MATHEMATICAL REASONING AND COMMUNICATION

MATHEMATICAL REASONING AND COMMUNICATION: ANALYZING SKILLS DEVELOPMENT IN UNIVERSITY STUDENTS AND HIGH-LEVEL MATH COMPETITORS

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Lay Abstract

Mathematics is often reduced to memorizing formulas and following procedures, yet true proficiency also demands the ability to reason through novel problems and explain ideas clearly. This thesis explores how students themselves perceive the link between mathematical reasoning (MR) and communication by gathering survey responses, interview insights, and discussion-activity reflections from both high school math competitors and McMaster University undergraduates. It examines their initial communication skills in mathematics, investigates whether participation in a formal MR course strengthens their ability to articulate and justify solutions, and considers if regular engagement in proof-writing and peer-explanation activities fosters deeper reasoning. Although most participants appreciate the importance of reasoning and clear expression for future success, they typically fail to connect these competencies until advanced coursework or extracurricular contexts make justification and collaboration explicit. These findings imply that, despite Canadian curricula aiming to develop both MR and communicative competence, standard instruction does not automatically integrate them, calling for more purposeful teaching strategies and further work to directly assess students' reasoning and communication abilities.

Abstract

Integrating mathematical reasoning and communication is essential for mastering and advancing mathematical learning, yet these competencies often develop along parallel tracks rather than in concert. This thesis investigates how students themselves perceive the link between reasoning and communication by studying three cohorts: first-year undergraduates in an introduction to mathematical reasoning course, upper-level mathematics majors, and high school competitors in math contests. Using surveys, semi-structured interviews, and a discussion-based "talking circle," it examines students' baseline communication skills, tracks their growth over a semester of formal reasoning instruction, and explores the role of proof-writing and collaborative problem solving in deepening their reasoning.

Findings reveal that, although nearly all participants recognize the importance of clear justification for their academic and professional futures, they seldom connect reasoning with communication until advanced coursework or competitive settings explicitly foreground the act of explaining and defending ideas. Introductory students gained confidence in constructing formal proofs but continued to struggle with audience-appropriate articulation; upper-level majors demonstrated precision in argumentation yet found it challenging to translate technical proofs for non-specialists; and high-school competitors excelled at adapting explanations under time pressure and in team contexts. These results underscore that standard curricula alone do not guarantee integrated skill development. Deliberate pedagogical strategies – such as structured peer-explanation exercises, scaffolded analogical reasoning tasks, and sustained discussion opportunities – are needed to cultivate students who are not only proficient problem solvers but also articulate mathematical communicators prepared for advanced STEM study and engaged citizenship.

Keywords: mathematics education, self-concept, mathematical reasoning, mathematical communication, communication skills, proof writing, problem-solving skills, reasoning and communication integration, skill development, math competitions, high-achieving students.

To all teachers that have shaped me into the person I am today.

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Abbreviations

#	Number
#*	Renumbering of the Technical Questions
AI	Artificial intelligence
CMS	Canadian Mathematical Society
Gr	Group
MR	Mathematical Reasoning
MSC	Mathematics Self-Concept
OECD	Organization for Economic Co-operation and Development
Q_i	i^{th} Question
Q-n	Question
STEM	Science, Technology, Engineering, and Mathematics

Declaration of Academic Achievement

I, Kateryna Tretiakova, declare this thesis to be my own work. I am the sole author of this document. No part of this work has been published or submitted for publication or for a higher degree at another institution. To the best of my knowledge, the content of this document does not infringe on anyone's copyright. My supervisor, Dr. Miroslav Lovric, and the members of my supervisory committee, which included Dr. Katherine Davies, Dr. Erin Clements, and Dr. David Lozinski, have provided guidance and support at all stages of this project. I completed all of the research work.

Chapter 1

Introduction

One of the main goals of formal education is to improve the quality of life of a human being [17]. Young learners have the opportunity to develop the knowledge, values, and behaviour models that will enable future success beyond the classroom. In the modern technology-driven economy, as highlighted in the OECD Skills Outlook 2023, a person's competitiveness and success depend on integrating cognitive, social, and communicative skills to solve complex problems [94]. The current labour market demands mastery of new technologies, adaptability to fast-changing conditions, and the ability to navigate vast information flows. Within this landscape, **mathematical reasoning** (MR) – defined here as a mental activity (process) of an individual governed by mathematical laws, involving abstraction, pattern recognition, and justification (e.g., generalizing, conjecturing, comparing, classifying, justifying, exemplifying, etc.) – and communication skills, particularly in science, become indispensable, as no individual can *know-it-all*. The development of mathematical thinking occurs through students' engagement in mathematical cognitive activity – defined as active cognition involving spatial representations and quantitative relations of the surrounding world. MR underpins both problem-solving and the ability to articulate, justify, and refine ideas collectively, while communication bridges individual understanding with communal validation.

Communication, broadly understood as the exchange of signs imbued with shared meaning [32], requires participants to decode messages through agreed-upon conventions. In mathematics, this involves translating abstract concepts into precise verbal, written, or symbolic forms — also known as the *mathematical register* [141]. Following Dr. Sfard's framework, this thesis defines communication as a patterned collective activity where individuals build on prior actions. Key objectives include the following *communicational* actions [109]:

1. Formulating one's thoughts accurately and articulating them aloud;
2. Formalizing ideas into written text;
3. Listening, comprehending, and posing questions;
4. Reading with understanding.

While the first "action" must be a certain type of *communicational* action, the reactions, or the actions that follow, may be of a communicational type or a practical type. In this case, an action that causes an object in the environment to physically change is said to be of a *practical* type. For example, in a classroom setting, a teacher explaining the Pythagorean theorem through verbal explanations and diagrams, or students asking questions and writing down the solution, are examples of communication actions. In contrast, when a student measures the sides of a triangle in a lab or at home, they engage practical actions. This interplay between communication (to share, explain, and question ideas) and practical application (to test, verify, and apply those ideas) underscores how MR relies on dialogue and iterative feedback.

Scientific communication, including its mathematical subset, presents unique challenges, particularly when participants lack shared expertise. Unlike casual discourse, it requires bridging gaps in background knowledge to ensure clarity and comprehension. For example, a mathematician might reference a theorem implicitly among peers but must justify it explicitly for broader audiences. This process, though often solitary in knowledge acquisition, is socially validated through communal discourse, as the verification and application of mathematical ideas depend on effective communication. The Mathematical Association of America emphasizes that successful mathematical communication balances formal rigor with audience awareness and logical flow [78], mirroring Aristotle's concept of *logos* – rational persuasion through structured reasoning [103]. This dynamic underscores the critical role of communication in linking individual reasoning with broader understanding within the mathematical community.

While listening, writing, and reading skills are practiced by students across subjects and everyday life, only mathematics and science classes have an opportunity to engage students in deriving results through pure logic, free from historical or contextual references. Unlike history or literature, where understanding relies on external narratives, such as past events, cultural contexts, or existing knowledge, mathematics focuses on abstract concepts and principles that can be applied universally. This makes mathematics unique among school subjects — while taught throughout most academic careers, it doesn't relate to tangible reality in the same concrete way as other disciplines. Rather, it reveals the hidden quantitative relationships and spatial forms underlying both physical phenomena and abstract concepts. This dual nature gives mathematical communication special significance: it serves not just as a practical tool for specialists, but as a fundamental framework for establishing and validating knowledge within intellectual communities [112]. The challenge lies in maintaining mathematical rigor while making these abstract relationships meaningful and applicable to real-world problems.

Although the Canadian education system comprises distinct provincial and territorial systems, the overarching objectives of high school mathematics curricula show remarkable consistency nationwide. These shared goals include [105, 83, 9]:

- to be able to reason mathematically and think critically in various given situations;
- to understand the meaning of various mathematical objects (numbers, shapes, etc.), their representations and properties;
- to recognize and understand the relationship between geometrical shapes and objects and numbers;
- to be able to communicate messages, considering the message's subject and purpose together with the intended audience.

Notably, the communication objective—emphasizing audience-appropriate conveyance of mathematical ideas—receives uneven attention across provinces, revealing prioritization differences within this common framework. This variation is particularly significant because while mathematical reasoning forms the core of the discipline, its development remains incomplete without parallel growth in communication skills.

The interplay between reasoning and communication creates a fundamental tension in mathematics education: students must learn both to construct logical arguments (mathematical reasoning) and to articulate them effectively (mathematical communication). Yet Canadian curricula often treat reasoning as an implicit byproduct of practice rather than an explicit learning objective, while communication skills frequently remain secondary concerns. This dual underemphasis creates a gap - students may grasp mathematical concepts internally yet struggle to present coherent arguments or evaluate others' mathematical thinking with precision.

In my experience working in math camps and classrooms, I have observed that students often conflate conceptual understanding with mastery, assuming that grasping an idea is sufficient. However, true mathematical competence requires both internal comprehension and the ability to articulate and justify reasoning. Students frequently dismiss logical steps as *obvious*, *clear*, or *trivial* – phrases that, as readers of the *Graduate Texts in Mathematics* series may recognize, often precede deceptively intricate arguments. Such oversimplifications reveal a gap between intuitive understanding and rigorous expression.

Bridging this gap demands explicit training in mathematical communication. Successfully explaining a solution involves structuring thoughts logically, providing precise justifications, and engaging in collaborative discussion. Research supports that students who regularly communicate their reasoning—whether through writing or dialogue—develop stronger conceptual connections, refine their problem-solving skills, and build the vocabulary needed to navigate mathematical discourse [24, 87]. These practices not only solidify individual understanding but also expose and correct misconceptions [24, 87].

Ultimately, mathematical proficiency mirrors language acquisition: mastering grammar and vocabulary alone is insufficient without the ability to communicate effectively [65, 55]. Similarly, mathematics education thrives when classrooms emulate language learning environments—prioritizing discussion, open-ended questioning, and peer exchange [104]. By fostering these habits, educators equip students to refine their reasoning, engage with diverse perspectives, and deepen their learning beyond procedural fluency.

University education, while building on high school foundations, also shapes independent, socially responsible thinkers capable of informed decisions [45]. As Dr. Joel Westheimer emphasized during the *What are universities for?* symposium:

> University education is a basic public good. [...] We have reframed universities as job-training institutions, and it is a very individualized view of the function of a university. [...] It is not just each of us wants an education, we all have a stake in living in an educated society.
> (University of Regina, 2023) [139]

This broader vision highlights the necessity of critical thinking and mathematical reasoning, skills that transcend disciplinary boundaries and are vital for evidence-based decision-making[49].

Yet, in high school and early university mathematics, instruction often emphasizes procedural fluency, which includes memorizing formulas and algorithms, over explicit reasoning development. While students may internalize some logical patterns through repetition, this passive approach leaves them ill-prepared for proof-based courses or real-world problem-solving.

The purpose of this study is to investigate the symbiotic relationship between MR and communication skills among university students of various levels and goals and high-school math competitions (e.g., Olympiad participants). By comparing these groups, the study addresses three questions:

- What is the current state of mathematical communication skills among the students in the groups mentioned above?
- To what extent does a course in mathematical reasoning enhance students' ability to communicate ideas effectively?
- Do students who engage in communication-oriented activities (e.g., explaining concepts, presenting proofs) exhibit better mathematical reasoning compared to those who do not?

By identifying key relationships between reasoning training and communication development, this study provides a foundation for designing curricula that cultivate both technical problem-solving skills and the ability to articulate mathematical thinking – critical capacities for advanced STEM education and careers.

Chapter 2

Literature Review

Mathematics has always been an essential component of school curricula. Strangely, however, years of math classes do not necessarily cultivate the skill of *mathematical reasoning* (MR). During the past few years, while working with students from various educational levels, I have encountered numerous individuals with close to perfect scores or grades in mathematics. Yet, it often became evident that their success stemmed primarily from memorizing material rather than truly understanding it. This reliance on rote learning directly contradicts one of the key goals outlined in the 2020 Ontario mathematics curriculum: "to develop critical and creative thinking skills." So, how did we get here?

2.1 Theoretical Foundations of Mathematical Communication

Since 2017, mathematical reasoning (MR) has increasingly been analyzed through *commognition* – a theory developed by Sfard (Citation) that redefines thinking as internalized communication rather than an isolated cognitive process [113]. This framework posits that all cognition, including MR, emerges from linguistic patterns and social interactions.

Applied to MR, commognition interprets reasoning as:

- Dialogic: A process of deriving new mathematical statements (e.g., proofs, solutions) through self-dialogue or collaborative discourse;
- Externally grounded: Internal thought validates itself by external expression (e.g., writing proofs, peer critique);
- Linguistically mediated: Dependent on mathematical discourse routines (notation, definitions, argumentation norms).

This view dissolves possible boundaries between reasoning and communication, as seen in studies of classroom practice [57, 109].

Despite widespread recognition of the importance of communication skills in mathematics, there is limited discussion on their systematic development within school and university settings. Mathematics graduates often do not receive formal training in mathematical communication during their education, instead acquiring these skills through trial and error and workplace observation [142]. This gap stems not only from the design of educational programs but also from the absence of clear procedures for accessing and cultivating these abilities. This mirrors challenges found in language communication, where the abstractness and non-observability of communicative competence complicates both its definition and assessment [30]. This disconnect between thought and verbal expression, as theorized by Dr. Vygotsky, further complicated the issue. In the same way that language learning requires guided practice to translate internal thought into verbal expression, mathematics education must similarly support students in bridging abstract reasoning with effective communication [134].

For instance, during the early years of language acquisition, children have ample opportunities to interact with adults and learn through their senses, such as sight, sound, and consistent exposure to speech. Similarly, as students encounter abstract mathematical concepts, such as numbers and operations, it becomes crucial for teachers and parents to support the development of mental, spatial, and numeracy intelligence [93]. Just as spatial thinking helps understand abstract concepts like ordered relations and Venn diagrams, effective communication in mathematics requires guided practice and structured learning. Since the path from thought to verbal expression is not always direct, particularly with abstract concepts, effective classroom management and targeted teaching about communication are essential [84].

Research on the interplay between language acquisition and learning of mathematics has been seen around the world, however, largely localized, and as result, falling short of driving systematic change in national curricula or education practices [141]. The 1968 *Hall-Dennis Report (Living and Learning)*, a landmark in Ontario educational reform, envisioned child-centered learning where “no boy or girl will be without a suitable place for learning” [96]. Released during Quebec’s Quiet Revolution

(1960–1966), which remarks a period of rapid secularization, modernization, and rejection of traditional clerical authority, the report championed educational reforms aligned with the era's transformative ideals: state secularization, linguistic inclusivity, and technological advancement in public institutions.The report did provide the first notes of the communication's connection to mathematics in Canadian education,

> The major essential for the achievement of virtually any curricular purpose is the acquisition of the skills of communication. Language is not the first or only means of communication [...] Together with simple mathematics, they constitute the one skill which must be measured and brought to an acceptable standard in keeping with the pupil's ability [96].

Yet, as historian Peter Hennessy notes, the report remains a *utopian dream* due to systemic implementation failures [26] – most critically, its neglect of teacher preparation. Despite 258 progressive recommendations (e.g., open-concept classrooms and individualized instruction), the report provided no clear strategies for training educators to adopt these methods. Teachers were expected to transition abruptly from traditional lecture-based instruction to student-centered pedagogies requiring refined communication skills, such as facilitating collaborative problem-solving or guiding metacognitive reflection. Without prior experience or schooling, professional development, or resources, experiments like team teaching and audio-visual instruction stumbled, leaving educators ill-equipped to foster the very communication competencies the reforms idealized.

This historical example underscores a persistent truth: pedagogical innovation requires investment in teacher capacity. As Dr. Liljedahl observes in *Building Thinking Classrooms*, even well-intentioned educators struggle to prioritize communication and critical thinking when systems prioritize curriculum coverage over competence:

> [...] she had a room full of students who weren't thinking, yet she had curriculum to get through and standards to meet. [...] Even teachers who, by traditional measures, are considered good teachers [...] face this dilemma. [71]

This observation raises a critical question: do we want to teach our children simply to follow directions, or do we aim to nurture their ability to think and communicate independently? In an environment where not all teachers are adequately prepared or supported — whether due to staff shortages or the overwhelming burden of administrative tasks — the tendency may be to assume that students lack the capacity or willingness to engage in deeper, independent thought. Tragically, over time, this assumption can solidify into reality, shaping the very environment that stifles the growth of critical thinking skills.

Nowadays, the boost of digital resources and extensive research offers teachers a broader array of tools to support their practice. Mathematical tasks have been analyzed and categorized into several types: curricular-based mathematical discourse, reasoning processes within that discourse, meta-discourse about proof, and integrated curricular-based and logic-related discourse [138]. Yet, the systematic development of mathematical communication seems to remain underexplored. This theoretical foundation, therefore, not only illuminates the intrinsic connection between thought and language in mathematics but also highlights the need for targeted teacher preparation and resource allocation to integrate communication skills into mathematics education.

2.2 The Role of Argumentation in Mathematics

Argumentation, a universal cognitive process, is deeply embedded in daily decision-making, where individuals weigh various pieces of information they received (competing claims and counterclaims) to form coherent conclusions [11, 61]. In mathematics, this process is formalized: arguments are defined as structured sequences of premises leading to a conclusion, mirroring the logical scaffolding of proofs [37]. While everyday argumentation assesses the *strength* of reasoning through dynamic discourse, mathematical proof demands *validity* through a given system of axioms. This distinction is contested: some researchers argue proof and argumentation are inseparable or even the same, differing primarily in formality rather than essence itself [73, 123].

This way, argumentation serves as the bridge between informal reasoning and formal proof. In textbooks, arguments are defined as a set of premises followed by a conclusion, often employing recursion, contradiction, contrapositive, or analogy to test conjectures [100]. These techniques align with proof methods (taught in various proof-oriented courses) but differ in scope: argumentation explores ideas fluidly, while proof organizes them into indisputable structure [34]. For instance, a student might intuitively argue that prime numbers are infinite but struggle to formalize this into Euclid's proof-by-contradiction, underscoring gaps in translating intuition into symbolic rigour. Such conflation may stem from cognitive difficulties in translating intuitive ideas into the precise language required by mathematical representation [34]. Duval further emphasizes the role of semiotic and representational hurdles, highlighting that difficulties in managing multiple modes of representation — verbal, symbolic, and visual — can impede students' ability to formalize their reasoning [35].

However, the progression from argumentation to proof is often disrupted in classrooms. Students conflate persuasive reasoning with formal validation, treating proofs as static artifacts rather than refined outcomes of discourse [73]. Curricula aggravate this by prioritizing procedural fluency over discursive depth, reducing argumentation to fragmented exercises, usually referred to as "communication exercises" [123]. Standardized assessments, which value efficiency over explanatory coherence, further sideline opportunities for iterative critique [130]. Although the transition from intuitive argumentation to formal proof deepens understanding of mathematical concepts and sharpens critical thinking skills [123], this shift remains a significant challenge.

Addressing these obstacles requires not only curricular adjustments but also pedagogical support. Teachers play a crucial role in shaping students' experiences with mathematical argumentation, yet many struggle to facilitate the transition from informal reasoning to proof. Effective argumentation requires logical precision alongside the ability to contextualize premises and conclusions within a broader narrative [11]. Unfortunately, pedagogical training often neglects these competencies, leaving educators ill-equipped to scaffold argumentation as a developmental process [147]. As a result, the discontinuity between argumentation and proof hinders students' recognition of the necessity for formal validation when they rely on empirical or intuitive reasoning [36].

Classroom discourse may thus default to algorithmic instruction, inadvertently reinforcing passive learning and resistance to ambiguity [71]. Although digital tools (e.g., dynamic geometry software such as *GeoGebra* or *Desmos*) offer promise by visualizing proof steps, their success depends on effective pedagogical framing [138]. Systemic reforms, including aligning curricula, teacher training, and assessments, are essential to nurture the development of both mathematical reasoning continuum from intuitive reasoning to formal proof [134].

Ultimately, argumentation transcends technical skill — it cultivates an intellectual disposition toward inquiry, much like language acquisition, where guided practice is essential to translate intuitive ideas into formal syntax [134]. Distinguishing the proper roles of both proofs and mathematical literacy, while still recognizing their interdependence, can transform pedagogy: emphasizing argumentation cultivates critical thinkers who view proof as the refinement of ideas, not just their validation. This alignment, echoing historical reforms' unmet aspirations [96], is vital for fostering resilient, inquiry-driven learners.

In summary, argumentation is not merely a preliminary step toward proof but an integral aspect of mathematical thought. Distinguishing between the process of developing arguments and the act of proving them can enhance teaching strategies and curriculum design, ultimately leading to improved mathematical literacy and critical reasoning abilities.

2.3 Curriculum, Assessment, and Global Contexts

A longstanding tension exists between innovative teaching methods and traditional assessment frameworks, creating barriers to meaningful educational reform. Schools grapple with a dual challenge: adopting progressive pedagogies that better serve modern learners while often lacking assessment tools capable of evaluating the broader competencies these approaches cultivate. This disconnect is especially pronounced in mathematics education, where a lot of *modern* instructional strategies (e.g., collaborative problem-solving, personalized learning, and flipped classrooms) coexist with conventional assessments dominated by restricted-response formats like multiple-choice questions. When such assessments fail to align with contemporary teaching practices, they risk measuring only fragmented knowledge rather than deeper conceptual understanding or applied skills, regardless of the instructional methods employed.

This problem is neither new nor isolated. As noted earlier, the *Hall-Dennis Report*'s lack of aligned evaluation procedures left teachers without tools to assess progressive learning goals. Decades later, this disconnect continues to shape education systems globally [47]. To illustrate this persistent challenge, consider the contrasting contexts of Canada and Ukraine, where standardized assessments wield significant influence over pedagogy, albeit through divergent mechanisms.

In Canada, provincial education systems often prioritize curricular flexibility, allowing educators to tailor instruction to student needs. However, this autonomy is increasingly overshadowed by standardized testing regimes. For instance, *Ontario's Education Quality and Accountability Office* (EQAO) assessments, while initially designed to ensure accountability, have inadvertently shifted classroom priorities. Teachers report dedicating substantial time to test-taking strategies (e.g., practicing time management, decoding question phrasing, and memorizing formulaic responses) rather than fostering analytical reasoning or creative problem-solving [46, 4]. Students themselves internalize this emphasis: many perceive their teachers as prioritizing test scores over authentic learning, fostering environments where performance anxiety eclipses intellectual curiosity [132].

The stakes are starkly visible in British Columbia, where standardized exams determine 40% of final grades in some year-long high-school courses. Students argue these tests fail to reflect their true abilities, as they emphasize rote recall over nuanced understanding [108]. Consequently, educators feel pressured to narrow their teaching methods, sidelining open-ended discussions or exploratory projects in favor of exam-centric drills [53]. While EQAO has bolstered reading and writing outcomes, stagnant math scores highlight the limitations of conflating assessment with instruction. Teachers' unions now argue that standardized testing, when elevated to the singular goal of education, erodes both pedagogical creativity and student motivation [101].

On the contrary, Ukraine's pre-war education system had traditionally emphasized rote learning and preparation for national standardized tests, like the *State Final Attestation* (SFA), which evaluated students' total knowledge in mathematics and served as a mandatory diagnostic tool for students in Grades 4, 9, and 11 [116], and the *External Independent Assessment* (EIA), whose high scores were required for university admission [102]. While all students had to take the mathematics SFA, the decision to take the mathematics EIA depended on the university program requirements. The SFA and EIA were the only assessments in Ukrainian schools that had explicitly formulated requirements and organizational norms controlled by the state institution, the *Ukrainian Center for Educational Quality Assessment* (UCEQA) [116].

Since the SFA was mandatory but lacked stakes for non-university-bound students, it often excluded open-ended questions requiring written explanations, limiting opportunities for students to demonstrate reasoning skills [115]. In contrast, the EIA results were publicly available, incentivizing schools to prioritize high scores to boost rankings and attract parents. This high-stakes pressure narrowed teaching practices, as educators often focused on test preparation over critical thinking. However, the EIA included open-format questions (e.g., essays in all language assessments and mathematics problems requiring explanations) to assess communication and reasoning, even if critics argued these were insufficient for evaluating deeper logical abilities [12, 116]. Detailed rubrics outlining specific expectations were provided to students and educators, ensuring consistency in evaluating complex tasks and emphasizing logical sequencing and evidence-based communication. While critics argued these questions still fell short of assessing deeper logical abilities [12, 116], the presence of clear expectations provided students with a shared framework for organizing ideas, contrasting with systems where vague rubrics lead to inconsistent outcomes. Even the mandatory Ukrainian language SFA essay, though formulaic, ensured all graduates could construct a coherent argument, reflecting the system's prioritization of standardized communication norms [12].

Canada and Ukraine exemplify differing approaches to balancing assessment and pedagogy. Canada's flexible curricula risk being overshadowed by high-stakes testing pressures, while Ukraine's system prioritized uniformity through structured guidelines. The Ukrainian case illustrates how standardized frameworks can shape student outcomes, whether through disciplined communication or constrained creativity, while Canada's challenges underscore the tension between innovation and accountability.

Metacognitive approaches have long been recognized as valuable tools in mathematics education. Research demonstrates their positive effects on reasoning skills across achievement levels, with structured reflection and self-regulation enhancing both higher- and lower-achieving students' problem-solving abilities [70]. The need for such strategies has become even more pressing in the post-COVID-19 era, where students' formative high school years were disrupted by online learning. While remote education mitigated health risks, studies highlight unintended consequences, including declines in social-emotional engagement and communication skills — factors crutical to cognitive development and academic performance [59].

Although fostering meaningful mathematical discussions is a well-established pedagogical goal, research suggests that designing a curriculum entirely reliant on conversational teaching — where dialogue and interpersonal engagement replace traditional lectures or written materials — is not fully supported. Studies indicate that conversation as a primary teaching tool must be deliberately taught and structured to be effective [114]. In mathematical reasoning, effective communication is not only productive but often necessary, influencing both students' conceptual understanding and their overall learning experience [28]. Students tend to learn more effectively from teachers who communicate well [62], and strong communication skills positively impact student-teacher relationships and student well-being [41, 7]. Furthermore, research suggests that effective communication serves as a mediator between interpersonal mindfulness and teachers' subjective well-being, which, in turn, facilitates better classroom discourse [39].

Complementing structured dialogue, open-ended questioning techniques — such as asking *how?*, *why?*, and *what if?* — have proven effective in deepening mathematical reasoning. These questions align with dialogic principles, encouraging exploration and scaffolding problem-solving strategies to cultivate independent thinking [28, 87]. Empirical studies illustrate their versatility: for example, seventh-grade students in Thailand studying polynomials developed critical thinking skills through open-ended tasks tailored to their achievement levels, with some lower-achieving students demonstrating notable gains [85]. Additionally, open-ended strategies can enhance self-efficacy and even engage parents in mathematical learning [126], further extending their impact beyond the classroom.

However, the effectiveness of open-ended questions and conversational teaching depends on the structural constraints imposed by the education system. Educators seeking to integrate these approaches must consider several factors: class size, the range of student ability levels, available instructional time, and the duration of student-teacher interactions (e.g., a single semester vs. multiple years). Additionally, the presence of high-stakes standardized exams may limit the extent to which such approaches can be implemented. These systemic factors create disparities in the feasibility of dialogic teaching, granting some schools and countries a significant advantage in adopting these methods effectively.

2.4 Modern Pedagogical Innovations

In addition to the education challenges, global migration patterns are reshaping the classroom demographics and the job market, prompting universities to reconsider their curricula [82]. With over 280 million people (around 3.5% of the world's population) migrating globally in 2024, institutions face increasingly diverse student bodies, including many with multilingual and multicultural backgrounds [79]. This diversity requires innovative teaching methods that would not just accommodate varying educational experiences but also address the increased need for proper communication skills.

In response, many universities worldwide have introduced programs to bridge gaps in students' prior learning. In Canada, for example, several institutions have implemented *university preparation* programs aimed at supporting students from diverse backgrounds or with special needs. Thompson Rivers University's *Department of University and Employment Preparation*, for instance, offers courses designed to "accommodate students' life experience and learning styles" [127]. While McMaster University does not have a dedicated program, individual departments have taken initiative; the Department of Mathematics offers MATH 1K03: *Advanced Functions* for students who have not completed the equivalent of Grade 12 *Calculus and Vectors* or *Advanced Functions*, thereby ensuring that students are adequately prepared for future university-level mathematics courses.

Beyond foundational knowledge, modern pedagogy increasingly prioritizes scientific communication as a core competency. This focus is especially critical in professional programs, where curricula are intentionally designed to align with workplace demands. For example, the University Medical Centre Hamburg-Eppendorf (Germany) integrated communication training into its medical curriculum in 2008. A longitudinal study found that students who participated in role-playing exercises with simulated and real patients reported significantly higher self-assessed communication competence, enabling them to refine these skills for clinical practice [48]. Similarly, the University of Technology Malaysia (UTM) implemented an active learning model for engineering mathematics courses in 2006. Though students initially struggled with collaborative activities, the structured emphasis on articulating technical concepts gradually improved their confidence and clarity — skills directly applicable to engineering teamwork and client interactions [106].

Reasoning skills are equally critical, particularly in STEM fields. An argument-driven inquiry model was introduced at Tallahassee Community College, U.S., in a chemistry laboratory course, requiring students to produce written reports and participate in peer-reviewed oral argumentation. This approach led to measurable improvements in both written and oral communication, underscoring the interplay between critical thinking and effective scientific discourse [135].

These principles extend to mathematics education, where specialized pedagogical innovations cater to both industry and teacher training. Adaptive learning platforms and simulation-based problem-solving were demonstrated to enhance mathematical instruction in specialized classes, creating personalized pathways that accommodate diverse backgrounds while maintaining rigour [13]. In parallel, active partnerships between educational institutions and industry have led to curricula that integrate real-world problems into mathematics courses. This integration helps bridge the gap between abstract theoretical concepts and their practical applications, while teacher training programs increasingly emphasize pedagogical strategies designed to contextualize mathematics for learners in technical and vocational tracks [50].

Collectively, these initiatives underscore the importance of tailoring pedagogy to both student needs and career trajectories. Whether through communication training in medicine, reasoning-focused labs in chemistry, or industry-aligned mathematics curricula, universities are increasingly adopting strategies that prepare graduates for the interdisciplinary, competency-driven demands of modern workplaces.

However, a noticeable gap remains in the literature: while extensive research exists on general communication skills and mathematical reasoning, few studies have addressed specialized mathematics communication. For instance, research on how mathematics students develop discipline-specific communication skills (e.g., presenting proofs in academia, explaining abstract concepts to non-specialists in teaching contexts, translating mathematical results for industry stakeholders, etc.) is underrepresented compared to analogous studies in fields like medicine or engineering. This gap highlights the need for further inquiry into targeted communication strategies that meet the distinct challenges of mathematics education and professional practice.

2.5 Self-Concept

Students may be reluctant to collaborate for various reasons, such as previous unsuccessful experiences, busy schedules, or personal preferences. To help students become more accustomed to collaboration and group work, teachers and professors have employed various methods, including direct instruction on communication skills and establishing group rules. A study from 2022 explored self-reflection as a tool to enhance communication skills [20]. While minimally impactful on teacher workloads, self-reflection's reliance on personal introspection limits its accessibility under universal design principles [81]. This highlights a tension between individualized reflection and inclusive pedagogy, particularly for neurodiverse learners who may require structured scaffolding to articulate their thoughts, a challenge worsened by the inherently personal nature of introspection.

These challenges underscore the interplay between internal and external modes of communication. Personal voice — the internal narrative through which students structure their reasoning — plays a pivotal role in mathematical development [28]. A a result, learners with underdeveloped communication skills (e.g., limited vocabulary or unstructured self-dialogue) may struggle to articulate ideas internally, hindering their ability to refine and advance their reasoning. Conversely, students who cultivate a coherent internal voice gain clarity not only in their understanding of mathematical concepts but also in their self-assessment. This metacognitive alignment — where clear thinking fosters accurate self-concept — strengthens both confidence and competence. Thus, the interplay between internal communication (self-talk) and external articulation underscores their inseparability: one cannot thrive without the other.

Mathematics self-concept (MSC), also known as *mathematics domain*, is defined as a student's perception of their own mathematics skills [120]. This self-concept has been shown to correlate with mathematics achievement [18], starting as early as elementary school [29]. Conversely, verbal self-concept, which pertains to one's understanding of their verbal or oral language skills, shows no direct relationship with math performance, a phenomenon explained by Marsh's Internal/External Frame of Reference Model [74]. This model posits that students evaluate their math abilities independently of verbal skills, creating compartmentalized self-perceptions. However, this dichotomy is complicated by cross-cultural research showing that self-concepts are socially constructed and context-dependent [99]. In fact, students from collectivist cultures (e.g., China, Russia, Ukraine, Japan, India, Mexico, etc.) may prioritize group harmony over individual articulation, which can affect their willingness to communicate in Western-style classrooms [82]. This might naturally affect their learning as, for instance, communication tasks in mathematics (e.g., explaining proofs) demand the integration of logical and linguistic competencies, a process mediated by cultural norms.

Building on this understanding of self-concept, it becomes crucial to explore how external factors, such as cultural norms and societal roles, further influence MSC calibration. In egalitarian, gender-flexible societies (e.g., Nordic countries), students' MSC aligns closely with actual achievement, fostering a virtuous cycle where clear self-assessment enhances both confidence and performance [23]. Conversely, in hierarchical or gender-rigid cultures (e.g., East Asia), cultural emphasis on humility or group harmony distorts self-perceptions [136]. This miscalibration reflects collectivist norms that prioritize contextual adaptation over individual self-enhancement, suppressing explicit MSC expression to maintain social cohesion.

These gendered dynamics are not innate but culturally contingent. For instance, 2009 PISA data shows boys in Confucian-heritage cultures (China, Korea, Japan, Vietnam, Singapore) report higher MSC despite similar performance to girls, while Nordic (Denmark, Finland, Iceland, Norway, Sweden) gender gaps in MSC are negligible [67]. This parallels broader findings that math anxiety and self-efficacy are universal predictors of achievement, but their gendered expression hinges on cultural narratives [22]. In rigid gender environments, boys' overconfidence correlates with lower achievement, mirroring the *confidence gap* observed in girls [144]. However, systemic equity in egalitarian societies mitigates these disparities, underscoring the plasticity of self-concept under differing sociocultural conditions.

These findings underscore that while cultural norms and societal roles significantly influence MSC calibration, they are not the only external factors at play. In parallel, a growing body of research highlights the pivotal role teachers have in shaping students' self-concept and their ability to communicate mathematical reasoning. Teachers' expectations [51, 125] and classroom practices [66, 110] not only inform students' internal self-assessments but also guide how effectively they articulate their understanding.

The current research focuses on the interaction between both types of achievement, using self-concept as a tool for evaluating both skills through surveys and interviews. Mathematics self-concept is often more comparison-based in individualistic cultures (e.g., Canada, Germany, the Netherlands, the United States, etc.), prompting interventions like the talking circle — a collectivist-oriented activity — to broaden self-assessment frameworks [129].

The interplay of all the factors discussed above underscores the need for interventions that dismantle comparison-driven hierarchies while nurturing communal articulation in Canada's individualistic educational context. Traditional self-assessment frameworks, rooted in individualistic competition, often worsen MSC anxieties by prioritizing ranking over growth (e.g., norm-referenced grading that emphasizes relative standing rather than mastery) [75]. Conversely, communal practices like *talking circles* leverage collectivist strengths to recalibrate self-concept through shared reflection. By decentralizing hierarchical classroom dynamics, these practices disrupt stratified participation patterns, based on either linguistic background, socioeconomic status, or prior math achievement, and foster equitable discourse [54], offering students safe spaces to externalize internal reasoning without fear of judgment.

In Canada, this shift aligns with Indigenous methodologies that redefine mathematics as a collective, culturally situated practice. *The Truth and Reconciliation Commission's Calls to Action* (62–65) spurred efforts to integrate Indigenous pedagogies, such as talking circles, into STEM education [117]. These initiatives build on the principle of Indigenous mathematics, which rejects Eurocentric binaries between *formal* and *informal* knowledge, instead framing mathematics as a dynamic, multi-generational endeavor shaped by diverse communities — from physicists to hobbyists [121]. Indigenous knowledge systems have always been mathematical, emphasizing relationality and holistic problem-solving over competitive individualism [42].

Talking circles exemplify this philosophy in action. Structured dialogues, guided by protocols like a talking piece, democratize participation and align with Indigenous principles of learning from place — connecting mathematical reasoning to local contexts and communal values [88]. For instance, talking circles in mathematics classrooms were shown to reduce participation gaps tied to gender or cultural background, enabling students to reframe perceived weaknesses as collaborative learning opportunities [54]. By situating mathematics within a relational rather than hierarchical framework, these circles counteract the isolating effects of comparison-based MSC, fostering metacognitive alignment through collective articulation.

2.6 High-Achieving Students

The inclusion of students identified in the literature as *gifted* or *mathematically promising* in this study warrants justification, particularly given the equity-oriented framing of such labels in contemporary research [68]. Learning opportunities are among the most crucial factors in fostering and realizing human intellectual potential, and a key feature of an effective learning environment for these students is the presence of *mathematical challenges*, a specific type of problem known as *non-standard* problems. These students, often termed *high-achieving*, are distinguished not merely by intrinsic ability but by their engagement with *non-standard* problems — tasks that defy algorithmic resolution and demand creative reasoning. Stolyar defines non-standard problems as those where “students do not know in advance either the method of solving or the necessary underlying material" [122]. Unlike routine exercises, these problems necessitate *curricular-based mathematical discourse* (applying known concepts in novel ways) and *meta-discourse about proof* (critiquing logical validity), both of which deepen reasoning skills [138].

Khinchin's assertion that problem-solving is an "act of creativity" [64], resonates with empirical findings. Studies using Multiple Solution Tasks (MSTs) reveal that gifted students outperform peers in fluency, flexibility, and originality, particularly for insight-based problems [69]. For instance, when tasked with proving a theorem through multiple approaches, gifted students demonstrate greater capacity to synthesize geometric, algebraic, and combinatorial perspectives — a hallmark of creative reasoning. This aligns with Sfard's communicative framework [113], where creativity emerges through iterative dialogue between internal exploration and external validation.

Critically, mathematical promise thrives in environments that balance challenge with community. This study focuses on students engaged in intersectional mathematical activities — competitions, Olympiads, and math clubs — that bridge formal curricula and self-directed learning [68]. Participants in Group 3, all national-level Canadian competitors, exemplify this dual engagement. These students are actively preparing to join the national team for international olympiads, such as the *International Mathematics Olympiad* (IMO), *European Girls' Mathematics Olympiad* (EGMO), *Asian-Pacific Mathematics Olympiad* (APMO), and others. These contests emphasize the nurturing of the following objectives, which inherently "fuse" reasoning and communication [118]:

- Clear and logical presentation, i.e., rigourous articulation of proofs under time constraints.
- Tenacity, i.e., persistence through unstructured problems.
- Academic sincerity, i.e., ethical rigour in attributing ideas and avoiding heuristic shortcuts.

While gifted students exhibit stronger reasoning skills than peers [10, 119], their communication abilities remain underexplored — a gap this study addresses.

Notably, competition participation fosters unexpected psychosocial benefits. Training camps and team contests create *intellectual communities* that counterbalance the isolation [19] and bullying risks often experienced by gifted students in conventional classrooms [76, 95]. Training programs bridge classrooms and academic spheres, exposing participants to mathematicians and peers through lectures, collaborative problem-solving, and mentorship [19].

The *Moscow Math Olympiad*'s (MMO) legacy exemplifies this: since 1935, its *Small Faculty of Mechanics and Mathematics* has connected Grades 5-11 students with professors, who deliver lectures, and university students, who lead study sessions [64]. Similar approaches can be seen worldwide, such as the University of Toronto *Math Circles* for Grades 9-12 students and *Canada/USA Mathcamp*, where students are offered opportunities to work with university professors and students on topics not covered in the standard curriculum, to “study with mathematicians who are passionate about their subject”, and to “make friends”.

In Canada, specialized invitation-only camps for top students (based on their performance in national competitions such as the *Canadian Open Mathematics Challenge* (COMC) and *Canadian Mathematical Olympiad* (CMO)) are organized by the

Canadian Mathematical Society alongside open local summer math programs usually operated by local universities and organizations. To address systemic inequities, many of these camps now reserve spots for female students, acknowledging barriers like underrepresentation and confidence gaps that persist in competitive mathematics [72, 80, 124, 144]. This equity-driven approach not only ensures diverse participation but also enriches the intellectual community by bringing together students from varied backgrounds, each contributing unique perspectives to mathematical discourse.

Confidence dynamics further illustrate this interplay. While initial Olympiad training may temporarily lower self-assurance due to exposure to higher standards, longitudinal studies show sustained participation stabilizes confidence through mastery experiences [77]. This aligns with Bandura's self-efficacy theory: repeated success in articulating solutions to peers and judges reinforces both technical competence and communicative fluency [6, 128]. Nonetheless, variability persists; Taiwanese Math Olympians exhibited stark differences in oral expression, underscoring the need for deliberate communication training even among the "elites" [143].

By examining students who navigate both classroom and competition contests, this study illuminates how structured challenge and community shape the dual development of mathematical reasoning and communication — the core skills critical for nurturing adaptable, articulate problem-solvers. Although much research has focused on reasoning skills, this work addresses the methodological gap in assessing mathematical communication, thereby contributing to a more holistic understanding of high-achieving students' development.

Chapter 3

Methodology

This study employs a mixed-methods design, integrating quantitative analysis of survey data (including both closed- and open-ended responses) and qualitative thematic analysis of interviews to investigate how students perceive the relationship between mathematical reasoning/proof skills and their mathematical communication abilities. By triangulating data from surveys, interviews, and observational insights from a talking circle, the research addresses gaps in understanding how diverse educational contexts shape math reasoning and communication competences.

3.1 Participants

This study involved 44 participants across three groups selected to represent distinct stages of mathematical development:

Group 1: Undergraduate students enrolled in MATH 1C03: *Introduction to Mathematical Reasoning* course (Fall 2024 semester, McMaster University, $n = 22$), with 5 students completing the interviews. Only 2 students participated in both pre- and post-course surveys. The pre/post comparisons excluded 4 participants who completed only the post-course survey.

Group 2: Upper-level mathematics students (McMaster University, $n = 8$), with 5 participating in interviews.

Group 3: High-achieving (gifted) high school students who participate in high-level mathematics competitions across Canada ($n = 12$), with 2 completing interviews.

Groups are hereafter denoted as *Gr1*, *Gr2*, and *Gr3* in tables and figures.

The study focused on students actively engaged in mathematical reasoning and communication practices, selecting three participant groups to represent distinct stages and contexts of this engagement. This stratification, conceptualized through the task categorization framework in [138], allows comparison across developmental trajectories, from novice to expert-level exposure.

- **Group 1**: Prerequisites for MATH 1C03 do not include any proof-based courses, and hence, most participants in this group, very likely, lack prior formal training in mathematical reasoning, having primarily encountered algorithmic or procedural tasks in earlier mathematics education. By focusing on this group, the study tracks how structured exposure to logical reasoning and proof-writing during the semester affected their perceptions of mathematical communication.

- **Group 2**: These students regularly engage with advanced mathematical concepts requiring both deep conceptual understanding and precise communication, for example, in coursework or research collaborations. Unlike Group 1, their experiences reflect sustained immersion in mathematical discourse, offering insights into how prolonged exposure to proof-intensive environments shapes communication skills.

- **Group 3**: Participants in this group encountered proof-based reasoning much earlier than Gr1 or Gr2, often through extracurricular competitions emphasizing *curricular-based reasoning* under time constraints. Their habitual engagement with complex, non-routine problems and the need to articulate solutions clearly in competition settings provide a contrasting perspective on how early training in mathematical communication influences self-perceived competence.

By comparing these groups, which are differentiated by their stage of exposure (introductory to sustained), learning context (classroom to extracurricular), and communication demands (structured proofs to time-pressured explanations), the study illuminates how varied developmental pathways shape mathematical engagement.

3.2 Data Collection

Data was collected through three primary methods: surveys, semi-structured interviews, and observational insights from a weekly talking circle.

Participants were recruited through a combination of classroom announcements (for Group 1), social media postings on *Discord* (server of the *McMaster Math & Stat Society*), institutional platforms (*Avenue to Learn*), and snowball sampling. This strategy ensured diverse representation across the three target groups.

A survey served as the primary data source, administered online via *Google Forms*. The survey began with an informed consent form, and the subsequent sections gathered demographic information, self-assessments of mathematical reasoning and communication skills using semantic differential scales, and open-ended prompts. More about the survey structure can be found in Section 3.2.1.

Moreover, semi-structured interviews were conducted with a subset of 12 interested participants across groups. Prior to each interview, oral consent was obtained, and sessions were audio-recorded and transcribed without including any identifying information. After that, the recordings were deleted; the interview transcriptions were kept until the end of April 2025. To mitigate potential risks, participants received a list of mental health support services (e.g., counselling, peer support), and the researcher adopted a flexible questioning approach, omitting topics if participants expressed discomfort. Interviews explored perceived linkages between reasoning and communication skills, contextualizing survey responses with narratives about challenges in articulating proofs or collaborative problem-solving. More about the interview structure can be found in Section 3.2.2.

Additionally, students in Group 1 were offered an opportunity to join a weekly talking circle, a group discussion aimed at supporting them through the course. Sessions were voluntary, unrecorded, and attended by 2–6 students (with 2 regular attendees), fostering a space for discussing course-related challenges, addressing students' anxieties, and celebrating triumphs. While no direct quotes were collected, the researcher documented general themes to complement survey and interview findings.

3.2.1 Survey Content

The survey, detailed in Appendix A, comprises six sections designed to gather demographic information, assess mathematical backgrounds, and evaluate written communication skills. Prior to engaging with research-specific questions, participants reviewed a *Letter of Information* and provided informed consent (see Section 3.5 for ethical protocols). The survey structure aligns with best practices in educational research design, emphasizing ethical transparency, participant agency, and triangulation of self-reported and observational data [27]. The sections are outlined below with Q denoting the question number:

1. Consent and Inclusion Criteria (Questions 1-2):

   First, participants were given a choice over how their responses would be used (quoted directly or paraphrased), addressing possible confidentiality concerns. Due to Section 2 being dynamically tailored to participant groups, Question 2 categorized respondents to ensure appropriate branching.

2. Background Information (Questions 3-7):

   Demographic and background questions contextualized participants' mathematical reasoning and communication skills, enabling analysis of potential correlations with developmental, educational, and motivational factors. For all groups, age (Question 3) was included to account for age-related neuroplasticity effects on cognitive skill acquisition [98], while time spent on deliberate practice (Questions 4-6 for Group 3) conceptualized Ericsson's theory of expertise development through structured, goal-oriented training [38].

For university participants (Groups 1 and 2): Parental education level (Question 5) was included based on evidence that academically educated parents engage in *academic socialization*, modeling abstract reasoning and metacognitive dialogue that indirectly shapes communication skills [145]. Intrinsic motivation (Question 6) was assessed through self-reported drive to engage with mathematics, grounded in Self-Determination Theory's emphasis on autonomous motivation as a predictor of deep learning [1, 2, 43, 92]. The study admits that there are more potential variables, which might affect the results, such as peer relationships, personal traits, resilience, and more.

For high school participants (Group 3): Questions 4-5 distinguished whether participants' skills emerged from formal curricula or extracurricular engagement, reflecting research on informal STEM programs as catalysts for problem-solving proficiency [91]. Study strategies (Question 7) probed metacognitive habits like self-questioning, aligning with Zimmerman's framework of self-regulated learning cycles (planning, monitoring, reflection) [146].

Additionally, participants from Group 1 were administered Questions 7* and 7** for tracking of their longitudinal development. The first noted question, 7*, generated unique identifiers linking pre- and post-course responses. Question 7** optionally collected contact information of participants interested in participation in the Talking Circle, with anonymization protocols detailed in Section 3.5.

3. Mathematical and Communication Experience and Background (Questions 8-24):

   Self-assessments of mathematical proficiency (e.g., confidence in problem-solving) provide preliminary insights into participants' perceived competence. While self-reports are subject to bias [63], they offer a baseline for triangulation with performance-based data from interviews (Appendix B).

4. Application and Understanding of Mathematical Concepts (Questions 25-30):
   Participants were asked to explain foundational mathematical concepts to a hypothetical group of children, a task grounded in pedagogical theories emphasizing *explanation* as a measure of deep conceptual understanding [21]. This mirrors the *Feynman technique*, where simplifying complex ideas reveals gaps in knowledge [107].

5. Additional Participation (Questions 31-32):
   Contact information was collected for voluntary follow-up interviews (see Section 3.5 for anonymization protocols and participant withdrawal rights).

6. Course-Specific Questions (Group 1, post-course survey only, Questions 33-37):
   Post-course questions evaluated the impact of MATH 1C03 on perceptions and skill development, drawing on backward design principles [140] to assess alignment between course objectives and outcomes.

3.2.2 Interview Content

The semi-structured interview protocol (Appendix B) complemented survey data by contextualizing participants' experiences and perceptions of communication-reasoning interdependencies through narrative inquiry. This design prioritized depth over breadth, allowing participants to articulate subtle connections between mathematical reasoning and communication that surveys alone could not capture [44].

To mitigate power imbalances inherent in researcher-participant dynamics [52], interviews began with rapport-building questions. This ethical scaffolding fostered psychological safety, encouraging candid reflections on technical topics [14]. Subsequent phases systematically explored four domains:

1. Communication Development (Questions 1-5):

   To distinguish between general and domain-specific communication, initial questions probed participants' perceptions of skill acquisition broadly before narrowing to mathematics. This progression mirrored Vygotsky's *scientific vs. spontaneous* concepts [133], testing whether participants viewed mathematical communication as transferable or context-bound.

2. Influence of Mathematical Reasoning (Questions 6-10):

   Questions 6–10 investigated perceived causal relationships between mathematical reasoning activities and communication habits.

3. Real - World Applications (Questions 11-17):

   Drawing on situated learning theory [3], these questions elicited narratives about collaborative problem-solving and career-aligned skills. This phase emphasized boundary-crossing between academic and professional environment, contextualizing communication as a transferable competence.

4. Mathematical Problem-Solving Demonstration (Questions 18-20):

   Participants solved a practical problem, adopting Schoenfeld's metacognitive interview model [111] to surface tensions between internal reasoning and external articulation. The problem's simplicity (area division) ensured accessibility, allowing focus on communicative processes rather than computational difficulty.

To preserve narrative authenticity, the researcher employed responsive interviewing: rephrasing questions or asking for clarifications, omitting sensitive topics, and prioritizing participant-led discourse.

3.2.3 Talking Circle

The talking circle, rooted in Indigenous traditions of egalitarian communication [18], was adapted to create a supportive environment for students in MATH 1C03 during the Fall 2024 semester. Weekly sessions (45–55 minutes) were offered voluntarily to students who completed the pre-course survey, with attendance ranging from 2 to 6 participants. To prioritize trust and confidentiality, sessions were not recorded, fostering open dialogue free from concerns about privacy. This approach directly addressed collaboration challenges outlined in the self-concept literature (Section 2.5), where reluctance to engage in group work often stems from comparison-based anxieties or past negative experiences.

Guided by pre-established protocols (Appendix C), the talking circle emphasized informal, reflective dialogue to counteract the isolating effects of traditional self-reflection methods critiqued in Section 2.5. By decentralizing hierarchical classroom dynamics, the structure aimed to reduce pressures tied to mathematics self-concept, such as fear of judgment or peer comparison. Participants shared experiences navigating the course, discussing challenges in articulating proofs or collaborating on problems. This communal exchange fostered a sense of community, helping students reframe perceived weaknesses as shared learning opportunities rather than individual shortcomings.

The methodology aligned with principles of Indigenous pedagogy [8], incorporating symbolic elements like a "talking piece" to regulate equitable participation and rituals to demarcate the space from conventional academic interactions. These design choices bridged the gap between internal reasoning and external communication highlighted in the self-concept analysis, offering real-time insights into how collaborative dialogue shapes both mathematical confidence and verbal clarity. Qualitative themes from anonymized field notes were later triangulated with survey and interview data, contextualizing the immediate impact of the course on skill development.

3.3 Data Analysis

The mixed-methods design necessitated distinct analytic approaches for quantitative (survey) and qualitative (interview, talking circle) data, with full quantitative results tabulated in Appendix D.

Survey responses were analyzed to examine patterns in mathematical communication skills across participant groups and, for Group 1 (MATH 1C03 students), changes over time. Given the ordinal nature of Likert-scale responses in Questions 8-23, analyses prioritized frequency distributions and modal responses rather than parametric statistics, with results visualized through 100% stacked bar graphs. For the binary yes / no questions, Pearson's chi-squared tests were used to evaluate whether response patterns were independent of group membership.

For Group 1's longitudinal data, pre- and post-course comparisons employed Wilcoxon signed-rank tests, chosen for their appropriateness with ordinal, paired data. Cross-group analyses utilized Kruskal-Wallis tests as a nonparametric alternative to ANOVA, better suited for comparing central tendencies across groups without assuming interval-level measurement.

Responses to open-ended questions in the *Application and Understanding of Mathematical Concepts* section (Questions 31–36) were scored using a 6-point rubric assessing mathematical accuracy (2 points), accessibility of explanations (2 points), use of examples/analogies (1 point), and balance between justification and definition (1 point). These category scores sum to a single numerical total (maximum 36 per participant), allowing the data to be treated as interval-scale scores. Hence, the mean (average) scores and standard deviations were analyzed within and across cohorts to quantify skill progression.

The analysis acknowledges several limitations inherent in the data characteristics and sample size. While parametric tests are reported for rubric-based interval scores, nonparametric alternatives were prioritized wherever assumptions of parametric tests were questionable, particularly for ordinal Likert-scale items. All statistical outcomes, including those with marginal significance, are presented transparently in Appendix D to facilitate evaluation and reproducibility. This approach maintains methodological rigor while appropriately accommodating the nature of the collected data.

Qualitative data from interviews and talking circles were analyzed using Braun and Clarke's reflexive thematic analysis (RTA) [15], a flexible yet systematic approach that prioritizes researcher subjectivity as a tool for meaning-making. The process began with immersion in the data: interview transcripts and field notes from talking circles were read repeatedly to identify patterns in participants' narratives about mathematical reasoning and communication. MAXQDA software [131] facilitated open coding, where initial codes were assigned to segments of the text. These codes were neither predefined nor purely emergent; instead, they reflected an inductive-deductive balance, allowing themes to surface organically while ensuring alignment with the study's focus on self-concept and skill transfer.

Thematic development progressed iteratively. Codes were clustered into candidate themes (e.g., "negotiating internal vs. external communication"), which were then refined through recursive comparison with the dataset.

The resulting method was inductive, latent, and (critically) realist reflexive thematic analysis [16]. However, a degree of deductive analysis was used to ensure that the open-coding of the data, which directed the codes themselves, still contributed to producing the themes meaningful to the research questions in the study.

Talking circle data were analyzed similarly, with field notes coded for common themes observed during sessions. Unlike interviews, which focused on individual narratives, talking circle themes emphasized collective experiences, such as participants collaboratively reframing challenges in MATH 1C03 as shared learning opportunities.

The analysis remained intentionally exploratory, prioritizing themes that directly addressed the research questions. While no formal member-checking was conducted, recurring patterns across interviews and talking circles — such as participants describing improved confidence after explaining proofs to peers — strengthened the validity of interpretations.

3.4 Researcher's Background

As the primary instrument of data collection and analysis in this qualitative study, the researcher's positionality –shaped by her academic, pedagogical, and competitive experiences — directly informs the interpretation of findings. Reflexive engagement with this positionality is critical to acknowledging potential biases and contextualizing insights [58].

The researcher has served as a teaching assistant in the Department of Mathematics and Statistics at Thompson Rivers University from 2019 until 2023 and at McMaster University since 2023, assisting with undergraduate courses in calculus, linear algebra, proofs, and more. This role fostered firsthand awareness of disparities in students' mathematical communication skills, particularly their struggles to articulate logical reasoning in written and oral formats. Simultaneously, her experiences as a student in proof-intensive courses sensitized her to the cognitive and communicative demands of advanced mathematics, motivating this study's focus on skill development trajectories.

The researcher's participation in mathematics competitions began during her primary education in Ukraine, where she engaged in various local and national problem-solving contests. Since relocating to Canada, she has organized training for high school competitors and judged national level competitions. This dual perspective — as both competitor and mentor — revealed stark contrasts between competition-driven learning (emphasizing concise, creative communication under time constraints) and classroom pedagogy (prioritizing procedural mastery). These observations underpinned the study's inclusion of high school competitors (Group 3) as a distinct cohort.

3.5 Ethical Considerations

This study was conducted in strict compliance with the ethical principles outlined in the Tri-Council Policy Statement (TCPS2) and received formal approval from the McMaster Research Ethics Board (REB Project 7096). Ethical safeguards were integrated at every stage of the research process to prioritize participant welfare, autonomy, and privacy.

To promote inclusivity, all materials — including surveys, interview scripts, and consent forms — were designed to be culturally sensitive and accessible. Acknowledging the non-representative sampling strategy, the study explicitly recognizes that findings reflect the perspectives of students in specialized educational contexts (introductory proofs courses, competition settings, and advanced mathematics programs) and may not generalize to broader populations. By design, participants were not randomly selected but recruited based on their enrollment in specific courses or competition involvement. This purposeful sampling limits generalizability but ensures alignment with the study's focus on specialized mathematical training contexts. The researcher's positionality — as a teaching assistant and competition organizer — was reflexively documented to contextualize potential biases in data interpretation (see Section 3.4).

Participants' consent was obtained through two mechanisms prior to their involvement in each component of the study:

- Digital consent: For online surveys, participants confirmed agreement via a clickable button after reviewing the *Letter of Information*.

- Oral consent: For interviews and talking circle participation, consent was verbally obtained and documented by the researcher before proceeding, adhering to REB guidelines for non-written consent in low-risk studies.

Participants retained the right to withdraw at any stage without penalty. Two participants from Group 1 opted to have their quotes paraphrased rather than directly cited, further safeguarding their anonymity.

All data were anonymized to protect participant identities. Survey responses were designed to ensure no identifying information was collected, with the exception of Group 1 participants, whose participant-created codes were collected as per Question 7*. Additionally, emails were collected exclusively from participants (across all groups) who expressed interest in interview/Talking Circle participation or receiving the study results. These emails were stored separately from the data and were deleted at the conclusion of the study. Interview recordings were transcribed verbatim, during which all personal identifiers were systematically removed. Audio files were deleted immediately after transcription to eliminate risks of accidental disclosure. Transcripts used non-identifiable pseudonyms following the format "Group X Participant Y". Data were stored on password-protected devices with access restricted to the primary researcher and supervisory team.

The study posed minimal risk, as questions focused on academic experiences rather than sensitive personal topics. To address potential discomfort during interviews (e.g., anxiety about discussing mathematical struggles), participants received a list of mental health support resources, including McMaster University Student Wellness Centre contacts and crisis hotlines. The researcher completed McMaster's Mental Health Training for Graduate Students, enabling recognition of and appropriate response to signs of participant distress.

Chapter 4

Quantitative Analysis: Surveys

The survey provided in Appendix A consists of multiple sections focusing on different aspects of mathematical reasoning and communication. The sections include demographic information, participants' experiences with mathematical communication, and their practical communication (technical questions), whose quantitative analysis is provided below in the same order. This chapter presents an analysis using descriptive statistics of both closed-ended and open-ended questions from all three groups — students with limited experience in proofs (Group 1), upper-level math students (Group 2), and high school students participating in mathematics competitions (Group 3). Afterwards, the responses of two participants from Group 1, who completed the study both early in the *Introduction to Mathematical Reasoning* course and after the completion of the course, are compared.

4.1 Background Questions

This section outlines the demographic and academic backgrounds of the three participant groups, highlighting key contrasts in age, educational history, and extracurricular engagement that contextualize their performance trends in the study.

The Group 1 consists predominantly of younger participants, with 74% aged 18 or younger, 16% aged 19, and minimal representation from older students (one participant each at 20 and 21 years). A majority (84.2%) graduated from Canadian high schools, aligning with standardized curricular exposure considered in the study, while 16% had different background. Notably, 89.5% are not first-generation university students, indicating familial academic experience, while 16% are first-generation students. Crucially, 56.6% reported no participation (or no specific memory of) in participation in high-level math competitions. These demographics suggest a cohort with age homogeneity, standardized Canadian academic preparation, and limited exposure to advanced math training, positioning them as a group likely reliant on foundational curricula rather than specialized problem-solving frameworks.

Group 2 is characterized by older participants, with 40% aged 25 or older, followed by younger cohorts aged 20–24 (60%). A majority (60%) graduated from Canadian high schools and a majority (70%) reported no involvement in high-level mathematics competitions. Notably, none of the students in this group are first-generation university students, indicating fairly universal familial academic experience. The absence of first-generation students and low competition participation suggests a moderately international and more mature cohort with stable academic support systems but limited competition training, also positioning them as relying on formal education rather than advanced problem-solving frameworks.

Group 3 comprises of participants primarily in Grade 10 (50%), with smaller proportions in Grade 11 (25%), Grade 12 or above (16.7%), and Grade 9 or below (8.3%). A majority (66.7%) are enrolled in advanced math courses at school, though 33.3% lack such formal curricular engagement. Strikingly, 91.7% participate in extracurricular advanced math programs, highlighting a strong commitment to math beyond school requirements. Competition experience is robust: 66.6% have engaged in math contests for at least four years, with 33.3% exceeding 5 years, while smaller fractions report 2–3 years (16.7%) or 1–2 years (8.3%). This group's profile reflects high extracurricular involvement and extensive competition exposure, contrasting with the moderate enrollment in school-based advanced courses. The data suggests a cohort deeply invested in math enrichment, leveraging external programs to supplement formal education, and possessing significant experience in competitive problem-solving frameworks.

4.2 Skill-Based Questions

The survey contained two back-to-back sections assessing participants' experiences and background in mathematics (Questions 8–13) and mathematical communication (Questions 14–23). These sections included both yes / no questions and 5-point semantic differential items. Given the ordinal nature of Likert-scale data, where the intervals between response categories cannot be assumed equal, the findings are presented through frequency distributions, showcasing the percentage of responses at each scale point within each group. These distributions are visualized in 100% stacked bar graphs (Figures 4.1 and 4.2), allowing for clear comparisons of how responses clustered across groups. For yes / no questions (Questions 21 and 22), Pearson's

Chi-squared tests assessed group differences (results in Table D.2). The complete frequency data for all questions appear in Appendix D (Table D.1).

The first part of the analysis examines participants' mathematical backgrounds, focusing on their self-reported skills, familiarity with mathematical reasoning, and prior experience with proof-writing. The second part explores mathematical communication, including perceptions of its importance, self-assessed abilities in explaining concepts, and evaluations of training effectiveness. Throughout, the emphasis remains on identifying proportional trends and modal responses—the most frequently selected ratings—rather than imposing parametric statistical techniques unsuitable for ordinal data. This approach ensures that the interpretations remain grounded in the data's inherent properties while still revealing meaningful patterns across the three participant groups.

Mathematical Experiences

Considering mathematical background of participants (Question 8), the result varied across groups. In Group 1 (MATH 1C03 students), 68.75% rated their general math experience as *good* (4 on the scale), with only 6.25% selecting "enjoyable" (5). In contrast, Groups 2 (upper-level math students) and 3 (math contest participants) showed higher proportions of *excellent* ratings (20% and 41.67%, respectively). These trends, visualized in Figure 4.1, confirm the background of Groups 2 and 3, suggesting that they entered the study with more positive perceptions of their mathematical skills.

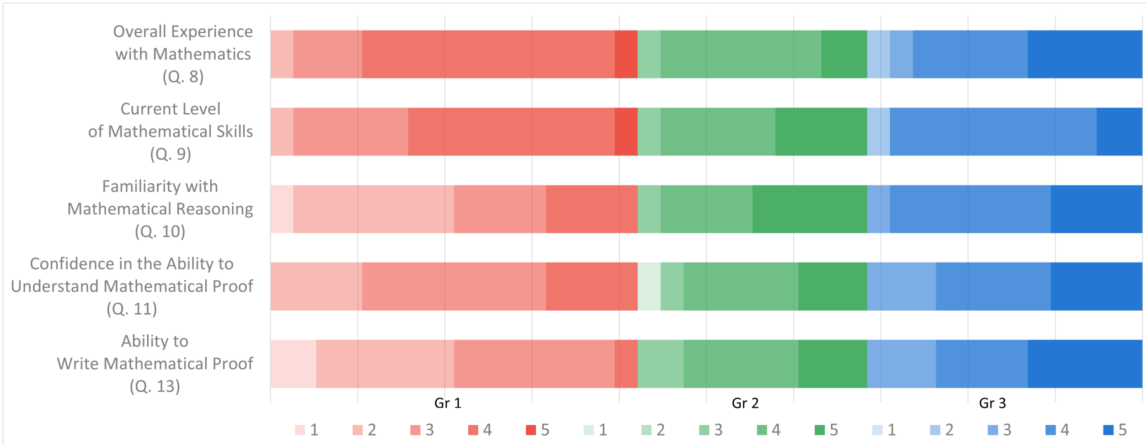

Figure 4.1: Self-Assessed Mathematical Reasoning and Proof-Related Competencies (Question 8-11, 13), Compared Across Three Participant Groups Using a 5-Point Likert Scale.

Familiarity with mathematical reasoning (Question 10) revealed stark contrasts: 43.75% of Group 1 selected *low* (2), while Groups 2 and 3 predominantly chose *high* (4) or *very high* (5) (90% and 91.66%, respectively). This aligns with prior exposure to proofs (Question 12), where only 50% of Group 1 had written a proof, compared to 100% in Groups 2 and 3. Similarly, self-rated proof-writing ability (Question 13) showed 43.75% of Group 1 selecting *neutral* (3), whereas Groups 2 and 3 skewed toward higher confidence (80% and 75% selecting 4 or 5).

Confidence in understanding proofs (Question 11) followed a similar trend: 50% of Group 1 reported *neutral* (3), while Groups 2 and 3 were more confident (83.33% and 75% selecting 4 or 5). Notably, 10% of Group 2 chose *very low* (1), possibly reflecting heightened self-awareness from advanced coursework.

Mathematical Communication Experiences

The interplay between valuing communication and executing it effectively emerges vividly in the data, with the accompanying graph (see Figure 4.2). Participants universally valued communication skills in mathematics (Question 14), with 50% of Group 1, 60% of Group 2, and 83.33% of Group 3 rating their importance as *very high* (5). Similarly, the importance of communication in academic life (Question 23) was widely acknowledged, with 81.25% of Group 1, 90% of Group 2, and 100% of Group 3 selecting *high* (4) or *very high* (5). Interestingly, while Group 2 showed a strong consensus about the value of communication skills in academic life, it also showed the highest variability about the role of communication in mathematics. This hints at mixed perspectives, possibly depending on personal preferences.

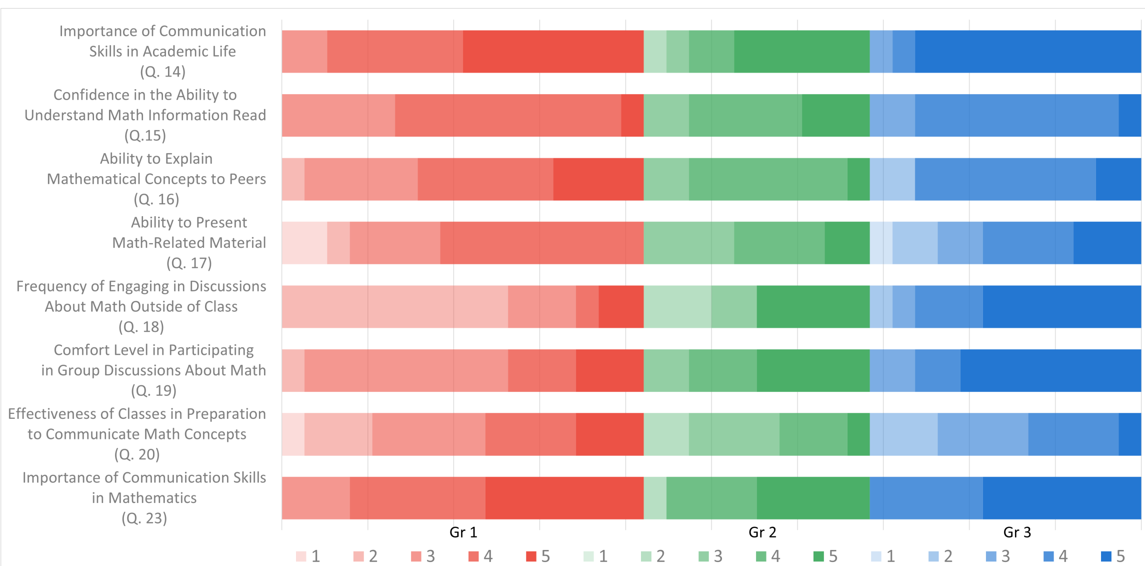

Figure 4.2: Self-Assessed Communication Skills in Mathematical Contexts (Questions 14–17, 19–20, 23), Compared Across Three Participant Groups Using a 5-Point Likert Scale.

Groups 2 and 3 participants reported consistently positive assessments of their ability to explain mathematical concepts (Question 16), with 66-70% selecting *good* (4). Group 1 showed greater diversity in self-ratings, which aligns with their differing mathematical backgrounds, where Groups 2 and 3 had more uniform training in advanced topics. Presenting math-related material (Question 17) revealed lower confidence: 56.25% of Group 1 chose *good* (4), but 12.5% selected *very low* (1), while Groups 2 and 3 had broader distributions (e.g., 20% of Group 2 selected *very high* (5), versus 25% of Group 3).

Critiques of math classes' communication training (Question 20) were notable. While 31.25% of Group 1 selected *neutral* (3), Groups 2 and 3 skewed toward lower ratings (40% and 33.33% chose *neutral*, respectively), with no group exceeding 25% for *good* (4). This aligns with participants' engagement in math discussions outside class (Question 18): 62.5% of Group 1 selected *rarely* (2), whereas 50% of Group 2 and 58.33% of Group 3 chose *very often* (5). Comfort in group discussions (Question 19) mirrored this divide: 56.25% of Group 1 rated themselves as *neutral* (3), while 50% of Group 2 and 66.67% of Group 3 selected *very high* (5).

A significant majority oa all participants (81.58%) admitted struggling to communicate mathematical ideas they understood well (Question 21), with no meaningful variation between groups ($\chi^2 = 0.0078$, $p = 0.996$). This widespread challenge contrasted with strong agreement (86.94%) that robust mathematical reasoning enhances communication skills (Question 22), again showing no group differences ($\chi^2 = 0.1601$, $p = 0.923$). This gap suggests that while respondents intellectually grasp the relationship between reasoning and communication, translating it into practice remains a challenge. The uniformity of responses across groups implies these patterns may reflect broader educational trends rather than cohort-specific experiences.

4.3 Technical Questions

The survey contains open-ended questions that were assessing participants' knowledge and understanding of main mathematical terms and their ability to communicate them to people who have never dealt with these terms before through writing (Questions 25-30). As noted in the methodology (see Section 3.2.1), technical questions were a way to check practical mathematical communication skills of the participants. The list of terms included the following:

Table 4.1: Definitions and Concepts Checked During the Survey (Questions 25-30).

	#*	**Concept**	**Grade**	**Definition**
Q_{25}	1	Variable	6-7	A letter or symbol used to represent an unknown quantity, a changing value, or an unspecified number [97].
Q_{26}	2	Function	8-9	A relation where a rule is defined to assign exactly one value to each element of a set (domain) of values [5].
Q_{27}	3	Equal vs. Equivalent	1-2 4-6	Having the same value [5]. Representing the same amount, number, comparison, etc [5].
Q_{28}	4	Exponential (product) rule	9-10	After considering the expanded forms of the given powers, the parenthesis can be dropped and the whole product can be considered as the expanded form of a single power [31].
Q_{29}	5	Points of intersection	K-3	The point, or coordinates, where two curves meet on a coordinate grid [97].
Q_{30}	6	Systems of equations	9-10	Finding the points of intersection of the curves determined by the equations [105].

Note: The questions have been renumbered for clarity and ease of reference in subsequent sections, with the new numbering format indicated by #.*

The evaluation of all 38 valid responses across three groups (MATH 1C03 students, upper-level math students, and mathematics competition high school students) based on four criteria — accuracy, accessibility, examples, and justification (see page 41) — reveals nuanced patterns in performance and consistency of practical mathematics communication skills. The analysis integrates quantitative findings (see Appendix D) with visual representations, including bar charts (see Figures 4.7 - 4.8) and a correlation heatmap (see Figure 4.8). All statistical data can be found in Appendix D, while this section presents numerical data, including mean scores (M), standard deviations (SD), and correlation coefficients (r), along with variables specific to statistical tests.

Note that in the following subsections, the term *average* refers to the arithmetic mean. Additionally, since the evaluated values can be interpreted as numeric grades, the statistical treatment (including means, standard deviations, and correlations) is appropriate.

Each bar graph displays the groups' average score for every question within a specific category, with horizontal lines indicating the overall category average score across all questions. This integrated approach highlights patterns, interdependencies, and statistically significant differences in mathematics communication skills of the study's participants.

4.3.1 Accuracy

Group 2 dominated with the highest average accuracy ($M = 94.79\%$, $SD = 11.03\%$), reflected in their consistently high trendline across questions, for example, 100% in both Questions 1, 2, and 5, as seen in Figure 4.3. Group 3 achieved moderate accuracy ($M = 84.03\%$, $SD = 18.38\%$), with scores clustering between 75% and 95%. Group 1 lagged ($M = 78.24\%$, $SD = 25.87\%$) with erratic performance (e.g., 36.11% in Question 1 vs. 94.44% in Question 2) visualized through wide error bands. Accuracy strongly correlates with accessibility ($r = 0.487$), suggesting groups excelling in one area often perform well in the other. Similarly, accuracy and justification exhibit a weaker but notable correlation ($r = 0.405$), implying that precise reasoning often accompanies thorough justification. This partially explains dual strengths of upper-level math students and stability of competitions participants.

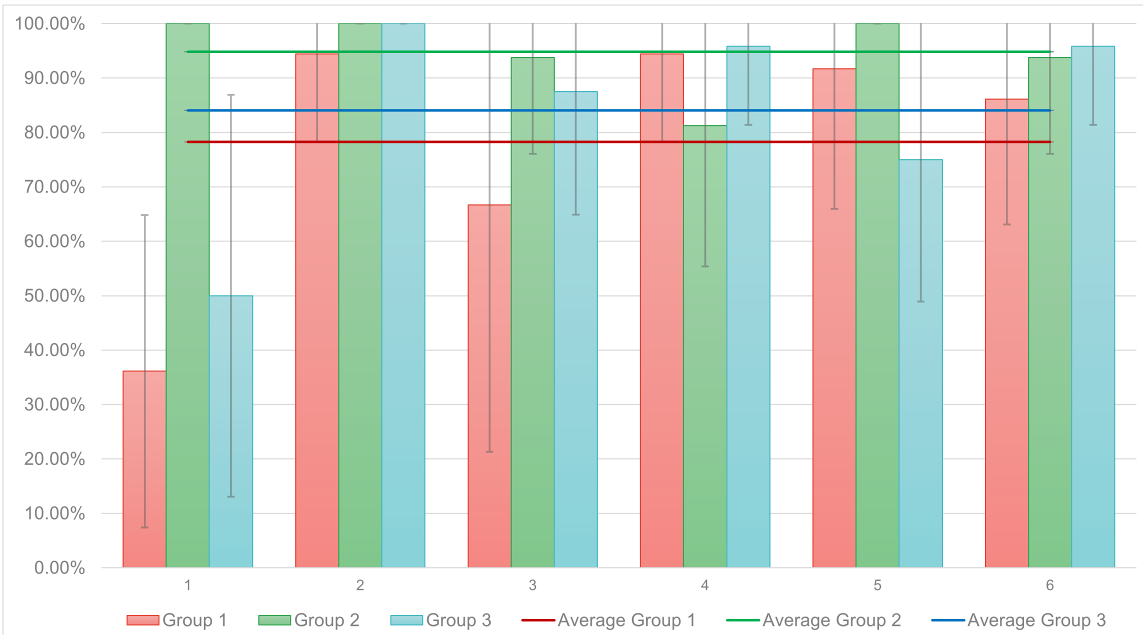

Figure 4.3: Participants' Accuracy Scores in Technical Questions (Questions 25-30) Across Different Groups.

4.3.2 Accessibility

Group 3 leads ($M = 87.5\%$, $SD = 22.91\%$), maintaining stable scores (e.g., 79.17% in Question 1 to 95.83% in Question 4), depicted as a smooth, high-lying trendline as seen on Figure 4.4. In contrast, Group 2 ($M = 84.38\%$, $SD = 23.06\%$) and Group 1 ($M = 82.41\%$, $SD = 28.72\%$) showed greater variability, with both groups experiencing sharp drops (e.g., Group 1's 66.67% in Question 3 and Group 2's 62.5% in Question 2). Although Group 3's responses were the most accessible overall, all groups performed relatively well in this criterion; indeed, the averages (82.41%, 84.38%,, and

87.50%,) are so similar that they largely obscure any substantive differences between the groups.

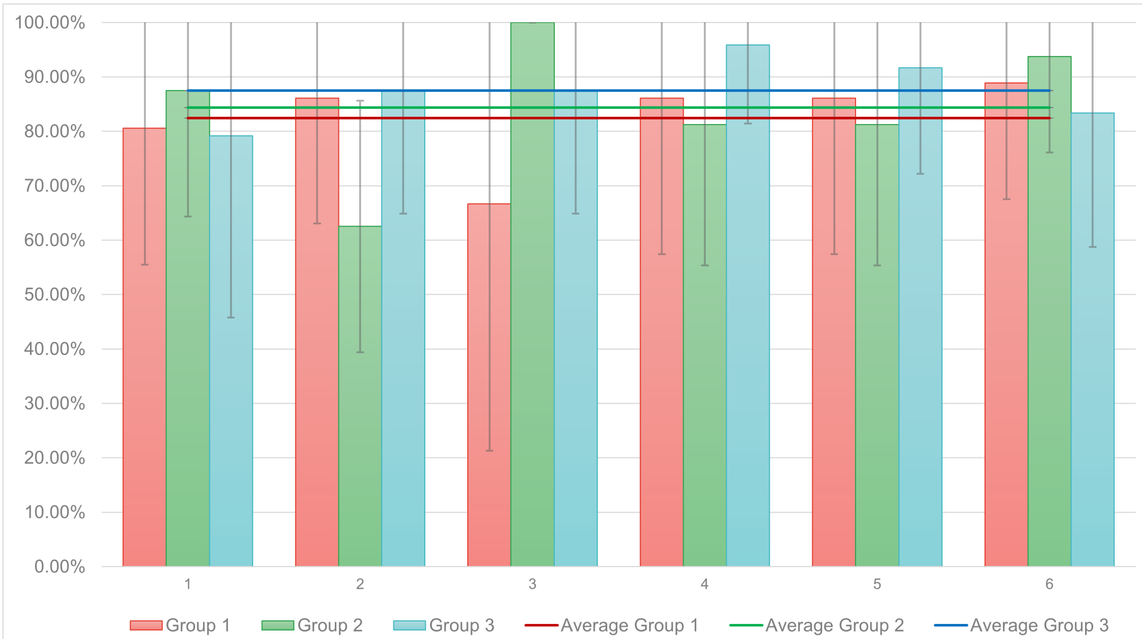

Figure 4.4: Participants' Accessibility Scores in Technical Questions (Questions 25-30) Across Different Groups.

4.3.3 Examples

Group 2 scored highest in the category (M = 33.33%, SD = 40.35%) but with extreme variability (0% in Question 4 vs. 62.50% in Question 5), visualized as a volatile line graph with dramatic peaks and valleys (see Figure 4.5). Group 3 (M = 23.61%, SD = 37.47%) and Group 1 (M = 20.83%, SD = 44.59%) struggled systemically, their bar graphs reflecting sporadic spikes (Group 1's 57.45% in Question 5) amid overall weakness. This suggests that Group 2 included more examples in their responses compared to the other groups. Use of examples show minimal ties to accuracy ($r = 0.136$) or accessibility ($r = 0.209$), underscoring its independence as a criterion.

The strongest inter-criteria relationship emerges between examples and justification ($r = 0.355$), hinting that groups providing examples may also deliver stronger logical justifications, though this connection remains modest.

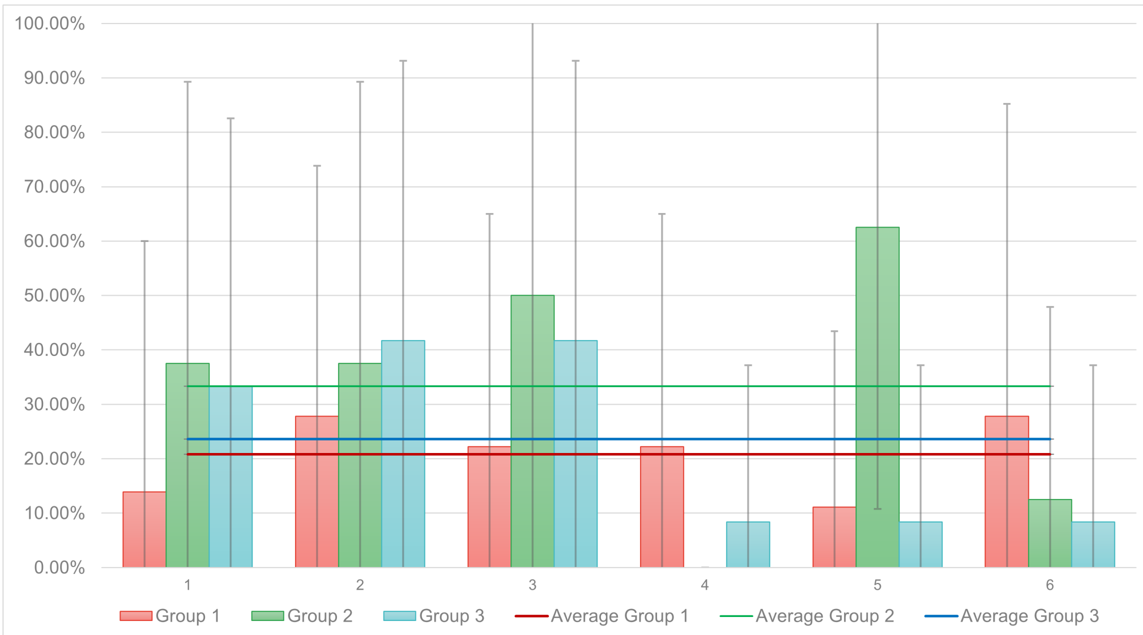

Figure 4.5: Participants' Example Scores in Technical Questions (Questions 25-30) Across Different Groups.

4.3.4 Justification

Group 2 achieved the highest overall performance (M = 72.92%, SD = 40.26%), achieving perfection in Questions 3 and 5 but collapsing to 62.5% in Question 4, as seen in Figure 4.6. Group 3 (M = 55.56%, SD = 39.64%) exhibited erratic fluctuations, starting at 25% in Question 1 and peaking at 100% in Question 6, but with unstable performance in between. Group 1 (M = 43.06%, SD = 44.93%) demonstrated a steady upward trajectory, rising from lower accuracy in earlier questions to

100% by Question 6, suggesting gradual improvement. The moderate correlation between justification quality and accuracy ($r = 0.405$), aligns with Group 2's peaks and Group 1's steady progression, emphasizing the role of precise reasoning in consistent outcomes.

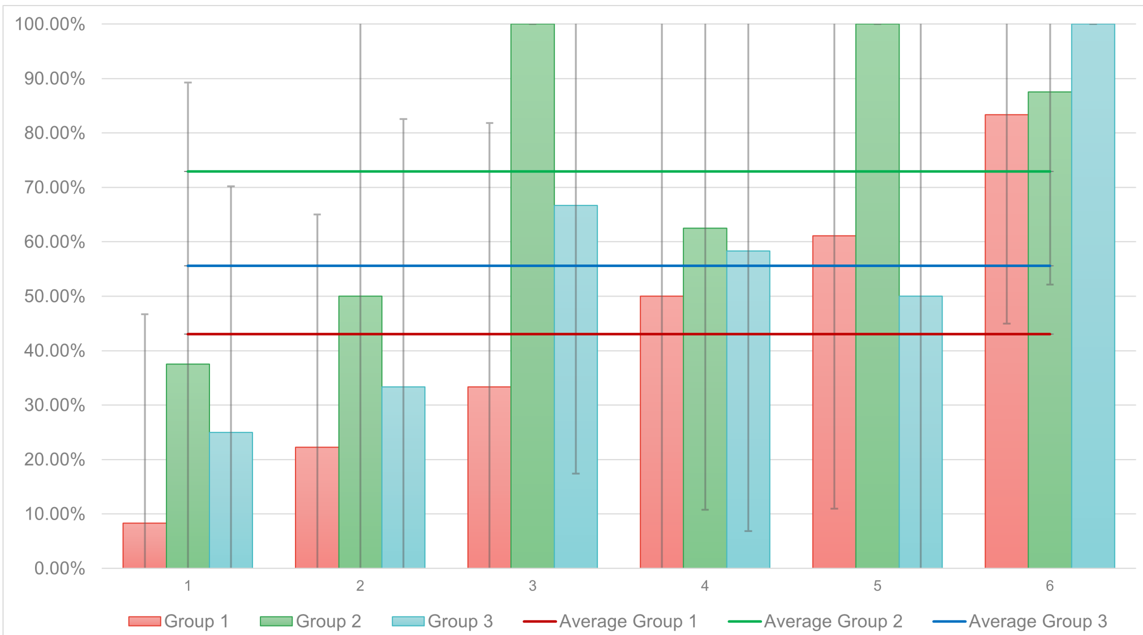

Figure 4.6: Participants' Justification vs. Definition Scores in Technical Questions (Questions 25-30) Across Different Groups.

4.3.5 Total Scores

The bar chart showing the total scores across all questions (see Figure 4.7) visually consolidates the trends from the sections detailing each of the score components. Academically high-experienced participants ($M = 77.43\%$, $SD = 18.49\%$) dominate, driven by accuracy and justification. High-experienced high school students ($M = 70.37\%$, $SD = 18.14\%$) occupy a middle ground, balancing moderate scores. Math 1C03 students ($M = 64.81\%$, $SD = 24.5\%$) trail due to weaknesses shown in examples and justification. The statistical findings underscore that total score differences are driven primarily by disparities in accuracy and justification, where Group 2 excels but struggles with consistency. Group 3's intermediate performance, while statistically indistinct from either extreme, reflects a compromise between moderate achievement and stability. Group 1's challenges, particularly in weakly correlated criteria like examples and justification, emphasize the need for targeted interventions to address isolated weaknesses.

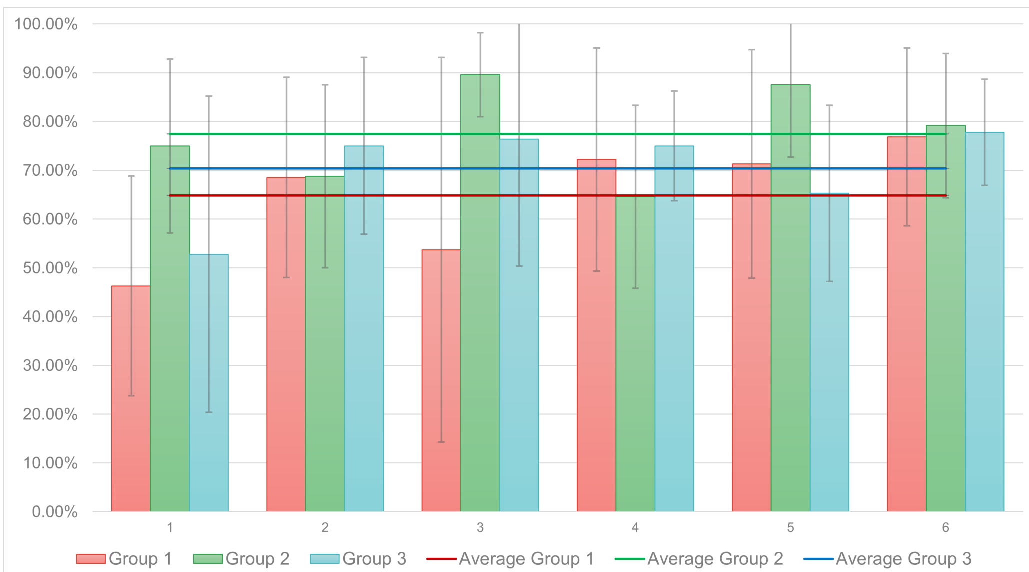

Figure 4.7: Participants' Total Scores in Technical Questions (Questions 25-30) Across Different Groups.

Error-bar charts for total scores illustrate Group 2's moderate variability (SD = 18.49%) against Group 1's higher instability (SD = 24.50%). While Group 2's peaks in accuracy and justification dominate the graphs, its inconsistency in examples — a criterion weakly tied to other metrics –poses operational risks, such as unpredictable performance in tasks requiring well-rounded execution. Group 3, with steadier performance (SD = 18.14%), may lack exceptional peaks but demonstrates reliability, making it suitable for contexts that prioritize consistent execution over sporadic excellence. Group 1's graphs, marked by erratic swings and the lowest averages, underscore systemic challenges, particularly in examples and justification.

Statistical tests confirm observable differences between groups performances on technical questions. The Welch ANOVA, employed dues to evidence of unequal variances (Levene's test: $p < 0.05$) revealed a statistically significant effect ($F(2, 20.82) = 3.08$, $p = 0.067$; Table D.7). Complementing this, the nonparametric Kruskal-Wallis test, robust to non-normality observed in Group 1 (due to a couple of outliers), also indicated significant group differences ($\chi^2 = 7.82$, $p = 0.020$; Table D.7). While this result approaches but does not reach conventional significance at $\alpha = 0.05$, the nonparametric Kruskal-Wallis test, which is less sensitive to non-normality in Group 1, supported a significant difference ($H = 5.98$, $p = 0.05$). Post-hoc Games-Howell tests revealed a significant difference between Group 1 and Group 2 ($p = 0.017$), with Group 2 outperforming Group 1 (mean difference = -4.55; Table D.8). However, no significant differences were observed between Group 1 and Group 3 or between Group 2 and Group 3 (both $p > 0.05$), though marginal trends suggest intermediate performance in Group 3. These results should be interpreted with caution, acknowledging the potential influence of borderline significance levels, non-normality in Group 1, and the relatively small sample sizes across groups (Table D.3).

This integrated analysis demonstrates that group performance is shaped not only by individual criterion strengths but also by the interplay between correlated competencies. While Group 2's high accuracy and justification explain their superior total scores, their inconsistency in examples and justification, which is evidenced by significant variability and weak inter-criteria correlations, limits their reliability. Group

3's steadiness, though unremarkable in peaks, offers a model of balanced execution. Group 1's trajectory, meanwhile, demands focused improvement in examples and justification to align with the interdependent strengths of accuracy and accessibility.

The data also highlights a preference for procedural explanations (accuracy and accessibility), achieving significantly higher average scores in these criteria (Group 1: 80.33%, Group 2: 89.59%, Group 3: 85.77%) compared to conceptual explanations (examples and justification), which lagged across all groups (Group 1: 31.95%, Group 2: 53.13%, Group 3: 39.59%). This disparity suggests greater comfort with step-based, formulaic reasoning over tasks requiring illustrative examples or deeper logical justification.

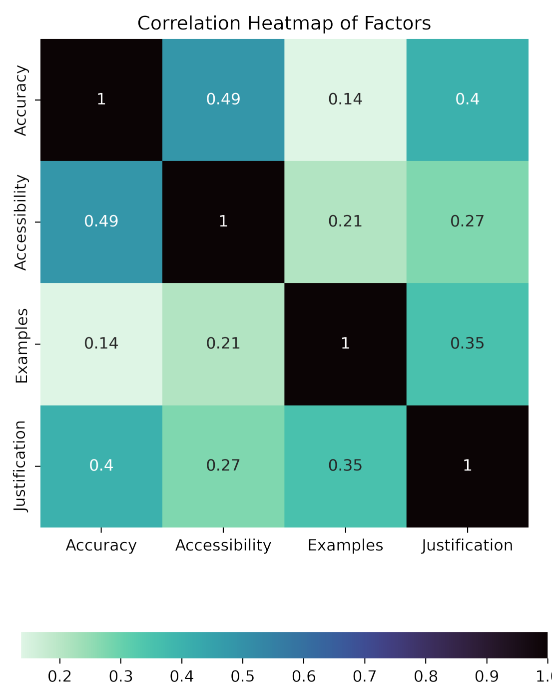

Figure 4.8: Correlation Heatmap of Scores Across the Criteria.

The moderate correlation between accuracy and accessibility ($r = 0.487$) further underscores this procedural fluency, as participants proficient in one criterion often excelled in the other. Though slightly below the conventional threshold for a *strong* correlation ($r \geq 0.5$), this relationship aligns with educational research norms where moderate correlations ($r \geq 0.3$) are often practically meaningful, particularly in skill-based tasks like procedural execution [25]. In contrast, the weaker performance in conceptual criteria — particularly examples, which showed minimal ties to accuracy ($r = 0.136$) — highlights challenges in articulating abstract understanding. These trends imply that while students adeptly execute predefined procedures, they struggle to independently generate or justify novel ideas, pointing to a need for more mathematical creativity.

4.4 Comparison: Shifts in Experiences

Additionally, the responses of two participants, who completed the study at both the beginning and end of their MATH 1C03 *Introduction to Mathematical Reasoning* course, were descriptively analyzed. Due to the low number of valid responses in this section, any formal statistical test is not meaningful, so the data was manually compared to note patterns and noteworthy changes. The average score of the participants with respect to a particular question is denoted AS below.

The *Introduction to Mathematical Reasoning* course elicited notable improvements in participants' confidence and ability to work with mathematical proofs: self-concept of proof-writing skills surged by 40% (AS: 1.5 to 3.5), while confidence in understanding proofs rose by 30% (AS: 3 to 4.5), underscoring the curriculum's focus on formal logic. Collaborative skills also strengthened, with comfort in group discussions increasing by 20% (AS: 3 to 4), though engagement in math-related discussions outside class remained static. While participants consistently valued communication skils (AS: 4.5 pre- and post-course), their ability to explain concepts improved modestly (AS: 3.5 to 4), and presentation skills stagnated (AS: 2.5). Notably, self-rated mathematical skills declined slightly (-10%, AS: 4 to 3.5), potentially reflecting participants' recalibrated self-assessment standards rather than skill loss, while familiarity with mathematical reasoning showed no growth. Communication preparedness saw minimal gains (+10%, AS: 2 to 2.5), highlighting a gap between classroom learning and real-world application.

Participants' individual experiences diverges: one reported greater comfort in group discussions and heightened awareness of communication's importance but noted slightly reduced reasoning familiarity and explanation ability. This suggests greater awareness of complexity, more honest self-assessment, or a new challenge encountered rather than skill loss. The other participant saw substantial gains in proof-related confidence coupled with a small dip in overall math satisfaction, alongside contradictory views on communication role — rating its general importance higher but its relevance within mathematics lower.

Coded results of the technical questions' responses of both of the participants before and after the course were compared using the Wilcoxon signed-rank test, which indicated that taking MATH 1C03 course on its own does not result in a statistically significant improvement in student performance. However, the data from previous sections suggests that continuous and consistent exposure to the techniques taught in the course can be beneficial over time. This is supported by the participants' improvement in specific criteria, particularly in accuracy (from the initial averaged total of 83% to 100% afterwards), which is closely correlated with justification and accessibility criteria. While examples also showed improvement (from the initial averaged total of 25% to 42% afterwards), accessibility and justification showed no change. This implies that students primarily refine their ability to produce accurate results, which may indirectly enhance their understanding and application of mathematical concepts. Overall, the findings highlight the importance of sustained practice and engagement with the material for meaningful progress.

Chapter 5

Qualitative Analysis

A thematic analysis has been completed on the survey and interview data to address the research questions. An analysis of the two data sets is outlined in this chapter, followed by an analysis of the talking circle, presented as a separate component.

5.1 Surveys

Open-ended survey questions allowed for more personalized responses. While these responses are challenging to interpret, they provide a richer set of variables and themes [40]. They also contributed to a larger pool of data, as they did not require additional time commitment from participants, unlike interviews and talking circle.

The open-ended questions in the survey (see Appendix A) can be categorized into two groups: personal questions (Questions 5, 11, and 30) and technical questions (Questions 31-36). Both of these groups are analyzed qualitatively below. Afterwards, the survey responses of two Group 1 participants – collected both early in and after the completion of the *Introduction to Mathematical Reasoning* course – were comparatively analyzed to evaluate changes in their perspectives over time.

5.1.1 Personal Questions

Personal questions provided insight into participants' backgrounds, allowing for a surface-level exploration of differences in their experiences. For example, elaborations of participants from Group 3 on their study habits (Question 11) suggested that math competition students primarily use study strategies that are not fundamentally different from those of non-competition students. Most follow a concept-first, then practice approach, relying on textbooks, notes, and problem-solving. Other commonly mentioned approaches included practice-based learning, memorization of key formulas, and passive review. However, while some incorporate advanced resources (e.g., Olympiad books) or develop personalized systems, the core methods — reading, practicing, reviewing mistakes, and refining understanding — are quite standard. The main distinction may lie in the depth and difficulty of the problems and the ways students engage with them, rather than in their overall study habits.

In response to Question 5, participants in Groups 1 and 2 elaborated on their motivations for pursuing a university degree. Since Group 3 consists of high school students, they were not asked this question. The responses highlight both similarities and differences between the two groups (see Figure 5.1). In both, job prospects and financial security are prominent factors, with many students acknowledging that a degree is essential for securing well-paying careers. Family influence is also a recurring theme, particularly in Group 1, where several students mention parental expectations as a significant factor in their decision to attend university.

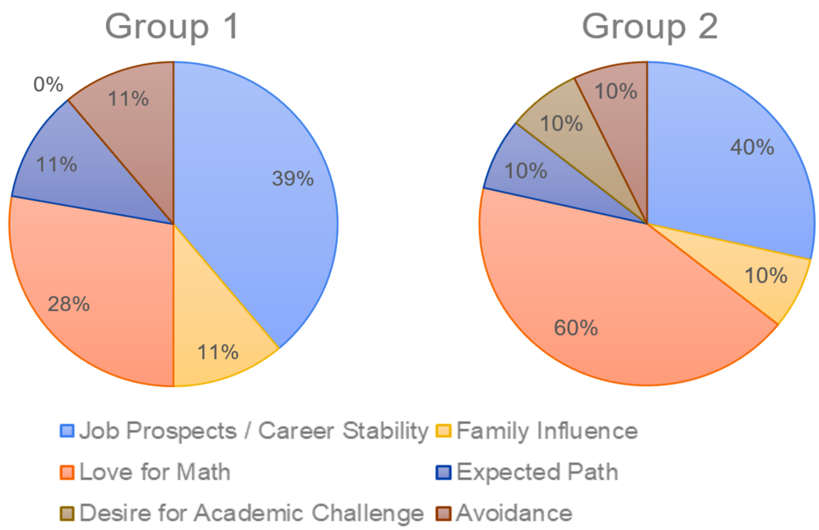

Figure 5.1: Comparison of Motivational Factors for Pursuing Higher Education Between First-Year Students (Gr 1) and Upper-Level Students (Gr 2).

In contrast, upper-level students (Group 2) often expressed a deeper intellectual engagement with mathematics. Many cite intrinsic motivation, curiosity, and a desire to challenge themselves with advanced topics as key reasons for continuing their studies. Unlike first-year students, who often view university as the 'logical next step,' upper-level students describe learning as a lifelong pursuit, emphasizing academic exploration and problem-solving. Some emphasize their enjoyment of academic exploration and problem-solving. Additionally, Group 2 includes students who have gained a greater awareness of academia and career pathways, with some humorously noting that their continued study of mathematics was, in part, a way to delay entering the workforce.

Over time, as students engage more deeply with mathematical reasoning and experience the intellectual rewards of doing so, they often develop a greater appreciation for the subject (see discussion on page 86).

Finally, all three groups were asked why they believe mathematicians need strong communication skills. Across all groups, participants emphasized that communication is essential for sharing ideas, refining work, and contributing to the mathematical community. They highlighted that effective communication enables the propagation of research, collaboration, and the application of mathematical findings to real-world problems(see Figure 5.2).

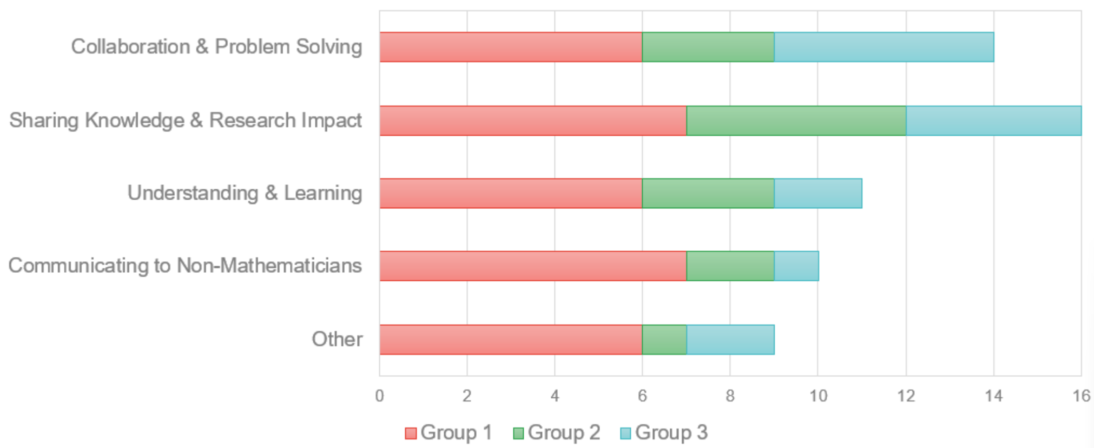

Figure 5.2: Reasons Why Mathematicians Should Have Good Communication Skills: Response Distribution by Group.

While responses from all groups generally emphasize the ability to share, verify, and build upon mathematical ideas, slight differences in perspective emerge (see Table 5.1). Participants in Groups 1 and 3 focus on the role of communication in group work, collaboration, and ensuring that mathematical findings are accessible and impactful. They argue that clear communication fosters mutual understanding, facilitates knowledge exchange, and ultimately accelerates problem-solving and innovation. Many also highlight that without effective communication, mathematical discoveries remain isolated, limiting their potential for verification and further development.

In contrast, participants in Group 2 placed greater emphasis on communication as a reflection of understanding. Many respondents argued that the ability to clearly explain a concept demonstrates true comprehension. Several responses suggest that if a mathematician cannot clearly explain a concept, they may not fully grasp it themselves. Respondents also underscore the importance of sharing ideas beyond academia, as mathematical knowledge gains value through dissemination.

Responses categorized as *Other* focused on personal, philosophical, or structural aspects of communication that, while less common, still play a role in mathematical discourse (see Table 5.1).

Table 5.1: *Other* Perspectives on Why It Is Important for Scientists to Be Effective Communicators (Question 30).

Gr	**Representative Quotes**
1	"Some sort of mutual standard to communicate ideas. E.g., formatting, what can be assumed." "Mathematics is a universal language that can unlock the secrets of the universe. It can explain any phenomenon. The way I see math is imagine the world is a simulation, and math is the coding language this simulation is built upon."
2	"One way to slowly starve a mathematical field of fresh minds to work on, is to write a textbook just bad enough that no one ever wants to read it, but just good enough that no one wants to write a new one." "Even within the sphere of academia, a student cannot be expected to *care* about an idea they cannot understand; this hinders the discipline if the student wants to pursue research in that area." "The way to create science is by communicating it to others. They can fill you up with new perspectives that you probably did not even consider." "Mathematicians and scientists speak a common language, one which enables them to pursue these often abstract and *high level* ideas. If these ideas are to leave the sphere of academia for practical use, for example, they better do so in a way understandable to the users."
3	"Probably a lot to a professional colleague who does not specialize in math, but less so to someone with limited educational background." "Having an ability to effectively communicate makes it easier for the communicator to consolidate new concepts, since they have to internalize a lot of the finer details to make their presentations make more sense."

Some responses highlighted personal growth and passion, such as one student who stated, “It feels good to tell someone about something you are passionate about and create interest in others.” Other responses addressed academic culture, accessibility, and expected communication standards. One participant in Group 1 stated, “stronger logical thinking ability and can express their opinions clearly,” directly linking communication to mathematical reasoning skills.

5.1.2 Technical Questions

Technical questions (originally numbered 25–30) are described in Section 4.3, where they have been renumbered as Questions 1–6 (see Table 4.1 for explicit renumbering correspondence). This section presents a comparative analysis of responses from all three groups– students with limited experience in proofs (Group 1), upper-level math students (Group 2), and high school students participating in mathematics competitions (Group 3) — regarding six mathematical concepts from high school (or earlier) curriculum presented in Table 4.1. The analysis evaluates the responses based on a few factors: clarity, use of examples, and depth of understanding of concepts by participants themselves.

The responses from high school students involved in mathematics competitions generally exhibited clarity and straightforwardness (see Table 5.2). They demonstrate a solid grasp of the fundamental concepts, often providing clear definitions and explanations, such as defining a function as “a mathematical variable whose value depends on one or more variables,” or explaining exponents as “multiplying a by itself n times.” This is expected as participants in this group are required to know these concepts and how to use them for mathematics competitions at various levels. Some participants also used relatable examples and analogies, such as apples, emojis, and machines, which help to make abstract concept more accessible.

However, while their explanations were clear, concise, and written in a simpler language, easily understandable by middle school students, there is room for improvement in providing more detailed explanations and context to enhance their overall understanding and communication. For example, one explanation states, "Equivalent is generally used on statements like equations and inequalities while equal applies to one-sided expressions," introducing undefined terms like *one-sided expressions* without defining them or providing examples. Similarly, another claims, "Equivalent means they can be interchanged and used the same way. Equal merely means they are the same in value," which conflates logical interchangeability with numerical equality. A more detailed explanation, such as clarifying that equivalence requires mutual implication (e.g., "two statements imply each other"), would help students grasp nuanced distinctions.

Table 5.2: Effectiveness of Math Competition High School Students in Explaining Mathematical Concepts from Table 4.1 (Questions 31-36).

Key Point	**Representative Quotes**	#*
Clarity and Simplicity	"A variable is a symbol that represents a data point; for example, x can represent the number 5. "	1
	"Set $f(x) = g(x)$ and solve for x, then put the value back into $f(x)$ to get y value."	5
	"Finding the intersection is the same as when their (x, y) are the same. Meaning, it is when $f(x) = g(x)$. "	5
	"To solve a system of equations means finding all the values of variables such that all the equations are satisfied. Alternatively, this is finding the point(s) of intersections of all the curves represented by each equation."	6
Relatable Examples	"A variable is like a symbol that represents a number (not necessarily known): like how an emoji can represent happiness, a variable x could represent the value 3."	1

Continuation of Table 5.2		
Key Point	**Representative Quotes**	#*
Relatable Examples	"A function is like a machine: you can put in different things, and it spits something out as output."	2
	"A function is a mathematical variable whose value depends on one or more other variables. For instance, we write $f(x) = 3x$ to be a variable $f(x)$ whose value is three times whatever x is. "	2
	"Let's say $a = 2$, $m = 3$, and $n = 4$. Multiplying 2^3 and 2^4 gives 2^7, like stacking seven 2's.	3
	"Solving a system of equations means finding when the runners [Frodie and Grodie] are at the same point at the same time.'	6
Effective Analogies	"A variable is a special box that can hold any number. Sometimes you don't know what's inside, and math helps you find out."	1
	"A function is like a grilled cheese sandwich machine: you input bread, cheese, and butter, and it outputs one sandwich. Similarly, functions take numbers as inputs and output numbers."	2
	"Equivalent means they can be interchanged and used the same way. Equal merely means they are the same in value, but not necessarily can be used in place of each other."	3
	"An exponent is multiplying a number by itself a bunch of times: I multiply a n times in a^n and I multiply a m times in a^m. Thus, when I multiply them together, I have a multiplied by itself $n + m$ times which is a^{n+m}."	4
	"The *intersection points* are all the conditions on x for which $f(x) = g(x)$. For instance, if $f(x) = x + 1$ and $g(x) = 2x - 3$, we can find their intersection points to be when $f(x) = g(x)$, which happens when $x + 1 = 2x - 3$. This turns out to be when $x = 4$."	5
End of Table		

In contrast, upper-level mathematics students demonstrated an advanced knowledge and a comprehensive understanding of the mathematical concepts (see Table 5.3). Their responses were detailed and precise, often including logical steps, underlying principles, and meticulous explanations. The use of extensive examples and analogies, both practical and abstract, by some of the participants further demonstrates their strong grasp of the topics. Their responses frequently emphasized generalization and abstraction, reflecting on their ability to extend concepts beyond specific cases. Overall, upper-level math students exhibited the highest level of clarity, depth, and detail in their responses, which is expected based on the highest level of expertise in the material and mathematics teaching among all three groups in the study.

Table 5.3: Effectiveness of Upper-Level Mathematics Students in Explaining Mathematical Concepts (Questions 31-36).

Key Point	**Representative Quotes**	#*
Definitions and Notation	"A function is a binary relation on a product of two sets that is left total and right unique.	2
	"A function is a rule which assigns to each element of a set X exactly one element of a set Y."	2
	"This is all about the notation '^'. Let's break it down: if we write a^m, we mean $a \cdot a \cdot \ldots \cdot a$ m times, and if we write [...]."	4
	"An equation is a statement with an equals sign and variables."	6
Detailed and Precise Explanations	"A variable is a symbol representing an unspecified element of a set."	1
	"The intersection point means that $f(x) = g(x)$, and so we just need to solve the resulting equation."	2
	"Equivalent things share a determining property, whereas equal things are identical in all relevant properties."	3
	"To solve a system of equations means finding all the inputs that will make all the equations true at the same time."	4
Real - World Analogies and Storytelling	"A variable is like a box that can hold different numbers. We don't know what's inside until we open it."	1
	"A function is like a magic hat — if you put one bunny inside, it always gives you 12 carrots back."	2
	"Two people with short hair are equivalent in terms of the barber shop of a small town because they both get sent to the same barber."	3
	"Finding the intersection point is like finding where two roads meet."	5
	"Solving a system of equations is like piecing together a puzzle, each equation giving a piece needed to see the whole picture."	6
Generalization and Abstraction	"A function is a general rule that relates each input to exactly one output, not just numbers but objects, shapes, or even ideas."	2
	"Two things are equivalent if they are the same in specific areas we care about, but equal if they are the same in every way."	3
	"Interpreting things in a plane, intersection points are all the points that have the same image at any given abscissa x, so the question is basically solving an equation $f(x) = g(x)$."	5
End of Table		

However, this level of sophistication can also introduce some challenges when the audience from the survey section set up is considered. Some participants used highly formal language and notation which may make their explanations less accessible. While upper-level students often assume a shared understanding of concepts– an assumption that may not hold for those still developing foundational mathematical reasoning skills — only a few participants tailored their responses accordingly. This could be due to either a lack of attention to the survey instructions or a failure to account for the audience's level. Additionally, while real-world analogies may appear conceptually appropriate, they often prove inadequate or lack relevance or comprehensibility for learners aged 11 to 14. As a result, some explanations, though suitable for undergraduate students, may not align with the cognitive and developmental levels of this younger group. There were also responses that represented the opposite side of the spectrum: more accessible explanations using simple language and even humor to engage their audience. For example, one explanation of how to find the intersection point of two functions was the following:

> *Taking functions as binary relations, their intersection is a set theoretic one. Just kidding!*
> *First we need to ask ourselves what does it mean for the graphs of two functions to be intersecting. Well, the graphs cross, or at least overlap, right? [Insert poorly drawn coordinate system with two scraggly drawn continuous functions.] Now what does that mean for our formal definitions? It means that at the same value of x both functions $f(x)$ and $g(x)$ have the same output. Now, that means that they are intersecting for x if $g(x) = f(x)$. Then you can solve the equation $g(x) = f(x)$ (or $g(x) - f(x) = 0$ if it's easier) and find all solutions x the intersection point then is $(x, g(x))$. If you can...*

The correct responses from students with limited experience in proofs were generally clear but simpler and less detailed (see Table 5.4). These students demonstrated a basic understanding of the fundamental concepts but often lacked the depth and comprehensive explanations seen in the other groups. Some responses suggested that these students relied more on memorization of rules than conceptual understanding, which sometimes led to incorrect explanations.

Table 5.4: Effectiveness of MATH 1C03 Students in Explaining Mathematical Concepts (Questions 31-36).

Key Point	**Representative Quotes**	#*
Basic and Incomplete Definitions	"A letter that represents a number(s)."	1
	"A function is a relationship between an input and an output."	2
	"Equal means the same, equivalent means similar."	3
Lack of Conceptual Understanding	"A variable is a number that can take on any possible value."	1
	"A function is a way to solve stuff."	2
	"Finding the intersection of f and g means setting them equal."	5
	"To solve a system of equations, find x and y."	6
Use of Everyday Language and Analogies	"A function is like a machine that takes an input, does something to it, and gives an output. Imagine you press a floor number (input), and the elevator(function) moves to that floor (output)."	2
	"Equivalent things are not copies of each other but do the same thing for all intensive purposes."	3
	"[It] is like finding out when two people have the same mood at the same time."	5
	"Solving a system of equations is like figuring out why your friend is having a bad day by piecing together different clues."	6
Rule-Based Thinking, No Justification	"When the base of two numbers with exponents is the same, you add the exponents."	4
	"[It] means setting the functions equal to each other and solving."	5
	"By the exponent rules."	4
Attempts at Formality with Misconceptions	"A function is a mapping of every element in a set into another set."	2
	"If two sets have the exact same elements, then these sets are said to be *equivalent*. Equivalence refers to sets with the same elements, but equality refers to two expressions being equal in value."	3
	"We will use proof by definition. Let $a \in \mathbb{R}$, and let $n \in \mathbb{R}$ such that the relationship between $a^n = a \cdot a \cdot \ldots \cdot a$ but n times.[...]"	4
Reliance on Numerical Examples	"If I had to find $2^3 \cdot 2^2$, I'd do $2^3 = 8$ and $2^2 = 4$, and then do $8 \cdot 4 = 32$. That would be the same as taking 2^{3+2}."	4
	"To solve a system of equations, take something like $x + y = 5$ and $x - y = 1$, and solve for x and y."	6
	End of Table	

Moreover, some responses were also oversimplified or incomplete, failing to capture essential aspects of the concepts. These responses demonstrate a lack of depth in understanding, as they fail to specify conditions, properties, or distinctions between related ideas. This is particularly evident in the *equal vs. equivalent* question, where many students provided answers based on intuition rather than rigorous mathematical reasoning. Conversely, some students attempted to use formal mathematical language; however, some of those responses ended up being incorrect or imprecise. This suggests that students recognize the importance of formalism but struggle with correctly applying it — indicating an attempt at reasoning that falls short because of the lack of knowledge. There were also some students who admitted to struggling with certain concepts, which might suggest that these students acknowledge gaps in their understanding and are open to learning. Responses such as "I don't know how to answer this," and "Not sure, would have to look it up on the web," were considered here.

Use of examples and analogies was the most limited among the participants in Group 1. Several responses illustrate concepts using specific numbers rather than abstract reasoning. This suggests that students may understand the concepts within a concrete context but struggle to generalize them into formal mathematical statements. They grasp the rule through examples rather than through logical deduction. Similarly, while students often attempted to explain concepts using real-world analogies, the effectiveness of these approaches varied.

Altogether, this analysis highlights different levels of both understanding and written communication skills among the three groups of students. Students in Group 1 show a basic understanding but would benefit from additional examples, deeper explanations, and more relatable analogies. Math competition high school students and upper-level math students generally exhibit strong understanding and clear communication of mathematical concepts. High school math competition students and upper-level math students displayed stronger understanding and communication skills, but Group 3 students—those with some competition experience—seemed to provide the most accessible explanations. Their use of simple language, relatable examples, and clarity made the concepts more engaging for audiences with no prior experience in the topic. However, by incorporating some of the detailed examples and step-by-step processes used by upper-level math students, their explanations could be further enhanced.

5.1.3 Comparison: Shifts in Perspectives

To explore how engagement with mathematical reasoning shapes evolving student perspectives on communication, a comparative analysis was conducted using pre- and post-course survey responses from two participants enrolled in the MATH 1C03 course. These participants' perspectives were compared to identify shifts in their understanding of communication's role in mathematics. Insights drawn from this comparison highlight how structured exposure to mathematical reasoning may refine students' communicative competencies.

Before completing the course, both students emphasized the importance of communication in disseminating mathematical discoveries and ensuring that ideas could be tested, expanded upon, and applied effectively. Student 1 framed communication as a means of making mathematics accessible and useful, drawing parallels between the communication of mathematical theorems and scientific advancements, as a doctor sharing a cure for a disease. Similarly, Student 2 underscored the necessity of a critique in mathematical work, arguing that without external feedback, discoveries would hold no long-term value.

By the end of the course, both students demonstrated a shift toward viewing communication not only as a tool for knowledge sharing but also as a means of deepening one's own understanding of mathematical concepts. Student 1, for example, transitioned from discussing communication as a means of efficiency and knowledge transfer to emphasizing that the ability to explain a concept clearly is indicative of one's true comprehension. This aligns with the common pedagogical assertion that teaching a concept is one of the ways to master it [33]. Similarly, Student 2 refined their initial view, shifting from a focus on external critique to an understanding of communication as an essential process for refining reasoning and ensuring that mathematical ideas hold persuasive power beyond the individual.

The changes in these students' responses highlight a growing awareness of the bidirectional relationship between communication and reasoning. Initially, their perspectives aligned with a pragmatic, outward-facing view of communication—mathematicians must communicate so that others can verify and use their work. However, over time, they developed a more self-reflective perspective, recognizing communication as a cognitive process that reinforces their own comprehension. This progression aligns with broader discussions in mathematics education that emphasize the role of discourse in developing mathematical understanding. It also suggests that engaging with rigorous reasoning tasks over time may help students appreciate communication not just as an obligation to the field but as a fundamental component of their own learning.

Analyzing the participants' responses to technical questions, several changes in writing, reasoning, and communication skills can be observed after students complete MATH 1C03 (see Table 5.5).

Table 5.5: Effectiveness of MATH 1C03 Students in Explaining Mathematical Concepts (Questions 31-36).

Before MATH 1C03	**After MATH 1C03**
"A variable is a parameter (ex. time) that can change. As it changes, it can influence something else. It is like a number that can take on different values in different situations."	"We can treat a variable like a mystery box that can hold many different things like numbers, words, or real life things, depending on the situation. We don't know how big our mystery box is, what kind of things it can hold, or what types of things it can hold, that's why it's a mystery! Here's a scenario: [...]"
"A function is a mathematical operation that takes input values and turns them into output values."	"A function is like a magic hat that you can put things inside of and the hat will give you some things back. The hat is magical, because it knows exactly what to give you based on what you put inside the hat! The magic hat has one important thing we need to remember, and that it will always follow the same rule. [Example following.]"
"If two things are equivalent, then they have a lot of the same properties, but they are not necessarily completely identical. For example, a quarter of a pie is equivalent to, but not equal to, a quarter of a cake."	"The best way to explore the concept of equivalent and equal is with an example: [...]"
"The exponent product rule is true because multiplying a^n is simply $a{*}a{*}a{*}a$ repeated n times, when a^n would be multiplied by another a, we are just increasing the initial n by 1 each time. Multiply a^n by a^m is the same idea, but increasing the initial n by m counts of a rather than 1."	"a^n simply represents a multiplied by itself an n number of times, and a^m simply is a multiplied by itself an m number of times, so if we multiply a by itself n times, and then multiply that product by a an additional m times, it is reasonable to say that the product of a^n and a^m is $a^(m + n)$"

Continuation of Table 5.5	
Before MATH 1C03	**After MATH 1C03**
"One can find the intersection points of $f(x)$ and $g(x)$ by equating $f(x)$ to $g(x)$ and solving for the x value(s) to satisfy the equation. One can then evaluate f or g at this/these point(s) to get the coordinate(s) of the intersection point(s)."	"We saw earlier that a magic hat could be seen as a function that gives us $f(x)$ depending on x, what we put inside the magic hat. In this case we have two magic hats, that give us $f(x)$ and $g(x)$ for what x we put into to each hat. We would like to know when these hats will give us the same thing, i.e., when $f(x) = g(x)$, but we don't know if they have the same rules. "
"To solve a system of equations is to find how many, if any, intersection points are in a function, and if there is a finite number, to provide the coordinates of said intersection points."	"To solve a system of equations is to find all possible values of all variables in a group of equations with the same variables relative to each other such that each equation in the system of equations remains true if the variables are replaced with any of the possible solutions found for the system of equations."
End of Table	

Before taking the course, students' definitions and explanations tended to be rigid and technical, with little effort to make concepts engaging or intuitive—especially for high school students. Their responses primarily stated facts without much explanation, often using vague or imprecise wording.

After the course, however, students demonstrated a shift toward making abstract ideas more accessible. Many incorporated analogies and specific examples, illustrating concepts in a way that suggests greater audience awareness. For instance, rather than simply stating that a function transforms inputs into outputs, they likened it to a "magic hat" that follows a consistent rule. Similarly, explanations of the exponent product rule moved beyond stating the rule itself to justifying *why* it holds.

At the same time, students' use of mathematical language became more precise. Many post-course responses began with an intuitive explanation or example before transitioning to a formal mathematical statement. Some also acknowledged that different representations of a concept may be useful in different contexts, demonstrating increased flexibility in reasoning—whereas pre-course responses often relied on a single fixed interpretation.

Overall, the post-course responses align more closely with those of upper-level mathematics students, indicating growth in both reasoning and communication skills. They also become more aware of their audience, using explanations that are clearer and more engaging. While there is still room for improvement in formal mathematical writing, the course appears to foster not only deeper conceptual understanding but also the ability to articulate mathematical ideas more effectively.

5.2 Interviews

Initially, the interview responses were separated into groups based on specific questions, and each group was then analyzed separately. Following the inductive approach of reflexive thematic analysis (RTA), which emphasizes a data-driven methodology, codes and patterns of meaning across the data were utilized to identify recurring themes in the responses. To align with the principles of latent RTA, the data was primarily retained in paragraph form to preserve the participants' sentiments and communication styles, ensuring these remained intrinsically connected to the meaning of their responses.

The deductive approach provided with several overarching themes and general remarks have emerged in the course of the interviews. They can be separated in three categories which are consistent with the line of the interview questions: *Communication as a skill*, *Cognitive skills from mathematics* and *Education aspects of* *communication skills.*

In the analysis below, tables were mainly used to represent the interview data, sometimes referring to the participant's Group, denoted **Gr** (see Section 3.1). Following the latent approach, participants' quotes were left in the initial format (unless participant specified the preference for paraphrasing).

5.2.1 Communication as a Skill

All interview respondents concurred on the pivotal role of communication in their lives. Various rationales were noted for the high prioritization of communication skills among the study participants. The everyday nature of communication, with its constantly shifting focus between areas such as comprehension, listening, writing, and articulation, depending on societal needs, coupled with concerns about miscommunication and misinformation, are recognized by participants as major factors (see Table 5.6).

Table 5.6: Participants' Perspectives on Why Communication Is Important (Question 1).

Theme	**Representative Quotes**	**Gr**
Everyday Nature & Relationships	"I have a lot of conversations every day." "Communication is especially important when it comes to relationships." "Being able to talk to others and express feelings, whether through words, body language, or writing, like emails to professors or texting, is super important."	1
	"It's quite important, considering social interaction is kind of what keeps humans alive." "I feel like nothing is done alone. The foundation of any community is communication."	2
	"It's essential in general interactions with colleagues, friends, and family."	3

Continuation of Table 5.6		
Theme	**Representative Quotes**	**Gr**
Miscommunication Issues	"Miscommunication can cause confusion or harm." "Nowadays, there is so much misinformation is going on. And not just in science."	1
	"It is a direct way of getting your point across. Your opinion and your words can get misconstrued, and then people might take your words out of contest." "If you're going to communicate with someone, you've got to do it right. Otherwise, it's like playing broken telephone. " "You could come up with an incredible result, prove it, and be correct, but if the proof isn't understandable to anyone else, it kind of defeats the purpose." "How are you supposed to convey what's going on inside your head to someone else if you don't have the right tools?"	2
	"In university, one needs to rely on communicating with classmates and teachers to solve the problem sets and to understand the class"	3
End of Table		

Respondents also agreed that communication skills are not inherently acquired but rather honed progressively through practice. The prevailing consensus suggests that these skills are developed *on-the-go* through repeated engagement over time. As one participant noted, "People who communicate are more comfortable doing so" (Group 1, Participant 2). Notably, only participants from Groups 1 and 2 mentioned the potential value of supplemental communication or rhetoric training, suggesting openness to such structured interventions. Several participants specified that while oral communication can be improved through careful, open-minded trial and error, reading comprehension and writing skills often require more formal guidance.

Regarding science and mathematics communication, some respondents advocated for formal training to address challenges in organizing and articulating complex ideas. For example, one participant from Group 1 stated, "The ability to organize your thoughts is crucial – one's thoughts can be all over the place, but being able to concisely describe them is a skill," a perspective shared by peers who similarly stressed the importance of clarity and structure. This response reflects the importance of creating structured opportunities to help individuals develop clear and effective science communication skills.

Additionally, psychological and cultural contexts were frequently cited by participants across various groups. This observation aligns with global variations in curricula and family dynamics. For instance, high school language classes in Germany often include tasks that emphasize logical reasoning and clear argumentation. Similarly, Ukrainian language curricula, as experienced by the main investigator during high school, incorporate analogous tasks.

5.2.2 Cognitive Skills from Mathematics

Before delving into the relationship between mathematical reasoning and communication skills, participants were asked to define mathematics or describe what it personally embodies for them. Problem-solving and communication emerged in the majority of responses, exceeding 50%, whereas logical thinking and mathematical creativity were mentioned less frequently (see Table 5.7). Additionally, logical thinking was only discussed by participants from Groups 2 and 3, consistent with these groups being the only ones familiar with proof-writing.

Table 5.7: Participants' Interpretations of Mathematics (Question 6).

Attribute	**Prevalence (Number of Respondents)**	**Representative Quotes**
Problem Solving	83.3% (10/12)	"[Mathematics] is more about problem-solving, tackling problems you haven't seen before, and using different tools to solve them." "[Math] involves applying real-world problems to mathematical models to find solutions,"
Logical Thinking	41.7% (5/12)	"[Mathematics] helps you become a better thinker. It helps you approach problems in other aspects of life in a unique way, compared to someone without a background in math." "Being able to provide a sound line of reasoning is essential. If your reasoning isn't logically sound, then it's not correct."
Communication	66.7% (8/12)	"[It] helps the ability to write effectively." "It's almost like storytelling: math gives you a solution, but then you need to communicate what that solution means in context. Te communication that happens after solving the math as just as important as the math itself."
Creativity	41.7% (5/12)	"Mathematicians do a lot of idea manipulation." "Well, obviously, every problem is different."

This is consistent with the observation that more than half of the participants who completed the interview portion of the study noted that their current perception of mathematics differs from their experience of it as a high school student. This discrepancy with curricular objectives was observed across all study groups, regardless of cultural context. However, there is a notable difference in sentiments. Consider the following excerpt from an interview with a participant from Group 3 where 'P' stands for *participant* and 'I' stands for *investigator*:

> *P: You see, usual math classes are very boring. People just talk about random things. I don't think this is a typical example of what a math class in school should look like. We don't really talk that much about math itself.*

> *I: So what do you do? Do you solve problems at least?*
> *P: It feels like our teacher explains things, but at some point, they just give up because they realize very few people are paying attention. [. . .] It kind of depends on the students in the class and how serious they are. Some classes probably do have discussions about math problems, guided by the teacher. But in others, students just don't want to talk about math and would rather do other stuff.*
>
> *It wasn't school math that made me love math. School math was really boring, and I understand why students would zone out. I got interested in math because I went to an extracurricular math place. They focus on competitions, that have more math than school, and the students there were more passionate about math.*
>
> *The usual math curriculum is tedious, just the same processes over and over. In elementary school, they even made us communicate every single step to a problem, even when it seemed obvious and unnecessary.*
> *I: Isn't that what you're supposed to do in math competitions — explain every step?*
> *P: It's not the same. In school, they made us write a lot of unnecessary things that didn't even help solve the problem. It was like writing an essay instead of just clearly explaining the steps. [. . .] And if you didn't do exactly what they wanted, even if you solved the problem correctly, you wouldn't get the highest grade. They required things like that for the top marks.*
> *(Group 3, Participant 1)*

This excerpt from the interview is corroborated by other participants who characterized high school mathematics as predominantly *procedural* and *algorithmic*. One participant described the process as, "Here's how you do it. Now, go do it," by one of the participants (Group 2, Participant 2).

Despite (or owing to) numerous changes to the mathematics curriculum over the years, communication has been explicitly incorporated into the course objectives in Canadian high schools. However, as one interviewee noted, "We had a *communication* section, but it mostly felt like following rules rather than truly explaining things" (Group 1, Participant 2). This noted distinction between the curricula guidelines and students' experiences is crucial in understanding the reasoning behind students' sentiments and attitude towards mathematics as a science. One participants from

Group 3 shared the following:

> *I never saw a proof in high school. I know some students in other Ontario schools did, which is great for them. Personally, I would have loved to see something like the proof for the quadratic formula. I remember watching a video about it later, and I thought, "This is amazing!" It had a beautiful geometric explanation, and seeing it in high school might have given me a greater appreciation for math. [...] The geometric approach with the squares is stunning. I only saw it for the first time in my third year, and I was blown away. By then, I had already used the quadratic formula in countless problems. If I had seen that proof earlier, I think it would've deepened my respect for math.*
> *(Group 2, Participant 6)*

While this thesis focuses more directly on the effect of proof-writing on communication skills, the quote allows for consideration of a more indirect line of influence: as students develop greater appreciation for the subject, they are more likely to enjoy it, engage with additional material, discuss it in conversations, and more.

When asked whether they had ever considered math classes as opportunities to enhance their communication skills, participants provided a range of responses. The majority of them supported the view that high school mathematics and lower-year university courses are predominantly computation-based. However, depending on students' study habits or attitudes, this focus might be shifted. One participant shared, "In calculus, for instance, I found myself writing out my reasoning much more than just using numbers and symbols. I had to fully justify my steps, and I found it really interesting and fun. It's been a different but helpful experience in improving my communication, even outside of math" (Group 1, Participant 2).

In contrast, one participant from Group 3 highlighted the ways their experiences with mathematics competitions directly influenced their communication skills. In particular, competition settings were noted to often require its participants to clearly articulate reasoning for review, improving both precision and succinctness in their communication. Another participant reflected, "When I started with Olympiad math, I would often say things vaguely and assume others would understand. But as I matured, I've learned to explain my thoughts more coherently" (Group 3, Participant 2).

Participants generally recognized that engaging in mathematical discourse and proof-writing can indeed affect their communication style due to its rigorous nature. Participants also noted that the discourse made their communication style more linear, akin to *IKEA instructions* (Group 2, Participant 3), while also enhancing comprehension skills and mindful self-talk. However, some participants did not directly associate these results with communication skills, despite acknowledging similar foundations and outcomes, "I guess so, but it hasn't really affected how I talk in everyday life. It has definitely improved my reading comprehension, though. In proofs, everything you need to know is often right there in the question or theory. This has made me more detail-oriented when I read things like literature or fiction. I tend to take things at face value more now and focus on what's explicitly there, rather than reading deeper into the text" (Group 2, Participant 2).

A similar response landscape is observed among participants regarding the impact of mathematics classes on their communication skills. While communication is the second most prevalent component in defining mathematics (see Table 5.7), students' experiences vary based on the length and extent of their studies. This raises the question of what exactly students perceive as mathematics upon leaving high school, having spent over a decade in mathematics classrooms. When asked if mathematics classes of various levels helped them become better communicators, participants belonging to different groups disagreed (see Table 5.8). For instance, students entering the MATH 1C03 course generally agreed that previous courses had little impact on their communication skills. However, interviews conducted later in the semester suggest a slight shift in opinions and even recognition of first-year university classes in this regard.

A more neutral consensus is seen among participants in Group 3, where students openly acknowledged that their experiences with proof-writing and mathematics competitions significantly influenced their ability to construct both written and oral arguments. One participant explained, "When communicating math, I can explain my ideas probably a bit more clearly because I have to do that a lot more when I do competitive math. In general communication, I think it depends, but the reasoning and step-by-step thinking definitely help" (Group 3, Participant 1). Another participant observed that debate clubs in Canadian high schools provided a similar preparation level for constructing arguments, but competition math uniquely emphasized precision and brevity.

Participants agreed that over time, engaging with mathematics has improved their thought organization and precision of language. This precision, a hallmark of mathematical communication, was reflected in their ability to convey ideas efficiently, making their communication both more careful and succinct.

Table 5.8: Participants' Views on Whether Mathematics Classes Enhanced Their Communication Skills (Question 6).

	Gr	**Representative Quotes**
Positive	1	"Yes, but maybe not from high school classes. In high school, I tutored friends, which forced me to learn how to teach people with different understandings. That wasn't something we learned in school." "Math 1C03, specifically, feels like a course focused on organizing thoughts, which I didn't experience much in earlier classes. So, yes, it helps."

Continuation of Table 5.8

	Gr	**Representative Quotes**
Positive	1	"I find myself structuring my thinking in an organized way. I think about what I know and what kind of effects various scenarios can have. This objective-based approach totally comes from my experience with math and science."
	2	"Math has definitely helped me in developing logical reasoning, which I've used to explain things more effectively." "I catch myself correcting people when they misinterpret statements like A *implies* B and assume it means B *implies* A. I definitely think more carefully about what I say." "Taking more courses makes it easier to communicate because you're getting more used to the ideas, and you're getting more used to talking about these complex, abstract things." "Working in study groups for math problems really strengthened my teamwork and communication skills." "I think math has made me better at structuring arguments and presenting evidence-based points in discussions." "Mathematics helped me to organize my thoughts clearly and logically before presenting them. It's a skill I use even outside math."
	3	"I learned how to articulate my arguments a bit better in terms of written communication from doing math. In general, I would say math has played a big role in that because it's better if you can get a bit of practice through math class learning how to write different types of proofs in depth." "If you do team math contests or if you're just working together on Olympiad problems, you're forced to actively and really efficiently communicate these very complicated ideas. Even in computational problems where you use different theorems, you have to be able to quickly explain your line of thought."
Neutral	1	"In high school, the program was restrictive. But in Grade 11, my teacher gave us oral exams if we messed up on tests. We'd have to explain concepts in our own words, and he'd correct our grade based on that. It was an interesting method of assessment that we didn't have much."
	2	"I suppose solving problems in math required some explanation, but I wouldn't say it was a huge focus." "We had presentations sometimes, but they were more about the math than improving how we talked about it. It was fine."
	3	"As for comprehension skills, yeah, because you have to read the problem and also know what it's asking for. You have to use all these skills in math, so I guess you practice all of them while doing it." "I think most of my proof knowledge comes from doing math competitions."

Continuation of Table 5.8

	Gr	Representative Quotes
Negative	1	"You don't learn much about communication in math; it's mostly procedural." "Not in high school, where the program was restrictive." "I don't think earlier courses fully prepared me in that regard."
	2	"Math classes didn't really help with communication. Most of the time, we just worked alone or handed in written work." "I never felt like explaining math was part of the goal. It was more about getting the answer right, not about how you got there."
	3	"Not really, at least not in school math classes. The format is usually just the teacher lecturing and then working on problems, often individually. I don't think my communication skills improve just by attending lectures." "It might have influenced my writing skills, like in literary analysis or critical essays, where I need to defend an idea. But I've never really connected proof writing to English essay writing; they feel like two separate skills."

End of Table

Regardless of their experiences, participants largely concur that mathematics instructors and teachers could enhance communication within their teaching, particularly during high school and the initial years of university education. This encompasses all forms of communication within the classroom: teacher to student, student to student, and student to teacher (see Table 5.9).

Table 5.9: Strategies for Enhancing Classroom Communication (Question 14).

Theme	Suggestions	Representative Quotes
Improving Explanations and Reasoning	Explain *why* concepts work, not just *how* to solve problems	"Instructors present something as true without explaining why."
	Focus on reasoning instead of rote patterns	"In high school… we weren't really learning the content so much as recognizing patterns."

Continuation of Table 5.9

	Group	Representative Quotes
Group Work and Discussions	Use group discussions as a lower-pressure alternative to presentations	"Group discussions would be less stressful than presentations." "Through talking with others, you collectively start to understand more. I wish more professors did this, especially in mathematics."
	Assign collaborative work	"We had assigned weekly homework in pairs, so you and a friend had to submit four homework problems each week. You were kind of forced to coordinate, even if you chose to split the exercises and work alone. By giving you another person to interact with, it forced some form of communication."
Active Engagement	Incorporate activities like solving problems collaboratively in class	"Instructors should get the audience to participate. . . Let's say, giving an idea or a problem, and then asking students to first discuss amongst each other how to go about solving it and then for people who have solved it to present the problem instead of just the instructor doing so."
	Explore edge cases and encourage student presentations	"Tutorials could be restructured to focus more on active communication, like the humanities often approach discussions. I had one teaching assistant who brought in a proof generated by ChatGPT and asked us to work in groups to dissect it and identify error. It was a great learning experience! I walked away from that session having learned far more than from a typical tutorial."
Integration with Curriculum	Include modules on logical reasoning and communication in high school	"There should be a course where you're taught how to think more creatively and present arguments. Then, you could build on that and also learn to communicate it well. So not only do you learn to communicate and write proofs creatively, but you could have a separate module to focus on communication and presentation skills."

Continuation of Table 5.9		
	Group	**Representative Quotes**
Integration with Curriculum	Combine proof-writing with presentation skills	"Pushing students towards explaining a concept helps them remember it better. The best way to learn something is to have to explain it to someone else."
Instructor Practices	Use visual aids and alternate between big-picture and technical explanations	"My undergraduate analysis instructor was a visual teacher; he'd draw sketches — even simple ones, like sets as *potatoes* with arrows between them. That helped a lot."
	Encourage participation by asking students to discuss ideas	"Some professors just ramble on or ask questions and then answer them themselves. That's not helpful. I think professors should encourage students to collaborate and throw out ideas, even if they're wrong. It helps everyone."
	Use math as a language	"Take our math building — people there constantly discuss some crazy math stuff. Right in the hallway!" "In university, once you start developing these skills, integrating them more into instruction would help with communication practice and overseeing. As a result from constant observation, you would also become better."
End of Table		

While the table above does not distinguish between suggestions for high school and university levels, such a distinction can be further elaborated upon. For instance, exploratory questions on the *why* are more appropriate for classrooms where students are still mastering the fundamentals of reasoning, whereas discussions about interesting edge cases and deeper analytical thinking are more suited to university settings, where students possess more extensive knowledge of the subject matter. Similarly, in high school, a course focusing on creative thinking and presenting arguments may be more fitting, whereas, in university, tutorials on dissecting proofs and explaining reasoning could be more beneficial.

Interestingly, none of the participants explicitly mentioned online communication tools, such as discussion forums, collaborative software, or interactive learning platforms. While these tools are often considered valuable for fostering engagement between students and teachers, participants strongly focused on in-person communication in their responses. Furthermore, their experiences with online communication were largely negative. One participant reflected:

> *I firmly believe math is best learned in person. Watching videos doesn't cut it, even for foundational courses like first-year calculus. There's so much value in discussing material with classmates. For me, communication skills didn't come from the classes themselves but from interacting with other students. When you're in person, you can exchange perspectives. For instance, if one person finds a problem easy, they might explain it in a way that clicks for someone else. That kind of mutual learning is invaluable and missing in an online environment. [...] It's not just about watching someone solve problems—it's about actively participating, reflecting, and communicating. (Group 2, Participant 6)*

Many of the suggestions had already been implemented by individual teachers or professors in students' experiences, which highlights their effectiveness. Although these suggestions are not *novel* to the field of education and have been examined to some extent previously (see [89, 137] for collaborations, [60, 90] for presentations), their positive reception indicates that students appreciate and recognize the benefits of diverse educational practices. Nevertheless, participants acknowledge the challenges and trade-offs associated with the proposed methods. Some participants noted that group work is *notoriously unpopular*, making it difficult for instructors to foster direct collaboration among students. To mitigate these challenges, participants suggested offering flexible options that cater to different preferences, such as allowing students to choose between completing the full workload individually or collaborating with peers to share the workload. Another highlighted approach was establishing clear expectations for collaboration and ensuring that instructors also adopt interactive teaching methods, such as discussing their thought processes and fostering open dialogue. This can create a more comfortable and less intimidating learning environment. Overall, participants expressed a positive sentiment towards this topic, noting that their most cherished memories of mathematics were formed when engaging in discussions or solving problems with friends.

5.2.3 Mathematics Communication Experiences

During the second phase of the interviews, students were specifically asked about their experiences in mostly oral communication activities, which often arise when engaging in mathematical practices. These activities — oral discussions, presentations, and collaborations — are analyzed independently. Oral communication encompasses activities involving the spoken exchange of ideas—such as discussions and debates—that enable participants to articulate thoughts and engage effectively. Collaborative work, meanwhile, involves tasks producing a shared tangible outcome (e.g., projects or homework assignments), where participants work collectively toward a common goal.

Oral Communication

All study participants have engaged in some form of oral math communication, whether to help a classmate or to participate in mathematical discourse. Many shared personal stories of explaining concepts, often highlighting the challenges and growth that came from these experiences. One participant reflected on how informal math communication helped them become more empathetic towards others:

> *Oh, I definitely didn't do this back in the day. Back then, my communication skills were not great. I would think, 'Why don't you get it? It's simple!' Now, I guide people to the answer by asking questions, nudging them in the right direction. (Group 2, Participant 6)*

Participants described various techniques they use to tailor their explanations to their audience's knowledge (see Table 5.10). These experiential techniques were also then observed in practice (see Section 5.2.5). The methods shared by participants underscore the importance of both mathematical reasoning and communication skills when explaining concepts. Mathematical reasoning is crucial for understanding the structure of concepts and breaking them down logically. Communication skills, however, are equally important for conveying this reasoning effectively. Participants described using strategies such as analogies, starting from the basics, and tailoring explanations to the audience's level of understanding. This highlights that, to teach mathematics effectively, knowing the material is not enough; one must also have the ability to communicate it clearly and adaptively.

Table 5.10: Participants' Approaches to Oral Mathematical Communication (Question 7).

Tactic	**Quote**
Breaking down a problem & Starting with the basics	"I'm good at breaking down concepts into manageable chunks, but I feel like there are still gaps where I could explain things better or articulate ideas more clearly."
	"I'll break the problem down, starting from the basics — definitions, theorems, or even axioms — and build back up from there"
Using intuition and simplicity	"When I explain math, I try not to assume that the person knows much about the topic. I avoid using difficult jargon and instead focus on the intuition behind the math rather than the technical details."
	"Discussing math with an upper-level university student is different from discussing it with a lower-level student or a high-school student. I aim to keep the explanation to the minimal logical steps necessary for them to understand."

Continuation of Table 5.10	
Tactic	**Quote**
Building from specific to general	"I like to begin with simple examples to illustrate a phenomenon and then abstract it to a broader concept."
	"I use a lot of analogies, and I like starting with examples to make abstract concepts more relatable."
Providing options	"I try to present multiple ways of understanding a concept. I might explain it one way, but another student might interpret it differently, so I offer different approaches and let them choose the one that makes the most sense for them. That helps build their mathematical intuition."
Encouraging self-discovery	"It's about helping them arrive at the solution themselves, rather than just telling them what it is."
End of Table	

Differences emerged between groups in their approaches, reflecting the participants' varying backgrounds. For instance, students in Group 1 mainly focused on personal interactions, emphasizing individual growth over developing strategies for a broader audience. In contrast, Group 2 participants emphasized audience-specific strategies, often mentioning the use of analogies and intuition, which suggests a more conceptual approach to explanations. Group 3 participants, however, demonstrated clear strategies for different types of audiences, emphasizing a balance between explanation and seeking feedback. This indicates a more reflexive approach to learning and teaching.

Participants highlighted the accessibility challenges they face when explaining concepts, especially without prior knowledge of their audience's background. Many described a learning curve in intuitive communication, emphasizing the difficulty of striking a balance between precision and clarity. A key step, they noted, was identifying where their explanations might lose the audience's attention. At the same time, participants acknowledged the personal benefits of teaching others, such as reinforcing their own understanding. As one participant remarked, "it makes you realize what you know" (Group 1, Participant 5).

Presenting an Argument

The ability to clearly articulate arguments and solutions was noted to be one of the most significant reasons as to why mathematics communication is an important skill. In those situations, presentation skills are an essential component of both mathematics education and professional development. Participants in this study were asked to reflect on their emotions (especially confidence) about presenting mathematical arguments or solutions in front of an audience, while also considering whether their comfort levels had changed over time (see Table 5.11).

Table 5.11: Thematic Analysis of Students' Experiences and Emotions with Presenting Mathematical Arguments (Question 13).

	Subtheme	**Representative Quotes**	**Gr**
Confidence	Confidence through practice	"The more you do it, the more comfortable you get, and it becomes more natural and more fluid over time."	2
		"At the beginning, they were abysmal. You're insecure, you don't know what to emphasize. But with more practice, it has improved."	2
	Audience - dependent confidence	"When the audience is very narrow, very specialized, it is one thing, but being able to explain your way of thinking — is totally different."	1
		"If I'm speaking to people who don't know much about math, I'd feel more confident because I know they won't ask difficult questions."	2
Challenges	Nervousness and performance anxiety	"I still get a bit nervous before presentations, especially a big one — I usually can't eat much beforehand."	2
		"When I explain something, I sometimes find myself changing my explanation halfway through and my presentation might come off as disjointed."	2

Continuation of Table 5.11			
	Subtheme	**Representative Quotes**	**Gr**
Challenges	Complexity of live presentations	"When you're writing, there's no one there with you. Whereas, if you're presenting live, there's no stopping and coming back. The audience might think, 'This guy doesn't know what he's talking about."	2
		"You have to think about how to phrase things so that the entire audience, with varying levels of understanding, can follow along."	3
Impact of MR	Improved communication skills	"As my understanding of the concepts has deepened, I can talk more freely about the material instead of just repeating definitions. My presentations have become more natural."	2
		"I think writing proofs specifically helped me out because prior, I'd just be going over a solution. But what's good about proof is that it makes you actually go through every single step of your thinking. You can't really take that many shortcuts. So then you just kind of automatically go through everything single step just by default because you're used to having to do that for proofs. It also let's you explain how the solution came together better."	3
	MR results in presentation clarity	"If you're able to explain what you've done to your grandmother... and then explain the same thing at a higher level... you know you've got it."	2
		"After you learn proof and you start explaining problems to people, you kind of automatically go through every single step just by default."	3
End of Table			

While the argument about mathematical reasoning enhancing the ability to communicate ideas clearly has been noted in various contexts and with regard to multiple questions, presentation skills get mostly improved thanks to proof-writing. Participants across all groups noted that their confidence is audience-dependent and a majority of them — more specifically, 9 out of 12 — either expect or have already experienced its growth with practice and exposure to presentations in various contexts.

Participants in Groups 2 and 3 have been exposed to some extent of mathematical presentations in the speaker's role, and as a result, they were able to report the confidence growth over time. At the same time, participants in Group 3 expressed a desire to improve presentation skills, with a hope that they will get better at it during university. Some participants also highlighted that the presentation skill is different from the usual writing skill and even oral casual communication because it requires a strict structure and linearity, similarly to the mathematical reasoning.

Collaborative Work

Collaboration has long been a cornerstone of mathematical progress, from historical partnerships between renowned mathematicians to modern team-based research projects. Beyond academia, collaboration is also integral to industry, where teamwork and effective communication are vital to success. Participants acknowledged this connection in their responses to Questions 12 and 17, emphasizing how communication and logical reasoning developed through mathematics extend far beyond the classroom. For example, one participant, who anticipated a future career in software engineering and was able to draw on their experiences from their paid internship, emphasized that much of the work involves communicating technical concepts to non-specialists, often through client consultations and team discussions. Another participant drew on their parents' professional experiences — one in business and the other in engineering — pointing out that, despite technical expertise, both spent a significant amount of time communicating ideas, be it through emails, meetings, or presentations.

Many participants noted that the logical thinking and problem-solving skills learned in mathematics are directly applicable to various careers. For instance, in industries like project management, communication is not only about conveying solutions but also about presenting complex ideas clearly and logically. As one participant remarked, the ability to map out a logical chain of reasoning is crucial for understanding systems, whether in math or in managing a supply chain. Furthermore, participants noted that, while mathematical communication tends to focus on clarity and precision, these same skills are essential for effective teamwork and problem-solving.

In educational settings, collaborative work — such as group projects and study groups — aims to mirror this dynamic. While not being particularly popular (see Table 5.9 and the discussion afterwards), participants do recognize a rationale for it (see Table 5.12). Almost all participants agree that communication is the cornerstone of effective collaboration. They emphasized that without good communication, group work quickly devolves into isolated individual tasks rather than a cooperative effort. Communication is also seen as a key means of building trust and ensuring that all participants are on the same page, enhancing the overall effectiveness of the collaboration. Several participants highlighted the role of communication in dealing with diverse groups, particularly in contexts where participants come from different cultural and educational backgrounds. Effective communication is necessary for ensuring that everyone is on the same page and that different approaches to problem-solving are understood and integrated. "If they don't follow your line of thought, they won't care about what you're working on," a participant from Group 2 noted. This is particularly important in international collaborations or when group members have varying levels of understanding or knowledge.

Table 5.12: Participants' Views on the Importance of Collaborative Work for Development of Communication Skills (Question 12).

Theme	**Representative Quotes**	**Gr**
Centrality of Communication	"If there is no communication, it's not really group work — it's just individual work happening in the same space"	1
Building Trust and Understanding	"In group projects, being able to say 'I'll finish this by this date' builds trust"	1
	"You have to figure out each person's strength and work together to solve problems"	1
Handling Diversity	"With people from different countries and educational backgrounds, it's important to be able to explain things clearly because they've learned concepts in different ways"	1
	"In research, clear communication becomes even more critical. If you can't communicate your findings well, no one will be able to understand or verify them."	2
Risk of Miscommunication	"It ends up being more like a patchwork than a seamless flow of work."	2
Critical in Problem-Solving	"You need to clearly explain your thought process to others so they can understand and contribute effectively"	3
	"If you can't explain your reasoning clearly, you might have made an error, and no one can catch it"	2
Social and Educational Growth	"Communication is the reason why we're able to form communities and social connections."	2
	"Communication helps everyone learn."	1
End of Table		

A notable sub-theme is the danger of poor communication, which can result in unclear or incomplete work. Some participants shared experiences where a lack of communication led to unsatisfactory or incomplete final projects or presentations, such as the "patchwork" effect in group projects. This highlighted the fact that without ongoing communication, the group's output can lack cohesion, and errors or misunderstandings may go unaddressed.

Participants also emphasized that in problem-solving contexts, particularly in mathematics competitions and collaborative research work, clear communication is not a courtesy but a structural requirement. In contests, solutions earn points only when every inferential step is transparent to judges, so explicit explanation functions as the primary error-checking mechanism; ambiguity can nullify an otherwise correct approach under tight time limits. In research teams, reasoning must be equally explicit so that collaborators can validate results, extend arguments, and integrate findings into larger projects. Thus, in both competitive and cooperative settings, the very design of the activity makes articulate reasoning indispensable for efficient, accurate problem solving.

Finally, communication is seen as a crucial factor for social interaction and educational development. Several participants noted that communication in group settings not only helps with task completion but also promotes social bonds and allows participants to learn from each other. Through collaborative work, individuals can expand their understanding and refine their communication skills, which contributes to personal and academic growth.

The differences across groups suggest that the context of mathematical collaboration significantly shapes how participants view and experience the role of communication. For instance, Group 1 participants focus more on everyday, informal collaborations and practical communication while referencing study groups and informal situations. Group 2, on the other hand, emphasized the challenges and nuances of communication in formal, high-stakes, and diverse settings. This aligns with their professional and academic experiences in more complex collaborations, such as projects involving multiple collaborators with different approaches or cultural perspectives. And finally, participants from Group 3 highlighted the urgency of collaboration in competitive or time-sensitive environments. They discussed high-pressure, goal-oriented team tasks, which showcases that they rely on communication more as a tool for real-time problem solving settings.

5.2.4 Reasoning $\Longleftrightarrow$ Communication

While the interviews were mostly centred around the influence of reasoning skills on communication, participants were also asked how they view the other direction of this relationship, i.e., the influence of communication skills on their reasoning skills. The key themes were identified amongst all responses (see Table 5.13) with some of the themes aligning with the topics discussed previously (see Section 5.2.3).

Table 5.13: Thematic Analysis of Participants' Experiences of Math Problem-Solving Through Effective Communication (Question 11).

Theme	**Representative Quotes**	**Gr**
Visual Communication	"When solving geometry problems, visualizing the information in a diagram helps me piece things together faster. It is just so much easier to represent things with drawings."	1
	"I was trying to learn an epsilon-delta proof… then I came across a visual representation on a website, and it clicked for me immediately. Communication isn't just verbal—that visual explanation really helped me understand the concept."	2
	"A colleague showed me another teammate's answer—a sequence represented by several small images. I immediately saw the pattern and the intended idea… The images weren't formal, but they conveyed the concept clearly and efficiently for me."	2
Conceptual Knowledge	"Being able to explain those different perspectives helped me understand things more deeply. Teaching really forces you to really understand the material."	1

Continuation of Table 5.13		
Theme	**Representative Quotes**	**Gr**
Conceptual Knowledge	"I talked it out with some friends... But it wasn't until I talked it out in words—really explaining it to myself and others—that I started to grasp it."	2
	"Self-talk is a classic example of something that helps enormously with mathematics. Computer scientists use it as 'rubber duck debugging,' if you're familiar."	2
Collaborations	"Most of my understanding and even research ideas have developed through discussion and conversation with others—like my supervisor, other researchers, and a lot of students."	2
	"A lot of times I've been discussing problems with people when I have no idea how to do it. And then they'll give me an idea, and I'll be like, 'Oh my God, that's so crazy,' and then I can understand it."	3
Language Matters	"Textbooks can be really wordy, so if I understand what they're trying to say, it's easier to grasp the math. If I don't understand the language, it's hard to visualize the math."	1
	"Some math books are dense and hard to read. A well-written book can make a huge difference. Take Dummit and Foote — some people find it dense, but I think it's a great book. On the other hand, I didn't like Judson's book much."	2
Problem-Solving Process	"Mathematical reasoning is finding an argument, and one way to find an argument is to argue. The act of communicating allows ideas to change a little and leads to new applications."	2
	"Even though the final proof used a lot of algebra and not much geometric intuition, getting the words out and talking it through helped me understand it better."	2
End of Table		

On the group-level, it also becomes clear what exactly groups consider as communication. For instance, Group 1 particularly focused on elements of visual communication, which is often used to explain various mathematical concepts. Group 2 provided the most refined reflections, often emphasizing self-communication and self-talk, which is needed in advanced coursework and problem-solving. Group 3, on the other hand, focused more on peer discussions as a way to break though problem-solving barriers. Their responses were more pragmatic, focusing more on communication's role in overcoming immediate challenges rather than conceptual understanding, which was seen with Groups 1 and 2.

At the very end of the discussion point about communication skills, participants were asked what they thought about the interconnection between mathematical reasoning and communication skills. Most of the participants, particularly from Groups 1 and 2, expressed a strong belief that that the two are deeply interconnected. They emphasized that being able to solve a lot of math problems, which includes breaking down problems, constructing logical arguments, and writing proofs clearly, inherently requires clear communication. Several participants likened the process of writing proofs or explaining mathematical concepts to storytelling, where presenting a coherent narrative is essential. These participants suggest that reasoning and communication reinforce each other, forming a symbiotic relationship.

Some participants from Groups 2 and 3 noted that the connection depends on context, audience, or individual preferences.

> *Whether you're writing a research paper or giving a presentation, you have to communicate your reasoning. But how much you break it down depends on your audience. For a more mathematically inclined group, you can be less explicit. For teaching or explaining concepts, you need to be more rigorous with your logic. (Group 2, Participant 1)*

One participant from Group 3 voiced skepticism about the relationship between mathematical reasoning and communication skills, both within the mathematics classroom and outside of it. They suggested that extracurricular activities, such as debate or public speaking, might play a larger role in developing oral communication skills than mathematics alone. This perspective indicates that while reasoning and communication can be linked, external factors and individual variability also play a role. However, they do not deny the possibility of a connection between the two.

> *I've never thought of them as related, but I do get feedback from my teachers that my writing is very clear, which might be a result of my logical reasoning skills from math competitions. However, it could also be from just practicing writing essays. I am not sure. (Group 3, Participant 2)*

5.2.5 Explicit Problem Solving

In the final segment of the interview, participants were presented with the following word problem:

> A farmer has a rectangular field that measures 150 meters by 100 meters. He wants to divide the field into smaller rectangular sections of equal size to plant different crops. Each section should have a length of 15 meters and a width of 10 meters. How many sections can the farmer create to maximize the use of the available space? (see Appendix B)

While the earlier parts of the interview focused on gaining insights into students' perceptions and experiences, this portion allowed for a closer examination of three key communication aspects: comprehension, self-talk, and oral communication.

The problem presented was a classic word problem and not intended to evaluate participants' mathematical abilities. This led to some humorous responses, such as the following:

> *Yeah, sure. Wow, that's a that's a real, real tough one. What do you mean? Wait, is this just a hundred? A hundred cause it's like 10 by 10. Am I being trolled? Wait, I think I got it, isn't it? Wait, I don't know isn't it just like? It is 10 one way and it's like 10 the other. It's 10 $\times$ 10. (Group 3, Participant 1)*

Participants were asked to describe their approach to solving the problem, walking the investigator through their reasoning process. The various approaches to the problem are illustrated in Figure 5.4, which traces the steps from reading the problem to submitting an answer. Those approaches support the opinions and theoretical knowledge participants have (see Section 5.2.3).

In Figure 5.4, A and B represent the dimensions of the large field, while a and b represent the dimensions of the smaller fields that subdivide it. The arrows indicate the order in which participants completed their problem-solving steps. The figures represent various purposes of the task: clouds denote purely mental actions, octagons with smoothed corners denote communication actions, and rectangles with smoothed corners denote mathematical actions. Dashed purple arrows highlight steps taken after reaching an initial answer to verify the uniqueness or correctness of the solution.

The term *standard tiling* describes the *horizontal stack* technique for floor tiling. This method involves arranging tiles in a grid-like pattern, where each tile is placed directly in line with the tiles above, below, and to its sides (see Figure 5.3). The result is a clean, uniform layout with straight lines and no staggered or offset rows.

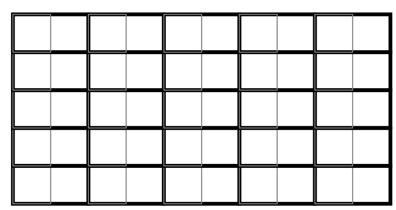

Figure 5.3: Example of the standard tiling of a 5×10 rectangle into smaller pieces of size 1×2.

Some participants were given the problem in written form, while others heard it verbally. All participants were provided with paper and a writing instrument, although only half chose to use them. Of those who used the paper, two participants completed actual calculations, while the others used it only for diagrams.

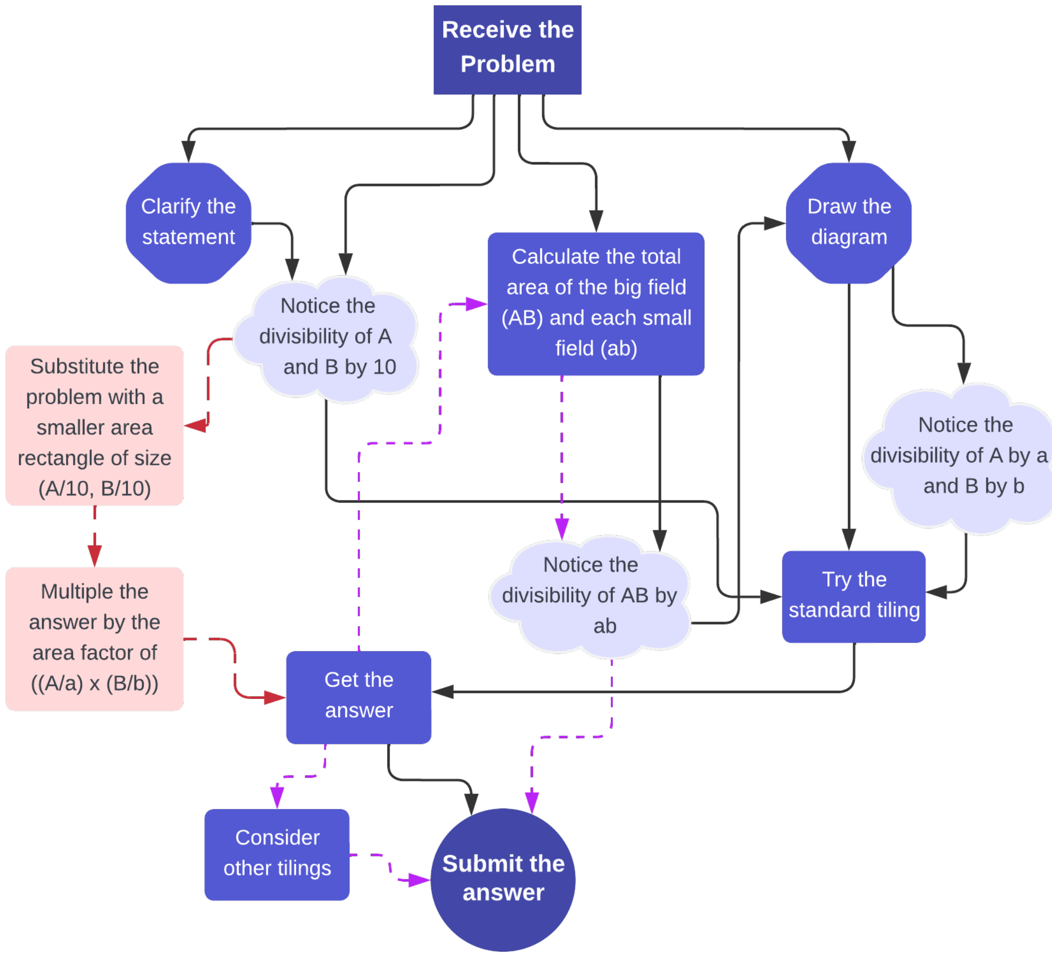

Figure 5.4: Participants' Problem-Solving Approaches (Question 18).

Most participants indicated that a visual representation was essential, particularly when explaining the problem or solution to someone else. However, not all participants used a diagram, and most (9 out of 12) did not consider alternative tiling methods. When asked why they chose the standard tiling, participants from Groups 2 and 3 provided logical justifications, while those from Group 1 reported relying more on intuition and pattern recognition.

The solution pathway without a diagram, depicted on the left side of Figure 5.4, begins with recognizing that the dimensions of the larger field are divisible by 10. The next steps (represented by red arrows) were added post-interview by the investigator to illustrate a possible reasoning path that bypassed the diagram or tiling methods.

Participants also anticipated a range of challenges while solving the problem (see Table 5.14). When analyzing these challenges in relation to the participants' groups, it becomes evident that their different mathematical experiences influenced their perspectives. For example, participants in Group 2 frequently mentioned conceptual difficulties and challenges related to problem comprehension, suggesting that they spent more time trying to understand the meaning of the problem. This group also explicitly identified computational challenges and was the only group to recognize numerical patterns, such as divisibility of both A and B by 10 or the divisibility of the total area AB by ab, indicating a tendency to leverage mathematical properties to simplify problems. These traits align with their exposure to both computational and proof-based university courses.

Conversely, participants from Group 1 highlighted challenges related to breaking down and reassembling problem components, suggesting a more fragmented, process-oriented approach to problem-solving. Notably, Group 1 was the only group to mention learning accommodations for students with disabilities, indicating diverse cognitive needs within the group, which reflects the group's varied background and level of problem-solving confidence.

Group 3 participants relied heavily on intuition, with a strong emphasis on visual reasoning and general discourse, rather than computation or detailed analysis. This suggests a preference for intuitive problem-solving strategies.

Table 5.14: Participants' Observed Challenges in Problem-Solving (Question 19).

Category	Subcategory	Issue	Gr
Problem Comprehension	Understanding of the setup	Difficulty visualizing the tiling process	3
	Incorrect interpretation of the problem	Ambiguity in whether the problem is about optimization, arrangement, or any other criteria	1 & 2
Conceptual Transitions	Switching between representations	Transitioning between area-based reasoning and discrete units	2
	Breaking down and reassembling ideas	Struggling with recombining components of the problem after breaking it into smaller parts	1
Intuitive and Visual Reasoning	Relying on intuition	Using intuitive approaches like grid-based tiling	3
	Importance of visual aids	Dependence on diagrams or visual representations for problem-solving	2
	Recognizing patterns	Identifying features like multiples of 10 but requiring validation for their role in solving the problem	2
Cognitive Accessibility	Educational support measures	Needing external tools, like calculators, to support problem-solving	1
Computational Challenges	Arithmetic and algebra	Adjusting for specific dimensions and performing numerical calculations	2
		Setting up equations to arrive at a solution	2
End of Table			

Overall, Group 1 displayed a wide range of approaches, with a tendency toward algorithmic methods. Group 2 participants consistently highlighted a broader array of issues, including conceptual transitions, problem comprehension, and computational challenges, pointing to a more analytical and detailed approach. Group 3 participants, on the other hand, focused more on intuitive reasoning, visual reasoning, and real-world analogies, with less emphasis on arithmetic or algebra.

Finally, based on the challenges participants anticipated, their explanations of the problem and communicating its solution to others varied, with each group emphasizing different aspects while still addressing the key components (see Table 5.15). Group 1 prioritized basic arithmetic and clear definitions, guiding the problem from simple to more complex concepts. Group 2 focused on structured, logical explanations, incorporating visual aids and diagrammatic representations to address logical connections. Group 3 emphasized intuitive reasoning and real-world analogies, transitioning to numerical specifics after establishing a practical understanding.

Table 5.15: Participant Approaches to Explaining a Math Problem (Question 20).

Category	**Quote**	**Gr**
Problem Comprehension	"They need to understand how the dimensions of a rectangle work and that the area of a rectangle is length times height. They need to grasp multiplication and division."	1
	"I would definitely go over the notion of dividing the field first, just in general. Given any dimension, how do we divide rectangles into equal little sections?"	2
	"Maybe I just say you have a box, and you have a bunch of boxes, how do you put them in? Intuitively, people can understand that if you want to maximize the space, you fill, you don't want any empty."	3
Visual Reasoning	"I'd suggest drawing a picture to visualize the problem and look for patterns."	3
	"Diagrams would definitely help illustrate the problem."	2
	"I'd suggest solving it visually at first—drawing it out—and then working backward to the math."	2
	"You put them as close together as you can—they just make a grid. So maybe I'd ask them to think about that and then... guide them through the process."	3
Practical Explanations	"A good way to explain it might be to start with a 4 by 4 square and show how to break it into smaller 2 by 2 squares."	1

Continuation of Table 5.15		
Category	**Quote**	**Gr**
Practical Explanations	"I'd start by clearly stating the measurements and maybe use a diagram to illustrate the field... Then, I'd work with them to understand the little jumps in logic that allow us to finish."	2
	"I think intuitively, people can understand that if you want to maximize the space you fill, you don't want any empty space. I'd ask them like how do we want to fit the most into a row?"	3
Addressing Misconceptions	"It's important to explain why we divide rather than add or subtract, especially if they have no prior knowledge."	1
	"Some might want to just think of the field as a single dimension, fitting sections by length alone. But since it's 2D, both dimensions and the overall rectangular shape matter."	2
	"You leave no empty space. Some might think you leave empty areas when aligning the pieces, but the goal is to maximize use of the space."	3
End of Table		

5.3 Talking Circle

During the Fall 2024 semester, the Talking Circle (see Section 3.2.3) was advertised to a total of approximately 150 students enrolled in MATH 1C03. Across the term, 11 unique students attended at least one session, though only two students met the minimum attendance criterion of 75% participation. Analysis of survey responses to Question 37 (see Appendix A) identified three primary reasons for limited attendance:

1. The demands of coursework during the semester became overwhelming.
2. Students who missed the initial weeks decided not to join thereafter.
3. A reluctance to attend alone, coupled with the absence of friends among participants, discouraged involvement.

Several challenges were noted by students as recurring difficulties during the semester (see Table 5.16). While the issue of untimely feedback does not depend on students and appeared to persist throughout the semester, the other two challenges — ambiguity in expectations for written work and difficulty in forming study groups — seemed to get resolved over time. For instance, as the semester progressed, both tracked participants became more comfortable communicating with their teaching assistants. Consequently, they were more inclined to ask questions and receive advice on their written solutions (see also Section 5.3.2). Moreover, by the semester's end, both regular Talking Circle attendees had joined study groups. It remains unclear, however, if this was coincidental or due to attending the Circle sessions, given the consistent encouragement to join or create a study group with at least one other student.

Table 5.16: Challenges for Students in MATH 1C03 during the Fall 2024 semester.

Challenge	**Explanation**
Ambiguity in expectations for written work	Students struggled to understand the expectations for writing solutions and effectively communicating their ideas in assignments and assessments.
Delayed feedback	Feedback on course materials was often released too late to be actionable, limiting its utility for subsequent assessments within the same module.
Difficulty in forming study groups	Many students found it challenging to establish connections with peers in the course.

5.3.1 Reflections from the Final Meeting

The concluding Talking Circle session provided an opportunity to reflect on the semester's experiences and outcomes. These reflections, augmented by post-course survey responses, highlighted four key themes:

1. Mathematics as more than computation:

   Participants recognized that MATH 1C03 emphasized logical reasoning and effective communication as integral components of mathematics, extending beyond mere computational skills.

2. Recognition of mathematics in everyday contexts:

   Students reported a heightened awareness of mathematical concepts and critical thinking in daily life. For instance, one student described estimating the number of parking spots at McMaster University using observed lot sizes and naming conventions.

3. Improved awareness of logical structure:

   Students became more attuned to identifying logical inconsistencies in both their own and others' communication. One participant noted an increased sensitivity to less structured communication styles in discussions with non-STEM peers.

4. Increased involvement in the mathematical and STEM community:

   Students reported greater engagement with the broader mathematical and STEM communities. For example, some participants noted creating or following social media accounts dedicated to math, as well as engaging with math-themed memes. These activities contributed to a sense of belonging and connection within the STEM field.

5.3.2 Impact and Feedback on the Talking Circle

Participants generally viewed the Talking Circle as beneficial for maintaining progress in the course. While course-specific questions were only discussed once, the sessions encouraged preparation akin to that required for MATH 1C03 lectures. They also provided opportunities for collaborative learning and exposure to diverse perspectives on mathematical concepts. One participant reflected:

> *The exercises during the sessions gave me a different perspective on many concepts and ideas by hearing others' takes on them. They also helped me understand how to verbalize and communicate my understanding of the subject. I didn't realize how important it was to explicitly define or explain a concept until I started doing it consistently. This made doing proofs while fully understanding the concepts much more comfortable.*

Another participant remarked:

> *The sessions helped me pause and consider how to phrase things before saying them. They also made me realize how mathematical terminology applies to non-math contexts. The sessions encouraged me to collaborate with classmates on homework, such as reading each other's proofs and giving feedback, which became a useful study strategy.*

An unanticipated benefit of the Talking Circle was the impact on students' perception of their teaching assistants. Regular interactions during the sessions helped students see their teaching assistants as approachable and relatable, reducing the intimidation often associated with seeking help from graduate students. This change fostered greater engagement and willingness to seek academic support.

Chapter 6

Conclusions

This study explores the relationship between mathematical reasoning (MR) and communication skills, in particular, in mathematics, across three distinct groups: first-year university students in an *Introduction to Mathematical Reasoning* course (MATH 1C03), upper-level mathematics students, and high-school math competitors. By integrating quantitative surveys, qualitative interviews, and observational data from a talking circle, the study offers a nuanced understanding of how these competencies develop in varied educational contexts. This chapter synthesizes the findings, contextualizes them within existing literature, and proposes actionable pathways for educators and researchers.

6.1 Synthesis of Findings

The study reveals that mathematical communication is neither a byproduct of mathematical skills nor a standalone skill but instead a dynamic process shaped by practice, purpose, and pedagogy. Upper-level students demonstrated refined abilities to structure and present logical arguments and justify conclusions, reflecting their prolonged exposure to high-level mathematics, abstract problem-solving, and peer discourse or collaborations. High school competitors excelled in translating complex ideas into accessible explanations — a skill developed through collaborative contests and out-of-classroom settings combined with the need to articulate solutions under time constraints.

For introductory students, who entered the study with minimal exposure to formal reasoning, the MATH 1C03 course served as a critical foundation, supporting previous research (see [56, 86]). Over the semester, the studied sample developed confidence in constructing proofs and navigating abstract concepts, shifting from algorithmic thinking to methodical problem dissection. Yet persistent struggles in tailoring explanations to others or integrating relatable examples underscored a disconnect between internal reasoning and external articulation. These findings highlight the importance of integrating communication-focused instruction into mathematics education.

As noted above, quantitative data indicated measurable improvements in students' confidence and engagement with mathematics, particularly in mathematical proofs. Structured instruction in proof-writing, common among upper-level and competition-experienced students, positively impacted reading and writing comprehension. However, oral communication skills, including collaborative discussions and presentations, showed only modest gains. This suggests a gap between theoretical understanding of mathematical communication and its practical application in teaching, especially given the professional demands of the field.

Qualitative data from interviews and the talking circle offered additional context to the numerical findings. Students reported greater awareness of the logical structure in their communication, both in mathematics and in everyday speech. Many participants noted that articulating mathematical ideas—whether to peers outside the classroom or in written work — helped solidify their understanding, reinforcing the idea that communication is a key tool for deeper learning. However, challenges such as ambiguity in written explanations and difficulty forming study groups highlighted barriers to effective collaborative learning.

These outcomes align with Sfard's commognitive theory [114], which suggests that mathematical thinking is inseparable from the language used to express it. The competitors' ability to make complex ideas understandable through analogies and simplified language mirrors the concept of exploratory discourse, while upper-level students' precision in proof-writing reflects the ritualized norms of academic mathematics. However, the study also complicates this framework: the talking circle's limited uptake among introductory students — despite its grounding in collaborative learning — highlights students' lack of motivation and engagement and systemic barriers, such as assessment pressures and institutional inertia, that hinder the very discourse essential for conceptual growth.

These findings emphasize the need to integrate explicit communication training into mathematical instruction, especially in specialized programs preparing students for STEM careers. While mathematical reasoning was not the primary focus of this study, participants noted improvements in their reasoning abilities, especially in proof-writing courses like MATH 1C03. Nevertheless, the limited gains in communication preparedness suggest the need for more targeted interventions, such as formal written exercises or structured peer discussions, to bridge the gap between reasoning and communication.

6.2 Discussion

The results both affirm and challenge prevailing narratives in mathematics education. On one hand, the superior communication skills of competitors and upper-level students support claims that non-routine problem-solving and sustained engagement with proofs foster clarity and logical rigour. This aligns with studies linking competition participation to metacognitive growth and collaborative reasoning. On the other hand, the limited impact of the introductory proofs course on holistic communication skills challenges the assumption that exposure to formal logic automatically leads to pedagogical proficiency. While students gained confidence in constructing proofs, their explanations remained formulaic and lacked sufficient examples—contrasting with literature that frames proof courses as vehicles for communicative fluency. This static view of communication preparedness suggests that implicit exposure to mathematical discourse may not be enough to develop strong communication skills without deliberate practice.

Furthermore, the findings reinforce the idea that sustained engagement in proof-oriented discussions benefits students over time. The results suggest that while students refine their ability to produce technically accurate solutions, their explanations do not necessarily become clearer to a broader audience without explicit guidance on accessibility and justification. This aligns with research on expertise development, which highlights the importance of both procedural fluency and the ability to convey complex ideas effectively.

Notably, the study uncovered a paradox in math competitions: although often criticized for promoting isolation, these environments unexpectedly fostered communication skills. Competitors' ability to tailor explanations to novices—using metaphors, visual aids, and iterative feedback—contrasts with portrayals of contests as purely performative. This suggests that the social dynamics of team-based competitions and preparation camps, rather than the contests themselves, may drive communicative growth, a nuance often overlooked in previous research.

An unexpected finding was the role of the talking circle in fostering a sense of community and reducing the intimidation of seeking help. This suggests that informal, peer-supported environments may be particularly effective in promoting engagement with mathematical discourse. Future research could explore whether structured discussion-based interventions have a measurable impact on students' ability to articulate mathematical arguments more effectively.

6.3 Directions for Future Research

This study highlights several key takeaways for mathematics education:

1. While mathematical reasoning improved, communication skills did not develop at the same pace. Future instructional strategies should include structured opportunities for students to articulate their reasoning, both in written and verbal formats. Peer explanation exercises, reflective writing, and translating proofs for non-specialists could help bridge the gap between procedural mastery and conceptual teaching.

2. Participants expressed a common experience of courses focused primarily on

material-based tasks with little emphasis on communication. Instructors can model metacognitive dialogue, verbalizing their problem-solving processes, and explicitly teach analogical reasoning, enabling students to see the *background work* behind thinking and reasoning. As one participant noted, "If you can't explain it to a friend, you don't really understand it" - a philosophy that should permeate classrooms.

3. Communication skills seem to evolve over extended periods, not within a single course. Longitudinal tracking of introductory students could determine if their communication skills improve in advanced courses or remain stagnant without targeted intervention.

4. Students who engaged in peer discussions and the talking circle reported greater comfort in expressing mathematical ideas. Implementing similar discussion-based initiatives in other courses could support the concurrent development of reasoning and communication skills. Additionally, creating low-stakes spaces for collaborative reasoning could help mitigate attendance barriers like workload or social anxiety.

5. Advocate for national standards that assess communication in high-stakes exams. While integrating communication-focused training into existing programs poses logistical challenges — particularly in large classes common in introductory courses — the findings underscore that *quality of engagement* matters more than *quantity of content*. For instance, scalable strategies like peer feedback systems, where students evaluate anonymized explanations using structured rubrics, or flipped classrooms that repurpose lecture time for small-group

discourse can foster communication skills without overwhelming instructors. Technology-assisted tools, such as AI-generated proof explanations for peer critique or discussion forums moderated by advanced students, offer additional pathways to prioritize depth over breadth.

Overall, this research contributes to the growing body of work on mathematical communication and self-concept, providing evidence that while formal instruction in mathematical reasoning is beneficial, additional interventions are needed to help students communicate mathematical ideas effectively.

This thesis argues that communication is not a peripheral skill in mathematics but a core component of disciplinary expertise. The ability to explain, justify, and adapt ideas for diverse audiences is as vital as doing mathematics itself—competencies that foster collaboration and innovation. In an era where interdisciplinary dialogue and public engagement are essential, nurturing these skills is not only an academic concern but a societal imperative. By reimagining curricula and classrooms as spaces where reasoning and communication develop in tandem, educators can cultivate not only skilled mathematicians but also engaged and communicative citizens.

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Appendix A

Survey Questions

The survey administered for this study utilized Google Forms to gather responses from participants. Each participant encountered a Letter of Information outlining the study's purpose and procedures, followed by a consent button to proceed.

Questions marked with an asterisk (*) were mandatory, while others were designed as open-ended responses to gather comprehensive insights. The survey below excludes Questions 7**, 31-32, where participants were asked to provide their email addresses for interview scheduling, talking circle scheduling, and study result dissemination. Based on initial survey responses, additional questions were tailored for specific participant groups to ensure relevance and inclusivity.

For a detailed breakdown of the survey sections and specific questions, please refer to Section 3.2.1. Below, Q_i for $i = 1, 2, \ldots$ denotes the question number i.

Q_1: Do you consent to your long answers being used as quotes? *

Yes, they can be used as full quotes. / No, I would like them to be paraphrased.

Q_2: Which ONE OF the following inclusion criteria do you meet? *

- *I am a student registered for MATH 1C03 course.*
- *I am an upper-level mathematics student.*
- *I am a high school student that actively participates in math competitions.*

This section aims to gather essential background and demographic information from participants. The questions in this section will help us understand the diverse characteristics of our study sample and how these factors may influence mathematical reasoning and communication skills.

University Participants (Groups 1 and 2):

Q_3: What is your age? *

18 or younger / 19 / 20 / 21 / 22 / 23 / 24 / 25 and higher

Q_4: Have you graduated from a high school in Canada? *

Yes / No

Q_5: Are you the first person in your family to attend university? *

Yes / No

Q_6: What motivated you to pursue higher education? *

Q_7: Have you participated in any high-level math competitions or programs (e.g., Euclid (Waterloo), Canadian Mathematical Olympiad (CMO))? *

Yes / No / Maybe

MATH 1C03 students only (Group 1):

Q_{7^*} To create a unique identifier for your response, please provide the first two letters of your parent's (guardian's) name followed by the last three digits of your phone number.

$Q_{7^{**}}$ If you would like to participate in a weekly group not recorded check-in with the primary investigator, please provide your email below. The check-ins are expected to happen weekly and take around 45-55 minutes during a scheduled agreed time. Participants will be expected to attend at least 10 sessions.

High School Participants (Group 3):

Q_3: What [high school] grade have you finished in 2024? *

9 or below / 10 / 11 / 12 or above

Q_4: Are you enrolled in any advanced math courses/programs in your school? *

Yes / No

Q_5: Are you enrolled in any extracurricular advanced math programs outside of your school? *

Yes / No

Q_6: How many years have you been involved in math competitions? *

Less than 1 / 1-2 / 2-3 / 3-4 / 4-5 / More than 5

Q_7: How do you typically approach studying for math exams or learning new mathematical concepts? *

This section aims to gather information about your mathematical experience and background. The questions in this section will help us understand the range and depth of mathematical exposure and training among our participants.

Q_8: How would you rate your overall experience with mathematics? *

1(Challenging) / 2 / 3 / 4 / 5 (Enjoyable)

Q_9: How would you rate your current mathematical skills (basic arithmetic, algebra, geometry)? *

1(Poor) / 2 / 3 / 4 / 5 (Excellent)

Q_{10}: How familiar are you with mathematical reasoning? We consider mathematical reasoning as a mental activity of an individual, subject to mathematical laws, aimed at studying the surrounding world and establishing patterns between various objects. *

1(Not at all) / 2 / 3 / 4 / 5 (Proficient)

Q_{11}: How confident are you in your ability to understand mathematical proof? *

1(Not at all) / 2 / 3 / 4 / 5 (Very confident)

Q_{12}: Have you written mathematical proof before?

Yes / No

Q_{13}: How would you rate your ability to write mathematical proof? For instance, consider proving the quadratic formula.

1(Poor) / 2 / 3 / 4 / 5 (Excellent)

This section aims to gather information about your communication, and in particular, mathematical communication experience and background. The questions in this section will help us understand the range and depth of communication channels used by participants.

Q_{14}: How important do you consider communication skills in your academic life? *

1(Not important) / 2 / 3 / 4 / 5 (Very important)

Q_{15}: How confident are you in your ability to understand/comprehend mathematical information read? *

1(Not at all) / 2 / 3 / 4 / 5 (Very confident)

Q_{16}: How would you rate your ability to explain mathematical concepts to peers? *

1(Poor) / 2 / 3 / 4 / 5 (Excellent)

Q_{17}: How would you rate your ability to present math-related material or give a presentation about a math-related topic? *

1(Poor) / 2 / 3 / 4 / 5 (Excellent)

Q_{18}: How often do you engage in discussions about mathematics outside of class? *

1(Never) / 2 / 3 / 4 / 5 (Very often)

Q_{19}: When working in groups, how comfortable are you in participating in discussions about mathematical problems? *

1(Not at all) / 2 / 3 / 4 / 5 (Very comfortable)

Q_{20}: How well do you think your math classes prepare you for communicating mathematical concepts? *

1(Very poorly) / 2 / 3 / 4 / 5 (Very well)

Q_{21}: Have you ever struggled to communicate a mathematical idea despite understanding it well yourself?

Yes / No

Q_{22}: Do you think that strong mathematical reasoning skills help improve your communication skills? *

Yes / No

Q_{23}: How important do you think communication skills are in learning and doing mathematics? *

1(Not at all) / 2 / 3 / 4 / 5 (Very important)

Q_{24}: Why do you think it's important for mathematicians and scientists to be able to effectively communicate their findings and ideas to others? *

Imagine you were asked to participate in a panel about mathematics in a middle school. The children in this school are quite eager to learn more about the world. They do not know higher math; however, they pick up simple sailing things quite easily and LOVE to generalize. How would you answer the following questions asked by the students in the audience?

Q_{25}: What is a variable? *

Q_{26}: What is a function? *

Q_{27}: What does it mean for two things to be equivalent? How is it different from equal? *

Q_{28}: Why is it true that the product of a^n and a^m is a^{n+m}? *

Q_{29}: Given functions $f(x)$ and $g(x)$, how does one find the intersection points of their graphs? Why? *

Q_{30}: What does it mean to "solve a system of equations"? *

The following section was administered exclusively to students of MATH 1CO3 course (Group 1) after December 1, 2024, coinciding with the conclusion of the course.

Q_{33}: What is your anticipated course grade?*

Below 60% / 60%-70% / 70%-80% / 80%-90% / Above 90%

Q_{34}: How well do you think a course like MATH 1CO3 prepares you to reason?*

1(Very poorly) / 2 / 3 / 4 / 5 (Very well)

Q_{35}: How well do you think a course like MATH 1CO3 prepares you to communicate about math?*

1(Very poorly) / 2 / 3 / 4 / 5 (Very well)

Q_{36}: Have you participated in the weekly check-ins with the investigator during the semester?*

Yes / Yes, but only once / No

Q_{37}: Elaborate your answer above*:

If you answered *yes* - were the sessions helpful/useful? How or why not?

If you answered *yes, but only once* - what was the reason that you didn't come more?

If you answered *no* - why not?

Appendix B

Interview Guide

The interviews conducted in this study aimed to complement the survey data by exploring participants' perspectives in depth. Each one-on-one session followed an open-ended format designed to encourage detailed responses. Prior to the interviews, participants completed a preliminary survey to introduce them to the research topic.

Participants provided consent at the outset of each interview, which was recorded and subsequently transcribed. The questions were intentionally varied to facilitate natural conversation, with the flexibility to adjust based on participants' responses. A detailed overview of the interview structure and question formulation can be found in Section 3.2.2. Below, Q_i for $i = 0, 1, 2, \ldots$ denotes the question number i.

Q_0: How are you feeling today? Is there anything I can do to make you feel more comfortable?

Q_1: How important do you consider communication in your ordinary life?

Q_2: Does everyone need to learn how to communicate?

Q_3: How do you think one develops such skills or gets better in communicating?

Q_4: Do you think math classes helped you become a better communicator in any way?

Q_5: Have you ever thought of a math class as a class which could help you be a better communicator?

Q_6: What is mathematics to you? What kind of skills do you think it teaches you?

Q_7: Can you describe a specific instance when you had to explain a mathematical concept to someone else? How did you approach it?

Q_8: Have you noticed any changes in how you communicate mathematical ideas since you started studying proofs and mathematical reasoning? If so, what changes?

Q_9: Do you feel that being able to construct a proof has influenced how you structure your arguments or explanations in other areas of life? Can you give an example?

Q_{10}: How do you think participating in mathematical competitions or advanced math courses has influenced your ability to communicate complex ideas?

Q_{11}: Have you had any experiences where good communication skills helped you solve a mathematical problem or understand a mathematical concept better? Please elaborate.

Q_{12}: In your opinion, what role does clear communication play in collaborative mathematical work, such as group projects or study groups?

Q_{13}: How do you feel about presenting mathematical arguments or solutions in front of an audience, such as your classmates or at a competition? Has this changed over time?

Q_{14}: Do you think that math instructors could do more to integrate communication skills into their teaching? If so, what suggestions would you have?

Q_{15}: How do you view the relationship between logical reasoning in math and the clarity of communication in general? Are they related in your experience?

Q_{16}: For high school participants: How do you feel about your communication skills compared to your peers who might not be involved in advanced math courses or competitions?

For university participants: Do you think that the skills you are developing in your current math courses will be useful in your future career, particularly in terms of communication?

For the last part of the interview, participants are given a math problem and a few minutes to work on it.

Problem: A farmer has a rectangular field that measures 150 meters by 100 meters. He wants to divide the field into smaller rectangular sections of equal size to plant different crops. Each section should have a length of 15 meters and a width of 10 meters. How many sections can the farmer create to maximize the use of the available space??

Q_{17}: How would you approach solving this problem? Please explain your thought process.

Q_{18}: Can you describe any challenges you faced while solving this problem and how you overcame them?

Q_{19}: How would you explain this problem and its solution to someone who is not familiar with this type of math problem?

Appendix C

Talking Circle Guide

Talking Circle in this study was mainly designed to help facilitate meaningful discussions and ensure that students can share their experiences, ask questions, and learn from each other. The main researcher facilitated the meetings while allowing students to lead the discussion as much as possible. The meetings were not recorded and lasted around 45-55 minutes. A detailed overview of the talking circle can be found in Section 3.2.3. The following guide and questions were used as a guideline:

- Initial Setup (ONLY during the first meeting): *Welcome to our weekly check-in! Let's start by introducing ourselves and sharing one interesting fact about ourselves or our experience with math so far. What are your first impressions of the course material? Is there anything that excites or worries you already?*

- Introduction: *Welcome, everyone. Let's start by sharing one thing you found interesting or challenging from this week's material.*

- Weekly Material Discussion: *What new concepts did you learn this week? How do you feel about your understanding of these concepts? Was there a particular* *problem or topic that you found especially difficult or confusing?*

- Application and Understanding: *Can someone explain [specific concept] in their own words? How would you teach this to a classmate who missed the lecture?*

- Collaborative Problem Solving: *Let's work through a problem together. Here's a question related to this week's material: [Present a problem]. How would you approach solving this? What steps would you take to ensure everyone understands the solution process? Let's discuss different approaches and clarify any misunderstandings.*

- Communication Skills: *Take a moment to explain a concept you found difficult to a partner. How did explaining it help you understand it better? What strategies do you use to communicate complex mathematical ideas effectively?*

- Self-Assessment and Goals: *Looking back at this week, what do you think you did well, and what areas do you think you need to improve?*

- Closing: *What is one key takeaway from today's discussion that you will apply to your studies next week? Remember, it's okay to find things challenging. Keep up the good work and continue to support one another.*

- Reflections (ONLY during the last meeting): *Looking back at the whole semester, what did you think about the course? If you had to describe MATH 1C03 course in one sentence, what would you say? How would you advertise the course for future students? What would you do different if you could take the course again? Were the talking circle sessions useful? Would you recommend others join similar activities? What exactly were the sessions doing for you?*

Appendix D

Quantitative Analysis: Statistics

This appendix provides the following numerical data discussed in Section 4:

1. Response frequencies for each rating-scale item, broken down by group.
2. χ^2-squared tests of association for categorical answers.
3. Means and standard deviations of the technical questions' scores.
4. Correlation matrix of technical-item the criteria for technical questions, indicating the strength and direction of relationships.
5. Welch ANOVA and Kruskal-Wallis analyses of technical questions' scores identifying differences in means across all three groups.
6. Post-hoc analyses results using Games-Howell tests for pairwise group mean comparisons of technical questions' scores.

Table D.1: Frequencies (in %) of Each Scale Response in Each Group for Questions 8–11, 13–20, and 23, With Percentages Calculated Relative to the Total Responses for That Question Within the Group.

#	**Gr 1**					**Gr 2**					**Gr 3**				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
8	0	6.25	18.75	68.75	6.25	0	0	10	70	20	0	8.33	8.33	41.67	41.67
9	0	6.25	31.25	56.25	6.25	0	0	10	50	40	0	8.33	0	75.00	16.67
10	6.25	43.75	25.00	25	0	0	0	10	40	50	0	0	8.33	58.33	33.33
11	0	25.00	50.00	25.00	0	10	0	10	50	30	0	0	25.00	41.67	33.33
13	12.50	37.50	43.75	6.25	0	0	0	20	50	30	0	0	25.00	33.33	41.67
14	0	0	12.50	37.50	50.00	0	10	10	20	60	0	0	8.33	8.33	83.33
15	0	0	31.25	62.50	6.25	0	0	20	50	30	0	0	16.67	75.00	8.33
16	0	6.25	31.25	37.50	25.00	0	0	20	70	10	0	16.67	0	66.67	16.67
17	12.50	6.25	25.00	56.25	0	0	0	40	40	20	8.33	16.67	16.67	33.33	25.00
18	0	62.50	18.75	6.25	12.50	0	30	20	0	50	0	8.33	8.33	25.00	58.33
19	0	6.25	56.25	18.75	18.75	0	0	20	30	50	0	0	16.67	16.67	66.67
20	6.25	18.75	31.25	25.00	18.75	0	20	40	30	10	0	25.00	33.33	33.33	8.33
23	0	0	18.75	37.50	43.75	0	10	0	40	50	0	0	0	41.67	58.33

Table D.2: Results of Chi-Squared Tests Between Groups for Questions 21 and 22.

	Gr	Yes	No	Yes	No	$\frac{(O-E)^2}{E}$
		(Observed)		(Expected)		
Q_{21}	1	13	3	13.05	2.95	0.0002
	2	8	2	8.16	1.84	0.0031
	3	10	2	9.79	2.21	0.0045
	Chi-square test statistic (χ^2)				0.0078	
	Degrees of Freedom				2	
	p-value				0.996	
Q_{22}	1	14	2	13.05	2.95	0.0692
	2	9	1	8.16	1.84	0.0865
	3	10	2	9.79	2.21	0.0045
	Chi-Square Test Statistic (χ^2)				0.1601	
	Degrees of Freedom				2	
	p-value				0.923	
End of Table						

Table D.3: Summary of the Total Scores for Technical Questions of the Survey.

Gr	Count	Sum	Mean(out of 36)	Standard Deviation	Median	Variance
1	18	420	23.333	4.07	25.00	15.8824
2	8	223	27.875	3.09	27.50	9.5536
3	12	304	25.333	3.42	25.50	10.2424

Table D.4: Mean Scores of Participant Responses Across All Criteria in Technical Questions of the Survey.

#	**Gr**	Criteria				**Total Score**
		Accuracy	Accessibility	Examples	Justification	
Q_{25}	1	36.11%	80.56%	13.89%	8.33%	46.30%
	2	100%	87.5%	37.5%	37.5%	75%
	3	50%	79.17%	33.33%	25%	52.78%
Q_{26}	1	94.44%	86.11%	27.78%	22.22%	68.52%
	2	100%	62.5%	37.5%	50%	68.75%
	3	100%	87.5%	41.67%	33.33%	75%
Q_{27}	1	66.67%	66.67%	22.22%	33.33%	53.70%
	2	93.75%	100%	50%	100%	89.58%
	3	87.5%	87.5%	41.67%	66.67%	76.39%
Q_{28}	1	94.44%	86.11%	22.22%	50%	72.22%
	2	81.25%	81.25%	0	62.5%	64.58%
	3	95.83%	95.83%	8.33%	58.33%	75%
Q_{29}	1	91.67%	86.11%	11.11%	61.11%	71.30%
	2	100%	81.25%	62.5%	100%	87.5%
	3	75%	91.67%	8.33%	50%	65.28%
Q_{30}	1	86.11%	88.89%	27.78%	83.33%	76.85%
	2	93.75%	93.75%	12.5%	87.5%	79.17%
	3	95.83%	83.33%	8.33%	100%	77.78%
Average	1	78.24%	82.41%	20.83%	43.06%	64.81%
	2	94.79%	84.38%	33.33%	72.92%	77.43%
	3	84.03%	87.5%	23.61%	55.56%	70.37%

Table D.5: Standard Deviations of Participant Responses Across All Criteria in Technical Questions of the Survey.

#	**Gr**	Criteria				**Total Score**
		Accuracy	Accessibility	Examples	Justification	
Q_{25}	1	28.73%	25.08%	46.09%	38.35%	22.55%
	2	0	23.15%	51.75%	51.75%	17.82%
	3	36.93%	33.43%	49.24%	45.23%	32.44%
Q_{26}	1	16.17%	86.11%	23.04%	46.09%	45.23%
	2	0	23.15%	51.75%	53.45%	18.77%
	3	0	22.61%	51.49%	49.24%	18.12%
Q_{27}	1	45.37%	45.37%	42.78%	48.51%	39.42%
	2	17.68%	0	53.45%	0	8.63%
	3	22.61%	22.61%	51.49%	49.24%	26.07%
Q_{28}	1	16.17%	28.73%	42.78%	51.45%	22.87%
	2	25.88%	25.88%	0	51.75%	18.77%
	3	14.43%	14.43%	28.87%	51.49%	11.24%
Q_{29}	1	25.72%	28.73%	32.34%	50.16%	23.44%
	2	0	25.88%	51.75%	0	14.77%
	3	26.11%	19.46%	28.87%	52.22%	18.06%
Q_{30}	1	23.04%	21.39%	57.45%	38.35%	18.20%
	2	17.68%	17.68%	35.36%	35.36%	14.77%
	3	14.43%	24.62%	28.87%	0	10.86%
Average	1	25.87%	28.72%	44.59%	44.93%	25.5%
	2	11.03%	23.06%	40.35%	40.26%	18.49%
	3	18.38%	22.91%	37.47%	39.64%	18.14%

Table D.6: Correlation Matrix for the Four Criteria Used to Evaluate Technical Questions in the Survey.

Criteria	Accuracy	Accessibility	Examples	Justification
Accuracy	1	0.487	0.135	0.4046
Accessibility	0.487	1	0.209	0.274
Examples	0.135	0.209	1	0.3546
Justification	0.4046	0.274	0.3546	1

Table D.7: Welch ANOVA and Kruskal-Wallis Tests Results for Technical Questions. Statistical Significance Was Assessed at $\alpha = 0.05$.

Test	Statistic	Degrees of Freedom	p-Value
Welch ANOVA	$F = 4.8043$	(2, 19.513)	0.02011
Kruskal-Wallis	$\chi^2 = 7.82$	2	0.020

Table D.8: Results of Post-Hoc Comparisons Using Games-Howell Tests for Technical Question Scores Across Group Pairs. Degrees of Freedom and Confidence Intervals Are Denoted by df and CI, Respectively.

Comparison	Mean Difference	95% CI	t-Statistic	df	p-Value
Gr 1 vs Gr 2	-4.55	[-8.38, -0.72]	-3.41	11.12	0.017
Gr 1 vs Gr 3	-2.00	[-5.36, 1.36]	-1.40	15.32	0.348
Gr 2 vs Gr 3	2.54	[-1.02, 6.10]	1.70	10.34	0.242