Non-collapsed Steady Ricci Solitons on a Cohomogeneity One-type Ansatz with a circle bundle over a product of Fano Kahler Einstein spaces as principal orbit

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1 Review and Summary of Results

We will ultimately be interested in an ODE system whose solutions give rise to complete metrics on a space whose differential structure is that of a complex line bundle P over a product of Fano Kähler Einstein manifolds as base. That is, we'll assume there are m Fano manifolds F_i such that P is given as a fibration:

$$\mathbb{C} \to P \to F_1 \times \ldots \times F_m$$

(see section 2 for a more thorough description of the setup)

Ansätze of this form have been thoroughly studied both from the perspective of Ricci Solitons and Einstein manifolds, with the earliest work on spaces of this form going all the way back to Calabi's original work on Kähler geometry. Meanwhile, Don Page [12], working at a similar time as Calabi, found one of the first non-Kähler examples of an Einstein metric (his was of positive curvature) when studying a similar Ansatz.

After this, Berard Bergery [3] systematized the examples of Calabi and Page via the cohomogeneity one approach. Working independently at a similar time, Page & Pope [13] found an explicit solution in the Einstein case.

Focusing on the one-factor case, in [17], Stolarski showed that, regardless of the Euler class, by taking $\ddot{u}(0)$ sufficiently small, there exists a complete steady soliton metric with this initial condition. Following this, Wink in [19] showed that a similar result holds in the case of a multi-factor base.

While all of the examples found by Wink and Stolarski were necessarily collapsed, Appleton, focusing on the one-factor case, found in [2] that there actually do exist non-collapsed steady Ricci solitons, but only when the Euler class is larger than the Chern class of the base. As only non-collapsed solitons can appear as singularity models of the Ricci flow, this was a very exciting find.

The main result of this paper may be summarized as follows (see section 2 for more details on the setup):

Theorem 1. *Consider a cohomogeneity one-type ansatz with principal orbit P a circle bundle over a product of Fano Kähler Einstein factors F_i:*

$$S^1 \to P \to F_1 \times \ldots \times F_m$$

Since the chern classes $c_1(F_i)$ are neccessarily integral classes, we may write $c_1(F_i) = p_i \eta_i$ with $p_i \in \mathbb{Z}^+$ and η_i an indivisible class.

We may then choose the above bundle P so that the Euler class is a $\mathbb{Z}$-linear combination of these indivisible classes

$$e(P) = \sum_{i=1}^{m} q_i \eta_i = \sum_{i=1}^{m} \frac{q_i}{p_i} c_1(F_i)$$

where $q_i \in \mathbb{Z}$. In this way, we say that the Euler class is a rational combination of the Chern classes.

If we make the assumption that $q_i \neq 0$ for each i, then there exists a complete, non-collapsed Steady Ricci Soliton on a manifold of the above type provided any one of the q_i are taken to be sufficiently large.

Remark: Following the physics terminology, we could just as easily call our Ansatz one of Kaluza-Klein type. Such metrics have been long studied in the physics literature as models of gravity coupled to electromagnetism. (since EM is described by $U(1)$ gauge theory, it corresponds geometrically to a circle bundle)

2 Detailing the setup: Cohomogeneity one metrics and our Fano Principal orbit

2.1 Introducing Ricci Solitons

The Ricci flow is the evolution of a Riemannian metric g_t according to the following partial differential equation:

$$\partial_t g_t = -2Ric\,(g_t)$$

While short time existence (and uniqueness) were already established in Hamilton's original work [10] (and was later simplified by DeTurk in [8]), the long-term behavior is far more subtle. Famously, Grigori Perelman (see [14], [15] and [16]) refined the picture of the flow in three dimensions, leading to amazing results, most famously the three-dimensional case of Poincaré's conjecture. In higher dimensions, though, much still remains mysterious about what happens near singularities.

Despite these difficulties, there are nevertheless some important results which help guide the way. In particular, it is known that singularity development is modeled by a particular class of solutions known as Ricci solitons. These solitons come in three flavours, expanding, steady and shrinking with only the latter two being relevant for the formation of singularities (although, expanding solitons do still help shed light in different ways as can be seen in, for example, the paper of Angenent and Knopf [1]).

The Soliton equation itself is given by:

$$Ric\,(g) + \frac{1}{2}\mathcal{L}_X g + \frac{\epsilon}{2} g = 0$$

Note the striking resemblance this has to the Einstein equation (albeit with a change in sign for the constant). Indeed, the Einstein equation is just the Ricci Soliton equation but with a zero (or Killing) vector field X. Because of this, we will see that the question of Einstein metrics is never far from mind when studying Ricci solitons and, indeed, many of the explicit examples of Ricci solitons which have thusfar been found were discovered by returning to an earlier found example of an Einstein metric (Our own example here is but one example of this).

The character of a Ricci soliton is determined by the constant ϵ and is called expanding, steady or shrinking depending on whether $\epsilon > 0$, $\epsilon = 0$ or $\epsilon < 0$ respectively. Note that, because of the sign flip, positive Einstein metrics are shrinking solitons.

While, in the general case, one may have quite complicated vector fields X, a particularly simple example is provided by the case where the vector field is the gradient of some function, so $X = grad(u)$ and $\mathcal{L}_X g$ is replaced with $Hess_g(u)$, so that the soliton equation now reads:

$$Ric\,(g) + Hess_g(u) + \frac{\epsilon}{2} g = 0$$

2.2 Introducing Cohomogeneity one

When approaching the problem of constructing a Ricci soliton, an Einstein metric, or any other geometric structure of interest, one is immediately confronted with the issue that nonlinear PDEs are an exceedingly difficult class of problems to solve. One hopes that by checking a more restrictive case, one might begin to make some progress on building an intuition for what to expect (or, perhaps more importantly, not expect) from the more general case of the problem.

One technique which has proven useful on many occasions is to consider a solution which is symmetric. Indeed, for much of the early period of investigation into the Einstein equations, many concepts could only be verified to exist on homogeneous spaces, where the PDE system reduces to a problem of real algebraic geometry. Homogeneous solutions are those which possess a transitive group action, which allows one to reduce the entire problem to an algebraic one at any chosen point of the space.

One step removed from this approach is to assume that the group action, instead of being transitive, is cohomogeneity one. This means that, at a generic point, the group action sweeps out a codimension one submanifold (which we'll call the principal orbit) of our space. In principle, this leaves us with a warped product over the line, although we achieve much greater generality by allowing for singular points, or "blow-downs". This is a point at which our codimension one submanifold reduces to something of lower dimension. To get a handle on this, let's consider the simplest possible case: we'll let the principal orbit be a circle, so that our space looks like a segment of a cylinder at a generic point. By shrinking the size of the circle more and more, it eventually reduces to a point, "capping off" the cylinder and leaving us with a disk. Indeed, this is exactly how we write the standard 2-dimensional euclidean metric in polar coordinates:

$$g_{\mathbb{E}} = dr^2 + r^2 d\theta^2$$

We see from the r^2 factor that the circle shrinks to zero at the origin and grows without bound as r grows (such asymptotic behaviour is called "asymptotically conical" in contrast to the "asymptotically cylindrical" space we would get if the function in front limited to a constant or the "asymptotically paraboidal" that occurs between these two).

In the most general case, we have:

$$g = dt^2 + g_P(t)$$

where g_P is a t-dependent metric on the principal orbit P.

Pinning down the beginning of the cohomogeneity one approach in general is extremely difficult as everything from constructing metrics on surfaces of revolution to the fundamental solutions of the laplace equation can all be argued to fit within this category. Instead, we'll begin our discussion at the systematic approach taken by Eschenburg and Wang in [9] . In their paper, they analysed the Einstein equation on a cohomogeneity one manifold and came up with the following terminology to keep track of all of the relevant pieces:

We consider a cohomogeneity one solution near a singularity to be described by a triple (G,H,K) of lie groups, with our principal orbit being acted on by G, with isotropy group K. The singular manifold this reduces to (in the same way the circle reduced to a point in our surface example) is then given by G/H. Thus, for the construction to make sense, we require the chain of inclusions $K \subseteq H \subseteq G$.

One key fact that constrains what is possible in the cohomogeneity one case is that, in order to ensure smoothness, the collapsing manifold must be a sphere. This then shows that the quotient H/K must be a sphere of some dimension. In our case, this will simply be a circle.

For our purposes, it will be more expedient to refer to [5], where the soliton case is treated, leading to the following set of equations:

$$\begin{aligned} -\mathrm{tr}(\dot{L}) - \mathrm{tr}(L^2) + \ddot{u} = 0 \\ -\dot{L} - (-\dot{u} + \mathrm{tr}(L))L + r = 0 \end{aligned} \tag{1}$$

Where L is the shape operator of the singular orbit satisfying $\dot{g} = g^{-1}L$, u is the soliton potential and r is the Ricci endomorphism of the metric on the principal orbit obtained by pullback.

For our case of interest, the principal orbit consists of a circle bundle over a product of Fano-Einstein manifolds. For simplicity, we'll assume that there are no symmetries relating different factors of the Fano product, meaning we may diagonalize the metric as:

$$g_P = f(t)^2(d\theta - A)^2 + \sum_{i=1}^{m} g_i(t)^2 g_{F_i}$$

where m is the number of Fano factors F_i and A is a multiple of a connection 1-form on $B = F_1 \times \ldots \times F_m$ whose curvature 2-form form Ω is a real representative for the Euler class of P, $[\Omega] = c_1(B) = \sum_{i=1}^{m} c_1(F_i)$.

This then immediately gives:

$$L = \mathrm{diag}\left(\frac{\dot{f}}{f}, \frac{\dot{g}_1}{g_1}, \ldots, \frac{\dot{g}_m}{g_m}\right)$$

where each $\frac{\dot{g}_i}{g_i}$ appears $d_i = \dim(F_i)$ times, giving a matrix of size $D = 1 + \sum_{i=1}^{m} d_i$ as expected.

2.3 Riemannian submersions and the O'Neill formulae

Now, returning to eq. (1), we are left with the question of how to compute the Ricci portion of the equations. This can be done by application of the O'Neill formulae for Riemannian submersions (as can be found in, for example, [4]) as applied to the fibre sequence:

$$S^1 \to P \to \prod_{i=1}^{m} F_i$$

Recalling that, the Fano-Einstein condition gives:

$$Ric(g_{F_i}) = p_i g_{F_i}$$

where $p_i \in \mathbb{Z}$ is the Chern class. By this, we mean that $p_i\eta$ is the Chern class where η is indivisible. Noting that the Euler class is a rational multiple of the chern class (see [18] for a more thorough discussion of this) allows us to define integers q_i by requiring $q_i\eta$ represents the Euler class

The O'Neill formulae for the Ricci tensor read: (see chapt 9 of [4] for more details)

$$r(U,V) = \hat{r} - (N, T_U V) + (AU, AV) + (\tilde{\delta}T)(U,V)$$

$$r(X,U) = ((\hat{\delta}T), X) + (D_U N, X) - ((\check{\delta}A)X, U) - 2(A_X, T_U)$$

$$r(X,Y) = \check{r}(X,Y) - 2(A_X, A_Y) - (TX, TY) + \frac{1}{2}((D_X N, Y) + (D_Y N, X))$$

Where X, Y are horizontal vectors, U, V are vertical vectors, T is the second fundamental form and A is related to the integrability of the horizontal distribution. ($\hat{r}$ and $\check{r}$ are on the fibre/base respectively) (N depends on T, which we'll see soon means it doesn't matter to our analysis). From here, the standard simplification to make the setup easier is to assume that the vertical fibres are totally geodesic so that $T = 0$ and furthermore take the connection associated to the horizontal distribution to be Yang-Mills so that $\check{\delta}A = 0$. This is the standard procedure as can be seen in, for example, [18]. Entering these assumptions into the O'Neill Formulae above gives:

$$r(U,V) = \hat{r} + (AU, AV) = 0 + (AU, AV)$$

$$r(X,U) = 0$$

$$r(X,Y) = \check{r}(X,Y) - 2(A_X, A_Y)$$

Where the first line uses the fact that all one-dimensional spaces are trivially flat (so that $\hat{r} = 0$).

Thus, we see that the only things to compute in this are the values of

$$(AU, AV) = \sum_{i=1}^{\dim(base)} (A_{X_i}U, A_{X_i}V)$$

and

$$(A_X, A_Y) = \sum_{i=1}^{\dim(base)} (A_X X_i, A_Y X_i) = \sum_{i=1}^{\dim(fibre)} (A_X U_i, A_Y U_i)$$

Where $\{X_i\}$ and $\{U_i\}$ are orthonormal bases of the base/fibre respectively. For the case of a single circle factor, we have the formula: (see 9.65 of Besse)

$$A_X Y = -\frac{1}{2}\omega(\check{X}, \check{Y})\hat{U}$$

While

$$(AU, AU) = \sum_{i=1}^{1}(AU_i, AU_i) = |A|^2 = \sum_{i=1}^{\dim(base)} (A_{X_i}, A_{X_i})$$

$$= \sum_{i,j}(A_{X_i}X_j, A_{X_i}X_j) = -\frac{1}{4}\sum_{i,j}\left(\omega(X_i, X_j)\right)^2 (U, U)$$

So assuming $(U, U) = \rho^2$, the above is equal to:

$$= \frac{1}{4}\sum_{i,j}\rho^2\left(\omega(X_i, X_j)\right)^2$$

While

$$(A_X, A_Y) = \sum_i (A_X X_i, A_Y X_i) = \frac{1}{4}\sum_i \omega(X, X_i)\omega(Y, X_i)(\hat{U}, \hat{U}) = \frac{1}{4}\sum_i \rho^2\omega(X, X_i)\omega(Y, X_i)$$

Now, when evaluating the above, we turn to [18]

We have that, the curvature form Ω above is given by $\Omega = \sum_{i=1}^{\dim(base)} q_i\omega_i$ where ω_i is the Kähler class of the i^{th} Fano factor.

In order to make sense of this, we turn to the standard equations for a Fano manifold

$$\omega(X, Y) = g(JX, Y)$$

Because of this, if we are to expand the above in an orthonormal basis $\{X_i\}$, we arrive at:

$$\omega(X_i, X_j) = g(JX_i, X_j)$$

Now, we can expand the endomorphism J in coordinates with respect to this basis:

$$JX_i = \sum_k J_i^k X_k$$

So that:

$$\omega(X_i, X_j) = \sum_k g(J_i^k X_k, X_j)$$

$$\omega(X_i, X_j)^2 = \left(\sum_k g(J_i^k X_k, X_j)\right)^2 = \sum_{k,m} g(J_i^k X_k, X_j)g(J_i^m X_m, X_j)$$

$$\sum_{k,m} J_i^k J_i^m g(X_k, X_j) g(X_m, X_j) = \sum_{k,m} J_i^k J_i^m \delta_{kj}\delta_{mj}$$

$$\implies \sum_{ij} \omega(X_i, X_j)^2 = \sum_{i,j,k,m} J_i^k J_i^m \delta_{kj}\delta_{mj} = \sum_{ij} J_i^j J_i^j = \mathrm{tr}(J^T J)$$

But, for a Kähler manifold we have that the complex structure is orthogoanal:

$$g(JX, JY) = g(X, Y) \implies J^T J = 1$$

This then gives us (remember: the norm of a two-form is the sum with $i < j$):

$$2\|\omega\|^2 = \sum_{ij} \omega(X_i, X_j)^2 = \mathrm{tr}I = d_i$$

Finally, we note that, in our case, we deal not with the original Fano metrics h_i when restricting to these parts of the tangent space, but instead use $g_i^2 h_i$. Since the norm of a two-tensor is given by double contraction with the metric, this then implies

$$\|\omega\|^2_{g_i^2 h_i} = \frac{1}{g_i^4}\|\omega\|^2_{h_i} = \frac{d_i}{2g_i^4}$$

Thus, combining all of the above with the fact that our bases are Einstein (so $\check{r}_i = p_i h_i$), we arrive at the following set of equations:

$$\begin{aligned} \ddot{u} &= \frac{\ddot{f}}{f} + \sum_{i=1}^{m} d_i \frac{\ddot{g}_i}{g_i} \\ \frac{d}{dt}\frac{\dot{f}}{f} &= -\,(\mathrm{tr}L - \dot{u})\frac{\dot{f}}{f} + \sum_{i=1}^{m} \frac{d_i q_i^2}{4}\frac{f^2}{g_i^4} \\ \frac{d}{dt}\frac{\dot{g}_i}{g_i} &= -\,(\mathrm{tr}L - \dot{u})\frac{\dot{g}_i}{g_i} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4} \end{aligned} \tag{2}$$

which will be the entire focus of the rest of the document.

2.4 Initial conditions and smoothness at the singular orbit.

The last piece of information which needs to be fixed to leave us with a well-defined ODE problem is the question of initial conditions. For the ray-type case we consider here, there is a single singular orbit corresponding to the zero of our geodesic variable. The work of Buzano in [5] gives an account of how to derive the initial conditions in the cohomogeneity one setting. Because of the singular orbit, it turns out that we also need to provide some second derivative information at $t = 0$, even though the ODE is second order. Namely, we only get a unique solution if we also prescribe $\ddot{u}(0)$.

Thankfully our case is sufficiently similar that her results still apply. In particular, she was able to show that both u and the g_i are even functions at

$t = 0$ while f is odd. This is already enough to give us $f(0) = \dot{u}(0) = \dot{g}_i(0) = 0$. In order to get something non-zero, we also need to require $\dot{f}(0), g_i(0) \neq 0$ and so, given the geometric picture, it is most natural to require $\dot{f}(0), g_i(0) > 0$. Since changing the metric by homothety doesn't change the geometry, it is common to find a choice of normalization. Often the condition is taken to be such that $vol(M) = 1$ however, in our case, a more convenient choice is to fix our homothety degree of freedom by requiring $\dot{f}(0) = 1$.

Putting all of this together, we see that the only initial conditions left to set are $u(0)$ and $\ddot{u}(0)$, which we will find later by careful analysis of a conserved quantity. Nevertheless, we collect all initial conditions here for later convenience:

$$\begin{gathered} f(0) = \dot{g}_i(0) = u(0) = \dot{u}(0) = 0 \\ \dot{f}(0) = 1, g_i(0) > 0 \\ \ddot{u}(0) = \frac{C}{2} \end{gathered} \tag{3}$$

3 The work of Appleton and Wink

What follows is hoped to be both an extension of the work of Appleton to a slightly broader case, while also serving as a fringe case of the setup studied by Wink. As such, in order to understand the context for what is being attempted here, it will be important to review both of their results.

3.1 Wink's work

We begin with the work of Wink. As I've decided to stick with his notation, we return to system 2. In [19], he approaches a number of Ansatz from the perspective of the soliton potential, including our own. In the course of his discussion, Wink considers both the steady and expanding cases. As my own material covers only the steady case, it is safe to assume $\epsilon = 0$ in what follows. For our case, he was able to prove the existence of a continuous family of complete Ricci solitons, surrounding Kähler-Einstein solutions which had been found earlier by Dancer and Wang in [7]. Crucially, though, Wink relied on a-priori estimates which resulted in all the solutions being collapsed.

In particular, in proving his theorem A, Wink assumes an upper bound on the quantity $\omega_i = \frac{f}{g_i}$. This can be seen explicitly in the statement of Proposition 2.15 from the paper:

Wink's Proposition 3.15. Let $\epsilon \geq 0$ and consider a maximal Einstein or Ricci soliton trajectory in the Dancer-Wang set-up with initial conditions given by eq. (3) and $\ddot{u}(0) \leq 0$. Suppose that $\frac{f^2}{g_i^2} < \frac{2p_i}{q_i^2}$ holds along the entire trajectory if $q_i \neq 0$.

Then this trajectory is defined for all $t \geq 0$ and corresponds to a complete Einstein or Ricci soliton metric.

Since we are requiring

$$\frac{f^2}{g_i^2} = \omega_i^2 < \frac{2p_i}{q_i^2}$$

we see that this set-up is specifically avoiding the critical phenomena which occurs as we let

$$\omega_i^2 = \frac{2p_i}{q_i^2}$$

This corresponds exactly to the $Q = 1$ case of Appleton's approach which we'll see later and in his case this value shows itself to be the critical value above which Q (our ω) is destined to grow without bound (eventually leading to an incomplete solution, whose existence is key to the existence of the noncollapsed trajectory).

3.2 Enter the conserved quantity

Before moving on to the work of Appleton, we'll first need to go over the role of the conserved quantity all (gradient) solitons share.

Recall that, for any C^3-regular gradient steady soliton, we must have

$$R + |\nabla u|^2 + \epsilon u + C = 0$$

for some constant C, function u and where R is the scalar curvature. This equation goes all the way back to Hamilton's original work (cf. [11]) and has proved a key way to gain insight to the geometry of solitons in just about every case found. The hope of this work is that it may help to cast Appleton's results within this broader framework of general solitons.

In our cohomogeneity one setup, this equation takes the form:

$$\ddot{u} + (-\dot{u} + \mathrm{tr}L)\dot{u} = C + \epsilon u \tag{4}$$

in the steady case, we have $\epsilon = 0$ and the equation no longer has any direct dependence on u, instead only depending on $\dot{u}$. As it turns out, the same is true of the steady equations generally, meaning that no matter which value of $u(0)$ we use, the dynamics are entirely unchanged. For this reason, we are safe to set $u(0) = 0$ (as our interest here will only be on the steady case).

Focusing more closely on the equation

$$\ddot{u} + (-\dot{u} + \mathrm{tr}L)\dot{u} = C$$

we note that we may relate the initial conditions to the conserved quantity simply by setting $t = 0$ in the above equation. There is, however, one wrinkle coming from the $\mathrm{tr}L$ term. While $\dot{u}(0) = 0$, we also have that $f(0) = 0$ and, since $\mathrm{tr}L$ contains a $\dot{f}/f$ term, we'll have to make sense of

$$\ddot{u}(0) + \frac{\dot{f}}{f}\dot{u}\bigg|_{t=0} = C$$

we can then evaluate this second term via l'Hopital, leaving us with:

$$C = \ddot{u}(0) + \frac{\dot{u}}{f}\dot{f}\bigg|_{t=0} = \ddot{u}(0) + \frac{\ddot{u}(0)}{\dot{f}(0)}\dot{f}(0) = 2\ddot{u}(0)$$

Thus, we are able to round off our initial conditions with what will turn out to be the most important one:

$$\ddot{u}(0) = \frac{C}{2}$$

In what follows (especially as we begin to discuss the new results of Appleton), we will see that the initial condition $\ddot{u}(0)$ will be central to the analysis and, indeed, to the production of new examples. When we see $\ddot{u}(0)$, we should have in mind the above relation, which tells us that we are equally studying the behaviour of the conserved quantity C. While the quantity $\ddot{u}$ is particular to our setup, this focus on the constant helps us contextualize things within the broader context of soliton solutions.

3.3 Translating Appleton's results to Wink's notation

In his work, Appleton made a different choice in normalizing the fibres which have the result of changing the presentation of the equations. Here we will give a quick account of how this difference comes about and, crucially, how to transfer between a result in Appleton's notation with one in Wink's.

To begin, let us return to our earlier discussion on how to derive the equations in the first place:

The steady soliton equations in [19] are:

$$\begin{aligned}
\ddot{u} &= \frac{\ddot{f}}{f} + \sum_{i=1}^{m} d_i \frac{\ddot{g}_i}{g_i} \\
\frac{d}{dt}\frac{\dot{f}}{f} &= -(\mathrm{tr}L - \dot{u})\frac{\dot{f}}{f} + \sum_{i=1}^{m} \frac{d_i q_i^2}{4}\frac{f^2}{g_i^4} \\
\frac{d}{dt}\frac{\dot{g}_i}{g_i} &= -(\mathrm{tr}L - \dot{u})\frac{\dot{g}_i}{g_i} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4}
\end{aligned} \tag{5}$$

While Appleton's [2] are:

$$\begin{aligned}
f'' &= \frac{a''}{a} + 2n\frac{b''}{b} \\
a'' &= 2n\left(\frac{a^3}{b^4} - \frac{a'b'}{b}\right) + a'f' \\
b'' &= \frac{2n+2}{b} - 2\frac{a^2}{b^3} - \frac{a'b'}{a} - (2n-1)\frac{(b')^2}{b} + b'f'
\end{aligned} \tag{6}$$

Note that, while we use the label t and denote our derivatives by $\dot{f}$, Appleton instead uses s (not to be confused with the s we'll use later on) and denotes derivatives as f', but other than the notation, these have the same meaning for us.

Rearranging the above then gives:

$$f'' = \frac{a''}{a} + 2n\frac{b''}{b}$$
$$\frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = 2n\left(\frac{a^2}{b^4} - \frac{a'b'}{ab}\right) - \left(\frac{a'}{a}\right)^2 + \left(\frac{a'}{a}\right)f' = -\left(-f' + \frac{a'}{a} - 2n\frac{b'}{b}\right)\frac{a'}{a} + 2n\frac{a^2}{b^4}$$
$$\frac{b''}{b} - \left(\frac{b'}{b}\right)^2 = \frac{2n+2}{b^2} - 2\frac{a^2}{b^4} - \frac{a'b'}{ab} - 2n\left(\frac{b'}{b}\right)^2 + \left(\frac{b'}{b}\right)f'$$

which ultimately gives:

$$\begin{aligned} f'' &= \frac{a''}{a} + 2n\frac{b''}{b} \\ \frac{d}{dt}\frac{a'}{a} &= -\left(\frac{a'}{a} + 2n\frac{b'}{b} - f'\right)\frac{a'}{a} + 2n\frac{a^2}{b^4} \\ \frac{d}{dt}\frac{b'}{b} &= -\left(\frac{a'}{a} + 2n\frac{b'}{b} - f'\right)\frac{b'}{b} + \frac{2n+2}{b^2} - 2\frac{a^2}{b^4} \end{aligned} \tag{7}$$

So,

$$2n\frac{a^2}{b^4} = \frac{dq^2}{4}\frac{f^2}{g^4}, \quad \frac{2n+2}{b^2} = \frac{p}{g^2}, \quad 2\frac{a^2}{b^4} = \frac{q^2}{2}\frac{f^2}{g^4}$$

From which we may derive:

$$\begin{aligned} f &\simeq u \\ a &\simeq \frac{q(n+1)}{p}f \\ b &\simeq \sqrt{\frac{2(n+1)}{p}}g \end{aligned} \tag{8}$$

The convenience of Appleton's setup is most apparant in the case where the Fano base is simply a complex projective space $\mathbb{CP}^n$ (and, indeed, his paper spends the most time with this case). It is a well-known fact in complex geometry that the first chern number of projective space is given by:

$$c_1(\mathbb{CP}^n) = n+1$$

Since p is just the chern class, we see that in this case $p = n+1$ so that the above relations simplify to:

$$\begin{aligned} f &\simeq u \\ a &\simeq qf \\ b &\simeq \sqrt{2}g \end{aligned}$$

Since our (Wink's) choices of normalization feature the condition $\dot{f}(0) = 1$, the middle relation above tells us that the initial condition for a (at least in the case of a projective base) is simply given by the Euler class of the bundle.

While this may seem a rather natural thing to do, it is clear that Appleton's choices here have no analogue in the multi-factor case and so Wink's equations are indeed the more natural presentation of the system for our interests.

3.4 Appleton's work

Appleton, in contrast, studied the same setup, but now with only a single Fano factor in the base of the principal orbit. Thanks to the existence of an explicit solution to the Einstein equation in this case, discovered initially by Berard Bergery (see [3]) and later expounded by Page and Pope in [13], Appleton was able to continue Wink's family to a larger domain of the parameters and showed that the corresponding Einstein trajectories would be forced to have finite time blow-up. He found that, right at the cusp of where the solutions cease to exist, there is in fact a noncollapsed solution. Unfortunately, his approach relies on the existence of an explicit solution to the Einstein equation and so is not readily extendable to the multi-factor case without a little work. Partial results in this direction have been achieved, and will be the focus of the final part of this document.

For now, let us return to the question of how Appleton's work diverges from the earlier results of Wink.

The primary deviation from Wink is that Appleton replaces the a-priori estimate on ω_i for a condition on the initial condition of the soliton potential $\ddot{u}(0)$. This can be seen clearly in the statement of his Lemma 7.1:

Lemma 7.1 Let $(f, a, b) : [0, s_0) \to \mathbb{R}^2, s_\infty \in \mathbb{R} \cup \{\infty\}$, be a maximal solution to the soliton equations with initial conditions $a'(0) > n + 1$ and $f''(0) = 0$. Then $Q > 1$ in finite distance s.

Here we can see how the a-priori estimate of Wink's analysis breaks down as the initial value is increased from the negative toward zero.

By the conserved quantity for Ricci solitons, this then transfers to a condition on the soliton constant C in a fairly direct way ($\ddot{u}(0) = \frac{C}{2}$). By reframing in this way, Appleton was able to make explicit the relationship between the collapsed complete solutions and the incomplete ones. Namely, the former exist while C is sufficiently negative and, on increasing C toward zero, we eventually cross over into the incomplete case. As the parameter is a multiple of one of the initial conditions, we are then able to use continuous dependence on initial conditions to argue that there must exist some particular solution at a critical value of C which somehow "interpolates" between the two families on either side. This is precisely the non-collapsed example.

He accomplishes this in his Theorem 7.2, whose essential features can still be seen in the present document (albeit with one slight change).

The key steps to Appleton's proof go through as follows: First, we define a quantity $\omega = \frac{f}{g}$ (Appleton uses $Q = \frac{a}{b}$). We see that if this quantity is bounded above, the solution continues to exist for some longer length of time. This is the content of Appleton's Lemma 5.3:

Lemma 5.3 Let $s_0 > 0$ and $(f, a, b) : [0, s_0) \to \mathbb{R}^3$ be a solution to the soliton equations with $f''(0) \leq 0$. If $Q < \sqrt{n+1}$ on $[0, s_0)$, the solution can be extended past s_0.

This is then contrasted against the beheviour of the solution when this bound is broken as can be seen in Lemma 6.5:

Lemma 6.5 There are no complete solutions $(f, a, b) : [0, \infty) \to \mathbb{R}^3$ to the soliton equations with $f''(0) < 0$ and $n + 1 < Q_\infty^2 < \infty$

Finally, in Lemma 7.1, he shows that a set of initial conditions exist which gaurantee the bound is broken in finite time, thus leading to an incomplete solution in this case. Theorem 7.2 then uses this nonexistence result in an essential way when establishing the existence of his new noncollapsed soliton. Thankfully, 7.2 still works with minor change and the final theorem of this document will be notably similar (albeit with one important change)

3.5 Asymptotics

In his section 6, Appleton covers the topic of Asymptotics proving:

Theorem 6.1 Let $(u, f, g) : [0, \infty) \to \mathbb{R}^3$ be a solution to the soliton equations with $\ddot{u}(0) < 0$. Then *either* $\lim_{t\to\infty} Q = 0$ or $\lim_{t\to\infty} Q = 1$. Furthermore

1. if $\lim_{t\to\infty} Q = 0$ we have $f \sim \text{const}$ and $g \sim \text{const}\sqrt{t}$
2. if $\lim_{t\to\infty} Q = 1$ we have $f \sim g \sim \text{const}\sqrt{t}$

as $t \to \infty$.

Naturally, we hope to derive an analogue of this for our case. For our purposes, the asymptotics of our new solutions were actually already covered in Wink's paper. Because his statement slightly differs, I have decided to include the argument though it should be noted there is nothing essentially new here (other than the trivial step at the end). See Proposition 3.18 for Wink's version.

Theorem 2. *Suppose the ω_i all have finite limits as $t \to \infty$.*

- *Then there are at most two possibilities for the limit of $\vec{\omega} = (\omega_1, \omega_m)$, one of which is the origin and the other has $\omega_i > 0$ for each i.*
- *Then, $g_i \to k_i\sqrt{t}$ asymptotically for some constants k_i.*

- *If, in addition, none of the ω_i have a limit of zero, then f has the same asymptotic behaviour (up to a constant)*

Proof. We may rewrite eq. (2) as:

$$\ddot{f} = -(-\dot{u} + \mathrm{tr}L)\dot{f} - \frac{\dot{f}^2}{f} + \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \frac{\omega_i^4}{f}$$

$$\ddot{g}_i = -(-\dot{u} + \mathrm{tr}L)\dot{g}_i - \frac{\dot{g}_i^2}{g_i} + \frac{p_i - \frac{q_i^2}{2}\omega_i^2}{g_i}$$

As in Wink, we rely on a result from Appleton's paper provided by Jon Wilkening (see Lemma 6.2 of Appleton).

As Wink does, we rely on a result from Appleton's paper. Namely, Lemma 6.2 of his paper (see pp. 12 & 24 of the paper) gives us the following:

Lemma 1. *Let $\alpha > 1$, $\epsilon > 0$ and $c_1^*, c_2^* > \epsilon$. Assume $c_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2$, are two positive smooth functions satisfying*

$$|c_i(t) - c_i^*| < \epsilon, i = 1, 2,$$

for all $t \geq 0$. Then for a solution $y : [0, \infty) \to \mathbb{R}$ to the ODE

$$y'' = \frac{c_1(t)}{2y} - \alpha\frac{(y')^2}{y} - c_2(t)y' \tag{9}$$

with initial conditions $y(0), y'(0) > 0$ there exists an $t_0 > 0$ such that for $t > t_0$

$$y^2(t_0) + \gamma_-(1+\epsilon)^{-1}(t - t_0) \leq y^2(t) \leq y^2(t_0) + \gamma_+(t - t_0),$$

where

$$\gamma_\pm = \frac{c_1^* \pm \epsilon}{c_2^* \mp \epsilon}$$

In particular, the Lemma requires that the functions c_1 and c_2 are bounded.

We will prove later on (see the beginning of section 3) that f is monotone increasing, while $\dot{g}_i = 0$ can only occur if $\omega_i \to \infty$. Putting these two together, the assumption of finite limit for ω_i is then enough to ensure the shape operator remains positive-definite along the trajectory and so $\mathrm{tr}L \to 0$ as $t \to \infty$. Meanwhile, Wink was able to show in his Proposition 2.3 that in such cases we must also have $-\dot{u} \to \sqrt{-C}$ where C is the Ricci soliton conserved quantity we saw earlier. All of this together is enough to account for the c_1, which is identical in both cases.

Meanwhile, dealing with c_2 will require a little more. First note that, for f, we cannot ensure c_2 is bounded away from zero as the hypotheses require when the $\omega_i \to 0$. Further, even if all ω_i are positive, we still need to make sure we have $p_i - \frac{q_i^2}{2}\omega_i^2 \neq 0$.

Nevertheless, applying the lemma to f and g_i then gives us (see Wink Prop 3.18 for the original argument) that for all small $\epsilon > 0$ there is $t_0 > 0$ such that

$$f(t_0)^2 + \gamma_-(1+\epsilon)^{-1}(t-t_0) \leq f^2(t) \leq f(t_0)^2 + \gamma_+(t-t_0)$$

$$g_i(t_0)^2 + \Gamma_{i,-}(1+\epsilon)^{-1}(t-t_0) \leq g_i^2(t) \leq g_i(t_0)^2 + \Gamma_{i,+}(t-t_0)$$

where

$$\gamma_\pm = \frac{\sum_{i=1}^m \frac{d_i q_i^2}{2}\omega_{i,\infty}^2 \pm \epsilon}{\sqrt{-C} \mp \epsilon} \text{ and } \Gamma_{i,\pm} = \frac{2p_i - q_i^2\omega_{j,\infty}^2 \pm \epsilon}{\sqrt{-C} \mp \epsilon}$$

This yields $\frac{\gamma_-}{\Gamma_{i,+}} \leq \omega_{i,max}^2 \leq \frac{\gamma_+}{\Gamma_{i,-}}$ for all small $\epsilon > 0$, hence,

$$\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4 \tag{10}$$

whose common locus of solution contains only two points, one being the origin and the other satisfying $\omega_{i,\infty} > 0$ for each i. We prove this by breaking into cases:

Zero case: Suppose any of the ω_i are zero. The corresponding equation

$$\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$$

then clearly has a zero LHS.

Thus,

$$\sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4 = 0$$

But, since ω_j appears to even power and the coefficients are all positive, it follows that we must then have $\omega_j = 0$ for every j.

Nonzero case: Now, instead suppose that $\omega_i > 0$ for each i. We begin by reshuffling our equation:

$$\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_i^2 = \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$$

$$\implies p_i\omega_i^2 - 2\frac{q_i^2}{4}\omega_{i,\infty}^4 = \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$$

$$\implies p_i\omega_i^2 - \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4 = 2\frac{q_i^2}{4}\omega_{i,\infty}^4$$

Now, since we are assuming $\omega_i > 0$ for each i, we must have that $\sum \frac{d_j q_j^2}{4}\omega_j^4 > 0$ too. Thus, we can divide to obtain:

$$p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1} - 1 = 2\frac{q_i^2}{4}\omega_{i,\infty}^4 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1}$$

$$\implies \left(p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1} - 1\right)\left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right) = 2\frac{q_i^2}{4}\omega_{i,\infty}^4$$

$$\left(p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1} - 1\right)\left[\sum \frac{d_j q_j^2}{4}\omega_j^4 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-2}\right] = 2\frac{q_i^2}{4}\omega_{i,\infty}^4 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-2}$$

$$\left(p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1} - 1\right)\left[\sum \frac{d_j q_j^2}{p_j^2}p_j^2\omega_j^4 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-2}\right] = 2\frac{q_i^2}{p_i^2}p_i^2\omega_{i,\infty}^4 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-2}$$

$$\begin{aligned}\left(p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1} - 1\right)&\left(\sum d_j \left(\frac{q_j}{p_j}\right)^2 \left[p_j\omega_j^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1}\right]^2\right) \\ &= 2\left(\frac{q_i}{p_i}\right)^2 \left[p_i\omega_{i,\infty}^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1}\right]^2\end{aligned}$$

Thus, if we define new variables s_i by:

$$s_i := p_i\omega_i^2 \left(\sum \frac{d_j q_j^2}{4}\omega_j^4\right)^{-1}$$

The above becomes:

$$(s_i - 1)\sum d_i \left(\frac{q_j}{p_j}\right)^2 s_j^2 = 2\left(\frac{q_i}{p_i}\right)^2 s_i^2$$

These equations go back to Wang and Ziller in [18] (cf. pp.222-223). There, when considering the question of Einstein metrics on a generalization of our principle orbit (they consider general torus bundles over a product Einstein-Fano base) they show the equations have unique positive solution.

Putting the above two together then gives us that there are only two possible finite limits for the ω_i, one being the origin, the other positive.

Wrapping up Thus, if we sit at the positive point, we can then see from eq. (10) that $p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2 \neq 0$ as follows: Since we know $\omega_j > 0$ for each j, the RHS of eq. (10) must be positive immediately contradicting the zero on the LHS. In this way, we see that the $\Gamma_{i,\pm}$ are still bounded away from zero in this case. Meanwhile, at the origin, this is just p_i and so, still, is nonzero. This retroactively justifies our use of the lemma by ensuring c_2 satisfies the relevant hypotheses (at least in the g_i case).

Thanks to this, we may apply the theorem directly to g_i to get the desired asymptotics.

Once we have the result for g_i, the following result for f simply comes from the fact that the quotient $\frac{f}{g_i}$ being asymptotically a (nonzero) constant implies the numerator must have the same asymptotics as the denominator.

□

4 The multi-factor case:

4.1 Initial estimates on f and g

Before beginning our analysis in earnest, we first establish some estimates on our functions which will prove helpful over and over.

The following proposition provides us a partial analogue of Appleton's Lemma 4.1 (we leave the g_i part of the lemma until after we've covered the behaviour of the ω_i in more detail as they play a larger role there).

Proposition 1. *Let $t_0 > 0$ and $(u, f, g_i) : [0, t_0) \to \mathbb{R}^{m+2}$ be a smooth solution to the soliton equations 2. Then f is a strictly increasing function on $[0, t_0)$*

Proof. By the initial conditions, $\dot{f}(0) > 0$, so if f is ever to decrease, we'd first have to have $\dot{f} = 0$. The evolution equation 2 of f implies

$$\ddot{f} = \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \frac{f^3}{g_i^4} > 0$$

whenever $\dot{f} = 0$ since f must clearly still be positive at this time. Meanwhile, since $g_i(0) > 0$, $1/g_i$ is initially positive and can only become zero if $g_i = \infty$, but this can't happen within the domain of our solution. Since the derivative of $\dot{f}$ is positive when it reaches zero from above, this leads to a contradiction and so $\dot{f} > 0$ so long as the solution exists.

□

We will also find it useful to have a relative bound, which gives us:

Lemma 2. *Along a solution to the soliton equations eq. (2), we have the following bound:*

$$\dot{f} \geq \frac{C}{\prod_i g_i^{d_i} e^{-u}}$$

for some positive constant C. In particular, $\dot{f} \to \infty$ if $g_i \to 0$ for any i.

Proof. In order to prove this, we'll need only the equation for f:

$$\frac{d}{dt}\frac{\dot{f}}{f} = -(\mathrm{tr}L - \dot{u})\frac{\dot{f}}{f} + \sum_{i=1}^{m}\frac{d_i q_i^2}{4}\frac{f^2}{g_i^4}$$

Since every non-constant term appears to an even power, it is easy to see we must have:

$$\sum_{i=1}^{m}\frac{d_i q_i^2}{4}\frac{f^2}{g_i^4} \geq 0$$

From which it immediately follows:

$$\frac{d}{dt}\frac{\dot{f}}{f} \geq -(-\dot{u}+\mathrm{tr}L)\frac{\dot{f}}{f} = -\left(-\dot{u} + \frac{\dot{f}}{f} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right)\frac{\dot{f}}{f} = -\left(-\dot{u} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right)\frac{\dot{f}}{f} - \left(\frac{\dot{f}}{f}\right)^2$$

$$\implies \frac{d}{dt}\frac{\dot{f}}{f} = \frac{\ddot{f}}{f} - \left(\frac{\dot{f}}{f}\right)^2 \geq -\left(-\dot{u} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right)\frac{\dot{f}}{f} - \left(\frac{\dot{f}}{f}\right)^2$$

$$\implies \frac{\ddot{f}}{f} \geq -\left(-\dot{u} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right)\frac{\dot{f}}{f}$$

Now, we divide by $\dot{f}/f$, which requires us to be able to claim this term is positive, which we just did in the previous lemma. Hence:

$$\frac{d}{dt}\ln(\dot{f}) = \frac{\ddot{f}}{\dot{f}} \geq -\left(-\dot{u} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right) = -\frac{d}{dt}\ln\left(e^{-u}\prod_i g_i^{d_i}\right)$$

Integrating the above from $t_0 > 0$ (to avoid issues with $f(0) = 0$) to t immediately yields our result. □

4.2 Evolution of the ω_i

Now we turn our attention to the ω_i, whose dynamics are central to what follows. Before going on to the multi-factor case, let us take a moment to go over the picture in the one-factor case.

As he derives in his section 3, the evolution of Appleton's Q quantity is given by the equation:

$$Q'' = \left(f' - (2n+1)\frac{b'}{b}\right)Q' + \frac{2n+2}{b^2}Q(Q^2-1)$$

So that, at a critical point of Q, we have:

$$Q''_{\mathrm{crit}} = \frac{2n+2}{b^2}Q(Q^2-1)$$

From this expression, we can see that the sign is entirely dependent on the size of Q and, in particular, whether $Q > 1$ or $0 < Q < 1$. Thus, any solution with a maximum for Q (as all collapsed solutions must have) must always lie in $0 < Q < 1$. Meanwhile, $Q^2 = n+1$ sits above this and whose role was expounded in section 2.4.

Using our formulae derived back in section 2.3, we can now relate the above to their equivalents in our normalization. We see that:

$$Q = \frac{a}{b} = \frac{\frac{q(n+1)}{p} f}{\sqrt{\frac{2(n+1)}{p}}\, g} = \frac{q(n+1)}{p} \frac{\sqrt{p}}{\sqrt{2(n+1)}} \frac{f}{g} = \frac{q\sqrt{n+1}}{\sqrt{2p}} \omega$$

$$\implies \omega = \frac{\sqrt{2p}}{q\sqrt{n+1}} Q$$

$$\implies \omega^2 = \frac{2p}{q^2(n+1)} Q^2 = \frac{4p}{q^2(d+2)} Q^2$$

Hence, it follows that Appleton's condition $Q = 1$ corresponds to $\omega^2 = \frac{4p}{(d+2)q^2}$, while $Q^2 = n+1$ corresponds to $\omega^2 = \frac{2p}{q^2}$.

Now, in the multi-factor case, we instead have (see Wink p.22):

$$\ddot{\omega}_i = \dot{\omega}_i \left(\dot{u} - (d_i+1)\frac{\dot{g}_i}{g_i} - \sum_{j \neq i} d_j \frac{\dot{g}_j}{g_j} \right) + \frac{\omega_i}{g_i^2} \left(\frac{q_i^2}{2}\omega_i^2 - p_i + \sum_{j=1}^{m} \frac{d_j q_j^2}{4} \frac{f^2}{g_j^2} \frac{g_i^2}{g_j^2} \right)$$

In the multi-factor setting, it is significantly less obvious to interpret this and, indeed, we will find it useful in the sequel to note it may also be written as:

$$\ddot{\omega}_i = \dot{\omega}_i \left(\dot{u} - (d_i+1)\frac{\dot{g}_i}{g_i} - \sum_{j \neq i} d_j \frac{\dot{g}_j}{g_j} \right) + \frac{\omega_i}{f^2} \left(\frac{q_i^2}{2}\omega_i^4 - p_i\omega_i^2 + \sum_{j=1}^{m} \frac{d_j q_j^2}{4} \omega_j^4 \right)$$

This form is useful as it replaces $Q(Q^2-1)$ above for polynomials only depending on the ω_i (note these are exactly the ones which appeared in our discussion of asymptotics earlier, whose common zero locus determines the possible limits of the ω_i). The zero set of this polynomial then acts as a non-linear analogue of the $Q = 1$ condition from Appleton. This is seen in the figure above, where we've also plotted the lines $\omega_i^2 = \frac{4p}{(d+2)q^2}$ and $\omega_i^2 = \frac{2p}{q^2}$, whose relation to Appleton's picture is seen above and whose relevance to the multi-factor setting will become clear as we continue.

Thus, along a soliton solution of eq. (2), a critical point of ω_i satisfies:

$$\ddot{\omega}_{i,\mathrm{crit}} = \frac{\omega_i}{g_i^2} \left(\frac{q_i^2}{2}\omega_i^2 - p_i + \frac{d_i q_i^2}{4} \frac{g_i^2 f^2}{g_i^4} + \sum_{j \neq i} \frac{d_j q_j^2}{4} \frac{g_i^2 f^2}{g_j^4} \right)$$

It is clear to see that the sum term must be nonnegative as all functions appear in even power. Thus the sign of this term is determined entirely by the sign of ω_i, whose positivity may be argued as follows: This is not true initially since $\omega_i(0) = 0$. However, since $\dot{\omega}_i(0) > 0$, we have $\omega_i > 0$ at least on $(0, \epsilon)$ for some small $\epsilon > 0$. Thus, in order for positivity to be broken, we'll need $\omega_i = 0$ in finite time, implying either $f = 0$ or $g_i = \infty$. Clearly, if the latter occurs, our solution cannot be defined for further times while the former may be ruled out by the monotonicity we proved for f in lemma 2 above.

Thus, we obtain the inequality:

$$\ddot{\omega}_i\big|_{t=t_{\mathrm{crit}}} \geq \frac{\omega_i}{g_i^2}\left(\frac{(d_i+2)q_i^2}{4}\omega_i^2 - p_i\right) = \frac{(d_i+2)\omega_i q_i^2}{4g_i^2}\left(\omega_i^2 - \frac{4p_i}{(d_i+2)q_i^2}\right)$$

which only makes sense assuming $q_i \neq 0$. But note that by the first inequality, the $q_i = 0$ case always leads to a negative on the RHS of the inequality. In such a case, the other factors dominate the discussion, so we focus on whichever of the q_i's are positive. If all are zero, then we clearly always have a max regardless and so such cases naturally lead to collapsed solutions.

So, thanks to the positivity of ω_i shown earlier, we see that if

$$\omega_i^2 > \frac{4p_i}{(d_i+2)q_i^2} \tag{11}$$

then $\ddot{\omega}_{i,\mathrm{crit}} > 0$ and we have a local minimum.

However, we know by smoothness that $\omega_i(0) = 0$, so the first time that this (eq. (11)) occurs, we must have $\dot{\omega}_i \geq 0$ (since it will have to increase to this point, we cannot have $\dot{\omega}_i < 0$) and this contradicts the local min. (i.e. ω_i cannot increase to a local min). So, if ω_i gets big enough, it remains monotone increasing for all further times.

But, recalling theorem 2, we know the only possible finite limits are given by the two solutions of the equations:

$$\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_j \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$$

Or, taking out the i part of the sum:

$$\left(p_i - \frac{(d_i+2)q_i^2}{4}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_{j\neq i} \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$$

But, if our condition eq. (11) is satisfied, then the LHS is < 0, and, as we said before the RHS is always ≥ 0 and so we get a contradiction.

This shows us that once eq. (11) is satisfied, it is no longer possible for ω_i to have a finite limit. Thus, in such cases, we must have $\omega_i \to \infty$ (note we haven't specified whether this occurs in finite time).

Putting all of our discussion above together, we arrive at the following:

Lemma 3. *Suppose we have a smooth solution* $(u, f, g_i) : [0, T) \to \mathbb{R}^k$ *(with* $k = \sum_i^m d_i + 2$*) to the soliton equations eq.* (2) *defined on some interval* $[0, T)$ *with* $T \in (0, \infty]$*. Then:*

1. If there exists a $t_0 \in [0,T)$ where eq. (11) holds, that is if:

$$\omega_i(t_0)^2 > \frac{4p_i}{(d_i+2)q_i^2}$$

Then ω_i is monotone increasing for all further $t \in (t_0, T)$.

2. If, in addition, $[0,T)$ is the maximal domain of definition of the solution above, then we must have:

$$\lim_{t \uparrow T} \omega_i(t) = +\infty$$

(Note we haven't shown $T < \infty$)

4.3 Returning to the g_i

Now that we've covered the basics of the ω_i and their dynamics, we can tighten up our control of the g_i, completing our Analogue to Appleton's Lemma 4.1 started in proposition 1:

Proposition 2. *Let $s_0 > 0$ and $(u, f, g_i) : [0, s_0) \to \mathbb{R}^{m+2}$ be a solution to the soliton equations. Then g_i are strictly increasing on any interval $0 \in I \subset [0, s_0)$ on which $\omega_i < \frac{\sqrt{2p_i}}{q_i}$. Moreover, $\dot{g}_i$ changes its sign at most once in the interval $[0, s_0)$.*

Proof. The evolution equation of g_i shows us that $\ddot{g}_i = \frac{p_i}{g_i} - \frac{q_i^2}{2}\frac{f^2}{g_i^3} = \frac{1}{g_i}\left(p_i - \frac{q_i^2}{2}\omega_i^2\right)$ whenever $\dot{g}_i = 0$. Applying l'Hopital's rule around $s = 0$ shows that

$$\frac{\ddot{g}_i(0)}{g_i(0)} = -\frac{\dot{f}}{f}\frac{\dot{g}_i}{g_i}\bigg|_{t=0} + \frac{p_i}{g_i(0)^2} = -\frac{\dot{f}(0)}{g_i(0)}\frac{\ddot{g}_i(0)}{\dot{f}(0)} + \frac{p_i}{g_i(0)^2}$$

so that $\ddot{g}_i(0) = \frac{p_i}{2g_i(0)} > 0$. This in conjunction with the boundary condition $\dot{g}_i(0) = 0$ implies that g_i is strictly increasing on any interval $I = [0, s]$, $s > 0$, where

$$p_i - \frac{q_i^2}{2}\omega_i^2 > 0 \implies \omega_i^2 < \frac{2p_i}{q_i^2} \implies \omega_i < \frac{\sqrt{2p_i}}{q_i}$$

Therefore, $\dot{g}_i$ can change its sign only when $\omega_i \geq \frac{\sqrt{2p_i}}{q_i}$. Since ω_i is strictly increasing when $\omega_i > \frac{2\sqrt{p_i}}{\sqrt{(d_i+2)q_i}}$ and $\ddot{g}_i = \frac{1}{g_i}\left(p_i - \frac{q_i^2}{2}\omega_i^2\right)$ whenever $\dot{g}_i = 0$, it follows that $\dot{g}_i$ changes sign at most once. □

4.4 Monotonicity properties of the soliton potential u

Thanks to earlier work on this ansatz (see e.g. prop 2.3 of [6] and prop 1.2 of [19]), we have the following properties for our soliton potential u:

Proposition 3. *Let $s_0 > 0$ and $(u, f, g_i) : [0, s_0) \to \mathbb{R}^{m+2}$ be a solution to the soliton equations. Then*

1. *if $\ddot{u}(0) < 0$ then u and $\dot{u}$ are strictly decreasing functions, in particular*
   (a) *$\dot{u} < 0$ for $s > 0$*
   (b) *$\ddot{u} < 0$ for $s \geq 0$*
2. *if $\ddot{u}(0) = 0$ then $u \equiv 0$*

From this general case, we then specialize to a corollary we will find helpful later on (note the similarity to Appleton's corollary 4.3):

Lemma 4. *Let $s_0 > 0$ and $(u, f, g_i) : [0, s_0) \to \mathbb{R}^{m+2}$ be a solution to the soliton equations with $\ddot{u} < 0$ and $\omega_i < \frac{\sqrt{2p_i}}{q_i}$. Then $\dot{u} \geq -\sqrt{-2\ddot{u}(0)}$.*

Proof. Recall we have the conserved quantity for a steady soliton:

$$\ddot{u} + (\mathrm{tr}L)\dot{u} - \dot{u}^2 = 2\ddot{u}(0) = C$$

Which is simply the conserved quantity equation, eq. (4), appearing back in section 2.2.

The bound on ω_i ensures $\dot{g}_i > 0$ which means $\mathrm{tr}L > 0$ and so it follows from $\dot{u}, \ddot{u} < 0$ that:

$$\begin{aligned} -\dot{u}^2 &> \ddot{u} + (\mathrm{tr}L)\dot{u} - \dot{u}^2 = 2\ddot{u}(0) \\ &\implies |\dot{u}| < \sqrt{-2\ddot{u}(0)} \\ &\implies \dot{u} > -\sqrt{-2\ddot{u}(0)} \end{aligned}$$

As desired. $\square$

4.5 Short-time Existence from the Wink bound

Now, we come to one of the main results of Appleton's paper. In essence, this information is already present in Wink (see his prop 3.15) and can be seen as the result of his choice of a-priori estimate. What is presented here is a local version (which can be seen to be essentially a recounting of Appleton's Lemma 5.3 in [2] albeit with the different normalizations).

Lemma 5. *Let $s_0 > 0$ and $(u, f, g_i) : [0, s_0) \to \mathbb{R}^{m+2}$ be a solution to the soliton equations with $\ddot{u}(0) \leq 0$. If $\omega_i < \frac{\sqrt{2p_i}}{q_i}$ on $[0, s_0)$, the solution can be extended past s_0.*

Proof. The monotonicity properties of u, f and the g_i derived in earlier imply that whenever $\omega_i < \frac{\sqrt{2p_i}}{q_i}$

$$\ddot{f} \leq \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \frac{f^3}{g_i^4} \leq \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \frac{\omega_i^3}{g_i} \leq \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \left(\frac{2\sqrt{p_i}}{q_i}\right)^3 \frac{1}{g_i(0)} = C_1$$

$$\ddot{g}_i \leq \frac{1}{g_i}\left(p_i - \frac{q_i^2}{2}\omega_i^2\right) \leq \frac{p_i}{g_i} \leq \frac{p_i}{g_i(0)} = C_2$$

which in turn shows that

$$\dot{f}(t) < \dot{f}(0) + C_1 t \implies f(t) < \dot{f}(0)t + \frac{C_1}{2}t^2$$

$$\dot{g}_i(t) < C_2 t \implies g_i(t) < g_i(0) + \frac{C_2}{2}t^2$$

as long as $\omega_i < \frac{\sqrt{2p_i}}{q_i}$ holds true. Moreover, since $\dot{f}, \dot{g}_i > 0$ for all $s \in (0, s_0)$, there exists a $c > 0$ such that $f, g_i > c$ for $s \in [\frac{s_0}{2}, s_0)$. Finally, recall that by Corollary 4.3 we have

$$-\sqrt{-2\ddot{u}(0)} \leq \dot{u} \leq 0$$

Applying the Picard-Lindelof theorem we conclude that the solution may be extended past s_0. $\square$

4.6 Existence of non-collapsed solutions

What follows is an attempt to recover a usable version of Appleton's Lemma 7.1 for the multi-factor setting. Unfortunately, this is where the lack of an explicit solution hurts the most and so the theorem, while more general, has the trade-off that we no longer have any quantitative information about what size of Euler class is required. Instead, a limiting argument will show there is some Euler class big enough to suit our purposes. Because of this, the following is still notably weaker in the case that all p_i are identical, where Appleton's result may be used (by interpreting the product Fano base as a Fano manifold with the single chern class p_i).

Remark: Given that we take $u \equiv 0$ in the below, we can also see this result on Ricci-flat metrics on our Ansatz. In particular, if one could show $\omega_i \to \infty$ can only occur in finite time, the below would suffice as a proof that no complete, Ricci-flat metrics exist if any of the q_i are taken sufficiently large.

In fact, since the condition $\omega_i < \mathit{const}$ is open, the below theorem is actually telling us such trajectories belong to a new open set in the $C = 2\ddot{u}(0)$ parameter space from the one found by Wink (giving his collapsed solutions). In this way, we see that Wink's open set must be a positive distance from 0 (and thus, crucially, so to is the supremum of the set which we will return to in the last section).

Theorem 3. *Fix the initial conditions as usual, ($\dot{f}(0) = 1$,$g_i(0) > 0$ and $\dot{g}_i(0) = f(0) = 0$) but now setting $\ddot{u}(0) = 0$ (recall by proposition 3 this implies $u \equiv 0$ and so we are on an Einstein trajectory).*

Fixing each of the p_j and fixing all of the $q_j \neq 0$ except for one (q_i, say), there exists a value of $\tilde{q}_i$ sufficiently large so that

$$\omega_i^2 > \frac{4p_i}{(d_i + 2)q_i^2}$$

in finite time for all $q_i \geq \tilde{q}_i$. (note we don't require $p_i \neq 0$ simply because this already follows from the Fano property)

Proof. Suppose toward a contradiction that given the initial condition $\ddot{u}(0) = 0$ we have

$$\omega_i^2 \leq \frac{4p_i}{(d_i + 2)q_i^2}$$

for each i, for as long as the solution is defined .

By lemma 5 , we have that so long as

$$\omega_i < \frac{2\sqrt{p_i}}{q_i} \implies \omega_i^2 < \frac{4p_i}{q_i^2}$$

our solution may be extended further, so since (we're assuming each $d_i \geq 2$)

$$\frac{4p_i}{(d_i + 2)q_i^2} < \frac{2p_i}{q_i^2}$$

we see that our assumption implies the solution can be extended to all of $[0, \infty)$. Thus, questions of existence are not a worry in what follows (this nice fact is the main reason for wanting to use a proof by contradiction). In particular, this tells us that f is bounded in finite time.

Our aim in what follows is to use the equations to find estimates for both the g_i and for f. We will then take this pair of estimates and combine them to give an estimate on ω_i which will then be enough to contradict the above thus showing, at least for certain choices of the Euler classes, that the Einstein ($\ddot{u}(0) = 0$) trajectory is no longer collapsed. This shows us that, while Wink's family exists for sufficiently negative values of C, the Einstein trajectory is not a member of the family (again, provided the Euler class is big enough).

Changing the independent variable: In the estimates we derive below, it is first useful to make a change of independent variable. The relevance of this choice will be seen by how the equations for g_i and f simplify below.

We begin with the (hopeful) algebraic relation:

$$ds = \frac{1}{f} dt$$

From this, it follows from the fundamental theorem of calculus that:

$$s(t_1) - s(t_0) = \int_{t_0}^{t_1} \frac{1}{f(t)} dt$$

In order to meaningfully make a change of variables, we must assure monotonicity of the proposed new independent variable on the initial variable t. In our case, we need to make sure that $\dot{s} > 0$. Thus, we are requiring that $\frac{1}{f} > 0$. But now we recall that, by proposition 1, f is always positive, thus ensuring monotonicity of s.

Note this relation above leaves us with the choice of which value to assign to $s(t_0)$ and to decide which value of t_0 is appropriate to take as a starting point. First off, as we'll see below, the limit $t \to 0$ corresponds to $s \to -\infty$. Thus, we'll want to begin our comparison from some $t_0 > 0$. We will choose to take $t_0 \in (0, \epsilon)$ for some choice of ϵ sufficiently small so that the initial conditions determine our functions. More specifically, we wish to use the identities

$$g_i(t_0), \dot{g}_i(t_0), f(t_0), \dot{f}(t_0) > 0$$

which by the initial conditions, must be satisfied at least on some small initial interval $(0, \epsilon)$. Note that, with this choice, we will necessarily have $\dot{g}_i(t_0), \dot{f}(t_0) \approx 0$ or, both quantities are positive and SMALL.

For the comparison we wish to use below, we see that

$$\frac{g_i'}{g_i}(s_0) = \frac{f\dot{g}_i}{g_i}(t_0) < M_\epsilon \tag{12}$$

where M_ϵ can be taken to be as small as we wish (with the requirement that ϵ be chosen small enough). This then fixes our value of t_0 as any choice of $t_0 \in (0, \epsilon)$.

Now that t_0 is fixed, we have determined s as a function if we can fix the integration constant $s(t_0)$. For the sake of clean formulas, the easiest choice for this constant is simply $s(t_0) = 0$. Thus, in what follows, one should understand that $s = 0$ corresponds to this choice of $t_0 \in (0, \epsilon)$.

Now, setting $t_0 = \epsilon > 0$ and $t_1 = t > 0$ with $\epsilon, t << 1$, so that $f \approx t$, we see that:

$$s(t) - s(\epsilon) = \int_\epsilon^t \frac{1}{t} dt = \ln|t| \Big|_\epsilon^t = \ln(t) - \ln(\epsilon) = \ln(t) + M$$

with M exceedingly large.

Thus, when we recover s as a function of t, we are left with the following approximation for small t:

$$s(t) \approx s(\epsilon) + M + \ln(t)$$

Since both $s(\epsilon)$ and M are ultimately constants of integration, we can just combine them into a single constant, let's say L so that:

$$s(t) = L + \ln(t)$$

From this, it's pretty clear that we must have $s(t) \to -\infty$ as $t \to 0$.

Estimate for the g_i: We begin by trying to establish an estimate for g_i. Since we're ultimately looking for a lower bound on ω_i and because g_i appears in the denominator, we will want a lower bound for g_i.

To do this, let us first consider the evolution equation for g_i:

$$\frac{d}{dt}\frac{\dot{g}_i}{g_i} = -\mathrm{tr}L\frac{\dot{g}_i}{g_i} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4}$$

Now, making the change of independent variable as above from t to s, we have:

$$\frac{d}{dt}\frac{\dot{g}_i}{g_i} = \frac{1}{f}\left(\frac{1}{f}\frac{g_i'}{g_i}\right)' = \frac{1}{f^2}\left(\frac{g_i'}{g_i}\right)' - \frac{g_i'}{g_i}\frac{f'}{f^3}$$

where we've denoted $\frac{d\varphi}{ds}$ by φ'. So,

$$\frac{1}{f^2}\left(\frac{g_i'}{g_i}\right)' - \frac{g_i'}{g_i}\frac{f'}{f^3} = -\mathrm{tr}L\frac{\dot{g}_i}{g_i} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4}$$

and,

$$-\mathrm{tr}L\frac{\dot{g}_i}{g_i} = -\left(\frac{\dot{f}}{f} + \sum_j d_j\frac{\dot{g}_j}{g_j}\right)\frac{\dot{g}_i}{g_i} = -\frac{1}{f^2}\left(\frac{f'}{f} + \sum_j d_j\frac{g_j'}{g_j}\right)\frac{g_i'}{g_i}$$

Cancelling the $-\frac{g_i'}{g_i}\frac{f'}{f^3}$ term from either side gives:

$$\frac{1}{f^2}\left(\frac{g_i'}{g_i}\right)' = -\frac{1}{f^2}\left(\sum_j d_j\frac{g_j'}{g_j}\right)\frac{g_i'}{g_i} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4}$$

$$\implies \left(\frac{g_i'}{g_i}\right)' = -d_i\left(\frac{g_i'}{g_i}\right)^2 - \left(\sum_{j\neq i} d_j\frac{g_j'}{g_j}\right)\frac{g_i'}{g_i} + p_i\omega_i^2 - \frac{q_i^2}{2}\omega_i^4$$

Now,

$$\frac{g_j'}{g_j} = \frac{f}{g_j}\dot{g}_j$$

So, to deal with the second term, we'll require $\dot{g}_j > 0$ for every j. If, instead, we had $\dot{g}_j = 0$ for some j, proposition 2 then shows us that we must have

$$\omega_j^2 \geq \frac{2p_j}{q_j^2} > \frac{4p_i}{(d_j+2)q_j^2}$$

contradicting our assumption on the ω_j.

Thus, it is safe to assume $\dot{g}_j > 0$ for each j (In this way, the approach of a proof by contradiction helps us again) and we have the following inequality:

$$-\left(\sum_{j\neq i} d_j\frac{g_j'}{g_j}\right)\frac{g_i'}{g_i} \leq 0$$

Which leads us to:

$$\left(\frac{g_i'}{g_i}\right)' \leq -d_i\left(\frac{g_i'}{g_i}\right)^2 + p_i\omega_i^2 - \frac{q_i^2}{2}\omega_i^4$$

Having focused on the first order terms in the equation, let us now look at the zero order terms. Completing the square gives:

$$p_i\omega_i^2 - \frac{q_i^2}{2}\omega_i^4 = -\frac{q_i^2}{2}\left(\omega_i^2 - \frac{p_i}{q_i^2}\right)^2 + \frac{1}{2}\frac{p_i^2}{q_i^2}$$

Clearly, the left term of the RHS is ≤ 0, so we can further simplify and get:

$$\left(\frac{g_i'}{g_i}\right)' \leq \frac{1}{2}\left(\frac{p_i}{q_i}\right)^2 - d_i\left(\frac{g_i'}{g_i}\right)^2$$

So, the relevant ODE to solve is:

$$\varphi' = A - k\varphi^2$$

where we've set

$$A = \frac{1}{2}\left(\frac{p_i}{q_i}\right)^2; \quad k = d_i$$

This can be done by a separation of variables, leading to:

$$\frac{d\varphi}{A - k\varphi^2} = ds \implies \int_{s_0}^{s} \frac{d\varphi}{\frac{A}{k} - \varphi^2} = k(s - s_0)$$

So, setting $\alpha^2 = \frac{A}{k}$, we're looking to solve the following type of integral:

$$\int_{s_0}^{s} \frac{d\varphi}{\alpha^2 - \varphi^2}$$

which we can do by partial fractions, ultimately giving:

$$\varphi(s) = \alpha\frac{\dfrac{\varphi_0 + \alpha}{\varphi_0 - \alpha}e^{2\alpha k(s - s_0)} + 1}{\dfrac{\varphi_0 + \alpha}{\varphi_0 - \alpha}e^{2\alpha k(s - s_0)} - 1}$$

where $\varphi_0 = \varphi(s_0)$.

Meanwhile,

$$\alpha^2 = \frac{r}{k} = \frac{1}{2d_i}\left(\frac{p_i}{q_i}\right)^2 \implies \alpha = \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}$$

Thus, it's safe to assume that $\alpha > \varphi_0$

$$\implies \frac{\varphi_0 + \alpha}{\varphi_0 - \alpha} = -\frac{\alpha + \varphi_0}{\alpha - \varphi_0}$$

and, since $\alpha - \varphi_0 > 0$, it's more natural to write φ as

$$\varphi \sim \frac{g_i'}{g_i} \sim \varphi(s) = \alpha\frac{\dfrac{\alpha + \varphi_0}{\alpha - \varphi_0}e^{2\alpha k(s - s_0)} - 1}{\dfrac{\alpha + \varphi_0}{\alpha - \varphi_0}e^{2\alpha k(s - s_0)} + 1}$$

where now we can clearly see that the denominator is always nonzero. Furthermore, since $2\alpha k = \sqrt{2d_i}\frac{p_i}{q_i} > 0$, we see that φ is a monotone function of s, so that:

$$\varphi(s \to -\infty) < \varphi < \varphi(s \to \infty)$$
$$\implies -\alpha < \varphi < \alpha$$

Now, since we see that φ can never be bigger than α, we must be certain this value is not so small that it breaks our attempt at comparison. Thankfully, eq. (12) provides us the necessary bound for us to conclude:

$$\varphi < \alpha = \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}$$
$$\implies \frac{d}{ds}\ln g_i < \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}$$

Since our s_0 from before has dropped out, we can reuse this label and integrate from s_0 to s:

$$\ln\left(\frac{g_i(s)}{g_i(s_0)}\right) < \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}(s - s_0)$$
$$\implies g_i(s) < g_i(s_0)\exp\left(\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}(s - s_0)\right)$$

Estimate for f: Now that we have established the necessary estimate on g_i, we turn our attention now to finding a similar estimate on f. As f appears on the denominator of ω_i, we will want a lower bound.

Ok, so now that we have an estimate for g_i, let's now turn our attention to f:

$$\frac{d}{dt}\frac{\dot{f}}{f} = -\mathrm{tr}L\frac{\dot{f}}{f} + \sum_i \frac{d_i q_i^2}{4}\frac{f^2}{g_i^4}$$

(where, again, we assume $u \equiv 0$)

In order to make a comparison with the estimate for g_i above, we'll need to put this f equation in terms of s:

$$\frac{d}{dt} = \frac{1}{f}\frac{d}{ds}$$

So,

$$\frac{d}{dt}\frac{\dot{f}}{f} = \frac{1}{f}\left(\frac{1}{f}\frac{f'}{f}\right)'$$
$$= \frac{1}{f^2}\left(\frac{f'}{f}\right)' - \frac{f'}{f^2}\frac{1}{f}\frac{f'}{f} = \frac{1}{f^2}\left[\left(\frac{f'}{f}\right)' - \left(\frac{f'}{f}\right)^2\right]$$

While:

$$-\mathrm{tr}L\frac{\dot{f}}{f} = -\left(\frac{\dot{f}}{f} + \sum_i d_i\frac{\dot{g}_i}{g_i}\right)\frac{\dot{f}}{f}$$

$$\implies \left(\frac{f'}{f}\right)' - \left(\frac{f'}{f}\right)^2 = -\left(\frac{f'}{f} + \sum_i d_i \frac{g_i'}{g_i}\right)\frac{f'}{f} + \sum_i \frac{d_i q_i^2}{4}\omega_i^4$$

where $\omega_i = \frac{f}{g_i}$

$$\implies \left(\frac{f'}{f}\right)' = -\sum_i d_i \frac{g_i'}{g_i}\frac{f'}{f} + \sum_i \frac{d_i q_i^2}{4}\omega_i^4$$

Clearly we must have $\frac{d_i q_i^2}{4}\omega_i^4 \geq 0$ regardless of ω_i

$$\implies \left(\frac{f'}{f}\right)' \geq -\sum_i d_i \frac{g_i'}{g_i}\frac{f'}{f}$$

$$\implies \frac{\left(\frac{f'}{f}\right)'}{\left(\frac{f'}{f}\right)} = \frac{d}{ds}\ln\frac{f'}{f} \geq -\sum_i d_i \frac{g_i'}{g_i}$$

But, as we saw earlier,

$$\frac{g_i'}{g_i} < \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}$$

$$\implies \left(\ln\frac{f'}{f}\right)' \geq -\sum_i \frac{d_i}{\sqrt{2d_i}}\frac{p_i}{q_i}$$

$$\implies \frac{d}{ds}\ln\frac{f'}{f} \geq -\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}$$

$$\implies \ln\left(\frac{f'}{f}\right) - \ln\left(\frac{f_0'}{f_0}\right) \geq -\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)$$

Where the last step follows by integrating from s_0 to s, $f_0 = f(s_0)$ and $f_0' = f'(s_0)$

$$\implies \ln\frac{f'}{f} \geq \ln\left(\frac{f_0'}{f_0}\right) - \sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)$$

$$\implies \frac{f'}{f} \geq \frac{f_0'}{f_0}\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)\right)$$

$$\implies (\ln f)' \geq \frac{f_0'}{f_0}\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)\right)$$

So, since we have a derivative on the left and a known function on the right, we can integrate again:

$$\implies \ln f - \ln f_1 \geq \frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)\right) - \exp\left(\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s_1-s_0)\right)\right]$$

where $s_1 \geq s_0$ is our second constant of integration and $f_1 = f(s_1)$.

Thus, we arrive at:

$$f(s) \geq f_1 \exp\left(\frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)\right) - \exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s_1-s_0)\right)\right]\right)$$

Which completes our estimate for f.

Putting the two together: Now, we have the following 2 estimates:

$$g_i(s) < g_i(\tilde{s}_0)\exp\left(\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}(s-\tilde{s}_0)\right)$$

and

$$f(s) \geq f_1 \exp\left(\frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s-s_0)\right) - \exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}(s_1-s_0)\right)\right]\right)$$

Recall we are ultimately trying to show eq. (11) is satisfied in finite time. Thus, we're hoping for a suitable lower bound on the quotient of our two functions above. (since g_i is the denominator, we needed an upper bound for that).

Now, we'd like to simplify a little. First off, let's take $\tilde{s}_0 = s_0 = s_1 = 0$. Recall this value of s corresponds to some choice of $t_0 \in (0, \epsilon)$ for ϵ small. (see the paragraph on changing independent variable for more information)

Note this choice implies $f_1 = f_0$, so we can use the same terms when we write:

$$g_i(s) < g_{i,0}\exp\left(\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}s\right)$$

and

$$f(s) \geq f_0 \exp\left(\frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}s\right) - 1\right]\right)$$

So, putting these two together yields

$$\omega_i(s) = \frac{f(s)}{g_i(s)} > \frac{f_0}{g_{i,0}}\exp\left(\frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(-\sum_i \sqrt{\frac{d_i}{2}}\frac{p_i}{q_i}s\right) - 1\right] - \frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}s\right)$$

Now, to find the largest value of the RHS, let's take an s-derivative, set it to zero, and see what we get.

First off, note that because our function is of the form $e^{\psi(s)}$,

$$\frac{d}{ds}e^{\psi(s)} = \psi'(s)e^{\psi(s)} = 0 \iff \psi'(s) = 0$$

So, we need only take the derivative of the exponent and try to solve:

$$\frac{d}{ds}\left(\frac{f_0'}{f_0}\left(-\frac{1}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}\right)\left[\exp\left(-\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}s\right)-1\right]-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}s\right)=0$$

$$\implies \frac{f_0'}{f_0}\exp\left(-\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}s\right)-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}=0$$

$$\implies \exp\left(-\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}s\right)=\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}$$

$$\implies -\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}s=\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)$$

$$\implies s=-\frac{\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)}{\sum_i \sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}}$$

Plugging this into the exponent above gives:

$$\frac{f_0'}{f_0}\left(-\frac{1}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}-1\right)-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\left(-\frac{1}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)$$

$$=\left(-\frac{1}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left[\frac{f_0'}{f_0}\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}-1\right)-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right]$$

$$=\left(-\frac{1}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left[\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}-\frac{f_0'}{f_0}-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right]$$

$$=\left(\frac{\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(-1+\frac{f_0'}{f_0}\sqrt{2d_i}\frac{q_i}{p_i}+\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right)$$

$$=\left(\frac{\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}}{\sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}+\sum_{j\neq i}\sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{f_0'}{f_0}\sqrt{2d_i}\frac{q_i}{p_i}-1+\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right)$$

So,

$$\omega_{i,max}>\frac{f_0}{g_{i,0}}\exp\left(\left(\frac{\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}}{\sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}+\sum_{j\neq i}\sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{f_0'}{f_0}\sqrt{2d_i}\frac{q_i}{p_i}-1+\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right)\right)$$

Now, we wish to get a handle of the RHS of this inequality above. This will be easiest if we return to the exponent in the following form:

$$\left(-\frac{1}{\sum_j \sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}-\frac{f_0'}{f_0}-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right)$$

$$=\left(\frac{1}{\sqrt{\frac{d_i}{2}\frac{p_i}{q_i}}+\sum_{j\neq i}\sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{f_0'}{f_0}-\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}+\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\ln\left(\frac{f_0}{f_0'}\frac{1}{\sqrt{2d_i}}\frac{p_i}{q_i}\right)\right)$$

So, letting $q_i \to \infty$, the above becomes:

$$\left(\frac{1}{\sum_{j\neq i}\sqrt{\frac{d_j}{2}\frac{p_j}{q_j}}}\right)\left(\frac{f_0'}{f_0}-0+0\right)$$

(since $x \ln x \to 0$ as $x \to 0$)

Which is a **positive** constant. Thus, $\omega_{i,max}$ has a positive lower bound even as $q_i \to \infty$. However, all we need is

$$\omega_i > \frac{2\sqrt{p_i}}{\sqrt{d_i + 2q_i}}$$

and, the RHS goes to zero as $q_i \to \infty$. From this, we obtain our contradiction and the theorem is proved.

□

4.7 Wrapping up

Remark on continuous dependence: In what follows, we will rely heavily on the continuous dependence of our solution on initial conditions and, particularly, on $\ddot{u}(0)$. This fact can be seen in (at least) two ways.

1. Direct proof using the methods of ODE theory. This is the approach taken by Appleton, as can be seen in his theorem 9.2.
2. Alternatively, the conserved quantity equation eq. (4) shows us that a choice of C is really just a choice of hypersurface to which our vector field must be restricted. From this perspective, recalling that $C = 2\ddot{u}(0)$, continuous dependence on initial conditions is really just the continuous dependence of a level set on the level.

Defining the vector-valued function $\vec{\omega}(t)$ by

$$\vec{\omega} = (\omega_1, \ldots, \omega_m)$$

we get the following equation for the evolution of the magnitude:

$$\frac{d^2}{dt^2}|\vec{\omega}|^2 = 4\left|\frac{d\vec{\omega}}{dt}\right|^2 - \mathcal{L}\frac{d}{dt}|\vec{\omega}|^2 + \sum_{k=1}^{m}\frac{2}{f^2}\omega_k^2\left(\frac{q_k^2}{2}\omega_k^4 + \sum_{l=1}^{m}\frac{d_l q_l^2}{4}\omega_l^4 - p_k\omega_k^2\right)$$

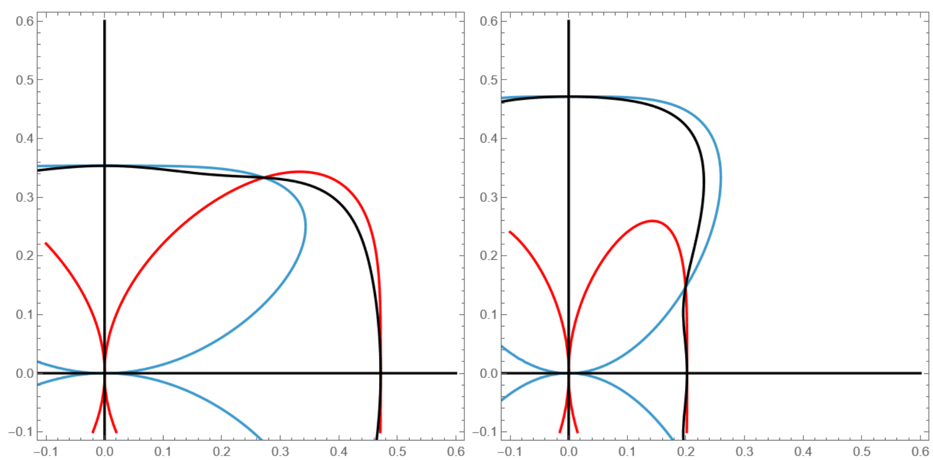

Figure 1: The above picture shows the contours of the relevant functions in the ω_i plane (we consider the 2-factor case for ease of imaging, but the general case is much the same). The red and blue show the lines $\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_j \frac{d_j q_j^2}{4}\omega_{j,\infty}^4$ for $i = 1, 2$ respectively. While the black line gives the contour corresponding to $\phi = 0$ (see below for definition of ϕ). Note how the black line crosses only where the red and blue curves intersect (this can be seen by $d_1 = d_2 = p_1 = p_2 = 2$ in both cases while $q_1 = 3, q_2 = 4$ in the first and $q_1 = 7, q_2 = 3$ for the second.

To more conveniently refer back to it later, I'll denote the non-derivative term as $\frac{2}{f^2}\phi$. We'll only care about the sign, so we don't need to worry about the $\frac{2}{f^2}$ factor as this is positive so long as the solution is defined. (Thanks to proposition 1 the only other possibility is if it is zero, which would correspond to $f \to \infty$ in finite time). So:

$$\phi = \sum_{k=1}^{m} \omega_k^2 \left(\frac{q_k^2}{2}\omega_k^4 + \sum_{l=1}^{m} \frac{d_l q_l^2}{4}\omega_l^4 - p_k\omega_k^2 \right)$$

Which we can see is exactly the sum of (ω_k^2 times) the polynomials defining the possible finite limits of ω_i we saw back in theorem 2. Also, as we see in the figures above, the locus $\phi = 0$ lies entirely inside the region where these polynomials are negative, except at the extremities (i.e. one of the ω_i's are zero) or the positive limit point (the only possible finite limit for $\vec{\omega}$ other than the origin).

Now, we wish to run an analogue of Appleton's theorem 7.2, with $\|\vec{\omega}\|$ taking the role his Q variable did there.

To do this, we define the quantity

$$u_0^* = \sup\{u_0 \in \mathbb{R} | \text{ for } \ddot{u}(0) \le u_0,\ \vec{\omega} \to 0 \text{ as } t \to \infty\} \tag{13}$$

Which serves as our analogue of the quantity Appleton gives the same name. Note, however, that this definition is quite a bit stronger than his, as it relies on exact knowledge of the limit. Nevertheless, Wink's earlier results are still sufficient to show us that $u_0^* > -\infty$ as he showed the set of collapsed solutions is never empty. Meanwhile, theorem 3 (and the remark before it) shows us that $u_0^* < 0$, at least in those cases where the q_i are large enough.

Now, from the evolution equation of the magnitude of $\vec{\omega}$, we can straight away determine 2 things:

1. Any critical point $\left(\frac{d}{dt}|\vec{\omega}|^2 = 0\right)$ with $\phi > 0$ is a local min. and so solutions that end up out here grow without bound (i.e. $|\vec{\omega}| \to \infty$).

2. If we have a critical point with $\phi = 0$, we either have $|\frac{d\vec{\omega}}{dt}| > 0$ in which case we repeat the above, Or, we have $\frac{d\vec{\omega}}{dt} = 0$, in which case the equation can only be solved by a constant. (this case is described in more detail below)

Returning to the equations for the evolution of ω_i,

$$\ddot{\omega}_i = \dot{\omega}_i \left(\dot{u} - (d_i+1)\frac{\dot{g}_i}{g_i} - \sum_{j \ne i} d_j \frac{\dot{g}_j}{g_j} \right) + \frac{\omega_i}{f^2} \left(\frac{q_i^2}{2}\omega_i^4 - p_i\omega_i^2 + \sum_{j=1}^{m} \frac{d_j q_j^2}{4}\omega_j^4 \right)$$

If we're in situation 2, we then have a critical point of this where, since we're inside the blue/red region, the left term is non-positive. If negative, we see that ω_i attains a maximum, while, if zero, we see that the equation reduces to $\ddot{\omega}_i = 0$, which is solved by a constant, contradicting our assumption that it is a solution to the equations starting from $\omega_i = 0$.

Thus, the only critical points for $|\vec{\omega}|^2$ all lie in the region where $\phi < 0$. Furthermore, since $\vec{\omega} \to 0$ when $\ddot{u}(0) < u_0^*$, $|\vec{\omega}| \to 0$ and $\forall \epsilon > 0$, $\exists$ some $T_\epsilon > 0$ such that $|\vec{\omega}| < \epsilon \; \forall \; t \in [T_\epsilon, \infty)$ Meanwhile, we may apply our earlier assertion in the range $[0, T_\epsilon]$ to assert that $|\vec{\omega}|$ attains a maximum somewhere where $\phi < 0$. Thus, if we assume that $\vec{\omega} \to 0$ even when $\ddot{u}(0) = u_0^*$, we must have that $\exists \max_{t \in \mathbb{R}^+} |\vec{\omega}|$ which occurs in a region where $\phi < 0$. Since the maximum lies in an open set, we can then argue (by continuous dependence on initial conditions) that $\ddot{u}(0) = u_0^* + \epsilon$ should also give rise to a solution with finite maximum inside the region $\phi < 0$. Then, the only possible terminal limit for such solutions is $\vec{\omega} \to 0$, contradicting the definition of u_0^*. Thus we see that $\vec{\omega} \not\to 0$ when $\ddot{u}(0) = u_0^*$

Since $\phi < 0 \; \forall \; t \neq 0$ along our solutions with $\vec{\omega} \to 0$, we can also assert (again, by continuous dependence on initial conditions), that our solution must at least satisfy $\phi \leq 0 \; \forall$ time. But, there is only one possible limit point on $\phi = 0$ that a solution may end up, so we must have that our solution ends here (asymptotically as $t \to \infty$).

Namely, as we saw in theorem 2, there is only one possible finite limit for the ω_i apart from zero. Because of this, we can guarantee our solution must end up here and, because the corresponding limits for the ω_i are finite and positive, we are left with a non-collapsed solution (i.e. since the different parts of the metric have the same asymptotic behaviour, the solution is thus non-collapsed).

Thus, we arrive at:

Lemma 6. *Take u_0^* as defined in eq. (13). We have the following conclusions, the first coming from our discussion above and the second from theorem 3:*

1. *if $u_0^* < 0$, then setting $\ddot{u}(0) = u_0^*$ leads to a non-collapsed steady Ricci soliton.*

2. *Given any choice of initial conditions (with $q_i \neq 0$ for each i) and a choice of factor i, there is some integer, $\tilde{q}_i$ say, for which $u_0^* < 0$ whenever $q_i > \tilde{q}_i$.*

Combining this with our asymptotics in theorem 2, we see that the resulting solutions are asymptotically paraboidal, and so we may finally conclude that theorem 1 is true, that is:

Theorem 4. *If we make the assumption that $q_i \neq 0$ for each i, then there exists a complete, non-collapsed Steady Ricci Soliton on our ansatz provided any one of the q_i are taken to be sufficiently large.*

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