hilbert algebras without fear.

Theorem 4.3.6. *The categories of weighted von Neumann algebras and Hilbert algebras are equivalent.*

The proof is technical and the result well-known. We will not recall the proof here, but the reader may find it in [65, Theorem 2.5 and Theorem 2.6]. We will, however, describe the process for moving from a weighted von Neumann algebra to a Hilbert algebra and vice versa. First, we consider the special case of σ-finite von Neumann algebra with a specified faithful normal state.

Example 4.3.7. Let $\mathcal{M}$ be a σ-finite von Neumann algebra and φ be a faithful normal state on $\mathcal{M}$. Consider the GNS representation π_φ on $\mathcal{H}_\varphi$ and the cyclic and separating vector ω such that $\varphi(x) = \langle \pi(x)\omega\,,\,\omega\rangle$ for all $x \in \mathcal{M}$. Then we can define the associated Hilbert algebra as $\mathcal{M}\omega$ together with multiplication and $^\#$ given by

$$(a\omega)(b\omega) = ab\omega = \pi(a)b\omega \qquad (a\omega)^\# = a^*\omega.$$

Note in the previous example that $\mathfrak{A}' = \mathcal{M}'\omega$. It is then trivial to show that $\mathfrak{A}$ is full. Generalizing the GNS representation, we define the semicyclic representation with respect to a weight.

Let $\mathcal{M}$ be a von Neumann algebra and let Φ be a faithful normal semifinite weight on $\mathcal{M}$. Recall that $P_\Phi := \{x \,:\, x \text{ is positive and } \Phi(x) < \infty\}$. We define the sets:

$$N_\Phi = \{x \in \mathcal{M} \,:\, x^*x \in P_\Phi\} \quad \text{and} \quad D_\Phi = N_\Phi^* N_\Phi.$$

One can show via a polarization argument (see [65]) that

$$D_\Phi = \text{span}\{a^*b \,:\, \Phi(a^*a) < \infty \text{ and } \Phi(b^*b) < \infty\}.$$

Notice that Φ can be extended to a linear functional on D_Φ and so we call D_Φ the **definition domain** of Φ.

We get an obvious embedding

$$\eta_\Phi : D_\Phi \to \mathcal{H}_\Phi$$

of D_Φ in its Hilbert space completion $\mathcal{H}_\Phi$.

Since $(yx)^*yx = x^*y^*yx \leq \|y\|x^*x$ and therefore $\Phi(x^*y^*yx) \leq \|y\|\Phi(x^*x)$, we have that D_Φ is a *-closed left ideal of $\mathcal{M}$. Since D_Φ is a *-closed left ideal of $\mathcal{M}$, we can define a representation of $\mathcal{M}$ on $\mathcal{H}_\Phi$ as follows. For any $a \in \mathcal{M}$, define the operator $\pi_\Phi(a) \in B(\mathcal{H}_\Phi)$ by $\pi_\Phi(a)\eta_\Phi(b) = \eta_\Phi(ab)$ for all $b \in D_\Phi$ and extending to $\mathcal{H}_\Phi$ by continuity.

Definition 4.3.8. Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{H}_\Phi$ and π_Φ be defined as above. Then π_Φ is called the **semicyclic representation** with respect to Φ.

Now we can associate a full Hilbert algebra to a weighted von Neumann algebra $(\mathcal{M}, \Phi)$ by taking $D_\Phi \cap D_\Phi^*$ together with the multiplication and involution of the given by

$$\eta_\Phi(a)\eta_\Phi(b) = \eta_\Phi(ab) \qquad \eta_\Phi(a)^\# = \eta_\Phi(a^*).$$

One can check (see [65]) that this defines a full Hilbert algebra. Conversely, given a full left Hilbert algebra $\mathfrak{A}$, the left von Neumann algebra $\mathcal{R}_l(\mathfrak{A})$ together with the faithful normal semifinite weight given for $x \in \mathcal{R}_l(\mathfrak{A})$ positive by

$$\Phi_l(x) = \begin{cases} \sqrt{\langle \xi \, , \, \xi \rangle} & \text{if } x^{1/2} = \xi \in \mathfrak{A} \\ \infty & \text{otherwise} \end{cases}.$$

We can also define a faithful normal semifinite weight on the commutant by

$$\Phi_r(x) = \begin{cases} \sqrt{\langle \xi \, , \, \xi \rangle} & \text{if } x^{1/2} = \xi \in \mathfrak{A}' \\ \infty & \text{otherwise .} \end{cases}.$$

We note the proof of the previous theorem shows that these processes are inverse to each other up to a unitary isomorphism.

4.4 Right Bounded and Totally Bounded Elements

Our axiomatization will build on the semicyclic representation with respect to a distinguished faithful normal semifinite weight. We will collect here some technical results on the semicyclic representation that will be useful in the sequel.

Denote by $\mathcal{M}'$ the commutant of $\pi(\mathcal{M})$ in $B(\mathcal{H})$.

Definition 4.4.1. We say $v \in \mathcal{H}$ is **right-bounded** if there is a $K \in \mathbb{R}$ such that

$$\|av\|_{\mathcal{H}} = \|\pi(a)v\|_{\mathcal{H}} \leq K\|\eta_\Phi(a)\|_{\mathcal{H}}$$

for all $a \in \mathcal{M}$. Denote by $\|v\|_{\text{right}}$ the infimum over such K. Denote by $\pi'(v)$ the operator on $\mathcal{H}$ induced by v.

Denote by $\mathcal{H}_{\text{right}}$ the set of right-bounded vectors. We define the set of **totally bounded elements** $\mathcal{H}_{\text{tb}} := (\mathcal{H}_{\text{right}} \cap D_\Phi) \cap (\mathcal{H}_{\text{right}} \cap D_\Phi)^*$. In other words, an element x of $D_\Phi \cap D_\Phi^*$ is totally bounded if x and x^* are furthermore right bounded. It is clear that $\mathcal{H}_{\text{right}}$ and $\mathcal{H}_{\text{tb}}$ are linear subspaces of $\mathcal{H}$. It is not clear at this point that $\mathcal{H}_{\text{right}}$ or $\mathcal{H}_{\text{tb}}$ has any more structure than this.

Lemma 4.4.2. *Let $v \in \mathcal{H}_{\text{right}}$ be given. Then $\pi'(v)\pi'(v)^* \in \mathcal{M}'$*

Proof. Let $x \in \mathcal{M}$ be given.

$$\begin{aligned} \pi'(v)\pi(x)\eta_\Phi(y) &= \pi'(v)\eta_\Phi(xy) \\ &= xyv \\ &= x\pi'(v)\eta_\Phi(y) \end{aligned}$$

So $\pi'(v)^*x = \pi(x)\pi'(v)^*$. Therefore

$$\begin{aligned} x\pi'(v)^*\pi'(v) &= \pi'(v)^*\pi(x)\pi'(v) \\ &= \pi'(v)^*\pi'(v)x \end{aligned}$$

and we are done. ∎

Theorem 4.4.3. $\mathcal{H}_{\text{tb}}$ *is a dense subspace of* $\mathcal{H}$.

Proof. Assume for contradiction that it is not dense. Denote by p the projection of $\mathcal{H}$ onto $\overline{\mathcal{H}_{\text{tb}}}$. By assumption, $p \neq 1$. Since $\overline{\mathcal{H}_{\text{tb}}}$ is fixed by $\mathcal{M}'$, we have that $p \in (\mathcal{M}')' = \mathcal{M}$. Then $\Phi(1-p) > 0$ and by Theorem 3.3.11, there is an increasing net $\varphi_{v_i}(a) = \langle av_i\,,\, v_i\rangle$ of vector states on $\mathcal{M}$ such that $\sum_{i\in I}\varphi_{v_i} = \Phi$. Therefore there is a v_k such that $\varphi_{v_k} \leq \Phi$ and $\varphi_{v_k}(1-p) > 0$. So $\varphi_{v_k}(1-p) = \langle (1-p)v_k\,,\, v_k\rangle$ and therefore $(1-p)v_k \neq 0$. Now since

$$\begin{aligned}\|xv_k\|_{\mathcal{H}} &= \varphi_{v_k}(x^*x)\\ &\leq \Phi(x^*x)\\ &= \|x\|_{\mathcal{H}},\end{aligned}$$

it follows that v_k is in $\mathcal{H}_{\text{right}}$ and hence $\mathcal{H}_{\text{tb}}$ so $(1-p)v_k = 0$. This is a contradiction. Therefore $\mathcal{H}_{\text{tb}}$ is a dense subspace of $\mathcal{H}$. □

Our next theorem and corollary will explain our use of the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^\#$. While the result is elementary, the author could not locate a proof in the literature, so one is provided.

Theorem 4.4.4. *Let* $(\mathcal{M}, \Phi)$ *be a von Neumann algebra equipped with a faithful, normal, semifinite weight and let* $\pi : \mathcal{M} \to \mathcal{H}$ *be the associated semicyclic representation. Then* $\|x\|_\Phi = \sqrt{\Phi(x^*x)}$ *metrizes the strong operator topology on* $D_\Phi \cap \mathcal{M}_1$*, the set of all* $x \in D_\Phi$ *such that* $\|x\| \leq 1$.

Proof. $\mathcal{M}$ and $B(\mathcal{H})$ are both von Neumann algebras and therefore, by Theorem 3.3.22, admit preduals. Thus $\mathcal{M}_1$ and $B(\mathcal{H})_1$ are WOT-compact by Banach-Alaoglu.

Since Φ is faithful and normal, π is a continuous bijection from $\mathcal{M}_1$ to $\pi(\mathcal{M}_1) \subset B(\mathcal{H})$. A continuous bijection between compact Hausdorff spaces is a homeomorphism, so π is a homeomorphism from $\mathcal{M}_1$ to $\pi(\mathcal{M}_1)$. We know that $x_n \to x$ converges in the strong operator topology if and only $(x_n - x)^*(x_n - x) \to 0$ in the WOT. Since π is a *-homomorphism, we conclude that a net (x_n) in $\mathcal{M}_1$ strong-converges if and only $\pi(x_n)$ strong-converges.

Now suppose that (x_n) is a net in $D_\Phi \cap \mathcal{M}_1$ such that $\|x_n - x\| \to 0$ for some $x \in D_\Phi \cap \mathcal{M}_1$. Let $\eta_\Phi(a)$ be a right bounded element of $\mathcal{H}$. Then there is $a := \pi'(\eta_\Phi(a)) \in \pi(\mathcal{M})'$ and we have:

$$\begin{aligned}
\|(x_n - x)\eta_\Phi(a)\|_\Phi^2 &= \langle (x_n - x)\eta_\Phi(a)\,,\,(x_n - x)\eta_\Phi(a)\rangle \\
&= \langle \pi'(a)\eta_\Phi(x_n - x)\,,\,\pi'(a)\eta_\Phi(x_n - x)\rangle \\
&= \langle \pi'(a)^*\pi'(a)\eta_\Phi(x_n - x)\,,\,\eta_\Phi(x_n - x)\rangle \\
&= \|a\|^2\|x_n - x\|_\Phi^2 \to 0
\end{aligned}$$

Since right-bounded elements of $\mathcal{H}$ are dense, for every $\epsilon > 0$ and $v \in \mathcal{H}$, there is $\eta_\Phi(a)$ right bounded such that $\|v - \eta_\Phi(a)\|_\Phi < \epsilon$. Whence

$$\begin{aligned}
|\|(x_n - x)\eta_\Phi(a)\|_\Phi - \|(x_n - x)v\|_\Phi| &\leq \|x_n - x\|_\Phi \|\eta_\Phi(a) - v\|_\Phi \\
&\leq 2\epsilon.
\end{aligned}$$

Thus $\|(x_n - x)v\| \to 0$ for all $v \in \mathcal{H}$. Hence $\pi(x_n) \to \pi(x)$ in the strong operator topology and consequently $x_n \to x$ in the strong operator topology. The converse is proven by the same considerations. □

Corollary 4.4.5. $\|x\|_\Phi^\sharp$ *metrizes the strong-* topology on norm bounded subsets of* D_Φ.

4.5 Tomita-Takesaki Theory and Standard Forms

Tomita-Takesaki theory is a powerful but extremely technical subject. As such, we do not have the room here to provide a comprehensive account of the theory. We will provide an overview of the primary setup of the theory, up to and including the definition of the modular automorphism group, but we will omit all details regarding the unbounded operators. We encourage the reader to consult [65] for a more thorough and complete exposition.

Let $(\mathcal{M}, \Phi)$ be a weighted von Neumann algebra and let $\mathcal{H}_\Phi$ be the semicyclic representation Hilbert space associated to Φ. Define the unbounded operator S_Φ on $\mathcal{H}_\Phi$ as the closure of

$$S_\Phi \eta_\Phi(x) = \eta_\Phi(x^*) \qquad \text{for } x \in \eta_\Phi(\mathcal{M}).$$

We can also define the unbounded operator F_Φ on $\mathcal{H}_\Phi$ as the closure of

$$F_\Phi \eta_\Phi(x) = \eta_\Phi(x^*) \qquad \text{for } x \in \eta_\Phi(\mathcal{M}')$$

and note that $S_\Phi^* = F_\Phi$. Define $\Delta_\Phi = S_\Phi^* S_\Phi = F_\Phi S_\Phi$ and consider the polar decomposition $S_\Phi = J_\Phi \Delta_\Phi^{1/2} = \Delta_\Phi^{-1/2} J_\Phi$. We call the positive densely-defined operator Δ_Φ the **modular operator** and we call the antilinear isometry J_Φ the **modular conjugation**.

Now for every $t \in \mathbb{R}$, we can define an operator Δ_Φ^{it} on $\mathcal{H}_\Phi$ by functional calculus. In fact, Δ_Φ^{it} is a unitary operator because Δ_Φ is positive and therefore self-adjoint. What follows is the main theorem of Tomita-Takesaki theory, known as Tomita's theorem. It has been proved multiple times by various authors using an array of techniques, for example: [64], [67], [45] and [62] among others.

Theorem 4.5.1 (Tomita's Theorem)**.** *For every* $t \in \mathbb{R}$*,*

$$\Delta_\Phi^{it}\mathcal{M}\Delta_\Phi^{-it} = \mathcal{M} \qquad\qquad J_\Phi \mathcal{M} J_\Phi = \mathcal{M}'.$$

It follows that $t \mapsto \sigma_t^\Phi$ where $\sigma_t^\Phi(x) = \Delta_\Phi^{it} x \Delta_\Phi^{-it}$ defines a one-parameter family of automorphisms of $\mathcal{M}$. In fact, one can prove that this is a strong*-continuous one-parameter group of automorphisms of $\mathcal{M}$. For any fixed $t \in \mathbb{R}$, the function σ_t^Φ is called a **modular automorphism**. The collection of all such modular automorphisms together with their structure as an action of $\mathbb{R}$ is called the **modular automorphism group**.

The correct notion of an embedding in the context of weighted von Neumann algebras is an injection of the underlying algebras which admits a conditional expectation onto its image.

Definition 4.5.2. Let $(\mathcal{N}, \Phi)$ be a weighted von Neumann algebra and let $\mathcal{M} \subseteq \mathcal{N}$ be a von Neumann subalgebra equipped with the weight given by restriction of Φ. A **conditional expectation** of $\mathcal{N}$ onto $\mathcal{M}$ with respect to Φ is a linear map satisfying:

- $\|x\| \geq \|E(x)\|$ for all $x \in \mathcal{N}$;
- $E(x) = x$ for all $x \in \mathcal{M}$; and

- $\Phi(x) = \Phi(E(x))$ for all $x \in \mathcal{N}$.

Fact 4.5.3. *[65, Volume II, Chapter IX, Section 4] A conditional expectation E of $\mathcal{N}$ onto $\mathcal{M}$ satisfies:*

- *E is a completely positive map;*
- *$E(x^*x) \geq 0$ for all $x \in \mathcal{N}$;*
- *$E(x^*x) \geq E(x)^*E(x)$ for all $x \in \mathcal{N}$; and*
- *$E(axb) = aE(x)b$ for all $x \in \mathcal{N}$ and $a, b \in \mathcal{M}$.*

In the next lemma, conditions 1 and 3 are from [65, Volume II, Chapter IX, Theorem 4.2.] and condition 2 is from [57, Theorem 5.13.]. The equivalence of conditions 1 and 3 is often referred to as **Takesaki's Theorem**.

Theorem 4.5.4. *Suppose that $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \Psi)$ are weighted von Neumann algebras and $f : \mathcal{M} \to \mathcal{N}$ is a state-preserving *-homomorphism. Then the following are equivalent:*

1. *f is an embedding of weighted von Neumann algebras.*
2. *f preserves distances to the sets of self-adjoint elements, that is, for all $a \in \mathcal{M}$, we have $d_\Phi(a, \mathcal{M}_{\mathrm{sa}}) = d_\Psi(f(a), \mathcal{N}_{\mathrm{sa}})$.*
3. *f commutes with the modular automorphism group, that is, for every $t \in \mathbb{R}$ and $a \in \mathcal{M}$, we have $f(\sigma_t^\Phi(a)) = \sigma_t^\Psi(f(a))$.*

In his thesis (see [37]), Haagerup gives a way to associate to any left Hilbert algebra, a selfdual cone that canonically captures the representation up to unitary equivalence. This gives rise to the notion of a standard form, which will play a significant role in the later chapters of this thesis. Here, we will collect the definitions and some important results related to standard forms.

Definition 4.5.5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{P} \subseteq \mathcal{H}$ be a nonempty subset. $\mathcal{P}$ is called a cone if $v \in \mathcal{P}$ and $r \in \mathbb{R}^+$ implies $rv \in \mathcal{P}$.

Definition 4.5.6. Let $\mathcal{P}$ be a cone in a Hilbert space $\mathcal{H}$. The **dual cone** $\mathcal{P}^\circ$ is defined by

$$\mathcal{P}^\circ = \{\xi \in \mathcal{H} \ : \ \langle \xi \, , \, \eta \rangle \geq 0 \text{ for all } \eta \in \mathcal{P}\}.$$

Then $\mathcal{P}$ is called **selfdual** if $\mathcal{P} = \mathcal{P}^\circ$.

Definition 4.5.7. A quadruple $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ is said to be a **standard form** if $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ such that $J : \mathcal{H} \to \mathcal{H}$ is an antilinear isometric involution and $\mathcal{P}$ is a selfdual cone in $\mathcal{H}$ all satisfying:

1. $J\mathcal{M}J = \mathcal{M}'$;

2. $JcJ = c^*$ for all $c \in Z(\mathcal{M})$, where $Z(\mathcal{M})$ is the center of $\mathcal{M}$;

3. $J\xi = \xi$ for all $\xi \in \mathcal{P}$; and

4. $aJaJ(\mathcal{P}) \subseteq \mathcal{P}$ for all $a \in \mathcal{M}$.

Let $\mathcal{M}$ be any von Neumann algebra and Φ a faithful normal semifinite weight on $\mathcal{M}$. Then the semifinite representation of $\mathcal{M}$ with respect to Φ becomes a standard form when equipped with its modular conjugation $J = J_\Phi$ and the selfdual cone

$$\mathcal{P} := \overline{\{\pi(\xi)(J\xi) \ : \ \xi \in \eta_\Phi(\mathcal{M})\}}$$

Standard forms are unique in the following sense.

Theorem 4.5.8. *[37, Theorem 2.3.] Let* $(\mathcal{M}_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ *and* $(\mathcal{M}_2, \mathcal{H}_2, J_2, \mathcal{P}_2)$ *be standard forms and let* $f : \mathcal{M}_1 \to \mathcal{M}_2$ *be a* **-isomorphism. Then there exists one and only one unitary* $u : \mathcal{H}_1 \to \mathcal{H}_2$ *such that:*

- $f(x) = uxu^*$ *for all* $x \in \mathcal{M}_1$*;*
- $J_2 = uJ_1u^*$*; and*
- $\mathcal{P}_1 = u(\mathcal{P}_2)$.

We end this section by recording two theorems that will be crucial to our analysis of generalized Ocneanu ultraproducts in Chapter 6.

Theorem 4.5.9. *[3, Theorem 3.18] Groh-Raynaud ultraproducts act standardly on the Hilbert space ultraproduct.*

Theorem 4.5.10. *[37, Corollary 2.5 and Lemma 2.6] If* $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})$ *is a standard form and* q *is a projection in* $\mathcal{M}$*, and* $p = qJqJ$ *then* $(p\mathcal{M}p, p(\mathcal{H}), pJp, p(\mathcal{P}))$ *is a standard form.*

Given a projection p of a von Neumann algebra $\mathcal{M}$, we call $p\mathcal{M}p$ a **corner** of $\mathcal{M}$.

4.6 Relative Modular Theory

A key tool in the use of Tomita-Takesaki theory is the observation by Connes that the modular automorphism group has a unique image in the outer automorphisms. For a comprehensive exposition of relative modular theory, we direct the reader to [65, Book 2, Chapter VIII, Section 3]. In fact, our presentation is an extremely condensed version of what can be found there.

Assume Φ and Ψ are faithful normal semifinite weights on $\mathcal{M}$ let $\mathcal{N} = M_2(\mathcal{M})$ be the algebra of 2×2 matrices with entries in $\mathcal{M}$. Fixing the standard matrix elements $e_{11}, e_{12}, e_{21}, e_{22}$ so that each element $x \in \mathcal{N}$ is represented by

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = x_{11}e_{11} + x_{12}e_{12} + x_{21}e_{21} + x_{22}e_{22},$$

we define the **balanced weight** $\rho = \Phi \oplus \Psi$ by

$$\rho\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \Phi(a_{11}) + \Psi(a_{22}).$$

Then ρ is a faithful normal semifinite weight and

$$D_\rho = \{x \in \mathcal{N} \ : \ x_{11}, x_{21} \in D_\Phi, x_{12}, x_{22} \in D_\Psi\}.$$

Now the GNS Hilbert space for ρ is given by

$$\mathcal{H}_\rho = \mathcal{H}_\Phi \oplus \mathcal{H}_\Psi \oplus \mathcal{H}_\Phi \oplus \mathcal{H}_\Psi$$

with the canonical injection

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}.$$

So left multiplication is given by

$$\pi_\rho\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} \pi_\Phi(x_{11}) & 0 & \pi_\Phi(x_{12}) & 0 \\ 0 & \pi_\Psi(x_{11}) & 0 & \pi_\Psi(x_{12}) \\ \pi_\Phi(x_{21}) & 0 & \pi_\Phi(x_{22}) & 0 \\ 0 & \pi_\Psi(x_{21}) & 0 & \pi_\Psi(x_{22}) \end{bmatrix}.$$

Letting $S_{\Phi,\Psi}$ be the closure of the map $a_{12} \mapsto a_{12}^*$ on $D_\Psi \cap D_\Phi^*$ and analogously letting $S_{\Psi,\Phi}$ be the closure of the map $a_{21} \mapsto a_{21}^*$ on $D_\Phi \cap D_\Psi^*$ we have

$$S_\rho = \begin{bmatrix} S_\Phi & 0 & 0 & 0 \\ 0 & 0 & S_{\Psi,\Phi} & 0 \\ 0 & S_{\Phi,\Psi} & 0 & 0 \\ 0 & 0 & 0 & S_\Psi \end{bmatrix}$$

and letting $S_{\Phi,\Psi} = J_{\Phi,\Psi}\Delta_{\Phi,\Psi}$ and $S_{\Psi,\Phi} = J_{\Psi,\Phi}\Delta_{\Psi,\Phi}$ be the respective polar decompositions, we have the polar decomposition $S_\rho = J_\rho \Delta_\rho$ where

$$J_\rho = \begin{bmatrix} J_\Phi & 0 & 0 & 0 \\ 0 & 0 & J_{\Psi,\Phi} & 0 \\ 0 & J_{\Phi,\Psi} & 0 & 0 \\ 0 & 0 & 0 & J_\Psi \end{bmatrix}, \quad \Delta_\rho = \begin{bmatrix} \Delta_\Phi & 0 & 0 & 0 \\ 0 & \Delta_{\Phi,\Psi} & 0 & 0 \\ 0 & 0 & \Delta_{\Psi,\Phi} & 0 \\ 0 & 0 & 0 & \Delta_\Psi \end{bmatrix}$$

Theorem 4.6.1.

$$\sigma_t^\rho\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^\Phi(x_{11}) & \sigma_t^{\Phi,\Psi}(x_{12}) \\ \sigma_t^{\Psi,\Phi}(x_{21}) & \sigma_t^\Psi(x_{22}) \end{bmatrix}$$

where $\sigma_t^{\Phi,\Psi}$ and $\sigma_t^{\Phi,\Psi}$ are strongly continuous one-parameter families of isometries satisfying

$$\sigma_t^{\Phi,\Psi}(x) = \sigma_t^{\Psi,\Phi}(x^*)^*$$

and also

$$\pi_\Phi(\sigma_t^{\Psi,\Phi}(x)) = \Delta_{\Psi,\Phi}^{it}\pi_\Phi(x)\Delta_\Phi^{-it} \qquad \pi_\Psi(\sigma_t^{\Phi,\Psi}(x)) = \Delta_{\Phi,\Psi}^{it}\pi_\Psi(x)\Delta_\Psi^{-it}$$

Theorem 4.6.2. *Setting $u_t = \sigma_t^{\Psi,\Phi}(1) \in \mathcal{M}$ for $t \in \mathbb{R}$, we have that u_t defines a continuous family of unitaries satisfying*

$$\sigma_t^\Psi(x) = u_t\sigma_t^\Phi(x)u_t^* \qquad u_{s+t} = u_s\sigma_s^\Phi(u_t)$$

for $s, t \in \mathbb{R}$ and $x \in \mathcal{M}$. The second condition is called the cocycle condition.

Definition 4.6.3. The map $t \mapsto u_t$ is called the **Connes cocycle derivative** or the **Connes Radon-Nikodym derivative** of Ψ with respect to Φ and is denoted by

$$u_t = (D\Psi \ : \ D\Phi)_t.$$

4.7 Group Algebras and Crossed Products

Let G be a countable discrete group.

Definition 4.7.1. The **group algebra** $\mathbb{C}[G]$ with respect to G is the vector space generated by elements δ_g for each $g \in G$ together with the multiplication defined by extending $(\delta_g)(\delta_h) = \delta_{gh}$ linearly.

Define an inner product on $\mathbb{C}[G]$ by $\langle \delta_g \, , \, \delta_h \rangle = 1$ if $g = h$ and $\langle \delta_g \, , \, \delta_h \rangle = 0$ otherwise. This extends to an inner product on the whole vector space. Notice that $\mathbb{C}[G]$ is naturally a *-algebra acting on itself by left multiplication. Denote the left multiplication by δ_g as $\pi(\delta_g)$.

Denote by $\ell^2(G)$ the Hilbert space completion of $\mathbb{C}[G]$ with respect to its inner product. Note that the action of $\mathbb{C}[G]$ on $\mathbb{C}[G]$ extends to an action of $\mathbb{C}[G]$ on $\ell^2(G)$ by continuity.

Definition 4.7.2. The **group von Neumann algebra** of G is the weak operator closure of $\mathbb{C}[G]$ in $B(\ell^2(G))$ where the identification of elements of $\mathbb{C}[G]$ as operators on $\ell^2(G)$ is as given above. The group von Neumann algebra is a tracial von Neumann algebra with respect to the trace $\tau(a) = \langle a\delta_e \, , \, \delta_e \rangle$ where e is the identity element of G.

Proposition 4.7.3. *The group von Neumann algebra of any infinite discrete group G is a finite von Neumann algebra. It is a factor if G has no non-trivial finite conjugacy classes. It is isomorphic to $\mathcal{R}$ if, in addition to the previous condition, it a union of its finite subgroups.*

Consider a locally compact group G. Take a left-invariant Haar measure μ together with its modulus Δ_μ. Consider the Hilbert space $L^2(G)$ of square integrable complex functions with respect to μ.

Let $(\mathcal{M}, \varphi)$ be a W*-probability space and define $\mathcal{H} = \mathcal{H}_\varphi$ the GNS space of $\mathcal{M}$ with respect to φ. Further assume that G acts on $\mathcal{M}$ by a strongly-continuous action α. Consider the Hilbert space $\mathcal{H} \otimes L^2(G) \cong L^2(G, \mathcal{H})$ of square integrable functions on G with codomain $\mathcal{H}$. $L^2(G, \mathcal{H})$ has the inner product

$$\langle \xi \, , \, \eta \rangle = \int_G \langle \xi(s) \, , \, \eta(s) \rangle \, d\mu$$

where $\xi, \eta \in L^2(G, \mathcal{H})$.

We define the crossed product as the von Neumann algebra $\mathcal{M} \rtimes_\alpha G$ acting on $L^2(G, \mathcal{H})$ as the algebra generated by

- $\pi(m)(\xi(s)) = (\alpha_{s^{-1}}(m))(\xi(s))$ for all $m \in M$, $\xi \in L^2(G, \mathcal{H})$ and $s \in G$; and
- $u_g(\xi(s)) = \xi(g^{-1}s)$ for all $\xi \in L^2(G, \mathcal{H})$ and $g, s \in G$.

It can be seen that:

- π is a faithful normal representation of $\mathcal{M}$ on $L^2(G, \mathcal{H})$;
- u_g is a strongly continuous unitary representation of G on $L^2(G, \mathcal{H})$; and

- $u_g \pi(m) u_g^* = \pi(\alpha_g(m))$ for all $m \in \mathcal{M}$ and $g \in G$.

We recall here the definition of the dual weight $\hat{\varphi}$ on $\mathcal{M} \rtimes_\alpha G$. Consider the involutive algebra $C_c(G, \mathcal{M})$ of compactly supported σ-strong*-continuous functions from G to $\mathcal{M}$ with the product defined by

$$a \cdot b = \int_G \alpha_t(a(st)b(t^{-1}))d\mu(t)$$

and involution defined by

$$b^\sharp = \Delta_\mu{}^{-1} \alpha_{s^{-1}}((b(s^{-1}))^*)$$

for all $a, b \in C_c(G, \mathcal{M})$. Consider the *-representation

$$r(a) = \int_G u_s \pi(a(s)) d\mu(s)$$

on $L^2(G, \mathcal{H})$. This defines a representation of $C_c(G, \mathcal{M})$ as a dense subalgebra of $\mathcal{M} \rtimes_\alpha G$. In general, the dual weight can then be constructed via taking the left Hilbert algebra associated to this representation and letting $\hat{\varphi}$ be the weight induced on the associated left von Neumann algebra. For concreteness, we will use Haagerup's construction of the dual weight for locally compact abelian groups. Consider $\hat{\mu}$, the Haar measure on $\hat{G}$ defined so that Plancherel's formula holds. Namely, if we define for $f \in L^1(G)$, the function $\hat{f}$ as

$$\hat{f}(p) = \int_G f(s)\overline{p(s)} d\mu(s)$$

then the following equation (the Plancherel formula) holds.

$$f(s) = \int_{\hat{G}} \hat{f}(p)p(s)d\hat{\mu}(p).$$

First, define an operator-valued faithful normal weight

$$Tx = \int_{\hat{G}} \hat{\alpha}_p(x)d\hat{\mu}(p)$$

for $x \in (\mathcal{M} \rtimes_\alpha G)_+$. Now define

$$\hat{\varphi} = (\varphi \circ \pi^{-1}) \circ T.$$

See [39] and [40] for more details.

The following facts about $\hat{\varphi}$ are important.

Proposition 4.7.4. *$\hat{\varphi}$ and $\sigma_t^{\hat{\varphi}}$ satisfy*

- $\hat{\varphi}(r(a^\sharp a)) = \varphi((a^\sharp a)(e))$ *for* $a \in C_c(G, \mathcal{M})$*;*
- $\sigma_t^{\hat{\varphi}}(\pi(x)) = \pi(\sigma_t^{\varphi}(x))$ *for* $x \in \mathcal{M}$ *and* $t \in \mathcal{R}$*; and*
- $\sigma_t^{\hat{\varphi}}(u_g) = (\Delta_\mu(g))^{it} u_g((D\varphi \circ \alpha_g : D\varphi)_t)$ *for* $g \in G$ *and* $t \in \mathcal{R}$*.*

Let G be a locally compact abelian group for the rest of this section. Let $(\mathcal{M}, \varphi, \alpha)$ a G-system. Denote by $\hat{G}$ the Pontryagin dual of G. There is a canonical $\hat{G}$-system $(\mathcal{M} \rtimes_\alpha G, \hat{\varphi}, \hat{\alpha})$ on $\mathcal{M} \rtimes_\alpha G$.

For every $p \in \hat{G}$, there is a unitary $v(p) \in B(L^2(G, \mathcal{H}))$ defined by:

$$v(p)\xi(s) = \overline{p(s)}\xi(s)$$

where $s \in G$ and $\xi \in L^2(G, \mathcal{H})$.

This defines an action $\hat{\alpha}$ of $\hat{G}$ on $\mathcal{M} \rtimes_\alpha G$ defined by:

$$\hat{\alpha}_p(\pi(x)) = \pi(x) \text{ and } \hat{\alpha}_p(u_g) = \overline{p(g)} u_g$$

for all $x \in \mathcal{M}$, $g \in G$ and $p \in \hat{G}$. It is easy to see that $\mathcal{M}$ is isomorphic to the fixed point algebra of $\hat{\alpha}$.

We call $(\mathcal{M} \rtimes_\alpha G, \hat{\varphi}, \hat{\alpha})$ the **dual system** for $(\mathcal{M}, \varphi, \alpha)$. The next theorem is [65, Volume II, Chapter X, Theorem 2.3.].

Theorem 4.7.5 (Takesaki Duality)**.** *There is an isomorphism* $\Gamma : \mathcal{M} \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G} \to \mathcal{M} \otimes B(\ell^2(G))$ *such that*

$$\Gamma(\hat{\pi}(\pi(x))) = \pi(x) \text{ and } \Gamma(\hat{\pi}(u_g)) = 1 \otimes u_g \text{ and } \Gamma(\hat{u}_p)(-) = 1 \otimes p(-).$$

It follows that if $\mathcal{M}$ *is properly infinite and* G *is second countable, that* $\mathcal{M} \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G} \cong \mathcal{M}$. *Moreover, the action* $\hat{\hat{\alpha}}$ *is transformed to the action* $\tilde{\alpha}_g = \alpha_g \otimes Ad(u_g)$.

Chapter 5

Model Theory of von Neumann Algebras

5.1 Overview

In [5], the present author, together with Goldbring, Hart and Sinclair, develop the model theory of σ-finite von Neumann algebras together with a choice of faithful normal state. Many von Neumann algebras, however, do not admit such a state and hence this excludes many von Neumann algebras of interest. Many applications, such as crossed product duality, necessitate the ability to work with more general faithful, normal, semifinite weights even in the σ-finite setting. It is not difficult to extend the methods of [5] to faithful normal semifinite weights and hence take this opportunity to present the model theory in full generality in this thesis. Since every von Neumann algebra admits a faithful normal semifinite weight, this allows us to work with arbitrary von Neumann algebras model-theoretically.

The existence of this axiomatization demonstrates further the utility of the approach given in [5]. In [16], Dabrowski provides axiomatizability results in both the σ-finite and general von Neumann algebra cases. However, where Dabrowski gives an explicit axiomatization in a language that stays close to everyday practice in the σ-finite case, they give only an abstract axiomatizability result, via Keisler-Shelah, in the general case in a language (so-called tracial matrix ordered operator spaces) that is not commonly seen in operator algebras. Our approach is explicit and designed to be easy to use by operator algebraists.

We point out that even in the σ-finite case, our results here and in [5] are new. While Dabrowski gives an explicit computable axiomatization in their so-called "minimal language", the extension to include the modular automorphisms is shown to be definable by more abstract methods. Therefore, it remains an open question whether this extension to include the modular automorphisms is computably axiomatizable. We give an explicit computable definition of the modular automorphisms. To do this, we use results and techniques from [57] which gives a bounded operator approach to Tomita-Takesaki theory on Hilbert algebras.

5.2 Preliminary Estimates and Bounds

We will collect some results we need from [57] as well as some estimates that follow from the techniques used therein. Throughout this section, let $\mathcal{M}$ be a von Neumann algebra and let Φ be a faithful normal semifinite weight on Φ. We will often use Theorem 4.3.6 to apply results about Hilbert algebras to the equivalent weighted von Neumann algebra.

Let $\mathcal{K}$ denote the real subspace of $\mathcal{H}$ defined by the closed span of elements of the form a^*a, where we consider $\mathcal{H}$ as a real Hilbert space with respect to the inner product given by the real part of $\Phi(b^*a)$. We define P to be the projection onto $\mathcal{K}$ and Q to be the projection onto $i\mathcal{K}$. By [57, Lemma 5.12.], $\mathcal{K}$ is the closure of the set of self-adjoint elements of $\eta_\Phi(\mathcal{M})$. We also define $R = P + Q$ and $TJ = P - Q$ as in [57].

Theorem 5.2.1. *If $x \in \mathcal{M}$ is left K-bounded, then $P\eta_\Phi(x)$ and $Q\eta_\Phi(x)$ are right $(2K^2 + 2)$-bounded. Moreover, since P is a projection, $\|P\eta_\Phi(x)\|_\Phi \leq \|\eta_\Phi(x)\|_\Phi$.*

Proof. Examine the proofs of [57, Lemma 5.4, Lemma 5.6 and Corollary 5.7]. There, it is seen that if $x \in \mathcal{K}$, then $\pi'(P\eta_\Phi(x)) = a^{-1}b$ where $a, b \in \mathcal{R}_r(\mathfrak{A}')$ are as defined in [57, Lemma 5.4]. It is clear from the definition that $\|b\| \leq 1$. It is shown in the proof of [57, Lemma 5.6] that $\|a^{-1}\| \leq 1 + \|\pi(x)\|^2$. Thus, by decomposing general x into real and imaginary parts, we see that if x is left K-bounded, then $P\eta_\Phi(x)$ is right $(2K^2 + 2)$-bounded. By the same reasoning, $Q\eta_\Phi(x)$ is right $(2K^2 + 2)$-bounded. □

Theorem 5.2.2. *If $x \in \mathcal{M}$ is totally K-bounded, then $P\eta_\Phi(x)$ and $Q\eta_\Phi(x)$ are totally $(5K^2 + 4)$-bounded.*

Proof. Now if x is totally K-bounded, then since $P' = (1 - Q)$, it follows that x is left $(5K^2 + 4)$-bounded. Combining with the previous theorem gives the result. The proof for Q is nearly identical. □

Corollary 5.2.3. *If $x \in \mathcal{M}$ is totally K-bounded, then $R\eta_\Phi(x)$ is totally $(10K^2 + 8)$-bounded. Moreover, $\|R\eta_\Phi(x)\|_\Phi \leq 2\|\eta_\Phi(x)\|_\Phi$.*

Note that we are being quite conservative in our estimates. So long as P, Q and R send totally K-bounded elements to totally L-bounded elements, where L is predictable in terms of K, we will not run into any issues (see Section 5.5).

Now we turn our attention to some conditions for fullness. We will need the following analysis for axiom (10) below. For $j > 1$, let $E(j^{-1}, j)$ be the spectral projection of Δ (and Δ^{-1}) corresponding to the interval (j^{-1}, j).

Following [45], define the following families of functions for $a \in \mathbb{R}$:

$$\begin{aligned} h_a(t) &= (\cosh(t-a))^{-1} = \frac{2}{e^{t-a} + e^{a-t}}; \\ f_a(t) &= e^{-|t-a|}; \\ g_a(t) &= e^{-|t|} - \frac{e^{-|t-a|} + e^{-|t+a|}}{e^a + e^{-a}}. \end{aligned}$$

The next lemma can be found in [65, Volume II, Lemma 1.17] and many other sources.

Lemma 5.2.4 (Bridging Lemma/Takesaki's Resolvent Lemma)**.** *For every $x \in \mathcal{M}$ and $\lambda \in \mathbb{C}/\mathbb{R}_+$, there exists $y' \in \mathcal{M}'$ such that*

$$(\Delta^{-1} - \lambda I)^{-1} \eta_\Phi(x) = \eta_\Phi(y').$$

Furthermore

$$\|y'\| \leq \frac{\|x\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}.$$

Similarly, for every $x' \in \mathcal{M}'$ and $\lambda \in \mathbb{C}/\mathbb{R}_+$, then there exists $y \in \mathcal{M}$ such that

$$(\Delta - \lambda)^{-1} \eta_\Phi(x') = \eta_\Phi(y)$$

and

$$\|y\| \leq \frac{\|x'\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}.$$

We show in the next lemma that the construction of the element y given above interacts well with totally bounded elements.

Lemma 5.2.5. *Let $x \in \mathcal{M}$ be given and let $y' \in \mathcal{M}'$ satisfy $(\Delta^{-1} - \lambda I)^{-1}\eta_\Phi(x) = \eta_\Phi(y')$ as in the previous lemma. If x is totally bounded as an element of $\mathcal{M}$, then y' as given above is totally bounded as an element of $\mathcal{M}'$.*

Proof. Let x and y' be as above. Write $\eta_\Phi(x) = (\Delta^{-1} - \lambda I)\eta_\Phi(y')$. Taking F of both sides gives

$$\begin{aligned} F\eta_\Phi(x) &= FSF\eta_\Phi(y') - \lambda F\eta_\Phi(y') \\ &= (\Delta - \lambda I)F\eta_\Phi(y'). \end{aligned}$$

Therefore

$$(\Delta - \lambda I)^{-1}F\eta_\Phi(x) = F\eta_\Phi(y).$$

The bounds from the previous lemma give a bound for the left bound of Fy' in terms of the right bound of x. A symmetric argument gives a left bound for y' in terms of the right bound of Sx. Explicitly

$$\begin{aligned} S\eta_\Phi(x) &= SSF\eta_\Phi(y') - \lambda S\eta_\Phi(y') \\ &= FSS\eta_\Phi(y') - \lambda S\eta_\Phi(y') \\ &= (\Delta - \lambda I)S\eta_\Phi(y') \end{aligned}$$

and the resolvent lemma gives the left bound of $S\eta_\Phi(y')$ which is equal to the left bound of $\eta_\Phi(y')$ in terms of the right bound of $S\eta_\Phi(x)$. □

The next lemma is [45, Lemma 4.11].

Lemma 5.2.6. *For every $a \in \mathbb{R}$ and $x \in \mathcal{M}$, there exists $y \in \mathcal{M}$ satisfying*

$$h_a(\log(\Delta))\eta_\Phi(x) = \eta_\Phi(y).$$

Furthermore, $\|y\| \leq \|x\|$.

Note that

$$\begin{aligned} h_a(\log(\Delta)) &= 2(e^{-a}\Delta + e^a\Delta^{-1})^{-1} \\ &= 2i(\Delta + ie^a I)^{-1}(\Delta^{-1} + ie^{-a}I)^{-1} \end{aligned}$$

which is a bounded injective operator on $\mathcal{H}$ of norm at most 1. By the bound given in the resolvent lemma, $h_a(\log(\Delta))\eta_\Phi(x) = \eta_\Phi(y)$ implies $\|y\| \leq \|x\|$. By symmetry under interchanging Δ and Δ^{-1}, we get decrease in right norm too. By Lemma 5.2.4, y is totally $\|x\|$-bounded.

Lemma 5.2.7. *For every $a \in \mathbb{R}$ and $x \in \mathcal{M}_{\text{tb}}$, there exists $y \in \mathcal{M}_{\text{tb}}$ such that we have $h_a(\log(\Delta))\eta_\Phi(x) = \eta_\Phi(y)$. Furthermore, $\|y\| \leq \|x\|$ and $\|\pi'(\eta_\Phi(y))\| \leq \|\pi'(\eta_\Phi(x))\|$.*

It is a routine calculation then to see that $h_a(\log(\Delta))\eta_\Phi(x) = \eta_\Phi(y)$ if and only if $2R(2-R)\eta_\Phi(x) = (e^{-a}(2-R)^2 + e^a R^2)\eta_\Phi(y)$.

Lemma 5.2.8 (Lemma 4.11 in [45])**.** *For all $a \in \mathbb{R}$ and $x \in \mathcal{M}$, there exists $y \in \mathcal{M}$ such that $f_a(\log(\Delta))\eta_\Phi(x) = \eta_\Phi(y)$. Furthermore, $\|y\| \leq \|x\|$.*

5.3 Spectral Subspaces

Some of the key ingredients of this chapter and the next make use of the so-called spectral subspaces of Arveson spectral theory. For the convenience of the reader, we will use this section to give a very brief exposition of those aspects of Arveson spectral theory we need. Arveson spectral theory was first introduced by William Arveson in [6], initially inspired by a desire to generalize to the context of C*-algebras a result of Marcel Riesz and Frigyes Riesz from harmonic analysis. It was noticed already in [6] that this framework is applicable to results on automorphism groups and derivations of operator algebras as well as in quantum field theory. Arveson works in a fair bit of generality by considering actions of arbitrary locally compact abelian groups. In the non-separable case, this necessitates the development of a generalization of the Bochner integral in that paper. However, we only consider actions of $\mathbb{R}$, which is separable, so we will ignore and use the Bochner integral. The reader who wishes to learn Arveson spectral theory in full generality may consult [6] or [65, Book II, Chapter XI].

We begin motivating the Arveson spectral theorem by recalling Stone's theorem on one-parameter unitary groups from functional analysis.

Theorem 5.3.1 (Stone's Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space. Strongly continuous one parameter groups of unitary operators U_t on $\mathcal{H}$ are in one-to-one correspondence with (unbounded) self-adjoint operators A on $\mathcal{H}$ under the relation $U_t = e^{itA}$. We call A the **infinitesimal generator** of U_t in this case. Furthermore, A is bounded if and only if U_t is norm-continuous.*

Notice that if we attempt to replace $\mathcal{H}$ with a Banach space $\mathcal{X}$, we have no hope of recovering such an infinitesimal generator in general. This limitation even holds for relatively nice Banach spaces, let alone the potentially pathological Banach spaces underlying C*-algebras and von Neumann algebras. One the other hand, it should be noted that A is completely determined as an unbounded operator on $\mathcal{H}$ by its spectral decomposition.

Let U_t be a strongly continuous one-parameter unitary group on $\mathcal{H}$ with infinitesimal generator A. Let E be the spectral decomposition of A. By functional calculus, for $t \in \mathbb{R}$, we have

$$U_t = e^{itA} = \int_{\mathbb{R}} e^{it\lambda} dE(\lambda).$$

The Arveson spectral theory, rather than finding an infinitesimal generator, instead finds a good enough analogue of what would be its spectral decomposition, were it to exist. The observant reader will notice that the integral in the above display ranges over characters $e^{it\lambda}$ of $\mathbb{R}$. The next well-known theorem from abstract harmonic analysis explains where this comes from.

Theorem 5.3.2. *Let G be a locally compact abelian group and let $\mathcal{H}$ be a Hilbert space. Every unitary representation $\alpha : G \to B(\mathcal{H})$ corresponds to a unique projection-valued measure on the dual group $\widehat{G}$ such that*

$$\alpha(g) = \int_{\widehat{G}} \beta(g) dE(\beta).$$

The proof goes roughly as follows. To each unitary representation $\alpha : G \to B(\mathcal{H})$, we can extend to a representation of $L^1(G)$ on $\mathcal{H}$ (see Definition 5.3.3). By the Spectral Theorem for Banach *-algebras, this representation is canonically associated to a unique projection-valued measure on the spectrum $\sigma(L^1(G))$. Finally, it is a classical property of the Fourier transform that $\sigma(L^1(G))$ is canonically isomorphic to $\widehat{G}$.

For the rest of this section, fix a von Neumann algebra $\mathcal{M}$. Further fix a strongly-continuous one-parameter automorphism group α_t, namely α is a continuous representation of $\mathbb{R}$ on $\mathcal{M}$, where $\mathcal{M}$ is equipped with the strong operator topology. Recall that the Pontryagin dual group $\widehat{\mathbb{R}}$ of $\mathbb{R}$ is homeomorphically isomorphic to $\mathbb{R}$. As such we will always implicitly make the identification $\widehat{\mathbb{R}} = \mathbb{R}$. Recall that for $f \in L^1(\mathbb{R})$, the Fourier transform $\widehat{f}$ of f is given by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} e^{it\lambda} f(t)dt \qquad \lambda \in \widehat{\mathbb{R}} = \mathbb{R}.$$

Definition 5.3.3. For $x \in \mathcal{M}$, we define

$$\alpha_f(x) = \int_{\mathbb{R}} f(t)\alpha_t(x)dt.$$

Notice this extends the representation α of $\mathbb{R}$ to a representation of $L^1(\mathbb{R})$. Next, we will define the spectrum of an element $x \in \mathcal{M}$ with respect to α.

Definition 5.3.4. For $x \in \mathcal{M}$, we define

$$\operatorname{Sp}_\alpha(x) = \{\lambda \in \widehat{\mathbb{R}} \ : \ \widehat{f}(\lambda) = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ such that } \alpha_f(x) = 0\}.$$

In other words, the spectrum $\operatorname{Sp}_\alpha(x)$ of x with respect to α is the subset of $\widehat{\mathbb{R}}$ associated to the ideal of functions $f \in L^1(\mathbb{R})$ such that α_f has x in its kernel. This can be interpreted as the simple oscillation component of the map $t \mapsto \alpha_t(x)$. Now we can define the Arveson spectrum.

Definition 5.3.5. We define **Arveson spectrum** $\mathrm{Sp}(\alpha)$ of α as

$$\mathrm{Sp}(\alpha) = \{\lambda \in \widehat{\mathbb{R}} \ : \ \widehat{f}(\lambda) = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ such that } \alpha_f = 0\}.$$

Now we make the most important definition of this section.

Definition 5.3.6. The **spectral subspace** of α corresponding to a closed subset $E \subseteq \widehat{\mathbb{R}}$ is defined as

$$M(\alpha, E) = \{x \in \mathcal{M} \ : \ \mathrm{Sp}_\alpha(x) \subseteq E\}.$$

The assignment $E \mapsto M(\alpha, E)$ is the analogue of spectral measure that was alluded to earlier in this section. We quote the following from [65, Book II, Chapter XI, Corollary 1.8].

Proposition 5.3.7. *Let $x, y \in \mathcal{M}$ and let $E, F \subseteq \mathbb{R}$ be given. Then*

1. $\mathrm{Sp}_\alpha(x^*) = -\mathrm{Sp}_\alpha(x)$.

2. $\mathrm{Sp}_\alpha(xy) \subseteq \overline{\mathrm{Sp}_\alpha(x) + \mathrm{Sp}_\alpha(y)}$.

3. $M(\alpha, E)^* = M(\alpha, -E)$.

4. $M(\alpha, E)M(\alpha, F) \subseteq M(\alpha, \overline{E + F})$.

The next theorem follows immediately from [65, Book II, Chapter XI, Proposition 1.24] and its proof.

Theorem 5.3.8. *Assume $\mathcal{H}$ is the Hilbert space corresponding semicyclic representation $\mathcal{M}$ with respect to some faithful normal semifinite weight Φ. Let $U_t = \Delta^{it}$*

be the one-parameter group of unitaries on $\mathcal{H}$ and σ_t be the one-parameter group of automorphisms of $\mathcal{M}$ given by Tomita-Takesaki theory. Then

1. $\mathrm{Sp}(U) = \mathrm{Sp}(\sigma)$.

2. *If $x \in \eta_\Phi(\mathcal{M})$ and $f \in L^1(\mathbb{R})$, then $\sigma_f(x) \in \eta_\Phi(\mathcal{M})$ and $U_f \eta_\Phi(x) = \eta_\Phi(\sigma_f(x))$.*

Moreover, it is well known that $\mathrm{Sp}(\sigma) = \mathrm{sp}(\log(\Delta)) \setminus \{0\}$. The above theorem will be indispensable to our proof of (and also the inspiration for the statement of) Theorem 6.2.6.

5.4 Axiomatization

We will only ever consider weights Φ that majorize a state. In particular, $\Phi(1) \geq 1$ or $\Phi(1) = \infty$. This is to avoid certain issues with ultraproducts becoming trivial. Namely, without this assumption, one might consider an ultraproduct of von Neumann algebras with scalings of a state which go to zero, whence the ultraproduct would be a singleton. This is fine in practice because one could always re-scale the weight. Very rarely in practice does one consider a bounded weight that is not a state, so this is not a particularly objectionable restriction.

We introduce the language $\mathcal{L}_{\mathrm{vNa}}$ for weighted von Neumann algebras, whose symbols include:

1. For each $n \in \mathbb{N}$, there is a sort S_n with bound $2n$, whose intended interpretation is the set of n-bounded, right n-bounded elements x of D_Φ with right n-bounded adjoint. We let d_n denote the metric symbol on S_n, whose intended interpretation is the metric induced by $\|\cdot\|_\Phi^\#$.

2. For each $n \in \mathbb{N}$, binary function symbols $+_n$ and $-_n$ with domain S_n^2 and range S_{2n} and whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols are addition and subtraction in the algebra restricted to the sort S_n.

3. For each $n \in \mathbb{N}$, a binary function symbol $\times_n$ with domain S_n^2 and range S_{n^2} and whose modulus of uniform continuity is $\delta(\epsilon) = \frac{\epsilon}{n}$. The intended interpretation of these symbols is multiplication in the algebra restricted to the sort S_n.

4. For each $n \in \mathbb{N}$, a unary function symbol $*_n$ whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols is for the adjoint restricted to each sort.

5. For each $n \in \mathbb{N}$, the constant symbol 0_n which lies in the sort S_n. The intended interpretation of these symbols is the element 0.

6. For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, there is a unary function symbol λ_n whose domain is S_n and range is S_{mn}, where $m = \lceil |\lambda| \rceil$ and with modulus of uniform continuity $\delta(\epsilon) = \frac{\epsilon}{|\lambda|}$ when $\lambda \neq 0$. The intended interpretation of these symbols is scalar multiplication by λ restricted to S_n. (If one is interested in keeping the language countable and computable, one may restrict to scalars whose real and imaginary parts are rational.)

7. For each $m, n \in \mathbb{N}$ with $m < n$, we have a unary function symbols $\iota_{m,n}$ with domain S_m and range S_n and whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols is the inclusion map between the sorts.

8. For each $n \in \mathbb{N}$, we have a unary predicate symbol Φ_n whose range is $[0, n]$ and whose modulus of uniform continuity is $\delta(\epsilon) = \frac{\epsilon}{\sqrt{2}}$. The intended interpretation of this symbol is the restriction of the weight to S_n. Technically speaking, there should really be two such symbols, one for the real and imaginary parts of the weight, but we content ourselves to abuse notation here.

9. For each $n \in \mathbb{N}$, there is a predicate symbol $\mathbf{A}_n$ on the sort S_n taking values in $[0, 4n]$ and with modulus of uniform continuity $\delta(\epsilon) = \frac{\epsilon}{\sqrt{2}}$. The intended interpretation of this symbol is the distance associated to the norm $\| \cdot \|_\Phi$ from an element to the set of self-adjoint elements in $S_{(5n^2+4)}$. The $\sqrt{2}$ in the denominator stems from the fact that $\|a\|_\Phi \leq \sqrt{2}\|a\|_\Phi^\#$ for all $a \in \mathcal{M}$.

Remark 5.4.1. Notice that the main difference between the above language and $\mathcal{L}_{\mathrm{W}^*}$ from [5] is that the above lacks an identity element. This is explained by the observation that [5] is really essentially restricting the above to unital full Hilbert algebras (where the vector corresponding to the unit is also assumed to have norm 1).

We now introduce the axioms T_{vNa} for weighted von Neumann algebras in the language given above. In axiom 9 below, we will use R to represent the R operator from [57]. This is merely a notational convenience (see the discussion after Corollary 5.6.2 for more details). Recall the notation $\dot{-}$ which we introduced just before Definition 2.2.33.

1. The usual algebraic axioms requiring the dissection of M into sorts S_n to be a *-algebra.

2. Axioms saying that Φ is a positive linear functional.

3. Axioms saying the connecting maps preserve addition, multiplication, adjoints and Φ.

4. An axiom for each n requiring that d_n is defined by the norm

$$\|x\|_{\Phi}^{\#} = \sqrt{\frac{\Phi(x^*x) + \Phi(xx^*)}{2}}.$$

5. Axioms requiring that the elements of S_n are n-bounded with n-bounded adjoint. To be explicit, for each k we have

$$\sup_{x \in S_n} \sup_{y \in S_k} \Phi((yx)^* yx) \dot{-} n^2 \Phi(y^* y) \qquad \sup_{x \in S_n} \sup_{y \in S_k} \Phi((xy)^* xy) \dot{-} n^2 \Phi(y^* y).$$

and similarly for x^*. We also add an axiom saying that elements x of S_n have $\|x\|_{\Phi}^{\sharp} \leq n$:

$$\sup_{x \in S_n} \Phi(x^* x + xx^*) \dot{-} 2n^2.$$

6. Axioms expressing that the inclusion maps are isometries and have the correct ranges. Specifically, for each n, and for all k we add:

$$\sup_{x \in S_1} \big| \|\iota_n(x)\| - \|x\| \big|$$

and

$$\sup_{x \in S_1} \sup_{z \in S_k} \big| \Phi((zx)^*(zx)) - \Phi((z\iota_n(x))^*(z\iota_n(x)) \big|.$$

7. Axioms expressing that the inclusion $\iota_n : S_1 \to S_n$ has range consisting of all

$x \in S_n$ where x is left and right 1-bounded with right 1-bounded adjoint:

$$\sup_{x\in S_n}\sup_{z\in S_k}\inf_{y\in S_1}\max\Big\{\|x-\iota_n(y)\| \dot{-} (\|x\| \dot{-} 1),\\ \|x-\iota_n(y)\| \dot{-} \big(\Phi((zx)^*(zx)) \dot{-} \Phi(z^*z)\big),\\ \|x-\iota_n(y)\| \dot{-} \big(\Phi((zx^*)^*(zx^*)) \dot{-} \Phi(z^*z)\big)\Big\}.$$

8. Axioms expressing that $\mathbf{A}_n$ represents the distance to the self-adjoint elements of $S_{(5n^2+4)}$. More precisely, for each $n \in \mathbb{N}$, we include the axiom

$$\sup_{x\in S_n}\left|\mathbf{A}_n(x) - \inf_{y\in S_{(5n^2+4)}}\left\|x-\frac{y+y^*}{2}\right\|_\Phi\right|.$$

9. For each $a \in \mathbb{N}$, we add the following $\forall\exists$ axioms saying that models are closed under $h_a(\log(\Delta))$. Defining $m = 2\lceil e^a\rceil(10n^2+8)^2$, we add

$$\sup_{x\in S_n}\inf_{y\in S_n} d_{S_m}(2R(2-R)x, (e^{-a}(2-R)^2+e^aR^2)y).$$

   Here, m reflects the natural sort for our terms to land in as compositions of functions. To see that this does indeed imply closure under $h_a(\log(\Delta))$, see the discussion following Lemma 5.2.7

10. We have been asking that all of our weights majorize a state. To this end, we add an axiom

$$\sup_{x\in S_1}(\sup_{y\in S_1}\max\{d_{S_1}(xy,y), d_{S_1}(yx,y)\} \dot{-} (d_{S_1}(x,0) \dot{-} 1)).$$

   Note that this says that if x is an element of S_1 that act like the identity on S_1

(and hence on all of $\mathcal{H}$ by linearity and density), then $\|x\|_\Phi = 1$. Notice that if there is no such x, then $\Phi(1) > 1$ since 1 has left and right norm 1.

5.5 Proof of Equivalence of Categories

This section is devoted to proving the following theorem. The most involved step is showing that the left Hilbert associated to any model is actually full. We point out that this step is done in the tracial setting using the Kaplansky density trick (see Fact 3.7.1 and the definition following it). In our setting, however, such an approach is insufficient as we cannot control left and right bounds simultaneously using spectral cut-down polynomials. Thus a whole new approach involving the spectral theory of the modular operator is necessary. Note that the approach here is adapted directly from [5] because the approach we developed there carries over very nicely.

Theorem 5.5.1. *The category of models of the theory defined above is equivalent to the category of weighted von Neumann algebras with morphisms being weight-preserving embeddings with a conditional expectation.*

As usual in continuous logic, we define the **dissection** associated to a weighted von Neumann algebra. This will define a functor from the category of weighted von Neumann algebras to $\mathrm{Mod}(T_{\mathrm{vNa}})$. Given a weighted von Neumann algebra $(\mathcal{M}, \Phi)$, form the associated semicyclic representation. The sort S_K of the dissection $\mathcal{D}(\mathcal{M}, \Phi)$ consists of the totally K-bounded elements of the semicyclic representation (or its associated full Hilbert algebra).

Theorem 5.5.2. *For any weighted von Neumann algebra $(\mathcal{M}, \Phi)$, the dissection $\mathcal{D}(\mathcal{M}, \Phi)$ is a model of T_{vNa}. Moreover, in any dissection $\mathcal{D}(\mathcal{M}, \Phi)$, $\mathbf{A}_K$ captures* *the distance from an element of $S_K(\mathcal{M})$ to $\mathcal{M}_{\mathrm{sa}}$.*

Now we must consider the "inverse" functor, called the **interpretation**. We associate to any model of T_{vNa} a Hilbert algebra as follows. We will later show that this Hilbert algebra is full and thereby can be considered as a weighted von Neumann algebra.

Suppose that we have a model $A \in \mathrm{Mod}(T_{\mathrm{vNa}})$ of the theory T_{vNa}. We begin by forming the direct limit $\mathfrak{A}_0$ of the sorts $S_K(A)$ for $K \in \mathbb{N}$ via the embeddings $i_{L,K}$. Using the interpretation of the function symbols on each sort, we see that $\mathcal{M}_0$ is naturally a complex *-algebra by axiom (1). Furthermore, using the predicate for the weight on each sort, one can define an inner product $\langle x, y\rangle := \Phi(y^*x)$ on $\mathfrak{A}_0$. We let $\mathcal{H}_0$ denote the Hilbert space completion of $\mathfrak{A}_0$ with respect to this inner product. For each $a \in \mathfrak{A}_0$, the maps $b \mapsto ab : \mathfrak{A}_0 \to \mathfrak{A}_0$ and $b \mapsto ba : \mathfrak{A}_0 \to \mathfrak{A}_0$ extend to unique bounded linear operators $\pi(a) : \mathcal{H}_0 \to \mathcal{H}_0$ and $\pi'(a) : \mathcal{H}_0 \to \mathcal{H}_0$ respectively; satisfying $\|\pi(a)\|, \|\pi'(a)\| \leq K$ if $a \in S_K(A)$ by axiom (5). Axiom (6) guarantees that each element belongs to the correct sort. In accordance with Theorem 4.3.2, we should show that $\mathcal{K} \cap i\mathcal{K} = \{0\}$. Denote by $\mathfrak{A}'$ the set of right bounded elements of $\mathcal{H}_0$. We need a small lemma.

Lemma 5.5.3. *Assume $\mathfrak{A}'$ is dense in $\mathcal{H}_0$. Then $\mathcal{K} \cap i\mathcal{K} = \{0\}$.*

Proof. By the discussion after [57, Proposition 5.3], we see that $\mathfrak{A}' \subseteq i\mathcal{K}^\perp + \mathcal{K}^\perp \subseteq (\mathcal{K} \cap i\mathcal{K})^\perp$. If $\mathfrak{A}'$ is dense, then so is $(\mathcal{K} \cap i\mathcal{K})^\perp$, in which case, $\mathcal{K} \cap i\mathcal{K} = \{0\}$, as desired. □

Now, since every element of $\mathfrak{A}_0$ is right bounded and $\mathfrak{A}_0$ is dense in $\mathcal{H}_0$ by assumption, the previous lemma implies $\mathcal{K} \cap i\mathcal{K} = \{0\}$. Thus $\mathfrak{A}_0$ is contained in a left Hilbert algebra.

To each model of our theory, we can define the interpretation to be the full left Hilbert algebra generated by the model. We will show soon that every model of our theory actually consists of all totally bounded elements of the generated von Neumann algebra and therefore determines the full left Hilbert algebra uniquely.

Definition 5.5.4. We define the **interpretation** $\mathfrak{A}_A$ of $A \in \mathrm{Mod}(T_{\mathrm{vNa}})$ to be the full left Hilbert algebra $\mathfrak{A}_A := (\mathfrak{A}_0(A))''$ on $\mathcal{H} := \mathcal{H}_0(A)$ generated by A as in the discussion above.

Note by definition that the direct limit $\mathfrak{A}_0(A)$ of the sorts of A is SOT-dense in the interpretation. For dissections and interpretations to define an equivalence of categories, we need to show that for any $A \in \mathrm{Mod}(T_{\mathrm{vNa}})$, the algebra $\mathfrak{A}_0(A)$ associated to A is precisely the set of totally bounded elements of the interpretation of A.

Theorem 5.5.5. *If $x \in \mathfrak{A}_0(A)$, then x is totally bounded in $\mathfrak{A}_A$. In fact, if x is totally K-bounded in $\mathfrak{A}_0(A)$, then x is totally K-bounded in $\mathfrak{A}_A$.*

Proof. This follows immediately from axiom (5) and SOT-density of $\mathcal{M}_0$ in $\mathcal{M}$. If $x \in S_K(A)$ and $\|x\| \leq 1$ and $\|\pi'(x)\|, \|\pi'(x^*)\| \leq 1$ then since the inclusion maps have the correct images by axiom (6), the inclusions are isometries. By axiom (7), and the fact that the sorts are complete, we have that $x \in S_1(A)$. □

Theorem 5.5.6. *$\mathfrak{A}_0(A)$ is closed under P, Q and R.*

Proof. $\mathcal{M}_0$ is SOT-dense in $\mathcal{M}$ and $\|\cdot\|_\Phi$ metrizes the strong operator topology. Thus the $\|\cdot\|_\Phi$-distance to the self-adjoint elements in $\mathcal{M}_0$ agrees with that in $\mathcal{M}$. Now, if $a \in S_K(\mathcal{M}_0)$ then, by the previous theorem, $\pi(a) \in S_K(\mathcal{M})$. Thus $P_\Phi(\pi(a))$ is totally bounded. Moreover, axiom (8) tells us that $\mathbf{A}_K(x)$ can be computed in

$S_{5K^2+4}(\mathcal{M}_0)$. Thus, by definition of inf, we have a sequence (a_n) of self adjoint operators in $S_{5K^2+4}(\mathcal{M}_0)$ such that $\|a_n - x\|_\Phi$ converges to $\mathbf{A}_K(x)$. Since $\|\cdot\|_\Phi$ metrizes the strong operator topology, we have that (a_n) converges to $P_\Phi(x)$. By the uniform total bound of $3K$, we have that (a_n) strong-* converges to $P_\Phi(x)$. By completeness of $S_{5K^2+4}(\mathcal{M}_0)$ in $\|\cdot\|_\Phi^\sharp$, we have that $P_\Phi(x) \in S_{5K^2+4}(\mathcal{M}_0)$. Thus $\mathcal{M}_0$ is closed under P_Φ and, in turn, under Q_Φ. Therefore $\mathcal{M}_0$ is closed under R_Φ. □

We now need to show that $\mathfrak{A}_0(A)$ contains all of the totally bounded elements of $\mathfrak{A}_A$. We will show that this is guaranteed by axiom (9), but first we need to prove some intermediate results. We will make the following definition to characterize the properties that we have now seen to be true of $\mathfrak{A}_0(A)$ for any model A of our theory.

Definition 5.5.7. Let $\mathcal{M}_0$ be a *-subalgebra $\mathcal{M}_0 \subseteq \mathcal{M}$ of a von Neumann algebra $\mathcal{M}$ such that:

- every element of $\mathcal{M}_0$ is totally bounded;
- the set of totally 1-bounded elements of $\mathcal{M}_0$ is $\|\cdot\|_\Phi^\#$-complete; and
- $\mathcal{M}_0$ is closed under $h_a(\log(\Delta))$ for all $a \in \mathbb{N}$.

Then we will call $\mathcal{M}_0$ a **good subalgebra**.

Note that the second condition above, together with scaling, implies that, in fact, the set of totally K-bounded elements of $\mathcal{M}_0$ is $\|\cdot\|_\Phi^\#$-complete for all $K \in \mathbb{R}$.

Proposition 5.5.8. *If $\mathcal{M}_0$ is a good subalgebra, then $\mathcal{M}_0$ is closed under $f_a(\log(\Delta))$ and $g_a(\log(\Delta))$ for all $a \in \mathbb{N}$.*

Proof. Write

$$e^{-|t|} = \sum_{n=0}^{\infty} a_n(\cosh(t))^{-(2n-1)},$$

wherein every a_n is a positive real and $\sum_{n=0}^{\infty} a_n = 1$ as in the proof of lemma 4.11 in [45]. Plugging in $\log(\Delta)$ gives

$$f_a(\log(\Delta)) = \sum_{n=0}^{\infty} a_n(h_a(\log(\Delta)))^{2n-1},$$

where the sum is norm-convergent. Thus for every $x \in \mathcal{M}_0$ which is totally K-bounded, closure under $h_a(\log(\Delta))$ implies $\eta_\Phi(y_n) := (h_a(\log(\Delta)))^{2n-1}x$ is in $\mathcal{M}_0$. Note $\eta_\Phi(y_n)$ is totally K-bounded for all n. Since each a_n is a positive real and $\sum_{n=0}^{\infty} a_n = 1$, it follows that the partial sums of

$$f_a(\log(\Delta))\eta_\Phi(x) = \sum_{n=0}^{\infty} a_n \eta_\Phi(y_n)$$

are all totally K-bounded and converge uniformly to the limit. Now, closure under $f_a(\log(\Delta))$ follows by completeness of totally bounded subsets. Closure under $g_a(\log(\Delta))$ follows from closure under $f_a(\log(\Delta))$ and $f_0(\log(\Delta))$ by closure under linear combinations. □

From now on, j denotes an element of $\mathbb{R}^{>1}$. We let $E(j^{-1}, j)$ denote the spectral projection of Δ (and Δ^{-1}) corresponding to the interval (j^{-1}, j) and set $E_j := E(j^{-1}, j)(\mathcal{H}_\Phi)$.

The following is [45, Lemma 4.12]:

Fact 5.5.9. *Suppose that $x \in \mathcal{M}$ and $\eta_\Phi(x) \in E_j$. Then:*

1. *For all $n \in \mathbb{Z}$, there is $x_n \in \mathcal{M}$ such that $\Delta^n(\eta_\Phi(x)) = \eta_\Phi(x_n)$ and $\|x_n\| \leq$*

$j^{|n|}\|x\|$; and

2. *x is right bounded and $\|x\|_{\text{right}} \leq j^{1/2}\|x\|$.*

The statement obtained by switching $\mathcal{M}$ and $\mathcal{M}'$ is also valid.

Corollary 5.5.10. *Suppose that $x \in \mathcal{M}$ and $\eta_\Phi(x) \in E_j$. Then x is totally $j^{3/2}\|x\|$-bounded.*

Proof. By Fact 5.5.9, $\|x\|_{\text{right}} \leq j^{\frac{1}{2}}\|x\|$. By Fact 5.5.9 again, we may write $\Delta(\eta_\Phi(x)) = \eta_\Phi(x_1)$ with $x_1 \in \mathcal{M}$ and $\|x_1\| \leq j\|x\|$. By invariance of E_j under Δ, we use the previous fact again to conclude that $\|x_1\|_{\text{right}} \leq j^{1/2}\|x_1\| \leq j^{3/2}\|x\|$. Note that $\Delta\eta_\Phi(x) \in \eta_\Phi(\mathcal{M}')$ by the previous fact and thus $S\eta_\Phi(x) = F\Delta\eta_\Phi(x)$. Since $\pi'(Fy) = \pi'(y)^*$ for all $y \in \mathcal{M}'$, it follows that

$$\begin{aligned}
\|x^*\|_{\text{right}} &= \|\pi'(S\eta_\Phi(x))\| \\
&= \|\pi'(S\eta_\Phi(x))^*\| \\
&= \|\pi'(FS\eta_\Phi(x))\| \\
&= \|\pi'(\Delta\eta_\Phi(x))\| \\
&\leq j^{\frac{3}{2}}\|x\|.
\end{aligned}$$

As we wanted. □

Our next main goal is the following.

Theorem 5.5.11. *Suppose that $\mathcal{M}_0$ is a good subalgebra. If $x \in \mathcal{M}$ is such that $\eta_\Phi(x) \in E_j$, then $x \in \mathcal{M}_0$.*

We begin with some lemmas. Until further notice, we write $\mathcal{M}_0$ to mean a good subalgebra of $\mathcal{M}$.

Lemma 5.5.12. *For all $x \in \mathcal{M}$ and $a \in \mathbb{N}$, we have that* $\|g_a(\log \Delta)(x)\|_{\text{right}} \leq 3e^{3a/2}\|x\|$ *and* $g_a(\log(\Delta))x \in \mathcal{M}_0$.

Proof. Without loss of generality, we may assume that $\|x\| = 1$. By Kaplansky density, we can find a sequence x_n of contractions in $\mathcal{M}_0$ strongly converging to x. Then $y_n := g_a(\log(\Delta))x_n$ strongly converges to $g_a(\log(\Delta))x$. Moreover, $\|y_n\| \leq 3$ for all n. Since $y_n \in E_j$, where $j = e^a$, it follows that y_n is totally $3j^{3/2}$-bounded. The claim now follows by completeness of totally bounded subsets of $\mathcal{M}_0$ and the fact that the strong and strong-* topologies are the same on totally bounded subsets. □

Lemma 5.5.13. *If $x \in \mathcal{M}$ is such that $\eta_\Phi(x) \in E_j$, then $\Delta^n g_j(\log(\Delta))\eta_\Phi(x) \in \eta_\Phi(\mathcal{M}_0)$ for all integers n.*

Proof. By assumption and Fact 5.5.9, $\Delta^n \eta_\Phi(x) \in \eta_\Phi(\mathcal{M})$. Lemma 5.5.12 thus yields that $\Delta^n g_j(\log(\Delta))(\eta_\Phi(x)) = g_j(\log(\Delta))(\Delta^n \eta_\Phi(x)) \in \eta_\Phi(\mathcal{M}_0)$. □

For a fixed j, we split E_j into pieces E_-, E_c and E_+ corresponding to the intervals $(\frac{1}{j}, 1)$, $\{1\}$ and $(1, j)$ respectively. Define $k_+(t) := g_a(2t - a)$ and $k_- := g_j(2t + a)$, where $a := \log(j)$.

Lemma 5.5.14. *$E_+(\mathcal{H}) \cap \eta_\Phi(\mathcal{M})$ is dense in $E_+(\mathcal{H})$ and $E_-(\mathcal{H}) \cap \eta_\Phi(\mathcal{M})$ is dense in $E_-(\mathcal{H})$.*

Proof. It is clear from construction that $k_+(\log(\Delta))$ maps $\eta_\Phi(\mathcal{M})$ to $\eta_\Phi(\mathcal{M})$. Note that $k_+(t)$ is strictly positive on $(0, a)$ and 0 everywhere else. Thus $k_+(\log(\Delta))x$ is contained in the spectral subspace of Δ corresponding to $(1, j)$ for all x. Since k_+ does not vanish on $(0, a)$, we have $k_+(\log \Delta)(\mathcal{H})$ is dense in $E_+(\mathcal{H})$. Moreover, since $\{E_+(\eta_\Phi(x)) : x \in \mathcal{M}\}$ is dense in $E_+(\mathcal{H})$, we have that $\{k_+(\log(\Delta))(E_+(\eta_\Phi(x))) : x \in \mathcal{M}\}$ is dense in $E_+(\mathcal{H})$. The proof for $E_-(\mathcal{H})$ is analogous. □

The following is a straightforward calculation:

Lemma 5.5.15. *If* $v \in E_c(\mathcal{H})$*, then*

$$g_j(\log \Delta)(v) = \frac{e^j - e^{-j}}{e^j + e^{-j}} v.$$

In connection with the next lemma, we note that $g_j(\log \Delta)$ is bounded and invertible (with bounded inverse) on E_j.

Lemma 5.5.16. *We have*

$$(g_j(\log \Delta)|E_-)^{-1}(\eta_\Phi(\mathcal{M}) \cap E_-) \subseteq \eta_\Phi(\mathcal{M}),$$

and

$$(g_j(\log \Delta)|E_+)^{-1}(\eta_\Phi(\mathcal{M}) \cap E_+) \subseteq \eta_\Phi(\mathcal{M}).$$

Proof. Suppose first that $\eta_\Phi(x) \in E_-(\mathcal{H})$. Then

$$\begin{aligned} g_j(\log(\Delta))(\eta_\Phi(x)) &= \left[\Delta - \frac{e^{-j}\Delta + e^{-j}\Delta^{-1}}{e^j + e^{-j}}\right](\eta_\Phi(x)) \\ &= \frac{\Delta(e^j + e^{-j}) - (e^{-j}\Delta + e^{-j}\Delta^{-1})}{e^j + e^{-j}}(\eta_\Phi(x)) \\ &= \frac{\Delta e^j - e^{-j}\Delta^{-1}}{e^j + e^{-j}}(\eta_\Phi(x)). \end{aligned}$$

It thus suffices to show that $(e^j\Delta - e^{-j}\Delta^{-1})^{-1}(\eta_\Phi(x)) \in \eta_\Phi(\mathcal{M})$. Since

$$(e^j\Delta - e^{-j}\Delta^{-1})^{-1} = \Delta(e^j\Delta^2 - e^{-j})^{-1}$$

and $\eta_\Phi(\mathcal{M}) \cap E_j$ is closed under Δ, it suffices to show that $(e^{2j}\Delta^2 - 1)^{-1}(\eta_\Phi(x)) \in \eta_\Phi(\mathcal{M})$; however, this follows by writing $(e^{2j}\Delta^2 - 1)^{-1}$ as a geometric series and using that $\eta_\Phi(x)$ is in the $(\frac{1}{j}, 1)$ spectral subspace of Δ.

Next suppose that $\eta_\Phi(x) \in E_+(\mathcal{H})$. Then

$$\begin{aligned} g_j(\log(\Delta))(\eta_\Phi(x)) &= \left[\Delta^{-1} - \frac{e^{-j}\Delta + e^{-j}\Delta^{-1}}{e^j + e^{-j}}\right](\eta_\Phi(x)) \\ &= \frac{\Delta^{-1}(e^j + e^{-j}) - (e^{-j}\Delta + e^{-j}\Delta^{-1})}{e^j + e^{-j}}(\eta_\Phi(x)) \\ &= \frac{\Delta^{-1}e^j - e^{-j}\Delta}{e^j + e^{-j}}(\eta_\Phi(x)). \end{aligned}$$

It thus suffices to show that $(e^{-j}\Delta - e^j\Delta^{-1})^{-1}(\eta_\Phi(x)) \in \eta_\Phi(\mathcal{M})$. This time, write $(e^{-j}\Delta - e^j\Delta^{-1})^{-1} = \Delta^{-1}(e^{-j} - e^j\Delta^{-2})^{-1}$ and argue as in the previous case. □

Proof of Theorem 5.5.11. Without loss of generality, we may assume that $j \in \mathbb{N}$ and $\|x\| \leq 1$. Write $\eta_\Phi(x) = v_- + v_c + v_+$, where $v_- \in E_-(\mathcal{H})$, $v_c \in E_c(\mathcal{H})$, and $v_+ \in E_+(\mathcal{H})$. By Lemma 5.5.14, we may write $v_- = \lim \eta_\Phi(a_n)$ with each $a_n \in \mathcal{M}$ and $\eta_\Phi(a_n) \in E_-(\mathcal{H})$ and $v_+ = \lim \eta_\Phi(b_n)$ with each $b_n \in \mathcal{M}$ and $\eta_\Phi(b_n) \in E_-(\mathcal{H})$. Moreover, we may assume that $\|a_n\|, \|b_n\| \leq 1$ for each n. By Lemma 5.5.16, for each n, we may find $\widehat{a_n}, \widehat{b_n} \in \mathcal{M}$ such that $g_j(\log\Delta)^{-1}(\eta_\Phi(a_n)) = \eta_\Phi(\widehat{a_n})$, $g_j(\log\Delta)^{-1}(\eta_\Phi(b_n)) = \eta_\Phi(\widehat{b_n})$, and $\eta_\Phi(\widehat{a_n}), \eta_\Phi(\widehat{b_n}) \in E_j$. By Corollary 5.5.10, we have that $\widehat{a_n}$ and $\widehat{b_n}$ are totally $j^{3/2}B$-bounded, where $B := \|(g_j(\log\Delta)|E_j)^{-1}\|$. Consequently, we have

$$\eta_\Phi(x) = g_j(\log\Delta)(\lim_n \eta_\Phi(\widehat{a_n})) + v_c + g_j(\log\Delta)(\lim_n \eta_\Phi(\widehat{b_n})).$$

By Lemma 5.5.12, we have $g_j(\log\Delta)(\eta_\Phi(\widehat{a_n})) := \eta_\Phi(a_n^\dagger)$ and $g_j(\log\Delta)(\eta_\Phi(\widehat{b_n})) = \eta_\Phi(b_n^\dagger)$ for some $a_n^\dagger, b_n^\dagger \in \mathcal{M}_0$; moreover, there is $K > 0$ such that each $a_n^\dagger$ and $b_n^\dagger$ are totally K-bounded. Therefore, by completeness of $S_K(\mathcal{M})$, the first and third terms of the display belong to $\eta_\Phi(\mathcal{M}_0)$. By Lemmas 5.5.12 and 5.5.15, the middle term also belongs to $\eta_\Phi(\mathcal{M}_0)$, whence the corollary is proved. □

In light of Theorem 5.5.11, if we want to prove that good subalgebras are unique, we must approximate arbitrary totally bounded elements by ones that belong to compact spectral subspaces. We use the method of Bochner integrals found, for instance, in [65, II, Lemma 2.4].

Let $v = \eta_\Phi(x)$. Let $r > 0$ be given. Set

$$v_r = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rt^2} \Delta^{it} v \, \mathrm{d}t \,.$$

For all $z \in \mathbb{C}$, the vector

$$v_r(z) = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-z)^2} \Delta^{it} v \, \mathrm{d}t$$

defines $\Delta^{iz} v_r$ and defines an entire function in the variable z. Therefore, for every r, $v_r \in D(\Delta^{iz})$. It follows that for all $r > 0$, the vector v_r is contained in a compact spectral subspace of Δ.

The proof of the following is in [37, Lemma 1.3] with details drawing from the proof of [64, Lemma 10.1].

Lemma 5.5.17. *$\pi(v_r)$ is bounded and satisfies $\|\pi(v_r)\| \leq \|\pi(v)\|$. Furthermore, as $r \to 0$, we have $\pi(v_r) \to \pi(v)$ in the strong-* topology.*

Since all of the expressions above are symmetric under interchanging $\mathcal{M}$ and $\mathcal{M}'$, we have that if $v = \eta_\Phi(x)$ is totally K-bounded, then v_r is totally K-bounded. We conclude the following.

Theorem 5.5.18. *If $\mathcal{M}_0$ is a good subalgebra, then $\mathcal{M}_0 = \mathcal{M}_{\mathrm{tb}}$.*

Proof. $\mathcal{M}_0 \subseteq \mathcal{M}_{\text{tb}}$ by definition. For the other direction, let $v \in \mathcal{M}_{\text{tb}}$ such that v is totally K-bounded. For any $r > 0$, we can construct v_r via the Bochner integral construction above. By the discussion after the previous lemma, we know that v_r is totally K-bounded. Importantly, we saw that each v_r is contained in a compact spectral subspace of Δ, and therefore is contained in E_j for some $j > 1$. By Theorem 5.5.11, it follows that $v_r \in \mathcal{M}_0$.

So consider the sequence $v_{1/n}$. Its terms are all K-bounded elements of $\mathcal{M}_0$. By the previous lemma, $v_{1/n}$ converges to v in the strong-* topology, and hence in $\|\cdot\|_\Phi^\#$. Therefore, by the assumption that totally bounded subsets of $\mathcal{M}_0$ are $\|\cdot\|_\Phi^\#$-complete, we conclude that $v \in \mathcal{M}_{\text{tb}}$, as desired. □

Now we can prove Theorem 5.5.1.

Proof of Theorem 5.5.1. At this point, we have shown that to a model A of T_{vNa}, we can associate a $*$-algebra $\mathcal{M}_0$ of operators on a Hilbert space $\mathcal{H}$ and a faithful $*$-representation $\pi : \mathcal{M}_0 \to \mathcal{B}(\mathcal{H})$ such that, setting $\mathcal{M}_A$ to be the strong closure of $\pi(\mathcal{M}_0)$ and letting Φ_A denote the weight on $\mathcal{M}_A$ corresponding to the inner product on $\mathcal{M}_0$, we have that $\mathcal{M}_0$ is a good subalgebra for $\mathcal{M}_A$. Hence we have $\mathcal{D}(\mathcal{M}_A, \Phi_A) = A$. In particular, if we start with a weighted von Neumann algebra $(\mathcal{M}, \Phi)$ and let $A := \mathcal{D}(\mathcal{M}, \Phi)$, we have (using that the totally bounded vectors in $\mathcal{M}$ are dense in $\mathcal{M}$) that $(\mathcal{M}_A, \Phi_A) = (\mathcal{M}, \Phi)$ and $\mathcal{D}(\mathcal{M}_A, \Phi_A) = \mathcal{D}(\mathcal{M}, \Phi)$.

Finally, we observe that embeddings between weighted von Neumann algebras correspond to embeddings between the corresponding dissections. One direction of this claim is obvious; to see the other, suppose that $\mathcal{D}(\mathcal{M}, \Phi) \subseteq \mathcal{D}(\mathcal{N}, \psi)$. By Lemma 4.5.4, we need to show that, for each $a \in \mathcal{M}$, the $\|\cdot\|_\Phi$-distance between a and the self-adjoint elements of $\mathcal{M}$ is the same as the $\|\cdot\|_\psi$-distance between a and the self-adjoint elements of $\mathcal{N}$. However this follows from axiom (7) and axiom (8), the density of bounded elements, and the fact from Theorem 5.2.2 that for $a \in S_K(\mathcal{M})$, the distance from a to $\mathcal{M}_{\text{sa}}$ is realized by an element of $S_{5K^2+4}(\mathcal{M})$. □

5.6 Definability of Modular Automorphisms

Recall that in the W*-probability space setting, we have the following theorem.

Theorem 5.6.1. *[5, Fact 7.1] Suppose that $(\mathcal{M}_i, \varphi_i)_{i\in I}$ is a family of* W*-*-probability spaces and $\mathcal{U}$ is an ultrafilter on I. Set $(\mathcal{M}, \varphi) = \prod_{\mathcal{U}}(\mathcal{M}_i, \varphi_i)$. Then, for any $t \in \mathbb{R}$ and $(x_i)^\bullet \in \mathcal{M}$, we have*

$$\sigma_t^{\varphi}(x_i)^\bullet = (\sigma_t^{\varphi_i}(x_i))^\bullet.$$

It follows by Beth definability that:

Corollary 5.6.2. *[5, Corollary 7.2] For all $t \in \mathbb{R}$, σ_t is a T_{W^*}-definable function. Moreover, if φ_t is a T_{W^*}-definable predicate defining σ_t, then the map $t \mapsto \varphi_t$ is continuous with respect to the logic topology.*

However, in the present setting, we do not yet know enough about the ultraproducts involved to prove such a theorem. It turns out that the direct analogue is in fact true, but we will need to prove this by other means. We will do this directly by providing an expansion by definitions $T_{\mathrm{vNa-mod}}$ of our theory such that the modular group is definable. We will further show that such an expansion is actually computably definable by explicitly demonstrating definable axioms.

We add new functions symbols $\mathbf{P}_n : S_n \to S_{(5n^2+4)}$ to our language and expand our dissections to interpret $\mathbf{P}_n$ as in Section 5.2 (see also [57, Section 5]) by adding the following axioms.

(11) $\sup_{x\in S_n} \mathbf{A}_n(\mathbf{P}_n(x))$; and

(12) $\sup_{x\in S_n} \sup_{y\in S_1} \varphi(y^*y(x-\mathbf{P}_n(x)))$.

We further expand our language with symbols $\mathbf{Q}_n : S_n \to S_{(5n^2+4)}$ and $\mathbf{R}_n : S_n \to S_{(10n^2+8)}$ and add the following axioms to our theory:

(13) $\sup_{x\in S_n} d_{(5n^2+4)}(\mathbf{Q}_n(x), i\mathbf{P}_n(-ix))$; and

(14) $\sup_{x\in S_n} d_{(10n^2+8)}(\mathbf{R}_n(x), \mathbf{P}_n(x)+\mathbf{Q}_n(x))$.

It now follows that models of axioms (1)-(14) are dissections of weighted von Neumann algebras with $\mathbf{P}$ interpreted in a dissection as the real projection onto the closure of $\mathcal{M}_{\mathrm{sa}}$ and $\mathbf{Q}$ as the corresponding projection onto the closure of $i\mathcal{M}_{\mathrm{sa}}$.

Our next task is to show that Δ_φ^{it} preserves our sorts. We recall that if $a \in \mathcal{M}_{\mathrm{tb}}$, then $\pi'_\Phi(a) \in \mathcal{M}'$, whence $J_\Phi \pi'_\Phi(a) J_\Phi \in \mathcal{M}$.

Lemma 5.6.3. *Suppose that $a \in \mathcal{M}_{\mathrm{tb}}$. Then $\|J_\Phi \pi'_\Phi(a) J_\Phi\| \le \|a\|_{\mathrm{right}}$.*

Proof. Fix $b \in \mathcal{M}'$ and set $d := J_\Phi b J_\Phi \in \mathcal{M}$. Then $J_\Phi \eta_\Phi(b) = \eta_\Phi(d)$ and so

$$\begin{aligned}\|(J_\Phi \pi'_\Phi(a) J_\Phi)\eta_\Phi(b)\|_\Phi &= \|\pi'_\Phi(a)\eta_\Phi(d)\|_\Phi \\ &\le \|a\|_{\mathrm{right}} \|\eta_\Phi(d)\|_\Phi \\ &= \|a\|_{\mathrm{right}} \|\eta_\Phi(b)\|_\Phi\end{aligned}$$

since J_Φ is an isometry. □

Proposition 5.6.4. *Suppose that $a \in \mathcal{M}_{\mathrm{tb}}$. Then for all $t \in \mathbb{R}$, we have $\Delta_\Phi^{it}\eta_\Phi(a) = \eta_\Phi(b)$ for unique $b \in \mathcal{M}_{\mathrm{tb}}$ with $\|b\|_{\mathrm{right}} = \|a\|_{\mathrm{right}}$.*

Proof. We know that $\Delta_\Phi^{it}\eta_\Phi(a) = \eta_\Phi(b)$ for a unique $b \in \mathcal{M}$. Since $a \in \mathcal{M}_{\mathrm{tb}}$, we have that $J_\Phi \pi'_\Phi(a) J_\Phi = \pi_\Phi(c)$ for some $c \in \mathcal{M}$. By Lemma 5.6.3 above, we have that $\|c\| \leq \|a\|_{\mathrm{right}}$. It follows that $\pi_\Phi(J_\Phi \eta_\Phi(a)) = J_\Phi \pi'_\Phi(a) J_\Phi = c$. Consequently, for any $d \in \mathcal{M}$, we have

$$\begin{aligned}(\Delta_\Phi^{it} J_\Phi \pi_\Phi(c) J_\Phi \Delta_\Phi^{-it})\eta_\Phi(d) &= \pi_\Phi(d)(\Delta_\Phi^{it} J_\Phi \eta_\Phi(c)) \\ &= \pi_\Phi(d)\eta_\Phi(b).\end{aligned}$$

Taking $\|\cdot\|_\Phi$ and using that Δ^{it} and J are isometries, we have

$$\begin{aligned}\|\pi_\Phi(d)\eta_\Phi(b)\|_\Phi &\leq \|\pi_\Phi(c)\|\|\eta_\Phi(d)\|_\Phi \\ &\leq \|a\|_{\mathrm{right}}\|\eta_\Phi(d)\|_\Phi.\end{aligned}$$

It follows that $\|b\|_{\mathrm{right}} \leq \|a\|_{\mathrm{right}}$. By applying Δ^{-it}, we achieve equality. □

Corollary 5.6.5. *If $a \in S_n(\mathcal{M})$, then for all $t \in \mathbb{R}$, we have $\Delta^{it}\eta_\Phi(a) = \eta_\Phi(b)$ for a unique $b \in S_n(\mathcal{M})$.*

Let $X \subseteq [0,2]$ denote the spectrum of R. For $t \in \mathbb{R}$, we let $f_t : X \to \mathbb{C}$ be the function defined by $f_t(x) = x^{it}$. Take polynomial functions $f_{t,n}$ on X with coefficients from $\mathbb{Q}(i)$ such that $\|f_t - f_{t,n}\|_\infty < \frac{1}{n}$. Moreover, these polynomials can be found effectively in the sense that the map which upon (t,n) returns the coefficients of $f_{t,n}$ from $\mathbb{N}^2$ to $\mathbb{Q}(i)^{<\omega}$ is a computable map.

The following lemma is straightforward from the definition of Δ^{it}:

Lemma 5.6.6. *For all $m, n \geq 1$, we have*

$$\|\Delta^{it} - f_{t,m}(2-R)f_{-t,n}(R)\| < \frac{1}{m}\|f_{-t}\|_\infty + \frac{1}{n}\|f_{t,m}\|_\infty.$$

Denoting the quantity on the right hand side of the inequality appearing in the previous lemma by $\delta_{t,m,n}$, we note that the map $(t,m,n) \mapsto \delta_{t,m,n}$ is computable. We also set $q_{t,n} \in \mathbb{N}$ to an integer such that $(f_{-t,n}(R_1))(S_1) \subseteq S_{q_{t,n}}$. For each $t \in \mathbb{Q}$, we add a symbol $\mathbf{\Delta}^{it} : S_1 \to S_1$ to the language and continue our enumeration of $T_{\mathrm{vNa-mod}}$ by adding the following axioms to our theory:

(15) $\sup_{x\in S_1}[\|\mathbf{\Delta}^{it}(x) - f_{t,m}(2-\mathbf{R}_{q_{t,n}})(f_{-t,n}(\mathbf{R}_1)(x)))\|_\Phi \dot{-} \delta_{t,m,n}]$.

We note that the above description of the axioms in (15) is a bit sloppy as the term involving the $\mathbf{R}$'s take values in some sort S_p with p predictably depending on t, m, and n, whence we should technically be plugging $\Delta^{it}(x)$ into an appropriate inclusion mapping.

We need one last lemma before we can complete our axiomatization; the proof follows immediately from Proposition 5.6.4.

Lemma 5.6.7. *For each $a \in S_n(\mathcal{M})$ and $t \in \mathbb{R}$, we have $\sigma_t^\Phi(a) \in S_n$.*

We are finally ready to complete the axiomatization $T_{\mathrm{vNa-mod}}$. We add function symbols σ_t to the language and add the following axioms:

(16) $\sup_{a,x\in S_1} d_1(\sigma_t(a)x, \mathbf{\Delta}^{it}(a \cdot \mathbf{\Delta}^{-it}(x)))$.

We have now described a language $\mathcal{L}_{\mathrm{vNa-mod}}$ extending the language $\mathcal{L}$ and an $\mathcal{L}_{\mathrm{vNa-mod}}$-theory $T_{\mathrm{vNa-mod}}$ extending T_{vNa} for which the following theorem holds:

Theorem 5.6.8. *The category of models of* $T_{\mathrm{vNa-mod}}$ *consists of the dissections of weighted von Neumann algebras with the symbols* $\mathbf{P}$*,* $\mathbf{Q}$*,* $\mathbf{R}$*, and* $\mathbf{\Delta^{it}}$ *interpreted as above (restricted to their appropriate sorts) and with the symbols* σ_t *interpreted as the modular automorphism group of the state (restricted to the sort* S_1*). Moreover, the theory* $T_{\mathrm{vNa-mod}}$ *is effectively axiomatized.*

The observation above that our axioms are effective is made in anticipation of, and crucial to, the computability theoretic results in Chapter 8.

Chapter 6

Ultraproducts of General von Neumann Algebras

6.1 Overview

There are several different ultraproduct constructions that arise in the study of operator algebras. In the (not necessarily tracial) von Neumann algebra setting, two prominent examples are the Groh-Raynaud ultraproduct and the Ocneanu ultraproduct. While the former is well-defined for arbitrary von Neumann algebras, the latter only makes sense as usually defined in the σ-finite setting. In [5], the present author, Goldbring, Hart, and Sinclair study the model theory of σ-finite von Neumann algebras in a language that naturally gives rise to the Ocneanu ultraproducts.

Traditionally, in continuous logic, axiomatizations are guided by pre-existing ultraproduct constructions which the axiomatizations aim to capture. In the case of weighted von Neumann algebras, however, such an ultraproduct does not exist. Our axiomatization thus gives us a whole new ultraproduct. We believe this ultraproduct is of independent interest and it makes sense to study its properties here. We give multiple characterizations of our new ultraproduct and relate it to the Groh-Raynaud ultraproduct, paralleling similar results for the Ocneanu ultraproduct given in [3].

6.2 Hilbert Algebra Ultraproducts

In this section, we will introduce the Hilbert algebra ultraproduct and then show that it agrees with the model-theoretic ultraproduct of our language. The notion of ultraproduct that our axiomatization seems to most immediately suggest is the following.

Definition 6.2.1. Let $(\mathfrak{A}_i, \mathcal{H}_i)$ be a family of full left Hilbert algebras and let Φ_i be the induced faithful normal semifinite weight on the left von Neumann algebra $\mathcal{R}(\mathfrak{A}_i)$. Define $\ell^\infty_\Phi(\mathfrak{A}_i)$ to be the set of all sequences (x_i) such that $\|x_i\|_{\Phi_i}$ and $\|\pi(x_i)\|$ are both uniformly bounded. Define the subalgebra

$$\mathcal{I}_\mathcal{U} = \left\{(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i) \ : \ \lim_{i \to \mathcal{U}} \|x_i\|^\sharp_{\Phi_i} = 0\right\}$$

and its two-sided normalizer

$$\mathcal{N}_\mathcal{U} = \{(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i) \ : \ (x_i)\mathcal{I}_\mathcal{U} \subseteq \mathcal{I}_\mathcal{U} \text{ and } \mathcal{I}_\mathcal{U}(x_i) \subseteq \mathcal{I}_\mathcal{U}\}.$$

Then the **Hilbert algebra ultraproduct** $\prod^\mathcal{U}_{\mathrm{HA}}(\mathfrak{A}_i, \mathcal{H}_i)$ is defined to be $\mathcal{N}_\mathcal{U}/\mathcal{I}_\mathcal{U}$ together with its Hilbert space completion. We can equivalently consider the left von Neumann algebra generated by this together with the associated faithful normal semifinite weight, which we will denote by $\prod^\mathcal{U}_{\mathrm{HA}}(\mathcal{M}_i, \Phi_i)$.

Theorem 6.2.2. *The Hilbert algebra ultraproduct agrees with the Ocneanu ultraproduct when every Φ_i is a state.*

Proof. Suppose every Φ_i is a state. Since $\|x_i\|_{\Phi_i} \leq \|\pi(x_i)\|$ by Cauchy-Schwartz, the condition that $\|x_i\|_{\Phi_i}$ uniformly bounded condition is follows from the condition that $\|x_i\|$ is uniformly bounded. Also, since when Φ_i is a state, we have that $\omega = 1$ is a cyclic and separating vector, it follows that $\mathfrak{A}_i = \mathcal{M}_i\omega$. Then this recovers exactly the definition of the Ocneanu ultraproduct. Furthermore, we know from [3] that in the state case, the Ocneanu ultraproduct acts standardly, it follows that the Hilbert space completion is isomorphic to $L^2(\prod_{\mathrm{HA}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_{\mathcal{U}}))$. □

Lemma 6.2.3. *[3, Lemma 4.14] Let $f \in L^1(\mathbb{R})$, and $(x_i) \in \mathcal{N}_{\mathcal{U}}(\mathcal{M}_i, \varphi_i)$ where each φ_i is a state. Then $(\sigma_f^{\varphi_i}(x_i)) \in \mathcal{N}_{\mathcal{U}}(\mathcal{M}_i, \varphi_i)$ and $\sigma_f^{\varphi^{\mathcal{U}}}((x_i)^\bullet) = (\sigma_f^{\varphi_i}(x_i))^\bullet$ holds.*

By the same proof, we have the following version of the above lemma for faithful normal semifinite weights.

Lemma 6.2.4. *Let $f \in L^1(\mathbb{R})$, and $(x_i) \in \mathcal{N}_{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$. Then $(\sigma_f^{\Phi_i}(x_i)) \in \mathcal{N}_{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$ and $\sigma_f^{\Phi^{\mathcal{U}}}((x_i)^\bullet) = (\sigma_f^{\Phi_i}(x_i))^\bullet$ holds.*

Now we state the main theorem of this section.

Theorem 6.2.5. *Suppose that $(\mathcal{M}_i, \varphi_i)$ is a family of weighted von Neumann algebras for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on I. Then $\mathcal{D}(\prod_{\mathrm{HA}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_i)) \cong \prod_{\mathcal{U}} \mathcal{D}(\mathcal{M}_i, \Phi_i)$.*

Before we prove the previous theorem, we need to prove some facts about spectral subspaces (recall Section 5.3).

Recall the Féjer kernel $F_a \in L^1(\mathbb{R})$ defined for $a > 0$ by

$$F_a(t) = \frac{a}{2\pi} 1_{\{t=0\}} + \frac{1 - \cos(at)}{\pi a t^2} 1_{\{t \neq 0\}}.$$

and its Fourier transform

$$\widehat{F_a}(\lambda) = \left(1 - \frac{|\lambda|}{a}\right) 1_{\{|\lambda| \leq a\}} + (0) 1_{\{|\lambda| > a\}}.$$

The following theorem is an adaptation of [3, Proposition 4.11] and its proof is nearly identical.

Theorem 6.2.6. *Let $(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i)$. Then the following statements are equivalent:*

1. $(x_i) \in \mathcal{N}_\mathcal{U} := \{(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i) \ : \ (x_i)\mathcal{I}_\mathcal{U} \subseteq \mathcal{I}_\mathcal{U}$ *and* $\mathcal{I}_\mathcal{U}(x_i) \subseteq \mathcal{I}_\mathcal{U}\}$.

2. *For any $\epsilon > 0$, there exists $a > 0$ and $(y_i) \in \ell^\infty_\Phi(\mathfrak{A}_i)$ such that*

   - $\lim_{i \to \mathcal{U}} \|x_i - y_i\|^\#_{\Phi_i} < \epsilon$*; and*
   - $y_i \in M(\sigma^{\Phi_i}, [-a, a])$ *for all* $i \in I$.

Proof. (1) $\implies$ (2): Let $(x_i) \in \mathcal{N}_\mathcal{U}$ and put $x := (x_i)_\mathcal{U}$. Also, define

$$x_a := \sigma^{\Phi^\mathcal{U}}_{F_a}(x) \in (\mathcal{M}_i, \Phi_i)^\mathcal{U}.$$

Define $f : t \mapsto \|x - \sigma^{\Phi^\mathcal{U}}_t(x)\|^\#_{\Phi^\mathcal{U}}$. Since f is continuous and bounded, we have

$$\begin{aligned} \|\eta_\Phi(x_a - x)\|_{\Phi^\mathcal{U}} &= \| \int_\mathbb{R} F_a(t)(\eta_\Phi(\sigma^{\Phi^\mathcal{U}}_t(x) - x)\,\mathrm{dt})\|^\#_{\Phi^\mathcal{U}} \\ &= \int_\mathbb{R} F_a(t)\|\eta_\Phi(x - \sigma^{\Phi^\mathcal{U}}_t(x))\|^\#_{\Phi^\mathcal{U}}\,\mathrm{dt}\,. \end{aligned}$$

This expression goes to 0 as $a \to \infty$ and therefore we have

$$\lim_{a \to \infty} \|x_a - x\|^\#_{\Phi^\mathcal{U}} = 0.$$

Therefore there exists $a > 0$ such that $y := \sigma_{F_a}^{\Phi^{\mathcal{U}}}(x)$ satisfies $\|\eta_\Phi(y - x)\|_{\Phi^{\mathcal{U}}} < \epsilon$. We have by Lemma 6.2.4, $y = (y_i)_{\mathcal{U}}$, where $y_i = \sigma_{F_a}^{\Phi_i}(x_i)$ for $i \in I$ and

$$\|y\|\|F_a\|_1\|x\| = \|x\|.$$

Therefore (y_i) is a sequence satisfying (2). Note also that $\|y_i\| \leq \|x_i\|$ for $i \in I$.

(2) $\implies$ (1): Take $x = (x_i) \in \ell^\infty_\Phi(\mathcal{M}_i)$ as in (2). Let $\epsilon > 0$. Then by Lemma 6.2.4 and by assumption, there is $y = (y_i) \in \mathcal{N}_{\mathcal{U}}$ such that $\lim_{i\to\mathcal{U}} \|x_i - y_i\|^{\#}_{\Phi_i} < \epsilon$. Taking $m = (m_i) \in \mathcal{I}_{\mathcal{U}}$ with $\sup_{n\geq 1} \|m_i\| \leq 1$, we have

$$\begin{aligned}\lim_{i\to\mathcal{U}} \|(x_i m_i)^*\|_{\Phi_i} &\leq \lim_{i\to\mathcal{U}} (\|m_i^*\|\|x_i^* - y_i^*\|_{\Phi_i} + \|m_i^* x_i^*\|_{\Phi_i}) \\ &\leq \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this proves $xm \in \mathcal{I}_{\mathcal{U}}$ so $x\mathcal{I}_{\mathcal{U}} \subseteq \mathcal{I}_{\mathcal{U}}$. By a similar proof we have, $\lim_{i\to\mathcal{U}} \|m_i x_i\|_{\Phi_i} = 0$ so $mx \in \mathcal{I}_{\mathcal{U}}$. We can now conclude that $x \in \mathcal{N}_{\mathcal{U}}$ as was required. □

Now we record a few more facts that we will need about spectral subspaces.

Proposition 6.2.7.

1. *If $x \in M(\sigma^\Phi, [-a, a])$ for some $a > 0$, then the map $t \mapsto \sigma_t^\Phi(x)$ extends to an entire function $\mathbb{C} \to \mathcal{M}$; moreover, for any $z \in \mathbb{C}$, there is a constant $C_{a,z}$ depending only on a and z such that $\|\sigma_z^\Phi(x)\| \leq C_{a,z}\|x\|$.*

2. *If $x \in M(\sigma^\Phi, [-a, a])$, then $x \in \mathcal{M}_{\text{tb}}$ and $\|x\|_{\text{right}}, \|x^*\|_{\text{right}} \leq C_{a,i/2}\|x\|$.*

Proof. The proof of item (1) is identical to the proof of [3, Lemma 4.13]. Item (2) is explicit in [3, Lemma 4.13], but we include the proof for the sake of completeness.

Suppose $x \in M(\sigma^\Phi, [-a, a])$ and take $y \in \eta_\Phi(\mathcal{M})$; we show that $\|y\eta_\Phi(x)\|_\Phi \le C_{a,i/2} \cdot \|x\| \cdot \|\eta_\Phi(y)\|_\Phi$. To see this, we calculate as follows:

$$\begin{aligned}
\|y\eta_\Phi(x)\|_\Phi &= \|(JyJ)(S\Delta^{-1/2}\eta_\Phi(x))\|_\Phi && \text{since } J \text{ is isom. and } JS\Delta^{-1/2} = 1\\
&= \|(JyJ)S\sigma^\Phi_{i/2}(\eta_\Phi(x))\|_\Phi && \text{by def. of } \sigma^\Phi_{i/2}\\
&= \|(\sigma^\Phi_{i/2}(x))^*(\eta_\Phi(Jy))\|_\Phi && \text{by Tomita's Theorem}\\
&\le \|\sigma_{i/2}(x)^*\| \cdot \|\eta_\Phi(Jy)\|_\Phi && \text{by def. of operator norm}\\
&\le C_{a,i/2} \cdot \|x\| \cdot \|\eta_\Phi(y)\|_\Phi.
\end{aligned}$$

The result for x^* is now immediate from (1). □

Proof of Theorem 6.2.5. First suppose that $(a_i)^\bullet \in \prod_\mathcal{U} S_n(M_i)$. It is clear that $(a_i) \in \ell^\infty(\mathcal{M}_i, I)$; we wish to show that $(a_i) \in \mathcal{M}$. To see this, suppose that $(m_i) \in \mathcal{I}$; we show that $(a_i m_i), (m_i a_i) \in \mathcal{I}$. Suppose, towards a contradiction, that $(a_i m_i) \notin \mathcal{I}$ and set $L := \lim_\mathcal{U} \|a_i m_i\|^\#_{\Phi_i} \neq 0$. Consider the set of $i \in I$ for which $\|m_i\|^\#_{\Phi_i} < L/2n$, which belongs to $\mathcal{U}$ since $(m_i) \in \mathcal{I}$. For a $\mathcal{U}$-large subset of these i, we have that $\|a_i m_i\|^\#_{\Phi_i} > L/2$; for these i, we see that the operator norm of a_i is greater than n or the right norm of a_i^* is greater than n, which is a contradiction. To see this, assuming $\|a_i\|, \|a_i^*\|_{\text{right}} \le n$, we have

$$\|a_i m_i\|^\#_{\Phi_i} = \sqrt{\frac{\|a_i m_i\|^2_{\Phi_i} + \|m_i^* a_i^*\|^2_{\Phi_i}}{2}} \le \sqrt{\frac{\|a_i\|^2\|m_i\|^2_{\Phi_i} + \|a_i^*\|^2_{\text{right}}\|m_i^*\|^2_{\Phi_i}}{2}} \le n\|m_i\|^\#_{\Phi_i},$$

as claimed. The proof that $(m_i a_i) \in \mathcal{I}$ proceeds similarly, using that each a_i is right n-bounded and a_i^* is left n-bounded. It is now clear that $(a_i)^\bullet = (a_i)^\star$ and that this element is in $S_n(\prod^\mathcal{U}_{\text{HA}}(\mathcal{M}_i, \Phi_i))$.

By Theorem 5.5.1, we now have that the von Neumann algebra associated to $\prod_{\mathcal{U}} \mathcal{D}(\mathcal{M}_i, \Phi_i)$ embeds into $\prod_{\mathrm{HA}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$; it suffices to show that this embedding is actually onto. To see this, it suffices to show that, given any $x \in \prod_{\mathrm{HA}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$ with $\|x\| = 1$ and any $\epsilon > 0$, there is $n > 0$ and $y \in \prod_{\mathcal{U}} S_n(M_i)$ such that $\|x - y\|^{\#}_{\Phi^{\mathcal{U}}} < \epsilon$. Write $x = (x_i)^{\star}$ and take $y = (y_i)^{\star}$ as in Theorem 6.2.6 for some $a > 0$. We show that $y \in \prod_{\mathcal{U}} S_n(M_i)$ for some $n > 0$. Indeed, as mentioned above, we may assume that $\|y_i\| \leq 1$ for all $i \in I$ whence, by Proposition 6.2.7(3), we have that $\|y_i\|_{\text{right}}, \|y_i^*\|_{\text{right}} \leq C_{a,i/2}$ for all $i \in I$ and thus $y \in \prod_{\mathcal{U}} S_n(M_i)$ when $n \geq C_{a,i/2}$. □

Theorem 6.2.8. *The modular automorphism group commutes with the Hilbert algebra ultraproduct.*

Proof. By Proposition 6.2.5, together with the definability of the modular automorphism group via Theorem 5.6.8, we have $(\sigma_t^{\Phi_i}(x_i))^{\bullet} = \sigma_t^{\Phi^{\mathcal{U}}}(x_i)^{\bullet}$ whenever x_i is totally K-bounded. Now by strong density of the totally bounded elements and strong continuity of modular automorphisms, we conclude the claim. □

Theorem 6.2.9. *The weight defined by*

$$\Phi_l(x) = \begin{cases} \sqrt{\langle \xi \,,\, \xi \rangle} & \textit{if } x^{1/2} = \xi \in \mathfrak{A} \\ \infty & \textit{otherwise} \end{cases}$$

on the right von Neumann algebra $\mathcal{R}_{\ell}(\prod_{\mathrm{HA}}^{\mathcal{U}}(\mathfrak{A}_i, \mathcal{H}_i))$ *of the Hilbert algebra ultraproduct is a faithful normal semifinite weight.*

Proof. By Theorem 6.2.5, the Hilbert algebra ultraproduct is a model of the theory T_{vNa}. By Theorem 5.5.1, this implies that it is a full left Hilbert algebra. Now the claim is true by the equivalence of categories between weighted von Neumann algebras and full left Hilbert algebras. □

6.3 Generalized Ocneanu Ultraproducts

While the Hilbert algebra ultraproduct from the previous section is nice from a Hilbert algebra perspective, it has the distinct disadvantage that it can only be understood in terms of vectors in the domain of definition. The von Neumann algebra we arrive at by taking the strong closure can be somewhat mysterious. In particular, it is difficult to know exactly what sequences from the factor algebras will appear in the strong closure of the Hilbert algebra ultraproduct if the sequence contains elements not in the respective domains of definition. It is also not prima-facie connected to more standard constructions in operator algebras. We will consider an alternative ultraproduct defined as a subset of the Groh-Raynaud ultraproduct and show that the former has many nice properties that generalize some key theorems found in [3]. This, in a way, serves as evidence that our language is the "correct" one. Many of the techniques we use will also be direct generalizations of or nearly identical to those used in [3]. Finally, we will exhibit an explicit spatial isomorphism between this new ultraproduct and the previous one.

Definition 6.3.1. Let $(\mathcal{M}_i, \Phi_i)$ be a family of weighted von Neumann algebras and let $\mathcal{H}_i$ be the associated family of semicyclic representations. Let (v_i) be a sequence of vectors such that $v_i \in \eta_{\Phi_i}(\mathcal{M}_i)$ for all i. We will call (v_i) a **σ-continuous point** if $(\Delta_{\Phi_i}^{it} v_i)^\bullet$ is a continuous function of t with respect to $\|\cdot\|_\Phi^\#$ on the Hilbert space ultraproduct $\prod^{\mathcal{U}} \mathcal{H}_i$.

We also define a version of this notion for the elements of the Banach space ultraproduct (and, by extension, the Groh-Raynaud ultraproduct) of the underlying von Neumann algebras.

Definition 6.3.2. Let $(\mathcal{M}_i, \Phi_i)$ be a family of weighted von Neumann algebras and let $\mathcal{H}_i$ be the associated family of semicyclic representations. Let (x_i) be a sequence of elements such that $x_i \in \mathcal{M}_i$ for all i. We will call (x_i) a **σ-continuous point** if $(\sigma_t^{\Phi_i}(x_i))^\bullet$ is a continuous function of t with respect to the strong* topology in the Groh-Raynaud ultraproduct $\prod_{\mathrm{GR}}^{\mathcal{U}} \mathcal{M}_i$.

Proposition 6.3.3. *Let (v_i) be a σ-continuous vector such that $\|\pi_{\Phi_i}(v_i)\|$ is uniformly bounded. Then $(\pi_{\Phi_i}(v_i))$ is a continuous element.*

We next state the following characterization of the usual Ocneanu ultraproduct in terms of σ-continuous vectors of the Hilbert space ultraproduct. This characterization and its proof can be found in [49, Theorem 1.5]. The proof of equivalence of the conditions (1) and (2) can be found in [3, Proposition 4.11]. We will show as a corollary that we can equivalently view the Ocneanu ultraproduct in terms of σ-continuous elements of the Groh-Raynaud ultraproduct.

Theorem 6.3.4. *Let $(x_i) \in \ell^\infty(\mathcal{M}_i)$. Then the following statements are equivalent:*

1. $(x_i) \in \mathcal{N}_{\mathcal{U}} := \{(x_i) \in \ell^\infty(\mathcal{M}_i) \ : \ (x_i)\mathcal{I}_{\mathcal{U}} \subseteq \mathcal{I}_{\mathcal{U}}$ *and* $\mathcal{I}_{\mathcal{U}}(x_i) \subseteq \mathcal{I}_{\mathcal{U}}\}$.
2. *For any $\epsilon > 0$, there exists $a > 0$ and $(y_i) \in \ell^\infty(\mathcal{M}_i)$ such that*
    - $\lim_{i \to \mathcal{U}} \|x_i - y_i\|_{\varphi_i}^{\#} < \epsilon$; *and*
    - $y_i \in M(\sigma^{\varphi_i}, [-a, a])$ *for all $i \in I$.*
3. $(x_i)^\bullet$ *is a σ-continuous vector.*

We also have:

Corollary 6.3.5. *Let* (x_i) *be as in the previous theorem. The conditions* (1), (2) *and* (3) *are in turn equivalent to:*

4. $(\pi_{\Phi_i}(x_i))^\bullet$ *is a* σ*-continuous element.*

Proof. We will show (3) $\iff$ (4) and the claim will follow by transitivity.

(3) $\implies$ (4) follows from Proposition 6.3.3.

(4) $\implies$ (3). Note $\|\cdot\|_\varphi^\#$ metrizes the strong* topology on norm bounded sets. Now since $\omega_{\varphi^\mathcal{U}} = (\omega_{\varphi_i})_\mathcal{U}$ is cyclic and separating for the Ocneanu ultraproduct, every element is represented by a vector. The claim now follows. □

We still need to define a corresponding faithful normal semifinite weight on this algebra. Recall that given a family $(\mathcal{M}_i, \varphi_i)$ of von Neumann algebras $\mathcal{M}_i$ together with uniformly bounded linear functionals φ_i on them, we can define a positive linear functional $\varphi_\mathcal{U}$ on the Groh-Raynaud (and, a fortiori, the Ocneanu) ultraproduct as

$$\varphi_\mathcal{U}(x_i) = \lim_{i\to\mathcal{U}} \varphi_i(x_i)$$

and in fact, this is how the ultraproduct of states is defined. We will use a slight generalization of [3, Corollary 3.25] that tells us that any positive linear functional on the ultraproduct can be achieved this way. First we recall:

Lemma 6.3.6. *[3, Corollary 3.25] Let* φ *be a state on the Groh-Raynaud ultraproduct* $\prod_{\mathrm{GR}}^\mathcal{U}(\mathcal{M}_i, \mathcal{H}_i)$*. Then there exists a sequence* φ_i *of states such that* $\varphi = (\varphi_i)_\mathcal{U}$*.*

The following generalization is clear by taking linear combinations.

Corollary 6.3.7. *Let φ be a positive linear functional on the Groh-Raynaud ultraproduct $\prod_{\mathrm{GR}}^{\mathcal{U}}(\mathcal{M}_i, \mathcal{H}_i)$. Then there exists a uniformly bounded sequence φ_i of positive linear functionals such that $\varphi = (\varphi_i)_{\mathcal{U}}$.*

Note that this definition does not make sense in the case of a weight. Indeed, consider $I = \mathbb{N}$ and take $\mathcal{M}_i = \mathcal{M}$ a fixed von Neumann algebra and let Φ be a faithful normal state on $\mathcal{M}$. Take the faithful normal semifinite weight $\Phi_i = i^2\Phi$ for all i. Now considering the sequence $x_i = \frac{1}{i}1$, we have that $(x_i)^\bullet = (0)^\bullet$ but

$$\lim_{i\to\mathcal{U}} \Phi_i(x_i) = \lim_{i\to\mathcal{U}} i = \infty$$

and therefore this construction is ill-defined.

On the other hand, Theorem 3.3.11 tells us that any faithful normal semifinite weight Φ is the supremum of all the positive linear functionals it majorizes. In other words $\Phi = \sup \varphi$ where the supremum ranges over $\varphi \in (\mathcal{M})_*^+$ such that $\varphi \leq \Phi$. Since every bounded linear functional on a Groh-Raynaud ultraproduct is realized as an ultraproduct of bounded linear functionals and it is clear that ultraproducts of bounded linear functionals preserve entry-wise majorization, we are led to the following definition.

Definition 6.3.8. Let $(\mathcal{M}_i, \Phi_i)$ be a family of weighted von Neumann algebras. The **ultraproduct weight** $\Phi_{\mathcal{U}}$ of Φ_i is the weight on the Groh-Raynaud ultraproduct $\prod_{\mathrm{GR}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$ defined by

$$\Phi_{\mathcal{U}} = \sup\{\varphi = (\varphi_i)_{\mathcal{U}} \ : \ \varphi_i \in (\mathcal{M}_i)_*^+ \text{ uniformly bounded such that } \varphi_i \leq \Phi_i\}.$$

While editing this thesis, the author became aware of the pre-print [10] due to

Martijn Caspers. The pre-print remains unpublished. Interestingly, Caspers defines the same weight we do, although for a larger class of weights. We also claim similar results about these weights, albeit with very different proofs and techniques. Our motivations for considering such weights also differ greatly. The fact that this definition was arrived at independently in such different ways suggests that it is a natural one to consider.

The following is immediate.

Proposition 6.3.9. *If Φ_i is a sequence of faithful normal states, then*

$$\Phi_{\mathcal{U}}(x_i) = \lim_{i \to \mathcal{U}} \Phi_i(x_i)$$

so that the ultraproduct weight agrees with the usual ultraproduct state when every Φ_i is a state.

We find it appropriate to remind the reader here that every weight we ever consider in this thesis majorizes a state. In the spirit of the Theorem 6.3.4 and the intervening discussion, we make the following definition.

Definition 6.3.10. Let $(\mathcal{M}_i, \Phi_i)$ be a family of weighted von Neumann algebras. We define the **generalized Ocneanu ultraproduct** $\prod_{\mathrm{Oc}}^{\mathcal{U}}(\mathcal{M}_i, \Phi_i)$ to be the subset of the Groh-Raynaud ultraproduct consisting of elements $(x_i)^\bullet$ for which $(\sigma_t^{\Phi_i}(x_i))^\bullet$ is strong*-continuous in the variable t. It is equipped with the restriction of the ultraproduct weight $\Phi_{\mathcal{U}}$ to it.

We will show that the generalized Ocneanu ultraproduct and the Hilbert algebra ultraproduct are actually equivalent. To do this, we will first identify the Hilbert algebra ultraproduct as a corner (see the statement after Theorem 4.5.10) of the Groh-Raynaud ultraproduct by a projection. The projection in question will be naturally isomorphic to the Hilbert space underlying the Hilbert algebra ultraproduct.

Theorem 6.3.11. *[3, Theorem 3.7] Let $(\mathcal{M}_i)$ be a sequence of σ-finite von Neumann algebras and let a normal faithful state φ_i on $\mathcal{M}_i$ be given for each $i \in I$. Assume that each $\mathcal{M}_i$ acts standardly on $\mathcal{H}_i = L^2(\mathcal{M}_i, \varphi_i)$, so that $\prod_{GR}^{\mathcal{U}} \mathcal{M}_i \subseteq B(\prod^{\mathcal{U}}(\mathcal{H}_i))$. Also let*

$$\mathcal{M}^{\mathcal{U}} = \prod^{\mathcal{U}}(\mathcal{M}_i, \varphi_i), \qquad \varphi^{\mathcal{U}} = (\varphi_i)^{\mathcal{U}},$$

and define $w : L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}}) \to \prod^{\mathcal{U}}(\mathcal{H}_i)$ by $w(x_i)^\bullet \omega_{\varphi^{\mathcal{U}}} := (x_i \omega_{\varphi_i})^\bullet$, $(x_i)^\bullet \in \mathcal{M}^{\mathcal{U}}$. Then w is an isometry, and $w^*(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)w = \mathcal{M}^{\mathcal{U}}$.

Inspired by the above, define $W : L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}}) \to \prod^{\mathcal{U}}(\mathcal{H}_i)$ by extending

$$W(x_i)^\bullet \omega_{\varphi^{\mathcal{U}}} := (x_i \omega_{\varphi_i})^\bullet \qquad \text{for } (x_i)^\bullet \in \mathcal{M}^{\mathcal{U}}$$

to its closure by density. Then W is an isometry by Proposition 6.3.9. Define also

$$p_\Phi : \prod^{\mathcal{U}}(\mathcal{H}_i) \to L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}})$$

to be the projection onto the corresponding subspace. Then we have

$$W^*(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)W = p_\Phi(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)p_\Phi$$

It is a simple calculation to see this agrees with the weight given by the inner product on $L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}})$.

We would like to show $p_\Phi(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)p_\Phi = \mathcal{M}^{\mathcal{U}}$. In other words, the Hilbert algebra ultraproduct is the corner of the Groh-Raynaud ultraproduct corresponding to p_Φ.

Theorem 6.3.12. *$p_\Phi(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)p_\Phi = \mathcal{M}^{\mathcal{U}}$.*

Proof. By Theorem 4.5.9, $\prod_{GR}^{\mathcal{U}} \mathcal{M}_i$ acts standardly on $\prod^{\mathcal{U}}(\mathcal{H}_i)$ with antilinear isometry $J := J_{\mathcal{U}} = (J_i)^{\mathcal{U}}$. Consider the projection $q = p_\Phi J$. It is clear that $q \in \prod_{GR}^{\mathcal{U}} \mathcal{M}_i$ since it is a projection that commutes with every element of $(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)' = \prod_{GR}^{\mathcal{U}} \mathcal{M}_i'$. Then since

$$\begin{aligned} p &= qJqJ \\ &= p_\Phi JJp_\Phi JJ \\ &= p_\Phi, \end{aligned}$$

we have by Theorem 4.5.10 that $p_\Phi(\prod_{GR}^{\mathcal{U}} \mathcal{M}_i)p_\Phi$ acts standardly on the subspace $p_\Phi(\prod^{\mathcal{U}}(\mathcal{H}_i)) \cong L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}})$.

We also have, by Theorem 5.5.1 and Proposition 6.2.5, that $\mathcal{M}^{\mathcal{U}}$ acts standardly on $L^2(\mathcal{M}^{\mathcal{U}}, \varphi^{\mathcal{U}})$. Note also that if $(v_i)^\bullet \in \eta_{\Phi^{\mathcal{U}}}(\mathcal{M}^{\mathcal{U}})$, then $\pi((v_i)^\bullet)$ is obviously in the Groh-Raynaud ultraproduct. Furthermore, $\pi((v_i)^\bullet)$ commutes with p_Φ by definition. Now the result follows by the fact that left Hilbert algebras give rise to unique standard forms up to unitary equivalence. □

Now we state and prove a generalization of Theorem 6.3.4 for the Hilbert algebra ultraproduct. The proof is nearly identical to that given in [49, Theorem 1.5] and is simply adapted here to the Hilbert algebra setting. Notice that in the following, we only work in the domain of definition, or the underlying Hilbert space. We will deal with the elements of infinite weight in Theorem 6.3.15.

Theorem 6.3.13. *Let* $(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i)$*. Then the following statements are equivalent:*

1. $(x_i) \in \mathcal{N}_\mathcal{U} := \{(x_i) \in \ell^\infty_\Phi(\mathfrak{A}_i) \ : \ (x_i)\mathcal{I}_\mathcal{U} \subseteq \mathcal{I}_\mathcal{U}$ *and* $\mathcal{I}_\mathcal{U}(x_i) \subseteq \mathcal{I}_\mathcal{U}\}$.

2. *For any* $\epsilon > 0$*, there exists* $a > 0$ *and* $(y_i) \in \ell^\infty_\Phi(\mathfrak{A}_i)$ *such that*

   - $\lim_{i\to\mathcal{U}} \|x_i - y_i\|^\#_{\Phi_i} < \epsilon$*; and*

   - $y_i \in M(\sigma^{\Phi_i}, [-a, a])$ *for all* $i \in I$*.*

3. $(x_i)^\bullet$ *is a* σ*-continuous vector.*

Proof. (1) $\iff$ (2). See Theorem 6.2.6.

(2) $\implies$ (3). Let $\epsilon > 0$. Take a and $y = (y_i)$ as in (2). Take $W \in \mathcal{U}$ so that if $i \in W$, then $\|\eta_\Phi(x_i - y_i)\|^\#_\Phi < \epsilon$. For all $t \in \mathbb{R}$ and $i \in W$, we have

$$\begin{aligned}
&\|\eta_\Phi(\sigma^\Phi_t(x_i) - x_i)\|^\#_\Phi \\
&\leq \|\eta_\Phi(\sigma^\Phi_t(x_i) - \sigma^\Phi_t(y_i))\|^\#_\Phi + \|\eta_\Phi(\sigma^\Phi_t(y_i) - y_i)\|^\#_\Phi + \|\eta_\Phi(y_i - x_i)\|^\#_\Phi && \text{by triangle ineq.} \\
&= 2\|\eta_\Phi(x_i - y_i)\|^\#_\Phi + \|\eta_\Phi(\sigma^\Phi_t(y_i) - y_i)\|^\#_\Phi && \text{since } \sigma^\Phi_t \text{ is isom.} \\
&< 2\epsilon + \|\eta_\Phi(\sigma^\Phi_t(y_i) - y_i)\|^\#_\Phi && \text{by assumption.}
\end{aligned}$$

Let $f \in L^1(\mathbb{R})$ satisfy $\widehat{f}(x) = 1$ if $|x| \leq a$. Then $\sigma^\Phi_f(y_i) = y_i$, and

$$\begin{aligned}
\|\eta_\Phi(\sigma^\Phi_t(y_i) - y_i)\|^\#_\Phi &= \|\eta_\Phi(\sigma^\Phi_{\lambda_t f - f}(y_i))\|^\#_\Phi \\
&\leq \|\lambda_t f - f\|_1 \|y_i\|^\#_\Phi
\end{aligned}$$

where λ_t denotes the left regular representation on $\mathbb{R}$. Therefore

$$\|\eta_\Phi(\sigma^\Phi_t(x_i) - x_i)\|^\#_\Phi < 2\epsilon + \|\lambda_t f - f\|_1 \|y_i\|^\#_\Phi$$

for all $t \in \mathbb{R}$, $i \in W$. Now, by dominated convergence, we can choose $\delta > 0$ such that $|t| < \delta$ implies $\|\lambda_t f - f\|_1 < \epsilon(\|y_i\|_\Phi^\#)^{-1}$. For such a δ, we have $\|\eta_\Phi(\sigma_t^\Phi(x_i) - x_i)\|_\Phi^\# < 3\epsilon$ for all $i \in W$, verifying σ-continuity.

(3) $\implies$ (1). Let $(x_i) \in \ell_\Phi^\infty(\mathcal{M})$ be a σ-continuous sequence. Let $\epsilon > 0$ and take $\delta > 0$ and $W \in \mathcal{U}$ so that if $|t| < \delta$ and $i \in W$, then we have $\|\sigma_t^\Phi(x_i) - x_i\|_\Phi^\# < \epsilon$.

For $r > 0$, set $g_r(t) := \sqrt{\frac{1}{\pi r}} e^{-t^2/r}$ for $t \in \mathbb{C}$. Note $g_r(t)$ is a Gaussian with L^1-norm of 1 for all $r > 0$. Furthermore, the smaller r is, the more concentrated $g_r(t)$ is around 0. Then there exists an r so that

$$\begin{aligned}\|\eta_\Phi(\sigma_{g_r}^\Phi(x_i) - x_i)\|_\Phi &\leq \int_{\mathbb{R}} g_r(t)\|\eta_\Phi(\sigma_t^\Phi(x_i) - x_i)\|_\Phi \, \mathrm{dt} \\ &< 2\epsilon\end{aligned}$$

for all $i \in W$. Now fix such an r and let $(y_i) \in \mathcal{I}_\mathcal{U}$ with $\|y_i\| \leq 1$ for all $i \in I$. Then for $i \in W$, we have

$$\begin{aligned}\|\eta_\Phi(y_i x_i)\|_\Phi &= \|y_i \eta_\Phi(x_i)\|_\Phi \\ &= \|y_i \eta_\Phi(x_i - \sigma_{g_r}^\Phi(x_i)) + y_i \eta_\Phi(\sigma_{g_r}^\Phi(x_i))\|_\Phi \\ &\leq \|y_i \eta_\Phi(x_i - \sigma_{g_r}^\Phi(x_i))\|_\Phi + \|y_i \eta_\Phi(\sigma_{g_r}^\Phi(x_i))\|_\Phi && \text{by triangle ineq.} \\ &\leq \|y_i\| \|\eta_\Phi(x_i - \sigma_{g_r}^\Phi(x_i))\|_\Phi + \|y_i \sigma_{g_r}^\Phi \eta_\Phi(x_i)\|_\Phi && \text{by def. of operator norm} \\ &< \epsilon + \|y_i \sigma_{g_r}^\Phi \eta_\Phi(x_i)\|_\Phi && \text{by choice of } r \text{ and } \|y_i\| \leq 1 \\ &= \epsilon + \|J_\Phi \sigma_{i/2}^\Phi(\sigma_{g_r}^\Phi(x_i))^* J_\Phi \eta_\Phi(y_i)\|_\Phi && \text{by Tomita's Theorem} \\ &\leq \epsilon + \|g_r(\cdot - i/2)\|_1 \|x_i\| \|\eta_\Phi(y_i)\|_\Phi && \text{by def. of } \sigma_f^\Phi(x).\end{aligned}$$

It now follows that $\lim_{i \to \mathcal{U}} \|\eta_\Phi(y_i x_i)\|_\Phi \leq \epsilon$ since $\|g_r(\cdot - i/2)\|_1 \leq 1$. By a similar proof $\lim_{i\to\mathcal{U}} \|\eta_\Phi(x_i y_i)\|_\Phi \le \epsilon$ which is what we required. □

Corollary 6.3.14. *An element $(x_i)^\bullet \in \ell^\infty_\Phi$ is in $\mathcal{N}_\mathcal{U}$ if and only if $(\Delta^{it}_{\Phi_i} x_i)^\bullet \in \mathcal{N}_\mathcal{U}$ for all $t \in \mathbb{R}$.*

Proof. First, note by translation invariance of continuous group homomorphisms, $(\sigma^{\Phi_i}_t(x_i))^\bullet$ is continuous at $t = 0$ if and only if it is continuous at all $t \in \mathbb{R}$. Thus since $\sigma^{\Phi_i}_t(x_i) = \pi(\Delta^{it} x_i)$, we conclude. □

Now we are ready to prove the equivalence between Hilbert algebra ultraproducts and generalized Ocneanu ultraproducts. The reader should notice at this point that, if we were working in the faithful normal state setting as in [3] and [49], we would be done. This is because, in that setting, every element of $\mathcal{M}$ is represented as a vector on the corresponding standard form and $\sigma_t(x) = \pi(\Delta^{it} x\omega)$. However, in our more general setting, we do not have a cyclic and separating vector, so we still need to take a strong closure. For this, we will make use of the corner characterization in Theorem 6.3.12.

It stands to reason now that an element $x \in \prod^{\mathcal{U}}_{\mathrm{GR}} \mathcal{M}_i$ is σ-continuous if and only if it sends σ-continuous vectors to σ-continuous vectors. This happens if and only if x commutes with p_Φ, which has already been seen to be equivalent to being an element of the Hilbert algebra ultraproduct. This is the essence of the proof to follow.

Theorem 6.3.15. *Let $x = (x_i)_\mathcal{U}$ be an element of the Groh-Raynaud ultraproduct where $x_i \in \mathcal{M}_i$ for all i. The following are equivalent.*

1. *$p_\Phi (x_i)^\bullet = (x_i)^\bullet p_\Phi$ so that x is an element of the Hilbert algebra ultraproduct.*

2. *$(x_i)^\bullet$ is a strong*-continuous point of $(\sigma^{\Phi_i}_t)_\mathcal{U}$ so that x is an element of the generalized Ocneanu ultraproduct.*

Proof. The (1) $\implies$ (2) direction follows immediately from Theorem 6.2.8 and the fact that the modular group is known to be strong*-continuous. In this case, the ultraproduct of the modular automorphism groups acts as the modular automorphism group of the weight induced on the Hilbert algebra ultraproduct. The latter agree with the ultraproduct weight on the Groh-Raynaud by the fact that p_Φ is an isometry.

Now we prove (2) $\implies$ (1). Assume $(x_i)^\bullet$ is a strong*-continuous point of $(\sigma_t^{\Phi_i})_{\mathcal{U}}$. Then for all $(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$, we have $(\sigma_t^{\Phi_i}(x_i)v_i)^\bullet$ strong* converges to $(x_iv_i)^\bullet$ as $t \to 0$.

Now

$$\begin{aligned}(\sigma_t^{\Phi_i}(x_i)v_i)^\bullet &= (\Delta_{\Phi_i}^{it}x_i\Delta_{\Phi_i}^{-it}v_i)^\bullet \\ &= (\Delta_{\Phi_i}^{it}x_i(\Delta_{\Phi_i}^{-it}v_i))^\bullet.\end{aligned}$$

We also have that $(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$ if and only if $(\Delta^{-it}v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$ by Theorem 6.3.13. So we have $(\Delta^{it}v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$. By parametrization we can replace $(\Delta^{it}v_i)^\bullet$ with v_i.

Thus $(\Delta^{it}x_iv_i)^\bullet$ strong* converges to $(x_iv_i)^\bullet$ as $t \to 0$ for all $(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$. Applying Theorem 6.3.13 again, we conclude $(x_i)^\bullet(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$ and $(x_i^*)^\bullet(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$ for all $(v_i)^\bullet \in \mathcal{N}_{\mathcal{U}}$. This is equivalent to $p_\Phi(x_i)^\bullet = (x_i)^\bullet p_\Phi$. Thus, we are done. □

Corollary 6.3.16. *The generalized Ocneanu ultraproduct is a von Neumann subalgebra of the Groh-Raynaud ultraproduct that is the image of a faithful normal conditional expectation.*

Proof. By Theorem 6.3.15 and Proposition 5.5.1, the generalized Ocneanu ultraproduct is a von Neumann subalgebra.

By Theorem 6.2.8 and since the ultraproduct weight is the restriction of the ultraproduct weight on the Groh-Raynaud ultraproduct, the modular automorphism group on the generalized Ocneanu ultraproduct agrees with the modular automorphism group on the Groh-Raynaud ultraproduct. Now since continuous elements are invariant under $\sigma_t^{\Phi^{\mathcal{U}}}$, we have by Takesaki's Theorem (Theorem 4.5.4), the desired conditional expectation. □

Thus, by Proposition 6.2.5 and Theorem 6.3.15, we conclude the ultimate result of this chapter.

Theorem 6.3.17. *For any family $(\mathcal{M}_i, \Phi_i)$ of weighted von Neumann algebras, the generalized Ocneanu ultraproduct acting on $p_\Phi(\prod^{\mathcal{U}} \mathcal{H}_i)$ is spatially isomorphic to the Hilbert algebra ultraproduct of the corresponding family of full Hilbert algebras $(\mathfrak{A}_i, \mathcal{H}_i)$. By transitivity, both of these are isomorphic to the model theoretic ultraproduct in our language.*

Given the evident utility of the Ocneanu ultraproduct in the study of σ-finite von Neumann algebras, there is now a wide array of obvious interesting questions to be asked and generalizations of known theorems to be proved. In the chapters to follow, we will focus on generalizations of some computable model theoretic results. We leave potentially many further developments with a more operator algebraic flavor to future work.

Chapter 7

Connes-Takesaki Decomposition and Imaginaries

7.1 Overview

An important landmark in the history of operator algebras was the 1976 publication of Alain Connes' classification of injective factors in [13]. Connes and this paper of his have inspired much of the work in operator algebras since then, in particular, the Elliot classification program for C*-algebras. Since a significant amount of the work in model theory of C*-algebra is related to the Elliot program, it is an interesting question to what extent model theory is already reflected in Connes' work in the von Neumann algebra setting. In this chapter, we will begin to answer this question by showing that the model theory of von Neumann algebras "remembers" the Connes-Takesaki decomposition, an important piece of the injective factors proof. To this end, along the way, we will develop a bit more of the definability theory for continuous logic. Much of the treatment here is based on [1].

7.2 Definability and Definable Groupoids

Definition 7.2.1. We define a **metric groupoid** to be a groupoid enriched in Met.

In other words, a groupoid C is a metric groupoid if $\mathrm{Ob}(C)$ forms a metric space and for any $a, b \in \mathrm{Ob}(C)$, we have the structure of a metric space on $\mathrm{Hom}(a, b)$. Notice this coincides with the usual notion of a groupoid if $\mathrm{Ob}(C)$ and $\mathrm{Hom}(a, b)$ has the discrete metric for all pairs of objects. Denote by $\mathrm{Mor}(C)$ the disjoint union of all the Hom sets. This naturally has a metric space structure.

Definition 7.2.2. We say a metric groupoid is **definable** in $\mathcal{M}$ if there exist isometries from $\mathrm{Ob}(C)$ and $\mathrm{Mor}(C)$ to definable sets A and B of $\mathcal{M}$ such that the source and target functions $s, t : B \to A$ and the composability relation comp on $B \times B$ are definable and the composition function $\circ : (B \times B) \to B$ is also definable.

We will simply call these definable groupoids for the remainder of this thesis. It should be noted that definable groupoids are much rarer in continuous logic than in classical logic. This is because continuous structures often have a paucity of definable sets. This same paucity of definable sets and the related difficulty in identifying which functions can be realized as a T-formula results in the continuous Beth definability theorem playing a much more central role in continuous logic than Beth definability does in classical logic. See, for example, [30, Chapter 2] for a discussion.

Theorem 7.2.3 (Continuous Beth Definability)**.** *Suppose that $\mathcal{L}' \subseteq \mathcal{L}$ are two continuous languages with the same sorts. Further, suppose T is an $\mathcal{L}$-theory. If the forgetful functor $F : \mathrm{Mod}(T) \to \mathrm{Str}(\mathcal{L}')$ given by restriction to $\mathcal{L}'$ is an equivalence of categories onto the image of F, then every $\mathcal{L}$-formula is T-equivalent to an $\mathcal{L}'$-formula.*

Considering the aforementioned centrality of the Beth definability theorem, it is remarkable that, to our knowledge, there is no attempt in the literature to prove the continuous logic analogues of the Svenonius or Chang-Makkai definability theorems. We will initiate these tasks now, to aid our discussion of the Connes-Takesaki decomposition, as well as for their independent interest to continuous model theorists more generally. Recall that in continuous logic, being definable is different from being expressible as a formula. It is worthwhile to define the following.

Definition 7.2.4. We say that an $\mathcal{L}'$ formula ϕ is **T-expressible in $\mathcal{L}$ up to disjunction** if there is a finite set $\theta_1, ..., \theta_n$ of $\mathcal{L}$-formulae such that

$$T \vDash \sup_x \left(\min_{1 \leq i \leq n} |\phi(\overline{x}) - \theta_i(\overline{x})| \right)$$

Definition 7.2.5. We say that an $\mathcal{L}'$ formula ϕ is **T-expressible in $\mathcal{L}$ up to parameters** if there is an $\mathcal{L}$-formula $\theta_1(\overline{x}, v_1...., v_n)$ with parameters $v_1, ..., v_n$ such that

$$T \vDash \sup_{\overline{x}} \left(\inf_{v_1,...,v_n} |\phi(\overline{x}) - \theta_i(\overline{x}, v_1, ..., v_n)| \right)$$

We begin with the following strengthening of Beth definability.

Theorem 7.2.6 (Continuous Svenonius' Theorem)**.** *Let U be a predicate symbol not in $\mathcal{L}$. Let ϕ be a sentence of $\mathcal{L} \cup \{U\}$. The following are equivalent:*

1. *For every model $\mathfrak{A}$ for $\mathcal{L}$, if $(\mathfrak{A}, P_1)$ and $(\mathfrak{A}, P_2)$ are expansions of $\mathfrak{A}$ to models of ϕ and $(\mathfrak{A}, P_1) \cong (\mathfrak{A}, P_2)$, then $P_1(\overline{x}) = P_2(\overline{x})$ for all $\overline{x} \in \mathfrak{A}^{\overline{x}}$.*

2. *U is T-expressible up to disjunction.*

Proof. (2) $\implies$ (1). Suppose that $\theta_1(\overline{x}), ..., \theta_k(\overline{x})$ are formulae of $\mathcal{L}$ such that

$$\phi \vDash \min_{1 \leq i \leq n} (\forall \overline{x}) |U(\overline{x}) - \theta_i(\overline{x})|.$$

Let $(\mathfrak{A}, P)$ be an expansion of $\mathfrak{A}$ which is a model of ϕ. This means that for some i, we have

$$P(\overline{a}) = \theta_i(\overline{a}) \text{ for all } \overline{a} \in \mathfrak{A}^{\overline{x}}.$$

So P remains fixed under any automorphism of $\mathfrak{A}$, and, in particular, under any isomorphism $(\mathfrak{A}, P)$ onto $(\mathfrak{A}, P')$.

(1) $\implies$ (2). Suppose that (2) does not hold. In other words, for no formulae $\theta_1(\overline{x}), ..., \theta_k(\overline{x})$ of $\mathcal{L}$ does

$$\phi \vDash \min_{1 \leq i \leq n} (\forall \overline{x}) |U(\overline{x}) - \theta_i(\overline{x})|$$

hold. Then there exists for every formula θ of $\mathcal{L}$, an $\epsilon_\theta > 0$ such that the set

$$\Sigma = \{\phi\} \cup \{\sup_{\overline{x}} |U(\overline{x}) - \theta(\overline{x}))| > \epsilon_\theta \ : \ \theta(\overline{x}) \text{ a formula of } \mathcal{L}\}$$

of sentences of $\mathcal{L} \cup \{U\}$ is consistent. Let T be any complete extension of Σ in $\mathcal{L} \cup \{U\}$. Note that T does not define U explicitly as there is no formula $\theta(\overline{x})$ of $\mathcal{L}$ such that $T \vDash \sup_{\overline{x}} |U(\overline{x}) - \theta(\overline{x})|$, whence by Beth definability, there is a model $\mathfrak{A}$ of $\mathcal{L}$ which has two different expansions into models of T:

$$(\mathfrak{A}, P_1) \vDash T, \qquad (\mathfrak{A}, P_2) \vDash T, \qquad P_1 \neq P_2.$$

Since T is complete, we also have

$$(\mathfrak{A}, P_1) \equiv (\mathfrak{A}, P_2).$$

Consider the model $(\mathfrak{A}, P_1, P_2)$. Let $(\mathfrak{B}, Y_1, Y_2)$ be a saturated extension of it in the language $\mathcal{L} \cup \{U, U'\}$. Clearly $(\mathfrak{B}, Y_1)$ and $(\mathfrak{B}, Y_2)$ are saturated, of the same cardinality and equivalent. So, because there exists ϵ such that

$$(\mathfrak{B}, Y_1, Y_2) \vDash \sup_{\overline{x}} |U(\overline{x}) - U'(\overline{x})| > \epsilon$$

we see that

$$(\mathfrak{B}, Y_1) \cong (\mathfrak{B}, Y_2), \qquad (\mathfrak{B}, Y_1, Y_2) \nvDash \sup_{\overline{x}} |U(\overline{x}) - U'(\overline{x})|$$

so $Y_1 \neq Y_2$, contradicting (1). □

We will not need to use the following continuous logic analogue of the Chang-Makkai theorem at any point in this thesis. For this reason, we will not include the proof. The proof will instead appear, alongside various elaborations of the contents of this section, in joint work with Bradd Hart.

Our reason for stating the theorem is for the role the classical Chang-Makkai theorem plays in Hrushovski's "generalized imaginaries". The study of generalized imaginaries is important in model theory. We suggest the paper [42] of Moosa-Haykazyan for more details on generalized imaginaries and their connection to definable groupoids. By studying definable metric groupoids in this thesis, we are thereby working with a continuous logic analogue of generalized imaginaries, so having a continuous logic analogue analogue of Chang-Makkai serves as a useful proof of concept for our work here.

Theorem 7.2.7 (Continuous Chang-Makkai). *Let ϕ be a sentence in $\mathcal{L} \cup \{U\}$. The following are equivalent.*

1. *For all infinite models $(M, P) \models \phi$ we have*

$$\chi(\{Q \ : \ (M,P) \cong (M,Q)\}) \leq \chi(M) := \kappa$$

   Where $\{Q \ : \ (M,P) \cong (M,Q)\}$ is considered as a subset of $[0,1]^M$ equipped with the metric arising from the sup-norm.

2. *ϕ is equivalent to the zeroset of a T-formula with parameters.*

7.3 Connes-Takesaki Decompositions

We now turn our attention to the model theoretic treatment of Connes-Takesaki decompositions.

Theorem 7.3.1. *[65] Let $\mathcal{M}$ be a von Neumann algebra of type III. Then there is a type II_∞ von Neumann algebra $\mathcal{N}$ with a faithful normal semifinite trace τ on $\mathcal{N}$ and a trace-scaling one-parameter group of automorphisms $\theta : \mathbb{R} \to \mathrm{Aut}(\mathcal{N})$ such that*

$$\mathcal{M} \cong \mathcal{N} \rtimes_\theta \mathbb{R}.$$

Furthermore, $(\mathcal{N}, \tau, \theta)$ is unique up to conjugacy.

$\mathcal{N}$ is often referred to as the **continuous core** of $\mathcal{M}$. If Φ is a faithful normal semifinite weight on $\mathcal{M}$, we can construct $(\mathcal{N}, \tau, \theta)$ as

$$\mathcal{N} \cong \mathcal{M} \rtimes_{\sigma^\Phi} \mathbb{R}$$

with τ the dual weight of Φ and θ the dual action of σ^Φ as given in Takesaki duality. We will then denote $\mathcal{N} = c_\Phi(\mathcal{M})$.

Theorem 7.3.2. *[66] Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra together with a faithful normal state and an action α by topological abelian group G. Consider also $\mathcal{M} \rtimes_\alpha G$ together with its dual action β by the dual group $\hat{G}$ and its dual weight ψ. Then the continuous points (with respect to β) of the Ocneanu ultrapower $(\prod^{\mathcal{U}} \mathcal{M} \rtimes_\alpha G)_c$ is isomorphic to the crossed product of the continuous part of the Ocneanu ultrapower $(\prod^{\mathcal{U}} \mathcal{M})$ by the ultraproduct action $\alpha^{\mathcal{U}}$.*

In a future work, we will further extend this result to general ultraproducts and with respect to more general groups. For our purposes though, we only need to note that we will work exclusively with $\mathbb{R}$ and the modular automorphism group. Since $\mathbb{R}$ is dual to itself and the Ocneanu ultraproduct is precisely, the continuous ultraproduct with respect to the modular automorphism group, we can copy Tomatsu's proof nearly exactly to get that Ocneanu ultraproducts commute with crossed products by the modular automorphism group.

Theorem 7.3.3. *$(c_\Phi(\mathcal{M}), \tau, \theta)$ as given above is definable in $(\mathcal{M}, \Phi)^{\mathrm{geq}}$.*

Proof. Consider the language $\mathcal{L}_{\text{vNa-Pairs}}$ for structures of the form $(\mathcal{M}, \Phi, \sigma, \mathcal{N}, \tau, \theta)$ where $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \tau)$ are weighted von Neumann algebras and σ and θ are one-parameter automorphism groups on $\mathcal{M}$ and $\mathcal{N}$ respectively.

Consider the class of structures in $\mathcal{L}_{\text{vNa-Pairs}}$ where $(\mathcal{M}, \Phi)$ and $(\mathcal{N}, \tau)$ satisfy the axioms of weighted von Neumann algebras, τ is tracial and σ^{Φ} is defined to be the modular automorphism group of $(\mathcal{M}, \Phi)$. This is axiomatizable by the discussion in Chapters 5 and 6. We claim that the subclass such that $\mathcal{N} = c_{\Phi}(\mathcal{M})$ and τ is the dual weight of Φ with θ the dual weight of σ^{Φ} is elementary. This follows from Theorem 7.3.2.

Now consider the reduct functor to the structures $(\mathcal{M}, \Phi)$. We claim this induces an equivalence of categories. This is true because σ is definable by the discussion in Chapter 5, and $\mathcal{N}$ is unique up to automorphism. The latter is true because $\mathcal{N}$ is unique up to unitary equivalence and since by [37] all automorphisms are implemented by unitaries in standard forms. Now the theorem follows by Beth definability. $\square$

A problem one runs into when trying to recover the underlying II_1 factor $\mathcal{M}$ from $\mathcal{M} \otimes B(\mathcal{H})$ is that there are many isomorphs of $\mathcal{M}$ contained therein. A parallel issue arises in the model theoretic study of differential fields. There, the Galois group is not canonically a definable group. Instead, there are multiple isomorphs of the Galois group that are related by a system of isomorphisms. Thus a need arises for a so-called binding groupoid (see, for example [42]).

If $\mathcal{N} = \mathcal{M} \otimes B(\mathcal{H})$ with the tracial weight τ is the II_∞ amplification of the II_1 factor $\mathcal{M}$ with trace τ_0, then there are copies of $\mathcal{M}$ associated to any trace 1 projection on $\mathcal{N}$. We will view this as a definable groupoid. The morphisms $\mathcal{M}_1 \to \mathcal{M}_2$ in this setting will be the partial isometry v with source projection $p_{\mathcal{M}_1}$ and range projection $p_{\mathcal{M}_2}$ if it exists in $\mathcal{N}$. We will show that this is a definable groupoid.

It is well-known that the trace 1 projections forms a (quantifier-free) definable set

A in $\mathcal{N}$ by

$$\max\{\|p^2 - p\|, \|p^* - p\|, |\tau(p) - 1|\}.$$

Denote by P_1 the definable set corresponding to the above. Taking the corner $p\mathcal{N}p$ for any trace 1 projection obviously gives a copy of $\mathcal{M}$. We will show now that the morphism set B is (existentially) definable. Consider the formula:

$$\max\{\inf_{p \in P_1} \|x^*x - p\|, \inf_{q \in P_1} \|xx^* - q\|\}.$$

Putting this all together yields the following theorem.

Theorem 7.3.4. *A and B defined above forms a definable groupoid. The source and target functions are the terms x^*x and xx^* respectively. The composability relation* comp *is given by* $\text{comp}(x, y) = \|x^*x - yy^*\|$ *and the composition function is given by* $\circ(x, y) = yx^*xy^*$.

Proposition 7.3.5. *The above formulae are almost-near and hence form definable sets.*

The next theorem can be summarized as saying that $(\mathcal{N}, \tau)$ is contained in the "generalized imaginaries" of our theory. A more thorough treatment of generalized imaginaries will appear in joint work with Bradd Hart.

Theorem 7.3.6. *There is a definable groupoid in $(\mathcal{M}, \Phi)$ such that every object is isomorphic to $(\mathcal{N}, \tau)$.*

We remark that the preceding theorem could have instead been seen to be an application of Theorem 7.2.7. However, we find our explicit construction of the definable groupoid more intuitive. More importantly, our explicit definition, and its evident computability, play a key role in our analysis of the universal theory of the hyperfinite II_∞ factor in Theorem 8.6.1.

A similar metric groupoid to the above can be defined where in the place of partial isometries, we use "partial unitaries". Instead of requiring the source and target to be projections then, we assume they are unitaries on the corresponding corners. Notice this will be much larger as a groupoid in general than the former definition. The former is a skeletal groupoid and the latter will in general have large Hom sets.

Theorem 7.3.3 and Theorem 7.3.6 together imply that the model theory type III von Neumann algebras retains information about their Connes-Takesaki decompositions.

Chapter 8

Undecidability of QWEP and Type III

8.1 Overview

The starting point for computable model theory of operator algebras was the paper [33] by Goldbring and Hart. Following the resolution of the Connes embedding problem in [43], it was discovered by Goldbring and Hart that some of the techniques therein can be used to prove that the universal theory of the hyperfinite II_1-factor $\mathcal{R}$ is undecidable. In turn, this implies an incredible strengthening of the refutation of the Connes embedding problem.

In [4], the present author, together with Goldbring and Hart, used the framework and results from [33] to prove various undecidability results about other hyperfinite factors as well as the classes of QWEP C*-algebras and QWEP W*-probability spaces. The results there on W*-probability spaces are proved in both the language given in [16] and that given in [35]. Here, we prove these results in a third language, that of

W*-probability spaces introduced in [5] or more generally, the language of weighted von Neumann algebras introduced in this thesis. Throughout this chapter, we will often use one name to refer to both a weighted von Neumann algebra and the metric structure representing it. This is unproblematic by our work in Section 5.5.

We point out that the languages we introduced here and in [5] serve to simplify some of the proofs in [4]. In particular, the proof of uncomputability of $\mathrm{Th}_\forall(\mathcal{R}_\lambda)$ for $\lambda \in (0,1)$ there involved introducing various types of functions, Fourier theory and smearing techniques on the modular automorphism group. The proof given here involves essentially only functional calculus for bounded operators and Takesaki's theorem (see Theorem 4.5.4).

Finally, we will use some of the technology introduced in Chapter 7 to show some new uncomputability results that were not possible in the faithful normal state setting.

8.2 Embedding Problems

In this section, we will introduce the embedding problem framework developed in [33] and some of the main theorems proved there.

Definition 8.2.1. [33, Definition 2.1] Given an $\mathcal{L}$-structure $\mathcal{M}$, we say the theory of $\mathcal{M}$ is **computable** if there is an algorithm that takes as input a restricted sentence σ and a positive rational δ and returns rationals $a < b$ with $b - a < \delta$ such that $\sigma^{\mathcal{M}} \in (a, b)$. We can obviously extend this definition to subsets of the theory such as the universal theory $\mathrm{Th}_\forall$ or the two quantifier theories $\mathrm{Th}_{\forall\exists}$ and $\mathrm{Th}_{\exists\forall}$.

We also define the much stronger notion of a decidable theory, corresponding to another view of theories, even though we will not use it much. The next two definitions are [33, Definition 2.2].

Definition 8.2.2. A theory T is **decidable** if there is an algorithm which takes as input a restricted sentence and outputs whether or not the sentence is in T.

Definition 8.2.3. A theory is **effectively enumerable** if there is an algorithm which enumerates the restricted sentences in T.

Definition 8.2.4. [33, Definition 2.5] Given an $\mathcal{L}$-structure M, we say the universal theory of M is **weakly effectively enumerable** if there is an algorithm which enumerates sentences of the form $\sigma \dot{-} r$ where σ is a restricted universal sentence, $r > 0$ is rational and $\sigma^M \leq r$.

The next definition is [33, Definition 5.1] and provides the key framework which we will use throughout the rest of this chapter.

Definition 8.2.5. Given a structure $\mathcal{M}$ in a language $\mathcal{L}$, we say that $\mathcal{M}$EP has a positive solution if there exists an effectively enumerable subset $T \subseteq \text{Th}(\mathcal{M})$ such that, for any $\mathcal{L}$-structure $\mathcal{N}$, if $\mathcal{N} \vDash T$, then $\mathcal{N}$ embeds into some ultrapower of $\mathcal{M}$.

Remark 8.2.6. [33, Remark 5.4] CEP is a weakening of $\mathcal{R}$EP, where $T \subseteq \text{Th}(\mathcal{R})$ would be taken to be the theory of II_1-factors. Therefore the fact that $\mathcal{R}$EP has a negative solution (see below) is a strengthening of the failure of CEP.

Proposition 8.2.7. *[33, Theorem 5.2] If $\mathcal{M}$EP has a positive solution, then* $\text{Th}_\forall(\mathcal{M})$ *is weakly effectively enumerable.*

Theorem 8.2.8. *[33, Corollary 5.3] $\mathcal{R}$EP has a negative solution.*

In fact, we have an even stronger result.

Theorem 8.2.9. *[33, Remark 5.4] There is no effectively enumerable theory T extending the theory of II_1-factors with the property that every model of T embeds into some ultrapower of $\mathcal{R}$.*

8.3 The Undecidability of QWEP

In [27], Goldbring shows that the class of C*-algebras with the QWEP is axiomatizable in the language of C*-algebras. The proof there is abstract, and thus no explicit axiomatization of this class is given. The class of QWEP C*-algebras is the same as the class of all C*-algebras if and only if the QWEP conjecture or, equivalently, the Connes embedding problem is affirmative. Thanks to [43], we know this is not the case. We show that, in fact, from a computability theoretic perspective, the class of QWEP C*-algebras is wildly different from the class of all C*-algebras. We will see that while the latter has an explicit computable axiomatization, the former admits no such axiomatization. We actually prove the much stronger result given in Theorem 8.3.2. This can serve as a post-hoc explanation for the lack of explicit axiomatization in [27]. We note that this section uses only the classical languages of C*-algebras and tracial von Neumann algebras.

We begin by recalling the following standard definition from C*-algebras.

Definition 8.3.1. Given $m \in \mathbb{N}$ and $0 < \gamma < 1$, we say that a unital C*-algebra $\mathcal{A}$ has the (m, γ)**-uniform Dixmier property** if, for all self-adjoint $a \in \mathcal{A}$, there are unitaries $u_1, \ldots, u_m \in U(\mathcal{A})$ and $z \in Z(\mathcal{A})$ such that

$$\left\| \sum_{i=1}^{m} \frac{1}{m} u_i a u_i^* - z \right\| \leq \gamma \|a\|.$$

We say that $\mathcal{A}$ has the **uniform Dixmier property** if it has the (m, γ)-Dixmier property for some m and γ.

This notion plays an important role in our proof for a constellation of reasons. Given m and γ, let $\theta_{m,\gamma}$ denote the following sentence:

$$\sup_a \inf_{u_1,\ldots,u_n} \inf_\lambda \max \left(\max_{i=1,\ldots,n} \|u_i u_i^* - 1\|, \| \sum_{i=1}^{m} \frac{1}{m} u_i a u_i^* - \lambda \| \ \dot{-}\ \gamma \|a\| \right)$$

in the language of C*-algebras. Here, the supremum is over self-adjoint contractions, the first infimum is over contractions, and the second infimum is over the unit disk in $\mathbb{C}$.

As mentioned in [33, Section 6], if $\mathcal{A}$ is a simple unital C*-algebra with the (m, γ)-uniform Dixmier property, then $\theta_{m,\gamma}^{\mathcal{A}} = 0$. On the other hand, if $\theta_{m,\gamma}^{\mathcal{A}} = 0$, then $\mathcal{A}$ is monotracial. The property of being monotracial is useful because monotracial QWEP C*-algebras admit a trace-preserving embedding in $\mathcal{R}^{\mathcal{U}}$ in a way that is compatible with the GNS construction. We will see this in the proof to follow.

Theorem 8.3.2. *[4, Theorem 2.1] There is no effectively enumerable theory T in the language of* C**-algebras with the following two properties:*

1. *All models of T have QWEP; and*

2. *There is an infinite-dimensional, simple model $\mathcal{A}$ of T that admits a trace and has the uniform Dixmier property.*

Proof. Suppose, towards a contradiction that T is such a theory. Take a model $\mathcal{A}$ of T as in the second condition in the statement of the theorem and fix a trace $\tau_{\mathcal{A}}$ on $\mathcal{A}$. Take m and γ such that $\mathcal{A}$ has the (m, γ)-Dixmier property. Consider the theory $T' = T \cup \{\theta_{m,\gamma} = 0\}$ in the language of tracial C*-algebras. It is clear that T' is effective and $(\mathcal{A}, \tau_{\mathcal{A}}) \models T'$.

Suppose that $(B, \tau_{\mathcal{B}}) \models T'$ and let $\mathcal{N} = \mathcal{B}''$ in the GNS representation with respect to $\tau_{\mathcal{B}}$. Since $\mathcal{B}$ has QWEP, $\mathcal{N}$ is a QWEP von Neumann algebra. Furthermore, since $\mathcal{B}$ is monotracial, $\mathcal{N}$ is a II_1 factor with unique trace. Therefore $\mathcal{N}$ admits a trace-preserving embedding into $\mathcal{R}^{\mathcal{U}}$. Since we have a sequence of embeddings

$$(\mathcal{R}, \tau_{\mathcal{R}}) \hookrightarrow (\mathcal{N}, \tau_{\mathcal{N}}) \hookrightarrow (\mathcal{R}^{\mathcal{U}}, \tau_{\mathcal{R}^{\mathcal{U}}}),$$

it follows that for any universal sentence σ in the language of tracial C*-algebras $\sigma^{(\mathcal{N}, \tau_{\mathcal{N}})} = \sigma^{(\mathcal{R}, \tau_{\mathcal{R}})}$. On the other hand, since $\mathcal{B}$ is simple, $\mathcal{B}$ embeds into $\mathcal{N}$ in a trace-preserving way. Furthermore, this embedding has SOT-dense image, so $\sigma^{(\mathcal{B}, \tau_{\mathcal{B}})} = \sigma^{(\mathcal{N}, \tau_{\mathcal{N}})}$. Now we can enumerate proofs from T' and, using the completeness theorem, find computable upper bounds on the values of $\sigma^{(\mathcal{R}, \tau_{\mathcal{R}})}$, contradicting Theorem 8.2.9. □

Corollary 8.3.3. *[4, Corollary 2.2] There is no effective theory T in the language of* C**-algebras such that a* C**-algebra has QWEP if and only if it is a model of T.*

We can now prove a "non-closure" under ultraproducts result for the class of C*-algebras without QWEP:

Corollary 8.3.4. *[4, Corollary 2.3] The class of* C**-algebras without the QWEP is not closed under ultraproducts.*

Proof. Suppose, towards a contradiction, that the class of C*-algebras without the QWEP is closed under ultraproducts. Let T_{C^*} denote the (effective) theory of C*-algebras. Since the class of C*-algebras with QWEP is axiomatizable and the language of C*-algebras is separable, there is a sentence σ_{QWEP} in the language of C*-algebras such that a C*-algebra $\mathcal{A}$ has QWEP if and only if $\sigma^{\mathcal{A}}_{QWEP} = 0$. Our contradiction assumption implies that there is some $r > 0$ such that $\sigma^{\mathcal{A}}_{QWEP} \geq r$ for all C*-algebras without the QWEP. Without loss of generality, $r \in \mathbb{Q}$. Since the language of C*-algebras is computable and the set of computable sentences is dense in the set of all sentences (see [33, Section 2]), there is a computable sentence ψ such that $d(\sigma_{QWEP}, \psi) < \frac{1}{3}$ in the usual metric on formulae. It then follows that $T_{C^*} \cup \{\psi \dot{-} \frac{r}{3}\}$ is an effective axiomatization of the class of C*-algebras with QWEP, contradicting Corollary 8.3.3. □

Using the analog of Corollary 8.3.3 for the class of tracial von Neumann algebras proven in [33], the exact same line of reasoning shows the following:

Corollary 8.3.5. *[4, Corollary 2.4] The class of tracial von Neumann algebras that do not admit a trace-preserving embedding in an ultrapower $\mathcal{R}^{\mathcal{U}}$ of the hyperfinite II_1 factor is not closed under ultraproducts.*

Recall the following definition.

Definition 8.3.6. A C*-algebra $\mathcal{A}$ is **pseudo-nuclear** if it is a model of the common theory of the class of nuclear C*-algebras. Alternatively, $\mathcal{A}$ is pseudo-nuclear if it is elementarily equivalent to an ultraproduct of nuclear C*-algebras.

[23, Problem 7.3.3.] asks if there is a natural characterization of the class of pseudo-nuclear C*-algebras. Our next corollary may be considered a negative solution to this problem if we take ”natural characterization” to mean effective axiomatization.

Corollary 8.3.7. *[4, Corollary 2.5] The elementary class of pseudo-nuclear* C*-*algebras is not effectively axiomatizable.*

Proof. This follows immediately from Theorem 8.3.2 together with the fact that pseudo-nuclear C*-algebras are QWEP. □

8.4 Failure of the $\mathcal{R}_\infty$EP

The following is a direct analogue of [4, Theorem 4.3] in our new language and the proof is nearly identical. Recall the notation $\mathcal{R}_\infty$ denoting the hyperfinite III_1 factor from Theorem 3.5.6.

Theorem 8.4.1. *The $\mathcal{R}_\infty EP$ is false in the language $\mathcal{L}_{\text{vNa}}$.*

Proof. Suppose, towards a contradiction, that the $\mathcal{R}_\infty EP$ is true as witnessed by the $\mathcal{L}_{\text{vNa}}$-theory $T \subseteq \text{Th}(\mathcal{R}_\infty)$. T is an effectively enumerable set of sentences by assumption. Now consider

$$T' := \{\bar{\theta} \ \dot{-}\ \epsilon \ : \ \epsilon \in \mathbb{Q},\ \theta \text{ is universal, and } T \vdash \theta \ \dot{-}\ \epsilon\}.$$

Note that T' is an effectively enumerable set of sentences in the language of tracial von Neumann algebras. Moreover, we have $T' \subseteq \text{Th}(\mathcal{R})$. Since $\mathcal{R}$ embeds in $\mathcal{R}_\infty^{\mathcal{U}}$ (as $\mathcal{L}_{\text{vNa}}$-structures), for any universal $\mathcal{L}_{\text{vNa}}$-sentence θ, we have that

$$\bar{\theta}^{\mathcal{R}} = \theta^{\mathcal{R}} \leq \theta^{\mathcal{R}_\infty}.$$

Moreover, if $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra which is a model of T', then it embeds, as a W*-probability space, into a model of T, which is QWEP by assumption. It follows that $\mathcal{M}$ is QWEP and thus admits a trace-preserving embedding into $\mathcal{R}^{\mathcal{U}}$. Consequently, T' witnesses that the $\mathcal{R}$EP has a positive solution, which is a contradiction. □

A fortiori, we conclude the following.

Theorem 8.4.2. *The universal theory of the hyperfinite III_1 factor* $\mathrm{Th}_\forall(\mathcal{R}_\infty)$ *is not effectively enumerable.*

A W*-probability space is QWEP if and only if it is a model of the universal theory of $\mathcal{R}_\infty$. Thus, we can conclude the following.

Theorem 8.4.3. *There is no effectively enumerable set of sentence in the language of* W*-*probability spaces that axiomatizes precisely the class of QWEP* W*-*probability spaces.*

Moreover, arguing just as in the case of Corollary 8.3.4, we have the following (see [4, Corollary 4.4]):

Corollary 8.4.4. *The class of* W*-*probability spaces without the QWEP is not closed under ultraproducts.*

8.5 Failure of the $\mathcal{R}_\lambda$EP

Our goal in this section is to show that, for any $\lambda \in (0, 1)$, the $(\mathcal{R}_\lambda, \varphi_\lambda)EP$ has a negative solution in the language $\mathcal{L}_{\mathrm{vNa}}$, where φ_λ is the Powers state on $\mathcal{R}_\lambda$.

Given a W*-probability space $(\mathcal{M}, \varphi)$, the **centralizer** of φ is

$$\begin{aligned}\mathcal{M}_\varphi &:= \{x \in \mathcal{M} \ : \ \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R}\} \\ &= \{x \in \mathcal{M} \ : \ \varphi(xy) = \varphi(yx) \text{ for all } y \in \mathcal{M}\}.\end{aligned}$$

Notice that $\mathcal{M}_\varphi$ is a finite von Neumann algebra with trace $\varphi|\mathcal{M}_\varphi$. The unit ball of $\mathcal{M}_\varphi$ is a zeroset in $(\mathcal{M}, \varphi)$, namely the zeroset of the quantifier-free formula $d(\mathbf{R}(x), x)$.

Definition 8.5.1. A faithful normal state φ on $\mathcal{M}$ is said to be **lacunary** if 1 is an isolated point of the spectrum of the modular operator Δ_φ.

Example 8.5.2. Suppose that $\mathcal{M}$ is a type III_λ factor and φ is a **periodic** faithful normal state on $\mathcal{M}$ with period $\frac{2\pi}{|\log(\lambda)|}$. Then $\sigma(\Delta_\varphi) \subseteq \{0\} \cup \lambda^{\mathbb{Z}}$. In particular, φ is a lacunary state on $\mathcal{M}$. Moreover, in this case, by a result of Connes (see [12, Theorem 4.2.6]), $\mathcal{M}_\varphi$ is a II_1 factor. In the special case of $\mathcal{R}_\lambda$ with the Powers state φ_λ, we have that $(\mathcal{R}_\lambda)_\varphi$ is the hyperfinite II_1 factor $\mathcal{R}$.

Proposition 8.5.3. *[3, Proposition 4.27] If φ is a lacunary faithful normal state on $\mathcal{M}$, then $(\mathcal{M}^\mathcal{U})_{\varphi^\mathcal{U}} = (\mathcal{M}_\varphi)^\mathcal{U}$.*

The following model theoretic consequence is immediate (see [4, Proposition 5.3]).

Proposition 8.5.4. *If φ is a lacunary faithful normal state on $\mathcal{M}$, then the unit ball of $\mathcal{M}_\varphi$ is a definable subset of the unit ball of $\mathcal{M}$.*

We remark that Ando-Haagerup state Proposition 8.5.3 for a faithful normal semifinite weight. However, for a few reasons, only this version of the statement for faithful normal states applies to our setting. Firstly, they take the Ocneanu ultrapower with respect to some arbitrary faithful normal state ψ and then consider an ultrapower weight on that. Since Ocneanu ultrapowers are independent of the choice of state and our definition of an ultrapower state matches theirs, this is fine in the state case. On the other hand, their definition of ultrapower weight does not agree with ours in general. They also do not define an Ocneanu ultrapower with respect to a weight. We will later show that a more germane version of this result holds in our setting, but for the moment, we will only use the state case.

If φ is a lacunary faithful normal state on $\mathcal{M}$, we say that φ is **λ-lacunary** if $\lambda \in (0,1)$ is such that $\sigma(\Delta_\varphi) \cap (\lambda, \frac{1}{\lambda}) = \{1\}$. (This terminology seems to be nonstandard but convenient.) In particular, note that if φ is a periodic faithful normal state on $\mathcal{M}$ with period $\frac{2\pi}{|\log(\lambda)|}$, then φ is λ-lacunary.

Note that $\Delta_\varphi = \mathbf{R}_\varphi^{-1}(2 - \mathbf{R}_\varphi)$ and that the spectra of both $\mathbf{R}_\varphi$ and $(2 - \mathbf{R}_\varphi)$ are both contained in $[0,2]$. By functional calculus, we have that Δ is λ-lacunary if and only if $\mathrm{sp}(\mathbf{R}_\varphi) \cap (2 - \frac{2}{\lambda+1}, \frac{2}{\lambda+1}) = \{1\}$. Suppose $f_\lambda(x)$ is a computable function such that $f_\lambda(1) = 1$ and f_λ is supported on $(2 - \frac{2}{\lambda+1}, \frac{2}{\lambda+1})$. For concreteness, one can choose f_λ to be a suitable horizontal scaling of $g(x)$ where

$$g(x) = \begin{cases} \exp\left(-\frac{1}{1-(1-x)^2} + 1\right) & \text{if } x \in (0,2) \\ 0 & \text{otherwise} \end{cases}.$$

It follows by functional calculus that $f_\lambda(\mathbf{R}_\varphi)$ is equal to the projection onto the $\lambda = 1$ eigenspace of $\mathbf{R}_\varphi$, namely the centralizer. By Takesaki's theorem, this projection is well-defined onto the centralizer. Our next theorem is the analogue in our language of [4, Theorem 5.14]. Notice that the proof here involves significantly less technical machinery than that given in [4]. Our proof only requires functional calculus for bounded operators and avoids the use of distributions and smearing of the modular automorphism group.

Theorem 8.5.5. *For any* $\lambda \in (0, 1)$*, the universal theory of* $(\mathcal{R}_\lambda, \varphi)$ *is not computable in the language* $\mathcal{L}_{\mathrm{vNa}}$ *where* φ *is the Powers state on* $\mathcal{R}_\lambda$*.*

Proof. Suppose, toward a contradiction that the universal theory of $(\mathcal{R}_\lambda, \varphi)$ is computable in the language $\mathcal{L}_{\mathrm{vNa}}$. Let $\sup_x \theta(x)$ be a universal sentence in the language of tracial von Neumann algebras (we will assume all variables range over the unit ball). Then

$$\sup_x \theta(f_\lambda(x))$$

is a universal sentence in our language. By assumption, given $\epsilon > 0$, we can compute the value of $\sup_x \theta(f_\lambda(x))$ up to ϵ. But the value of $\sup_x \theta(f_\lambda(x))$ is exactly the value of $\sup_x \theta(x)$ computed in $(\mathcal{R}_\lambda)_\varphi \cong \mathcal{R}$. This contradicts the uncomputability of the universal theory of $\mathcal{R}$. □

Since the $(\mathcal{R}_\lambda, \varphi_\lambda)EP$ would imply that the universal theory of $(\mathcal{R}_\lambda, \varphi_\lambda)$ is computable, we immediately get the following analogue of [4, Corollary 5.15]:

Corollary 8.5.6. *The* $(\mathcal{R}_\lambda, \varphi_\lambda)EP$ *is false in the language* $\mathcal{L}_{\mathrm{vNa}}$*.*

The proof of Theorem 8.5.5 shows something more general than what was proven in [4]:

Theorem 8.5.7. *Suppose that* $(\mathcal{M}, \varphi)$ *is a* W**-probability space such that* $\mathcal{M}$ *is QWEP and* φ *is a lacunary faithful, normal state on* $\mathcal{M}$ *for which* $\mathcal{M}_\varphi$ *contains* $\mathcal{R}$*. Then the universal theory of* $(\mathcal{M}, \varphi)$ *is not computable in the language* $\mathcal{L}_{\mathrm{vNa}}$*.*

Proof. Since $\mathcal{M}_\varphi$ is invariant under the modular automorphism group, we have that $(\mathcal{M}_\varphi, \varphi|\mathcal{M}_\varphi)$ embeds in $(\mathcal{M}, \varphi)$ with conditional expectation. Consequently, $\mathcal{M}_\varphi$ is also QWEP. Now since $\mathcal{M}_\varphi$ is a QWEP tracial von Neumann algebra there is a trace-preserving embedding $\mathcal{M}_\varphi \hookrightarrow \mathcal{R}^\mathcal{U}$. On the other hand, we also assumed that $\mathcal{M}_\varphi$ contains $\mathcal{R}$. Combining the last two statements, we have $\mathrm{Th}_\forall(\mathcal{R}) = \mathrm{Th}_\forall(\mathcal{M}_\varphi)$. Now we argue as in the proof of Theorem 8.5.5 above to reach a contradiction. □

It is well known that every II_1 factor contains a copy of $\mathcal{R}$. We also have that if φ is a periodic state, then $\mathcal{M}_\varphi$ is a II_1 factor. Therefore, we have the following corollary.

Corollary 8.5.8. *Suppose $\mathcal{M}$ is a QWEP* W^**-probability space and φ be a lacunary, faithful, normal, periodic state on $\mathcal{M}$. Then the universal theory of $(\mathcal{M}, \varphi)$ is not computable in the language $\mathcal{L}_{\mathrm{vNa}}$.*

8.6 The Universal Theory of the Hyperfinite II_∞ Factor is Undecidable

Notice that the proof of undecidability of the universal theory $\mathcal{R}_\lambda$, for $\lambda \neq 0, 1$ above involves giving a computable definition of the centralizer which, being $\mathcal{R}$, has undecidable universal theory. We will use this to show that the hyperfinite II_∞ factor $\mathcal{R}_{0,1} = \mathcal{R} \otimes B(\mathcal{H})$ together with its canonical tracial weight has undecidable universal theory. In this case, $\mathcal{R}$ is not a definable set, but we can find a definable family of isomorphic copies of $\mathcal{R}$. The material in this section requires the machinery we have built to handle general faithful normal semifinite weights and, as such, is original to this thesis.

Theorem 8.6.1. *The universal theory of the hyperfinite II_∞ factor $\mathcal{R}_{0,1}$ is not computable in the language $\mathcal{L}_{\mathrm{vNa}}$.*

Proof. Let $\sup_x \theta(x)$ be a universal sentence in the language of tracial von Neumann algebras where again we assume all variables range over the unit ball. Let P_1 denote the set of trace 1 projections in $\mathcal{R}_{0,1}$. As noted in the previous chapter, P_1 is a quantifier-free definable set by the formula

$$\max\{d(p^2, p), d(p^*, p), |\Phi(p) - 1|\}.$$

Note that for every $p \in P_1$, the set

$$p\mathcal{R}_{0,1}p := \{pxp \ : \ x \in \mathcal{R}_{0,1}\}$$

is a copy of $\mathcal{R}$ and the restriction of the Φ on $\mathcal{R}_{0,1}$ to this set is the canonical trace on $\mathcal{R}$. Furthermore, the map $x \mapsto pxp$ sends the unit ball of $\mathcal{R}_{0,1}$ to the unit ball of the corresponding copy of $\mathcal{R}$. Thus, computing the $\sup_x \theta(x)$ in any one copy of $\mathcal{R}$ is the same as computing it in any other. So consider the universal sentence in $\mathcal{L}_{\mathrm{vNa}}$ given by

$$\psi = \sup_{p \in P_1} \sup_x \theta(pxp).$$

By assumption, ψ is computable in $\mathcal{R}_{0,1}$. But

$$\psi^{\mathcal{R}_{0,1}} = (\sup_x \theta(x))^{\mathcal{R}},$$

contradicting the uncomputability of the universal theory of $\mathcal{R}$. □

The techniques used in the previous proof may be leveraged to resolve other embedding problems and characterize the computability of universal theories of other weighted von Neumann algebras. In fact, the proof above can easily be seen to prove the following, more general, statement.

Theorem 8.6.2. *Let* $(\mathcal{M}, \tau_0)$ *be a* II_1 *factor equipped with its canonical trace. Assume* $(\mathcal{M}, \tau_0)$ *has undecidable universal theory. If* $\mathcal{N} = \mathcal{M} \otimes B(\mathcal{H})$ *and the tracial weight* τ *is the* II_∞ *amplification of the trace* τ_0*, then* $(\mathcal{N}, \tau)$ *also has undecidable universal theory.*

We will now prove an analogue of Proposition 8.5.3 in our setting.

Theorem 8.6.3. *Let* $(\mathcal{M}, \Phi)$ *be a weighted von Neumann algebra such that* Φ *is lacunary and the restriction of* Φ *to* $\mathcal{M}_\Phi$ *is a semifinite weight. Then* $(\mathcal{M}_\Phi)^{\mathcal{U}} \cong (M^{\mathcal{U}})_{\Phi_{\mathcal{U}}}$.

Proof. First we prove $(\mathcal{M}_\Phi)^{\mathcal{U}} \subseteq (\mathcal{M}^{\mathcal{U}})_{\Phi_{\mathcal{U}}}$. By Takesaki's theorem, there is a conditional expectation

$$E_\Phi : \mathcal{M} \to \mathcal{M}_\Phi,$$

which is furthermore implemented by a projection on the underlying Hilbert spaces

$$P_\Phi : \mathcal{H}_{\mathcal{M}} \to \mathcal{H}_{\mathcal{M}_\Phi}.$$

Therefore we have a natural identification $\ell^\infty_\Phi(\mathcal{M}_\Phi) \hookrightarrow \ell^\infty_\Phi(\mathcal{M})$. So consider $x = (x_i) \in (\mathcal{M}_\Phi)^{\mathcal{U}}$. Then by Theorem 6.2.8, for all $t \in \mathbb{R}$, we have

$$\sigma_t^{\Phi_{\mathcal{U}}}(x) = (\sigma_t^{\Phi_i}(x_i))^\bullet = (x_i)^\bullet = x$$

whence the claim follows by taking strong closures.

Now we want to prove $(\mathcal{M}^{\mathcal{U}})_{\Phi_{\mathcal{U}}} \subseteq (\mathcal{M}_\Phi)^{\mathcal{U}}$. Arguing as in the discussion before Theorem 8.5.5, the lacunary assumption implies the projection from $\mathcal{H}_{\mathcal{M}}$ onto the closure of $\eta_\Phi(\mathcal{M}_\Phi)$ is definable. Therefore we have

$$\eta_{\Phi_{\mathcal{U}}}((\mathcal{M}^{\mathcal{U}})_{\Phi_{\mathcal{U}}}) \subseteq \eta_{\Phi_{\mathcal{U}}}((\mathcal{M}_\Phi)^{\mathcal{U}}).$$

Since $1 \in \mathcal{M}_\Phi$, we have that $\mathcal{M}_\Phi$ has the same identity as $\mathcal{M}$. We also assumed that Φ is semifinite on $\mathcal{M}_\Phi$, and normality and faithfulness on $\mathcal{M}_\Phi$ is automatic. Thus by equivalence of faithful normal semifinite weights and full left Hilbert algebras, by taking strong closures, we are done. □

We have an immediate model theoretic consequence.

Theorem 8.6.4. *Let $(\mathcal{M}, \Phi)$ be a weighted von Neumann algebra such that Φ is lacunary and the restriction of Φ to $\mathcal{M}_\Phi$ is a semifinite weight. Then the centralizer $\mathcal{M}_\Phi$ is definable.*

Bibliography

[1] J.-M. Albert and B. Hart. Metric Logical Categories and Conceptual Completeness for First Order Continuous Logic. *arXiv preprint arXiv:1607.03068*, 2016.

[2] W. Ambrose. Structure Theorems for a Special Class of Banach Algebras. *Transactions of the American Mathematical Society*, 57(3):364–386, 1945. URL http://www.jstor.org/stable/1990182.

[3] H. Ando and U. Haagerup. Ultraproducts of von Neumann algebras. *Journal of Functional Analysis*, 266(12):6842–6913, 2014.

[4] J. Arulseelan, I. Goldbring, and B. Hart. The Undecidability of Having the QWEP. *J. Operator Theory*, 92(2):349–362, dec 2024.

[5] J. Arulseelan, I. Goldbring, B. Hart, and T. Sinclair. Totally Bounded Elements in W*-probability Spaces. *arXiv:2501.14153v1*, 2025.

[6] W. Arveson. On Groups of Automorphisms of Operator Algebras. *Journal of Functional Analysis*, 15(3):217–243, 1974. doi: https://doi.org/10.1016/0022-1236(74)90034-2. URL https://www.sciencedirect.com/science/article/pii/0022123674900342.

[7] S. Atkinson, I. Goldbring, and S. K. Elayavalli. Factorial Relative Commutants and the Generalized Jung Property for II_1 Factors. *Advances in Mathematics*, 396:108107, 2022.

[8] B. Blackadar. *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras*. Number v. 13 in Encyclopaedia of Mathematical Sciences. Springer, 2006. URL https://books.google.ca/books?id=6b_T1j3Ib8oC.

[9] H. J. Borchers. On the use of modular groups in quantum field theory. *Annales de l'I.H.P. Physique théorique*, 63(4):331–382, 1995. URL http://eudml.org/doc/76702.

[10] M. Caspers. Weight Theory for Ultraproducts, 2016. URL https://arxiv.org/abs/1605.07435v2.

[11] C. Chang and H. Keisler. *Model Theory*. Studies in logic and the foundations of mathematics. North-Holland Publishing Company, 1977. URL https://books.google.ca/books?id=hnorAAAAYAAJ.

[12] A. Connes. Une Classification des Facteurs de Type III. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 6(2):133–252, 1973. doi: 10.24033/asens.1247. URL http://www.numdam.org/articles/10.24033/asens.1247/.

[13] A. Connes. Classification of Injective Factors Cases $II_1, II_\infty, III_\lambda, \lambda \neq 1$. *Annals of Mathematics*, 104(1):73–115, 1976. URL http://www.jstor.org/stable/1971057.

[14] A. Connes and M. Marcolli. A Walk in the Noncommutative Garden, 2006. URL https://arxiv.org/abs/math/0601054v2.

[15] A. Connes and E. J. Woods. Approximately Transitive Flows and ITPFI Factors. *Ergodic Theory and Dynamical Systems*, 5(2):203–236, 1985. doi: 10.1017/S0143385700002868.

[16] Y. Dabrowski. Continuous model theories for von neumann algebras. *Journal of Functional Analysis*, 277, 08 2015. doi: 10.1016/j.jfa.2019.108308.

[17] K. Davidson. *C*-Algebras by Example*. Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs. American Mathematical Society, 1996. URL `https://books.google.ca/books?id=PjpgCgAAQBAJ`.

[18] J. Dixmier. Algèbres Quasi-unitaires. *Commentarii Mathematici Helvetici*, 26:275–322, 1952. URL `https://api.semanticscholar.org/CorpusID:118676399`.

[19] H. Enderton. *A Mathematical Introduction to Logic*. Academic Press, 1972. URL `https://books.google.ca/books?id=LHjNCgAAQBAJ`.

[20] I. Farah, B. Hart, and D. Sherman. Model Theory of Operator Algebras II: Model Theory. *Israel Journal of Mathematics*, 201:477–505, 2010. URL `https://api.semanticscholar.org/CorpusID:45495734`.

[21] I. Farah, B. Hart, and D. Sherman. Model Theory of Operator Algebras I: Stability. *Bulletin of the London Mathematical Society*, 45(4):825–838, 2013.

[22] I. Farah, B. Hart, and D. Sherman. Model Theory of Operator Algebras III: Elementary Equivalence and II_1 Factors. *Bulletin of the London Mathematical Society*, 46(3):609–628, 2014.

[23] I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, and W. Winter. Model Theory of C*-algebras. *Memoirs of the American Mathematical Society*, 271, 05 2021. doi: 10.1090/memo/1324.

[24] A. Fox, I. Goldbring, and B. Hart. Locally Universal C*-algebras with Computable Presentations. *Journal of Functional Analysis*, 287(12):110652, 2024.

[25] T. Fritz. Tsirelson's Problem and Kirchberg's Conjecture. *Reviews in Mathematical Physics*, 24(05):1250012, 2012. doi: 10.1142/S0129055X12500122. URL https://doi.org/10.1142/S0129055X12500122.

[26] R. Godement. Theorie Des Caracteres. I Algebres Unitaires. *Annals of Mathematics*, 59(1):47–62, 1954. URL http://www.jstor.org/stable/1969832.

[27] I. Goldbring. Model Theory and the QWEP Conjecture. *Illinois Journal of Mathematics*, 59(3):795 – 799, 2015. doi: 10.1215/ijm/1475266408. URL https://doi.org/10.1215/ijm/1475266408.

[28] I. Goldbring. Enforceable Operator Algebras. *Journal of the Institute of Mathematics of Jussieu*, 20(1):31–63, 2021.

[29] I. Goldbring. The Connes Embedding Problem: A Guided Tour. *Bulletin of the American Mathematical Society*, 59(4):503–560, June 2022. doi: 10.1090/bull/1768. URL https://www.ams.org/bull/2022-59-04/S0273-0979-2022-01768-5/.

[30] I. Goldbring, editor. *Model Theory of Operator Algebras*. De Gruyter, Berlin, Boston, 2023. ISBN 9783110768282. doi: doi:10.1515/9783110768282. URL https://doi.org/10.1515/9783110768282.

[31] I. Goldbring. Undecidability and Incompleteness in Quantum Information Theory and Operator Algebras. *Monatshefte für Mathematik*, pages 1–34, 2025.

[32] I. Goldbring and B. Hart. Operator Algebras with Hyperarithmetic Theory. *Journal of Logic and Computation*, 31(2):612–629, 09 2020. doi: 10.1093/logcom/exaa059. URL `https://doi.org/10.1093/logcom/exaa059`.

[33] I. Goldbring and B. Hart. The Universal Theory Of The Hyperfinite II_1 Factor Is Not Computable. *The Bulletin of Symbolic Logic*, 30(2):181–198, 2024. doi: 10.1017/bsl.2024.7.

[34] I. Goldbring and C. Houdayer. Existentially Closed W^*-probability Spaces. *Mathematische Zeitschrift*, 301(4):3787–3816, 2022.

[35] I. Goldbring, B. Hart, and T. Sinclair. Correspondences, Ultraproducts and Model Theory. *arXiv*, 2018. URL `https://arxiv.org/abs/1809.00049v3`.

[36] I. Goldbring, D. Jekel, S. K. Elayavalli, and J. Pi. Uniformly Super McDuff II_1 Factors. *Mathematische Annalen*, 391(2):2757–2781, 2025.

[37] U. Haagerup. The Standard Form of von Neumann Algebras. *Mathematica Scandinavica*, 37(2):271–283, 1975. URL `http://www.jstor.org/stable/24490737`.

[38] U. Haagerup. Normal Weights on W*-algebras. *Journal of Functional Analysis*, 19:302–317, 1975. URL `https://api.semanticscholar.org/CorpusID:121452425`.

[39] U. Haagerup. On the Dual Weights for Crossed Products of von Neumann Algebras I: Removing Separability Conditions. *Mathematica Scandinavica*, 43(1):99–118, 1978. URL `http://www.jstor.org/stable/24491344`.

[40] U. Haagerup. On the Dual Weights for Crossed Products of von Neumann Algebras II. *Mathematica Scandinavica*, 43:119–140, 1978. URL http://eudml.org/doc/166608.

[41] U. Haagerup. On the Uniqueness of Injective III_1 Factor. *Documenta Mathematica*, 21(2016):1193–1226, 2016.

[42] L. Haykazyan and R. Moosa. Functoriality and Uniformity in Hrushovski's Groupoid-cover Correspondence. *Annals of Pure and Applied Logic*, 169(8):705–730, 2018.

[43] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen. MIP*=RE, Nov. 2022. URL http://arxiv.org/abs/2001.04383v3.

[44] R. Kadison and J. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume I*. Fundamentals of the Theory of Operator Algebras. American Mathematical Society, 1997. URL https://books.google.ca/books?id=Q3J6TV6euVYC.

[45] R. V. Kadison. Similarity of Operator Algebras. *Acta Mathematica*, 141(none): 147 – 163, 1978. doi: 10.1007/BF02545746. URL https://doi.org/10.1007/BF02545746.

[46] E. Kirchberg. On Non-semisplit Extensions, Tensor Products and Exactness of Group C*-algebras. *Inventiones mathematicae*, 112:449–489, 1993. URL https://api.semanticscholar.org/CorpusID:120162185.

[47] C. Lance. Direct Integrals of Left Hilbert Algebras. *Mathematische Annalen*, 216:11–28, 1975. URL http://eudml.org/doc/162719.

[48] B. MacCluer. *Elementary Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 2008. URL https://books.google.ca/books?id=8KB-ALmOSagC.

[49] T. Masuda and R. Tomatsu. Classification of Actions of Discrete Kac Algebras on Injective Factors. *arXiv*, 2013. URL https://arxiv.org/abs/1306.5046v2.

[50] F. J. Murray and J. von Neumann. On Rings of Operators. *Annals of Mathematics*, 37(1):116–229, 1936. URL http://www.jstor.org/stable/1968693.

[51] F. J. Murray and J. von Neumann. On Rings of Operators. II. *Transactions of the American Mathematical Society*, 41(2):208–248, 1937. URL http://www.jstor.org/stable/1989620.

[52] F. J. Murray and J. von Neumann. On Rings of Operators. IV. *Annals of Mathematics*, 44(4):716–808, 1943. URL http://www.jstor.org/stable/1969107.

[53] H. Nakano. Hilbert Algebras. *Tohoku Mathematical Journal, Second Series*, 2 (1):4–23, 1950.

[54] N. Ozawa. About the QWEP Conjecture. *International Journal of Mathematics*, 15(05):501–530, 2004. doi: 10.1142/S0129167X04002417. URL https://doi.org/10.1142/S0129167X04002417.

[55] G. Pedersen. *Analysis Now*. Graduate Texts in Mathematics. Springer New York, 1996. URL https://books.google.ca/books?id=a1R0livwR9AC.

[56] G. Pedersen and M. Takesaki. The Radon-Nikodym Theorem for von Neumann Algebras. *Acta Mathematica*, 130:53–87, 07 1973. doi: 10.1007/BF02392262.

[57] M. Rieffel and A. Van Daele. A Bounded Operator Approach to Tomita-Takesaki Theory. *Pacific Journal of Mathematics*, 69(1):187–221, Mar. 1977. doi: 10.2140/pjm.1977.69.187. URL `http://msp.org/pjm/1977/69-1/p17.xhtml`.

[58] W. Rudin. *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill, 1991. URL `https://books.google.ca/books?id=Sh_vAAAAMAAJ`.

[59] S. Sakai. *C*-Algebras and W*-Algebras*. Classics in mathematics. Springer, Berlin, Germany, Dec. 1997.

[60] I. E. Segal. Irreducible Representations of Operator Algebras. *Bull. Amer. Math. Soc*, 1947.

[61] I. E. Segal. A Non-Commutative Extension of Abstract Integration. *Annals of Mathematics*, 57(3):401–457, 1953. URL `http://www.jstor.org/stable/1969729`.

[62] J. Sorce. An Intuitive Construction of Modular Flow. *JHEP*, 12:079, 2023. doi: 10.1007/JHEP12(2023)079.

[63] K. Strung and F. Perera. *An Introduction to C*-Algebras and the Classification Program*. Advanced Courses in Mathematics - CRM Barcelona. Springer International Publishing, 2020. URL `https://books.google.ca/books?id=TzMPEAAAQBAJ`.

[64] M. Takesaki. *Tomita's Theory of Modular Hilbert Algebras and its Applications*. Lecture Notes in Mathematics. Springer-Verlag, 1970. doi: 10.1007/bfb0065832.

[65] M. Takesaki. *Theory of Operator Algebras II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003. doi: 10.1007/978-3-662-10451-4. URL http://link.springer.com/10.1007/978-3-662-10451-4.

[66] R. Tomatsu. Ultraproducts of Crossed Product von Neumann Algebras. *Illinois Journal of Mathematics*, 61(3-4):275 – 286, 2017. doi: 10.1215/ijm/1534924828. URL https://doi.org/10.1215/ijm/1534924828.

[67] A. Van Daele. A New Approach to the Tomita-Takesaki Theory of Generalized Hilbert Algebras, 1974-01-01.

[68] J. von Neumann. Zur Algebra der Funktionaloperationen und Theorie der Normalen Operatoren. *Mathematische Annalen*, 102:370–427, 1930. URL http://eudml.org/doc/159384.

[69] J. von Neumann. On Rings of Operators. III. *Annals of Mathematics*, 41(1): 94–161, 1940. URL http://www.jstor.org/stable/1968823.