# THE EFFECT OF PAIRING CORRELATIONS ON NUCLEAR TRANSITION RATES

# THE EFFECT OF PAIRING CORRELATIONS ON NUCLEAR TRANSITION RATES

By

Abel Ferreira de Miranda, B.Sc.

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

September 1962

DOCTOR OF PHILOSOPHY (1962) McMASTER UNIVERSITY
(PHYSICS) Hamilton, Ontario

TITLE: The Effect of Pairing Correlations on Nuclear Transition Rates

AUTHOR: Abel Ferreira de Miranda, B.Sc. (Imperial College, University of London)

SUPERVISOR: Professor M. A. Preston

NUMBER OF PAGES: v, 113

SCOPE AND CONTENTS:

An attempt is made to study the influence of pairing correlations in some $\gamma$ and $\beta$ decay processes, using wave functions that are eigenfunctions of the number operator, and that have extremely large overlaps with the exact solutions of the pairing Hamiltonian. The techniques used are straight-forward, and it is not much more difficult to obtain accurate numerical results with the new wave functions than it is with the usual Bardeen-Cooper-Schrieffer wave functions. The results show appreciable differences between the two types of wave functions, which in some $\beta$-decay cases studied can be as much as 10% to 25%, when the same single particle level structure and pairing force parameter are used. Improvement is obtained in some cases. In others, the projected wave functions indicate that pairing correlations can account for only part of the configuration mixing required to explain the large deviations from single particle values that are experimentally observed.

ACKNOWLEDGEMENTS

I sincerely wish to express my gratitude to Professor M. A. Preston for his very valuable criticism and suggestions, and for his guidance throughout this work.

I am also grateful to Dr. D. J. Kenworthy, of the Department of Mathematics, for having written a very elegant program that essentially permitted the calculation to a sufficient accuracy of the various reduction factors given by the projected wave functions.

The numerical work was performed using the Bendix - G15D digital computer at McMaster University.

I am very thankful to the Department of Physics of McMaster University for a maintenance grant for the academic terms 1959 - 1961 and to the National Research Council of Canada for a studentship for the year 1961 - 1962.

TABLE OF CONTENTS

LIST OF TABLES

## INTRODUCTION

The independent-particle model of the nucleus provides the starting-point of our efforts to solve approximately the many-body nuclear problem.

It was originally inspired by the Hartree method of treating the similar problem at the atomic level. In the spirit of this method, each nucleon in a nucleus is supposed to move independently in some sort of average potential field generated by all the other nucleons.

The wave-function for the nucleus in this approximation is then taken to be a properly anti-symmetrized product of single-particle wave functions, these being the eigenfunctions of the single-particle Hamiltonian corresponding to the average nuclear potential assumed.

For a true Hartree-Fock approximation, the average potential should be self-consistent, but no attempt is usually made to actually solve the Hartree-Fock equations for the nucleus, in order to obtain the self-consistent field and wave functions. Instead, a spherical potential of the form

$$V_1(r) + V_2(r)\ \vec{l}\cdot\vec{s}$$

is postulated in first approximation and the parameters contained therein are so chosen that the magic number sequence, ground state spins and other observables are obtained.

The successes and failures of this basic shell-model picture, when its predictions are compared to experimental data, are well-known. The more serious discrepancies that are found to exist can be traced to the fact that any interactions between the nucleons which in principle cannot be incorporated in the average field, are simply ignored. As we shall soon illustrate, these so-called residual interactions can induce considerable mixing of the single particle configurations and thus largely destroy the simplicity of the model. Attempts have often been made to treat as perturbations the residual interactions, and in many cases, a much better agreement with experiment was actually obtained. Although this approach to the problem is often capable of yielding rather accurate results, it evidently becomes impracticable as the number of nucleons that must be treated (generally only those outside of closed shells) becomes large. Therefore, the need arises for developing a method capable of handling to a good approximation these residual interactions, without running into insuperable computational difficulties.

On the other hand, it soon was clear that besides single particle properties so well described by the shell-model, many nuclei also exhibit collective properties, requiring a co-operative effort of many nucleons acting together. These effects are made evident in a variety of ways, for instance, in studies of static quadrupole moments

and enhanced E2 transitions, or in the existence of energy levels that can be explained only in terms of rotations and vibrations of the nuclear body as a whole.

The collective model was developed out of attempts to fit all these properties together, and for some time it existed side by side, in some sense complementing the shell model. It is well known that these two models were finally tied together in the unified description developed by Bohr, Mottelson, Nilsson and other authors.

This new approach keeps the spirit of the original shell model but allows the self-consistent field to have a deformed equilibrium shape. The nucleons are then assumed to move independently of one another in the available single particle states of the deformed nuclear field. From basic quantum mechanical theorems, the nucleus will then exhibit rotational states, as well as vibrational and single-particle states. This model has known a very remarkable success in most of its applications. Thus, the low-energy level systematics, $\gamma$ and $\beta$ selection rules and trends in transition rates, decoupling parameters, magnetic g-factors and all the single particle properties in general are very satisfactorily explained within its framework. It is believed that it now provides an excellent picture of the single-particle properties of odd nuclei. When, however, one tries to apply it to even-even nuclei, one immediately encounters a basic difficulty. It is a very conspicuous feature of these nuclei that their energy spectrum exhibits a gap just above the ground state in which without exception no intrinsic excitations are found. This energy gap is of the order of 1 Mev for heavy deformed

nuclei and therefore more than one order of magnitude greater than the single-particle energy spacings.

The independent particle model would predict that the low-lying intrinsic excitations in even-even nuclei should have an average energy spacing comparable to the empirically observed single-particle level density in odd-A nuclei. That this is not the case and that there is a relatively large energy gap in the intrinsic excitation spectrum clearly points to a breakdown of the independent particle picture, and at the same time shows that residual forces, which cannot be incorporated in the average field considered by the shell-model, are at work among the shell-model particles. These forces somehow prevent the occurrence of low-lying single-particle excitations in even-even nuclei.

As was first pointed out by Bohr, Mottelson and Pines (BMP 58), the existence of such an energy gap suggests a strong analogy with the superconducting state of electrons in a metal: basically, both electrons in a metal and nucleons in a nucleus are fermion systems which to a very good first approximation can be described by an independent particle model. Moreover, both systems exhibit an energy gap in their single particle excitation spectrum.

As pointed out by these authors, this suggests that one can assume that residual interactions exist among nucleons which are similar in some sense to the residual interactions among electrons that cause superconductivity in a metal. The basic mechanism that leads to superconductivity is, according to the Bardeen-Cooper-Schrieffer Theory, a short-range two-electron attractive force that

strongly binds together any two electrons moving with opposite momenta and in singlet spin states (Cooper pairs). By virtue of this force, electron pairs are continually scattered from one state of zero total momentum to another. Two-body correlations of this type are considered to be so important for superconductivity that all other correlations are ignored at least in first approximation. As shown by Bardeen-Cooper-Schrieffer these particular correlations directly lead to the observed gap in the electron excitation spectrum and give a natural explanation of most phenomena associated with superconductivity. In the nuclear case, one postulates similarly two-body correlations and the existence of an attractive short-range part of the two-nucleon residual force, which cannot be incorporated in the average nuclear field, and which strongly binds together any two nucleons moving in degenerate single-particle orbitals, which differ only in the sign of their magnetic quantum numbers (conjugate states).

At this stage, it may be well to recall that Mayer (M 50) had already found that, in the case of spherical nuclei, one could give an interpretation to the observed systematic differences in the binding energies of even-even and odd-A nuclei (a phenomenon intimately related to the energy gap in the intrinsic excitation spectrum) in terms of a short-range diagonal pairing force, which could provide a large additional binding energy to pairs of identical nucleons moving in conjugate states. Similarly, as was pointed out by Bohr and Mottelson (BM 55) in the case of deformed axially symmetric nuclei, we have

to postulate short-range pairing forces binding identical nucleons filling in pairwise each of the doubly degenerate Nilsson's orbitals.

It was observed, however, that although these diagonal pairing forces are sufficient to explain the even-odd mass differences, they cannot account for the energy gap. The reason is that although they prevent the breaking up of such bound pairs, they cannot prevent the occurrence of low-lying two-particle excitations at about twice the single particle energies, corresponding to exciting a pair as a whole.

Since we already know that no excitations of this type are observed even to energies up to the order of six times the single particle energies, we conclude that off-diagonal matrix elements of the residual two-body force must be taken into account: these residual forces should then correlate not only the members of a particular conjugate state (K, -K) but also a great number of such conjugate states, just as in the superconducting case. That is to say, one expects that the residual two-body force should be capable of scattering a pair of particles from the conjugate state (K, -K) to another conjugate state (K', -K'), compatible with the conservation laws. One can thus suspect that an excitation of a pair as a whole would involve a general redistribution of all similarly correlated pairs, making the excitation energy several times larger than the single particle energy, as required (P 62).

We shall presently see how these residual pairing forces can explain a whole set of nuclear properties that could not be understood even qualitatively from the independent-particle model viewpoint.

But before mentioning these developments, we must recall another important progress made at about the time Bohr, Mottelson and Pines suggested the application of the theory of superconductivity to the nuclear structure problem.

In fact, Elliott (El 58a) was able to obtain states of a distinctly rotational character from a typical shell-model calculation. He assumed that the average nuclear field could be described by an harmonic oscillator potential, with the nucleons moving independently in its allowed states. He further assumed that there are residual forces between any two nucleons of the quadrupole type, i.e.,

$$V_{ik} = - r_i^2 \, r_k^2 \, P_2 (\cos \theta_{ik})$$

He then showed that the nucleus will exhibit energy levels which are distinctively collective rotational in character, in fact essentially the same as obtained by the rotational model. Elliott's result was shown (Be 59) to be true even if the potential were not harmonic, or if a different radial dependence for $V_{ik}$ were assumed, provided that we insist that the angular dependence of the $V_{ik}$ be of the quadrupole type. This result of Elliott's clearly suggests the qualitative relationship between the actual two-nucleon force and the short-range attractive forces that we found necessary to postulate in order to explain the even-odd mass differences and the energy gap (Be 59). Let us expand the two-nucleon interaction potential in spherical harmonics

$$V(\vec{r}_1 - \vec{r}_2) = \sum_{\ell=0}^{\infty} V_\ell (r_1, r_2) \, P_\ell (\cos \theta_{12})$$

In this expansion, only even harmonics occur if we assume that V is a parity-conserving interaction. It is reasonable to assume that the $l = 0$ term gives the isotropic self-consistent potential considered in the early versions of the shell-model. On the other hand, Elliott's result indicates that the $l = 2$ (quadrupole) term will produce the ellipsoidal self-consistent field of the unified model. We can then assume that the higher harmonics $l \geq 4$ will give rise to a residual force which cannot be incorporated in the self-consistent field, and which will produce the effects that we have associated with the pairing force. Since the quadrupole force can couple only those states having angular momenta differing at most by 2 units, we see that it will contribute very little to the pairing force, which is capable of scattering pairs among states with widely different angular momentum quantum numbers. Conversely, the terms for which $l \geq 4$ will make negligible contributions to the self-consistent field.

These considerations suggest then that one should be able to understand the low-energy properties of nuclei starting from the description of an independent-particle motion in a spherical self-consistent field and adding to it the combined effects of the short-range pairing force (high harmonics) and the long-range quadrupole force (low harmonics) operating among shell-model particles.

For deformed nuclei, it is clear that we can always incorporate most of the effects of the quadrupole force in the construction of the self-consistent field. The pairing force, however, has to be treated on a different footing, since it cannot be incorporated in the self-consistent field.

It can be shown (Ra 58) that the quadrupole force alone will give rise to a distribution of nucleons outside closed shells (supposed to account essentially for the low-energy properties of nuclei) such that each nucleon tends to adjust its motion to the shape of the average field produced by all the other nucleons (N 60). The shell model implies that a single particle outside closed shells has a spatial distribution mostly confined to a plane passing through the center of the nucleus and thus highly anisotropic. The model then indicates that even in this case a non-spherical equilibrium shape will be preferred by the nucleus (B 58b). We can then define a quantization axis $Z$ perpendicular to the plane on which the nucleon is mostly confined. If we add another nucleon, then the residual quadrupole force will correlate it with the already present nucleon in such a way that the spatial distribution of the added nucleon will be concentrated as nearly to the same plane as possible, i.e., with the maximum possible $|m_z|$ value; and so forth. This is because a force

$$-P_K(\cos\theta_{ij})$$

is most attractive when the angular distance $\theta ij$ between any two nucleons $i$ and $j$ differs from 0 or $\pi$ by less than about (Ba 58),

$$\sqrt{\frac{2}{K(K+1)}}$$

So the best possible correlation brought about by an attractive quadrupole force ($K=2$) will result when the nucleons move in single-particle states with the highest $|m_z|$ values. This

ensures that each particle feels the effects of the long-range force due to all the other particles. The best wave function embodying this type of correlation will then be a properly antisymmetrized product of single-particle wave functions with the highest possible $n_z$ values.

We are thus led, in the case of a pure attractive quadrupole force, to the so-called aligned coupling scheme (Mottelson) in which the nucleons are distributed in states which best fit the average deformed field which they themselves generate. This of course gives rise to the well known rotational states.

Considering now the pairing force alone, a quite different coupling scheme results.

The nucleons under its influence tend to form virtual bound pairs (Cooper pairs) which, as we shall see (cf. Chapter I) must have spherical spatial distribution for the best possible correlation. In this sense, its effects are opposed to the long-range correlations brought about by the quadrupole force.

It is the interplay of these two forces that can explain, among many other things, the equilibrium deformations of nuclei, and the observed sudden onset of a permanent deformation as the number of particles outside of closed shells increases (Be 59).

For nuclei in the region nearest to the closed shells, one finds that the pairing force is dominant, while the quadrupole force can be treated as a perturbation (Mo 60). The pairing force, which favors spherical symmetry, is responsible for the spherical equilibrium shape of these nuclei but the quadrupole force perturbs the

nucleons in their paired states in such a way that slow quadrupole-type shape vibrations result, as observed (Bar 60).

For nuclei with strong equilibrium deformations (farthest from closed shells), the quadrupole force dominates, and although the pairing force little affects the equilibrium shape, it is by no means negligible. For we have already seen that it must account for the energy gap in the intrinsic spectrum.

On the other hand, the pairing force explains why the moments of inertia of deformed nuclei evaluated on the basis of the assumption of independent motion of the nucleons is substantially larger than the observed values. In fact, the value obtained corresponds exactly to the value that would result if the nucleus were a rigid rotator. Bohr and Mottelson (BM 55) on the basis of the cranking model, have pointed out that the inclusion of an additional pairing energy brought about by the type of pairing correlations that we are considering will reduce by an order of magnitude at least the rigid rotation estimates. Actual calculations carried out by Beliaev, Migdal et al (Be 59, MP 61) introducing the pairing force, bring the computed moments of inertia a great deal closer to the experimental values.

Following this general line concerning the various effects of the pairing force on the nuclear structure, we propose here to make a study of its effects on the nuclear $\beta$ and $\gamma$ decay rates. Some calculations done by Kisslinger and Sorensen (KS 60) and Soloviov (S61) show that the strong single-particle configuration mixing brought about by the pairing force explains in a way

consistent with experiment the reduction of these transition rates from the single-particle estimates.

We shall be concerned here solely with the effect of the pairing force, but before we go on discussing it in detail, it may well be pointed out that efforts to diagonalize simultaneously the quadrupole force and the pairing force generally in a realistic case have not been successful so far. Lipkin (L 61) has succeeded in doing an exact diagonalization for a two-dimensional model using an harmonic oscillator field, but although he regained some of the basic results mentioned above, he failed to obtain any effect of the pairing force on the moment of inertia of the deformed system, and even the existence of low-lying vibrational excitations. This may evidently be due to the over-simplifications introduced in his model.

More recently, however, Moshinsky (Mo 61) was able to obtain a numerical diagonalization for a particular three-dimensional case. These efforts, if not anything else, have the merit of showing more clearly than before the nature of the relationship between the quadrupole and the pairing force.

CHAPTER I

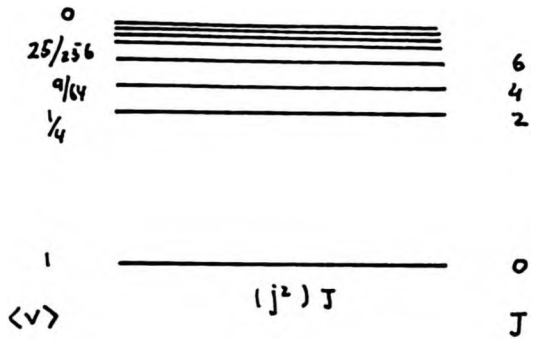
## 1. Completely Degenerate Case: The Quasi-Spin Model

Our task now is to formulate explicitly that part of the Hamiltonian corresponding to the pairing force alone and then diagonalize it. We consider spherical nuclei first. We ignore short-range neutron-proton correlations, since in heavy nuclei (to which we refer in this work) the neutrons and protons occupy different energy shells and can be assumed to have different Fermi levels. There is no satisfactory theory to this date giving the solution to the problem of n - p short-range pairing correlations, which should evidently be important in light nuclei.

Since we are assuming basically two-body correlations, we must first look at the spectrum of two identical shell-model particles without interaction and see how it is modified by a residual short range force between them (M 50). We need consider only equivalent particles (same n, l, j quantum numbers), since for non-equivalent particles in heavy nuclei the short-range force will not be appreciably effective.

Calculations done by Mayer (Ma 50) using a $\delta$-function force show that one essentially gets the result that the unperturbed $j^2$ degenerate configurations are modified in such a way that the state

with total angular momentum $J = 0$ is quite depressed whereas the degeneracy of the $J = 2, 4, \ldots 2j-1$ states is appreciably lifted (Fig. 1) (after M 58).




Proceeding a step further, we define the pairing force as that force causing the $J = 0$ state to be depressed, leaving all the other states $J = 2, 4, \ldots 2j-1$ unperturbed. That is to say, the pairing force couples only the states

$$|jm, j-m; J=0\ M=0\rangle$$

and

$$\langle j^n m', j-m'; J'=0\ M'=0|$$

its matrix elements between all the other configurations being taken as zero. The Cooper pairs then have spherical symmetry.

From the basic assumptions of the theory of superconductivity (BCS 56), one then is assured that, also in the nuclear case, when we next consider many equivalent particles, the pairing force will produce a considerable energy gap in the single particle excitation spectrum, if the number of particles is even.

The pairing force cannot of course be represented in configuration space as a local potential, although from its definition we see that it resembles somewhat a δ-function force. But, whereas a δ-function force can couple all kinds of single-particle configurations, the pairing force connects conjugate states only (or, in general, any set of states which differ from another set of states only by time-reversal, such as the two-degenerate states of a Nilsson's orbital).

It is most convenient to use a second quantized representation, since it automatically takes in account the exclusion principle. Our basic set of states are the single-particle states of the spherical self-consistent field $|n\ell jm\rangle$ (later we shall use the single particle states of an axially symmetric deformed field). In this chapter we shall write down explicitly only the angular momentum j and its z-component m. We define the vacuum state $|0\rangle$ as the state having no nucleons present. We introduce the fermion creation and annihilation operators $a^+_{jm}$ and $a_{jm}$ such that

$$a^+_{jm}\,|0\rangle \equiv |jm\rangle \qquad \text{I.1.1}$$

$$a_{jm}\,|0\rangle \equiv 0 \qquad \text{I.1.2}$$

These operators satisfy the usual anticommutation rules. We define

$$a^+_{j-m}\,|0\rangle = |j-m\rangle = (-1)^m\,|jm\rangle^* \qquad \text{I.1.3}$$

where * means complex conjugate, and $(-1)^m\,|jm\rangle^*$ is the time-reversed state of $|jm\rangle$.

The general Hamiltonian including two-body forces is then

$$H = \sum_{jm} \varepsilon_j\, a^+_{jm}\, a_{jm} + \frac{1}{2} \sum_{\substack{j_1 j_2 j_1' j_2' \\ m_1 m_2 m_1' m_2'}} \langle j_1 m_1\, j_2 m_2 | V | j_1' m_1'\, j_2' m_2' \rangle\, a^+_{j_1 m_1}\, a^+_{j_2 m_2}\, a_{j_2' m_2'}\, a_{j_1' m_1'}$$

$$= H_1 + H_2 \qquad \text{I.1.4}$$

We call $\varepsilon_j$ the energy of the single particle state $|jm\rangle$. We must now write H in such a way that its invariances under rotations (conservation of angular momentum) and under space inversions (conservation of parity) are explicit. To do this we couple the angular momentum states $\langle j_1 m_1 | \langle j_2 m_2 |$ to give $\langle JM|$ and $|j_1' m_1'\rangle |j_2' m_2'\rangle$ to give $|JM\rangle$ and express $H_2$ as a scalar product. We then get

$$H = \sum_{jm} \varepsilon_j \, a^+_{jm} a_{jm} \; + \;$$

$$+ \frac{1}{2} \sum_{\substack{j_1 j_2 j_1' j_2' \\ m_1 m_2 m_1' m_2'}} \sum_{JM} (j_1 m_1, j_2 m_2 | JM)(j_1' m_1' j_2' m_2' | JM) \times$$

$$\times \langle j_1, j_2 ; JM | V | j_1' j_2' ; JM \rangle \, a^+_{j_1 m_1} a^+_{j_2 m_2} a_{j_2' m_2'} a_{j_1' m_1'} \qquad \text{I.1.5}$$

where

$$(j_1 m_1 \; j_2 m_2 | JM) \; ; \; (j_1' m_1' \; j_2' m_2' | JM)$$

are Clebsch-Gordan coefficients (we are using Condon-Shortley's phases).

We shall now effectively truncate our Hamiltonian, keeping only those terms that correspond to the pairing force, as we defined it.

So our model Hamiltonian is

$$H_0 = \sum_{jm} \varepsilon_j \, a^+_{jm} a_{jm} \; +$$

$$+ \frac{1}{2} \sum_{\substack{jm \\ j'm'}} (jm \; j-m | 00)(j'm' \; j'-m' | 00) \langle jj ; 00 | V | j'j' ; 00 \rangle \; + \;$$

$$a^+_{jm} a^+_{j-m} a_{j'-m'} a_{j'm'} \qquad \text{I.1.6}$$

Recalling that

$$(jm \; j-m | 00) = \frac{(-1)^{j-m}}{\sqrt{2j+1}} \qquad \text{I.1.7}$$

We got

$$H_0 = \sum_{jm>0} \epsilon_j \left( a_{jm}^{+} a_{jm} + a_{j-m}^{+} a_{j-m} \right) \quad +$$

$$+ \frac{1}{2} \sum_{\substack{jm>0 \\ j'm'>0}} 2 \frac{\langle jj;00 \mid V \mid j'j';00 \rangle}{\sqrt{(j+\frac{1}{2})(j'+\frac{1}{2})}} (-1)^{j-m} a_{jm}^{+} a_{j-m}^{+} (-1)^{j'-m'} a_{j'-m'} a_{j'm'}$$

I · 1.8

As Kisslinger and Sorensen have done (KS 60), we further simplify the problem by replacing the above matrix element by some sort of an average, G, which is to be treated as a parameter to be fixed from the empirical evidence. Thus we set

$$\frac{\langle jj;00 \mid V \mid j'j';00 \rangle}{\sqrt{(j+\frac{1}{2})(j'+\frac{1}{2})}} = -G$$

I · 1.9

The minus sign indicates that we are taking the interaction to be attractive, as in the case of superconducting electrons. Thus we have finally our much simplified pairing Hamiltonian,

$$H_0 = \sum_{jm>0} \epsilon_j \left( a_{jm}^{+} a_{jm} + a_{j-m}^{+} a_{j-m} \right) \quad -$$

$$- G \sum_{\substack{jm>0 \\ j'm'>0}} (-1)^{j-m} a_{jm}^{+} a_{j-m}^{+} (-1)^{j'-m'} a_{j'-m'} a_{j'm'}$$

I · 1.10

We should note that the step involved in I.1.9 is not really essential for the logical development of the model (Be 59).

We are doing it, however, because in practice it very much simplifies the calculations, and because we can always treat G as a parameter to be adjusted empirically (for instance, from the even-odd mass data). What is essential, however, is that the two-body matrix element should have predominantly one sign (in our case, negative, indicating attractive forces). It is precisely this that causes the peculiar coherent two-body correlations that leads to the splitting off of a single state from a large set of degenerate states, leaving the remaining states unperturbed(BCS 58).

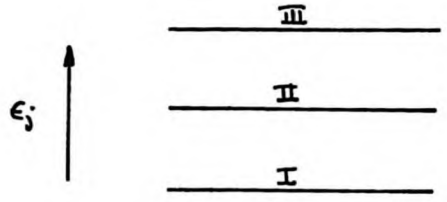
The resulting energy gap may then be very large compared to the average single particle level-spacing, even if the residual pairing force be quite weak, since the energies of the remaining states are not changed. The wave function corresponding to the depressed state will then be a linear combination of the available single particle states, with coefficients having predominantly one sign.

Before going into the discussion of the methods available to solve approximately the Hamiltonian 1.16, we must consider a particular case for which an exact solution can be given (Racah, Lowman, Mottelson et al).

This solution already contains the fundamental features of the effects associated with the pairing force.

We consider the following very simplified model of the nucleus, which we shall call the "quasi-spin" model (2 ). We divide the nucleus in three regions. Region I contains all the closed j-shells, and region III contains all the empty j-shells. In region II we assume

that there is a single j-shell in the process of being filled. Regions I and III are supposed to be purely passive, i.e., they do not influence in any way the dynamics of the nucleons in the j-shell in




region II. Further, we assume that the sublevels jm in region II are completely degenerate in energy, and we set their energy $\epsilon_j$ to zero. Then 1.10 becomes

$$H^j_{deg} = - G \sum_{\substack{m>0 \\ m'>0}} (-1)^{j-m} a^+_{jm} a^+_{j-m} (-1)^{j-m'} a_{j-m'} a_{jm'}$$

I . 1. 11

This then describes an idealized system of N nucleons, say, in the j-shell (region II) free from configuration mixing from region I and III (the passive region). The problem in this form was completely solved by Racah (1940) from a different viewpoint, and led him to introduce the concept of the "seniority coupling scheme" appropriate to the pairing force, just as the "aligned coupling scheme" is to the quadrupole force (cf. Introduction).

We then see that the full Hamiltonian 1.10 will simply generalize the concept of seniority, when mixing from all the regions I, II and III must be taken into account.

Following Kerman (K 61), we introduce three operators

$$\mathcal{J}^{+}_{jm} \equiv (-1)^{j-m}\, a^{+}_{jm}\, a^{+}_{j-m}$$

$$\mathcal{J}^{-}_{jm} \equiv (\mathcal{J}^{+}_{jm})^{\dagger} \equiv (-1)^{j-m}\, a_{j-m}\, a_{jm} \qquad m>0$$

$$\mathcal{J}^{0}_{jm} \equiv \frac{1}{2}\left( a^{+}_{jm}\, a_{jm} + a^{+}_{j-m}\, a_{j-m} - 1 \right)$$

I · 1. 12

where † means Hermitean adjoint.

These operators transform as the three spherical components of an irreducible tensor of rank 1, or in other words, the vector $\vec{\mathcal{J}}_{jm}$ behaves as an angular momentum (K 61). Furthermore, $\vec{\mathcal{J}}_{jm}$ and $\vec{\mathcal{J}}_{jm'}$ $(m \neq m')$ commute with each other.

We define

$$\vec{S}_j = \sum_{m>0} \vec{S}_{jm} \qquad \text{I} \cdot 1 \cdot 13$$

We note that

$$S^{0}_{j} = \frac{1}{2}(N - \Omega) \qquad \text{I} \cdot 1 \cdot 14$$

where $\Omega$ is the number of the pairing levels, $\Omega = j + \frac{1}{2}$. Therefore, when the shell is empty $(N = 0)$ we have

$$S^{0}_{j}\,(\text{empty}) = -\frac{\Omega}{2}$$

and when it is full $(N = 2\Omega)$

$$S^{0}_{j}\,(\text{full}) = +\frac{\Omega}{2}$$

That is to say, when all the pairing levels are unoccupied, each elementary $\mathcal{J}^{0}_{jm}$ "points down" and is equal to $-\frac{1}{2}$, and since these are $\Omega$ of them, the total $S^{0}_{j}$ is $-\frac{\Omega}{2}$. Similarly, when each pairing level $(m, -m)$ is occupied, every elementary $\mathcal{J}^{0}_{jm}$ "points up"

and is equal to $+\frac{1}{2}$ , and again, since these are $\Omega$ pairing levels, the total $S_j^\circ$ is $+\frac{\Omega}{2}$ . Also note that the expectation values of $S_{jm}^+$ and $S_{jm}^-$ in the states having $S_{jm}^\circ = \pm\frac{1}{2}$ vanishes identically.

This then shows that the elementary $\vec{S}_{jm}$ has the properties of a spin $\frac{1}{2}$ when the conjugate levels (m, -m) are occupied by an even number of particles (0 or 2).

On the other hand, if there is only one particle occupying one of the pairing levels, the $S_{jm}^\circ$ for that particular level vanishes, as can easily be verified from the third expression in I.1.12.

Note that $S_j^\circ$ is an integer or half-integer depending upon whether $N-\Omega$ is even or odd; also, from elementary angular momentum theory

$$\left| \frac{N-\Omega}{2} \right| = |S_j^\circ| \leq S_j \qquad \text{I . 1 . 15}$$

It then follows that $S_j$ is an integer or half-integer corresponding to whether N is even or odd.

The expression I.1.11 becomes

$$H'_{deg}{}^{j} = -G\, S_j^+ S_j^- \qquad \text{I . 1 . 16}$$

From the properties of an angular momentum operator, the eigenvalues of $H_{deg}$ are simply

$$E_{deg} = -G\left\{ S_j(S_j+1) - S_j^{\circ 2} + S_j^\circ \right\} \qquad \text{I . 1 . 17}$$

Therefore, the eigenstates of $H_{deg}$ will be labelled by the quantum numbers corresponding to the total quasi-spin $S_j$ and its zero-component $S_j^\circ$ .

The total quasi-spin quantum number $S_j$ is related to Racah's seniority $\upsilon$ by

$$\upsilon = \Omega - 2S_j \qquad \text{I . 1 . 18}$$

In terms of seniority quantum number, the energy spectrum is, as found by Racah,

$$E_{deg} = -G\left[\frac{N\Omega}{2}\left(1-\frac{N-2}{2\Omega}\right) - \frac{\Omega v}{2} + \frac{v}{2}\left(\frac{v}{2}-1\right)\right] \qquad \text{I}\cdot 1\cdot 19$$

Since $E_{deg}$ for a given $j^N$ configuration depends solely upon $v$, and is degenerate in all the other quantum numbers (except N itself), it is called the "seniority spectrum".

We recall that the seniority $v$ is the smallest number of particles needed for building a state with a given set of properties, and therefore it specifies the simplest configuration which contains such a state (Racah). Adding a number of saturated pairs, i.e., pairs of identical nucleons coupled to $J=0$ angular momentum, to the $j^N$ configuration does not change its seniority $v$, or the related quasi-spin $S_j$. This is also evident from the fact that $S_j^+$ (or $S_j^-$) operating on the vacuum creates (destroys) a saturated pair, without changing the magnitude of $S_j$, from elementary angular momentum theory.

Thus we see that if we have saturated pairs only in the $j^N$ configuration, then the seniority quantum number $v=0$. If we have, besides saturated pairs, $v$ particles which do not form saturated pairs, then the seniority of the state equals the number of particles that remain unsaturated, $v$. The energy level $E_{deg}(v)$ will occur only if there are at least $v$ particles in the configuration.

For a given N and $v$ the level $E_{deg}$ is $D_v$-fold degenerate, where

$$D_v = \frac{\Omega!}{\frac{v}{2}!\left(\Omega-\frac{v}{2}+1\right)!}\left(\Omega-v+1\right) \qquad \text{I}\cdot 1\cdot 20$$

as can easily be shown (T 61).

Note that $D_\nu$ rapidly increases with $\nu$.

The level spacing is

$$E(\nu) - E(\nu+2) = -2G(S_j) \qquad \text{I} \cdot 1 \cdot 21$$

We note that it is independent of N, although the absolute value of $E_{\alpha_1}$ depends strongly upon N.

It then follows that the level density near the ground state is independent of N. For an even-N system, all the particles form saturated pairs in the ground state. The energy needed to break a pair is independent of N, and therefore the single particle level density near the ground state does not increase with N in spite of the rapidly increasing number of degrees of freedom available as N increases.

For an even N system, this corresponds to having a gap in the single particle excitation spectrum.

For an odd-N system, because the odd particle does not interact with the saturated pairs, there is no gap, since it requires arbitrarily small energies to excite it. We shall see, however, that the average single particle level density near the ground state is appreciably reduced compared to the case where $G = 0$ (no pairing interaction) (BK 58).

Also notice that, if $N \ll \Omega$, the ground state energy increases linearly with N; on the other hand, we recall that with the residual

long-range quadrupole force the particles are best correlated when they are all as near to the equatorial plane of the deformed nucleus as possible. We then have to consider interactions between all the possible pairs and, as the actual calculation shows, the ground state energy increases with N roughly as $N^2$. We thus substantiate the remarks already made in the Introduction that near the closed shells the pairing force will dominate, whereas far away from closed shells the quadrupole force will take over.

We can now construct explicitly the eigenvectors of $H^j_{deg}$. We label them by the two quantum numbers $S_j$ and $S_j^\circ$:

$$H^j_{deg} \; | S_j \, S_j^\circ \rangle = E_{deg} \, | S_j \, S_j^\circ \rangle \qquad \text{I} \cdot 1 \cdot 22$$

The vacuum state is then easily seen to be

$$|0\rangle \equiv \left| \frac{\Omega}{2} \; -\frac{\Omega}{2} \right\rangle \qquad \text{I} \cdot 1 \cdot 23$$

The completely filled shell ($N = 2\Omega$) is described by

$$\left| \frac{\Omega}{2} \; +\frac{\Omega}{2} \right\rangle \qquad \text{I} \cdot 1 \cdot 24$$

For an even N, any state $|S_j S_j^\circ\rangle$ can be obtained by operating on the vacuum by the saturated pair creation operator $S_j^+$. Thus

$$\left| \frac{\Omega}{2} \; S_j^\circ \right\rangle \equiv \left| \frac{\Omega}{2} \; \frac{N-\Omega}{2} \right\rangle = \sqrt{\frac{(\Omega - p)!}{p! \, \Omega!}} \; (S_+)^p \, |0\rangle \qquad \text{I} \cdot 1 \cdot 25$$

where $p = \frac{N}{2}$ is the number of pairs and

$$\sqrt{\frac{(\Omega - p)!}{p! \, \Omega!}}$$

is a normalization factor.

Similarly, for an odd-N system, the state with the odd particle in the sublevel $m_1$ is

$$\left| \frac{\Omega - 1}{2} \; S_0 \right\rangle \equiv \left| \frac{\Omega - 1}{2} \; \frac{N-\Omega}{2} \right\rangle \equiv \sqrt{\frac{\left(\Omega - \frac{N+1}{2}\right)!}{\left(\frac{N-1}{2}\right)! \, (\Omega - 1)!}} \; (S_+)^{\frac{N-1}{2}} \, a^+_{j m_1} \, |0\rangle$$

$$\text{I} \cdot 1 \cdot 26$$

We note that the angular momentum of these states can easily be obtained: for N even, all the particles form saturated pairs in the ground state, so $J = 0$; for odd-N, the angular momentum is given by the last unpaired particle. We thus regained the well-known shell-model coupling rules (fig. 50).

Excited states can also be constructed: for N even, the first excited states will have seniority 2, corresponding to breaking up a pair.

The resulting two unsaturated particles must couple to an angular momentum with values restricted by the exclusion principle. Thus we must construct the generalized pair-creation operator:

$$S_+(JM) = \sum_{m>0} (jm\ jM-m|JM)\ a^+_{jm}\ a^+_{jM-m} \qquad I.1.27$$

which creates a state with good angular momentum $J$ (z-component M). So, the first excited state will be, apart from normalization factors

$$\left|\frac{\Omega}{2}-1\ S_j^0\right\rangle \sim (S_+)^{\frac{N-2}{2}}\ S_+(JM)\,|0\rangle \qquad I.1.28$$

This state is degenerate, of course, since the energy is independent of $J$ and M: from (I.1.20) we see that its degeneracy is $\Omega - 1$.

Higher excited states can be obtained with the help of operators such as $S_+(JM)$: however, the problem of obtaining states with good angular momentum will be more difficult.

Also, the problem of investigating the orthogonality of these states to the ground state is nontrivial (Kerman).

## 2. "Ansatz" Solution for the General Pairing Hamiltonian

In the previous paragraph we have shown how the problem of a completely degenerate $j^N$ configuration with residual pairing forces can be exactly solved. The seniority coupling scheme was thus obtained. We saw that for an even-N system an energy gap occurs. Its magnitude was found to be (I.1.19)

$$\Delta = G\Omega \qquad \text{I.2.1}$$

where $\Omega$ is the number of pairing levels among which the Cooper pairs scatter under the action of the pairing forces.

We must now look at the general pairing Hamiltonian. First, consider the limit of a very strong coupling parameter $G$, which implies that the spacings between the single particle levels $\epsilon_j$ are very small compared to the energy gap $\Delta$. We can then take the solutions of the last paragraph to be exact in zeroth order, and consider as a perturbation the slight deviation from complete degeneracy of the single particle levels. If we call $\epsilon$ the barycentre of the single particle levels $\epsilon_j$ (in the last paragraph these levels were degenerate and we took $\epsilon = 0$), then the perturbation is

$$V = \sum_{jm} (\epsilon_j - \epsilon)\, a^+_{jm}\, a_{jm} \qquad \text{I.2.2}$$

The perturbed ground state can be shown to be to first order (apart from normalization factors)

$$|\psi\rangle \sim \left( \sum_{jm>0} \left[ 1 - \frac{2(\epsilon_j - \epsilon)}{G\Omega} \right] (-1)^{j-m}\, a^+_{jm}\, a^+_{j-m} \right)^p |0\rangle \qquad \text{I.2.3}$$

On the other hand, in the opposite limit $G \to 0$ (pairing force negligible), we evidently recover the simple shell-model many-particle wave function

$$|\psi\rangle \sim a^+_{jm_1} a^+_{j-m_1} \cdots a^+_{jm_p} a^+_{j-m_p} |0\rangle \qquad \text{I} \cdot 2.4$$

In the region where $G$ is comparable to the average single-particle level-spacing, no exact eigenvector of $H_0$ can be obtained but the above limiting expressions suggest that we use as a trial wave function for the ground state

$$|\psi\rangle \sim \left( \sum_{jm>0} c_{jm} a^+_{jm} a^+_{j-m} \right)^p |0\rangle \qquad \text{I} \cdot 2.5$$

where the $c_{jm}$'s are to be determined by a variational calculation.

The above limiting wave-functions are obviously simple cases corresponding to particular choices of the $c_{jm}$. The exact solution discussed in the last paragraph corresponds to the case where all the two-particle configurations contribute to the wave function exactly with the same weighting factor. For systems with large degeneracies, this will be very approximately the situation. But for systems with small degeneracies, the weighting factors depend on the details of the single-particle states and we must look at the more general methods available for handling the pairing Hamiltonian.

We shall see, as already mentioned, that all the basic features of the solutions to the completely degenerate case are preserved.

Before closing this Chapter we remark that the "ansatz" wave function I.2.5, giving the correct limiting cases, is not easy to handle. The reason is that it is not a product of wave-functions and

therefore it lacks fermion-like properties. We cannot then say that the probability of finding two pairs occupied in the ground state is equal to the product of the probabilities of finding each pair in the ground state (C 59). In order to have this property we shall see that it will be necessary to relax the condition that the number of particles N in the system be fixed and insist that only the average number of particles in the system be equal to N.

CHAPTER II

## 1. The Independent Quasi-Particle Model (Bogoliubov, Beliaev)

The methods that we will now consider were designed to solve the pairing Hamiltonian in general. These methods make possible the study of the effects of the pairing force in systems for which the degeneracy of the basic set of states is reduced to a minimum, as in the case of strongly deformed nuclei. In the rare earth region, for instance, and at least for large deformations, the degeneracy has almost completely disappeared. The residual two-fold degeneracy of the Nilsson states is due to the axial symmetry of the average field chosen. Therefore, a division of the nucleus in three regions (cf. Chapter I) excluding mutual configuration mixing becomes untenable.

On the other hand, it is evident that we should regain essentially the solution presented in Chapter I in the limit of large degeneracies associated with spherical nuclei.

We shall modify slightly our labelling of the eigenstates of the self-consistent field by writing $|\nu m\rangle$ for a given eigenstate in which $m$ is the component along the z-axis of the angular momentum (we shall take the body-fixed z-axis along the symmetry axis for an axially symmetric deformed nucleus); $\nu$ stands for all the remaining

quantum numbers required to make the representation complete. Specifically, states with positive projections on the z-axis will be represented by $|\nu+m\rangle$ or simply $|\nu m\rangle$ ; states with negative projections will be denoted by $|\nu-m\rangle$.

The pairing Hamiltonian (I.1.10) then becomes

$$H_o = \sum_{\nu m>0} \epsilon_\nu \left( a^+_{\nu m} a_{\nu m} + a^+_{\nu -m} a_{\nu -m} \right) -$$

$$- G \sum_{\substack{\nu\nu' \\ m>0 \\ m'>0}} a^+_{\nu m} a^+_{\nu -m} a_{\nu' m'} a_{\nu' m'} \qquad \text{II} \cdot 1 \cdot 1$$

The phase factors occurring in I.1.10 are supposed to have been absorbed in the definition of the single particle states. This is done simply for convenience since these phase factors are unimportant for our purposes (they may be important in other contexts - see L 61).

As pointed out in Chapter I, it will prove to be more convenient to remove the restriction that the number of particles $n$ in the system be fixed, and instead impose the condition that the average number of particles in the ground state be equal to the actual number of particles in the system which we wish to describe. That is to say, we are going to discuss the system from a grand canonical ensemble point of view. We thus introduce a Lagrange multiplier $\lambda$, and write

$$H = H_o - \lambda N \qquad \text{II} \cdot 1 \cdot 2$$

as the new Hamiltonian, where $N$ is the number operator

$$N = \sum_{\nu m} a^+_{\nu m} a_{\nu m} \qquad \text{II} \cdot 1 \cdot 3$$

and $\lambda$ has the character of a chemical potential, to be fixed by the constraint

$$\langle \phi_o | N | \phi_o \rangle = n \qquad \text{II} \cdot 1 \cdot 4$$

where $|\phi_0\rangle$ is the ground state of the system.

The solution satisfying these conditions will then describe an ensemble of nuclei centered at the desired nucleus. We shall discuss later the accuracy of such a description.

As shown by Bogoliubov and Valàtin, the next step is to introduce a system of new particles that will embody in their definition all the effects of the pairing force between the old particles. That is to say, the pairing force, introducing strong correlations between nucleons in conjugate states, modify in a very definite manner the shell model particles. The spirit of the Bogoliubov-Valàtin method is then to make a canonical transformation incorporating these effects and introducing a modified shell-model particle or "quasi-particle". It is next required to have a system of indepent quasi-particles, just as originally the shell-model particles were. This can be achieved to some approximation by a proper choice of the canonical transformation. In this spirit we may argue that the original shell-model particles are not the actual nucleons but already a system of "quasi-particles" which are nucleons modified by the long-range field-producing part of the actual two-nucleon force. The new transformation is then just another step in the chain of canonical transformations leading to a better and more accurate description of the system in terms of normal modes.

We introduce the new quasi-particle operators

$$\alpha_{\nu m} = U_\nu\, a_{\nu m} - V_\nu\, a^{+}_{\nu,-m} \qquad \text{II.1.5}$$

$$\beta_{\nu m} = U_\nu\, a_{\nu,-m} + V_\nu\, a^{+}_{\nu m} \qquad m>0$$

where $u_\nu$ and $v_\nu$ are real numbers, satisfying the normalization condition

$$v_\nu^2 + u_\nu^2 = 1 \qquad \text{II}.1.6$$

Furthermore, we insist that

$$v_\nu = v_{\nu m} = v_{\nu -m}$$
$$u_\nu = u_{\nu m} = u_{\nu -m} \qquad \text{II}.1.7$$

It is then easily shown that with this choice of the $u_\nu$'s and $v_\nu$'s the usual anticommutation rules for fermion operators hold for "quasi-particle" operators as well. It also follows easily that, if $\alpha_{\nu m}$ is a quasi-particle in the $m$ state then $\beta_{\nu m}$ is a quasi-particle in the $-m$ state.

The new operators are a linear combination of a particle and hole operators. In the case of an independent fermion system they become actually unoccupied, if we choose $u_\nu = 1$, $v_\nu = 0$ for the states above the Fermi level and $u_\nu = 0$, $v_\nu = 1$ for the states below. In this particular case then the only difference between quasi-particle and particle operators lies in the definition of their vacuum state: for particle operators it is the state having no particles present and for the quasi-particle operators it is the state corresponding to having a filled Fermi sea with no particles above the Fermi level. This is readily verified from the defining relationships I.1.5.

As we shall presently see, the pairing correlation mixes conjugate pair states in such a way that states near the Fermi level become partly occupied and partly empty, with probabilities $v_\nu^2$ and $u_\nu^2$ respectively. That is to say, the occupation distribution number $v_\nu^2$, instead of having a sharp cut-off at the Fermi level, because

actually smeared out to an extent depending on the strength of the coupling parameter G. With the actually observed strengths the occupation number distribution becomes rapidly what it would be in the case of an independent particle system for states fairly removed in energy from the Fermi level (i.e., 1, for states fairly below and 0 for states fairly above this level).

For $G = 0$ we go back exactly to the familiar step-function distribution of a non-interacting system.

On the other hand, for $G \neq 0$, the quasi-particle vacuum, just as in the case of a non-interacting system, should correspond to the Fermi sea for the particles modified by the pairing correlations (see below).

Using the properties II.1.6 and II.1.7 we can invert the relationships II.1.5 and obtain

$$a_{\nu m} = U_\nu \alpha_{\nu m} + V_\nu \beta^+_{\nu m}$$
$$a_{\nu -m} = U_\nu \beta_{\nu m} - V_\nu \alpha^+_{\nu m} \qquad \text{II.1.8}$$

Substituting this into II.1.2, and using the Dyson-Wick's theorem for expanding a product of operators in terms of their normal products, we find that H can be written as (Be 59)

$$H = U + H_{20} + H_{11} + H_{int.} \qquad \text{II.1.9}$$

$U$ is the term containing all the possible contractions and is therefore a pure number representing the ground state energy corresponding to H:

$$U = \sum_\nu (\epsilon_\nu - \lambda) 2 V_\nu^2 - \frac{\Delta^2}{G} - G \sum_\nu V_\nu^4 \qquad \text{II.1.10}$$

where, by definition,

$$\Delta = G \sum_{\nu} u_\nu v_\nu \qquad \text{II} \cdot 1 \cdot 11$$

The terms $H_{20}$ and $H_{11}$ contain normal products of two creation and annihilation operators; their expressions are

$$H_{20} = \sum_{\nu m>0} \Big[ (\epsilon_\nu - \lambda) 2 u_\nu v_\nu - \Delta (u_\nu^2 - v_\nu^2) - 2 G u_\nu v_\nu^3 \Big] (\alpha^+_{\nu m} \beta^+_{\nu m} + \beta^+_{\nu m} \alpha_{\nu m}) \qquad \text{II} \cdot 1 \cdot 12$$

and

$$H_{11} = \sum_{\nu m>0} \Big[ (\epsilon_\nu - \lambda)(u_\nu^2 - v_\nu^2) + 2 u_\nu v_\nu \Delta - G v_\nu^3 (u_\nu^2 - v_\nu^2) \Big] (\alpha^+_{\nu m} \alpha_{\nu m} + \beta^+_{\nu m} \beta_{\nu m}) \qquad \text{II} \cdot 1 \cdot 13$$

Since we want a system of independent quasi-particles we must now do two things, first, neglect the interaction term $H_{int}$ (containing the normal products of 4 quasi-particle operators) and next obtain an Hamiltonian of the form

$$H' = U + \sum_{\nu m>0} E_\nu (\alpha^+_{\nu m} \alpha_{\nu m} + \beta^+_{\nu m} \beta_{\nu m}) \qquad \text{II} \cdot 1 \cdot 14$$

which clearly represents a system of free quasi-particles moving in states $\nu$ with energy $E_\nu$.

To achieve this we must set $H_{20} = 0$ identically. This gives the condition

$$(\epsilon_\nu - \lambda) 2 u_\nu v_\nu - \Delta (u_\nu^2 - v_\nu^2) - 2 G u_\nu v_\nu^3 = 0 \qquad \text{II} \cdot 1 \cdot 15$$

and defining

$$\tilde{\epsilon}_\nu = \epsilon_\nu - G v_\nu^2 \qquad \text{II} \cdot 1 \cdot 16$$

we have

$$(\tilde{\epsilon}_\nu - \lambda)\, 2 u_\nu v_\nu - \Delta (u_\nu^2 - v_\nu^2) = 0 \qquad \text{II} \cdot 1 \cdot 17$$

which yields

$$v_\nu^2 = \frac{1}{2}\left(1 - \frac{\tilde{\epsilon}_\nu - \lambda}{\sqrt{(\tilde{\epsilon}_\nu - \lambda)^2 + \Delta^2}}\right) \qquad \text{II} \cdot 1 \cdot 18a$$

$$u_\nu^2 = \frac{1}{2}\left(1 + \frac{\tilde{\epsilon}_\nu - \lambda}{\sqrt{(\tilde{\epsilon}_\nu - \lambda)^2 + \Delta^2}}\right) \qquad \text{II} \cdot 1 \cdot 18b$$

From these equations it follows that for $\Delta = 0$ (or $G = 0$, no pairing interaction), we obtain

$$\begin{cases} v_\nu^2 = 1 \\ u_\nu^2 = 0 \end{cases} \qquad \epsilon_\nu < \lambda \qquad \text{II} \cdot 1 \cdot 19a$$

$$\begin{cases} v_\nu^2 = 0 \\ u_\nu^2 = 1 \end{cases} \qquad \epsilon_\nu > \lambda \qquad \text{II} \cdot 1 \cdot 19b$$

which is the well-known Fermi distribution for the non-interacting system. When $G \neq 0$, the probability distribution is modified, especially in an energy region around the Fermi level and width about $2\Delta$ (see Page 33). Also, since there is now a finite probability for finding a pair above the Fermi level of the non-interacting system $\lambda_0$, we expect (as proven by a simple calculation) that the Fermi energy for the superconducting system $\lambda$ be slightly greater than $\lambda_0$.

From II.1.13 it also follows that the quasi-particle energy in the state $|\nu m\rangle$ is

$$E_\nu = \sqrt{(\tilde{\epsilon}_\nu - \lambda)^2 + \Delta^2} \qquad \text{II} \cdot 1 \cdot 20$$

From this relationship one can see that no quasi-particle excitations may occur with energies less than $\Delta$ : for the $E_\nu$ vs $\tilde{\epsilon}_\nu$ curve has a positive minimum at $\tilde{\epsilon}_\nu = \lambda$ equal to $\Delta$ . This quantity is then naturally called the energy gap.

Replacing $U_\nu$ and $V_\nu$ from II.1.18 into the defining equation for the energy gap, II.1.11, we get

$$\Delta = \frac{G}{2} \sum_\nu \frac{\Delta}{\sqrt{(\tilde{\epsilon}_\nu - \lambda)^2 + \Delta^2}} \qquad \text{II.1.21}$$

This equation has the trivial solution $\Delta = 0$. From II.1.18 we again see that this solution corresponds to the normal state of the system, without pairing interactions.

The non-trivial solution corresponds to the energy gap introduced by the pairing correlation:

$$\frac{2}{G} = \sum_\nu \frac{1}{\sqrt{(\tilde{\epsilon}_\nu - \lambda)^2 + \Delta^2}} \qquad \text{II.1.22}$$

From the constraint condition II.1.4 it also follows that

$$n = 2 \sum_\nu V_\nu^2 \qquad \text{II.1.23}$$

This confirms the interpretation given to $V_\nu^2$ as being the probability that the conjugate states $|\nu u\rangle$ and $|\nu \bar{u}\rangle$ are occupied by a pair.

Equations II.1.22 and II.1.23 must be solved together in order to determine $\lambda$ and $\Delta$ for any given nucleus. The position of $\lambda$ depends of course on the effective number of pairing levels taken to be appreciably mixed by the pairing interaction and whose occupation number is therefore sensibly different from 1. A spherical nucleus, for instance, exhibits a well-developed shell structure. Matrix elements of the pairing force between states of the completely filled shells and the states of the region of partly filled shells are vanishingly small, so most of the contributions to the sum II.1.22 comes from the states outside of closed shells. We may then consider only those shells in the actual solution of II.1.22 and II.1.23.

Similarly, the shells that lie well above the Fermi level will give negligibly small matrix elements with states of the partially filled shells.

It can be shown (Bo 59) that if the various shells outside the shell of interest are taken into account the effect is to renormalize the value of the strength parameter G. We have then to fit G according to the actual cut-off placed both below and above the Fermi level.

In order to solve II.1.22 and II.1.23 one needs firstly the energies $\epsilon_\nu$ and the strength parameter G. We shall discuss below how to find G empirically.

We notice from II.1.15 (or II.1.16) that the pairing interaction contributes a term to the self-consistent field, viz. $G u_\nu v_\nu$ and does in a sense cause a renormalization of the single particle energies $\epsilon_\nu$ of the deformed field. It originates from the diagonal part of the pairing Hamiltonian and therefore has the character of a self-energy term. From its nature, we see that it affects mostly those energy levels close to the Fermi level, but otherwise is small compared to the second term in II.1.15.

We have then two alternative choices, viz. either incorporate the self-energy term in the definition of the single particle levels, thus working with effectively renormalized single particle energies, or just ignore it, for the reason explained above.

From the first viewpoint, we may use the known experimental single particle energies available in trying to find the best fit for the strength parameter G and the level distribution. We have, however, used here the second approach. In what follows we shall discuss this point somewhat further.

One should notice that from Equation II.1.21 it follows that a non-trivial solution is possible only when the matrix elements of the pairing interaction have predominantly one sign over sufficiently many pairing levels: we have, in fact, already ensured this coherent behaviour by replacing the matrix elements of the pairing force by an average, -G. This is because otherwise we would have no guarantee that the expression II.1.21, involving algebraic sums over many matrix elements would not be vanishingly small due to cancellations, giving no solution other than that corresponding to the normal solution.

We thus see that the more pairing levels there are for the Cooper pairs to scatter into, the more justifiable and accurate will be the description of the effects of the short-range interactions between nucleons by means of the pairing force.

Following Nilsson and Prior (NP 61) we then define an effective degeneracy $\Omega_{eff}$ by the expression

$$\Omega_{eff} = \frac{2\Delta}{G} = \sum_{\nu} \frac{1}{\sqrt{1+\left(\frac{\varepsilon_\nu - \lambda}{\Delta}\right)^2}} \qquad \text{II} \cdot 1.24$$

and we take it as a measure of the accuracy of the description by means of the pairing force. As we shall see, these solutions reproduce the exact results developed in Chapter I for the completely degenerate case to an accuracy of the order of $\Omega_{eff}^{-1}$.

On the other hand, if we look at the last two terms of II.1.15 and rewrite them, thus

$$\Delta\left(u_\nu^2 - v_\nu^2\right) + 2G u_\nu v_\nu^3 \equiv \Delta - 2\Delta v_\nu^2 + 2G u_\nu v_\nu v_\nu^2 =$$

$$= \Delta - 2G\left[\sum_{\nu'} u_{\nu'} v_{\nu'} - u_\nu v_\nu\right] v_\nu^2 \qquad \text{II} \cdot 1.25$$

we see that the neglect of the last term in the last expression when compared to the second term will involve an error of the order of magnitude $\Omega_{eff}^{-1}$ , since there are effectively $\Omega_{eff}$ terms in the sum.

It turns out (HP 61) that, among the various contributions involved in the neglected interaction term $H_{int}$ in II.1.9, there is a term containing 4 quasi particle operators that gives non-zero matrix elements between the vacuum and 4 quasi particle states, and also between 1 and 5 quasi particle states. Again, the effect of this coupling on the ground state is of the order of magnitude $\Omega_{eff}^{-1}$ , which is then a good measure of the validity of the independent quasi particle approximation.

## 2. Energy Levels and Wave Functions for Low-Lying Quasi-Particle States

As was mentioned in the last paragraph, we now wish to introduce a vacuum state $|\phi_0\rangle$ for the quasi particles that would correspond naturally to the ground state of the old particles modified by the pairing correlations. Moreover, this state $|\phi_0\rangle$ must be such that

$$\alpha_{\nu m}|\phi_0\rangle = 0 \qquad \text{II} \cdot 2.1a$$

$$\beta_{\nu m}|\phi_0\rangle = 0 \qquad \text{II} \cdot 2.1b$$

From the definitions of the quasi-particle operators II.1.15, it is easily seen that the state vector

$$|\phi_0\rangle = \prod_{\nu m>0} \left( u_\nu + v_\nu \, a^+_{\nu m} \, a^+_{\nu,-m} \right)|0\rangle \qquad \text{II} \cdot 2.2$$

satisfies the conditions II.2.1. The state $|0\rangle$ represents here the true vacuum of the system.

We thus take $|\phi_0\rangle$ as being the state with no quasi-particles present, and corresponding to having the old particles (at least those occupying the states lying within about $\Delta$ from the Fermi surface) in the form of Cooper pairs. We thus see that the concept of a quasi-particle is actually a generalization of the concept of seniority introduced in the last chapter.

We note that the product over $\nu m$ runs over all the single particle states of the self-consistent field. However, with the practical values of the pairing force, only those states lying within a band of width of about $2\Delta \approx 2$ Mev centered at the Fermi surface have occupation probabilities sensibly different from 1 (or zero).

So we might consider in practice

$$|\phi_0\rangle \approx \prod_{\nu_1 \omega_1 > 0}^{\nu_2} \left(u_\nu + v_\nu a^+_{\nu\omega} a^+_{\nu,-\omega}\right)|0\rangle \qquad \text{II}\cdot 2.2$$

where $\nu_1 (\nu_2)$ is the lowest (highest) level for which the probability number $v_\nu^2 (u_\nu^2)$ differs from 1 by a pre-assigned amount, say by less than .01.

Particles below $\nu_1$ would be described by a simple product wave functions ( $v_\nu^2 = 1$) and above the limit $\nu_2$, we would have true vacuum ( $u_\nu^2 = 1$).

Clearly a state of seniority one, or a state with one quasi-particle present, is simply

$$|\phi_1\rangle = \alpha^+_{\nu_0\omega_0} |\phi_0\rangle = a^+_{\nu_0\omega_0} \prod_{\nu\omega \neq \nu_0\omega_0} \left(u_\nu + v_\nu a^+_{\nu\omega} a^+_{\nu,-\omega}\right)|0\rangle \qquad \text{II}\cdot 2.3$$

or

$$|\phi_1\rangle = \beta^+_{\nu_0\omega_0} |\phi_0\rangle = a^+_{\nu_0,-\omega_0} \prod_{\nu\omega \neq \nu_0\omega_0} \left(u_\nu + v_\nu a^+_{\nu\omega} a^+_{\nu,-\omega}\right)|0\rangle \qquad \text{II}\cdot 2.3$$

where $|\nu_0\omega_0\rangle$ (or $|\nu_0 - \omega_0\rangle$) is the state occupied by the odd particle. Similarly, states of seniority two are

$$|\phi_2\rangle = \alpha^+_{\nu_1\omega_1} \alpha^+_{\nu_2\omega_2} |\phi_0\rangle =$$

$$= a^+_{\nu_1\omega_1} a^+_{\nu_2\omega_2} \prod_{\nu\omega \neq \nu_1\omega_1, \nu_2\omega_2} \left(u_\nu + v_\nu a^+_{\nu\omega} a^+_{\nu,-\omega}\right)|0\rangle \qquad \text{II}\cdot 2.4$$

and so on. All these states are, of course, normalized and are exact eigenstates of the quasi-particle Hamiltonian, provided that the interaction term $H_{int}$ can be neglected. Again, the problem of investigating the orthogonality of these states is non-trivial, just as in the case of a completely degenerate system.

We note that none of these states describe a system with a definite number of particles; in other words, they are not eigenstates of the number operator. $|\phi_0\rangle$ is actually a superposition of 0, 2, 4, 6 ... particle states, whereas $|\phi_1\rangle$ describes a superposition of 1, 3,

5 ... particle states. Therefore $|\phi_0\rangle$ can describe only even-n systems and $|\phi_1\rangle$ odd-n systems. We shall later see how we can extract from these a set of wave functions containing a definite number of particles and how well can they describe the system.

We now note the following: the state $|\phi_0\rangle$ can be taken as a trial wave function for the full pairing Hamiltonian II.1.1.

In the usual manner we evaluate the expectation value of $H$ and minimize it with respect to $V_\nu$, say,

$$\delta \langle \phi_0 | H | \phi_0 \rangle = 0 \qquad \text{II} \cdot 2 \cdot 5$$

subject to the condition

$$u_\nu^2 + v_\nu^2 = 1 \qquad \text{II} \cdot 2 \cdot 6$$

This yields

$$(\epsilon_\nu - \lambda) 2 u_\nu v_\nu - \Delta (u_\nu^2 - v_\nu^2) - 2 G u_\nu v_\nu^3 = 0 \qquad \text{II} \cdot 2 \cdot 7$$

which we see to be identical to II.1.15. Therefore the choices for $V_\nu$ and $U_\nu$ given by II.1.18a and II.1.18b guarantees that $|\phi_0\rangle$ gives the minimum ground state energy of the particle system. $|\phi_0\rangle$ is then the best product wave function available, containing the maximum possible correlations among the particles which are favourable to the pairing force.

We can take similarly one quasi-particle states as trial functions for the ground state and low-lying excited states of the exact pairing Hamiltonian of an odd-n system. However, all basic equations previously developed have to be modified slightly to take into account the blocking of the orbital occupied by the odd nucleon. Using the single quasi particle state II.2.3 as a trial wave function, a variational calculation yields the equation

$$n-1 = \sum_{\nu \neq \nu'} 2 v_\nu^2 \qquad \text{II.2.8}$$

$$\Delta_0 = G \sum_{\nu \neq \nu'} u_\nu v_\nu \qquad \text{II.2.9}$$

where n is the total number of particles in the system and $\nu'$ is the orbital occupied by the odd particle. Nilsson and Prior (NP 61) have estimated the difference between $\Delta_e$ (when there is no blocking of an orbital, or when it is ignored) and $\Delta_0$ given by the above equations, and they find that

$$\Delta_0 \simeq \Delta_e - \frac{1}{\Delta_e^2} \left( \sum_{\nu \neq \nu'} \frac{1}{\epsilon_\nu^3} \right)^{-1} \qquad \text{II.2.10}$$

where

$$E_\nu = \sqrt{(\epsilon_\nu - \lambda)^2 + \Delta_e^2} \qquad \text{II.2.11}$$

assuming that states well below and well above the Fermi level are not affected by the blocking. This difference can be quite appreciable (see for instance Table XI) and produces quite marked changes in the occupation numbers, at least near enough to the Fermi level.

The excitation energy of a quasi-particle in the state $|\nu_1 u_1\rangle$ is simply

$$E_{\nu_1} = \sqrt{(\epsilon_{\nu_1} - \lambda)^2 + \Delta^2} \qquad \text{II.2.12}$$

as can easily be seen from II.1.13 and II.1.15. If the quasi particle ground state energy is $E_{\nu_0}$, then the relative energy of the excitation is

$$E_{\nu_1} - E_{\nu_0} = \sqrt{(\epsilon_{\nu_1} - \lambda)^2 + \Delta^2} - \sqrt{(\epsilon_{\nu_0} - \lambda)^2 + \Delta^2} \qquad \text{II.2.13}$$

which to 1st order is

$$(\epsilon_{\nu_1} - \epsilon_{\nu_0}) \frac{\epsilon_{\nu_1} + \epsilon_{\nu_0} - 2\lambda}{2\Delta} \qquad \text{II.2.14}$$

The energy of two quasi-particles in the states $\nu_1$ and $\nu_2$ is seen to be

$$E = \epsilon_{\nu_1} + \epsilon_{\nu_2} = \sqrt{(\epsilon_{\nu_1} - \lambda)^2 + \Delta^2} + \sqrt{(\epsilon_{\nu_2} - \lambda)^2 + \Delta^2} \gtrsim 2\Delta \qquad \text{II.2.15}$$

Therefore, odd-A nuclei can have arbitrarily low excitation energies, whereas even-even nuclei cannot have intrinsic excitations below $2\Delta$, in the independent quasi-particle approximation.

One should also notice that the pairing force implies that the average level density of intrinsic states near the ground state of odd nuclei is sensibly higher than that predicted by the independent particle model (M 58).

This is a consequence of the Pauli principle. In fact, if the orbital $\nu_a$, say, is blocked by the odd particles, then the correlated pairs are unable to scatter into $\nu_a$, due to the exclusion principle. However, if $\nu_a$ has a fairly high energy, then the pairs can correlate much better under the action of the pairing force, since they will have more available pairing levels. This implies that the experimentally observed low-lying single particle levels are slightly pushed up in energy relative to their positions according to the independent particle model, whereas the higher lying levels are not much affected. These indeed are the observed facts (Bk 58).

The energy gap may be obtained from the known nucleon separation energies, or from an estimate of the position of the first excited two quasi-particle states, although in this latter case one should make allowances for residual forces not taken into account by the pairing Hamiltonian, and which may affect considerably the two quasi-particle states.

The method followed here is based on estimates of even-odd mass differences. This is defined for neutrons, say, by

$$P_n = 2\epsilon(Z,N-1) - \epsilon(Z,N) - \epsilon(Z,N-2) \quad \text{II}.2.16$$

where the $E$'s are total binding energies of the even - odd neighbours. A more exact relationship is given by (NP 61)

$$P_n = \frac{1}{4}\left\{ - S_n(Z,N+1) + 2S_n(Z,N) - S_n(Z,N-1)\right\} \quad \text{II} \cdot 2.17$$

where $S_n(Z,N)$ is the neutron separation energy in the nucleus From the expressions previously developed for the ground state energies of an even - n and odd - n systems, it can be shown (NP 61) that the theoretical even - odd mass difference is

$$P_n \simeq \Delta_e + \frac{1}{\Delta_e^2}\left(\sum_{\nu\neq\nu'} \frac{1}{\epsilon_\nu^3}\right)^{-1} + \frac{G}{4}\left(1 - \frac{2}{\Delta_e}\,\frac{\sum_{\nu\neq\nu'} \frac{(\epsilon_\nu - \lambda)^2}{\epsilon_\nu^4}}{\sum_{\nu\neq\nu'} \frac{1}{\epsilon_\nu^3}}\right) \quad \text{II} \cdot 2.18$$

where the self-energy terms discussed previously have been included. If the blocking effect were ignored, which is tantamount to assuming the same vacuum for zero and one quasi-particle states, we would simply have

$$P_n \simeq \Delta_e \quad \text{II} \cdot 2.19$$

Nilsson and Prior calculations show that the third term in the expression for Pn reduces the contribution of the second to make Pn actually smaller than $\Delta_e$ by about 10% on the average, both for neutrons and protons.

Following these authors, we have tried to adjust the pairing force parameters $G_n$ and $G_p$ to give

$$P_n^{exp} \simeq \Delta_e^n \quad \text{II} \cdot 2.20$$

$$P_p^{exp} \simeq \Delta_e^p \quad \text{II} \cdot 2.21$$

where $P_n^{exp}$ and $P_p^{exp}$ are the experimentally obtained even - odd mass differences. The results are given in Tables X and XI, and are further referred to in Chapter V.

## CHAPTER III

### 1. Projected Wave Functions (Baymn)

As we have seen, the BCS wave functions are trial wave functions for the pairing Hamiltonian corresponding to an ensemble of nuclei.

The minimization of the energy was seen to give rise to definite expressions for the occupation number amplitudes such that the ensemble really consists of a few nuclei with slightly different numbers of particles. More precisely, if we define a mean square deviation for the number of particles in the ground state of an even system by

$$\sigma_N^2 = \langle N^2 \rangle - \langle N \rangle^2 \qquad \text{III. 1.1}$$

where N is the number operator, then it is easily seen that

$$\sigma_N^2 = 4 \sum_\nu u_\nu^2 v_\nu^2$$

$$= \sum_\nu \frac{1}{1 + \left( \frac{\epsilon_\nu - \lambda}{\Delta} \right)^2} \qquad \text{III. 1.2}$$

Bolinev (Bo 59) has estimated this quantity assuming a uniform single particle level density $\rho$ and a pairing force parameter $G$ large compared to the average single particle level spacing, and found it to be

$$\sigma_N^2 \simeq \pi \rho \Delta \qquad \text{III. 1.3}$$

Both conditions are verified to a sufficient approximation for nuclei in the rare earth region. We found that for neutrons, say, in the nuclei we considered in this region, the value of ranges from about 10 to about 5, which is much smaller than the effective number of neutrons considered (see Chapter II) (a typical number is 70).

For an odd system, due to the blocking of a single particle orbital $\nu'$, the mean square deviation is

$$\sigma_N^{o\,2} = 4 \sum_{\nu \neq \nu'} u_\nu^2 v_\nu^2 \qquad \text{III. 1.4}$$

$\sigma_N^{o}$ turns out to be smaller than $\sigma_N^{e}$ by about 5%.

Another consequence of this lack of definiteness in the number of particles in the independent quasi-particle picture is the appearance of spurious states (R 58). To illustrate this point let us restrict ourselves for a moment to the case of a single degenerate j-shell containing $\Omega = j + 1/2$ pairing levels. Since quasi-particles are supposed to be independent we can easily construct a state with total angular momentum $JM$ ; for instance,

$$|JM\rangle = \sum_{m>0} (j m\, j M-m \,|\, JM)\, \alpha_{jm}^{+}\, \alpha_{jM-m}^{+}\, |\phi_o\rangle \qquad \text{III. 1.5}$$

where $|\phi_o\rangle$ is the quasi-particle vacuum (i.e., the BCS ground state for an even-n system). Because there are $\Omega = j + 1/2$ pairing levels, there are obviously $\Omega$ degenerate two quasi-particle states. Since we know that the concept of a quasi-particle is a simple generalization of the concept of seniority, we conclude that there is one extra quasi-particle state, because there are only $\Omega - 1$ degenerate states of

seniority 2 $\left( = \frac{\Omega!}{\left(\frac{2}{2}\right)!\left(\Omega - \frac{2}{2} + 1\right)!} \quad (\Omega - 2 + 1) = \mathcal{D}_2 \; ; \; \text{see I. 1. 2a} \right)$

III. 1. 6a

Similarly, there are $\frac{1}{2}\Omega(\Omega-1)$ four quasi-particle states, and only

$$\mathcal{D}_4 = \frac{\Omega!}{\left(\frac{4}{2}\right)!\left(\Omega - \frac{4}{2} + 1\right)!} \quad (\Omega - 4 + 1) = \frac{\Omega!}{2(\Omega-1)!} \quad (\Omega - 3) = \frac{1}{2}\Omega(\Omega - 3)$$

III. 1. 6b

degenerate states of seniority 4. We could go on this way to show that there are always too many quasi-particle states compared to the number of states of real particles of seniority equal to the number of quasi-particles.

This is a direct consequence of the fact that we are attempting to describe the system in the independent quasi-particle approximation by introducing wave functions that are not eigenstates of the number operator. That is to say, expressing N in terms of quasi-particle operators $\alpha$, $\beta$, we have

$$N|\phi_0\rangle = \Big[ \sum_{m>0} 2V_m^2 + \sum_{m>0} \left(U_m^2 - V_m^2\right)\left(\alpha_m^+ \alpha_m + \beta_m^+ \beta_m\right) + 2\sum_{m>0} U_m V_m \left(\alpha_m^+ \beta_m^+ + \beta_m \alpha_m\right)\Big] |\phi_0\rangle$$

$$= n|\phi_0\rangle + 2\sqrt{\frac{n}{2\Omega}}\sqrt{\left(1 - \sqrt{\left(\frac{n}{2\Omega}\right)^2}\right)} \sum_{m>0} \alpha_m^+ \beta_m^+ |\phi_0\rangle$$

III. 1. 6c

Since, in the case we are now considering (degenerate $j^n$ configuration), the occupation numbers are

$$V_m^2 = \frac{n}{2\Omega} \quad , \quad U_m^2 = 1 - \frac{n}{2\Omega}$$

as can be easily seen.

The extra component

$$\sum_{\mu>0} \alpha_\mu^+ \beta_\mu^+ |\phi_0\rangle \qquad \text{III} \cdot 1.7$$

is, apart from numerical factors, a state in which formally two quasi-particles are coupled to $J = 0$. Its presence, however, precludes the possibilities of making $|\phi_0\rangle$ an eigenstate of $N$. It represents the extra state that we found when mentioning the degeneracy of a quasi-particle pair state.

Unfortunately, these spurious states have non-vanishing components on the various two quasi-particle configurations that one can form, which of course is undesirable since the spurious states cannot have physical significance.

As Kayman suggested (Ba 59), we can try to work with wave functions having a definite number of particles if we accept the BCS wave functions as a set of generating functions for the various matrix elements of physical interest. That is to say, we define (for the ground state, for instance) the generating function

$$|\psi(z)\rangle = \prod_{\alpha\,\mu>0} \left( u_\alpha + \sqrt{z}\, v_\alpha\, b_{\alpha\mu}^+\, b_{\alpha,-\mu}^+ \right) |0\rangle \qquad \text{III} \cdot 1.8$$

We can then extract from this the states containing definite numbers of particles by simply expanding it in powers of $z$ and picking up the coefficient of $z^{p/2}$, if $p$ is the number of pairs in the system that we wish to describe.

These projected (and properly normalized) functions will be taken than as a presumably better approximation to the actual nuclear state. The coefficient of $z^{p/2}$, for instance, is

$$\frac{1}{p!} \left( \prod_\beta u_\beta \right) \left( \sum_{\substack{\alpha_1, \alpha_2 \ldots \alpha_p \\ \mu_1, \mu_2 \ldots \mu_p > 0}} \frac{v_{\alpha_1}}{u_{\alpha_1}} \frac{v_{\alpha_2}}{u_{\alpha_2}} \cdots \frac{v_{\alpha_p}}{u_{\alpha_p}} b_{\alpha_1 \mu_1}^+ b_{\alpha_1,-\mu_1}^+ \cdots b_{\alpha_p \mu_p}^+ b_{\alpha_p,-\mu_p}^+ \right) |0\rangle$$

$$\text{III} \cdot 1.9$$

there being $p$ pair operators in the sum.

Similarly one can easily find the matrix elements of any operator by expanding

$$\langle \psi_1(z) | O | \psi_2(z) \rangle \qquad \text{III. 1.10}$$

in powers of $z$ and picking up the coefficient of $z^p$, if $p$ is the number of pairs, and if the number of pairs remains unchanged (see App. I).

If we take the $v$'s and $u$'s as parameters for a variation calculation, the problem is to minimize the expectation value of the pairing Hamiltonian with respect to these parameters (with the normalization condition $V_\nu^2 + U_\nu^2 = 1$), using the projected wave functions. As Bayman pointed out, this is equivalent to evaluating the contour integrals

$$E = \frac{\oint z^{-p-1} \langle \psi(z) | H | \psi(z) \rangle dz}{\frac{1}{2\pi i} \oint z^{-p-1} \langle \psi(z) | N | \psi(z) \rangle dz} \qquad \text{III. 1.11}$$

where E is the expectation value of the energy and the contours are to enclose the origin. This expression is exact, but in practice of course we can evaluate it only approximately. This is done by using the saddle-point method, i.e., essentially replacing the integrals by the value of their integrands evaluated at the saddle point $z_0$. The resulting expression for E is then minimized subject to the condition that the number of pairs in the system be equal to $p$. As shown by Bayman, when $G$ is large compared to the average single particle level spacing, the result is the familiar fundamental equation of the BCS theory, viz., the ground state expectation energy to be minimized (cf. II.1.10)

$$U = \sum_\nu (\epsilon_\nu - \lambda) 2 V_\nu^2 - \frac{\Delta^2}{G} - G \sum_\nu V_\nu^4 \qquad \text{II. 1.10}$$

and the equation yielding λ

$$p = \sum_{\nu} V_{\nu}^{2} \qquad \text{II. 1.23}$$

The error made in using $U_\nu$'s and $V_\nu$'s determined by the BCS eq. II.1.10 and II.1.25 is then formally equivalent to that made in replacing the ratio of the contour integrals by the ratio of their integrands evaluated at the saddle point. Baranger concludes that this error is small provided that, firstly, the contributions to the integrand come mainly from a small neighbourhood of the saddle point and, secondly, that the integrands be approximately proportional over the entire contour. The first condition is shown to imply that $\sigma_N^2$ be large, more precisely, that

$$\frac{1}{\sigma_N^2} \ll \frac{1}{2}\pi \qquad \text{III. 1.12}$$

Therefore, when this condition holds, the saddle point approximation will be valid. It might seem a little surprising at first that, as the mean square root number deviation increases, the accuracy of projected wave functions as found by the above saddle point method should become better. But, in fact, as the mean square root increases, so does the effective number of particles, and the effective degeneracy

$$\Omega_{eff} = \sum_{\nu} \frac{1}{\sqrt{1 + \left(\frac{\epsilon_\nu - \lambda}{\Delta}\right)^2}}$$

Then the quantitative differences between the physical predictions of the Bardeen and projected wave functions tend to be reduced, essentially because the number of pairing states admixed becomes so large. We find, for instance, that for M4 transitions in Pb isotopes, the quantitative differences between the Bardeen and projected wave functions is quite negligible as the number of neutron holes increases (see Chapter V).

It is also shown that for large $\sigma_N^2$ the conditions II.1.10 and II.1.23 alone guarantee that the excited states obtained from a variational solution of these equations are nearly orthogonal to one another and to the ground state.

Bayman further shows that the second requirement mentioned above is satisfied in the limit of non-interacting particles. No direct proof has yet been given in the general case.

## 2. Projected Wave Functions (Kerman, McFarlane, Lawson)

We consider again the full pairing Hamiltonian I.1.10

$$H_0 = \sum_{jm>0} \epsilon_j \left( a_{jm}^+ a_{jm} + a_{j-m}^+ a_{j-m} \right) - \\ - G \sum_{jj'mm'} (-1)^{j-m} a_{jm}^+ a_{j-m}^+ (-1)^{j'-m'} a_{j'-m'} a_{j'm'} \qquad \text{I.1.10}$$

and re-write it in terms of the quasi-spin operator I.1.12

$$H_0 = \sum_j \epsilon_j 2 S_j^0 + 2 \sum_j \epsilon_j \Omega_j - \\ - G \sum_{jj'} S_j^+ S_{j'}^- \qquad \text{III. 2.1}$$

where by definition

$$S_j^0 = \sum_{m>0} S_{jm}^0 \qquad \text{III. 2.2}$$

$$S_j^+ = \sum_{m>0} (-1)^{j-m} a_{jm}^+ a_{j-m}^+$$

$$S_j^- = \sum_{m>0} (-1)^{j-m} a_{j-m} a_{jm}$$

The quasi-spin vector $\vec{S}_j$ refers to the j-shell; we introduce the total quasi-spin for the system

$$\vec{S} = \sum_j \vec{S}_j \qquad \text{III. 2.3}$$

Corresponding to the total seniority

$$V = \sum_j v_j = \sum_j \left( \Omega_j - 2 S_j \right) \qquad \text{III. 2.4}$$

Then III.2.1 becomes

$$H_0 = \sum_j 2 \epsilon_j S_j^0 + 2 \sum_j \Omega_j \epsilon_j - G S^+ S^- \qquad \text{III. 2.5}$$

Kerman et al (McL 61) proceed to diagonalize $H_0$ numerically by choosing a convenient representation. The basic set of states most useful in this connection are $|\alpha S S_0\rangle$ introduced in Chapter I. We have merely to make sure that the set of states is complete and therefore $\alpha$ denotes all the additional quantum numbers required. The expressions for the eigenvectors $|S_j S_j^0\rangle$ were given in Chapter I. While the

quantum numbers $S_j'$ and $S_j^\circ$ completely define the state of an even number of nucleons in a single j-shell, when we have several shells we must couple the several $S_j'$ and $S_j^\circ$ to give the total $S$ and $S^\circ$, following the usual rules of the algebra of irreducible tensors.

Once all the linearly independent states (for the ground state of an even system only the states of seniority zero) have thus been constructed, we can then use them as our basis set to make the representation of $H_o$, after which the diagonalisation is done numerically. Similarly, one can proceed to represent $H_o$ when the system has an odd number of particles. One has to remember only that the odd particle, say in the j shell, blocks a state in that shell and therefore reduces $S_j'$ by 1/2 (cf. Chapter I). All the states must of course have seniority one. The diagonalisation will then yield the states of angular momentum j.

The most important result that emerges from this analysis, and which has a direct relevance here, is that the projected wave functions introduced in the previous paragraph give overlap integrals with the exact solution, obtained by the method outlined above, better than 99% in most cases discussed by Kerman et al (for states of seniority 0, 1 and 2). For instance, in the case of $Ni^{59}$, with 3 neutrons outside a closed shell, the $p_{3/2}$ state, say given by the numerical diagonalisation of the energy matrixes, was found to be

$$0.812\ (p^3_{3/2})_{3/2} + 0.526\ (f^2_{5/2})_0\ p_{3/2} +$$

$$+ 0.187\ (p^2_{1/2})_0\ p_{3/2} + 0.170\ (g^2_{9/2})_0\ p_{3/2}$$

whereas the corresponding amplitudes in the projected and normalised wave functions containing 3 particles are (0.757, 0.614, 0.216, 0.163).

The overlap integral is seen to be 0.995. For all the other states the overlap is 0.999.

From a general quantum mechanical theorem we know that if we have a trial wave function for the ground state which is good to order $\epsilon$, the energy will be good to order $\epsilon^2$. We expect then, since the overlap integrals are so close to unity, that the energies obtained by using the projected wave functions should be in excellent agreement with those obtained exactly, and this indeed is the case: for instance, taking again the Ni isotope quoted above, the energies obtained for the various seniority one states using the projected wave functions differ from the exact value by a small fraction of a per cent.

These results are indeed remarkable, since the projected wave functions describing a definite number of particles constitutes only about 40% of the original BCS wave functions, for typical cases.

This excellent overlap between the projected wave functions and the exact solutions should encourage one to use these wave functions to calculate matrix elements. In Chapter V we make an attempt to test the wave functions on some $\gamma$ and $\beta$ decay cases of interest.

Another important conclusion reached by Hecman et al is that for states of seniority one in spherical nuclei the wave functions and energies are not sensibly changed if the blocking effect due to the odd particle is ignored in evaluating $\lambda$ and $\Delta$, as was done by Kisslinger and Sorensen (KS 60). Similar conclusions seem to hold for states of seniority two.

We have found, however, that there is an appreciable change in the occupation numbers $v^2$ and $u^2$ when blocking is taken into account for deformed nuclei.

## CHAPTER IV

### 1. Matrix Elements for Allowed and First Forbidden β-Decay Operators

In developing the formula for the β-decay matrix elements using the pairing model we shall explicitly consider neutron decay, since proton decay can be obtained by simply taking the corresponding hermitean conjugate operators.

Moreover, since we are interested in the reduced transition probabilities, the Wigner-Eckart theorem permits us to consider only the transitions in which the initial and final magnetic quantum numbers are positive.

If $\Theta$ is an operator representing some dynamical variable associated with a single nucleon, its second quantisation representation is

$$\Theta = \sum_{\substack{\mu m \\ \mu' m'}} \langle \mu m | \Theta | \mu' m' \rangle \, C^{+}_{\mu m} C_{\mu' m'} \qquad \text{IV. 1.1}$$

The $C^{+}_{\mu m}$ ( $C_{\mu' m'}$) are single particle creation (annihilation) operators, creating (destroying) particles in states $|\mu m\rangle$ and $|\mu' m'\rangle$ respectively. The matrix element weighting the operator $C^{+}_{\mu m} C_{\mu' m'}$ in IV.1.1 is just the configuration space representation of $\Theta$, if the single particle states are given in configuration space.

The operator $\theta$ in the cases of interest to us here is

$$\theta = \xi \tau_- \qquad \text{IV. 1.2}$$

This $\tau_-$ annihilates a neutron state ( $\tau_z = +1/2$) and creates a proton state ( $\tau_z = -1/2$), and is one of the spherical components of the i-spin operator $\vec{\tau}$ . In IV.1.2 $\xi$ is a tensor the exact nature of which depends on the classification of the decay process (for discussion of $\beta$ decay theory see for instance F. 61, pg 394 ff, note that we are using here the opposite convention for the i-spin operators).

Thus IV.1.1 becomes

$$\xi = \sum_{\substack{\mu m \\ \mu' m'}} \sum_{p} \langle \mu m | \xi | \mu' m' \rangle_n \, b^+_{\mu m} a_{\mu' m'} \qquad \text{IV. 1.3}$$

where we are using the labels $p$ and $n$ for quantities referring to protons and neutrons respectively. Similarly, the $b$'s will stand for proton operators and the $a$'s for neutron operators.

We discuss first the case when the decay results from the transformation of the odd particles, and there is no change in the number of Cooper pairs.

Let there be $N = 2n + 1$ neutrons and $Z = 2p$ protons in the parent nucleus. We recall that we are assuming that there are no pairing correlations between neutrons and protons. Then the generating function for this nucleus in its initial state in the pairing approximation is

$$|\psi_i(N,Z)\rangle = |\psi_i(N)\rangle \, |\psi_i(Z)\rangle =$$

$$= \left[ a^+_{\alpha_1 m_1} \prod_{\substack{\alpha m \neq \\ \alpha_1 m_1}} \left( u_\alpha + \sqrt{z}\, v_\alpha \, a^+_{\alpha m} \, a^+_{\alpha -m} \right) |0\rangle \right]_n \times$$

$$\times \left[ \prod_{\alpha m > 0} \left( u_\alpha + \sqrt{z}\, v_\alpha \, b^+_{\alpha m} \, b^+_{\alpha -m} \right) |0\rangle \right]_p \qquad \text{IV. 1.4}$$

Here $|\alpha_1 u_1>$ is the odd neutron orbital in the initial configuration. If after the decay the resulting proton goes into the orbital $<\alpha_2 u_2|$, we have similarly for the final state

$$<\psi_f\ (N-1, Z+1)| = \left[<0|\prod_{\alpha u \neq \alpha_2 u_2} b_{\alpha_2 u_2}\ (A_\alpha + \sqrt{Z}\ B_\alpha\ b_{\alpha,-u}\ b_{\alpha u})\right]_p \times$$

$$\times \left[<0|\prod_{\alpha u>0}\ (A_\alpha + \sqrt{Z}\ B_\alpha\ a_{\alpha,-u}\ a_{\alpha u})\right]_n \qquad \text{IV. 1.5}$$

where the coefficients A and B denote the occupation amplitudes for the daughter nucleus. As remarked above, we restrict ourselves to $u_1>0$, $u_2>0$. The matrix element of IV.1.3 between these states is then

$$<\psi_f\ (N-1, Z+1)\ |\xi|\psi_i\ (N, Z)> = \sum_{\substack{kk' \\ \mu\mu'}} S_p^{\mu k}(Z)\ S_n^{\mu' k'}(Z) <\mu k|\xi|\mu' k'> \qquad \text{IV. 1.6}$$

where, by definition,

$$S_p^{\mu k}(Z) = <0|\sum_{\substack{\alpha u \neq \\ \alpha_2 u_2}} b_{\alpha_2 u_2}\ (A_\alpha + \sqrt{Z}\ B_\alpha\ b_{\alpha,-u}\ b_{\alpha u})\ b_{\mu k}^+ \prod_{\alpha u>0}(U_\alpha +$$

$$+ \sqrt{Z}\ V_\alpha\ b_{\alpha m}^+\ b_{\alpha,-u}^+)|0> \qquad \text{IV. 1.6a}$$

and

$$S_n^{\mu' k'}(Z) = <0|\prod_{\alpha u>0}(A_\alpha + \sqrt{Z}\ B_\alpha\ a_{\alpha,-u}\ a_{\alpha u})\ a_{\mu' k'}\ a_{\alpha_1 u_1}^+ \prod_{\substack{\alpha u \neq \\ \alpha_1 u_1}}(U_\alpha +$$

$$+ \sqrt{Z}\ V_\alpha\ a_{\alpha m}^+\ a_{\alpha,-u}^+)|0> \qquad \text{IV. 1.6b}$$

It is seen that (cf. App. Ia)

$$S_p^{\mu k}(Z) = \delta_{\mu\alpha_2}\ \delta_{k u_2}\ U_{\alpha_2} \prod_{\substack{\alpha u \neq \\ \alpha_2 u_2}} (A_\alpha U_\alpha + Z B_\alpha V_\alpha) \qquad \text{IV. 1.7a}$$

and

$$S_n^{\mu' k'}(Z) = \delta_{\mu'\alpha_1}\ \delta_{k' u_1}\ A_{\alpha_1} \prod_{\substack{\alpha u \neq \\ \alpha_1 u_1}} (A_\alpha U_\alpha + Z B_\alpha V_\alpha) \qquad \text{IV. 1.7b}$$

Thus

$$<\psi_f\ (N-1, Z+1)|\xi|\psi_i\ (N, Z)> =$$

$$= \sum_{\substack{\mu k \\ \mu' k'}} <\mu k|\xi|\mu' k'>\ \delta_{\mu\alpha_2}\ \delta_{k u_2}\ \delta_{\mu'\alpha_1}\ \delta_{k' u_1}\ R_p(Z) R_n(Z) \qquad \text{IV. 1.8}$$

where $R_p(Z) = \left[U_{\alpha_2} \prod_{\alpha u \neq \alpha_2 u_2} (A_\alpha U_\alpha + Z V_\alpha B_\alpha)\right]_p$ IV. 1.9a

and

$$R_n(Z) = \left[A_{\alpha_1} \prod_{\substack{\alpha u \neq \\ \alpha_1 u_1}} (A_\alpha U_\alpha + Z B_\alpha V_\alpha)\right]_n \qquad \text{IV. 1.9b}$$

Thus the matrix element is just

$$\langle \psi_f\,(N-1, Z+1) \,|\, \xi \,|\, \psi_i\,(N,Z)\rangle = \langle \alpha_2 \omega_2 \,|\, \xi \,|\, \alpha_1 \omega_1\rangle\; R_p(Z) R_n(Z) \qquad \text{IV. 1.10}$$

The Bardeen results would correspond to making $Z \equiv 1$ in these formulae. Thus the Bardeen "reduction" factors are

$$R'_p\,(Z \equiv 1) = \Big[ u_{\alpha_2} \prod_{\substack{\alpha\omega \neq \\ \alpha_2\omega_2}} (A_\alpha u_\alpha + B_\alpha v_\alpha) \Big]_p \qquad \text{IV. 1.10a}$$

and

$$R'_n\,(Z \equiv 1) = \Big[ A_{\alpha_1} \prod_{\substack{\alpha\omega \neq \\ \alpha_1\omega_1}} (A_\alpha u_\alpha + B_\alpha v_\alpha) \Big]_n \qquad \text{IV. 1.10b}$$

The product terms in these expressions are clearly very close to unity and would have been exactly 1 if the blocking of the single particle orbitals $|\alpha_1 \omega_1\rangle$ and $|\alpha_2 \omega_2\rangle$ could be ignored. The reduction factor for the transition probability is then

$$R\,(Z \equiv 1) = \Big[ R'_p\,(Z \equiv 1)\; R'_n\,(Z \equiv 1) \Big]^2 \qquad \text{IV. 1. 11}$$

From the generating functions for the reduction factors IV.1.9a and IV.1.9b we can easily get (cf. App. Ia) the reduction factors when projected wave functions are used:

$$R'_p = \left\{ \frac{\sum_{\alpha\beta\cdots\gamma \neq \alpha_2} c_\alpha\, c_\beta \cdots c_\gamma}{\Big[\sum_{\alpha\beta\cdots\gamma} a_\alpha a_\beta \cdots a_\gamma\Big]^{1/2} \Big[\sum_{\substack{\alpha\beta\cdots\gamma \\ \neq \alpha_2}} b_\alpha b_\beta \cdots b_\gamma\Big]^{1/2}} \right\}_p \qquad \text{IV. 1.12a}$$

and

$$R'_n = \left\{ \frac{\sum_{\alpha\beta\cdots\gamma \neq \alpha_1} c_\alpha\, c_\beta \cdots c_\gamma}{\Big[\sum_{\alpha\beta\cdots\gamma} b_\alpha b_\beta \cdots b_\gamma\Big]^{1/2} \Big[\sum_{\substack{\alpha\beta\cdots\gamma \\ \neq \alpha_1}} a_\alpha a_\beta \cdots a_\gamma\Big]^{1/2}} \right\}_n \qquad \text{IV. 1.12b}$$

and

$$R = \Big[ R'_p\; R'_n \Big]^2 \qquad \text{IV. 1.12c}$$

We have put

$$a_\alpha = \frac{v_\alpha^2}{u_\alpha^2}$$

$$b_\alpha = \frac{\beta_\alpha^2}{\alpha_\alpha^2} \qquad \text{IV. 1.13}$$

$$c_\alpha = \sqrt{a_\alpha b_\alpha} = \frac{\beta_\alpha}{\alpha_\alpha}\frac{v_\alpha}{u_\alpha}$$

In IV.1.12 no two single particle indices $\alpha$, $\beta$ ... can be repeated, due to the exclusion principle. If there are p proton pairs and n neutron pairs (which in the case we are now discussing remains unchanged) the sums are to be taken over products of p factors $a_\alpha$, $b_\alpha$, $c_\alpha$ respectively for the protons, and n similar factors for the neutrons. The more explicit form given in App. Ia (see for instance, eq. 1 (5), 1 (6), 2 (7), 2(8)) makes clear what meaning is to be attached to this formula when any of the u's or v's vanishes.

To simplify the writing we introduce

$$S(k) = \sum_{\alpha\beta\cdots\gamma} 1_\alpha \; 1_\beta \cdots 1_\gamma \qquad \text{IV. 1.14}$$

with the convention that k stands for the number of factors in each product in the sum. If one of the single particle states is blocked, we define

$$S(k;\alpha_1) = \sum_{\substack{\alpha\beta\cdots\gamma \\ \neq \alpha_1}} 1_\alpha \; 1_\beta \cdots 1_\gamma \qquad \text{IV. 1.15}$$

where $\alpha_1$ is the label for the blocked orbital.

With this notation, the reduction factors IV.1.12 become

$$R_p' = \frac{C(p;\alpha_2)}{\sqrt{A(p)\,B(p;\alpha_2)}} \qquad \text{IV. 1.12c}$$

and

$$R'_n = \frac{C(n;\alpha_1)}{\sqrt{A(n;\alpha_1)\,B(n)}} \qquad \text{IV. 1.12d}$$

There S (k) satisfy simple algebraic properties (cf Ap. Ia). Since the blocking of a single particle orbital actually reduces the effective number of pairing states, it follows that

$$S(k;\alpha) < S(k) \qquad \text{IV. 1.16}$$

Let us now consider the case when the number of pairs in each system changes, i.e., the $(2n, 2p+1)$ parent decays into the $(2n-1, 2p+2)$ daughter. Since the sign of the magnetic quantum numbers do not influence the final results, we assume that the proton orbital in the parent nucleus is $|\nu_1 - \mu_1\rangle$ and that the odd particle in the daughter occupies the orbital $|\nu_2 - \mu_2\rangle$.

The generating function for the initial state is then

$$|\psi_i(2n, 2p+1)\rangle = |\psi_i(2n)\rangle\,|\psi_i(2p+1)\rangle =$$
$$= \Big[\prod_{\nu\mu>0}\left(u_\nu + \sqrt{z}\,v_\nu\, a^+_{\nu\mu}\, a^+_{\nu-\mu}\right)|0\rangle\Big]_n \times \Big[\prod_{\substack{\nu\mu>0\\ \neq\nu_1\mu_1}}\left(u_\nu + \sqrt{z}\,v_\nu\, b^+_{\nu\mu}\, b^+_{\nu-\mu}\right) b^+_{\nu_1-\mu_1}\,|0\rangle\Big]_p \qquad \text{IV. 1.17}$$

For the final state we have

$$\langle\psi_f(2n-1, 2p+2)| = \langle\psi_f(2n-1)|\,\langle\psi_f(2p+2)| =$$
$$= \Big[\langle 0|\prod_{\nu\neq\nu_2} a_{\nu_2-\mu_2}\left(A_\nu + \sqrt{z}\,B_\nu\, a_{\nu-\mu}\, a_{\nu\mu}\right)\Big]_n \times \Big[\langle 0|\prod_{\nu\mu}\left(A_\nu + \sqrt{z}\,B_\nu\, b_{\nu-\mu}\, b_{\nu\mu}\right)\Big]_p \qquad \text{IV. 1.18}$$

We are of course assuming that the proton system is unexcited after the decay. The the matrix element is just

$$M^K_M(\nu_1 \to \nu_2; z) = \sum_{\substack{\mu k\\ \mu' k'}} \langle \mu k\,|\,T^K_M\,|\mu' k'\rangle \Big\{\langle 0|\prod_{\substack{\nu\mu\neq\\ \nu_2\mu_2}} a_{\nu_2-\mu_2}\left(A_\nu + \sqrt{z}\,B_\nu\, a_{\nu-\mu}\, a_{\nu\mu}\right) a_{\mu' k'}\prod_{\nu\mu>0}\left(u_\nu + \sqrt{z}\,v_\nu\, a^+_{\nu\mu}\, a^+_{\nu-\mu}\right)|0\rangle\Big\}_n \times$$
$$\times \Big\{\langle 0|\prod_{\nu\mu>0}\left(A_\nu + \sqrt{z}\,B_\nu\, b_{\nu-\mu}\, b_{\nu\mu}\right) b^+_{\mu k}\prod_{\substack{\nu\mu>0\\ \neq\nu_1\mu_1}}\left(u_\nu + \sqrt{z}\,v_\nu\, b^+_{\nu\mu}\, b^+_{\nu-\mu}\right) b^+_{\nu_1-\mu_1}|0\rangle\Big\}_p \qquad \text{IV. 1.19}$$

We show in Appendix I.A that this reduces to

$$M^K_M(\nu_1 \to \nu_2 ; z) = \langle \nu_1 m_1 | T^K_M | \nu_2 m_2 \rangle \{ [\sqrt{z}\, V_{\nu_2} \prod_{\nu m \neq \nu_2 m_2} (A_\nu U_\nu + z B_\nu V_\nu)]_n \times [\sqrt{z}\, B_{\nu_1} \prod_{\nu m \neq \nu_1 m_1} (A_\nu U_\nu + z B_\nu V_\nu)]_p \quad \text{IV. 1.20}$$

With Bardeen wave functions, the reduction factors are then simply

$$R'_n(z=1) = [V_{\nu_2} \prod_{\nu m \neq \nu_2 m_2} (A_\nu U_\nu + B_\nu V_\nu)]_n \quad \text{IV. 1.21a}$$

$$R'_p(z=1) = [B_{\nu_1} \prod_{\nu m \neq \nu_1 m_1} (A_\nu U_\nu + B_\nu V_\nu)]_p \quad \text{IV. 1.21b}$$

and the single particle reduced probabilities are decreased by a factor

$$R(\text{Bardeen}) = [R'_n \times R'_p]^2 \quad \text{IV. 1.22}$$

For the projected wave functions, it is shown in the Appendix I.A that we get

$$R'_n = \left[ \frac{V_{\nu_2}}{U_{\nu_2}} \sqrt{n}\, \frac{C(n-1 ; \nu_2)}{\sqrt{A(n)\, B(n-1;\nu_2)}} \right]_n \quad \text{IV. 1.23a}$$

and

$$R'_p = \left[ \frac{B_{\nu_1}}{A_{\nu_1}} \sqrt{p+1}\, \frac{C(p ; \nu_1)}{\sqrt{B(p+1)\, A(p;\nu_1)}} \right]_p \quad \text{IV. 1.23b}$$

Since these transitions always occur near the Fermi level where the occupation number $V_\nu^2$ and $U_\nu^2$ have the same order of magnitude, and also in virtue of IV.1.16, the reduction factors are again less than unity. The factors $\sqrt{n}$ and $\sqrt{p+1}$ are, of course, trivial and will cancel out because S (k) involves k! permutations of the k factors $1_\alpha, 1_\beta, 1_\gamma$.

## 2. Matrix Elements for Single-Particle Operators for Electromagnetic Transitions

We consider a transition of a nucleon from the orbital $|\alpha_1 \mu_1\rangle$ to the orbital $|\alpha_2 \mu_2\rangle$, and since we are interested only in reduced transition probabilities, we may assume that $\mu_1 > 0$, $\mu_2 > 0$.

The generating function for the matrix element is then, if we denote by $T^K_M$ the single particle part of the corresponding electromagnetic operator

$$M^K_M(\alpha_1 \to \alpha_2; z) = \sum_{\substack{\mu k \\ \nu k'}} \langle \mu k | T^K_M | \nu k' \rangle \Big[ \langle 0 | \prod_{\substack{\alpha_q > 0 \\ \neq \alpha_2 \mu_2}} b_{\alpha_2 \mu_2} (A_\alpha + \sqrt{z}\, B_\alpha b_{\alpha -q} b_{\alpha q}) b^+_{\mu k} b^+_{\nu k'} \prod_{\alpha q \neq \alpha_1 \mu_1} (U_\alpha + \sqrt{z}\, V_\alpha b^+_{\alpha q} b^+_{\alpha -q}) b^+_{\alpha_1 \mu_1} | 0 \rangle \quad \text{IV.2.1}$$

It is then easily shown that the only non-vanishing contributions to IV.2.1. are

$$M^K_M(\alpha_1 \to \alpha_2; z) = \sum_{\substack{\mu k \\ \nu k'}} \langle \mu k | T^K_M | \nu k' \rangle \Big\{ \Big[ \delta_{\mu \alpha_2} \delta_{\nu \alpha_1} \delta_{k \mu_2} \delta_{k' \mu_1} \times A_{\alpha_1} U_{\alpha_2} \prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} (A_\alpha U_\alpha + z B_\alpha V_\alpha) \Big] + z \Big[ B_{\alpha_1} V_{\alpha_2} \delta_{\mu \alpha_1} \delta_{\nu \alpha_2} \delta_{k -\mu_1} \delta_{k' \mu_2} \times \prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} (A_\alpha U_\alpha + z B_\alpha V_\alpha) \Big] \Big\} \quad \text{IV.2.2}$$

But

$$\langle \alpha_1 -\mu_1 | T^K_M | \alpha_2 -\mu_2 \rangle = (-1)^\tau \langle \alpha_2 \mu_2 | T^K_M | \alpha_1 \mu_1 \rangle \quad \text{IV.2.3}$$

from time-reversal properties, and where $\tau$ is even or odd depending on whether the operator $T^K_M$ is even or odd under time reversal. Thus for electric transitions $\tau = 1$, and for magnetic transitions $\tau = 0$.

$$M^K_M(\alpha_1 \to \alpha_2; z) = \langle \alpha_2 \mu_2 | T^K_M | \alpha_1 \mu_1 \rangle \Big[ A_{\alpha_1} U_{\alpha_2} + z (-1)^\tau B_{\alpha_1} V_{\alpha_2} \Big] \prod_{\alpha \neq \alpha_1, \alpha_2} (U_\alpha A_\alpha + z V_\alpha B_\alpha) \quad \text{IV.2.4}$$

The Bardeen result would correspond to making $z \equiv 1$:

$$M^K_M(\alpha_1 \to \alpha_2; z \equiv 1) = \langle \alpha_2 \mu_2 | T^K_M | \alpha_1 \mu_1 \rangle \Big[ A_{\alpha_1} U_{\alpha_2} + (-1)^\tau B_{\alpha_1} V_{\alpha_2} \Big] \prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} (U_\alpha A_\alpha + V_\alpha B_\alpha) \quad \text{IV.2.5}$$

The reduction relative to the single particle estimate would then be

$$R_{\text{Random}}(\alpha_1 \to \alpha_2) = \left(A_{\alpha_1} U_{\alpha_2} + (-1)^{\tau} B_{\alpha_1} V_{\alpha_2}\right)^2 \left[\prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} (U_\alpha A_\alpha + V_\alpha B_\alpha)\right]^2 \qquad \text{IV.2.6}$$

If the system has p pairs, then the coefficient of $Z^p$ in the expansion of IV.2.2 in powers of $Z$ is then

$$A_{\alpha_1} U_{\alpha_2} \frac{1}{p!} \left(\prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} U_\alpha A_\alpha\right) \sum_{\substack{\alpha,\beta \cdots \nu_p \\ \neq \alpha_1 \alpha_2}} \left(\frac{V_\alpha}{U_\alpha} \frac{V_\beta}{U_\beta} \cdots \frac{V_{\nu_p}}{U_{\nu_p}}\right)\left(\frac{B_\alpha}{A_\alpha} \frac{B_\beta}{A_\beta} \cdots \right. \qquad \text{IV.2.7}$$

$$\left. \cdots \frac{B_{\nu_p}}{A_{\nu_p}}\right) + (-1)^{\tau} \frac{1}{(p-1)!} B_{\alpha_1} V_{\alpha_2} \left(\prod_{\substack{\alpha \neq \alpha_1 \\ \alpha \neq \alpha_2}} U_\alpha A_\alpha\right) \sum_{\substack{\alpha,\beta \cdots \nu_{p-1} \\ \neq \alpha_1 \alpha_2}} \left(\frac{V_\alpha}{U_\alpha} \frac{V_\beta}{U_\beta} \cdots \frac{V_{\nu_{p-1}}}{U_{\nu_{p-1}}}\right) \times \left(\frac{B_\alpha}{A_\alpha} \cdots \frac{B_{\nu_{p-1}}}{A_{\nu_{p-1}}}\right)$$

Dividing this by the normalization factors (cf Ap. I.B) we get

$$M^K_{M}(\alpha_1 \to \alpha_2) = \langle \alpha_2 \omega_2 | T^K_M | \alpha_1 \omega_1 \rangle \left[ \frac{\sum_{\neq \alpha_1, \alpha_2} \left(\frac{V_\alpha}{U_\alpha} \frac{B_\alpha}{A_\alpha}\right) \cdots \left(\frac{V_{\nu_p}}{U_{\nu_p}} \frac{B_{\nu_p}}{A_{\nu_p}}\right)}{\left[\sum_{\neq \alpha_1} \frac{V_\alpha^2}{U_\alpha^2} \cdots \frac{V_{\nu_p}^2}{U_{\nu_p}^2}\right]^{1/2} \left[\sum_{\neq \alpha_2} \frac{B_\alpha^2}{A_\alpha^2} \cdots \frac{B_{\nu_p}^2}{A_{\nu_p}^2}\right]^{1/2}} + \right.$$

$$\left. + (-1)^{\tau} p \left(\frac{B_{\alpha_1}}{A_{\alpha_1}} \frac{V_{\alpha_2}}{U_{\alpha_2}}\right) \frac{\sum_{\neq \alpha_1, \alpha_2} \left(\frac{V_\alpha}{U_\alpha} \frac{B_\alpha}{A_\alpha}\right) \cdots \left(\frac{V_{\nu_{p-1}}}{U_{\nu_{p-1}}} \frac{B_{\nu_{p-1}}}{A_{\nu_{p-1}}}\right)}{\left[\sum_{\neq \alpha_1} \frac{V_\alpha^2}{U_\alpha^2} \cdots \frac{V_{\nu_p}^2}{U_{\nu_p}^2}\right]^{1/2} \left[\sum_{\neq \alpha_2} \frac{B_\alpha^2}{A_\alpha^2} \cdots \frac{B_{\nu_p}^2}{A_{\nu_p}^2}\right]^{1/2}} \right] \qquad \text{IV.2.8}$$

If we assume that the blocking has a negligible effect in the wave functions (which is the case in spherical nuclei (see KMcF 61)) then the reduction factors simplifies to

$$R = \left[\frac{S(p; \alpha_1 \alpha_2) + (-1)^{\tau} p \sqrt{\alpha_1 \alpha_2}\, S(p-1; \alpha_1 \alpha_2)}{\sqrt{S(p; \alpha_1) S(p; \alpha_2)}}\right]^2$$

where

$$\alpha_i = \frac{V_i^2}{U_i^2} \qquad \text{IV.2.9}$$

Correspondingly the Bardeen reduction factor would be just

$$R_{\text{Bardeen}}(\alpha_1 \to \alpha_2) = \left(A_{\alpha_1} U_{\alpha_2} + (-1)^{\tau} B_{\alpha_1} V_{\alpha_2}\right)^2 \quad \text{IV.2.10}$$

For a magnetic transition, $\tau$ is even, and for an electric transition $\tau$ is odd. We notice again that the reduction factors in both cases will always be less than unity.

CHAPTER V

## 1. Results and Discussions for some Electromagnetic Transitions

The application of the pairing model to electromagnetic transitions, in particular to some isomeric transitions in spherical odd-A nuclei, has already been discussed by Kisslinger and Sorensen (KS 61). They find that the configuration mixing of the type introduced by the pairing force can explain certain features of these transitions, although the detailed quantitative comparison with experimental results was not considered to be of significance, due to large uncertainties involved in the evaluation of single particle matrix elements.

We re-examined here their results for odd-A nuclei, taking the same pairing force and single particle energies for both projected and BCS wave functions.

In Table I we give the various probability numbers for Pb isotopes with the observed single-hole levels in $Pb^{207}$, and using $G = \frac{30}{A}$ Mev. This parameter was found to fit reasonably well the even-odd mass differences observed (KS 61), although KS have used $\frac{23}{A}$ Mev in their calculations of levels of Pb isotopes. However, the value $\frac{30}{A}$ Mev seems to give a slightly improved agreement for the M4 data for Pb.

In Table II, similar quantities are listed for the two Sn isotopes considered. The single particle levels, however, are taken from

the experimental data obtained by Cohen et al (C 61), and represent average values over the various Sn isotopes studied by these authors by means of stripping and pick-up reactions. They differ appreciably from the Kisslinger and Sorensen data (KS 61). We have used their value for the pairing force parameter, assuming that the different level order and spacing does not affect appreciably this parameter.

Table III lists similar quantities for three N = 50 isotones, taking the single particle levels and the pairing force strength as given by Kisslinger and Sorensen.

Table IV gives the transition energies, level changes and reduction factors for Pb isotopes. In column 4 are listed the reduction factors for Pb isotopes obtained with the BCS wave functions, using formula IV.2.10 (Chapter IV).

The salient feature, as pointed out by Kisslinger and Sorensen, is their smooth variation as the isotopic number changes from 205 to 197, i.e., when the Fermi level moves from close to zero ($p_{1/2}$ state) up to about the position of the $p_{3/2}$ level, going past the final level of the transition ($f_{5/2}$).

In column 6 are listed the experimental reduction factors, defined by the ratio $P_{exp}/P_{s.p.}$ of the observed probability of transition per unit time to the corresponding quantity evaluated on the basis of the single particle model. One observes that they are nearly constant. These reduction factors are actually Moszkowski's estimates with the average nuclear radius taken as

$$R = 1.2 \times A^{1/3} \quad \text{fermis}$$

and are normalized to the $Pb^{207}$ value in column 9, since the reduction factor given by the pairing model for $Pb^{207}$ is obviously unity.

This feature of the M4 transitions in Pb isotopes, as explained by Kisslinger and Sorensen, cannot be accounted for without bringing in configuration mixing that appreciably smears out the probability distribution of states near the Fermi level: the smooth changes in the occupation amplitudes across the Fermi surface are responsible for the rather gradual variations of the reduction factors, as one adds particles to the system, i.e., as the Fermi level moves past the $f_{5/2}$ state. Such a behavior cannot be obtained on the basis of an independent particle model, or even the shell model with diagonal pairing forces, because of the sharp cut-off at the Fermi energy.

One also notes that when the Fermi level is well below both the levels involved in the transition, the occupation numbers have practically their independent particle values and the matrix elements are essentially the same as given by the independent particle model.

In column 5 we list the reduction factors obtained using the projected wave functions (see formula IV.2.9, Chapter IV). In column 10 we list the experimental reduction factors using harmonic oscillator wave functions (which essentially implies here that we should multiply the number in column 9 corresponding to a given A by the factor $\frac{A}{207}$). We note that there is an overall slight improvement in the reduction factors. The new results tend rapidly to the BCS reduction factors as the number of holes in the shell increases, becoming practically indistinguishable from them already at $Pb^{197}$. This is to be expected because, as already mentioned in Chapter IV, the more particles we add to the system the more complicated the actual nuclear configurations become, and the more accurately they are described by the large configuration mixing introduced

by the pairing forces. At the same time, configuration mixing from neighbouring nuclei introduced by the BCS wave functions tend to become increasingly less important, as the root mean square deviation of the number of particles becomes negligible when compared to the actual number of particles in the system. Therefore, the BCS reduction factors rapidly approach those given by the projected wave functions.

The fact that even when the number of holes is relatively small there is only a slight difference in the results predicted by the two types of wave functions shows that M4 transitions are not very sensitive to the differences in the structure of the two types of wave functions. When the number of holes in the open shell is small compared to the total number of pairing states available, the projected wave functions tend to reduce the single particle matrix elements less drastically than do the BCS wave functions, because the terms corresponding to levels lying below the Fermi level ($u^2 \rightarrow 0$) weigh more heavily in the sums involved in eq. IV.2.9 (Chapter IV).

As the number of holes increases (but the number of levels does not), the contribution of the levels above the Fermi surface becomes increasingly important, and the overall result is, of course, that given by the BCS wave functions.

Table V lists similar quantities for Sn isotopes and the N = 50 isotones considered. There is hardly any difference at all, for both $Sn^{117}$ and $Sn^{119}$, between the calculated reduction factors using the two types of wave functions. The matrix elements they give are very close to the single particle matrix elements, because, in both cases, the Fermi

level lies well below the two energy levels involved in the transition, viz., $h_{11/2}$ and $d_{3/2}$.

One may notice, incidentally, that the BCS reduction factors given are very close to the values quoted in KS 61, although the values for the energy levels are quite different. This is because, as pointed out by those authors, the energy difference involved in the transition is quite small compared to the gap.

The last three cases listed in Table V show some marked differences between the BCS reduction factors and those calculated by means of projected wave functions, although these differences tend to decrease as the number of protons in the open shells increases, as one could expect. Note that the projected wave functions reduce less drastically the single particle matrix elements. Since they must give a better estimate of the configuration mixing introduced by pairing forces alone (Chapter III) than the BCS wave functions (which introduce irrelevant configuration mixing from neighbouring nuclei) we conclude that, in these three cases, there are important forces that have not been taken into account, such as residual quadrupole forces, and that considerably mix quasi particle configurations. This would easily explain why the projected wave functions, although capable of following the trend observed when one goes from $Y^{89}$ to $Te^{92}$, do not reduce sufficiently the single particle matrix elements.

## 2. A. Single Particle Levels

The basic set of energy levels we have used were taken from the original paper by Nilsson (NIL 55), with a few modifications added, in line with the results reported by Mottelson and Nilsson (MN 59), and Nilsson and Prior (NP 61). The single particle Hamiltonian chosen by Nilsson contains two adjustable parameters, $\eta$ the deformation parameter and $\mu$, independent of the deformation. The latter determines the sequence of levels within a given major oscillator shell N. The total energy spread of levels within a major shell is determined by yet another parameter $\varkappa$, which is fixed from the observed level sequence in spherical nuclei. It is related to the strength of the spin-orbit component in the potential. The deformation parameter is related to $\varkappa$ by the relationship (NIL 55)

$$\eta = \frac{\delta_N}{\varkappa} \frac{\omega_0(\delta)}{\omega_0^0} = \frac{\delta_N}{\varkappa}\left(1 - \frac{4}{3}\delta_N^2 - \frac{16}{27}\delta_N^3\right)^{-\frac{1}{6}} \qquad 5.2.1$$

where $\delta_N$ is independent of $\varkappa$ and represents the deformation when the coupling between two shells with different N is neglected. It is related to the deformation parameter defined by Mottelson and Nilsson (MN 59) $\delta_{NN}$ approximately by the equation

$$\delta_N \simeq \sqrt{9 + 6\delta_{NN}} - 3 \qquad 5.2.2$$

In eq. 5.2.1 $\omega_0^0$ is the frequency of the isotropic harmonic oscillator field ( $\delta_N = 0$). It determines the energy scale and is chosen, according to Nilsson, by the condition (NIL 55)

$$\frac{5}{3}\langle r^2 \rangle = R_0^2 \qquad 5.2.3$$

where $R_0 = 1.2 \times A^{1/3}$ Fermis. That is to say, the mean value of the radius vector for all nucleons is set equal to the mean value of the square of the charge distribution. This gives

$$\hbar\omega_0 = 41 \times A^{-1/3} \text{ Mev} \qquad 5.2.4$$

We have obtained $\delta_N$ from the data collected in NM 59. We have included all the neutron levels belonging to the N = 4, 5, 6 shells (56) and all the proton levels belonging to the N = 3, 4, 5 shells (46) as given in Nilsson's paper (SGM 55). A few shifts of individual groups of levels were introduced, as indicated in Table VI and Table VII, following Nilsson and Prior (NP 61) and Szymanski and De's (SB 61). Note the somewhat larger shifts for proton levels, which are expected to take into account the corrections due to the Coulomb forces among protons. These level shifts were found to give a better representation of empirically found levels than the one obtained originally by Nilsson (SGM 55).

B. Choice of the Pairing Force Parameter

Using the levels determined as indicated in the previous paragraph, the eq. for an even system, viz.

$$\begin{cases} n = 2\sum_{\nu} V_{\nu}^2 \\ \frac{2}{G} = \sum_{\nu} \frac{1}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \end{cases} \qquad n \text{ even}$$

were solved simultaneously for all the nuclei listed in Tables X and XI. For odd n, of course, the equations have to include the blocking of a single particle orbital $\nu'$ (cf Chapter II)

$$\begin{cases} n-1 = 2\sum_{\nu \neq \nu'} V_{\nu}^2 \\ \frac{2}{G} = \sum_{\nu \neq \nu'} \frac{1}{\sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2}} \end{cases} \qquad n \text{ odd}$$

The parameter G for neutrons, for instance, was adjusted until the value $\Delta_n \simeq P_n^{exp}$ was obtained, where $P_n^{exp}$ is the experimentally observed even - odd mass difference (see Chapter II). We recall that theoretically the even - odd mass difference Pn is related to the binding energies E to a sufficient accuracy by the formula

$$P_n = \frac{1}{4}\left\{-S_n(Z, N+1) + 2S_n(Z, N) - S_n(Z, N-1)\right\}$$

where $S_n$ (Z, N) is the neutron separation energy. Similar formulae hold for protons.

The results of the calculations are given in Tables X and XI.

The parameters $G_n$ and $G_p$ depend, of course, on the cut-offs placed below and above the Fermi level because of re-normalizing effects (Chapter II).

For the number of levels chosen, we found that

$$G_n = \frac{18.5}{A} \text{ Mev} \qquad \delta_N \geq 0.28$$

$$G_n = \frac{19}{A} \text{ Mev} \qquad \delta_N < 0.28$$

and

$$G_p = \frac{27.2}{A} \text{ Mev}$$

give a reasonably good fit for the gap parameter $\Delta_e^n$ and $\Delta_e^p$ respectively. Note that the pairing force strength is a little higher for neutrons than that found by Nilsson and Prior (NP 61). This is due to the fact that only part of the $N = 6$ levels were taken into account (the higher levels in the shell are not given in Nilsson's paper), whereas Nilsson and Prior report that they have considered all the $N = 6$ levels. Also note that G is appreciably higher for protons, leading to quite larger energy gaps (of Table X).

## C. Results and Discussion

The reduction factors for neutrons and protons, and the total reduction factor $R = R_N'^2 \times R_Z'^2$, obtained with the BCS wave functions, using all the 56 neutron levels and 46 proton levels (formulae IV.1.10 and IV.1.21, Chapter IV) are given in Table XII, columns 4, 5 and 6 respectively. Column 7 lists the reduction factors obtained by Soloviev (S 61). One observes that, in general, there is a rather good agreement, in spite of the fact that our energy levels near the ground states differ appreciably from those calculated by Soloviev. It appears that there is a large discrepancy between the total reduction factors in the case of the decay of $_{64}Gd^{161}_{97} \rightarrow {}_{65}Tb^{161}_{96}$ (Table XII).

A possible explanation is that here the differences in the level structure caused by the blocking of the odd proton orbital, which are most marked near the ground state, or the Fermi level (cf Chapter II), are of importance. In fact, let us look at the levels given by Soloviev (S 61) for protons in the region of interest ($63 \leq Z \leq 70$) and the levels we have used here, corresponding to the deformation $\delta_N = 0.30$ (in Mottelson and Nilsson's paper, MN 59, they give $\delta_{MN} = 0.31$ for $_{65}Tb^{161}_{96}$; we have assumed through this work that there is no change in the deformations of the parent and daughter nuclei that can cause appreciable differences in the level structure).

The Fermi level for $_{64}Gd^{161}_{97}$ (Table X; zero re-adjusted to be the same as Soloviev's) is 0.0275, which is slightly above the $5/2 + [413]$ level. The blocked proton level in this transition is $7/2 - [523]$, which in our case lies well below the Fermi level. Soloviev does not give the

| Soloviev (levels in units of $\hbar\omega_0$) | | Ours (levels in units of $\hbar\omega_0$) | |
|---|---|---|---|
| Orbital | Energy | Orbital | Energy |
| 5/2 +[413] | 0 | 7/2 - [523] | -0.04 |
| 3/2 +[411] | 0.04 | 5/2 + [413] | 0 |
| 7/2 - [523] | 0.12 | 3/2 + [411] | 0.01 |
| 1/2 +[411] | 0.20 | 1/2 + [411] | 0.16 |
| 9/2 - [514] | 0.22 | 9/2 - [514] | 0.20 |
| 7/2 + [404] | 0.31 | 1/2 - [530] | 0.34 |
| 5/2 +[402] | 0.36 | 7/2 + [404] | 0.375 |
| 1/2 +[400] | 0.52 | | |

position of the Fermi level in this case, but it presumably lies below the 7/2 - [523] state. Note that the Fermi level, being very close to the 3/2 + [411] state, is well below Soloviev's 7/2 - [523] level. One would not expect that the re-normalization effect would alter very much the position of the Fermi level.

This would explain the large differences obtained in the reduction factors, because the proton reduction factor is proportional to $U_{\nu_\perp}^2$ ($\nu_\perp$ being the blocked state in the daughter), which in our case is small, because the level in question lies well below the Fermi surface.

In Table XIV, columns 7, 8 and 9, and Table XV, columns 6, 7 and 8, we give the various reduction factors for protons and neutrons obtained when the levels the occupation amplitudes of which are less than 0.01 were removed, in order to save computing time. A comparison with the

corresponding amplitudes given in Table XII shows that the difference is indeed insignificant. Columns 11, 12 and 13 of Table XIV and columns 10, 11 and 12 of Table XV give the reduction factors obtained with the projected wave functions.

The last two columns in Table XIV are the theoretical $\log_{ft}$ values when the BCS correction factors and the correction factors given by the projected wave functions have been made to the single particle estimates (columns 14, 15 and 6 respectively).

The last three columns of Table XV give $\log (ft_{exp} \times R)$, i.e, the logarithm of the experimental ft value times the correction factor: column 13 corresponds to our reduction factors, evaluated using the BCS wave functions, column 14 corresponds to Soloviev's reduction factors, and finally column 15 corresponds to the reduction factors obtained with the projected wave functions.

The first 6 rows of Table XV contain 3 pairs of decay cases in which the initial and the final single particle quantum numbers are the same.

Therefore, assuming that the average fields are not sensibly different for the nuclei involved in each pair, we ought to have nearly constant values in the last three columns for each case. A comparison of the numerical values given show that for the 1st pair and 3rd pair considered, the projected wave functions give values that differ slightly less than the corresponding values given by the BCS wave functions (our results). For the second pair, the opposite is the case. We would then conclude that the pairing correlations are more important in the 1st and 3rd pairs than they are in the 2nd pair considered.

Looking at Table XIV, at the decay of $Ag^{101} \longrightarrow Pd^{101}$, it is hard to see the actual effect of the pairing correlations, since this decay violates quite strongly an asymptotic selection rule ($|\Delta_z| = 3$). One should perhaps try to eliminate the influence of the average field, as it was attempted in Table IV.

The next two cases in Table XIV indicate that correlations introduced by the pairing forces have actually not even the importance that the BCS wave functions would lead one to believe. These are cases where some strong additional configuration mixing is clearly needed. The last case in Table XIV seems to indicate that the pairing forces account for most of the reduction of the single particle estimates.

Note that the BCS and projected reduction factors are practically the same, which indicates that components having slightly wrong number of particles have negligible effects in this case.

## 5. Conclusions

We have made an attempt to apply the parts of BCS wave functions containing the correct number of particles to some $\gamma$ and $\beta$ decay processes.

These components having $\pi$ as a good quantum number have been shown to have extremely large overlaps with the exact solutions of the pairing Hamiltonian (at least for states of seniority 0, 1 and 2). The techniques used here for computing with these wave functions are straight-forward, and it turns out that it is not much harder to obtain numerical results from them than is the case with the BCS wave functions. However, due to the very complicated structure of these wave functions, which somewhat resemble the wave functions used in the ordinary configuration mixing calculations of the shell model, one should perhaps use the BCS wave functions first in order to have an idea of the qualitative features of the problem. But in order to have a reliable quantitative estimate of the relative importance of the pairing correlations on the decay processes (or on the matrix elements of any physical operator) one should use the projected wave functions with techniques such as used here.

The projected wave functions were found, in all the cases studied, to have a less drastic reduction effect on the single particle matrix elements than the BCS wave functions, whenever the number of particles is small relatively to the number of pairing levels effectively considered; as the number of particles increases, with the same number of levels, the BCS and projected wave functions tend to give very nearly the same results.

The single particle Nilsson transitions considered were found to be rather insensitive to the detailed structure of the two types of wave functions used. Nevertheless, the β-decay reduction factors are not insensitive to the wave functions assumed. In the first place, the differences between BCS and projected wave functions commonly lie in the region 10% to 25% when the same level structure and coupling parameters are used. In the second place, the difference between different level schemes is illustrated by comparing our BCS calculations with those of Soloviev et al, since in neither of these cases is a projected wave function used; the differences are again of the same order normally 15% to 25%, but with some exceptionally sensitive cases such as $Gd^{161}$ discussed above. For most of the β-decay processes considered here, it was found that the pairing correlations do not seem to be sufficient to explain the reductions from single particle estimates, even taking into account the uncertainties associated with the lack of precise knowledge of the average field. We also found that, in general, our reduction factors, obtained with the BCS wave functions, and using slightly modified Nilsson's levels, do not differ too much from those quoted by Soloviev (S 61), taking full account of the contributions to the average field from the pairing force, which is most noticeable near the ground state.

We propose to make a further study of β-decay processes using the projected wave functions, and the improved level scheme proposed by Soloviev and collaborators.

APPENDIX I

## A. Method of Generating Functions

Let $|\psi_1(z)\rangle$ be a generating function for projected states. We shall restrict ourselves to states with zero and one quasi particles.

If the system we want to describe has k pairs, we write (cf Chapter III)

$$|\psi_1(z)\rangle = z^{k/2}\,|\varphi_1(k)\rangle + \sum_{n\neq k} z^{n/2}\,|\varphi_1(n)\rangle \qquad 1.(1)$$

where $|\varphi_1(k)\rangle$ is not normalized. If A is a single particle operator and if there is no change in the number of pairs, then obviously the transition amplitude between the states $|\varphi_1(k)\rangle$ and $\langle\varphi_2(k)|$ is proportional to just the coefficient $C_k$ of $z^k$ in the expansion of

$$\langle\psi_2(z)\,|A|\,\psi_1(z)\rangle \qquad 1.(2)$$

in powers of $z$. The transition amplitude is therefore equal to

$$\frac{C_k}{\sqrt{\langle\varphi_1(k)|\varphi_1(k)\rangle\,\langle\varphi_2(k)|\varphi_2(k)\rangle}} \qquad 1.(3)$$

where $\langle\varphi_1(k)|\varphi_1(k)\rangle$ is the coefficient of $z^k$ in the power expansion of $\langle\psi_1(z)|\psi_1(z)\rangle$ and $\langle\varphi_2(k)|\varphi_2(k)\rangle$ is the coefficient of $z^k$ in the expansion of $\langle\psi_2(z)|\psi_2(z)\rangle$.

Similarly, if the number of pairs changes, say from k to $k \pm 1$, it is evident that the matrix element is proportional to the coefficient of $z^{k\pm 1/2}$ in the expansion of the generating function (2) in powers of $z$.

For example, in discussing the matrix elements considered in Chapter IV, we need the normalization integrals; these we can obtain from the following expansions by the procedure outlined above:

$$\langle\psi(z)|\psi(z)\rangle = \langle 0| \prod_{\alpha u>0} \left(u_\alpha + \sqrt{z}\, v_\alpha\, b_{\alpha,-u}\, b_{\alpha u}\right) \times$$

$$\times \prod_{\alpha u>0} \left(u_\alpha + \sqrt{z}\, v_\alpha\, b^+_{\alpha u}\, b^+_{\alpha,-u}\right)|0\rangle = \prod_\alpha \left(u_\alpha^2 + z\, v_\alpha^2\right) = \prod u_\alpha^2 \left[1 + \right.$$

$$+ z \sum_\alpha \frac{v_\alpha^2}{u_\alpha^2} + \frac{1}{2!} z^2 \sum_{\alpha,\beta} \frac{v_\alpha^2}{u_\alpha^2} \frac{v_\beta^2}{u_\beta^2} + \cdots + \frac{1}{k!} z^k \sum_{\alpha,\beta,\gamma\ldots} \frac{v_\alpha^2}{u_\alpha^2} \frac{v_\beta^2}{u_\beta^2} \cdots \frac{v_\gamma^2}{u_\gamma^2} \left. + \cdots \right]$$

and a similar expression for the one-quasi particle state, provided we block the state occupied by the odd particle. So the normalizations of the state having k pairs is

$$\left(\prod_\alpha u_\alpha\right) \left[\frac{1}{k!} \sum_{\alpha,\beta\ldots\gamma} \frac{v_\alpha^2}{u_\alpha^2} \frac{v_\beta^2}{u_\beta^2} \cdots \frac{v_\gamma^2}{u_\gamma^2}\right]^{1/2} \qquad 1.(5)$$

and of the state having k pairs and an odd particle in the orbital $(\alpha_1, u_1)$

$$\left(\prod_{\nu \neq \alpha_1} u_\alpha\right) \left[\frac{1}{k!} \sum_{\substack{\alpha,\beta,\ldots\gamma \\ \neq \alpha_1}} \frac{v_\alpha^2}{u_\alpha^2} \frac{v_\beta^2}{u_\beta^2} \cdots \frac{v_\gamma^2}{u_\gamma^2}\right]^{1/2} \qquad 1.(6)$$

It is clear that we are assuming that no single particle label $\alpha, \beta \ldots$ appears twice in the same expression. It is also clear that these expressions reduce to the normalizations found in Chapter I where

$$v_\alpha^2 = v_\beta^2 = \cdots = v_\gamma^2 = v^2 = \frac{n}{2\Omega}$$

i.e., in the case of complete degeneracy (of Chapter I).

In this connection it may be of some interest to note the following: consider the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \qquad 1.(7)$$

with $n$ roots $b_{\alpha_1}, b_{\alpha_2} \ldots b_{\alpha_n}$, and construct $f(x)$ $\quad i = 1, 2, \ldots n$

$$g(x) = \sum_{\alpha_i} \frac{f(x)}{x - b_{\alpha_i}} \qquad 1.(8)$$

Then it is easily seen that $g(x) \equiv f'(x)$. By explicitly writing down the expression for $g(x)$ making use of 1.8 and equating it identically to $f'(x)$ it follows at once that

$$S_k + a_1 S_{k-1} + a_2 S_{k-2} + \cdots + k a_k = 0 \qquad 1.(9)$$

$$k = 1, 2, 3 \ldots$$

where by definition

$$S_j = \sum_{\alpha_j} b_{\alpha_j}^{\,j} \qquad j = 1, 2 \ldots n \qquad 1.(10)$$

The evaluation of the coefficients $a_k$ has interest because they are related to the roots of the polynomial 1 (7) by the well-known relations

$$a_i = (-1)^i \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_i} b_{\alpha_1} b_{\alpha_2} \cdots b_{\alpha_i} \qquad 1.(11)$$

These sums have precisely the same structure as our normalization integrals and other matrix elements and are of help in writing a program for an electronic computer. Thus we can first define, for a given integer k,

$$C_i = d_{\alpha_i} - b_{\alpha_{k+2}}\, d_{\alpha_{i-1}} \qquad \begin{matrix} i = 1, 2, \ldots k \\ k = 1, 2, 3 \ldots \end{matrix} \qquad 1.(12)$$

and store initially $d_{\alpha_0} = 1$, $d_{\alpha_i} = 0$ $(i \neq 0)$, $d_{\alpha_{-1}} = 0$, $C_{\alpha_1} = 0$, $C_{\alpha_0} = 1$ $(i \neq 0)$. We then set $d_{\alpha_i} = C_i$ and iterate. The coefficients $a_i$ can then be obtained from 1.12:

$$a_i = d_{\alpha_i} - (b_{\alpha_1} + b_{\alpha_2})\, d_{\alpha_{i-1}} + b_{\alpha_1} b_{\alpha_2}\, d_{\alpha_{i-2}} \qquad 1.(13)$$

Thus all the sums we require can be easily obtained with sufficient accuracy.

## B. Reduction Factors for β and γ Transition Rates

In Chapter IV, we found that the matrix element of a β decay operator $O = \sum \zeta -$, when the number of pairs remain unchanged, depended on the matrices

$$S_p^{\mu k}(z) = \langle 0 | \prod_{\alpha u \neq \alpha_2 u_2} b_{\alpha_2 u_2} \left( A_\alpha + \sqrt{z}\, B_\alpha b_{\alpha -u} b_{\alpha u} \right) b^+_{\mu k \, \alpha u > 0} \left( u_\alpha + \sqrt{z}\, V_\alpha b^+_{\alpha u} b^+_{\alpha -u} \right) | 0 \rangle \qquad 2.(1)$$

and

$$S_n^{\mu' k'}(z) = \langle 0 | \prod_{\alpha u > 0} \left( A_\alpha + \sqrt{z}\, B_\alpha a_{\alpha -u} a_{\alpha u} \right) a_{\mu' k'} a^+_{\alpha_1 u_1} \prod_{\neq \alpha_1 u_1} \left( u_\alpha + \sqrt{z}\, V_\alpha a^+_{\alpha u} a^+_{\alpha -u} \right) | 0 \rangle \qquad 2.(2)$$

These can easily be evaluated by a systematic application of the anti-commutation rules for the operators a and b, and orthogonality conditions. For instance, 1 can be written

$$S_{pn}^{\mu k}(z) = \delta_{\mu u_2}\, \delta_{k u_2} \langle 0 | \prod_{\alpha u \neq} \left( A_\alpha + \sqrt{z}\, B_\alpha b_{\alpha -u} b_{\alpha u} \right) \times \left( u_{\alpha_2} + \sqrt{z}\, V_{\alpha_2} b^+_{\alpha_2 u_2} b^+_{\alpha_2 -u_2} \right) \left( u_{\alpha_1} + \sqrt{z}\, V_{\alpha_1} b^+_{\alpha_1 u_1} b^+_{\alpha_1 -u_1} \right) \times \prod_{\substack{\alpha u \neq \alpha_1 u_1 \\ \neq \alpha_2 u_2}} \left( u_\alpha + \sqrt{z}\, V_\alpha b^+_{\alpha u} b^+_{\alpha -u} \right) | 0 \rangle - \langle 0 | \prod_{\alpha u \neq \alpha_2 u_2} \left( A_\alpha + \sqrt{z}\, B_\alpha b_{\alpha -u} b_{\alpha u} \right),$$

$$\times b^+_{\mu k} b_{\alpha_2 u_2} \left( \sqrt{z}\, V_{\alpha_2} b^+_{\alpha_2 u_2} b^+_{\alpha_2 -u_2} \right) \left( u_{\alpha_1} + \sqrt{z}\, V_{\alpha_1} b^+_{\alpha_1 u_1} b^+_{\alpha_1 -u_1} \right) \times \prod_{\neq \alpha_1 u_1, \alpha_2 u_2} \left( u_\alpha + \sqrt{z}\, V_\alpha b^+_{\alpha u} b^+_{\alpha -u} \right) | 0 \rangle$$

The orthogonality between states having different numbers of particles reduces this simply to

$$S_p^{\mu k}(z) = \delta_{\mu \alpha_2}\, \delta_{k u_2}\, u_{\alpha_2} \prod_{\alpha u \neq \alpha_2 u_2} \left( A_\alpha u_\alpha + z B_\alpha V_\alpha \right) \qquad 2.(3)$$

Exactly the same steps lead us to the neutron factor

$$S_n^{\mu' k'}(z) = \delta_{\mu' \alpha_1}\, \delta_{k' u_1}\, A_{\alpha_1} \prod_{\alpha u \neq \alpha_1 u_1} \left( A_\alpha u_\alpha + z B_\alpha V_\alpha \right) \qquad 2.(4)$$

We must then consider the generating functions for the reduction factors

$$R_p(z) = \left[ u_{\alpha_2} \prod_{\substack{\alpha u \neq \\ \alpha_2 u_2}} \left( A_\alpha u_\alpha + z B_\alpha V_\alpha \right) \right]_p \qquad 2.(5)$$

and $$R_n(z) = \left[ A_{\alpha_1} \prod_{\alpha_m \neq \alpha_1, m_1} \left( A_\alpha u_\alpha + z B_\alpha V_\alpha \right) \right]_n \qquad 2.(6)$$

From the considerations at the beginning of this appendix, we expand these in powers of $z$ and pick up the coefficient of $z^p$ (for p proton pairs) and $z^n$ (for n proton pairs). We then get

$$R_p' = u_{\alpha_2} \left( \prod_{\alpha \neq \alpha_2} A_\alpha u_\alpha \right) \frac{1}{p!} \sum_{\neq \alpha_2} \left( \frac{B_\alpha}{A_\alpha} \frac{V_\alpha}{u_\alpha} \right) \cdots \left( \frac{B_\nu}{A_\nu} \frac{V_\nu}{u_\nu} \right) \qquad 2.(7)$$

and

$$R_n' = A_{\alpha_1} \left( \prod_{\neq \alpha_1} A_\alpha u_\alpha \right) \frac{1}{n!} \sum_{\neq \alpha_1} \left( \frac{B_\alpha}{A_\alpha} \frac{V_\alpha}{u_\alpha} \right) \cdots \left( \frac{B_\nu}{A_\nu} \frac{V_\nu}{u_\nu} \right) \qquad 2.(8)$$

Using the normalizations given by 1.5 and 1.6 we finally obtain

$$R_p = \frac{\sum_{\neq \alpha_2} \left( \frac{B_\alpha}{A_\alpha} \frac{V_\alpha}{u_\alpha} \right) \cdots \left( \frac{B_\nu}{A_\nu} \frac{V_\nu}{u_\nu} \right)}{\left[ \sum_{\alpha\beta\cdots\gamma} \frac{V_\alpha^2}{u_\alpha^2} \frac{V_\beta^2}{u_\beta^2} \cdots \frac{V_\delta^2}{u_\delta^2} \right]^{1/2} \left[ \sum_{\neq \alpha_2} \frac{B_\alpha^2}{A_\alpha^2} \cdots \frac{B_\delta^2}{A_\delta^2} \right]^{1/2}} \qquad 2.(9)$$

where the terms in the sums have p factors if there are p proton pairs. The coefficients $u$, $V$ refer to the parent nucleus and A, B to the daughter nucleus, in which the single proton orbital $|\alpha_2 m_2\rangle$ is blocked.

Similarly the neutron part is

$$R_n = \frac{\sum_{\neq \alpha_1} \left( \frac{B_\alpha}{A_\alpha} \frac{V_\alpha}{u_\alpha} \right) \cdots \left( \frac{B_\gamma}{A_\gamma} \frac{V_\gamma}{u_\gamma} \right)}{\left[ \sum_{\alpha\beta\cdots\gamma} \frac{B_\alpha^2}{A_\alpha^2} \frac{B_\beta^2}{A_\beta^2} \cdots \frac{B_\gamma^2}{A_\gamma^2} \right]^{1/2} \left[ \sum_{\neq \alpha_1} \frac{V_\alpha^2}{u_\alpha^2} \cdots \frac{V_\gamma^2}{u_\gamma^2} \right]^{1/2}} \qquad 2.(10)$$

where the symbols have the same meaning as above, and there are n factors in each product, if there are n neutron pairs present. The neutron orbital $|\alpha_1 u_1\rangle$ is blocked, of course.

Now suppose that there is a change in the number of pairs, say from 2n neutrons and 2p+1 protons to 2n-1 neutrons and 2p+2 protons. Let $|\nu_1 -u_1\rangle$ be the odd proton orbital in the parent nucleus and $|\nu_2 -u_2\rangle$ the odd neutron orbital in the daughter nucleus. Then the initial state of the decay can be obtained from

$$|\psi_i\ (2n, 2p+1)\rangle = \Big[\prod_{\nu u>0} \left(u_\nu + \sqrt{z}\ V_\nu\ a^+_{\nu u}\ a^+_{\nu -u}\right)|0\rangle\Big]_n$$
$$\Big[\prod_{\neq \nu_1 u_1} \left(u_\nu + \sqrt{z}\ V_\nu\ b^+_{\nu u}\ b^+_{\nu -u}\right) b^+_{\nu_1 -u_1}\ |0\rangle\Big]_p$$

The final state is given by

$$\langle \psi_f\ (2n-1, 2p+1)| = \Big[\langle 0| \prod_{\neq \nu_2 u_2} a_{\nu_2 -u_2}\ \left(A_\nu + \sqrt{z}\ B_\nu\ a_{\nu -u}\ a_{\nu u}\right)\Big]_n \times$$
$$\times\ \Big[\langle 0| \prod_{\nu u>0} \left(A_\nu + \sqrt{z}\ B_\nu\ b_{\nu -u}\ b_{\nu u}\right)\Big]_p$$

assuming that the proton system is in its lowest state.

The generating function for the matrix element is then

$$M^K_\mu\ (\nu_1 \to \nu_2\ ;\ z) = \sum_{\substack{\mu k \\ \mu' k'}} \langle \mu k |\ T^K_\mu\ |\mu' k'\rangle \Big\{\langle 0| \prod_{\neq \nu_2 u_2} a_{\nu_2 u_2}$$
$$\left(A_\nu + \sqrt{z}\ B_\nu\ a_{\nu -u}\ a_{\nu u}\right)\ a_{\mu' k'} \prod_{\nu u>0} \left(u_\nu + \sqrt{z}\ V_\nu\ a^+_{\nu u}\ a^+_{\nu -u}\right)|0\rangle$$
$$\times \Big\{\langle 0| \left(A_\nu + \sqrt{z}\ B_\nu\ b_{\nu -u}\ b_{\nu u}\right) \prod_{\nu u>0} b^+_{\mu k} \prod_{\neq \nu_1 u_1} \left(u_\nu + \sqrt{z}\ V_\nu\ b^+_{\nu u_1}\ b^+_{\nu -u}\right.$$
$$\left. \times b^+_{\nu_1 u_1}\ |0\rangle\Big\}_p$$

The neutron part can be written as

$$S_n(z) = \sqrt{z}\ V_{\nu_2}\ \langle 0| \prod_{\neq \nu_2 u_2} \left(A_\nu + \sqrt{z}\ B_\nu\ a_{\nu -u}\ a_{\nu u}\right) \times$$
$$\times \prod_{\neq \nu_2 u_2} \left(u_\nu + \sqrt{z}\ V_\nu\ a^+_{\nu u}\ a^+_{\nu -u}\right) a_{\nu_2 -u_2}\ a_{\mu' k'}\ a^+_{\nu_2 u_2}\ a^+_{\nu_2 -u_2}\ |0\rangle$$

This is simply

$$S_n(z) = \sqrt{z}\ \Big\{ V_{\nu_2}\ \delta_{\mu' \nu_2}\ \delta_{k' u_2} \prod_{\substack{\nu u \neq \\ \nu_2 u_2}} \left(A_\nu\ u_\nu + z\ B_\nu\ V_\nu\right)\Big\}_n$$

Similarly the proton part is

$$S_p(z) = \sqrt{z}\left\{ B_{\nu_1}\ \delta_{\mu\nu_1}\ \delta_{\omega_1 k} \prod_{\neq \nu_1\omega_1} (A_\nu U_\nu + z\, B_\nu V_\nu)\right\}_p$$

The BCS reduction factor ($z \equiv 1$) are then

$$S_n(z=1) = V_{\nu_2} \prod_{\neq \nu_2\omega_2} (A_\nu U_\nu + B_\nu V_\nu)\ \delta_{\mu'\nu_2}\ \delta_{k'\omega_2}$$

$$S_p(z=1) = B_{\nu_1} \prod_{\neq \nu_1\omega_1} (A_\nu U_\nu + B_\nu V_\nu)\ \delta_{\mu\nu_1}\ \delta_{k\omega_1}$$

and the matrix element is

$$M^K_{\mu}(\nu_1 \to \nu_2;\ z=1) = \langle \nu_1\omega_1 |\, T^K_\mu\, | \nu_2\omega_2\rangle \left\{ V_{\nu_2} \prod_{\neq \nu_2\omega_2} (A_\nu U_\nu + B_\nu V_\nu)\right\}_n \times \left\{ B_{\nu_1} \prod_{\neq \nu_1\omega_1} (A_\nu U_\nu + V_\nu B_\nu)\right\}_p = (-1)^\tau \langle \nu_2'-\omega_2 |\, T^K_\mu\, | \nu_1'-\omega_1\rangle \times$$

$$\times \left\{ V_{\nu_2} \prod_{\neq \nu_2\omega_2} (A_\nu U_\nu + B_\nu V_\nu)\right\}_n \times \left\{ B_{\nu_1} \prod_{\neq \nu_1\omega_1} (A_\nu U_\nu + V_\nu B_\nu)\right\}_p$$

where $\tau$ is even or odd depending on whether the operator $T^K_\mu$ is even or odd under time reversal.

Therefore, the single particle reduced probabilities are decreased by a factor

$$R_n = \left\{ V_{\nu_2} \prod_{\neq \nu_2\omega_2} (A_\nu U_\nu + B_\nu V_\nu)\right\}^2_n$$

and

$$R_p = \left\{ B_{\nu_1} \prod_{\neq \nu_1\omega_1} (A_\nu U_\nu + B_\nu V_\nu)\right\}^2_p$$

The projected reduction factor for the neutron can be obtained from the coefficients of $z^{n-1}$, i.e.,

$$\prod_{\nu\neq\nu_2} A_\nu U_\nu \frac{1}{(n-1)!} \sum_{\neq\nu_2\omega_2} \left(\frac{B_\alpha}{A_\alpha}\frac{V_\alpha}{U_\alpha}\right) \cdots \left(\frac{B_\gamma}{A_\gamma}\frac{V_\gamma}{U_\gamma}\right)$$

these being n - 1 factors in each product in the sum.

The normalization integrals are, for parent and daughter respectively,

$$\sqrt{\left(\prod_\nu U_\nu^2\right) \frac{1}{n!} \sum_{\alpha\beta\cdots\gamma} \frac{V_\alpha^2}{U_\alpha^2} \cdots \frac{V_\gamma^2}{U_\gamma^2}}$$

and

$$\sqrt{\left(\prod_{\nu\neq\nu_2} A_\nu^2\right) \frac{1}{(n-1)!} \sum_{\neq\nu_2\omega_2} \frac{B_\alpha^2}{A_\alpha^2} \cdots \frac{B_\gamma^2}{A_\gamma^2}}$$

The neutron reduction factor is therefore

$$R_n = \left\{ \frac{V_{\nu_2}}{U_{\nu_2}} \sqrt{n} \; \frac{\sum_{\neq \nu_2} C_\alpha C_\beta \cdots C_\gamma}{\left[\sum_{\alpha,\beta\cdots\gamma} a_\alpha a_\beta \cdots a_\gamma\right]^{1/2} \left[\sum_{\neq \nu_2} b_\alpha b_\beta \cdots b_\gamma\right]^{1/2}} \right\}^2$$

Exactly the same procedure yields for protons

$$R_p = \left\{ \frac{B_{\nu_1}}{A_{\nu_1}} \sqrt{p+1} \; \frac{\sum_{\neq \nu_1} C_\alpha C_\beta \cdots C_\gamma}{\left[\sum_{\neq \nu_1} a_\alpha a_\beta \cdots a_\gamma\right]^{1/2} \left[\sum_{\alpha\beta\cdots\gamma} b_\alpha b_\beta \cdots b_\gamma\right]^{1/2}} \right\}^2$$

We have set

$$\frac{V_\alpha^2}{U_\alpha^2} = a_\alpha \qquad \frac{B_\alpha^2}{A_\alpha^2} = b_\alpha \qquad \frac{V_\alpha}{U_\alpha} \frac{B_\alpha}{A_\alpha} = C_\alpha$$

APPENDIX II

## Single Particle Matrix Elements for Allowed β-Transitions.

The β decay transition rates are usually measured by their comparative half-lives ft, which, in the case of allowed transitions ($4.5 \lesssim \log ft < 7.5$) also provide a direct estimate of the nuclear matrix elements.

Using the natural units system, in which we set $\hbar = m = c = 1$ and in which, therefore, transition probabilities are measured in units of $\frac{mc^2}{\hbar} = 7.7 \times 10^{20}\ sec^{-1}$, we have for allowed transitions

$$ft = \frac{B}{(1-x)\,D_F(0) + x\,D_{GT}(0)}$$

where $D_F(0)$ and $D_{GT}(0)$ are the reduced transition probabilities for the Fermi and Gamow-Teller operators respectively, viz.

$$D_F(0) = \sum_{M_f} \sum_{k=1}^{A} |\langle \psi_f | \tau_k^{\pm} | \psi_i \rangle|^2 \qquad 2a$$

and

$$D_{GT}(0) = \sum_{M_f} \sum_{k=1}^{A} |\langle \psi_f | \vec{\sigma}_k \tau_k^{\pm} | \psi_i \rangle|^2 \qquad 2b$$

Here the $M_f$ is the component of the total angular momentum of the daughter nucleus along the space fixed z-axis, $\vec{\sigma}_k$ and $\tau_k^{\pm}$ is respectively the Pauli spin operator and the i-spin operator for the $k$th nucleon.

The wave functions $\psi_f$ and $\psi_i$ refer to the final and initial nuclear states.

In the eq. 1, B and $\chi$ are universal constants which are found to be (R 60)

$$B = 2787 \pm 70 \text{ sec}^{-1}$$

$$\chi = 0.560 \pm 0.012$$

B is related to the $\beta$-decay constants by the defining equation

$$B = \frac{2\pi^2 f_n^2}{g^2}$$

where $g\sqrt{1-\chi}$ and $g\sqrt{\chi}$ are respectively the Fermi and Gamow-Teller coupling constants.

In the strong coupling limit for the nuclear wave functions, the collective and intrinsic parts separate out and the reduced transition probabilities can be evaluated very simply. For odd-A nuclei the situation is further simplified because it is usually the odd nucleon that undergoes decay.

Thus if we denote the initial total angular momentum by I, its z-component in the space fixed system by M and along the body fixed z axis by K, and the similar quantities in the daughter by I', K', M', we can write

$$|\psi_i\rangle \sim |IKM\rangle \quad |\chi^N_\Omega\rangle$$

$$\langle\psi_f| \sim \langle I'K'M'| \quad \langle\chi^{N'}_{\Omega'}|$$

Here N and $\Omega$ are respectively the total oscillator quantum number and the body fixed z-component of the intrinsic angular momentum, and N' and $\Omega'$ are the corresponding quantities for the daughter nucleus.

Then for allowed transitions we can deduce that (SCH 55)

$$D_{GT}(0) = \left| (IK\,1K'-K \mid I'K') + \beta_1 (-1)^{I'+K'} (IK\,1-K'-K \mid I'-K') \right|^2 \gamma_1^2$$

where, by definition

$$\gamma_1 = \delta_{NN'} \sum_{\substack{\ell\ell' \\ \Sigma\Sigma'}} \sum_{\Lambda\Lambda'} \delta_{\ell\ell'}\, \delta_{\Lambda\Lambda'}\, a'_{\ell'\Lambda'}\, a_{\ell\Lambda} \left[ \delta_{\Sigma\Sigma'} (-1)^{\Sigma - \frac{1}{2}} + \right.$$

$$\left. + \sqrt{2}\, \delta_{-\frac{1}{2}\Sigma'}\, \delta_{\frac{1}{2}\Sigma} - \sqrt{2}\, \delta_{\frac{1}{2}\Sigma'}\, \delta_{-\frac{1}{2}\Sigma} \right]$$

and

$$\beta_{1,} = (-1)^{K'+\frac{1}{2}+l'} \frac{\sqrt{2}}{\gamma_1} \delta_{NN'} \sum_{\ell} a'_{\ell 0}\, a_{\ell 0}\, \delta_{\Omega \frac{1}{2}}\, \delta_{\Omega' \frac{1}{2}}$$

Primed quantities refer to the daughter nucleus.

In the case of axial symmetry, we have

$$\Omega = K$$
$$\Omega' = K'$$

The wave functions $|\chi^N_\Omega\rangle$ and $\langle \chi^{N'}_{\Omega'}|$ have been expanded in terms of the eigenfunctions of anisotropic three-dimensional oscillator well (SGN 55), e.g.,

$$|\chi^N_\Omega\rangle = \sum_{\ell} \left\{ a_{\ell\Omega-\frac{1}{2}} |N\ell\, \Omega-\tfrac{1}{2}\, \tfrac{1}{2}\rangle + a_{\ell\Omega+\frac{1}{2}} |N\ell\, \Omega+\tfrac{1}{2}\, -\tfrac{1}{2}\rangle \right\}$$

where

$$\sum_{\ell\Lambda} a^2_{\ell\Lambda} = 1$$

and $\Lambda + \Sigma = \Omega$, $\Lambda$ being the body fixed z-component of the orbital angular momentum and $\Sigma$ the corresponding component for the spin.

For transitions involving $\Omega \neq 1/2$ and (or) $\Omega' \neq 1/2$, this further simplifies to

$$\mathcal{D}_{GT}(0) = |(I K\, 1\, K'-K | I' K')|^2\, \gamma_1^2$$

and the ft value for a pure Gamow-Teller transition, for instance, is

$$(ft)_{s.p.} = \frac{K}{\pi} \left[ \frac{1}{|(I K\, 1\, K'-K | I' K')|^2} \frac{1}{\gamma_1^2} \right]$$

The entire dependence on the details of the single particle orbitals for the initial and final nuclei is contained in $\gamma_1^2$. In these circumstances, if we take the ratio of two ft values for transitions

occurring between states which can be characterized by the same set of single particle quantum numbers, the deviations from unity would give a measure of particle correlations, which are not taken in account by this simple model. This holds, of course, if we make the reasonable assumption that the $f_i^2$'s are essentially the same for both cases, which is obviously true if the deformations of the average nuclear field are nearly the same.

The rules for associating the description of an orbital assymptotic quantum numbers $[\Lambda' n_z \Lambda\Omega]$ with the orbital number in the Nilsson's scheme are given in the paper referred above (NIM 55).

TABLE I

Fermi Levels $\lambda$, Energy Gaps $\Delta$ and Occupation Numbers $v^2$ for the Pb Isotopes Considered, Using $G = \frac{30}{A}$ Mev and Single Particle Energies (in Mev)

$p_{1/2} = 0$, $f_{5/2} = 0.57$, $p_{3/2} = 0.90$, $i_{13/2} = 1.634$, $f_{7/2} = 2.35$

| A (Z = 82) | $\lambda$ (Mev) | $\Delta$ (Mev) | $v^2\ i_{13/2}$ (initial | $v^2\ f_{5/2}$ (final) | $v^2\ p_{1/2}$ | $v^2\ p_{3/2}$ | $v^2\ f_{7/2}$ |
|---|---|---|---|---|---|---|---|
| 205 | 0.072 | 0.561 | 0.02943 | 0.16807 | 0.56365 | 0.08606 | 0.01450 |
| 203 | 0.278 | 0.730 | 0.05974 | 0.3143 | 0.6779 | 0.1757 | 0.02841 |
| 201 | 0.477 | 0.856 | 0.09805 | 0.4460 | 0.7434 | 0.2705 | 0.04524 |
| 199 | 0.676 | 0.954 | 0.1457 | 0.5552 | 0.7890 | 0.3857 | 0.06559 |
| 197 | 0.875 | 1.033 | 0.2039 | 0.6416 | 0.8232 | 0.4879 | 0.09045 |

TABLE II

Fermi Levels $\lambda$, Energy Gaps $\Delta$ and Occupation Numbers $v^2$ for the Sn Isotopes Considered, Using $G = \frac{19}{A}$ Mev and Single Particle Energies (in Mev)

$\varepsilon_{1/2} = 0.95$, $\varepsilon_{3/2} = 1.23$, $\varepsilon_{5/2} = 0$, $\varepsilon_{7/2} = -0.16$, $\varepsilon_{11/2} = 1.41$

| A (Z = 50) | $\lambda$ (Mev) | $\Delta$ (Mev) | $v^2_{11/2}$ (initial) | $v^2_{3/2}$ (final) | $v^2_{5/2}$ | $v^2_{7/2}$ | $v^2_{1/2}$ |
|---|---|---|---|---|---|---|---|
| 117 | 0.84 | 1.10 | 0.2699 | 0.3329 | 0.0035 | 0.0363 | 0.4502 |
| 119 | 1.07 | 1.07 | 0.3486 | 0.4260 | 0.0535 | 0.0772 | 0.5557 |

TABLE III

Fermi Levels λ, Energy Gaps Δ and Occupation Numbers for the N = 50 Nuclei Considered, Using G = 0.291 Mev and Single Particle Levels (in Mev)

$f_{5/2} = 0$ $p_{3/2} = 0.6$ $p_{1/2} = 1.8$ $g_{9/2} = 3.4$

| A (N = 50) | λ (Mev) | Δ (Mev) | $v^2\ g_{9/2}$ | $v^2\ p_{1/2}$ | $v^2\ p_{3/2}$ | $v^2\ f_{5/2}$ |
|---|---|---|---|---|---|---|
| $_{39}Y^{89}$ | 1.83 | 0.83 | 0.05797 | 0.5180 | 0.9145 | 0.9554 |
| $_{41}Nb^{91}$ | 2.59 | 0.93 | 0.1716 | 0.8257 | 0.9529 | 0.9706 |
| $_{43}Tc^{93}$ | 3.08 | 1.04 | 0.3529 | 0.8880 | 0.9611 | 0.9737 |

TABLE IV

Reduction Factors for M4 Transitions in Pb Isotopes Using BCS Wave Functions ($R_{KS}$) and Projected Wave Functions ($R_{projected}$) with Pairing Force Parameter and Single Particle Levels Given in Table I

| A | Transition Energy (Mev) | Level Change | $R_{(KS)}$ | $R_{(projected)}$ | $P_{exp}$ $sec^{-1}$ [a] | $P_{s.p.}$ [a] | $F_o$ | $F_1$ ($a_o = 1.2$) | $F_2$ ($a_o = 1.2$) |
|---|---|---|---|---|---|---|---|---|---|
| 207 | 1.064 | $i_{13/2} \rightarrow f_{5/2}$ | 1.0000 | 1.0000 | $7.72 \times 10^{-1}$ | 1.96 | $3.94 \times 10^{-1}$ | 1.00 | 1.000 |
| 205 | | $i_{13/2} \rightarrow f_{5/2}$ | 0.9390 | 0.9681 | | | | | |
| 203 | 0.825 | $i_{13/2} \rightarrow f_{5/2}$ | 0.8834 | 0.9042 | $8.76 \times 10^{-2}$ | $1.92 \times 10^{-1}$ | $4.56 \times 10^{-1}$ | 1.16 | 1.13 |
| 201 | 0.629 | $i_{13/2} \rightarrow f_{5/2}$ | 0.8390 | 0.8501 | $6.3 \times 10^{-3}$ | $1.63 \times 10^{-2}$ | $3.87 \times 10^{-1}$ | 0.982 | 0.953 |
| 199 | 0.426 | $i_{13/2} \rightarrow f_{5/2}$ | 0.8115 | 0.8157 | $1.91 \times 10^{-4}$ | $4.61 \times 10^{-4}$ | $4.14 \times 10^{-1}$ | 1.05 | 1.00 |
| 197 | 0.235 | $i_{13/2} \rightarrow f_{5/2}$ | 0.8026 | 0.8029 | $9.12 \times 10^{-7}$ | $2.14 \times 10^{-6}$ | $4.26 \times 10^{-1}$ | 1.08 | 1.03 |

$F_1$ Moszkowski estimate

$F_2$ Estimate using an isotropic harmonic oscillator wave functions

[a] After BA 57

TABLE V

Reduction Factors for E4 Transitions Using BCS Wave Functions ($R_{BCS}$) and Projected Wave Functions ($R_{projected}$) with Pairing Force Parameter and Single Particle Levels Given

| A | Transition Energy (Mev) | Level Change | $R_{(BCS)}$ | $R_{(projected)}$ | $F_{exp}$ [a] ($a_0 = 1.2$) | $F_{exp}$ [a] ($a_0 = 1.1$) |
|---|---|---|---|---|---|---|
| $_{50}Sn^{117}$ | 0.159 E4 | $h_{11/2} \rightarrow d_{3/2}$ | 0.994 | 0.995 | 0.41 | 0.66 |
| $_{50}Sn^{119}$ | 0.065 E4 | $h_{11/2} \rightarrow d_{3/2}$ | 0.993 | 0.9938 | 0.57 | 0.95 |
| $_{39}Y^{89}_{50}$ | 0.913 E4 | $g_{9/2} \rightarrow p_{1/2}$ | 0.68 | 0.857 | 0.26 | 0.43 |
| $_{41}Nb^{91}_{50}$ | 0.105 E4 | $g_{9/2} \rightarrow p_{1/2}$ | 0.48 | 0.579 | 0.20 | 0.33 |
| $_{43}Tc^{92}_{50}$ | 0.390 E4 | $p_{1/2} \rightarrow g_{9/2}$ | 0.65 | 0.710 | 0.35 | 0.59 |

[a] After KS 61

TABLE VI

Shifts of Neutron Levels in Units of $\hbar\omega_c$
Relative to those Obtained by Nilsson (1955)

| Shell | $\kappa$ | $\mu$ | Shifts |
|---|---|---|---|
| N = 4 | 0.05 | 0.45 | 0 |
| N = 5 | 0.05 | 0.45 | 0 |
| N = 6 $\pm i_{13/2}$ | 0.05 | 0.45 | + 0.15 |
| N = 6 $-i_{13/2}$ | 0.05 | 0.45 | 0 |

TABLE VII

Shifts of Proton Levels in Units of $\hbar\omega_c$
Relative to those Obtained by Nilsson (1955)
(The N = 4 Shifts Relative to those
Given by Mottelson and Nilsson (1959))

| Shell | $\kappa$ | $\mu$ | Shifts |
|---|---|---|---|
| N = 3 | 0.05 | 0.35 | - 0.23 |
| N = 4 | 0.05 | 0.45 | 0 |
| N = 5 $\pm h_{11/2}$ | 0.05 | 0.45 | + 0.10 |
| N = 5 $-h_{11/2}$ | 0.05 | 0.45 | - 0.20 |
| N = 6 | 0.05 | 0.45 | + 00 |

TABLE VIII

Allowed Transitions Considered (After NP 59 and Nuclear Data Sheets)

| Parent Nucleus | Daughter Nucleus | Excitation Energy of the Daughter | Type of Transition | Orbit Assignment | | Classification | $\log_{10} ft$ |
|---|---|---|---|---|---|---|---|
| | | | | Parent | Daughter | | |
| $_{72}Hf^{181}_{109}$ | $_{73}Ta^{181}_{108}$ | 958 | $\beta^-$ | [510 1/2-] | [541 1/2-] | ah | $\sim 6.5$ |
| $_{74}W^{181}_{107}$ | $_{73}Ta^{181}_{108}$ | 0 | ec | [624 9/2+] | [404 7/2+] | ah | $\sim 6.6$ |
| $_{70}Yb^{177}_{107}$ | $_{71}Lu^{177}_{106}$ | 0 | $\beta^-$ | [624 9/2+] | [404 7/2+] | ah | 6.2 |
| $_{68}Er^{171}_{103}$ | $_{69}Tm^{171}_{102}$ | 425 | $\beta^-$ | [512 5/2-] | [523 7/2-] | ah | 6.3 |
| $_{64}Gd^{159}_{95}$ | $_{65}Tb^{159}_{94}$ | 364 | $\beta^-$ | [521 3/2-] | [532 5/2-] | ah | 6.7 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | 700 | $\beta^-$ | [523 7/2-] | [523 5/2-] | au | $\sim 4.8$ |
| $_{64}Gd^{161}_{97}$ | $_{65}Tb^{161}_{96}$ | 418 | $\beta^-$ | [523 5/2-] | [523 7/2-] | au | $\sim 4.9$ |

TABLE IX

1st Forbidden Transitions Considered (After MI 59)

| Parent Nucleus | Daughter Nucleus | Excitation Energy in Daughter | Type of Transition | Orbit Assignment | | Classification | $\log_{10} ft$ |
|---|---|---|---|---|---|---|---|
| | | | | Parent | Daughter | | |
| $_{70}Yb^{175}_{105}$ | $_{71}Lu^{175}_{104}$ | 0 | $\beta^-$ | [ 514 7/2- ] | [ 404 7/2+ ] | 1u | 6.4 |
| $_{66}Dy^{165}_{99}$ | $_{67}Ho^{165}_{98}$ | 0 | $\beta^-$ | [ 633 7/2+ ] | [ 523 7/2- ] | 1u | ~ 6.2 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | 0 | $\beta^-$ | [ 523 7/2- ] | [ 633 7/2+ ] | 1u | ~6.0 |
| $_{63}Eu^{155}_{92}$ | $_{64}Gd^{155}_{91}$ | 0 | $\beta^-$ | [ 413 5/2+ ] | [ 521 3/2- ] | 1h | 8.7 |
| $_{63}Er^{157}_{94}$ | $_{64}Gd^{157}_{93}$ | 0 | $\beta^-$ | [ 413 5/2+ ] | [ 521 3/2- ] | 1h | 8.0 |

TABLE X

Pairing Force Parameter G, Fermi Levels λ and Energy Gaps Δ for Protons in Nuclei Considered (In Units of $\hbar\omega_0(\delta)$). The Zeros of Energy are Arbitrarily Taken at the Observed Ground State Levels in Nuclei Marked by *.

| A | $\delta$ B | $\hbar\omega_0$ (Mev) | $G = \frac{27.2}{\hbar\omega_0 A}$ | $\lambda_e$ | $\Delta_e$ | Blocked Orbital | $\lambda_o$ | $\Delta_o$ | $\rho_p^{exp}$ | Mean Square Deviation $\sigma_N^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $_{76}W^{181}$ | 0.23 | 7.35 | 0.02044 | 0.139 | 0.1018 | | | | | 3.887 |
| $*_{73}Ta^{181}$ | 0.23 | 7.35 | 0.02044 | 0.0945 | 0.1075 | [404 7/2+] | 0.1315 | 0.0685 | 0.1182 | |
| | | | | | | [541 1/2-] | 0.055 | 0.1042 | | |
| $_{72}Hf^{181}$ | 0.23 | 7.35 | 0.02044 | 0.0600 | 0.1145 | | | | | 5.1448 |
| $_{71}Lu^{177}$ | 0.25 | 7.42 | 0.02070 | -0.037 | 0.1190 | [404 7/2+] | -0.0517 | 0.0751 | | |
| $_{70}Yb^{177}$[a] | 0.25 | 7.42 | 0.02070 | -0.074 | 0.1157 | | | | | 4.839 |
| $_{70}Yb^{171}$ | 0.27 | 7.53 | 0.02113 | -0.102 | 0.1163 | | | | | |
| $_{68}Er^{171}$ | 0.27 | 7.53 | 0.02113 | -0.183 | 0.1191 | | | | | 4.960 |

[a] Energies referred to last filled proton level in $Hf^{178}$

TABLE X

(CONTINUED)

| A | $\xi$ II | $\hbar\omega_0$ (Mev) | $\varepsilon = \frac{27.2}{\hbar\omega_0 A}$ | $\lambda_e$ | $\Delta_e$ | Blocked Orbital | $\lambda_o$ | $\Delta_o$ | $\rho_p^{exp}$ | Mean Square Deviation $\sigma_\lambda^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $_{69}Tm^{171}$ | 0.27 | 7.53 | 0.02113 | | | [523 7/2-] | -0.1145 | 0.0732 | | |
| $_{69}Tm^{169}$ | 0.27 | 7.56 | 0.02129 | -0.1442 | 0.1200 | [411 1/2+] | -0.1205 | 0.0701 | 0.1167 | |
| $_{70}Yb^{175}$ | 0.27 | 7.47 | 0.0208 | -0.101 | 0.1088 | | | | | 4.213 |
| *$_{71}Lu^{175}$ | 0.27 | 7.47 | 0.0208 | | | [404 7/2+] | -0.0832 | 0.0680 | 0.1083 | |
| *$_{68}Er^{167}$ | 0.28 | 7.60 | 0.02143 | 0.008 | 0.1221 | | | | | 5.002 |
| $_{67}Ho^{167}$ | 0.28 | 7.60 | 0.02143 | | | [523 7/2-] | -0.049 | 0.0785 | | |
| $_{66}Dy^{165}$ | 0.29 | 7.65 | 0.02155 | -0.0719 | 0.1216 | | | | | 4.889 |
| *$_{67}Ho^{165}$ | 0.29 | 7.65 | 0.02155 | -0.033 | 0.1205 | [523 7/2-] | -0.054 | 0.0767 | 0.1202 | |

TABLE X

(CONTINUED)

| A | $\delta_N$ | $\hbar\omega_o$ (Mev) | $g = \frac{27.2}{\hbar\omega_o A}$ | $\lambda_e$ | $\Delta_e$ | Blocked Orbital | $\lambda_o$ | $\Delta_o$ | $p_p^{exp}$ | Mean Square Deviation $\sigma_N^{e2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $_{64}Gd^{155}$ | 0.30 | 7.82 | 0.02244 | -0.0107 | 0.1316 | | | | | |
| $_{63}Eu^{155}$ | 0.30 | 7.82 | 0.02244 | | | $[413\frac{5}{2}+]$ | -0.066 | 0.0856 | | |
| $_{65}Tb^{159}$ | 0.30 | 7.75 | 0.02206 | | | $[532\frac{5}{2}-]$ | 0.0585 | 0.1071 | | |
| $_{64}Gd^{159}$ | 0.30 | 7.75 | 0.02206 | -0.0105 | 0.1242 | | | | | 5.057 |
| $_{64}Gd^{161}$ | 0.30 | 7.72 | 0.02187 | -0.0105 | 0.12092 | | | | | 4.94 |
| $_{65}Tb^{161}$ | 0.30 | 7.72 | 0.02187 | | | $[523\frac{7}{2}-]$ | -0.0005 | 0.09422 | | |
| $_{63}Eu^{157}$ | 0.30 | 7.78 | 0.02225 | | | $[413\frac{5}{2}+]$ | -0.0662 | 0.08122 | | |
| $_{64}Gd^{157}$ | 0.30 | 7.78 | 0.02225 | -0.0106 | 0.12794 | | | | | 5.193 |

TABLE XI

Pairing Force Parameter $g$, Fermi Levels $\lambda$ and Energy Gaps $\Delta$ for Neutrons in Nuclei Consider (in Units of $\hbar\omega_0$. The Zeros for Energy are Arbitrarily Taken at the Observed Ground State Levels in Nuclei Marked by *.

| | A | N | (Mev) | $g = \frac{18.5}{A}$ | $g = \frac{19}{A}$ | $\lambda_e$ | $\Delta_e$ | Blocked Orbital | $\lambda_o$ | $\Delta_o$ | $\rho_{\nu}^{2}$ | Mean Square Deviation $\sigma_{\nu}^{2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| * | $W_{109}^{183}$ | 0.20 | 7.30 | 0.01384 | 0.01422 | -0.0565<br>-0.0560 | 0.1030<br>0.0940 | [510 1/2-] | -0.0701<br>-0.0695 | 0.07906<br>0.06945 | 0.1004 | |
| | $W_{107}^{181}$ | 0.23 | 7.35 | | 0.01428 | | | [624 9/24] | -0.008 | 0.0641 | | |
| * | $Te_{108}^{181}$ [b] | 0.23 | 7.35 | | 0.01428 | 0.023 | 0.1037 | | | | 0.103 | 6.421 |
| | $Hf_{109}^{181}$ | 0.23 | 7.35 | | 0.01428 | | | [510 1/2-] | 0.0395 | 0.08225 | | |
| | $Lu_{106}^{177}$ | 0.25 | 7.42 | | 0.01446 | -0.0285 | 0.0935 | | | | | 5.267 |
| * | $Hf_{107}^{179}$ | 0.25 | 7.39 | | 0.01436 | 0.0015 | 0.09136 | | | | 0.0934 | |
| | $Yb_{107}^{177}$ | 0.25 | 7.42 | | 0.01446 | | | [624 9/24] | 0.065 | 0.0494 | | |

[a] After NP 61

[b] Zero at the last filled level

TABLE XI

(CONTINUED)

| A | $\delta_N$ | $\hbar\omega_0$ (Mev) | $\varepsilon = \frac{18.5}{\hbar\omega_0 A}$ | $\varepsilon = \frac{19}{\hbar\omega_0 A}$ | $\lambda_0$ | $\Delta_0$ | Blocked Orbital | $\lambda_0$ | $\Delta_0$ | $P_{n+a}$ | Mean Square Deviation $\sigma N^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *$Hf^{177}_{105}$ | 0.26 | 7.43 | | 0.01444 | 0.00595 | 0.09313 | [514 7/2-] | 0.01 | 0.05584 | 0.0867 | |
| $Yb^{171}_{101}$ | 0.27 | 7.53 | | 0.01476 | -0.062 | 0.1067 | [521 1/2-] | -0.046 | 0.08041 | 0.09794 | |
| $Tm^{171}_{102}$ | 0.27 | 7.53 | | 0.01476 | -0.0345 | 0.1033 | | | | | 5.929 |
| $Er^{171}_{103}$ | 0.27 | 7.53 | | 0.01476 | | | [512 5/2-] | -0.011 | 0.06410 | | |
| $Yb^{175}_{105}$ | 0.27 | 7.47 | | 0.01453 | | | [514 7/2-] | 0.0525 | 0.06042 | | 6.004 |
| *$Lu^{175}_{106}$ | 0.27 | 7.47 | | 0.01453 | 0.0235 | 0.1043 | | | | | |
| *$Er^{167}_{99}$ | 0.28 | 7.60 | 0.01457 | | -0.0145 | 0.1039 | [633 7/2+]<br>[523 5/2-] | -0.0249<br>0.0012 | 0.0860<br>0.09135 | 0.103 | |
| $Ho^{167}_{100}$ | 0.28 | 7.60 | 0.01457 | | 0.010 | 0.1057 | | | | | 6.363 |

TABLE XI

(CONTINUED)

| A | $\delta$ N | $\hbar\omega_{osc}$ (Mev) | $\varepsilon = \frac{18.5}{\hbar\omega_0 A}$ | $\varepsilon = \frac{19}{\hbar\omega_0 A}$ | $\lambda_e$ | $\Delta_e$ | Blocked Orbital | $\lambda_o$ | $\Delta_o$ | $\rho_\nu^{exp}$ | Mean Square Deviation $\sigma_\nu^{*2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Dy_{95}^{161}$ | 0.29 | 7.71 | 0.01490 | | -0.081 | 0.1335 | [642 5/2+] | -0.067 | 0.1124 | 0.1182 | |
| $^{*}Ho_{98}^{165b}$ | 0.29 | 7.65 | 0.01465 | | -0.0095 | 0.1174 | | | | | 7.397 |
| $Dy_{99}^{165}$ | 0.29 | 7.65 | 0.01465 | | | | [633 7/2+] | 0.0085 | 0.08965 | | |
| $^{*}Gd_{91}^{155}$ | 0.30 | 7.82 | 0.01526 | | -0.0100 | 0.1475 | [521 3/2-] | -0.0155 | 0.1264 | 0.1464 | |
| $Eu_{92}^{155}$ | 0.30 | 7.82 | 0.01526 | | 0.01050 | 0.1446 | | | | | 9.834 |
| $Gd_{93}^{157}$ | 0.30 | 7.78 | 0.01514 | | 0.0310 | 0.1382 | [521 3/2-] | 0.0332 | 0.1142 | 0.1272 | |
| $Eu_{94}^{157}$ | 0.30 | 7.78 | 0.01514 | | 0.0535 | 0.1347 | | | | | 0.8783 |
| $Tb_{96}^{159}$ | 0.30 | 7.75 | 0.01500 | | 0.0530 | 0.1311 | | | | | 0.8554 |
| $Gd_{95}^{159}$ | 0.30 | 7.75 | 0.01500 | | | | [521 3/2-] | 0.089 | 0.1054 | | |
| $Tb_{96}^{161}$ | 0.30 | 7.72 | 0.01488 | | 0.0995 | 0.1217 | | | | | 7.659 |
| $Gd_{97}^{161}$ | 0.30 | 7.72 | 0.01488 | | | | [523 5/2-] | 0.1389 | 0.09885 | | |

[b] Zero at the last filled level

TABLE XII

Reduction Factors Using BCS Wave Functions

| Parent Nucleus | Daughter Nucleus | Classification | $R_N$(neutrons) | $R_Z$(protons) | $R_1 = R_N \times R_Z$ | $R_2 = R_N \times R_Z$ (Soloviev) |
|---|---|---|---|---|---|---|
| $_{72}Hf^{181}_{109}$ | $_{73}Ta^{181}_{108}$ | ah | 0.758 | 0.971 | 0.736 | - |
| $_{74}W^{181}_{107}$ | $_{73}Ta^{181}_{106}$ | ah | 0.487 | 0.834 | 0.406 | 0.34 |
| $_{70}Yb^{177}_{107}$ | $_{71}Lu^{177}_{106}$ | ah | 0.503 | 0.747 | 0.376 | 0.41 |
| $_{68}Er^{171}_{103}$ | $_{69}Tm^{171}_{102}$ | ah | 0.551 | 0.347 | 0.191 | 0.15 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | au | 0.870 | 0.441 | 0.387 | 0.52 |
| $_{64}Gd^{159}_{95}$ | $_{65}Tb^{159}_{94}$ | ah | 0.277 | 0.225 | 0.0623 | 0.07 |
| $_{64}Gd^{161}_{97}$ | $_{65}Tb^{161}_{96}$ | au | 0.277 | 0.220 | 0.0609 | 0.20 |

Column 6 gives the reduction factors obtained when using 46 proton levels and 56 neutron levels

Column 7 gives the reduction factors obtained by Soloviev (361)

TABLE XII

(CONTINUED)

| Parent Nucleus | Daughter Nucleus | Classification | $R_N$(neutrons) | $R_Z$(protons) | $R_1 = R_N \times R_Z$ | $R_2 = R_N \times R_Z$ (Soloviev) |
|---|---|---|---|---|---|---|
| $_{70}Yb^{175}_{105}$ | $_{71}Lu^{175}_{104}$ | 1 u | 0.503 | 0.747 | 0.375 | 0.32 |
| $_{66}Dy^{165}_{99}$ | $_{67}Ho^{165}_{98}$ | 1 u | 0.574 | 0.666 | 0.382 | 0.33 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | 1 u | 0.504 | 0.440 | 0.222 | 0.23 |
| $_{63}Eu^{155}_{92}$ | $_{64}Cd^{155}_{91}$ | 1 h | 0.511 | 0.417 | 0.213 | |
| $_{63}Eu^{157}_{94}$ | $_{64}Cd^{157}_{93}$ | 1 h | 0.654 | 0.298 | 0.195 | |

Column 6 gives the reduction factors obtained when using 46 proton levels and 56 neutron levels

Column 7 gives the reduction factors obtained by Soloviev (S 61 )

TABLE XIV

Reduction Factors for Allowed Transitions

| Parent | Daughter | Orbit Assignment | | $\log_{10} ft$ | | Red. Factors (BCS) | | | $R_{Solv.}$ | Red. Factors (proj.) | | | $\log_{10} ft$ (calc.) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Parent | Daughter | Exp. | S.Part. | $R_N$ | $R_Z$ | $R_1 = R_N R_Z$ | | $R_N$ | $R_Z$ | $R_2 = R_N R_Z$ | $R_1$ | $R_2$ |
| $_{72}Hf^{181}_{109}$ | $_{73}Ta^{181}_{108}$ | [510 1/2-] | [541 1/2-] | ~6.5 | ~8.00 | 0.761 | 0.971 | 0.739 | | 0.810 | 0.979 | 0.793 | | |
| $_{68}Er^{171}_{103}$ | $_{69}Tm^{171}_{102}$ | [512 5/2-] | [523 7/2-] | 6.3 | 4.64 | 0.559 | 0.351 | 0.196 | 0.15 | 0.642 | 0.383 | 0.246 | 5.35 | 5.24 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | [523 7/2-] | [523 5/2-] | ~4.8 | 3.74 | 0.880 | 0.446 | 0.392 | 0.52 | 0.998 | 0.488 | 0.487 | 4.14 | 4.05 |
| $_{64}Gd^{159}_{95}$ | $_{65}Tb^{159}_{94}$ | [521 3/2-] | [532 5/2-] | 6.7 | 4.47 | 0.279 | 0.227 | 0.0633 | 0.07 | 0.282 | 0.228 | 0.0643 | 5.66 | 5.66 |
| $_{64}Gd^{161}_{97}$ | $_{65}Tb^{161}_{96}$ | [523 5/2-] | [523 7/2-] | 4.8 | 3.52 | 0.279 | 0.222 | 0.0618 | 0.26 | 0.274 | 0.223 | 0.0612 | 4.73 | 4.73 |

TABLE XV

Reduction Factors for Allowed and 1st Forbidden Transitions

| Parent | Daughter | Orbit Assignment | | $\log_{10} ft$ | Red. Factors (SCS) | | | $R_{Solv.}$ | Red. Factors (proj) | | | $\log (ft)_{exp}^{R}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Parent | Daughter | | $R_H$ | $R_A$ | $R_H R_A$ | | $R_H$ | $R_A$ | $R_H R_A$ | SCS | proj | Sol. |
| $_{74}W^{181}_{107}$ | $_{73}Ta^{181}_{108}$ | [624 9/2+] | [404 7/2+] | 6.6 ah | 0.495 | 0.841 | 0.416 | 0.34 | 0.607 | 0.906 | 0.549 | 6.22 | 6.34 | 6.13 |
| $_{70}Yb^{177}_{107}$ | $_{71}Lu^{177}_{106}$ | [624 9/2+] | [404 7/2+] | 6.2 ah | 0.506 | 0.692 | 0.350 | 0.41 | 0.617 | 0.776 | 0.478 | 5.74 | 5.87 | 5.81 |
| $_{66}Dy^{165}_{99}$ | $_{67}Ho^{165}_{98}$ | [633 7/2+] | [523 7/2-] | ~6.2 | 0.576 | 0.674 | 0.389 | 0.53 | 0.625 | 0.767 | 0.479 | 5.79 | 5.88 | 5.72 |
| $_{67}Ho^{167}_{100}$ | $_{68}Er^{167}_{99}$ | [523 7/2-] | [633 7/2+] | ~6.0 | 0.506 | 0.446 | 0.225 | 0.23 | 0.513 | 0.488 | 0.250 | 5.55 | 5.40 | 5.56 |
| $_{68}Eu^{155}_{92}$ | $_{64}Gd^{155}_{91}$ | [413 5/2+] | [521 3/2-] | 8.7 | 0.512 | 0.320 | 0.164 | | 0.613 | 0.521 | 0.319 | 7.91 | 8.20 | |
| $_{63}Eu^{157}_{94}$ | $_{64}Gd^{157}_{93}$ | [413 5/2+] | [521 3/2-] | 8.0 | 0.657 | 0.300 | 0.197 | | 0.692 | 0.548 | 0.383 | 7.30 | 7.60 | |
| $_{70}Yb^{175}_{105}$ | $_{71}Lu^{175}_{104}$ | [514 7/2-] | [404 7/2+] | 6.4 | 0.515 | 0.758 | 0.390 | 0.32 | 0.500 | 0.849 | 0.432 | | | |

| | |
|---|---|
| BA 57 | I. Bergstrom, S. Anderson, Arkiv. för Fysik. Vol. 12, No. 19 (1957). |
| Ba 58 | B. F. Bayman, Proc. of the Intern. Conf. on Nucl. Phys., Paris (1958). |
| Ba 59 | B. F. Bayman, Nucl. Phys. 15, 33 (1960). |
| Bar 60 | H. Barangor, Phys. Rev. 120, 957 (1960). |
| BCS 58 | J. Bardeen, L. Cooper, R. Schrieffer, Phys. Rev. 108, 1175 (1957). |
| Be 59 | S. T. Beliaev, Dan. Mat. Fys. Medd. 31, No. 11 (1959). |
| Bk 58 | F. H. Bakke, Proc. of the Intern. Conf. on Nucl. Phys., Paris (1958). |
| BM 55 | A. Bohr, B. R. Mottelson, Dan. Mat. Fys. Medd. 30, No. 1 (1955). |
| BMP 58 | A. Bohr, B. R. Mottelson, D. Pines, Phys. Rev. 110, 936 (1958). |
| BMJ 61 | Bhanot, Johnson and Nier, Phys. Rev. 120, 235 (1960). |
| BS 61 | D. Bes, Z. Szymanski, Nucl. Phys. 28, 42 (1961). |
| C 59 | L. Cooper, Lectures at Brandeis University, Summer Institute in Theoretical Physics, Vol. II (1959). |
| E 58a | J. P. Elliott, Proc. Roy. Soc. A245, 128 (1958). |
| E 58b | J. P. Elliott, Collective Motion in Nuclei, University of Rochester, N. Y. (1958). |
| K 61 | A. Kerman, Annals of Physics 12, 300 (1961). |

KMcL 61 A. Kerman, R. Lawson, M. MacFarlane, Phys. Rev. 124, 162 (1961).

KS 61 L. Kisslinger, R. Sorenson, Dan. Mat. Fys. Medd. 32, No. 9 (1960).

L 61 H. J. Lipkin, Nucl. Phys. 26, No. 1 (1961).

M 58 B. R. Mottelson et al, The Many Body Problem, John Wiley and Sons, New York (1958).

M 60 B. R. Mottelson, Proc. of the Int. Conf. on Nuclear Structure, Kingston, Canada (1960).

Ma 50 M. Mayer, Phys. Rev. 78, 22 (1950).

MN 59 B. R. Mottelson, S. G. Nilsson, Dan. Mat. Fys. Skr. 1, No. 8 (1959).

No 61 M. Moshinsky, Proc. of the Intern. Conf. on Nuclear Structure, Manchester (1961).

NP 61 S. G. Nilsson, O. Prior, Dan. Mat. Fys. Medd. 32, No. 16 (1961).

P 62 M. A. Preston, Physics of the Nucleus, Addison-Wesley Publishing Co., Mass. (1962).

R 60 M. E. Rose et al, Nuclear Spectroscopy, Part B, Edited by F. Ajzenberg-Selove, Academic Press, New York (1960).

SGN 55 S. G. Nilsson, Dan. Mat. Fys. Medd. 29, No. 16 (1955).

S 61 V. G. Soloviev, Dan. Mat. Fys. Skr. 1, No. 61 (1961).

T 61 Dr. J. Thouless, The Quantum Mechanics of Many Body Systems, Academic Press, New York (1961).

Co 61 B.L. Cohen, Phys. Rev. 125, 1358 (1962

ERRATA:

After this work was completed, it was found that the $N = 5$ $h_{11/2}$ proton levels for $\delta_N = 0.30$ had been depressed by too large an amount. Correct parameters (in line with Table VII) yield the results shown in Table X (pg. 104). In addition, the following corrections must be made:

| Where it is | | It should be |
|---|---|---|
| Table XII (pg. 108) | 0.225 | 0.0701 |
| | 0.220 | 0.8107 |
| | 0.0623 | 0.0196 |
| | 0.0609 | 0.225 |
| (pg. 109) | 0.298 | 0.413 |
| | 0.195 | 0.271 |
| Table XIV | 0.227 | 0.0703 |
| | 0.222 | 0.816 |
| | 0.0633 | 0.0196 |
| | 0.0618 | 0.227 |
| | 0.228 | 0.0591 |
| | 0.223 | 0.867 |
| | 0.0643 | 0.0167 |
| | 0.0612 | 0.237 |
| | 5.66 ($R_1$) | 6.20 ($R_1$) |
| | 4.73 ($R_1$) | 4.16 ($R_1$) |
| | 5.66 ($R_2$) | 6.25 ($R_2$) |
| | 4.73 ($R_2$) | 4.14 ($R_2$) |

| | Where it is | It should be |
|---|---|---|
| Table XV | 0.320 | 0.424 |
| | 0.300 | 0.429 |
| | 0.164 | 0.217 |
| | 0.197 | 0.282 |
| | 0.521 | 0.468 |
| | 0.568 | 0.467 |
| | 0.388 | 0.318 |
| | 7.91 | 8.03 |
| | 7.30 | 7.45 |
| | 8.20 | 8.16 |
| | 7.60 | 7.50 |

The large deviation from Soloviev's results for the case of $Gd^{161}$ (see pg. 76) is thereby explained.