Algebraic Solitons in the Massive Thirring Model

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A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Master of Science

McMaster University
Master of Science (2025)
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TITLE: Algebraic Solitons in the Massive Thirring Model
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NUMBER OF PAGES: vi, 43

Abstract

This thesis presents exact solutions describing dynamics of N identical algebraic solitons in the massive Thirring model. Each algebraic soliton corresponds to a simple embedded eigenvalue in the Kaup–Newell spectral problem and attains the maximal mass among solitary waves traveling with the same speed. In the case of $N = 2$ solitons, we use expressions for two exponential solitons and find a new solution in the singular limit for the algebraic double-soliton which corresponds to a double embedded eigenvalue. To systematically derive the rational solutions for N identical algebraic solitons for any $N \geq 1$, we employ the double-Wronskian method, a determinant-based approach that generates solitons to Hirota's bilinear equations. While traditional stability techniques fail for algebraic solitons due to their embedded spectral nature, the exact solutions obtained here suggest persistence of algebraic solitons under time evolution.

Acknowledgements

I would like to express my deepest gratitude to my supervisor Dr. Dimitry Pelinovsky, for his invaluable guidance and unwavering support throughout the course of this research. His expertise and encouragement have been instrumental in shaping this work, and I am truly grateful for the opportunity to learn under his mentorship.

I am deeply grateful to my mother for her constant encouragement and patience, I would not have been able to pursue graduate studies without her support.

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Chapter 1

Introduction

The massive Thirring model (MTM) describes the interaction of two counter-propagating waves and serves as a relativistically invariant analog of the Dirac equation in one spatial dimension. It has been widely studied due to its integrability and its applications in photonics, Bose–Einstein condensates, and nonlinear optics. While conventional solitons in this model exhibit exponential spatial decay, algebraic solitons arise as a special class of solutions characterized by power-law asymptotics, raising significant questions regarding their stability and dynamical behavior.

The spectral properties and stability of solitary waves in nonlinear Dirac equations have been the subject of extensive research. Monograph [3] provides a rigorous analysis of the spectral stability of solitary waves in nonlinear Dirac-type models, developing a functional-analytic framework to examine the behavior of localized structures in relativistic field theories. Monograph [25] provides a comprehensive study of the Gross–Pitaevskii equation in periodic potentials which reduces to the nonlinear Dirac-type models for solitary waves.

This thesis is devoted to the algebraic solitons in the MTM written in laboratory coordinates:

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v \end{cases} \tag{1.1}$$

where $(u, v) \in \mathbb{C}^2$ and subscripts denote partial derivatives in $(x, t) \in \mathbb{R}^2$. The MTM system (1.1) is a prototypical Dirac equation which belongs to the class of integrable equations associated with the Kaup–Newell (KN) spectral problem [16, 20, 24].

Algebraic solitons are traveling solitary waves with the power rather than exponential spatial decay rate at infinity. Such solutions are common for integrable nonlinear equations with nonlocal terms such as the Benjamin–Ono and Kadomtsev–Petviashvili equations, where they are associated with isolated eigenvalues of the linear Lax equations [1]. However, algebraic solitons are special for local integrable nonlinear equations since they arise as the limiting points in the family of exponential solitons and they are associated with embedded eigenvalues in the continuous spectrum of the linear Lax equations [19, 26]. Physically relevant examples of the algebraic solitons as special limits of exponential solitons appear in the modified Korteweg–de Vries equation [6, 31], the derivative nonlinear Schrödinger equation [11, 29, 32], and the nonlinear Dirac equation [12].

Stability of algebraic solitons is a notoriously difficult mathematical problem, where every method of nonlinear analysis known in the theory of integrable systems fails. Coercivity of the energy function required for the proof of Lyapunov stability holds for exponential solitons [28] but fails for algebraic solitons because the spectral gap between the zero eigenvalue and the continuous spectrum in the linearized MTM system closes up in the limit to the algebraic soliton. Stability of exponential solitons in the MTM system can be proven with the Darboux transformation [7] which constructs exponential solitons from isolated eigenvalues of the KN spectral problem. However, the Darboux transformation does not generate algebraic solitons because the embedded eigenvalues have to be defined inside the continuous spectrum of the KN spectral problem, where both eigenfunctions are bounded. Finally, the inverse scattering transform (IST) method requires fast spatial decay of solutions of the MTM system at infinity in order to ensure smoothness properties of the scattering data and solvability of the associated Riemann–Hilbert problems [14, 27]. Algebraic solitons decay too slowly and violate the requirements of the fast spatial decay.

The modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{1.2}$$

has the algebraic soliton

$$u_1(x,t) = 1 - \frac{4}{1 + 4(x-6t)^2} \tag{1.3}$$

which also emerge as the limiting case of exponential solitons, leading to its instability under perturbations. Previous studies [9, 17] have demonstrated that algebraic solitons in mKdV and NLS do not satisfy the standard stability conditions. Further insight into this instability is provided by the second-order rational solution of the mKdV equation (1.2) derived in [6]:

$$u_2(x,t) = 1 + 12\frac{G}{D}, \tag{1.4}$$

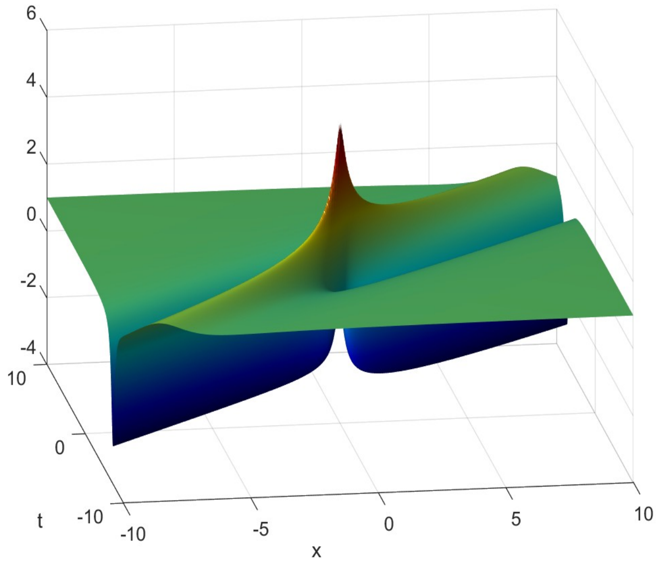

Figure 1.1: The solution surface for the second order mKdV rational solution (1.4).

where

$$G = 3 - 8(-x+t)[2(-x+t)^3 + 3(-\frac{11}{3}x + t)],$$
$$D = -8x[48x^4(-\frac{1}{6}x+t) - 2x^3(60t^2 - 13) + 8x^2t(20t^2 - 9) - \frac{1}{2}x(240t^4 - 120t^2 + 139) + t(48t^4 - 8t^2 + 51)] + 64t^6 + 48t^4 + 108t^2 + 9.$$

This second-order rational solution exhibits a more complex dynamics shown in Figure 1.1, featuring how the soliton is flipped due to the growth of perturbations. The presence of such solutions implies that the algebraic soliton (1.3) of the mKdV equation (1.2) is unstable.

Unlike the mKdV equation (1.2), where instability of the algebraic soliton has been rigorously established, the orbital stability of the traveling algebraic soliton in the MTM system (1.1) remains an open question. While algebraic solitons in the mKdV equation is associated with the formation of higher-order rogue wave structures, the MTM system exhibits distinct spectral and dynamical properties that suggest a different stability behavior. In particular, the spectral analysis in [19] suggests that the embedded eigenvalues characterizing algebraic solitons in the MTM system do not necessarily induce instability. Unlike in the mKdV equation, where embedded eigenvalues bifurcate into isolated eigenvalues for both exponential solitons and time-periodic breathers, in the MTM system embedded eigenvalues only bifurcate into isolated eigenvalues for exponential solitons.

In this thesis, we derive exact solutions to the MTM system (1.1) which are second-order and higher-order rational solutions generating the algebraic soliton.

In Chapter 2 we explore direct methods of solutions of the MTM system (1.1) to study the interactions of two algebraic solitons. Hirota's bilinear formulation of the MTM system (1.1) was recently developed in [4, 5] to obtain exponential multi-solitons. By using the analytical expressions for two exponential solitons, we obtain the exact solutions for two algebraic solitons which scatter fast from each other with two different wave speeds. In the limit when the wave speeds coincide, we obtain the algebraic double-soliton solution which describes a slow interaction of two identical algebraic solitons. The content of Chapter 2 was published in [13].

A powerful approach for constructing soliton solutions in integrable systems involves the bilinear formalism and determinant-based methods. In Chapter 3, the double-Wronskian representation provides a systematic framework for generating higher-order rational solutions corresponding to algebraic solitons. This approach has been used successfully in other integrable models, such as the derivative NLS equation [30].

As the main outcome of the new rational solutions, we conclude in Chapter 4 that the algebraic solitons of the MTM system (1.1) display stable dynamics under small perturbations. A rigorous proof of stability of the algebraic soliton in the MTM system (1.1) is still open.

Chapter 2

Algebraic Soliton Solutions via Hirota Method

2.1 Preliminaries

To simplify the presentation of soliton solutions of the MTM system (1.1), we shall use the basic symmetries of this Hamiltonian system. These include the translational and rotational symmetries

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} u(x+x_0,t+t_0)e^{\mathrm{i}\theta_0} \\ v(x+x_0,t+t_0)e^{\mathrm{i}\theta_0} \end{bmatrix}, \qquad x_0,t_0,\theta_0 \in \mathbb{R}, \tag{2.1}$$

as well as the Lorentz symmetry

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} \left(\frac{1-c}{1+c}\right)^{1/4} u\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \\ \left(\frac{1+c}{1-c}\right)^{1/4} v\left(\frac{x+ct}{\sqrt{1-c^2}}, \frac{t+cx}{\sqrt{1-c^2}}\right) \end{bmatrix}, \qquad c \in (-1,1). \tag{2.2}$$

Without loss of generality, each solution of the MTM system (1.1) can be extended with three translational parameter in (2.1) and the speed parameter $c \in (-1,1)$ in (2.2).

A general family of solitary waves of the MTM system (1.1) is obtained from the normalized standing wave solutions

$$\begin{bmatrix} u_{\text{sol}}(x,t) \\ v_{\text{sol}}(x,t) \end{bmatrix} = \sin\gamma \begin{bmatrix} \operatorname{sech}\left(x\ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \operatorname{sech}\left(x\ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix} e^{\mathrm{i}t\cos\gamma}, \quad \gamma \in (0,\pi), \tag{2.3}$$

after the translations (2.1) and the Lorentz transformation (2.2). The family (2.3) corresponds to the gap $(-1,1)$ in the frequency spectrum $\omega := \cos(\gamma)$ of the linear

Dirac operator

$$\mathcal{D} := \begin{bmatrix} \mathrm{i}\partial_x & 1 \\ 1 & -\mathrm{i}\partial_x \end{bmatrix}$$

which defines the time evolution of the MTM system (1.1).

The limits $\omega \to \pm 1$ are referred to as the nonrelativistic limits of the MTM system (1.1). It is well-known (see, e.g., [2, 3, 8]) that the nonlinear Dirac equations such as the MTM system (1.1) can be reduced to the focusing NLS equation as $\omega \to 1$ and to the defocusing NLS equation as $\omega \to -1$ given in the normalized form as

$$\mathrm{i}\psi_t + \psi_{xx} + \sigma|\psi|^2\psi = 0, \qquad \sigma = \mathrm{sgn}(\omega) = \pm 1. \tag{2.4}$$

The family (2.3) reduces to the small-amplitude, long-scale, sech-shaped soliton of the focusing NLS equation (2.4) with $\sigma = +1$ as $\omega \to 1$ ($\gamma \to 0$) and to the finite-amplitude, finite-scale, algebraic soliton

$$\gamma = \pi : \quad \begin{bmatrix} u_{alg}(x,t) \\ v_{alg}(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{1+2\mathrm{i}x} \\ \dfrac{2}{1-2\mathrm{i}x} \end{bmatrix} e^{-\mathrm{i}t} \tag{2.5}$$

as $\omega \to -1$ ($\gamma \to \pi$). Note that the algebraic soliton (2.5) does not satisfy the defocusing NLS equation (2.4) with $\sigma = -1$ as $\omega \to -1$, because its amplitude is finite (not small).

The algebraic soliton (2.5) has the largest mass among the exponential solitons in the family (2.3), where the mass for the MTM system (1.1) is defined by

$$Q(u,v) := \int_{\mathbb{R}} (|u|^2 + |v|^2) dx. \tag{2.6}$$

It follows from (2.3) that

$$|u_{\rm sol}(x,t)|^2 + |v_{\rm sol}(x,t)|^2 = \frac{4\sin^2\gamma}{\cos\gamma + \cosh(2x\ \sin\gamma)},$$

which implies that $Q_{\rm sol}(\gamma) := Q(u_{\rm sol}, v_{\rm sol}) = 4\gamma$ with the largest mass at $Q_{\rm sol}(\pi) = Q(u_{\rm alg}, v_{\rm alg}) = 4\pi$.

2.2 New parameterization of the exponential two-soliton solutions

The MTM system (1.1) can be transformed to a system of bilinear equations by the following transformation [4],

$$u = \frac{g}{\bar{f}}, \qquad v = \frac{h}{f}, \tag{2.7}$$

where $\bar{f}$ is the complex conjugate of f. Substituting (2.7) into (1.1) yields the following system of bilinear equations for f, h, and g:

$$\left.\begin{array}{r} \mathrm{i}f(g_t + g_x) - \mathrm{i}g(f_t + f_x) + h\bar{f} = 0, \\ \mathrm{i}\bar{f}(h_t - h_x) - \mathrm{i}h(\bar{f}_t - \bar{f}_x) + gf = 0, \\ \mathrm{i}\bar{f}(f_x + f_t) - \mathrm{i}f(\bar{f}_t + \bar{f}_x) - |h|^2 = 0, \\ \mathrm{i}f(\bar{f}_t - \bar{f}_x) - \mathrm{i}\bar{f}(f_t - f_x) - |g|^2 = 0. \end{array}\right\} \tag{2.8}$$

It was proven in [4] that the system (2.8) is satisfied by the following two-soliton solutions in the general form:

$$\left\{\begin{array}{l} f = 1 + c_{11}e^{\zeta_1+\bar{\zeta}_1} + c_{12}e^{\zeta_1+\bar{\zeta}_2} + c_{21}e^{\bar{\zeta}_1+\zeta_2} + c_{22}e^{\zeta_2+\bar{\zeta}_2} + c_{1212}e^{\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2}, \\ h = \bar{\alpha}_1 e^{\zeta_1} + \bar{\alpha}_2 e^{\zeta_2} + c_{121}e^{\zeta_1+\zeta_2+\bar{\zeta}_1} + c_{122}e^{\zeta_1+\zeta_2+\bar{\zeta}_2}, \\ g = \frac{\mathrm{i}\bar{\alpha}_1}{p_1}e^{\zeta_1} + \frac{\mathrm{i}\bar{\alpha}_2}{p_2}e^{\zeta_2} - \frac{\mathrm{i}\bar{p}_1}{p_1 p_2}c_{121}e^{\zeta_1+\zeta_2+\bar{\zeta}_1} - \frac{\mathrm{i}\bar{p}_2}{p_1 p_2}c_{122}e^{\zeta_1+\zeta_2+\bar{\zeta}_2}, \end{array}\right. \tag{2.9}$$

where

$$\zeta_j = \frac{1}{2}\left(p_j + \frac{1}{p_j}\right)x + \frac{1}{2}\left(p_j - \frac{1}{p_j}\right)t$$

and

$$\begin{aligned} c_{ij} &= -\frac{\mathrm{i}p_i\bar{\alpha}_i\alpha_j}{(p_i + \bar{p}_j)^2}, \\ c_{12j} &= (p_1 - p_2)\bar{p}_j\left[\frac{\bar{\alpha}_2 c_{1j}}{p_1(p_2 + \bar{p}_j)} - \frac{\bar{\alpha}_1 c_{2j}}{p_2(p_1 + \bar{p}_j)}\right], \\ c_{1212} &= |p_1 - p_2|^2\left[\frac{c_{11}c_{22}}{(p_1 + \bar{p}_2)(p_2 + \bar{p}_1)} - \frac{c_{12}c_{21}}{(p_1 + \bar{p}_1)(p_2 + \bar{p}_2)}\right], \end{aligned}$$

whereas parameters $p_1, p_2, \alpha_1, \alpha_2 \in \mathbb{C}$ are arbitrary.

In order to represent the 2-soliton solutions in a meaningful way where each soliton resembles the 1-soliton solution (2.3), we will use the following parameterization:

$$p_j = \mathrm{i}\delta_j e^{-\mathrm{i}\gamma_j}, \quad \alpha_j = 2\sqrt{\delta_j}\sin\gamma_j e^{\frac{\mathrm{i}\gamma_j}{2}+\sin\gamma_j x_j - \mathrm{i}\cos\gamma_j t_j}, \quad j=1,2, \tag{2.10}$$

with arbitrary parameters $\gamma_j \in (0,\pi)$, $\delta_j > 0$, and $(x_j, t_j) \in \mathbb{R}^2$. By using the parameterization (2.10) for p_j, we obtain

$$\zeta_j = \sin\gamma_j \left[\frac{1}{2}(\delta_j + \delta_j^{-1})x + \frac{1}{2}(\delta_j - \delta_j^{-1})t\right] + \mathrm{i}\cos\gamma_j \left[\frac{1}{2}(\delta_j - \delta_j^{-1})x + \frac{1}{2}(\delta_j + \delta_j^{-1})t\right].$$

This representation resembles the Lorentz transformation (2.2) with

$$\frac{1}{2}(\delta_j + \delta_j^{-1}) = \frac{1}{\sqrt{1-c_j^2}}, \quad \frac{1}{2}(\delta_j - \delta_j^{-1}) = \frac{c_j}{\sqrt{1-c_j^2}},$$

where we have introduced the wave speeds

$$c_j := \frac{\delta_j^2 - 1}{\delta_j^2 + 1} \in (-1, 1), \quad j = 1, 2. \tag{2.11}$$

Due to parameterization (2.10), we obtain

$$c_{jj} = e^{-\mathrm{i}\gamma_j + 2\sin\gamma_j x_j}, \quad j = 1, 2,$$

and, more generally,

$$c_{ij} = -\frac{4\sqrt{\delta_i\delta_j}\sin\gamma_i\sin\gamma_j\delta_i}{(\delta_i e^{-\frac{\mathrm{i}}{2}(\gamma_i+\gamma_j)} - \delta_j e^{\frac{\mathrm{i}}{2}(\gamma_i+\gamma_j)})^2} e^{-\frac{\mathrm{i}}{2}(\gamma_i+\gamma_j)+\sin\gamma_i x_i + \sin\gamma_j x_j + \mathrm{i}\cos\gamma_i t_i - \mathrm{i}\cos\gamma_j t_j},$$

so that we can introduce the following two real-valued coordinates

$$\xi_j := \sin\gamma_j \left(\frac{x + c_j t}{\sqrt{1-c_j^2}} + x_j\right), \quad \eta_j := \cos\gamma_j \left(\frac{t + c_j x}{\sqrt{1-c_j^2}} + t_j\right),$$

where $(x_j, t_j) \in \mathbb{R}^2$ play the role of translational parameters in (2.1).

Next, we deduce the explicit expressions with the parameterization (2.10) for c_{12j} and c_{1212}. It follows from

$$c_{121} = (p_1 - p_2)\bar{p}_1 \left[\frac{\bar{\alpha}_2 c_{11}}{p_1(p_2 + \bar{p}_1)} - \frac{\bar{\alpha}_1 c_{21}}{p_2(p_1 + \bar{p}_1)}\right] = \frac{\mathrm{i}\bar{p}_1|\alpha_1|^2\bar{\alpha}_2(p_1 - p_2)^2}{(p_1 + \bar{p}_1)^2(\bar{p}_1 + p_2)^2}$$

that

$$c_{121} = \frac{(p_1 - p_2)^2}{(\bar{p}_1 + p_2)^2} \bar{\alpha}_2 e^{i\gamma_1 + 2\sin\gamma_1 x_1}, \quad c_{122} = \frac{(p_1 - p_2)^2}{(p_1 + \bar{p}_2)^2} \bar{\alpha}_1 e^{i\gamma_2 + 2\sin\gamma_2 x_2}.$$

Similarly, we obtain

$$c_{1212} = e^{-\mathrm{i}\gamma_1 - \mathrm{i}\gamma_2 + 2\sin\gamma_1 x_1 + 2\sin\gamma_2 x_2} A_{12},$$

where

$$\begin{aligned} A_{12} &= \frac{|p_1 - p_2|^2}{(p_1 + \bar{p}_2)(p_2 + \bar{p}_1)} \left[1 - \frac{16\delta_1^2\delta_2^2 \sin^2\gamma_1 \sin^2\gamma_2}{(p_1 + \bar{p}_1)(p_2 + \bar{p}_2)(p_1 + \bar{p}_2)(\bar{p}_1 + p_2)}\right] \\ &= -\frac{|p_1 - p_2|^2}{(p_1 + \bar{p}_2)^2 (p_2 + \bar{p}_1)^2} \left[(\delta_1 e^{-\mathrm{i}\gamma_1} - \delta_2 e^{\mathrm{i}\gamma_2})(\delta_2 e^{-\mathrm{i}\gamma_2} - \delta_1 e^{\mathrm{i}\gamma_1}) + 4\delta_1\delta_2 \sin\gamma_1 \sin\gamma_2\right] \\ &= \left(\frac{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2\cos(\gamma_1 - \gamma_2)}{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2\cos(\gamma_1 + \gamma_2)}\right)^2. \end{aligned}$$

This representation allows us to rewrite the three components of the 2-soliton solution (2.9) in the explicit form:

$$\begin{aligned} f &= 1 + e^{2\xi_1 - \mathrm{i}\gamma_1} + e^{2\xi_2 - \mathrm{i}\gamma_2} + A_{12} e^{2\xi_1 + 2\xi_2 - \mathrm{i}\gamma_1 - \mathrm{i}\gamma_2} - 4\sqrt{\delta_1\delta_2} \sin\gamma_1 \sin\gamma_2 e^{\xi_1 + \xi_2 - \frac{\mathrm{i}}{2}\gamma_1 - \frac{\mathrm{i}}{2}\gamma_2} \\ &\quad \times \left[\frac{\delta_1 e^{\mathrm{i}(\eta_1 - \eta_2)}}{(\delta_1 e^{-\frac{\mathrm{i}}{2}(\gamma_1 + \gamma_2)} - \delta_2 e^{\frac{\mathrm{i}}{2}(\gamma_1 + \gamma_2)})^2} + \frac{\delta_2 e^{-\mathrm{i}(\eta_1 - \eta_2)}}{(\delta_1 e^{\frac{\mathrm{i}}{2}(\gamma_1 + \gamma_2)} - \delta_2 e^{-\frac{\mathrm{i}}{2}(\gamma_1 + \gamma_2)})^2}\right], \\ h &= \bar{\alpha}_1 e^{\zeta_1} \left[1 + \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2}\right)^2 e^{2\xi_2 + \mathrm{i}\gamma_2}\right] + \bar{\alpha}_2 e^{\zeta_2} \left[1 + \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2}\right)^2 e^{2\xi_1 + \mathrm{i}\gamma_1}\right], \end{aligned}$$

and

$$g = \frac{\mathrm{i}\bar{\alpha}_1}{p_1} e^{\zeta_1} \left[1 + \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2}\right)^2 e^{2\xi_2 + 3\mathrm{i}\gamma_2}\right] + \frac{\mathrm{i}\bar{\alpha}_2}{p_2} e^{\zeta_2} \left[1 + \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2}\right)^2 e^{2\xi_1 + 3\mathrm{i}\gamma_1}\right].$$

The one-soliton solution appears from this formula by taking $\xi_2 \to -\infty$:

$$u = \lim_{\xi_2 \to -\infty} \frac{g}{\bar{f}} = \frac{\mathrm{i}\bar{\alpha}_1 e^{\zeta_1}}{p_1(1 + e^{2\xi_1 + \mathrm{i}\gamma_1})} = \sin\gamma_1 \delta_1^{-1/2} \mathrm{sech}\left(\xi_1 + \frac{\mathrm{i}}{2}\gamma_1\right) e^{\mathrm{i}\eta_1}$$

and similarly,

$$v = \lim_{\xi_2 \to -\infty} \frac{h}{f} = \frac{\bar{\alpha}_1 e^{\zeta_1}}{1 + e^{2\xi_1 - \mathrm{i}\gamma_1}} = \sin\gamma_1 \delta_1^{1/2} \mathrm{sech}\left(\xi_1 - \frac{\mathrm{i}}{2}\gamma_1\right) e^{\mathrm{i}\eta_1},$$

from which we recognize the exact solution (2.3) with the account of the symmetry transformations (2.1) and (2.2).

2.3 Limit to the two algebraic solitons

Each soliton in the two-soliton solution has four arbitrary parameters $\delta_j > 0$, $\gamma_j \in (0, \pi)$, and $(x_j, t_j) \in \mathbb{R}^2$ for $j = 1, 2$. In order to get two algebraic solitons, we need to take the singular limit $\gamma_j \to \pi$ for each $j = 1, 2$. Hence, we set

$$\gamma_j = \pi - \epsilon_j, \quad j = 1, 2$$

and expand to the leading order

$$\sin \gamma_j = \epsilon_j + \mathcal{O}(\epsilon_j^3), \quad \cos \gamma_j = 1 + \mathcal{O}(\epsilon_j^2).$$

We can then define

$$X_j := \frac{x + c_j t}{\sqrt{1 - c_j^2}} + x_j, \quad T_j := \frac{t + c_j x}{\sqrt{1 - c_j^2}} + t_j$$

and expand

$$\left(\frac{p_1 - p_2}{p_1 + \bar{p}_2} \right)^2 = \left(\frac{\delta_1 e^{\mathrm{i}\epsilon_1} - \delta_2 e^{\mathrm{i}\epsilon_2}}{\delta_1 e^{\mathrm{i}\epsilon_1} - \delta_2 e^{-\mathrm{i}\epsilon_2}} \right)^2 = 1 - \frac{4\mathrm{i}\epsilon_2 \delta_2}{\delta_1 - \delta_2} + \mathcal{O}(\epsilon_1^2, \epsilon_2^2)$$

and

$$A_{12} = \left(\frac{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2 \cos(\epsilon_1 - \epsilon_2)}{\delta_1^2 + \delta_2^2 - 2\delta_1\delta_2 \cos(\epsilon_1 + \epsilon_2)} \right)^2 = 1 - \frac{8\delta_1\delta_2\epsilon_1\epsilon_2}{(\delta_1 - \delta_2)^2} + \mathcal{O}(\epsilon_1^2\epsilon_2^2).$$

This yields the expansions:

$$\begin{aligned}
f = {} & 1 - e^{\epsilon_1(2X_1+\mathrm{i})+\mathcal{O}(\epsilon_1^3)} - e^{\epsilon_2(2X_2+\mathrm{i})+\mathcal{O}(\epsilon_2^3)} + A_{12} e^{\epsilon_1(2X_1+\mathrm{i})+\mathcal{O}(\epsilon_1^3)+\epsilon_2(2X_2+\mathrm{i})+\mathcal{O}(\epsilon_2^3)} \\
& + 4\sqrt{\delta_1\delta_2\epsilon_1\epsilon_2} \left[\frac{\delta_1 e^{-\mathrm{i}(T_1-T_2)} + \delta_2 e^{\mathrm{i}(T_1-T_2)}}{(\delta_1 - \delta_2)^2} + \mathcal{O}(\epsilon_1, \epsilon_2) \right], \\
h = {} & -2\mathrm{i}\delta_1^{1/2}\epsilon_1 e^{-\mathrm{i}T_1} \left[1 - \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2} \right)^2 e^{\epsilon_2(2X_2-\mathrm{i})+\mathcal{O}(\epsilon_2^3)} \right] \left[1 + \mathcal{O}(\epsilon_1^2) \right] \\
& - 2\mathrm{i}\delta_2^{1/2}\epsilon_2 e^{-\mathrm{i}T_2} \left[1 + \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2} \right)^2 e^{\epsilon_1(2X_1-\mathrm{i})+\mathcal{O}(\epsilon_1^3)} \right] \left[1 + \mathcal{O}(\epsilon_2^2) \right],
\end{aligned}$$

$$g = 2\mathrm{i}\delta_1^{-1/2}\epsilon_1 e^{-\mathrm{i}T_1}\left[1 - \left(\frac{p_1 - p_2}{p_1 + \bar{p}_2}\right)^2 e^{\epsilon_2(2X_2 - 3\mathrm{i}) + \mathcal{O}(\epsilon_2^3)}\right]\left[1 + \mathcal{O}(\epsilon_1^2)\right]$$
$$+ 2\mathrm{i}\delta_2^{-1/2}\epsilon_2 e^{-\mathrm{i}T_2}\left[1 - \left(\frac{p_1 - p_2}{\bar{p}_1 + p_2}\right)^2 e^{\epsilon_1(2X_1 - 3\mathrm{i}) + \mathcal{O}(\epsilon_1^3)}\right]\left[1 + \mathcal{O}(\epsilon_2^2)\right].$$

Hence, we get the power expansions

$$f = \epsilon_1\epsilon_2[F + \mathcal{O}(\epsilon_1, \epsilon_2)], \quad h = \epsilon_1\epsilon_2[H + \mathcal{O}(\epsilon_1, \epsilon_2)], \quad g = \epsilon_1\epsilon_2[G + \mathcal{O}(\epsilon_1, \epsilon_2)]$$

with

$$F = (2X_1 + \mathrm{i})(2X_2 + \mathrm{i}) + \frac{4\sqrt{\delta_1\delta_2}}{(\delta_1 - \delta_2)^2}\left[\sqrt{\delta_1}e^{-\frac{\mathrm{i}}{2}(T_1 - T_2)} - \sqrt{\delta_2}e^{\frac{\mathrm{i}}{2}(T_1 - T_2)}\right]^2, \tag{2.12}$$

$$H = 2\mathrm{i}\delta_1^{1/2}e^{-\mathrm{i}T_1}\left[2X_2 - \mathrm{i} - \frac{4\mathrm{i}\delta_2}{\delta_1 - \delta_2}\right] + 2\mathrm{i}\delta_2^{1/2}e^{-\mathrm{i}T_2}\left[2X_1 - \mathrm{i} + \frac{4\mathrm{i}\delta_1}{\delta_1 - \delta_2}\right], \tag{2.13}$$

and

$$G = -2\mathrm{i}\delta_1^{-1/2}e^{-\mathrm{i}T_1}\left[2X_2 + \mathrm{i} - \frac{4\mathrm{i}\delta_1}{\delta_1 - \delta_2}\right] - 2\mathrm{i}\delta_2^{-1/2}e^{-\mathrm{i}T_2}\left[2X_1 + \mathrm{i} + \frac{4\mathrm{i}\delta_2}{\delta_1 - \delta_2}\right]. \tag{2.14}$$

The algebraic two-soliton solution of the MTM system (1.1) appears in the Hirota form as

$$u = \frac{G}{\bar{\bar{F}}}, \quad v = \frac{H}{F}. \tag{2.15}$$

It describes two algebraic solitons traveling with the speeds $c_{1,2}$ obtained from $\delta_{1,2}$ by (2.11). A single algebraic solution appears by taking $X_2 \to \infty$:

$$u = \lim_{X_2\to\infty}\frac{G}{\bar{\bar{F}}} = \frac{2\delta_1^{-1/2}}{1 + 2\mathrm{i}X_1}e^{-\mathrm{i}T_1}$$

and similarly,

$$v = \lim_{X_2\to\infty}\frac{H}{F} = \frac{2\delta_1^{1/2}}{1 - 2\mathrm{i}X_1}e^{-\mathrm{i}T_1},$$

from which we recognize the exact solution (2.5) with the account of the symmetry transformations (2.1) and (2.2).

Figure 2.1 shows the solution surfaces which suggest that the algebraic two-soliton solution given by (2.15) describes normal scattering of two algebraic solitons. When the wave speeds c_1 and c_2 are very different from each other (top panels), the scattering is quick and the trajectories of the two solitons are almost straight lines. When the wave speeds approach to each other (bottom panels), the scattering becomes slow and the trajectories of the two solitons are curved near the soliton overlapping regions.

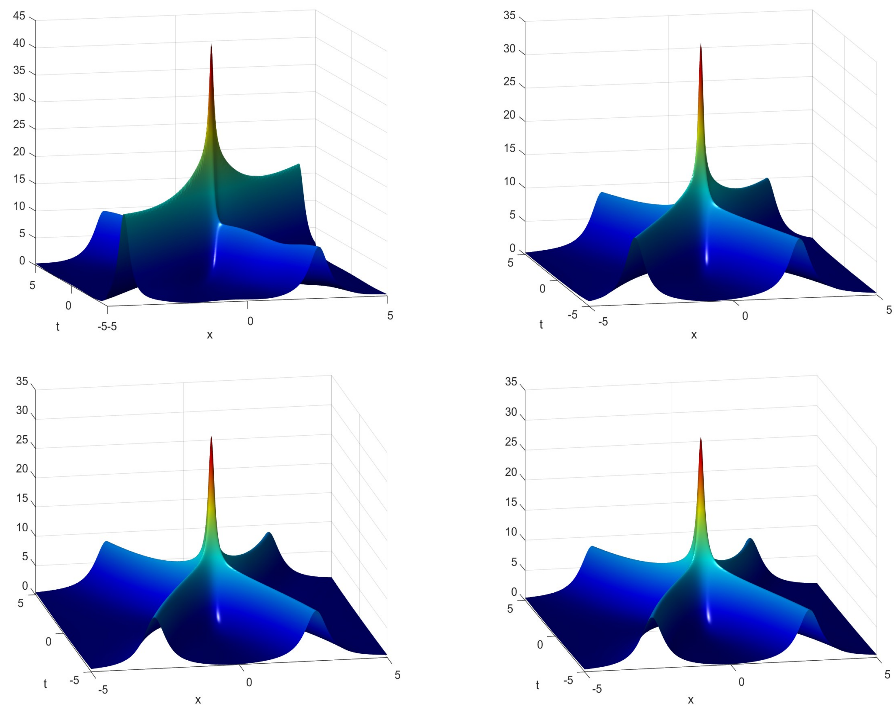

FIGURE 2.1: The solution surface for $|u|^2 + |v|^2$ versus (x, t) for the family (2.15) with $x_1 = x_2 = t_1 = t_2 = 0$ and $\delta_1 = 1 + \varepsilon$, $\delta_2 = 1 - \varepsilon$ with $\varepsilon = 0.75$ (top left), $\varepsilon = 0.5$ (top right), $\varepsilon = 0.25$ (bottom left), and $\varepsilon = 0.01$ (bottom right).

2.4 Limit to the algebraic double-soliton

Each algebraic soliton in the two-soliton solution (2.15) has three arbitrary parameters $\delta_j > 0$ and $(x_j, t_j) \in \mathbb{R}^2$ for $j = 1, 2$. We shall now consider the limit $\delta_1 \to \delta_2$ to obtain the algebraic double-solitons. Due to the Lorentz transformation (2.2), it is sufficient to set

$$\delta_1 = 1 + \varepsilon, \quad \delta_2 = 1 - \varepsilon$$

and take the limit $\varepsilon \to 0$, this gives the algebraic double-soliton with $c = 0$. Expanding $X_{1,2}$ and $T_{1,2}$ in the first powers of ε, we write

$$\begin{cases} X_1 = x + \varepsilon t + \frac{1}{2}\varepsilon^2(x-t) - \frac{1}{2}\varepsilon^3(x-t) + x_1 + \mathcal{O}(\varepsilon^4), \\ X_2 = x - \varepsilon t + \frac{1}{2}\varepsilon^2(x-t) + \frac{1}{2}\varepsilon^3(x-t) + x_2 + \mathcal{O}(\varepsilon^4), \\ T_1 = t + \varepsilon x - \frac{1}{2}\varepsilon^2(x-t) + \frac{1}{2}\varepsilon^3(x-t) + t_1 + \mathcal{O}(\varepsilon^4), \\ T_2 = t - \varepsilon x - \frac{1}{2}\varepsilon^2(x-t) - \frac{1}{2}\varepsilon^3(x-t) + t_2 + \mathcal{O}(\varepsilon^4). \end{cases}$$

In view of the translational symmetry (2.1), it is also sufficient to set

$$\begin{cases} x_1 = \varepsilon a_1 + \frac{1}{2}\varepsilon^2 a_2 - \frac{1}{2}\varepsilon^3 a_3, + \mathcal{O}(\varepsilon^4), \\ x_2 = -\varepsilon a_1 + \frac{1}{2}\varepsilon^2 a_2 + \frac{1}{2}\varepsilon^3 a_3, + \mathcal{O}(\varepsilon^4), \\ t_1 = \varepsilon b_1 - \frac{1}{2}\varepsilon^2 b_2 + \frac{1}{2}\varepsilon^3 b_3, + \mathcal{O}(\varepsilon^4), \\ t_2 = -\varepsilon b_1 - \frac{1}{2}\varepsilon^2 b_2 - \frac{1}{2}\varepsilon^3 b_3, + \mathcal{O}(\varepsilon^4), \end{cases}$$

with arbitrary parameters a_1, a_2, a_3, b_1, b_2, and b_3. This gives the algebraic double-soliton with zero translational parameters. The double-soliton can be extended to three additional parameters by using (2.1) and (2.2).

For expansions of F, we use

$$(2X_1 + \mathrm{i})(2X_2 + \mathrm{i}) = (2x + \mathrm{i})^2 + \varepsilon^2[2(2x + \mathrm{i})(x - t + a_2) - 4(t + a_1)^2] + \mathcal{O}(\varepsilon^4)$$

and

$$\begin{aligned} \sqrt{\delta_1}e^{-\frac{\mathrm{i}}{2}(T_1-T_2)} - \sqrt{\delta_2}e^{\frac{\mathrm{i}}{2}(T_1-T_2)} &= -2\mathrm{i}\left(1 - \frac{\varepsilon^2}{8}\right)\sin\left(\varepsilon(x+b_1) + \frac{\varepsilon^3}{2}(x-t+b_3)\right) \\ &\quad + 2\left(\frac{\varepsilon}{2} + \frac{\varepsilon^3}{16}\right)\cos\left(\varepsilon(x+b_1)\right) + \mathcal{O}(\varepsilon^5) \\ &= \varepsilon(1 - 2\mathrm{i}(x+b_1)) + \varepsilon^3[\frac{\mathrm{i}}{3}(x+b_1)^3 - \mathrm{i}(x-t+b_3) + \frac{\mathrm{i}}{4}(x+b_1) \\ &\quad - \frac{1}{2}(x+b_1)^2 + \frac{1}{8}] + \mathcal{O}(\varepsilon^5). \end{aligned}$$

Substituting expansions into (2.12) yields $F = F_0 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^4)$ with

$$\begin{aligned} F_0 &= -4b_1(2x + \mathrm{i} + b_1), \\ F_2 &= 2(2x+\mathrm{i})(x-t+a_2) - 4(t+a_1)^2 - \frac{1}{2}(1 - 2\mathrm{i}(x+b_1))^2 \\ &\quad + 2(1-2\mathrm{i}(x+b_1))\left[\frac{\mathrm{i}}{3}(x+b_1)^3 - \mathrm{i}(x-t+b_3) + \frac{\mathrm{i}}{4}(x+b_1) - \frac{1}{2}(x+b_1)^2 + \frac{1}{8}\right]. \end{aligned}$$

If $b_1 \neq 0$, then the limit $\varepsilon \to 0$ recovers the single algebraic soliton in the form (2.5). However, if $b_1 = 0$, then we get

$$\begin{aligned}F_2 &= (1-2\mathrm{i}x)\left[\frac{2\mathrm{i}}{3}x^3 + 3\mathrm{i}x - x^2 - \frac{1}{4} + 2\mathrm{i}(a_2-b_3)\right] - 4(t+a_1)^2 \\ &= -\frac{1}{12}\left[3 - 24\mathrm{i}x - 24x^2 - 32\mathrm{i}x^3 - 16x^4 + 48(t+a_1)^2 + 24(b_3-a_2)(2x+\mathrm{i})\right].\end{aligned}$$

For expansion of H, we use

$$\begin{aligned}\delta_1^{1/2}e^{-\mathrm{i}T_1}(2X_2-\mathrm{i}) + \delta_2^{1/2}e^{-\mathrm{i}T_2}(2X_1-\mathrm{i}) = e^{-\mathrm{i}t+\frac{\mathrm{i}}{2}\varepsilon^2(x-t+b_2)}\{2(2x-\mathrm{i}) \\ +\varepsilon^2\left[2(x-t+a_2)+2(t+a_1)(2\mathrm{i}(x+b_1)-1)\right] \\ -\varepsilon^2(2x-\mathrm{i})\left[(x+b_1)^2+\mathrm{i}(x+b_1)+\frac{1}{4}\right]\Big\} + \mathcal{O}(\varepsilon^4)\end{aligned}$$

and

$$\begin{aligned}&\delta_2^{1/2}e^{-\mathrm{i}T_1} - \delta_1^{1/2}e^{-\mathrm{i}T_2} = e^{-\mathrm{i}t+\frac{\mathrm{i}}{2}\varepsilon^2(x-t+b_2)}\times \\ &\left[-2\mathrm{i}\left(1-\frac{\varepsilon^2}{8}\right)\sin\left(\varepsilon(x+b_1)+\frac{\varepsilon^3}{2}(x-t+b_3)\right) - 2\left(\frac{\varepsilon}{2}+\frac{\varepsilon^3}{16}\right)\cos\left(\varepsilon(x+b_1)\right)+\mathcal{O}(\varepsilon^5)\right] \\ &= e^{-\mathrm{i}t+\frac{\mathrm{i}}{2}\varepsilon^2(x-t+b_2)}\{-\varepsilon(1+2\mathrm{i}(x+b_1)) \\ &\quad +\varepsilon^3\left[\frac{\mathrm{i}}{3}(x+b_1)^3+\frac{\mathrm{i}}{4}(x+b_1)-\mathrm{i}(x-t+b_3)+\frac{1}{2}(x+b_1)^2-\frac{1}{8}\right]+\mathcal{O}(\varepsilon^5)\Big\}.\end{aligned}$$

Substituting expansions into (2.13) yields $H = e^{-\mathrm{i}t+\frac{\mathrm{i}}{2}\varepsilon^2(x-t+b_2)}\left[H_0+\varepsilon^2 H_2+\mathcal{O}(\varepsilon^4)\right]$ with

$$\begin{aligned}H_0 &= -8\mathrm{i}b_1, \\ H_2 &= 2\mathrm{i}\left[2(x-t+a_2)+2(t+a_1)(2\mathrm{i}(x+b_1)-1)-(2x-\mathrm{i})\left[(x+b_1)^2+\mathrm{i}(x+b_1)+\frac{1}{4}\right]\right] \\ &\quad +4\left[\frac{\mathrm{i}}{3}(x+b_1)^3+\frac{\mathrm{i}}{4}(x+b_1)-\mathrm{i}(x-t+b_3)+\frac{1}{2}(x+b_1)^2-\frac{1}{8}\right]+2\left[1+2\mathrm{i}(x+b_1)\right].\end{aligned}$$

We confirm again that if $b_1 \neq 0$, then the limit $\varepsilon \to 0$ recovers the single algebraic soliton in the form (2.5). However, if $b_1 = 0$, then we get

$$\begin{aligned}H_2 &= 2\mathrm{i}\left[2(a_2-b_3)+2(t+a_1)(2\mathrm{i}x-1)-\frac{4}{3}x^3-2\mathrm{i}x^2+x-\frac{\mathrm{i}}{2}\right] \\ &= -\frac{1}{3}\left[-3-6\mathrm{i}x-12x^2+8\mathrm{i}x^3+12(t+a_1)(2x+\mathrm{i})+12\mathrm{i}(b_3-a_2)\right].\end{aligned}$$

Similarly, we obtain in the case of $b_1 = 0$ that $G = \varepsilon^2 e^{-\mathrm{i}t} G_2 + \mathcal{O}(\varepsilon^4)$ with

$$G_2 = -\frac{1}{3}\left[-3 + 6\mathrm{i}x - 12x^2 - 8\mathrm{i}x^3 - 12(t + a_1)(2x - \mathrm{i}) - 12\mathrm{i}(b_3 - a_2)\right].$$

The limit $\varepsilon \to 0$ yields a new solution for the algebraic double-soliton in the form:

$$\begin{bmatrix} u_{\text{double}}(x,t) \\ v_{\text{double}}(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{4(-3 + 6\mathrm{i}x - 12x^2 - 8\mathrm{i}x^3 - 12(t+\alpha)(2x - \mathrm{i}) - \mathrm{i}\beta)}{3 + 24\mathrm{i}x - 24x^2 + 32\mathrm{i}x^3 - 16x^4 + 48(t+\alpha)^2 + 2\beta(2x - \mathrm{i})} \\ \dfrac{4(-3 - 6\mathrm{i}x - 12x^2 + 8\mathrm{i}x^3 + 12(t+\alpha)(2x + \mathrm{i}) + \mathrm{i}\beta)}{3 - 24\mathrm{i}x - 24x^2 - 32\mathrm{i}x^3 - 16x^4 + 48(t+\alpha)^2 + 2\beta(2x + \mathrm{i})} \end{bmatrix} e^{-\mathrm{i}t}. \tag{2.16}$$

where $\alpha := a_1$ and $\beta := 12(b_3 - a_2)$ are two real-valued parameters of the solution. Due to the symmetry transformation (2.1), the parameter α is trivial and can be set to 0. The parameter β is nontrivial and gives the asymmetry of the algebraic double-soliton.

Note that we have confirmed the validity of (2.16) by searching for polynomial solutions of the bilinear equations (2.8) with f being polynomial in x of degree 4 and in t of degree 2 and with h and g being polynomials in x of degree 3 and in t of degree 1. The only parameters of the polynomial solutions were found to be $\alpha, \beta \in \mathbb{R}$ as in (2.16) and the translational parameters in (2.1).

The algebraic double-soliton given by (2.16) describes a slow scattering of two identical algebraic solitons. The parameter β describes the distance between the two solitons. Figure 2.2 illustrates the solution surface for $|u|^2 + |v|^2$ versus (x,t) for the family of solutions (2.16) with $\beta = 0, 1, 10, 100$. The solution with $\beta = 0$ is symmetric with the global maximum at $(0,0)$. Since $|u_{\text{double}}(0,0)|^2 + |v_{\text{double}}(0,0)|^2 = 32$ for (2.16) and $|u_{\text{alg}}(0,0)|^2 + |v_{\text{alg}}(0,0)|^2 = 8$ for (2.5), the double-soliton has the quadruple magnification factor for the squared amplitudes compared to the single algebraic soliton.

As β increases, the symmetry is broken and the magnification factor becomes smaller. For sufficiently large β, the two solitons do not overlap but slowly scatter at a distance from each other. As $\beta \to \infty$, one soliton goes to infinity and the other soliton is located near the origin. Indeed, the family of solutions (2.16) converges as $\beta \to \infty$ to a single algebraic soliton (2.5).

We will prove in the next section that

$$Q(u_{\text{double}}, v_{\text{double}}) = 8\pi = 2Q(u_{\text{alg}}, v_{\text{alg}}), \tag{2.17}$$

which implies that the double-soliton (2.16) has a double mass compared to the single algebraic soliton (2.5).

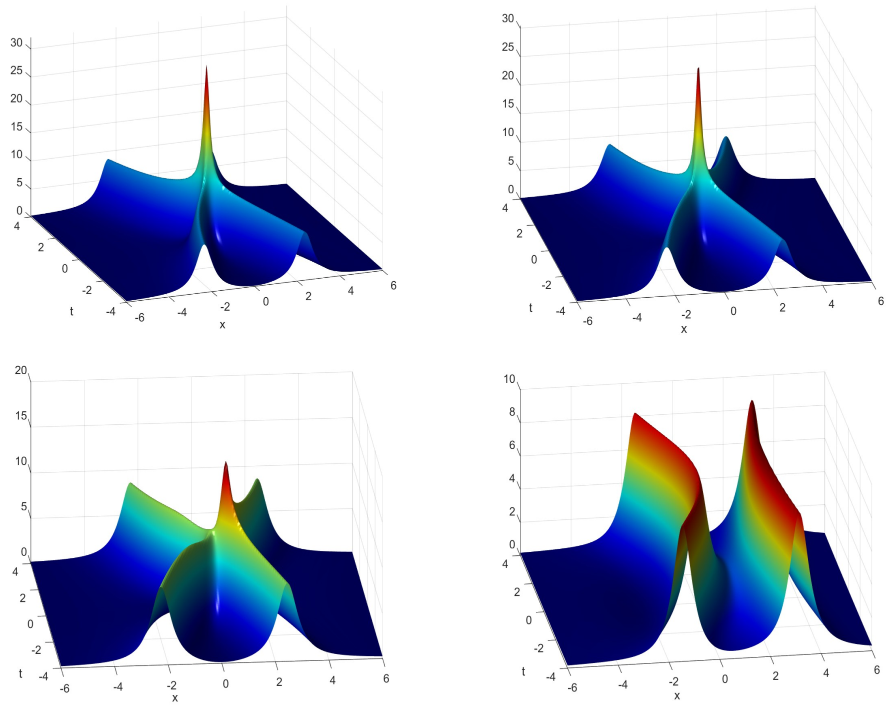

FIGURE 2.2: The solution surface for $|u|^2 + |v|^2$ versus (x, t) for the family (2.16) with $\beta = 0$ (top left), $\beta = 1$ (top right), $\beta = 10$ (bottom left), and $\beta = 100$ (bottom right).

2.5 Mass of the algebraic double-soliton

It follows from (2.7) and (2.8) that

$$|u|^2 + |v|^2 = \frac{|g|^2 + |h|^2}{|f|^2} = 2\mathrm{i}\left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right),$$

where

$$f = 16x^4 + 32\mathrm{i}x^3 + 24x^2 + 24\mathrm{i}x - 3 - 48t^2 - 2\beta(2x + \mathrm{i}),$$

where we have set $\alpha = 0$. We claim that f has no zeros on $\mathbb{R}$ in x for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Indeed, if $x, t, \beta \in \mathbb{R}$, zeros of f must satisfy

$$\begin{cases} 16x^4 + 2x^2 - 3 - 48t^2 - 4\beta x = 0, \\ 32x^3 + 24x - 2\beta = 0. \end{cases}$$

Expressing $\beta = 16x^3 + 12x$ yields $-48x^4 - 24x^2 - 3 - 48t^2 = 0$, which cannot be satisfied for $x, t \in \mathbb{R}$. Hence, there exist no roots of f on $\mathbb{R}$ in x for every $t \in \mathbb{R}$

and $\beta \in \mathbb{R}$. This and the fast decay at infinity,

$$\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}} = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as} \quad |x| \to \infty,$$

justify the applications of Jordan's lemma and the argument principle to compute the integral on $\mathbb{R}$ with techniques of complex analysis:

$$\begin{aligned}\int_{\mathbb{R}} (|u|^2 + |v|^2) dx &= \lim_{R\to\infty} \int_{[-R,R]\cup C_R^+} (|u|^2 + |v|^2) dz \\ &= 2\mathrm{i} \lim_{R\to\infty} \int_{[-R,R]\cup C_R^+} \left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right) dz \\ &= 4\pi (N_{\bar{f}} - N_f),\end{aligned}$$

where C_R^+ is a semicircle of radius R in the upper half of the complex extension of x denoted by $\mathbb{C}_+$, N_f is the number of zeros of f in $\mathbb{C}_+$ and $N_{\bar{f}}$ is the number of zeros of $\bar{f}$ in $\mathbb{C}_+$. Since f has no zeros on $\mathbb{R}$, we have

$$N_{\bar{f}} = \deg(f) - N_f.$$

Since $\deg(f) = 4$, we only need to show that $N_f = 1$ to obtain (2.17). However, this is true as $|t| \to \infty$ due to the representation of f in the equivalent form

$$f = (2x + \mathrm{i})^4 + 12(2x + \mathrm{i})^2 - 4\mathrm{i}(2x + \mathrm{i}) - 2\beta(2x + \mathrm{i}) + 4 - 48t^2,$$

from which we have

$$(2x + \mathrm{i}) = \sqrt[4]{12}\sqrt{|t|}e^{\frac{\mathrm{i}\pi n}{2}} + \mathcal{O}\left(\frac{1}{\sqrt{|t|}}\right) \quad \text{as} \quad |t| \to \infty,$$

where $n = 0, 1, 2, 3$. There is only one root in $\mathbb{C}_+$ which corresponds to $n = 1$. Since the number N_f cannot change in the continuation of f in $t \in \mathbb{R}$, we have $N_f = 1$ for every $t \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Hence $Q(u_{\text{double}}, v_{\text{double}}) = 8\pi$ and (2.17) holds for every $\beta \in \mathbb{R}$.

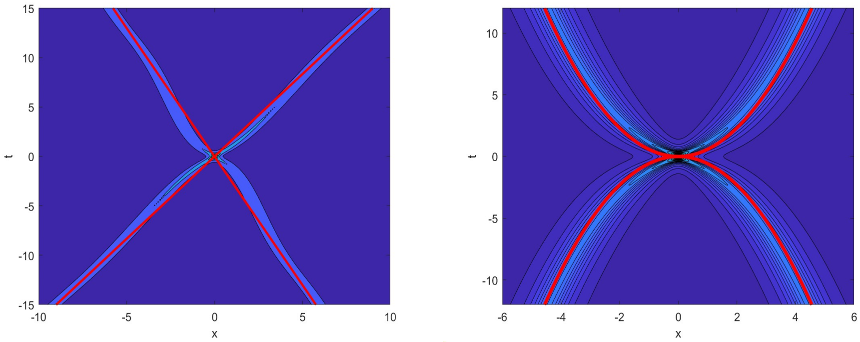

FIGURE 2.3: The contour plots for the solution surfaces from Fig. 2.1 with $\varepsilon = 0.5$ (left) and from Fig. 2.2 with $\beta = 0$ (right). The red lines show the straight lines $x + c_1 t = 0$ and $x + c_2 t = 0$ (left) and the parabolas $x^2 = \sqrt{3}|t|$ (right).

For the normal scattering of two algebraic solitons given by the two-soliton solution (2.15), the algebraic solitons move along straight lines before and after interaction in the overlapping region. No phase shift arises as a result of the soliton interaction, which is a standard feature of algebraic multi-soliton solutions, see [10, 15]. This is illustrated on the contour plot of Figure 2.3 (left panel), where we showed the solution from Figure 2.1 with $\varepsilon = 0.5$ together with the straight lines $x + c_1 t = 0$ and $x + c_2 t = 0$. On the other hand, the slow scattering of two identical solitons given by (2.16) results in the solitons propagating along a curve on the (x, t) plane. Figure 2.3 (right panel) shows the solution from Figure 2.2 with $\beta = 0$ together with the parabolas $x^2 = \sqrt{3}|t|$. The free solitons would be standing waves with $c = 0$ but their slow interaction results in the dynamics along the trajectories at $x^2 \approx \sqrt{3}|t|$ as $|t| \to \infty$ with nonzero but asymptotically vanishing velocities $\frac{dx}{dt} \approx \pm \frac{\sqrt{3}}{2\sqrt{\sqrt{3}|t|}}$.

Chapter 3

Algebraic Soliton Solutions via Double-Wronskian

3.1 Preliminaries

The MTM system (1.1) is a compatibility condition for the Lax pair [24]:

$$\partial_x \vec{\Phi} + L(u,v,\zeta)\vec{\Phi} = 0, \quad \partial_t \vec{\Phi} + A(u,v,\zeta)\vec{\Phi} = 0, \tag{3.1}$$

where $\zeta \in \mathbb{C}$ is the spectral parameter, $\vec{\Phi} = \vec{\Phi}(x,t) \in \mathbb{C}^2$ is the wave function, and the 2-by-2 matrices $L(u,v,\zeta)$ and $A(u,v,\zeta)$ are given by

$$L = \frac{\mathrm{i}}{4}\left(|u|^2 - |v|^2\right)\sigma_3 + \frac{\mathrm{i}}{2}\zeta\begin{pmatrix} 0 & \bar{v} \\ v & 0 \end{pmatrix} + \frac{\mathrm{i}}{2\zeta}\begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} + \frac{\mathrm{i}}{4}\left(\zeta^2 - \zeta^{-2}\right)\sigma_3$$

and

$$A = -\frac{\mathrm{i}}{4}\left(|u|^2 + |v|^2\right)\sigma_3 + \frac{\mathrm{i}}{2}\zeta\begin{pmatrix} 0 & \bar{v} \\ v & 0 \end{pmatrix} - \frac{\mathrm{i}}{2\zeta}\begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} + \frac{\mathrm{i}}{4}\left(\zeta^2 + \zeta^{-2}\right)\sigma_3.$$

For simplicity of computations, we rewrite the MTM system (1.1), the bilinear equations (2.8), and the Lax pair (3.1) in the characteristic variables

$$\begin{cases} \xi = \frac{t+x}{4}, \\ \eta = \frac{t-x}{4}, \end{cases} \quad \Rightarrow \quad \begin{cases} t = 2(\xi+\eta), \\ x = 2(\xi-\eta), \end{cases} \quad \Rightarrow \quad \begin{cases} \partial_\xi = 2(\partial_t + \partial_x), \\ \partial_\eta = 2(\partial_t - \partial_x). \end{cases} \tag{3.2}$$

The MTM system (1.1) transforms into

$$\begin{cases} \mathrm{i}u_\xi + 2v = 2|v|^2 u, \\ \mathrm{i}v_\eta + 2u = 2|u|^2 v. \end{cases} \tag{3.3}$$

The bilinear equations (2.8) transform into

$$\begin{cases} \mathrm{i}D_\xi(g\cdot f)+2h\bar{f}=0,\\ \mathrm{i}D_\eta(h\cdot \bar{f})+2gf=0,\\ \mathrm{i}D_\xi(f\cdot \bar{f})-2h\bar{h}=0,\\ \mathrm{i}D_\eta(\bar{f}\cdot f)-2g\bar{g}=0. \end{cases} \tag{3.4}$$

The Lax pair (3.1) transforms into

$$\partial_\xi\vec{\Phi}+\left(-\mathrm{i}|v|^2\sigma_3+2i\zeta\begin{pmatrix} 0 & \bar{v}\\ v & 0\end{pmatrix}+i\zeta^2\sigma_3\right)\vec{\Phi}=0 \tag{3.5}$$

and

$$\partial_\eta\vec{\Phi}+\left(-\mathrm{i}|u|^2\sigma_3-2i\zeta^{-1}\begin{pmatrix} 0 & \bar{u}\\ u & 0\end{pmatrix}+i\zeta^{-2}\sigma_3\right)\vec{\Phi}=0. \tag{3.6}$$

3.2 Double-Wronskian solutions

To construct the double-Wronskian solutions, we note that solutions of the linear system (3.5)–(3.6) with $(u,v)=(0,0)$ are given in the explicit form

$$\vec{\Phi}=e^{-\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)\sigma_3}\vec{c}, \tag{3.7}$$

where $\vec{c}\in\mathbb{C}^2$. By writing $\vec{\Phi}=(\psi_0,\phi_0)^T$, this yields

$$\psi_0=e^{-\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}c_1,\qquad \phi_0=e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}c_2. \tag{3.8}$$

If $\zeta\in\mathbb{C}$ is an eigenvalue in the first quadrant, so are $\bar{\zeta}$, $-\zeta$, and $-\bar{\zeta}$ in the other three quadrants of the complex plane. Hence, we also obtain another relevant solution of the linear system (3.5)–(3.6) with $(u,v)=(0,0)$:

$$\tilde{\psi}_0=e^{-\mathrm{i}(\bar{\zeta}^2\xi+\bar{\zeta}^{-2}\eta)}\tilde{c}_1,\qquad \tilde{\phi}_0=e^{\mathrm{i}(\bar{\zeta}^2\xi+\bar{\zeta}^{-2}\eta)}\tilde{c}_2. \tag{3.9}$$

one-soliton solution (2.3) can be obtained from (3.8) and (3.9) with the one-fold Darboux transformation. In order to obtain N-soliton solutions with N-fold Darboux transformations and to degenerate the N-soliton solutions in the limit of algebraic solitons, we introduce now the double-Wronskian solutions of the MTM system (3.3) based on the functions generalizing (3.8) and (3.9).

Let $A \in \mathbb{M}^{2N \times 2N}$ be a complex-valued invertible matrix for $N \in \mathbb{N}$. We define two vectors $\phi, \psi \in \mathbb{C}^{2N}$ from solutions of the linear equations

$$\begin{cases} \partial_\xi \phi = \mathrm{i} A \phi, \\ \partial_\eta \phi = \mathrm{i} A^{-1} \phi, \end{cases} \quad \text{and} \quad \begin{cases} \partial_\xi \psi = -\mathrm{i} A \psi, \\ \partial_\eta \psi = -\mathrm{i} A^{-1} \psi, \end{cases} \tag{3.10}$$

which generalize the solutions (3.8) and (3.9). We note that

$$\partial_\eta \partial_\xi \phi = -\phi, \qquad \partial_\eta \partial_\xi \psi = -\psi.$$

We assume that A can be factorized by an invertible matrix $S \in \mathbb{M}^{2N \times 2N}$ as follows

$$A = -S\bar{S}, \quad \bar{A} = -\bar{S}S. \tag{3.11}$$

Furthermore, we related the two vectors $\phi, \psi \in \mathbb{C}^{2N}$ by using

$$\psi = S\bar{\phi}. \tag{3.12}$$

It follows from (3.11) that $AS = S\bar{A}$ and $-I = \bar{S}A^{-1}S$. We recall the conventional notations for double-Wronskian determinants of $(2N) \times (2N)$ matrices:

$$\begin{aligned} |\widehat{N-1}; \widehat{N-1}| &:= |\phi, \phi', \dots, \phi^{(N-1)}, \psi, \psi', \dots, \psi^{(N-1)}|, \\ |\widetilde{N}; \widetilde{N}| &:= |\phi', \phi'', \dots, \phi^{(N)}, \psi', \psi'', \dots, \psi^{(N)}|, \\ |\overline{N+1}; \overline{N+1}| &:= |\phi'', \phi''', \dots, \phi^{(N)}, \psi'', \psi''', \dots, \psi^{(N+1)}|, \end{aligned}$$

where $|A| = \det(A)$ and the prime stands for the derivative with respect to ξ. Similarly, we introduce notations for modifications of the double-Wronskian determinants, e.g.

$$\begin{aligned} |0, \overline{N}; \widehat{N-1}| &= |\phi, \phi'', \dots, \phi^{(N)}, \psi, \psi', \dots, \psi^{(N-1)}|, \\ |\widetilde{N}; -1, \widetilde{N-1}| &= |\phi', \phi'', \dots, \phi^{(N)}, \partial_\xi^{-1}\psi, \psi', \dots, \psi^{(N-1)}|. \end{aligned}$$

The following theorem gives the double-Wronskian solutions of the bilinear equations (3.4).

Theorem 1. *Under the assumptions (3.10), (3.11), and (3.12), the following double-Wronskian*

$$\begin{cases} f = |\widetilde{N}; \widehat{N-1}|, \\ \bar{f} = C|\widetilde{N}; \widetilde{N}|, \end{cases} \quad \begin{cases} g = |\widehat{N}; \widetilde{N-1}|, \\ \bar{g} = \mathrm{i}C|\overline{N}; \widehat{N}|, \end{cases} \quad \begin{cases} h = \mathrm{i}C^{-1}|\widehat{N}; \widehat{N-2}| \\ \bar{h} = C\bar{C}^{-1}|\widetilde{N-1}; \widehat{N}| \end{cases} \tag{3.13}$$

represent exact solutions of the bilinear equations (3.4) with $C := (-\mathrm{i})^N / |S|$.

For the proof of Theorem 1, we recall the following two lemmas used in [23, 30]. Lemma 1 is a restatement of Liouville's theorem for differential equations. Lemma 2 is proven by using properties of determinants.

Lemma 1. *Let $A \in \mathbb{M}^{n\times n}$ and $\{x_1, x_2, \ldots, x_n\} \in \mathbb{C}^n$ for any $n \in \mathbb{N}$. Then*

$$\mathrm{tr}(A)|x_1, x_2, \ldots, x_n| = |Ax_1, x_2, \ldots, x_n| + |x_1, Ax_2, \ldots, x_n| + \cdots + |x_1, x_2, \ldots, Ax_n| \tag{3.14}$$

Lemma 2. *Let $M \in \mathbb{M}^{n\times n-2}$ and $\{a, b, c, d\} \in \mathbb{C}^n$ for any $n \in \mathbb{N}$. Then*

$$|M, a, b||M, c, d| - |M, a, c||M, b, d| + |M, a, d||M, b, c| = 0. \tag{3.15}$$

Proof of Theorem 1. Let us first verify the complex-conjugate symmetry in (3.13). By using (3.12), we have

$$f = |\widetilde{N}; \widehat{N-1}| = |\phi', \phi'', \ldots, \phi^{(N)}, S\bar{\phi}, S\bar{\phi}', \ldots, S\bar{\phi}^{(N-1)}|$$

Taking complex conjugation and using (3.10), (3.11), and (3.12), we get

$$\begin{aligned}
\bar{f} &= |\bar{\phi}', \bar{\phi}'', \ldots, \bar{\phi}^{(N)}, \bar{S}\phi, \bar{S}\phi', \ldots, \bar{S}\phi^{(N-1)}| \\
&= (-1)^N |\bar{S}\phi, \bar{S}\phi', \ldots, \bar{S}\phi^{(N-1)}, \bar{\phi}', \bar{\phi}'', \ldots, \bar{\phi}^{(N)}| \\
&= (-1)^N |S|^{-1} |S\bar{S}\phi, S\bar{S}\phi', \ldots, S\bar{S}\phi^{(N-1)}, S\bar{\phi}', S\bar{\phi}'', \ldots, S\bar{\phi}^{(N)}| \\
&= (-1)^N |S|^{-1} |-A\phi, -A\phi', \ldots, -A\phi^{(N-1)}, \psi', \psi'', \ldots, \psi^{(N)}| \\
&= (-\mathrm{i})^N |S|^{-1} |\phi', \phi'', \ldots, \phi^{(N)}, \psi', \psi'', \ldots, \psi^{(N)}|,
\end{aligned}$$

which confirms $\bar{f} = C|\widetilde{N}; \widetilde{N}|$ with $C = (-\mathrm{i})^N/|S|$. Similarly, we start with

$$g = |\widehat{N}; \widetilde{N-1}| = |\phi, \phi', \ldots, \phi^{(N)}, S\bar{\phi}', S\bar{\phi}'', \ldots, S\bar{\phi}^{(N-1)}|$$

and obtain

$$\begin{aligned}
\bar{g} &= |\bar{\phi}, \bar{\phi}', \ldots, \bar{\phi}^{(N)}, \bar{S}\phi', \bar{S}\phi'', \ldots, \bar{S}\phi^{(N-1)}| \\
&= (-1)^{N+1} |\bar{S}\phi', \bar{S}\phi'', \ldots, \bar{S}\phi^{(N-1)}, \bar{\phi}, \bar{\phi}', \ldots, \bar{\phi}^{(N)}| \\
&= (-1)^{N+1} |S|^{-1} |S\bar{S}\phi', S\bar{S}\phi'', \ldots, S\bar{S}\phi^{(N-1)}, S\bar{\phi}, S\bar{\phi}', \ldots, S\bar{\phi}^{(N)}| \\
&= (-1)^{N+1} |S|^{-1} |-A\phi', -A\phi'', \ldots, -A\phi^{(N-1)}, \psi, \psi', \ldots, \psi^{(N)}| \\
&= (-\mathrm{i})^{N-1} |S|^{-1} |\phi'', \phi''', \ldots, \phi^{(N)}, \psi, \psi', \ldots, \psi^{(N)}|,
\end{aligned}$$

which yields $\bar{g} = \mathrm{i}C|\overline{N};\widehat{N}|$ with the same $C = (-\mathrm{i})^N/|S|$. Finally, we start with

$$h = \mathrm{i}C^{-1}|\widehat{N};\widehat{N-2}| = \mathrm{i}C^{-1}|\phi,\phi',\dots,\phi^{(N)},S\bar{\phi},S\bar{\phi}',\dots,S\bar{\phi}^{(N-2)}|$$

and obtain

$$\begin{aligned}
\bar{h} &= -\mathrm{i}\bar{C}^{-1}|\bar{\phi},\bar{\phi}',\dots,\bar{\phi}^{(N)},\bar{S}\phi,\bar{S}\phi,\dots,\bar{S}\phi^{(N-2)}| \\
&= -\mathrm{i}\bar{C}^{-1}(-1)^{N+1}|\bar{S}\phi,\bar{S}\phi',\dots,\bar{S}\phi^{(N-2)},\bar{\phi},\bar{\phi}',\dots,\bar{\phi}^{(N)}| \\
&= -\mathrm{i}\bar{C}^{-1}(-1)^{N+1}|S|^{-1}|S\bar{S}\phi,S\bar{S}\phi',\dots,S\bar{S}\phi^{(N-2)},S\bar{\phi},S\bar{\phi}',\dots,S\bar{\phi}^{(N)}| \\
&= -\mathrm{i}\bar{C}^{-1}(-1)^{N+1}|S|^{-1}|-A\phi,-A\phi',\dots,-A\phi^{(N-2)},\psi,\psi',\dots,\psi^{(N)}| \\
&= (-\mathrm{i})^N|S|^{-1}\bar{C}^{-1}|\phi',\phi'',\dots,\phi^{(N-1)},\psi,\psi',\dots,\psi^{(N)}|,
\end{aligned}$$

which yields $\bar{h} = C\bar{C}^{-1}|\widetilde{N-1};\widehat{N}|$ with the same $C = (-\mathrm{i})^N/|S|$. It remains to check validity of the four bilinear equations (3.4).

<u>Validity of $\mathrm{i}D_\eta(\bar{f}\cdot f) - 2g\bar{g} = 0$.</u>

By using expression for f and $\bar{f}$ in (3.13), we get

$$\begin{aligned}
\mathrm{i}D_\eta(\bar{f}\cdot f) &= \mathrm{i}(\bar{f}_\eta f - \bar{f}f_\eta) \\
&= \mathrm{i}C|\widetilde{N};\widetilde{N}|\left(|0,\overline{N};\widehat{N-1}| + |\widetilde{N};-1,\widetilde{N-1}|\right) - \mathrm{i}C|\widetilde{N};\widehat{N-1}|\left(|0,\overline{N};\widetilde{N}| + |\widetilde{N};0,\overline{N}|\right) \\
&= 2\mathrm{i}C|\widetilde{N};\widetilde{N}||0,\overline{N};\widehat{N-1}| - 2\mathrm{i}C|\widetilde{N};\widehat{N-1}||0,\overline{N};\widetilde{N}|.
\end{aligned}$$

To get the second equality, we have used

$$\begin{aligned}
\mathrm{tr}(A^{-1})|\widetilde{N};\widehat{N-1}| &= i\left(|0,\overline{N};\widehat{N-1}| - |\widetilde{N};-1,\widetilde{N-1}|\right), \\
\mathrm{tr}(A^{-1})|\widetilde{N};\widetilde{N}| &= i\left(|0,\overline{N};\widetilde{N}| - |\widetilde{N};0,\overline{N}|\right),
\end{aligned}$$

which follow from the identity (3.14) in Lemma 1 with A^{-1}. Combining with $-2g\bar{g}$ from (3.13), we get

$$\mathrm{i}D_\eta(\bar{f}\cdot f) - 2g\bar{g} = 2\mathrm{i}C\left(|\widetilde{N};\widetilde{N}||0,\overline{N};\widehat{N-1}| - |\widetilde{N};\widehat{N-1}||0,\overline{N};\widetilde{N}| - |\widehat{N};\widetilde{N-1}||\overline{N};\widehat{N}|\right).$$

To show that the expression in brackets is identically zero, we use identity (3.15) of Lemma 2 with $M := |\widehat{N};\widetilde{N-1}|$, $a = \phi'$ in the first column, $b = \psi^{(N)}$ in the last column, $c = \phi$ in the first column, and $d = \psi$ in the $(N+1)$-th column. The identity (3.15) holds after rearrangement of the columns provided that the order of vector a, b, c, d appear to be the same in each determinant. Thus, the bilinear equation $\mathrm{i}D_\eta(\bar{f}\cdot f) - 2g\bar{g} = 0$ is verified for the double-Wronskian solution (3.13).

Validity of $\mathrm{i}D_\xi(g\cdot f) + 2h\bar{f} = 0$.
By using expression for g and f in (3.13), we have

$$\begin{aligned}
\mathrm{i}D_\xi(g\cdot f) =& \mathrm{i}(g_\xi f - g f_\xi) \\
=& \mathrm{i}|\widetilde{N}; \widehat{N-1}| \left(|\widehat{N-1}, N+1; \widetilde{N-1}| + |\widehat{N}; \widetilde{N-2}, N| \right) \\
&- \mathrm{i}|\widehat{N}; \widetilde{N-1}| \left(|\widetilde{N-1}, N+1; \widehat{N-1}| + |\widetilde{N}; \widehat{N-2}, N| \right) \\
=& 2\mathrm{i} \left(|\widetilde{N}; \widehat{N-1}||\widehat{N}; \widetilde{N-2}, N| - |\widehat{N}; \widetilde{N-1}||\widetilde{N}; \widehat{N-2}, N| \right).
\end{aligned}$$

To get the second equality, we have used

$$\begin{aligned}
\mathrm{tr}(A)|\widehat{N}; \widetilde{N-1}| &= -\mathrm{i} \left(|\widehat{N-1}, N+1; \widetilde{N-1}| - |\widehat{N}; \widetilde{N-2}, N| \right), \\
\mathrm{tr}(A)|\widetilde{N}; \widehat{N-1}| &= -\mathrm{i} \left(|\widetilde{N-1}, N+1; \widehat{N-1}| - |\widetilde{N}; \widehat{N-2}, N| \right),
\end{aligned}$$

which follows from the identity (3.14). Together with $2h\bar{f}$, we have

$$\begin{aligned}
&\mathrm{i}D_\xi(g\cdot f) + 2h\bar{f} \\
&\quad = 2\mathrm{i} \left(|\widetilde{N}; \widehat{N-1}||\widehat{N}; \widetilde{N-2}, N| - |\widetilde{N}; \widehat{N-2}, N||\widehat{N}; \widetilde{N-1}| + |\widehat{N}; \widehat{N-2}||\widetilde{N}; \widetilde{N}| \right).
\end{aligned}$$

To show that the expression in brackets is identically zero, we use identity (3.15) with $M := (\widetilde{N}; \widetilde{N-2})$, $a = \psi$ in the $(N+1)$-th column, $b = \psi^{(N-1)}$ in the last column, $c = \phi$ in the first column, and $d = \psi^{(N)}$ in the last column. This completes the verification of the bilinear equation $\mathrm{i}D_\xi(g\cdot f) + 2h\bar{f} = 0$ with the double-Wronskian solution (3.13).

Validity of $\mathrm{i}D_\eta(h\cdot\bar{f}) + 2gf = 0$.
By using expression for h and $\bar{f}$ in (3.13), we obtain

$$\begin{aligned}
\mathrm{i}D_\eta(h\cdot\bar{f}) &= \mathrm{i}(h_\eta\bar{f} - h\bar{f}_\eta) \\
&= |\widetilde{N}; \widetilde{N}|(|-1, \widehat{N}; \widehat{N-2}| + |\widehat{N}; -1, \widetilde{N-2}|) - |\widehat{N}; \widehat{N-2}|(|0, \overline{N}; \widetilde{N}| + |\widetilde{N}; 0, \overline{N}|) \\
&= 2|\widetilde{N}; \widetilde{N}||\widehat{N}; -1, \widetilde{N-2}| - 2|\widehat{N}; \widehat{N-2}||\widetilde{N}; 0.\overline{N}|
\end{aligned}$$

To get the second equality, we have used

$$
\begin{aligned}
\operatorname{tr}(A^{-1})|\widehat{N};\widehat{N-2}| &= \mathrm{i}\left(|-1,\widetilde{N};\widehat{N-2}| - |\widehat{N};-1,\widetilde{N-2}|\right), \\
\operatorname{tr}(A^{-1})|\widetilde{N};\widetilde{N}| &= \mathrm{i}\left(|0,\overline{N};\widetilde{N}| - |\widetilde{N};0,\overline{N}|\right),
\end{aligned}
$$

which follow from the identity (3.14) with A^{-1}. Combining with $2gf$, we get

$$
\mathrm{i}D_\eta(h\cdot\bar{f}) + 2gf = 2\left(|\widehat{N};-1,\widetilde{N-2}||\widetilde{N};\widetilde{N}| - |\widehat{N};\widehat{N-2}||\widetilde{N};0,\overline{N}| + |\widetilde{N};\widehat{N-1}||\widehat{N};\widetilde{N-1}|\right).
$$

To show that the expression in brackets is identically zero, we can not use identity (3.15) directly. However, we can write

$$
\begin{aligned}
|\widehat{N};-1,\widetilde{N-2}| &= |\partial_\xi^{-1}\phi',\partial_\xi^{-1}\phi'',\dots,\partial_\xi^{-1}\phi^{(N+1)};\partial_\xi^{-1}\psi,\partial_\xi^{-1}\psi'',\dots,\partial_\xi^{-1}\psi^{(N-1)}| \\
&= |-\partial_\eta\phi',-\partial_\eta\phi'',\dots,-\partial_\eta\phi^{(N+1)};-\partial_\eta\psi,-\partial_\eta\psi'',\dots,-\partial_\eta\psi^{(N-1)}| \\
&= |-\mathrm{i}A^{-1}\phi',-\mathrm{i}A^{-1}\phi'',\dots,-\mathrm{i}A^{-1}\phi^{(N+1)};\mathrm{i}A^{-1}\psi,\mathrm{i}A^{-1}\psi'',\dots,\mathrm{i}A^{-1}\psi^{(N-1)}| \\
&= (-\mathrm{i})^{N+1}\mathrm{i}^{N-1}|A^{-1}||\phi',\phi'',\dots,\phi^{(N+1)};\psi,\psi'',\dots,\psi^{(N-1)}| \\
&= -|A^{-1}||\widetilde{N+1};0,\overline{N-1}|
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
|\widehat{N};\widehat{N-2}| &= -|A^{-1}||\widetilde{N+1};\widetilde{N-1}|, \\
|\widehat{N};\widetilde{N-1}| &= -|A^{-1}||\widetilde{N+1};\overline{N}|.
\end{aligned}
$$

Hence, we rewrite the formula in the equivalent way:

$$
\begin{aligned}
&\mathrm{i}D_\eta(h\cdot\bar{f}) + 2gf \\
&\quad = -2|A^{-1}|\left(|\widetilde{N+1};0,\overline{N-1}||\widetilde{N};\widetilde{N}| - |\widetilde{N+1};\widetilde{N-1}||\widetilde{N};0,\overline{N}| + |\widetilde{N+1};\overline{N}||\widetilde{N};\widehat{N-1}|\right).
\end{aligned}
$$

We can now use identity (3.15) with $M := (\widetilde{N},\overline{N-1})$, $a = \phi^{(N+1)}$ in the $(N+1)$-th column, $b = \psi$ in the $(N+2)$-th column, $c = \psi'$ in the $(N+1)$-th column, and $d = \psi^{(N)}$ in the last column. This yields zero and verifies the bilinear equation $\mathrm{i}D_\eta(h\cdot\bar{f}) + 2gf = 0$ with the double-Wronskian solution (3.13).

Validity of $\mathrm{i}D_\xi(f\cdot\bar f)-2h\bar h=0$.
By using expression for f and $\bar f$ in (3.13), we find

$$\begin{aligned}
\mathrm{i}D_\xi(f\cdot\bar f) =&\mathrm{i}(f_\xi\bar f - f\bar f_\xi)\\
=&\mathrm{i}C|\widetilde{N};\widetilde{N}|\left(|\widetilde{N-1},N+1;\widehat{N-1}|+|\widetilde{N};\widehat{N-2},N|\right)\\
&-\mathrm{i}C|\widetilde{N};\widehat{N-1}|\left(|\widetilde{N-1},N+1;\widetilde{N}|+|\widetilde{N},\widetilde{N-1},N+1|\right)\\
=&2\mathrm{i}C\left(|\widetilde{N};\widetilde{N}||\widetilde{N-1},N+1;\widehat{N-1}|-|\widetilde{N};\widehat{N-1}||\widetilde{N-1},N+1;\widetilde{N}|\right).
\end{aligned}$$

To get the second equality, we have used

$$\begin{aligned}
\mathrm{tr}(A)|\widetilde{N};\widetilde{N}| &= -\mathrm{i}\left(|\widetilde{N-1},N+1;\widetilde{N}|-|\widetilde{N};\widetilde{N-1},N+1|\right),\\
\mathrm{tr}(A)|\widetilde{N};\widehat{N-1}| &= -\mathrm{i}\left(|\widetilde{N-1},N+1;\widehat{N-1}|-|\widetilde{N};\widehat{N-2},N|\right),
\end{aligned}$$

which follow from the identity (3.14). Together with the term $-2h\bar h$, we have

$$\begin{aligned}
\mathrm{i}D_\xi(f\cdot\bar f)-2h\bar h = 2\mathrm{i}C&\left(|\widetilde{N};\widetilde{N}||\widetilde{N-1},N+1;\widehat{N-1}|-|\widetilde{N};\widehat{N-1}||\widetilde{N-1},N+1;\widetilde{N}|\right)\\
&-2\mathrm{i}\bar C^{-1}|\widehat{N};\widehat{N-2}||\widetilde{N-1};\widehat{N}|
\end{aligned}$$

In order to use the identity (3.15), we need to rewrite the last term in the equivalent way. Since

$$\begin{aligned}
|\widehat{N};\widehat{N-2}| &= |\phi,\phi',\ldots,\phi^{(N)};\psi,\psi',\ldots,\psi^{(N-2)}|\\
&= |A^{-1}|(-\mathrm{i})^{N+1}i^{N-1}|iA\phi,iA\phi',\ldots,iA\phi^{(N)};-\mathrm{i}A\psi,-\mathrm{i}A\psi',\ldots,-\mathrm{i}A\psi^{(N-2)}|\\
&= -|A^{-1}||\widetilde{N+1};\widetilde{N-1}|,
\end{aligned}$$

we use $|A^{-1}|=(|S||\bar S|)^{-1}$ and $C=(-\mathrm{i})^N/|S|$ to rewrite

$$\begin{aligned}
&\mathrm{i}D_\xi(f\cdot\bar f)-2h\bar h\\
&=2\mathrm{i}C\Big(|\widetilde{N};\widetilde{N}||\widetilde{N-1},N+1;\widehat{N-1}|-|\widetilde{N};\widehat{N-1}||\widetilde{N-1},N+1;\widetilde{N}|\\
&\quad+|\widetilde{N+1};\widetilde{N-1}||\widetilde{N-1};\widehat{N}|\Big).
\end{aligned}$$

We can now use identity (3.15) with $M:=(\widetilde{N-1};\widetilde{N-1})$, $a:=\phi^{(N)}$ in the N-th column, $b:=\psi^{(N)}$ in the last column, $c:=\phi^{(N+1)}$ in the N-th column, and $d:=\psi$ in the $(N+1)$-th column. This yields zero in the brackets and verifies the bilinear equation $\mathrm{i}D_\xi(f\cdot\bar f)-2h\bar h=0$ with the double-Wronskian solution (3.13). $\square$

3.3 one-soliton solutions via double-Wronskian

We shall recover the one-soliton solution (2.3) by using the double-Wronskian solutions (3.13) generated from (3.10), (3.11), and (3.12). For $N=1$, we define

$$A := \begin{bmatrix} e^{\mathrm{i}\gamma} & 0 \\ 0 & e^{-\mathrm{i}\gamma} \end{bmatrix}, \quad \text{and} \quad \mathrm{S} := \begin{bmatrix} 0 & e^{\frac{\mathrm{i}\gamma}{2}} \\ -e^{\frac{-\mathrm{i}\gamma}{2}} & 0 \end{bmatrix}, \tag{3.16}$$

where $\gamma\in(0,\pi)$ is an arbitary parameter. The choice of A agrees with the solutions (3.8) and (3.9) for $\zeta=e^{\frac{i\gamma}{2}}$ in the first quadrant of the complex plane. We confirm that S satisfies the identity $A=-S\bar S$ in (3.11). Using (3.10) and (3.12), we get

$$\phi = \begin{bmatrix} c_1 e^{\mathrm{i}e^{\mathrm{i}\gamma}\xi+\mathrm{i}e^{-\mathrm{i}\gamma}\eta} \\ c_2 e^{\mathrm{i}e^{-\mathrm{i}\gamma}\xi+\mathrm{i}e^{\mathrm{i}\gamma}\eta} \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \bar c_2 e^{\frac{\mathrm{i}\gamma}{2}-ie^{\mathrm{i}\gamma}\xi-\mathrm{i}e^{-\mathrm{i}\gamma}\eta} \\ -\bar c_1 e^{-\frac{\mathrm{i}\gamma}{2}-ie^{-\mathrm{i}\gamma}\xi-\mathrm{i}e^{\mathrm{i}\gamma}\eta} \end{bmatrix}$$

Then, we get from (3.13) with $C=-\mathrm{i}/|S|=-\mathrm{i}$ that

$$\begin{cases} f &= |\phi';\psi| &= -\mathrm{i}\left(|c_1|^2 e^{\frac{\mathrm{i}\gamma}{2}-2(\xi-\eta)\sin\gamma}+|c_2|^2 e^{-\frac{\mathrm{i}\gamma}{2}+2(\xi-\eta)\sin\gamma}\right), \\ g &= |\phi,\phi'| &= 2c_1c_2\sin\gamma e^{2\mathrm{i}(\xi+\eta)\cos\gamma}, \\ h &= -|\phi,\phi'| &= -2c_1c_2\sin\gamma e^{2\mathrm{i}(\xi+\eta)\cos\gamma}, \end{cases}$$

which generate due to (2.7) and (3.2) for $c_1=1$ and $c_2=\mathrm{i}$ the exact solution identical to (2.3). In addition, we check the validity of the complex-conjugate equations (3.13):

$$\begin{cases} \bar f &= -\mathrm{i}|\phi';\psi'| &= \mathrm{i}\left(|c_1|^2 e^{-\frac{\mathrm{i}\gamma}{2}-2(\xi-\eta)\sin\gamma}+|c_2|^2 e^{\frac{\mathrm{i}\gamma}{2}+2(\xi-\eta)\sin\gamma}\right), \\ \bar g &= |\psi,\psi'| &= 2\bar c_1\bar c_2\sin\gamma e^{-2\mathrm{i}(\xi+\eta)\cos\gamma}, \\ \bar h &= -|\psi,\psi'| &= -2\bar c_1\bar c_2\sin\gamma e^{-2\mathrm{i}(\xi+\eta)\cos\gamma}. \end{cases}$$

Thus, the validity of Theorem 1 for a diagonal 2×2 matrix A has been verified by comparison with the exact one-soliton solutions obtained via exponential functions.

Remark 1. *If $A=\mathrm{diag}(a,b)$ is a more general diagonal matrix with $a,b\in\mathbb{C}$, then we show that $b=\bar a$. Indeed, if*

$$A := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \text{and} \quad \mathrm{S} := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

then constraint $A = -S\bar{S}$ *implies*

$$\begin{aligned} -a &= |\alpha|^2 + \beta\bar{\gamma}, \\ 0 &= \alpha\bar{\beta} + \beta\bar{\delta}, \\ 0 &= \bar{\alpha}\gamma + \bar{\gamma}\delta, \\ -b &= \gamma\bar{\beta} + |\delta|^2. \end{aligned}$$

If $\beta, \gamma \neq 0$*, then the second and third equations imply that* $|\alpha|^2 = |\delta|^2$*. It follows then from the first and fourth equations that* $a = \bar{b}$*. By defining* $a = e^{i\gamma}$ *and* $b = e^{-\mathrm{i}\gamma}$ *as in (3.16), we satisfy this constraint and normalize* $|a| = |b| = 1$ *for one-solitons with zero speed.*

To recover the algebraic one-soliton solution (2.5) by using the double-Wronskian solution (3.13) with $N = 1$, we define

$$A := \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad \mathrm{S} := \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \tag{3.17}$$

which satisfies $A = -S\bar{S}$. The choice of A agrees with the solution (3.8) and its derivative with respect to ζ^2 at $\zeta = i$. Using (3.10) and (3.12) with these A and S, we get

$$\phi = \begin{bmatrix} c_1 e^{-\mathrm{i}(\xi+\eta)} \\ (c_2 + \mathrm{i}(\xi - \eta)c_1)e^{-\mathrm{i}(\xi+\eta)} \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \bar{c}_1 e^{\mathrm{i}(\xi+\eta)} \\ (\bar{c}_2 - \mathrm{i}(\xi - \eta)\bar{c}_1 - \frac{1}{2}\bar{c}_1)e^{\mathrm{i}(\xi+\eta)} \end{bmatrix}$$

Then, we get from (3.13) with $C = -\mathrm{i}/|S| = -\mathrm{i}$ that

$$\begin{cases} f &= |\phi'; \psi| &= \mathrm{i}(\bar{c}_1 c_2 - c_1 \bar{c}_2) - 2(\xi - \eta)|c_1|^2 - \frac{i}{2}|c_1|^2, \\ g &= |\phi, \phi'| &= \mathrm{i}c_1^2 e^{-2\mathrm{i}(\xi+\eta)}, \\ h &= -|\phi, \phi'| &= -\mathrm{i}c_1^2 e^{-2\mathrm{i}(\xi+\eta)}, \end{cases}$$

which generate due to (2.7) and (3.2) for $c_1 = 1$ and $c_2 = 0$ the exact solution identical to (2.5). The validity of the complex-conjugate equation (3.13) for $\bar{f}$, $\bar{g}$, and $\bar{h}$ is obtained by similar computations. Thus, the validity of Theorem 1 for a 2×2 Jordan block of the matrix A associated with eigenvalue $\zeta = \mathrm{i}$ has been verified by comparison with the exact algebraic one-soliton solutions obtained via polynomial functions.

3.4 Hierarchy of rational solutions

For every $N \geq 1$, let I be the identity $(2N) \times (2N)$ matrix and L be the nilpotent $(2N) \times (2N)$ matrix with the only nonzero entries being ones on the first lower diagonal. The j-th power of L has ones at the j-th lower diagonal for $j = 1, 2, \dots, 2N-1$, whereas $L^{2N} = 0$.

For the Nth-order rational solution which corresponds to the N-multiple eigenvalue at $\zeta = i$, we define

$$A = -I + L, \qquad A^{-1} = -I - L - L^2 - \dots - L^{2N-1}. \tag{3.18}$$

This choice generalizes (3.17) for $N \geq 1$.

Vector $\phi \in \mathbb{C}^{2N}$ satisfies the first system in (3.10), from which we derive the following recurrent equations for components of ϕ:

$$\partial_\xi \phi_j = -\mathrm{i}\phi_j + \mathrm{i}\phi_{j-1}, \quad j = 1, 2, \dots, 2N \tag{3.19}$$

closed with $\phi_0 \equiv 0$ and

$$\partial_\eta \phi_j = -\mathrm{i}\phi_j - \mathrm{i}\phi_{j-1} - \mathrm{i}\phi_{j-2} - \dots - \mathrm{i}\phi_1, \quad j = 1, 2, \dots, 2N. \tag{3.20}$$

The other vector $\psi \in \mathbb{C}^{2N}$ is defined by $\psi = S\bar{\phi}$ as in (3.11) and (3.12), where the matrix S is obtained in the following lemma.

Lemma 3. *Solution of the matrix equation $-S^2 = A = -I + L$ is given by*

$$S = I - \frac{1}{2}L - \frac{1}{2^3}L^2 - \dots - \frac{(2m-3)!!}{m!2^m}L^m - \dots - \frac{(4N-5)!!}{(2N-1)!2^{2N-1}}L^{2N-1}. \tag{3.21}$$

Proof. For the Jordan block $J = \lambda I + L \in \mathbb{M}^{n\times n}$ associated with any value of the spectral parameter $\lambda \in \mathbb{C}$, we use the following Taylor expansion for every smooth function $f : \mathbb{R} \to \mathbb{R}$ extended to matrices as $f : \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$:

$$\begin{aligned} f(J) &= f(\lambda)I + f'(\lambda)L + \frac{1}{2!}f''(\lambda)L^2 + \dots + \frac{1}{m!}f^{(m)}(\lambda)L^m + \dots \\ &= \begin{pmatrix} f(\lambda) & 0 & 0 & \dots & 0 \\ f'(\lambda) & f(\lambda) & 0 & \dots & 0 \\ \frac{1}{2!}f''(\lambda) & f'(\lambda) & f(\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{1}{(n-1)!}f^{(n-1)}(\lambda) & \frac{1}{(n-2)!}f^{(n-2)}(\lambda) & \frac{1}{(n-3)!}f^{(n-3)}(\lambda) & \dots & f(\lambda) \end{pmatrix} \end{aligned}$$

We apply this formula for $A = -I + L$ with $f(\lambda) = \sqrt{\lambda}$ and $\lambda = -1$. We get recursively

$$f(-1) = \mathrm{i}, \quad f'(-1) = -\frac{\mathrm{i}}{2}, \quad f''(-1) = -\frac{\mathrm{i}}{2^2}, \quad f'''(-1) = -\frac{\mathrm{i}3!!}{2^3}, \quad f''''(-1) = -\frac{\mathrm{i}5!!}{2^4},$$

and generally,

$$f^{(m)}(-1) = -\frac{\mathrm{i}(2m-3)!!}{2^m}, \quad m \in \mathbb{N}.$$

Defining $S = -\mathrm{i}\sqrt{A}$ and dividing $f^{(m)}(-1)$ by $m!$ yields (3.21). □

Let us define the fundamental solution of equations (3.19) and (3.20) by using the generating function as in

$$\phi_j = \frac{1}{(j-1)!}\partial_{\zeta^2}^{j-1} e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}|_{\zeta^2=-1}, \quad j = 1, 2, \ldots, 2N. \tag{3.22}$$

The example for $N = 3$ yields

$$\begin{aligned}
\phi_1 &= e^{-\mathrm{i}(\xi+\eta)}, \\
\phi_2 &= \mathrm{i}(\xi-\eta)e^{-\mathrm{i}(\xi+\eta)}, \\
\phi_3 &= \left[\frac{\mathrm{i}^2}{2!}(\xi-\eta)^2 - \mathrm{i}\eta\right] e^{-\mathrm{i}(\xi+\eta)}, \\
\phi_4 &= \left[\frac{\mathrm{i}^3}{3!}(\xi-\eta)^3 + \eta(\xi-\eta) - \mathrm{i}\eta\right] e^{-\mathrm{i}(\xi+\eta)}, \\
\phi_5 &= \left[\frac{\mathrm{i}^4}{4!}(\xi-\eta)^4 + \frac{\mathrm{i}}{2!}\eta(\xi-\eta)^2 + \eta(\xi-\eta) - \frac{1}{2!}\eta^2 - \mathrm{i}\eta\right] e^{-\mathrm{i}(\xi+\eta)}, \\
\phi_6 &= \left[\frac{\mathrm{i}^5}{5!}(\xi-\eta)^5 + \frac{\mathrm{i}^2}{3!}\eta(\xi-\eta)^3 + \frac{\mathrm{i}}{2!}\eta(\xi-\eta)^2 - \frac{\mathrm{i}}{2!}\eta^2(\xi-\eta) + \eta(\xi-\eta) - \eta^2 - \mathrm{i}\eta\right] e^{-\mathrm{i}(\xi+\eta)}.
\end{aligned}$$

For parameterization of a general rational solution, we take a linear combination of the fundamental solutions (3.22) with $2N$ complex parameters:

$$\phi_j = \sum_{k=1}^{j} \frac{c_k}{(j-k)!}\partial_{\zeta^2}^{j-k} e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}|_{\zeta^2=-1}, \quad j = 1, 2, \ldots, 2N. \tag{3.23}$$

The following lemma gives the exact count of arbitrary real parameters in the general rational solution.

Lemma 4. *Let ϕ be defined by (3.23) with $c_1, c_2, \ldots, c_{2N} \in \mathbb{C}$ and $\psi = S\bar{\phi}$ be defined by (3.21). The double-Wronskian solutions (3.13) depend on $2N$ arbitrary* *real parameters.*

Proof. The representation (3.23) can be rewritten in the equivalent form

$$\phi_j = \frac{1}{(j-1)!}\partial_{\zeta^2}^{j-1}\left(\sum_{k=1}^{j} a_k(\zeta^2+1)^{k-1}\right) e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta+\sum_{k=1}^{j} b_k(\zeta^2+1)^{k-1})}|_{\zeta^2=-1}, \quad (3.24)$$

where $a_1, a_2, \ldots, a_{2N}, b_1, b_2, \ldots, b_{2N} \in \mathbb{R}$. Without loss of generality, we can set $a_1 = 1$ because a_1 can be scaled out by choosing $a_j = a_1\tilde{a}_j$ with new $\tilde{a}_j$ for $j = 2, 3, \ldots, 2N$ and the parameter a_1 is canceled in the quotients (2.7) with (3.13). With $a_1 = 1$, we have the recursive structure

$$\phi_j = \tilde{\phi}_j + a_2\tilde{\phi}_{j-1} + \cdots + a_j\tilde{\phi}_1, \quad j = 1, 2, \ldots, 2N,$$

where $\tilde{\phi}_j$ is obtained from (3.24) with $a_1 = 1$ and $a_2 = \cdots = a_{2N} = 0$. Similarly, we have

$$\psi_j = \tilde{\psi}_j + a_2\tilde{\psi}_{j-1} + \cdots + a_j\tilde{\psi}_1, \quad j = 1, 2, \ldots, 2N,$$

where $\tilde{\psi}_j = S\bar{\tilde{\phi}}_j$. Due to the row structure of the double-Wronskian solutions, all terms with $a_2, \ldots, a_{2N}$ give no contribution in the determinants. Hence, we can set $a_2 = \cdots = a_{2N} = 0$ without loss of generality. The double-Wronskian solutions only depends on $2N$ real parameters $b_1, b_2, \ldots, b_{2N}$ which are generally irreducible. □

Remark 2. *The result of Lemma 4 suggests that the Nth-order rational solution represents N copies of identical algebraic solitons, where each soliton has its own pair of two translational parameters generated by the translational symmetries in (x,t). No additional parameters arise due to the rotational phase symmetry, see (2.1).*

The following theorem characterizes the Nth-order rational solution obtained from the fundamental solution (3.22) with zero values of the arbitrary parameters in Lemma 4.

Theorem 2. *Let A and S be given by (3.18) and (3.21) with ϕ defined by (3.22) and $\psi = S\bar{\phi}$. The double-Wronskian solutions (3.13) generate the rational solutions of the MTM system (1.1) in the form:*

$$u = \frac{Q_N(x,t)}{\bar{P}_N(x,t)}e^{-\mathrm{i}t}, \quad v = \frac{R_N(x,t)}{P_N(x,t)}e^{-\mathrm{i}t}, \quad (3.25)$$

where P_N is a polynomial of degree N^2 in x and Q_N, R_N are polynomials of degree N^2-1 in x.

Proof. We observe from (3.22) that

$$\phi_j(\xi,\eta) = \left[\frac{\mathrm{i}^{j-1}}{(j-1)!}(\xi-\eta)^{j-1} + \eta p_{j-3}(\xi-\eta,\eta)\right] e^{-\mathrm{i}(\xi+\eta)}, \quad j = 1,2,\dots,2N, \tag{3.26}$$

where p_{j-3} is a polynomial in variables $\xi - \eta = \frac{1}{2}x$ and $\eta = \frac{1}{4}(t-x)$. The degree of polynomials P_N, Q_N, R_N in x can be obtained by inspecting the leading order of f, g, h with the first dominant term in (3.26). By (3.13) with (3.18), we have

$$\begin{aligned} f &= |\phi', \phi'', \dots, \phi^{(N)}, \psi, \psi', \dots, \psi^{(N-1)}| \\ &= |\phi', -\mathrm{i}\phi' + \mathrm{i}L\phi', \dots, \phi^{(N)}, \psi, \mathrm{i}\psi - \mathrm{i}L\psi, \dots, \psi^{(N-1)}| \\ &= |\phi', L\phi', \dots, \phi^{(N)}, \psi, L\psi, \dots, \psi^{(N-1)}|. \end{aligned}$$

where the factor i disappears due to compensation between $\mathrm{i}L\phi'$ and $-\mathrm{i}L\psi$. Continuing by induction, we reduce this expression to

$$f = |\phi', L\phi', \dots, L^{N-1}\phi', \psi, L\psi, \dots, L^{N-1}\psi'|.$$

Similarly, we reduce g and h to

$$\begin{aligned} g &= |\phi, \phi', \dots, \phi^{(N)}, \psi', \psi'', \dots, \psi^{(N-1)}| \\ &= \mathrm{i}^{2N-1}|\phi, L\phi, \dots, L^N\phi, \psi', L\psi', \dots, L^{N-2}\psi'| \end{aligned}$$

and

$$\begin{aligned} h &= \mathrm{i}C^{-1}|\phi, \phi', \dots, \phi^{(N)}, \psi, \psi', \dots, \psi^{(N-2)}| \\ &= \mathrm{i}C^{-1}\mathrm{i}^{2N-1}|\phi, L\phi, \dots, L^N\phi, \psi, L\psi, \dots, L^{N-2}\psi|, \end{aligned}$$

where the factor i^{2N-1} is due to the two columns with $\phi^{(N-1)}$ and $\phi^{(N)}$ which are not compensated by the columns from ψ.

Since $S\bar{\phi} = \bar{\phi} + \mathcal{O}(L\bar{\phi})$ and $\phi' = (-\mathrm{i})\phi + \mathcal{O}(L\phi)$ at the leading polynomial order, see (3.21) and (3.26), the leading-order part of the polynomial f in variable $z := \mathrm{i}(\xi-\eta) = \frac{\mathrm{i}}{2}x$ is given by

$$f = (-\mathrm{i})^N \left|B_N(z); B_N(-z)\right| \left[1 + \mathcal{O}(z^{-1})\right],$$

where $B_N(z)$ is the block of size $2N \times N$ given by

$$B_N(z) := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ z & 1 & 0 & 0 & \dots \\ \frac{1}{2!}z^2 & z & 1 & 0 & \dots \\ \frac{1}{3!}z^3 & \frac{1}{2!}z^2 & z & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Due to the hierarchic structure of $B_N(\pm z)$ in powers of z, it follows that

$$(B_N(z); B_N(-z)) = D_-(z)\,(B_N(1); B_N(-1))\,D_+(z),$$

where

$$D_- = \mathrm{diag}(z^{-N+1}, z^{-N+2}, \dots, 1; z, z^2, \dots, z^N),$$
$$D_+ = \mathrm{diag}(z^{N-1}, z^{N-2}, \dots, 1; z^{N-1}, z^{N-2}, \dots, 1).$$

This yields the result

$$|B_N(z); B_N(-z)| = z^{N^2} |B_N(1); B_N(-1)|$$

since

$$\sum_{j=1}^{N} j + \sum_{j=1}^{N-1} j = \frac{N(N+1)}{2} + \frac{N(N-1)}{2} = N^2.$$

If the numerical coefficient $|B_N(1); B_N(-1)|$ is nonzero, the degree of P_N in x is given by the leading-order term z^{N^2}, which completes the proof of the assertion.

To show that $|B_N(1); B_N(-1)| \neq 0$, we use elementary row operations:

$$|B_N(1); B_N(-1)| = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots & -1 & 1 & 0 & 0 & \dots \\ \frac{1}{2!} & 1 & 1 & 0 & \dots & \frac{1}{2!} & -1 & 1 & 0 & \dots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \dots & -\frac{1}{3!} & \frac{1}{2!} & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$= 2^N(-1)^N \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2!} & 0 & 1 & 0 & \dots & 0 & 1 & 0 & 0 & \dots \\ 0 & \frac{1}{2!} & 0 & 1 & \dots & \frac{1}{3!} & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

where we first added each j-th and $(N+j)$-th columns for $1 \le j \le N$, then extracted the factor of 2 from the first N columns, and finally subtracted the resulting j-th column from the $(N+j)$-th column for $1 \leq j \leq N$ and multiplied the last N columns by the negative signs. Continuing the elementary row operations we obtained a general formula:

$$|B_N(1); B_N(-1)| = \frac{2^N(-1)^N}{1^{2N-1}3^{2N-3}5^{2N-5}7^{2N-7}\dots(2N-1)^1}, \tag{3.27}$$

which was confirmed for $1 \leq N \leq 15$ by using numerical computations.

Expressions for g and h are not polynomials but they are given by the polynomial multiplied by $e^{-2\mathrm{i}(\xi+\eta)} = e^{-\mathrm{i}t}$. Therefore, we can compute the leading-order parts of g and h as

$$g = \mathrm{i}^{3N-2}e^{-\mathrm{i}t}\,|B_{N+1}(z); B_{N-1}(-z)|\left[1 + \mathcal{O}(z^{-1})\right]$$

and

$$h = \mathrm{i}^{2N}C^{-1}e^{-\mathrm{i}t}\,|B_{N+1}(z); B_{N-1}(-z)|\left[1 + \mathcal{O}(z^{-1})\right],$$

with the same determinants which consist of two blocks of nonequal sizes $(2N) \times (N+1)$ and $(2N) \times (N-1)$. Again, we can factorize the matrix as

$$(B_{N+1}(z); B_{N-1}(-z)) = \tilde{D}_-(z)\,(B_{N+1}(1); B_{N-1}(-1))\,\tilde{D}_+(z),$$

where

$$\begin{aligned}\tilde{D}_- &= \mathrm{diag}(z^{-N}, z^{-N+1}, \dots, 1; z, z^2, \dots, z^{N-1}),\\ \tilde{D}_+ &= \mathrm{diag}(z^N, z^{N-1}, \dots, 1; z^N, z^{N-1}, \dots, z^2).\end{aligned}$$

This yields the result

$$|B_{N+1}(z); B_{N-1}(-z)| = z^{N^2-1}\,|B_{N+1}(1); B_{N-1}(-1)|$$

since

$$\sum_{j=1}^{N-1} j + \sum_{j=1}^{N} j - 1 = \frac{N(N-1)}{2} + \frac{N(N+1)}{2} - 1 = N^2 - 1.$$

We have again verified numerically for $1 \leq N \leq 15$ that

$$\begin{aligned}|B_{N+1}(1); B_{N-1}(-1)| &= \frac{N}{2}|B_N(1); B_N(-1)|\\ &= \frac{2^{N-1}(-1)^N N}{1^{2N-1}3^{2N-3}5^{2N-5}7^{2N-7}\dots(2N-1)^1}.\end{aligned} \tag{3.28}$$

Since the numerical coefficients $|B_N(1); B_N(-1)|$ and $|B_{N+1}(1); B_{N-1}(-1)|$ are nonzero, the degrees of P_N, Q_N and R_N in x are given by the leading-order terms z^{N^2} and z^{N^2-1}. □

Remark 3. *By using expressions in the proof of Theorem 2, we can compute the leading-order behavior of the Nth-order rational solutions in (3.25) as $|x| \to \infty$. Since $z = \frac{i}{2}x$, we get*

$$\frac{Q_N(x,t)}{\bar{P}_N(x,t)} \sim \frac{i^{3N-2} z^{N^2-1} |B_{N+1}(1); B_{N-1}(-1)|}{i^N \bar{z}^{N^2} |B_N(1); B_N(-1)|}$$
$$= \frac{2i|B_{N+1}(1); B_{N-1}(-1)|}{|B_N(1); B_N(-1)| x}$$
$$= \frac{iN}{x}.$$

This agrees with the particular result (2.16) for $N = 2$. In view of the theory of embedded eigenvalues in [19], the leading-order behavior of $|u(x,t)| \sim \frac{N}{|x|}$ as $|x| \to \infty$ suggests that the corresponding rational solutions are related to the multiple embedded eigenvalue $\lambda = i$ of geometric multiplicity one and algebraic multiplicity N. For $N = 2$, this has been proven in [22].

The following lemma ensures that the Nth-order rational solution of Theorem 2 is bounded for all real values of (x,t).

Lemma 5. *Let P_N, Q_N, and R_N be polynomials in the solution (3.25) of Theorem 2. Then, (u,v) are real analytic functions of (x,t).*

Proof. If $P_N(x,t) \neq 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}$, then zeros of the polynomial P_N of degree N^2 in x are bounded away from the real axis for every $t \in \mathbb{R}$ so that (u,v) are real analytic functions of (x,t).

Assume now that P_N has a zero at $(x,t) = (x_0,t_0) \in \mathbb{R} \times \mathbb{R}$ of multiplicity M. Without loss of generality, we can fix $\eta = \eta_0$ and consider the behavior of $P_N = (\xi - \xi_0)^M \tilde{P}_N$, where $\tilde{P}_N$ is a polynomial of degree $N^2 - M$ and $\tilde{P}_N(\xi_0,\eta_0) \neq 0$. Due to reality of (ξ_0,η_0), both $f = P_N$ and $\bar{f} = \bar{P}_N$ have a zero at (ξ_0,η_0) of the same multiplicity M so that $iD_\xi(f \cdot \bar{f})$ in the third bilinear equation of the system (3.4) has a zero at (ξ_0,η_0) of multiplicity $2M$. Hence, $h = R_N e^{-it}$ has a zero at (ξ_0,η_0) of multiplicity M. The first bilinear equation in the system (3.4) implies that $g = Q_N e^{-it}$ has also a zero at (ξ_0,η_0) of multiplicity N. Thus, $Q_N = (\xi - \xi_0)^M \tilde{Q}_N$ and $R_n = (\xi - \xi_0)^M \tilde{R}_N$ with polynomial $\tilde{Q}_N, \tilde{R}_N$ satisfying $\tilde{Q}_N(\xi_0,\eta_0) \neq 0$ and $\tilde{R}_N(\xi_0,\eta_0) \neq 0$. Hence ξ_0 is a removable singularity of the rational functions $Q_N/\bar{P}_N$ and R_N/P_N so that (u,v) are real analytic functions of (x,t). The same analysis holds for fixed $\xi = \xi_0$ with respect to η, but we use the second and fourth bilinear equations in the system (3.4) to prove that η_0 is a removable singularity of the rational functions. □

Remark 4. *We have conjectured that* $P_N(x,t) \neq 0$ *for all* $(x,t) \in \mathbb{R} \times \mathbb{R}$ *but we were not able to prove it directly for* $N \geq 3$. *As Lemma 5 shows, this property is not important since the bilinear equations (3.4) ensure that even if* P_N *vanishes at some* $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}$, *then* (x_0, t_0) *is a removable singularity of the real analytic functions of* (x, t).

3.5 Examples for $N = 2, 3$

Based on the explicit computations of the rational solutions with $N = 2, 3$, we conjecture that the Nth-order rational solution describes an interaction of N identical algebraic solitons for every $N \in \mathbb{N}$. Moreover, for every $t \in \mathbb{R}$, P_N admits no zeros on the real axis of x, $\frac{N(N-1)}{2}$ poles in the upper half-plane, and $\frac{N(N+1)}{2}$ poles in the lower half-plane. The argument principle used in Section 2.5 for $N = 2$ suggests that the mass conservation yields

$$\int_{\mathbb{R}} (|u|^2 + |v|^2) dx = 4\pi N, \tag{3.29}$$

which is exactly N multiple of the mass of a single algebraic soliton. Moreover, as $|t| \to \infty$, we have exactly N algebraic solitons diverging from each other with the distance growing as $\sqrt{|t|}$. These conjectures are illustrated with the explicit examples for $N = 2, 3$.

3.5.1 Double algebraic solitons with $N = 2$

To recover the algebraic double solitons (2.16) by using the double-Wronskian solution (3.13) with $N = 2$, we define

$$A := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 \end{bmatrix},$$

which satisfies $A = -S\bar{S}$. Using (3.10) and (3.12) with these A and S, we get

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

Then, we get from (3.13) with $C = (-\mathrm{i})^2/|S| = -1$ that

$$\begin{cases} f &= |\phi', \phi''; \psi, \psi'|, \\ g &= |\phi, \phi', \phi''; \psi'|, \\ h &= -\mathrm{i}|\phi, \phi', \phi''; \psi|, \end{cases}$$

which generate due to (2.7) and (3.2) for $c_1 = 1$ and $c_j = 0$ for $j = 2, 3, 4$.

$$u = \frac{g(x,t)}{\bar{f}(x,t)},\ v = \frac{h(x,t)}{f(x,t)}$$

where

$$\begin{aligned} f(x,t) &= \frac{1}{192}(-16x^4 - 32\mathrm{i}x^3 - 24x^2 - 24\mathrm{i}x + 48t^2 + 3), \\ g(x,t) &= -\frac{1}{48}(-8\mathrm{i}x^3 - 12x^2 + 6\mathrm{i}x - 24tx + 12\mathrm{i}t - 3)e^{-\mathrm{i}t}, \\ h(x,t) &= \frac{1}{48}(-8\mathrm{i}x^3 + 12x^2 + 6\mathrm{i}x - 24tx - 12\mathrm{i}t + 3)e^{-\mathrm{i}t}. \end{aligned}$$

This exact solution recovers the expressions in (2.16) for $\alpha = \beta = 0$. Consequently, the solution surface is shown in Figure 2.2 (top left), and the contour plot is shown in Figure 2.3 (right).

3.5.2 Triple algebraic solitons with $N = 3$

We set $N = 3$ to derive the triple soliton solution. In this case, we construct the matrices:

$$A := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{5}{128} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ -\frac{7}{256} & -\frac{5}{128} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 \end{bmatrix},$$

which satisfies $A = -S\bar{S}$. The components of ϕ and ψ are given by

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$$

Using the Wronskian determinant (3.13) with $C = (-\mathrm{i})^3/|S| = i$, we obtain

$$\begin{cases} f &= |\phi', \phi'', \phi'''; \psi, \psi', \psi''|, \\ g &= |\phi, \phi', \phi'', \phi'''; \psi', \psi''|, \\ h &= |\phi, \phi', \phi'', \phi'''; \psi, \psi'|, \end{cases}$$

which yield the exact triple algebraic soliton solution. For $c_1 = 1$ and $c_j = 0$ for $j = 2, \ldots, 6$, the exact solution is given by

$$u = \frac{g(x,t)}{\bar{f}(x,t)}, \; v = \frac{h(x,t)}{f(x,t)} \tag{3.30}$$

where

$$\begin{aligned} f(x,t) = \frac{1}{4423680}\Big(& -135\mathrm{i} + 512x^9 + 2304\mathrm{i}x^8 + 4608x^7 + 23808\mathrm{i}x^6 + 576(-16t^2 - 21)x^5 \\ & + 1440\mathrm{i}(-16t^2 + 15)x^4 + 1440(48t^2 - 19)x^3 + 2160\mathrm{i}(16t^2 + 3)x^2 \\ & + 270(-256t^4 + 32t^2 - 9)x - 34560\mathrm{i}t^4 - 12960\mathrm{i}t^2 \Big), \\ g(x,t) = -\frac{1}{64}e^{-\mathrm{i}t}\Big(& \frac{\mathrm{i}x^8}{45} + \frac{4x^7}{45} + (\frac{\mathrm{i}}{15} + \frac{8t}{45})x^6 + (-\frac{8\mathrm{i}t}{15} + \frac{7}{15})x^5 + (-\frac{2\mathrm{i}t^2}{3} + \frac{3\mathrm{i}}{8} - \frac{2t}{3})x^4 \\ & + (-\frac{4t^2}{3} + \frac{3}{4})x^3 + (-\mathrm{i}t^2 - \frac{5\mathrm{i}}{16} - \frac{t}{2})x^2 + (-\frac{\mathrm{i}t}{2} - t^2 - \frac{1}{16})x + \mathrm{i}t^4 + \frac{\mathrm{i}t^2}{8} - \frac{3\mathrm{i}}{256} - \frac{t}{8}\Big), \\ h(x,t) = \frac{1}{64}e^{-\mathrm{i}t}\Big(& \frac{\mathrm{i}}{45}x^8 - \frac{4x^7}{45} + (\frac{\mathrm{i}}{15} + \frac{8t}{45})x^6 + (\frac{8\mathrm{i}t}{15} - \frac{7}{15})x^5 + (-\frac{2\mathrm{i}t^2}{3} + \frac{3\mathrm{i}}{8} - \frac{2t}{3})x^4 \\ & + (\frac{4t^2}{3} - \frac{3}{4})x^3 + (-\mathrm{i}t^2 - \frac{5\mathrm{i}}{16} - \frac{t}{2})x^2 + (\frac{\mathrm{i}t}{2} + t^2 + \frac{1}{16})x + \mathrm{i}t^4 + \frac{\mathrm{i}t^2}{8} - \frac{3\mathrm{i}}{256} - \frac{t}{8}\Big). \end{aligned}$$

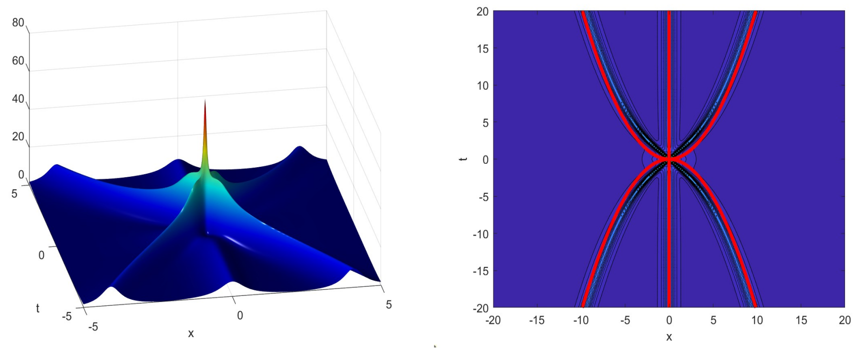

FIGURE 3.1: The solution surface for $|u|^2 + |v|^2$ versus (x, t) for the solution (3.30) (left). The contour plot for the 3-soliton solution surface together with $x^2 = \sqrt{9 + 6\sqrt{6}}|t|$ (right).

Figure 3.1 illustrates the solution surface for $|u|^2 + |v|^2$ versus (x, t) for the solution (3.30) and the contour plot for the solution surface together with the parabolas $x^2 = \sqrt{9 + 6\sqrt{6}}|t|$. The explicit expressions for the parabolas is given by the leading-order terms in the denominator of (3.30), which are

$$512x^9 - 9216t^2x^5 - 69120t^4x.$$

Making this equal to zero yields three solutions: $x = 0$, $x^4 = (9 + 6\sqrt{6})t^2$, $x^4 = (9 - 6\sqrt{6})t^2$, where the last solution only gives complex values of x, whereas the second solution yields $x^2 = \sqrt{9 + \sqrt{6}}|t|$. The solution is symmetric with the global maximum $|u(0,0)|^2 + |v(0,0)|^2 = 72$, which is nine times larger than the squared amplitudes of the single algebraic soliton (2.5). The algebraic solitons exhibit slower separation of the individual algebraic solitons, implying their long-range interactions. The solitons do not exhibit phase shift after the interaction.

Chapter 4

Conclusion and Open Questions

We have constructed the exact solutions of the MTM system (1.1) which describe the dynamics of algebraic solitons. By employing both the Hirota bilinear method and the double-Wronskian approach, we have systematically derived the Nth-order rational solutions. The exact solutions suggest that the algebraic solitons are stable coherent structures arising in a more complicated evolution of the MTM system (1.1).

The functional-analytic proof for orbital stability of algebraic solitons is still open with only partial progress obtained within the derivative NLS equation in [21] and recently in [18]. Beyond the stability proof, several open questions arise from this work. First, a similar algebraic double-soliton and a similar hierarchy of higher-order rational solutions must exist in other nonlinear equations associated with the KN spectral problem, among which the most significant model is the derivative NLS equation [11, 32]. Second, development of the IST methods and the Darboux transformation methods for the algebraic solitons associated with the embedded eigenvalues is still a challenging mathematical problem for future research. Third, the analytical proof of the general expressions (3.27) and (3.28) requires advanced combinatorial analysis and is left open.

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