MONOMIAL IDEALS WITH A PRESCRIBED WALDSCHMIDT CONSTANT

MONOMIAL IDEALS WITH A PRESCRIBED WALDSCHMIDT CONSTANT

By CRAIG KOHNE, M.Sc.

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AUTHOR:	Craig Kohne M.Sc. (Mathematics), McMaster University, Hamilton, Canada
SUPERVISOR:	Adam Van Tuyl, Antoine Deza
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Lay Abstract

Let I be a monomial ideal in $R = K[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field K. The Waldschmidt constant of I, denoted $\hat{\alpha}(I)$, is a numeric invariant of I. The Waldschmidt constant manifests in many ways in commutative algebra and algebraic geometry, and is related to open problems such as the ideal containment problem and Nagata's conjecture. For a monomial ideal, the computation of $\widehat{\alpha}(I)$ reduces to solving a linear optimization problem. This thesis shows how to construct a monomial ideal with $\widehat{\alpha}(I)$ equal to any rational number greater than or equal to 1. The family of monomial ideals investigated are intersections of powers of prime monomial ideals (in Chapter 3) and square-free principal Borel ideals (in Chapter 4).

Abstract

Let I be a homogeneous ideal in $R = K[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field K with characteristic zero. Define the *m-th symbolic power* of I as the ideal $I^{(m)} := \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R)$.

The *Waldschmidt constant* of I is defined as

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

Here, $\alpha(J)$ denotes the smallest degree of a generator of the ideal J. The Waldschmidt constant is interpreted as an asymptotic invariant of I. The ratio $\frac{\alpha(I)}{\widehat{\alpha}(I)}$ gives a measure of the growth of the symbolic power $I^{(m)}$ compared to the ordinary power I^m as $m \to \infty$.

The focus of this thesis is answering this question: for a given rational number $\frac{q}{p} \geq 1$, how can we construct a monomial ideal I such that $\widehat{\alpha}(I) = \frac{q}{p}$? Computing the Waldschmidt constant for a monomial ideal reduces to solving a linear optimization problem. We study two families of monomial ideals in depth.

In Chapter 3 we study ideals of the form

$$I = \langle x_2, \ldots, x_n \rangle^{e_1} \cap \langle x_1, x_3, \ldots, x_n \rangle^{e_2} \cap \cdots \cap \langle x_1, \ldots, x_{n-1} \rangle^{e_n}.$$

Our main result in this chapter (Corollary 3.3.4) is that by choosing the e_i's appropriately, we can construct an ideal I with Waldschmidt constant for "almost all" $\frac{q}{p} \geq 1$.

In Chapter 4 we study square-free principal Borel ideals, denoted $\mathrm{sfBorel}(m)$, where m is the generating square-free monomial. We give upper and lower bounds for the Waldschmidt

constant of sfBorel(m) in terms of the support of m, and in some cases, exact values. Our main result (Corollary 4.3.2) is that for any $\frac{q}{p} \geq 1$, we show that there exists a square-free principal Borel ideal with Waldschmidt constant equal to $\frac{q}{p}$.

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Declaration of Academic Achievement

I, Craig Kohne, declare that this thesis titled "Monomial ideals with a prescribed Waldschmidt constant" is my own work, with the exception of the Chapter 4 which is the result of collaboration. The content of Chapter 4 first appears in [CMKSVT22], and my collaborators were: Eduardo Camps Moreno, Eliseo Sarmiento, and Adam Van Tuyl.

Chapter 1

Introduction

In Section 1.1 we introduce symbolic powers and the Waldschmidt constant, the main objects of study of this thesis. In Section 1.2 we give an overview of this thesis, and highlight its main results.

1.1 Symbolic powers and the Waldschmidt constant

We begin by giving a brief overview of the development and position of symbolic powers and the Waldschmidt constant in mathematics.

Let I be a homogeneous ideal in $R = K[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field K with characteristic zero, and denote I^m to be m-th (ordinary) power of the ideal. Noether showed that any ideal in a Noetherian ring can be written as the intersection of primary ideals [Noe21]. This primary decomposition allows for another type of power of I, the symbolic power of an ideal, denoted $I^{(m)}$.

Formally, the m-th symbolic power of I is defined as

$$I^{(m)} := \bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P \cap R)$$

where R_P is the ring R localized at the prime ideal P, and $\mathrm{Ass}(I)$ is the set of associated primes from the primary decomposition of I.

Zariski [Zar49] explored how the symbolic power $I^{(m)}$ provides more differential and geometric information about I than its ordinary powers. In terms of the algebraic variety $V(I)$ defined by radical I we can write the symbolic power as:

$$I^{(m)} = \{f \in R \mid f \text{ vanishes at } V(I) \text{ to order } m\}.$$

Symbolic powers, rather than ordinary powers, seem more amenable to study for algebraic geometry. In general, the symbolic power of an ideal does not equal the ordinary power, and the symbolic power is more difficult to compute. Comparing and measuring the difference between symbolic and ordinary powers has proven useful in many areas of algebraic geometry and commutative algebra.

The first question in comparing symbolic and ordinary powers is to ask if and when one contains the other. It is known that we have the containment $I^r \subseteq I^{(m)}$ if any only if $r \geq m$. The more interesting question is the following:

Question 1.1.1. (Ideal containment problem) *For a fixed integer m, what is the smallest integer r such that $I^{(r)} \subseteq I^m$ holds?*

The ideal containment problem is difficult because symbolic powers are difficult to compute. There is a large literature available which give bounds and solutions for certain families of ideals. The papers [ELS02], [HH02], [BH10], [DHSTG11], [HT19], and [BJZ21] are a small sample of the research available.

Introduced in the 1970's by Waldschmidt ([Wal77],[Wal79]) to study points in $\mathbb{C}^n$, the Waldschmidt constant plays a pivotal role in the study of the asymptotic properties of homogeneous ideals. For a given homogeneous ideal $I \subseteq R = K[x_1, \ldots, x_n]$, the *Waldschmidt constant* of I is defined as

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

Here, $\alpha(J)$ denotes the smallest degree of a generator of the ideal J. The Waldschmidt constant is interpreted as an asymptotic invariant of I. The ratio $\frac{\alpha(I)}{\widehat{\alpha}(I)}$ gives a measure of the growth of the symbolic power $I^{(m)}$ compared to the ordinary power I^m as $m \to \infty$. Bocci and Harbourne were among the first to use the Waldschmidt constant to study Question 1.1.1 (ideal containment problem). In their paper [BH10] Bocci and Harbourne also introduced the *resurgence* of I, defined as $\rho(I) = \sup\left\{\frac{r}{m} \mid I^{(r)} \not\subset I^m\right\}$, which provides the bound $\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I)$. For more on the Waldschmidt constant see [BDRH+19], [BH20], and [NH21].

A major open problem in connection with the Waldschmidt constant is Nagata's conjecture. In 1959 Nagata (a collaborator of Zariski) provided counterexamples to Hilbert's fourteenth problem [Nag59] and proposed a follow-up conjecture on complex curves. This conjecture (known as Nagata's conjecture), which remains open, can be rephrased as a statement involving the Waldschmidt constant of the ideal associated to a set of points.

Theorem 1.1.2 (Nagata's conjecture re-phrased)**.** *[GHVT13] Let $I \subset K[\mathbb{P}^2]$ be the ideal of $s \geq 10$ generic points of $\mathbb{P}^2$. Then $\widehat{\alpha}(I) = \sqrt{s}$.*

1.2 Monomial ideals with a prescribed Waldschmidt constant

We now give an overview of the purpose and content of this thesis. The focus of this thesis will be answering the following natural questions about the Waldschmidt constant.

Question 1.2.1. *What are the possible values of the Waldschmidt constant?*

Question 1.2.2. *For a prescribed number q, how can we construct an ideal I such that $\widehat{\alpha}(I) = q$?*

The Waldschmidt constant for a general ideal is difficult to compute because it involves taking the limit of symbolic powers (which in themselves involve intersections and localizations). In this thesis we will focus on monomial ideals (polynomial ideals which are generated by monomials). Fortunately, computing the Waldschmidt constant for monomial ideals is simpler than the general case; we can transform the analytic problem into one of convex geometry and linear optimization.

We now give an overview of the sections of the thesis and highlight its main results.

Section 2.1. Here we give the definition of the Waldschimdt constant, and basic properties and terminology for monomial ideals, square-free monomial ideals, and linear optimization problems.

Section 2.2. The goal of this section is to show how the computation of the Waldschmidt constant for monomial ideals can be recast as a linear optimization program. We make use of the work of Cooper, Embree, Hà, Hoefel in [CEHH17] which shows that the Waldschmidt constant of a monomial ideal can be viewed as an invariant of a convex polyhedron called the symbolic polyhedron (see Theorem 2.2.8, [Corollary 6.3 in [CEHH17]]).

The work of Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl, and Vu in [BCG$^+$16] explicitly shows how the primary decomposition of a square-free monomial ideal can be used to construct a linear optimization problems whose value is the Waldschmidt constant of ideal. (see Theorem 2.2.11, [Theorem 3.2 in [BCG$^+$16]]).

Section 2.3. In this section we expand on the result of [BCG$^+$16] to give an explicit procedure for constructing a linear optimization problems whose value is the Waldschmidt constant for any monomial ideal without embedded primes presented in irreducible form. The main result of this section is given below.

Theorem 1.2.3. *(Theorem 2.3.4) Suppose I is a monomial ideal in $R = K[x_1, \ldots, x_n]$ with minimal irreducible primary decomposition*

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_s$$

where $Q_i = \langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \ldots, x_{i_k}^{a_{i_k}} \rangle$. Furthermore, suppose this I does not contain any embedded primes. For each Q_i associate an n-vector $\boldsymbol{r}^{(i)}$ with the i_j-th entry equal to $\frac{1}{a_{i_j}}$ for $1 \leq j \leq k$ and otherwise equal to 0. Let A be the $s \times n$ matrix with row i equal to $\boldsymbol{r}^{(i)}$. Then

$\widehat{\alpha}(I)$ is the value of the linear optimization problem

$$\min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}.$$

Section 2.4. This section uses the work of Camarneiro, Drabkin, Fragoso, Frendreiss, Hoffman, Seceleanu, Tang, and Yang in [CDF+22] to give a more complete framework for the correspondence between the algebra of the monomial ideal and its associated polyhedra.

The following lemma gives the Newton polyhedra (see Definition 2.4.1) of the power of prime monomial ideal. This will prove useful in Section 3.

Lemma 1.2.4 (Lemma 2.4.7)**.** *Let $B = \{b_1, b_2, \ldots, b_r\} \subseteq \{1, 2. \ldots, n\}$ with $|B| = r$. Let $I = \langle x_{b_1}, x_{b_2}, \ldots, x_{b_r}\rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$. Then for $I^d = \langle x_{b_1}, x_{b_2}, \ldots, x_{b_r}\rangle^d$ we have*

$$NP(I^d) = \left\{ \sum_{i=1}^{r} \frac{1}{d}\boldsymbol{y}_{b_i} \geq 1 \;\middle|\; \boldsymbol{y}_{b_i} \in \mathbb{R}_{\geq 0} \right\}.$$

Section 2.5. Here we begin to answer Question 1.2.1 by looking at ideals in just two variables. By analysing the generators of any monomial ideal I in $K[x_1, x_2]$, we find that the Waldschmidt constant $\widehat{\alpha}(I)$ equals the initial degree $\alpha(I)$:

Theorem 1.2.5 (Theorem 2.5.3)**.** *Let I be a monomial ideal in $K[x_1, x_2]$. Then $\widehat{\alpha}(I) = \alpha(I)$.*

Since $\alpha(I)$ is the degree of the smallest generator of I, the Waldschmidt constant can only take on a natural number in two variables. To find an ideal with a non-integer Waldschmidt constant, we need add more variables.

Chapter 3 - Waldschmidt constant for monomial ideals in $n > 2$ variables

To attempt to answer Question 1.2.1 and Question 1.2.2 we want to choose a family of monomial ideals that are described by few parameters while producing a large range of Waldschmidt constants. In Chapter 3 we look at monomial ideals that are intersections of powers of prime monomial ideals.

Section 3.1. In this section we focus on the family ideals in $R = K[x_1, x_2, x_3]$ of the form

$$I = \langle x_2, x_3\rangle^{e_1} \cap \langle x_1, x_3\rangle^{e_2} \cap \langle x_1, x_2\rangle^{e_3}$$

with $e_1 \geq e_2 \geq e_3 > 0$. By analysing the feasible region of the linear optimization problem associated to I, the following corollary shows how we can choose e_1, e_2, and e_3 to construct an ideal with Waldschmidt constant equal to $\frac{q}{2} > 1$ for a prescribed positive integer $q > 2$.

Corollary 1.2.6 (Corollary 3.1.2)**.** *Fix a positive integer $q > 2$. Then write the integer partition $q = e_1 + e_2 + e_3$ where*

$$\begin{cases} e_1 = \frac{q}{3}, e_2 = \frac{q}{3}, e_3 = \frac{q}{3} & \text{if } q \equiv 0 \mod 3 \\ e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lfloor \frac{q}{3} \rfloor, e_3 = \lfloor \frac{q}{3} \rfloor & \text{if } q \equiv 1 \mod 3 \\ e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lceil \frac{q}{3} \rceil, e_3 = \lfloor \frac{q}{3} \rfloor & \text{if } q \equiv 2 \mod 3. \end{cases}$$

Then

$$I = \langle x_2, x_3\rangle^{e_1} \cap \langle x_1, x_3\rangle^{e_2} \cap \langle x_1, x_2\rangle^{e_3}$$

is a monomial ideal in $R = K[x_1, x_2, x_3]$ with $\widehat{\alpha}(I) = \frac{q}{2}$.

Section 3.2. In this section we focus on the family ideals in $R = K[x_1, x_2, x_3, x_4]$ of the form

$$I = \langle x_2, x_3, x_4\rangle^{e_1} \cap \langle x_1, x_3, x_4\rangle^{e_2} \cap \langle x_1, x_2, x_4\rangle^{e_3} \cap \langle x_1, x_2, x_3\rangle^{e_4}$$

with $e_1 \geq e_2 \geq e_3 \geq e_4 > 0$. We derive a similar result as the above corollary with a notable exception: we can choose e_1, e_2, e_3 and e_4 to create an ideal I with Waldschmidt constant $\frac{q}{3} > 1$ *except* for when $q = 5$.

Section 3.3. In this section we focus on the family of ideals in $R = K[x_1, \ldots, x_n]$ of the form

$$I = \langle x_2, \ldots, x_n\rangle^{e_1} \cap \langle x_1, x_3, \ldots, x_n\rangle^{e_2} \cap \cdots \cap \langle x_1, \ldots, x_{n-1}\rangle^{e_n}$$

with $e_1 \geq e_2 \geq \cdots \geq e_n > 0$. The following result allows us to choose the e_i's so that we can attain a Waldschmidt constant equal to $\frac{q}{p}$ for almost all rational numbers greater than 1. By "almost all" we mean for a given denominator p, there are only finitely many numerators q such that $\widehat{\alpha}(I) = \frac{q}{p}$ is not attainable by this method. Our result is summarized below.

Corollary 1.2.7 (Corollary 3.3.4)**.** *Consider the fraction $\frac{q}{p} \geq 1$ for some positive integers q, p. Let $n = p+1$ and $q \equiv k \mod n$. Then write the integer partition $q = e_1 + e_2 + \cdots + e_n$ where we set*

$$e_i = \begin{cases} \lceil \frac{q}{n} \rceil & \text{for } 1 \leq i \leq k \\ \lfloor \frac{q}{n} \rfloor & \text{for } k+1 \leq i \leq n. \end{cases}$$

Now suppose $n^2 - (k+1)n + k \leq q$ is true. Then

$$I = \langle x_2, \dots, x_n \rangle^{e_1} \cap \langle x_1, x_3, \dots, x_n \rangle^{e_2} \cap \cdots \cap \langle x_1, \dots, x_{n-1} \rangle^{e_n}$$

is a monomial ideal in $R = K[x_1, ..., x_n]$ with $\widehat{\alpha}(I) = \frac{q}{p}$.

Chapter 4. The content of Chapter 4 first appeared in the paper *On the Waldschmidt Constant of Square-free Principal Borel Ideals* [CMKSVT22] in collaboration with Camps Moreno, Sarmiento, and Van Tuyl.

We investigate the Waldschmidt constant of a family of monomial ideals called *square-free principal Borel ideals*. We denote these ideals by sfBorel(m), where $m = x_{i_1} \cdots x_{i_s}$ is the square-free monomial which generates the ideal. Given a monomial m, if $x_i | m$ and $j < i$, then we call $x_j \cdot \frac{m}{x_i}$ a *Borel move* of m. A monomial ideal is a *Borel ideal* (or a strongly stable ideal) if for every $m \in I$, all of the Borel moves of m are also in I. For example, let $m = x_2x_5$ and let $I = \text{sfBorel}(x_2x_5) \in K[x_1, \dots, x_5]$. The set of monomials attained via every Borel move on $m = x_2x_5$ is the following set:

$$B = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5\}.$$

The generators of $\mathrm{sfBorel}(x_2x_5)$ are the square-free monomials of B:

$$\mathrm{gens}(\mathrm{sfBorel}(x_2x_5)) = \{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5\}.$$

Since these ideals are square-free monomial ideals, the method of Section 2.2 for computing $\widehat{\alpha}(\mathrm{sfBorel}(m))$ applies. In practice, solving the linear optimization problem for square-free principal Borel ideals can be especially difficult due to the large number of inequalities involved. As an example, suppose we wish to compute $\widehat{\alpha}(\mathrm{sfBorel}(m))$ where the generating monomial is

$$m = x_{33215}x_{33216}\cdots x_{104348} \in K[x_1, \ldots, x_{104348}]. \tag{1.2.1}$$

To apply the method of Theorem 2.2.11 to find the Waldschmidt constant for this ideal would involve solving a linear optimization problem in 104348 variables and $\binom{104348}{33215} \approx 5.1 \times 10^{28347}$ inequalities defining its feasible region. By contrast, the ideals of Section 3.3 have only n variables and n inequalities.

We provide an upper bound (Theorem 4.2.4) and a lower bound (Theorem 4.3.4) for $\widehat{\alpha}(\mathrm{sfBorel}(m))$, expressed in terms of the support of $m = x_{i_1}\cdots x_{i_s}$, that is, $\{i_1, \ldots, i_s\}$. Our analysis relies on the work of Francisco, Mermin, and Schweig [FMS11] which describes the associated primes of $\mathrm{sfBorel}(m)$.

The following corollary is the highlight of the chapter: when the monomial $m = x_{i_1}x_{i_2}\cdots x_{i_k}$ has no "gaps" or "jumps" in the i_j's, we can compute $\widehat{\alpha}(\mathrm{sfBorel}(m))$ exactly.

Corollary 1.2.8 (Corollary 4.3.2)**.** *Let $I = \mathrm{sfBorel}(x_ix_{i+1}\cdots x_{i+l})$. Then*

$$\widehat{\alpha}(I) = \frac{i+l}{i}.$$

Consequently, let $\frac{q}{p} \geq 1$ be a rational number. Then there exists a square-free principal Borel ideal I such that $\widehat{\alpha}(I) = \frac{q}{p}$.

The above result is constructive in the sense that we can explicitly choose the required monomial m such $\widehat{\alpha}(\mathrm{sfBorel}(m)) = \frac{q}{p} \geq 1$. So this result gives an answer to Question 1.2.2

for rational numbers. Also, observe the monomial m of (1.2.1) was chosen to approximate π up to nine digits

$$\widehat{\alpha}(\text{sfBorel}(m)) = \frac{104348}{33215} = 3.14159265392. \tag{1.2.2}$$

Chapter 5. In this final chapter we show that we can construct a monomial ideal I in three variables whose Waldschmidt constant has a prescribed denominator (Example 5.1.1). We observe that while the square-free principal Borel ideals of Chapter 4 can attain any rational Waldschmidt constant greater or equal to 1, they do so "inefficiently" compared to the ideals from Section 3.3. By "inefficient" we mean that an ideal from Chapter 4 will often require far more variables than an ideal from Section 3.3 with the same Waldschmidt constant. The thesis concludes by proposing questions for further inquiry, including:

Question 1.2.9. *For a prescribed Waldschmidt constant $\frac{q}{p} \geq 1$, what is the monomial ideal I that most "efficiently" (e.g., in terms of minimal number of variables or associated primes required) that attains $\widehat{\alpha}(I) = \frac{q}{p}$?*

Chapter 2

Background

2.1 Monomial ideals, primary decomposition

In this thesis we will consider ideals in the polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with characteristic zero.

The Hilbert Basis Theorem implies that an ideal I in R can be described by a finite generating set of polynomials (Corollary 7.7, [AM69]):

$$I = \langle f_1, f_2, \dots, f_s \rangle := \{g_1 f_1 + \cdots + g_s f_s \mid g_i \in R\}.$$

We denote the set of generators by $\mathrm{gens}(I) = \{f_1, f_2, \dots, f_s\}$. A set of generators for I is called a **minimal generating set** if no element of $\mathrm{gens}(I)$ can be omitted. In general, $\mathrm{gens}(I)$ is not a unique set.

An ideal I is **prime** if $fg \in I$ implies either $f \in I$ or $g \in I$. An ideal I is **primary** if $fg \in I$ implies either $f \in I$ or $g^m \in I$ for some $m > 0$. The **radical** ideal of an ideal I is the set

$$\sqrt{I} = \{f \in R \mid f^m \in I \text{ for some } m \geq 1\}.$$

If I is primary, then $\sqrt{I}$ is the smallest prime ideal containing I.

Theorem 2.1.1 (Theorem 7.3 [AM69], [Noe21])**.** *Every ideal I in R can be written as a finite intersection of primary ideals. That is, we can write*

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$$

*where the Q_i's are primary ideals. Additionally, this decomposition is called **minimal** or **irredundant** if $\sqrt{Q_i}$ are all distinct and $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$.*

Definition 2.1.2. Suppose $I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ is a primary decomposition. Then the set of **associated primes** of I is

$$\operatorname{Ass}(I) = \{\sqrt{Q_1} = P_1, \sqrt{Q_2} = P_2, \ldots, \sqrt{Q_r} = P_r\}.$$

The associated primes of I which are not minimal (with respect to set-theoretic inclusion) in $\operatorname{Ass}(I)$ are called the **embedded associated primes** of I.

Definition 2.1.3. A ring R is called a **graded ring** if it can be decomposed into a direct sum $R = \bigoplus_{i=0}^{\infty} R_i$ of additive groups where $R_m R_n \subseteq R_{m+n}$. Since R is a direct sum, any non-zero element a of R can be uniquely written as $a = a_0 + a_1 + \cdots + a_n$, where each a_i is either 0 or homogeneous of degree i (the non-zero a_i are called **homogeneous components** of a). A non-zero element of R_n is called a **homogeneous element** of degree n. An ideal $I \subseteq R$ is called a **homogeneous ideal** if for every $a \in I$, the homogeneous components of a are also in I.

Definition 2.1.4. For a homogeneous ideal I, denote $\alpha(I)$ to be the smallest degree of an element in a minimal set of homogeneous generators for I.

Definition 2.1.5. The ***m*-th symbolic power** of $I \subseteq R$, denoted $I^{(m)}$, is the ideal

$$I^{(m)} := \bigcap_{P \in \operatorname{Ass}(I)} (I^m R_P \cap R)$$

where R_P is the ring R localized at the prime ideal P.

Remark 2.1.6. The definition of symbolic powers is not uniform in the literature, where in some references, the indexing set is only over the minimal associated primes as in (Definition 4.3.22, [Vil15]) In this thesis we use the version as written in Definition 2.1.5 above.

We now define the main invariant of study for this thesis.

Definition 2.1.7. For any homogeneous ideal I, define the **Waldschmidt constant** of I as

$$\widehat{\alpha}(I) := \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

Remark 2.1.8. The limit in the above definition converges because $\alpha(I^{(m)})$ is an a subadditve function: $\alpha(I^{(m_1+m_2)}) \leq \alpha(I^{(m_1)}) + \alpha(I^{(m_2)})$ (see Lemma 2.3.1, [BH10]).

2.1.1 Monomial ideals

An ideal I is a **monomial ideal** if it can be generated by a set of monomials (polynomials with one term). A **prime monomial ideal** is generated by a subset of the variables, i.e., of the form $\langle x_{i_1}, \ldots, x_{i_k}\rangle$. The **height** of a prime monomial ideal P, denoted $\operatorname{ht}(P)$, is the largest number h such that there exists a chain of distinct prime ideals $P_0 \subset P_1 \subset \ldots P_h = P$. For a prime monomial ideal $P = \langle x_{i_1}, \ldots, x_{i_k}\rangle$ we have $\operatorname{ht}(P) = k$.

The following proposition summarizes how we can compute a set of monomial generators under standard ideal operations.

Proposition 2.1.9 (Section 1.2.1, [HH11])**.** *Let I, J be monomial ideals in $K[x_1, \ldots, x_n]$ with generating sets* $\operatorname{gens}(I)$ *and* $\operatorname{gens}(J)$*, respectively. Then I, J under the following standard ideal operations are again monomial ideals.*

1. $I + J = \langle\{m \mid m \in \operatorname{gens}(I) \cup \operatorname{gens}(J)\}\rangle$.

2. $IJ = \langle\{m_1m_2 \mid m_1 \in \operatorname{gens}(I), m_2 \in \operatorname{gens}(J)\}\rangle$.

3. $I \cap J = \langle\{\operatorname{lcm}(m_1, m_2) \mid m_1 \in \operatorname{gens}(I), m_2 \in \operatorname{gens}(J)\}\rangle$,

where $\operatorname{lcm}(m_1, m_2)$ *denotes the monomial that is the least common multiple of m_1 and m_2.*

A monomial is often denoted by m but we may also write a monomial as

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

and refer to $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ as the **exponent vector** of the monomial. A monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is **square-free** when $a_i \in \{0, 1\}$ for all i. A monomial ideal is **square-free** if each of its generators is a square-free monomial.

The following theorem summarizes useful properties of square-free monomial ideals and their symbolic powers.

Theorem 2.1.10 (Theorem 10.4, [CHHVT20])**.** *Let I be a square-free monomial ideal in $R = K[x_1, \ldots, x_n]$.*

1. *There exist unique prime monomial ideals of the form $P_i = \langle x_{i_1}, \ldots, x_{i_{t_i}} \rangle$ such that $I = P_1 \cap \cdots \cap P_s$.*
2. *With the P_i's as above, the m-th symbolic power of I is given by $I^{(m)} = P_1^m \cap \cdots \cap P_s^m$.*
3. *For all integers $m \geq 1$,*

$$\alpha(I^{(m)}) = \min\{a_1 + \cdots + a_n \mid x_1^{a_1} \cdots x_n^{a_n} \in I^{(m)}\}.$$

2.1.2 Linear optimization problems

Definition 2.1.11. Let S be a subset of $\mathbb{R}^n$. Then S is a **convex set** if for all $x, y \in S$, the line segment connecting x to y is also in S. The **convex hull** of S, denoted conv(S) is the smallest convex set containing S. A **convex polyhedron** is a convex set which equals the solution set to a system of linear inequalities. An **extreme point** or **vertex** of a convex set S is a point in S which does not lie in any open line segment joining two points of S.

Notation 2.1.12. Suppose A is an $m \times n$ matrix, $\mathbf{b}$ is a n-vector, $\mathbf{c}$ is an m-vector, and $\mathbf{0}$ is the zero n-vector. The standard form of a **linear optimization problem** (or

linear program) is expressed as:

$$\text{Solve } \min\{\mathbf{b}^{\mathbf{T}}\mathbf{y} \mid A\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} \tag{2.1.1}$$

where $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n]^T$ is a n-vector of variables.

In (2.1.1), the vector order $\mathbf{d} \geq \mathbf{e}$ means the i-th coordinate of $\mathbf{d}$ is greater than or equal to the i-th coordinate of $\mathbf{e}$ for all i. Any vector $\mathbf{y}$ satisfying $A\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ is called a **feasible solution**. The set of feasible solutions, denoted $\mathcal{F}$, is a convex polyhedron called the **feasible region**. The **objective function** is the expression $\mathbf{b}^{\mathbf{T}}\mathbf{y}$.

If $\mathbf{y}'$ is a feasible solution which minimizes the objective function $\mathbf{b}^{\mathbf{T}}\mathbf{y}$, then $\mathbf{y}'$ is called an **optimal solution** or **minimizer** and $\mathbf{b}^{\mathbf{T}}\mathbf{y}'$ is the **value** of the linear optimization problem.

Theorem 2.1.13 (Theorem 2.3 (Fundamental Theorem of Linear Programming), [FP93])**.** *For a linear optimization program in its standard form with a non-empty feasible region $\mathcal{F}$, the minimum value of the objective function $\mathbf{b}^{\mathbf{T}}\mathbf{y}$ over $\mathcal{F}$ is either unbounded below or attained by a vertex of $\mathcal{F}$.*

If we can compute the vertices of $\mathcal{F}$, then solving the linear optimization program amounts to testing which vertex minimizes the objective function. The following characterization of the vertices of $\mathcal{F}$ will prove helpful in Chapter 3.

Definition 2.1.14 (Definition 2.9, [BT97])**.** The feasible region $\mathcal{F}$ associated to the linear optimization problem (2.1.1) is the intersection of $m+n$ half-spaces defined by inequalities. Let $(A)_{i,*}$ be the i-th row of A. There are m inequalities of the form

$$(A)_{i,*} \cdot \mathbf{y} = a_{i,1}\mathbf{y}_1 + a_{i,2}\mathbf{y}_2 + \cdots + a_{i,n}\mathbf{y}_n \geq \mathbf{c}_i$$

and n inequalities of the form $\mathbf{y}_i \geq 0$. A vector $\mathbf{z} \in \mathbb{R}^n$ is called a **basic solution** if it is the unique solution to any n of the inequalities changed to equalities (i.e., hyperplane equations). If a basic solution is feasible, i.e., it further satisfies the remaining m inequalities (those not changed to hyperplane equations), then $\mathbf{z}$ is called a **basic feasible solution** or **vertex** of $\mathcal{F}$.

Definition 2.1.15. The **dual** of the standard linear optimization problem (2.1.1) is a linear optimization problem of the form

$$\text{Solve } \max\{\mathbf{c^T x} \mid A^T\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \tag{2.1.2}$$

where $A, \mathbf{b}$ and $\mathbf{c}$ are the same as in (2.1.1), and $\mathbf{x} \in \mathbb{R}^m$.

The following theorem which relates (2.1.1) and (2.1.2) will be useful in Chapter 4.

Theorem 2.1.16. *1.* (Theorem 4.2 (Weak duality), [BT97]) *For each feasible solution $\boldsymbol{y'}$ to (2.1.1) and each feasible solution $\boldsymbol{x'}$ to (2.1.2) we have $\mathbf{c}^T\boldsymbol{x'} \leq \boldsymbol{b}^T\boldsymbol{y'}$.*

2. (Theorem 4.4 (Strong duality), [BT97]) *If either (2.1.1) or (2.1.2) have an optimal solution, then so does the other and both linear optimization problems have the same value.*

2.2 Symbolic polyhedrons and the Waldschmidt constant as a linear optimization program

We now introduce the symbolic polyhedron of a monomial ideal and show that computing the Waldschimdt constant amounts to solving a linear optimization problem.

Definition 2.2.1. Let M be a monomial ideal or finite set of monomials. Define

$$\mathcal{L}(M) = \{\mathbf{a} \in \mathbb{N}^n \mid \mathbf{x^a} \in M\}$$

to be the set of lattice points (i.e., elements of $\mathbb{Z}^n_{\geq 0}$) corresponding to the exponent vectors of monomials in M.

Definition 2.2.2. Let $A, B \subseteq \mathbb{R}^n$. Then the **Minkowski sum** of A and B is formed by adding each vector in A to each vector in B, that is, $A + B = \{a + b \mid a \in A, b \in B\}$.

Lemma 2.2.3 (Lemma 5.2, [CEHH17])**.** *Let I be a non-zero monomial ideal. The convex hull of $\mathcal{L}(I)$ is*

$$\operatorname{conv}(\mathcal{L}(I)) = \operatorname{conv}(\mathcal{L}(\operatorname{gens}(I))) + \mathbb{R}^n_+,$$

where $+$ is the Minkowski sum.

Example 2.2.4. We will compute $\operatorname{conv}(\mathcal{L}(I))$ for $I = \langle x_1^2, x_2^3 \rangle$ in $R = K[x_1, x_2]$. The set of monomials in I is the set

$$\{x_1^{k_1} x_2^{k_2} \mid k_1 \geq 0, k_2 \geq 3, k_1, k_2 \in \mathbb{N}\} \cup \{x_1^{k_1} x_2^{k_2} \mid k_1 \geq 2, k_2 \geq 0, k_1, k_2 \in \mathbb{N}\}.$$

The exponent vectors for the generators of I are: $\mathcal{L}(\operatorname{gens}(I)) = \{(2,0), (0,3)\}$.
The convex hull of the two exponent vectors is the line segment joining $(2,0)$ and $(0,3)$: $\operatorname{conv}(\mathcal{L}(\operatorname{gens}(I))) = \{(\mathbf{y}_1, \mathbf{y}_2) \mid \frac{1}{2}\mathbf{y}_1 + \frac{1}{3}\mathbf{y}_2 = 1, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}^2_{\geq 0}\}$. Then taking the Minkowski sum with $\mathbb{R}^2_+$ gives:

$$\operatorname{conv}(\mathcal{L}(I)) = \{(\mathbf{y}_1, \mathbf{y}_2) \mid \frac{1}{2}\mathbf{y}_1 + \frac{1}{3}\mathbf{y}_2 \geq 1, (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}^2_{\geq 0}\}.$$

See Figure 2.1.

The following lemma generalizes Example 2.2.4 for any monomial ideal generated by powers of the variables.

Lemma 2.2.5. *Let $I = \langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \ldots, x_{i_k}^{a_{i_k}} \rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$. Let $\boldsymbol{r}$ be an n-vector with entry the i_j-th equal to $\frac{1}{a_{i_j}}$ for $1 \leq j \leq k$ and every other entry equal to 0. Then*

$$\operatorname{conv}(\mathcal{L}(I)) = \{\mathbf{y} \mid \mathbf{r} \cdot \mathbf{y} \geq 1, \mathbf{y} \geq \mathbf{0}\},$$

where $\boldsymbol{y} = [\boldsymbol{y}_1 \ \boldsymbol{y}_2 \ \cdots \boldsymbol{y}_n]$ is a vector of real variables.

Proof. For each of the k generators of I we have the corresponding exponent vector $(0, \ldots, a_{i_j}, \ldots, 0)$, i.e., an n-vector with a_{i_j} in its i_j-th coordinate and 0 elsewhere. Let $\mathbf{y} \in \mathbb{R}^n$ and observe that the k exponent vectors lie on the hyperplane $H = \sum_{j=1}^{k} \frac{1}{a_{i_j}} \mathbf{y}_{i_j} = 1$.

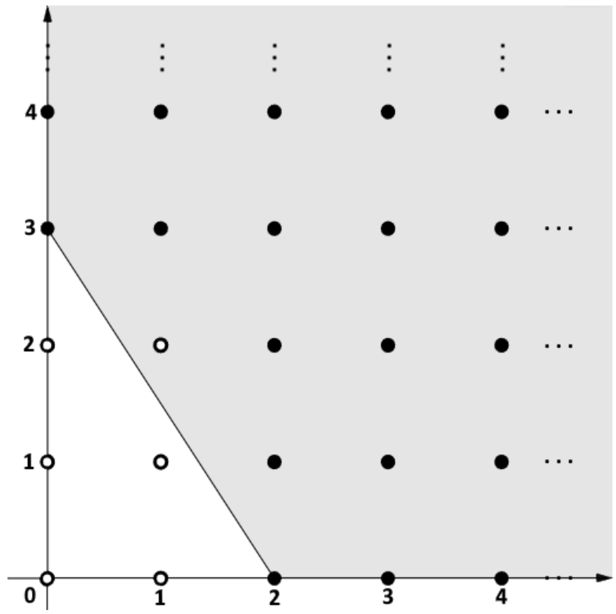


Figure 2.1: Convex hull of Example 2.2.4
The filled in dots represent exponent vectors for monomials in I. The grey area equals $\mathrm{conv}(\mathcal{L}(I))$.

Then the convex hull of the k exponent vectors is the intersection of the hyperplane H with the non-negative orthant of $\mathbb{R}^n$:

$$\mathrm{conv}(\mathcal{L}(\mathrm{gens}(I))) = \left\{\mathbf{y} \;\Big|\; \sum_{j=1}^{k} \frac{1}{a_{i_j}} \mathbf{y}_{i_j} = 1, \mathbf{y} \geq \mathbf{0}\right\}$$

where $\mathbf{y} = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \mathbf{y}_n]$ is a vector of real variables. Now let $\mathbf{r}$ be an n-vector with the i_j-th entry equal to $\frac{1}{a_{i_j}}$ for $1 \leq j \leq k$ and every other entry equal to 0. Then we get

$$\mathrm{conv}(\mathcal{L}(\mathrm{gens}(I))) = \{\mathbf{y} \mid \mathbf{r} \cdot \mathbf{y} = 1, \mathbf{y} \geq \mathbf{0}\}.$$

Now taking the Mikowski sum with $\mathbb{R}^n_+$ equals the set

$$\mathrm{conv}(\mathcal{L}(I)) = \{\mathbf{y} \mid \mathbf{r} \cdot \mathbf{y} \geq 1, \mathbf{y} \geq \mathbf{0}\},$$

giving the result. □

Definition 2.2.6. [Definition 5.3, [CEHH17]] The **symbolic polyhedron** of a monomial ideal I is the polyhedron $SP(I) \subseteq \mathbb{R}^n$ given by

$$SP(I) = \bigcap_{P \in \max\mathrm{Ass}(I)} \mathrm{conv}(\mathcal{L}(Q_{\subseteq P})),$$

where $Q_{\subseteq P}$ is the intersection of all primary ideals Q_i in the primary decomposition of I with $\sqrt{Q_i} \subseteq P$. We denote $\max\mathrm{Ass}(I)$ to be the subset of $\mathrm{Ass}(I)$ that are maximal with respect to set-theoretic inclusion.

Definition 2.2.7. For any polyhedron $\mathcal{P} \subset \mathbb{R}^n$, define

$$\alpha(\mathcal{P}) = \min\{\mathbf{1}^T\mathbf{y} \mid \mathbf{y} \in \mathcal{P}\}$$

where $\mathbf{1}$ is the column vector of appropriate size with all entries being 1.

Theorem 2.2.8 (Corollary 6.3, [CEHH17])**.** *Let I be a monomial ideal and $SP(I)$ be its symbolic polyhedron. Then $\alpha(SP(I)) = \widehat{\alpha}(I)$.*

The above two definitions and theorem taken together tell us: For a monomial ideal I, computing the Waldschmidt constant $\widehat{\alpha}(I)$ is equivalent to solving a linear optimization problem where the symbolic polyhedron $SP(I)$ is the feasible region $\mathcal{F}$.

2.2.1 Waldschmidt constant of a square-free monomial ideal

In this section we compute the Waldschmidt constant for square-free monomial ideals.

Theorem 2.2.9 (Corollary 1.3.4, [HH11])**.** *A square-free monomial ideal is an intersection of prime monomial ideals.*

Example 2.2.10. Consider the following ideal in $R = K[x_1, x_2, x_3, x_4]$ with primary decomposition

$$I = \underbrace{\langle x_1, x_2, x_3 \rangle}_{P_1} \cap \underbrace{\langle x_1, x_2, x_4 \rangle}_{P_2} \cap \underbrace{\langle x_1, x_3, x_4 \rangle}_{P_3} \cap \underbrace{\langle x_2, x_3, x_4 \rangle}_{P_4}.$$

Each P_i is a prime monomial ideal since it is generated by variables. So by Theorem 2.2.9 I is a square-free monomial ideal. By inspection no associated prime contains another so

$\mathrm{Ass}(I) = \mathrm{maxAss}(I) = \{P_1, P_2, P_3, P_4\}$. This means we have $Q_{\subseteq P_i} = P_i$ for each i. Then applying Lemma 2.2.5 we have:

$$\begin{cases} \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_1})) = \mathrm{conv}(\mathcal{L}(P_1)) = \{\mathbf{y} \mid [1\ 1\ 1\ 0]^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \\ \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_2})) = \mathrm{conv}(\mathcal{L}(P_2)) = \{\mathbf{y} \mid [1\ 1\ 0\ 1]^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \\ \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_3})) = \mathrm{conv}(\mathcal{L}(P_3)) = \{\mathbf{y} \mid [1\ 0\ 1\ 1]^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \\ \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_4})) = \mathrm{conv}(\mathcal{L}(P_4)) = \{\mathbf{y} \mid [0\ 1\ 1\ 1]^T\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}. \end{cases}$$

Letting

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

we get

$$SP(I) = \bigcap_i \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_i})) = \{\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}.$$

By Definition 2.2.7, we have $\alpha(SP(I)) = \min\{\mathbf{1}^T\mathbf{y} \mid \mathbf{y} \in SP(I)\}$. Then by Theorem 2.2.8 and the form of $SP(I)$ computed above we get

$$\widehat{\alpha}(I) = \alpha(SP(I)) = \min\{\mathbf{1}^T\mathbf{y} \mid \mathbf{y} \geq \mathbf{0}, A\mathbf{y} \geq \mathbf{1}\}.$$

We have expressed the Waldschmidt constant as the value of a linear optimization problem in standard form (2.1.1). The unique optimal solution to the linear optimization program is $\mathbf{z} = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$ (in Section 3.2 we will show how to compute the optimal solution for this family of ideals). So the Waldschmidt constant is $\widehat{\alpha}(I) = \mathbf{1}^T\mathbf{z} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$. □

The following theorem shows that the above example of computing the Waldschmidt constant as a linear optimization problem generalizes to any square-free monomial ideal.

Note this method only requires knowing $\mathrm{Ass}(I)$ and importantly avoids computing localization and limits.

Theorem 2.2.11. *Let $I \subseteq R = K[x_1, \ldots, x_n]$ be a square-free monomial ideal with minimal primary decomposition $I = P_1 \cap P_2 \cap \cdots \cap P_s$. Define the matrix the $s \times n$ matrix*

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$$

Then $\widehat{\alpha}(I)$ is the value of the linear optimization problem

$$\min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}.$$

Proof. Note that since I is a square-free monomial ideal we cannot have $P_i \subset P_j$, otherwise the primary decomposition has a redundant component and is not minimal. So we have $\mathrm{Ass}(I) = \mathrm{maxAss}(I) = \{P_1, \ldots, P_s\}$ and $Q_{\subseteq P_i} = P_i$ for each i. Define the $s \times n$ matrix

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$$

Then by Lemma 2.2.5 we can write

$$\mathrm{conv}(\mathcal{L}(Q_{\subseteq P_i}) = \mathrm{conv}(\mathcal{L}(P_i)) = \{\mathbf{y} \mid (A)_{i,*} \cdot \mathbf{y} \geq 1, \mathbf{y} \geq \mathbf{0}\}$$

where $(A)_{i,*}$ is the i-th row of the matrix A. Then

$$SP(I) = \bigcap_i \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_i})) = \{\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}$$

and so the result follows from Definition 2.2.7 and Theorem 2.2.8. □

Note that a different proof for Theorem 2.2.11 is given as Theorem 3.2 in [BCG$^+$16] which does not use the symbolic polyhedron.

2.3 Waldschmidt constant of a monomial ideal without embedded primes

With the method of Theorem 2.2.11 in hand we can provide a partial answer to the following question.

Question 2.3.1 (10.23, [CHHVT20])**.** Is there a procedure similar to (Theorem 2.2.11) to find $\widehat{\alpha}(I)$ for non-square-free monomial ideals?

Definition 2.3.2. An ideal I is called **irreducible** if whenever we can write $I = I_1 \cap I_2$, then $I = I_1$ or $I = I_2$. An **irreducible decomposition** of an ideal I is $I = I_1 \cap I_2 \cap \cdots \cap I_s$ where each I_i is an irreducible ideal.

We require the following facts on irreducible monomial ideals.

Theorem 2.3.3. *1.* (Theorem 1.3.1, [HH11]). *Let $I \subseteq R = K[x_1, \dots, x_n]$ be a monomial ideal. Then $I = \bigcap_{i=1}^{s} Q_i$, where each Q_i is generated by pure powers of variables. In other words, each Q_i is of the form $\langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \dots, x_{i_k}^{a_{i_k}} \rangle$. Moreover, a minimal (or irredunant) presentation of this form is unique.*

2. (Corollary 1.3.2, [HH11]). *A monomial ideal is irreducible if and only if it is generated by pure powers of the variables.*

3. (Proposition 1.3.7, [HH11]). *The irreducible ideal $\langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \dots, x_{i_k}^{a_{i_k}} \rangle$ is $\langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle$-primary. That is, $\sqrt{\langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \dots, x_{i_k}^{a_{i_k}} \rangle} = \langle x_{i_1}, x_{i_2}, \dots, x_{i_k} \rangle$.*

The following theorem gives a similar procedure for computing the Waldschmidt constant non-square-free monomial ideals, provided they are presented in their minimal irreducible form and do not contain embedded primes.

Theorem 2.3.4. *Suppose I is a monomial ideal in $R = K[x_1, \dots, x_n]$ with minimal irreducible primary decomposition*

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_s$$

where $Q_i = \langle x_{i_1}^{a_{i_1}}, x_{i_2}^{a_{i_2}}, \ldots, x_{i_k}^{a_{i_k}} \rangle$. *Furthermore, suppose this* I *does not contain any embedded primes. For each* Q_i *associate an* n*-vector* $\mathbf{r}^{(i)}$ *with the* i_j*-th entry equal to* $\frac{1}{a_{i_j}}$ *for* $1 \leq j \leq k$ *and otherwise equal to* 0. *Let* A *be the* $s \times n$ *matrix with row* i *equal to* $\mathbf{r}^{(i)}$. *Then* $\widehat{\alpha}(I)$ *is the value of the linear optimization problem*

$$\min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}.$$

Proof. Since I has no embedded primes we have

$$\mathrm{Ass}(I) = \mathrm{maxAss}(I) = \{\sqrt{Q_1} = P_1, \sqrt{Q_2} = P_2, \ldots, \sqrt{Q_s} = P_s\},$$

and $Q_{\subseteq P_i} = Q_i$ for each $P_i \in \mathrm{Ass}(I)$. For each Q_i associate an n-vector denoted $\mathbf{r}^{(i)}$ with the i_j-th entry equal to $\frac{1}{a_{i_j}}$ for $1 \leq j \leq k$ and every other entry equal to 0. Let $\mathbf{y} \in \mathbb{R}^n$. Then by Lemma 2.2.5 we have

$$\mathrm{conv}(\mathcal{L}(Q_{\subseteq P_i})) = \mathrm{conv}(\mathcal{L}(Q_i)) = \{\mathbf{y} \mid \mathbf{r}^{(i)} \cdot \mathbf{y} \geq 1, \mathbf{y} \geq \mathbf{0}\}.$$

Define the matrix A to be the $s \times n$ matrix with row i equal to $\mathbf{r}^{(i)}$. Then we get

$$SP(I) = \bigcap_i \mathrm{conv}(\mathcal{L}(Q_{\subseteq P_i})) = \{\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\}$$

and so the result follows from Definition 2.2.7 and Theorem 2.2.8. □

In both Theorem 2.2.11 and Theorem 2.3.4 we require the assumption that $\mathrm{Ass}(I) = \mathrm{maxAss}(I)$. To see why, consider the following example.

Example 2.3.5. Consider the monomial ideal $I = \langle x_1^3 \rangle \cap \langle x_1^4, x_2 \rangle \subseteq K[x_1, x_2]$.
This ideal is in its minimal irreducible presentation, but note that $\mathrm{Ass}(I) = \{\langle x_1 \rangle, \langle x_1, x_2 \rangle\}$ $\neq \mathrm{maxAss}(I) = \{\langle x_1, x_2 \rangle\}$. Computing the symbolic polyhedron gives

$$SP(I) = \bigcap_{P \in \mathrm{maxAss}(I)} \mathrm{conv}(\mathcal{L}(Q_{\subseteq P})) = \mathrm{conv}(\mathcal{L}(Q_{\subseteq \langle x_1, x_2 \rangle})) = \mathrm{conv}(\mathcal{L}(\langle x_1^3 \rangle \cap \langle x_1^4, x_2 \rangle)).$$

In general, computing the minimal generators of an intersection of ideals is not trivial. Additionally, computing the convex hull does not commute with intersecting ideals. So when we have embedded primes, it is not possible to "read off" the entries of A as in

Theorem 2.2.11 and Theorem 2.3.4.

2.4 Newton, symbolic, and irreducible polyhedra

In general, a monomial ideal I may have embedded primes and may not be presented in its minimal irreducible form. The definitions and theorems of this section allow us to describe the Waldschmidt constant, and other asymptotic properties associated to I, for a monomial ideal in general.

The following definition is another notation for $\operatorname{conv}(\mathcal{L}(I))$:

Definition 2.4.1. [CDF$^+$22] Let I be a monomial ideal. Then the **Newton polyhedra** of I, denoted $NP(I)$, is the convex hull of the exponent vectors for all monomials in I. That is, $NP(I) = \operatorname{conv}(\mathcal{L}(I))$. We will often write the symbolic polyhedron as $SP(I) = \bigcap_{P \in \operatorname{maxAss}(I)} NP(Q_{\subseteq P})$.

Definition 2.4.2. [CDF$^+$22] The **irreducible polyhedron** of a monomial ideal I with irreducible decomposition $I = Q_1 \cap \cdots \cap Q_s$ is

$$IP(I) = NP(Q_1) \cap \cdots \cap NP(Q_s).$$

Solving a linear optimization problem over each of these polyhedra gives a constant associated to I.

Theorem 2.4.3. (Equation (4.1), [CDF$^+$22]) *The initial degree of a monomial ideal I, denoted $\alpha(I)$, is the value of the following linear optimization problem with feasible region given by its Newton polyhedron:*

$$\begin{array}{ll} \text{minimize} & y_1 + y_2 + \cdots + y_n \\ \text{subject to} & (y_1, y_2, \ldots, y_n) \in NP(I). \end{array}$$

Theorem 2.4.4. (Corollary 4.6, [CDF$^+$22]) *The Waldschmidt constant of a monomial ideal I, denoted $\widehat{\alpha}(I)$, is the value of the following linear optimization problem with feasible region given by its symbolic polyhedron:*

$$\begin{array}{ll} \text{minimize} & y_1 + y_2 + \cdots + y_n \\ \text{subject to} & (y_1, y_2, \ldots, y_n) \in SP(I). \end{array}$$

Definition 2.4.5. (Corollary 4.7, [CDF+22]) The **naive Waldschmidt constant** of a monomial ideal I, denoted $\tilde{\alpha}(I)$, is the value of the following linear optimization problem with feasible region given by its irreducible polyhedron:

$$\begin{array}{ll} \text{minimize} & y_1 + y_2 + \cdots + y_n \\ \text{subject to} & (y_1, y_2, \ldots, y_n) \in IP(I). \end{array}$$

The following theorem gives the containment of the above three polyhedra and ordering of their associated constants:

Theorem 2.4.6. (Theorem 3.9, Proposition 4.8, [CDF+22]) *For any monomial ideal I we have*

$$NP(I) \subseteq SP(I) \subseteq IP(I).$$

which gives

$$\tilde{\alpha}(I) \leq \widehat{\alpha}(I) \leq \alpha(I).$$

The following lemma describes the Newton polyhedra of the power of a prime monomial ideal. This will prove useful in Chapter 3.

Lemma 2.4.7. *Let $B = \{b_1, b_2, \ldots, b_r\} \subseteq \{1, 2. \ldots, n\}$ with $|B| = r$. Let $I = \langle x_{b_1}, x_{b_2}, \ldots, x_{b_r} \rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$. Then for $I^d = \langle x_{b_1}, x_{b_2}, \ldots, x_{b_r} \rangle^d$ we have*

$$NP(I^d) = \left\{ \sum_{i=1}^{r} \frac{1}{d} \boldsymbol{y}_{b_i} \geq 1 \;\middle|\; \boldsymbol{y}_{b_i} \in \mathbb{R}_{\geq 0} \right\}.$$

Proof. Suppose $I_1, \ldots, I_d$ are monomial ideals. Then their product is a monomial ideal (see Proposition 2.1.9) that can be written as:

$$I_1 I_2 \cdots I_d = \langle \{ m_1 m_2 \cdots m_d \mid m_i \in \text{gens}(I_i) \text{ for } i = 1, \ldots, d \} \rangle.$$

Now suppose $I = \langle x_{b_1}, x_{b_2}, \dots, x_{b_r} \rangle$. Then I^d can written as:

$$I^d = \langle \{ m_1 m_2 \cdots m_d \mid m_i \in \mathrm{gens}(I) \text{ for } i = 1, \dots, d \} \rangle \text{ or,}$$

$$I^d = \langle \{ x_{b_1}^{a_1} x_{b_2}^{a_2} \cdots x_{b_r}^{a_r} \mid a_1 + a_2 + \cdots + a_r = d \} \rangle.$$

So each generator of I^d is a product of $x_{b_1}, x_{b_2}, \dots, x_{b_r}$ with degree d. Let $\mathbf{y} \in \mathbb{R}^n$. Then since the exponents of the generators satisfy $a_1 + a_2 + \cdots + a_r = d$, the exponent vector of each generator of I^d lies on the hyperplane $\mathbf{y}_{b_1} + \mathbf{y}_{b_2} + \cdots + \mathbf{y}_{b_r} = d$ or $\frac{1}{d}\mathbf{y}_{b_1} + \frac{1}{d}\mathbf{y}_{b_2} + \cdots + \frac{1}{d}\mathbf{y}_{b_r} = 1$. The convex hull of the generators is the intersection of this hyperplane with $\mathbb{R}^n_{\geq 0}$, i.e.,

$$\mathrm{conv}(\mathcal{L}(\mathrm{gens}(I^d))) = \Big\{ \sum_{i=1}^{r} \frac{1}{d} \mathbf{y}_{b_i} = 1 \;\Big|\; \mathbf{y}_{b_i} \in \mathbb{R}_{\geq 0} \Big\}.$$

Taking the Minkowski sum of $\mathrm{conv}(\mathcal{L}(\mathrm{gens}(I^d)))$ with $\mathbb{R}^n_+$ gives the result. $\square$

Example 2.4.8. Let $I = \langle x_1^4, x_1 x_2^2, x_2^3 \rangle \subseteq R = K[x_1, x_2, x_3, x_4]$. Then we have the minimal irreducible primary decomposition

$$I = \langle x^4, y^2 \rangle \cap \langle x, y^3 \rangle.$$

Since $\sqrt{\langle x_1^4, x_2^2 \rangle} = \sqrt{\langle x_1, x_2^3 \rangle} = \langle x_1, x_2 \rangle$, we have $\mathrm{Ass}(I) = \mathrm{maxAss}(I) = \{\langle x_1, x_2 \rangle\}$. Observe this means $Q_{\subseteq \langle x_1, x_2 \rangle} = \langle x_1^4, x_2^2 \rangle \cap \langle x_1, x_2^3 \rangle = I$ which gives

$$SP(I) = \bigcap_{P \in \mathrm{maxAss}(I)} NP(Q_{\subseteq P}) = NP(Q_{\subseteq \langle x_1, x_2 \rangle}) = NP(I).$$

Since $\alpha(I)$ is the smallest degree of a generator of I (Definition 2.1.4), we know that $\alpha(I) = 3$. We showed $SP(I) = NP(I)$ so by Theorem 2.4.6 we get $\widehat{\alpha}(I) = \alpha(I) = 3$.

2.5 The Waldschmidt constant for monomial ideals in two variables

In Example 2.4.8 we saw that $\alpha(I)$, the minimal degree of a generator of I, was equal to the Waldschmidt constant. In this section we will show this is true for any monomial ideal in two variables. We will use the following two lemmas:

Lemma 2.5.1 (The Splitting Lemma, [EGSS02])**.** *If m is a minimal generator of a monomial ideal I with $m = m_1m_2$ and $gcd(m_1, m_2) = 1$, then*

$$I = (I + \langle m_1\rangle) \cap (I + \langle m_2\rangle).$$

Lemma 2.5.2. *Let I be a monomial ideal in $K[x_1, x_2]$ of the form*

$$I = \langle x_1^{a_1}x_2^{b_1}, x_1^{a_2}x_2^{b_2}, \dots, x_1^{a_r}x_2^{b_r}\rangle$$

with $a_1 > a_2 > \cdots > a_r$ and $b_1 < b_2 < \cdots < b_r$. Then I has irreducible primary decomposition

$$I = \langle x_1^{a_r}\rangle \cap \langle x_2^{b_1}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_1^{a_2}, x_2^{b_3}\rangle \cap \cdots \cap \langle x_1^{a_{r-1}}, x_2^{b_r}\rangle.$$

Proof. Start with $I = \langle x_1^{a_1}x_2^{b_1}, x_1^{a_2}x_2^{b_2}, \dots, x_1^{a_r}x_2^{b_r}\rangle$. Split I on the monomial $x_1^{a_1}x_2^{b_1}$ (i.e., apply the Lemma 2.5.1 on $m = x_1^{a_1}x_2^{b_1}$) to get

$$I = \langle x_1^{a_1}, x_1^{a_2}x_2^{b_2}, \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_2^{b_1}, \cancel{x_1^{a_2}x_2^{b_2}, \dots, x_1^{a_r}x_2^{b_r}}\rangle$$

$$I = \langle x_1^{a_1}, x_1^{a_2}x_2^{b_2}, \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_2^{b_1}\rangle.$$

Note $b_1 < b_2 < \cdots < b_r$ gives cancellation in the second component.

Now split the first component of I on the monomial $x_1^{a_2}x_2^{b_2}$ to get

$$I = \langle \cancel{x_1^{a_1}}, x_1^{a_2}, x_1^{a_3}x_2^{b_3}, \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}, \cancel{x_1^{a_3}x_2^{b_3}, \dots, x_1^{a_r}x_2^{b_r}}\rangle \cap \langle x_2^{b_1}\rangle$$

$$I = \langle x_1^{a_2}, x_1^{a_3}x_2^{b_3}, \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_2^{b_1}\rangle.$$

Note $a_1 > a_2$ gives cancellation in the first component.

Now split the first component of I on the monomial $x_1^{a_3}x_2^{b_3}$ to get

$$I = \langle \cancel{x_1^{a_2}}, x_1^{a_3}, x_1^{a_4}x_2^{b_4}, \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_1^{a_2}, x_2^{b_3}, \cancel{x_1^{a_4}x_2^{b_4}, \dots, x_1^{a_r}x_2^{b_r}}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_2^{b_1}\rangle$$

$$I = \langle x_1^{a_3}, x_1^{a_4}x_2^{b_4} \dots, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_1^{a_2}, x_2^{b_3}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_2^{b_1}\rangle.$$

Continue splitting the first component of I on the monomial $x_1^{a_i}x_2^{b_i}$ up to $i = r-1$ to get

$$I = \langle x_1^{a_{r-1}}, x_1^{a_r}x_2^{b_r}\rangle \cap \langle x_1^{a_{r-2}}, x_2^{b_{r-1}}\rangle \cap \cdots \cap \langle x_1^{a_2}, x_2^{b_3}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_2^{b_1}\rangle.$$

Then split on $x_1^{a_r}x_2^{b_r}$ to get

$$I = \langle \cancel{x_1^{a_{r-1}}}, x_1^{a_r}\rangle \cap \langle x_1^{a_{r-1}}, x_2^{b_r}\rangle \cap \langle x_1^{a_{r-2}}, x_2^{b_{r-1}}\rangle \cap \cdots \cap \langle x_1^{a_2}, x_2^{b_3}\rangle \cap \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_2^{b_1}\rangle.$$

giving the result. □

Theorem 2.5.3. *Let I be a monomial ideal in $K[x_1, x_2]$. Then $\widehat{\alpha}(I) = \alpha(I)$.*

Proof. Let

$$I = \langle x_1^{a_1}x_2^{b_1}, x_1^{a_2}x_2^{b_2}, \ldots, x_1^{a_r}x_2^{b_r}\rangle$$

with $a_1 > a_2 > \cdots > a_r$ and $b_1 < b_2 < \cdots < b_r$.

First suppose $r = 1$. If $a_1 \neq 0$ and $b_1 \neq 0$, then $I = \langle x_1^{a_1}x_2^{b_1}\rangle = \langle x_1^{a_1}\rangle \cap \langle x_2^{b_1}\rangle$. So $\operatorname{Ass}(I) = \operatorname{maxAss}(I) = \{\langle x_1\rangle, \langle x_2\rangle\}$. This gives

$$SP(I) = \bigcap_{P \in \operatorname{maxAss}(I)} NP(Q_{\subseteq P}) = NP(Q_{\subseteq \langle x_1\rangle}) \cap NP(Q_{\subseteq \langle x_2\rangle}) = NP(I).$$

Since $SP(I) = NP(I)$ we have $\widehat{\alpha}(I) = \alpha(I)$. The argument is similar if one of a_1, b_1 is zero. Now suppose $r \geq 2$.

Case 1: Suppose $a_r = 0$ and $b_1 = 0$. Then by Lemma 2.5.2 the primary decomposition of I is

$$I = \langle x_1^{a_1}, x_2^{b_2}\rangle \cap \langle x_1^{a_2}, x_2^{b_3}\rangle \cap \cdots \cap \langle x_1^{a_{r-1}}, x_2^{b_r}\rangle.$$

and so $\operatorname{Ass}(I) = \operatorname{maxAss}(I) = \{\langle x_1, x_2\rangle\}$. Note that each primary component Q of I is of the form $\langle x_1^{a_i}, x_2^{b_{i+1}}\rangle$, so we have $\sqrt{Q} = \sqrt{\langle x_1^{a_i}, x_2^{b_{i+1}}\rangle} \subseteq \langle x_1, x_2\rangle$. This gives

$$SP(I) = \bigcap_{P \in \operatorname{maxAss}(I)} NP(Q_{\subseteq P}) = NP(Q_{\subseteq \langle x_1, x_2\rangle}) = NP(I),$$

and since $SP(I) = NP(I)$ we have $\widehat{\alpha}(I) = \alpha(I)$.

Case 2: Suppose $a_r \neq 0$ and $b_1 \neq 0$. Then by Lemma 2.5.2 the primary decomposition of

I is

$$I = \langle x_1^{a_r} \rangle \cap \langle x_2^{b_1} \rangle \cap \langle x_1^{a_1}, x_2^{b_2} \rangle \cap \langle x_1^{a_2}, x_2^{b_3} \rangle \cap \cdots \cap \langle x_1^{a_{r-1}}, x_2^{b_r} \rangle.$$

Thus $\mathrm{Ass}(I) = \{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_1, x_2 \rangle\}$ and $\max\mathrm{Ass}(I) = \{\langle x_1, x_2 \rangle\}$. Note that $\sqrt{\langle x_1^{a_r} \rangle} = \langle x_1 \rangle \subset \langle x_1, x_2 \rangle$ and $\sqrt{\langle x_2^{b_1} \rangle} = \langle x_2 \rangle \subset \langle x_1, x_2 \rangle$. So for each primary component Q of I we have $\sqrt{Q} \subseteq \langle x_1, x_2 \rangle$, meaning $Q_{\subseteq \langle x_1, x_2 \rangle} = I$. So we have

$$SP(I) = \bigcap_{P \in \max\mathrm{Ass}(I)} NP(Q_{\subseteq P}) = NP(Q_{\subseteq \langle x_1, x_2 \rangle}) = NP(I)$$

and since $SP(I) = NP(I)$ we have $\widehat{\alpha}(I) = \alpha(I)$.

Case 3: Suppose $a_r \neq 0$ and $b_1 = 0$. Then by Lemma 2.5.2 the primary decomposition of I is

$$I = \langle x_1^{a_r} \rangle \cap \langle x_1^{a_1}, x_2^{b_2} \rangle \cap \langle x_1^{a_2}, x_2^{b_3} \rangle \cap \cdots \cap \langle x_1^{a_{r-1}}, x_2^{b_r} \rangle.$$

Thus $\mathrm{Ass}(I) = \{\langle x_1 \rangle, \langle x_1, x_2 \rangle\}$ and $\max\mathrm{Ass}(I) = \{\langle x_1, x_2 \rangle\}$. Note that $\sqrt{\langle x_1^{a_r} \rangle} = \langle x_1 \rangle \subset \langle x_1, x_2 \rangle$. So for each primary component Q of I we have $\sqrt{Q} \subseteq \langle x_1, x_2 \rangle$, meaning $Q_{\subseteq \langle x_1, x_2 \rangle} = I$. So we have

$$SP(I) = \bigcap_{P \in \max\mathrm{Ass}(I)} NP(Q_{\subseteq P}) = NP(Q_{\subseteq \langle x_1, x_2 \rangle}) = NP(I)$$

and since $SP(I) = NP(I)$ we have $\widehat{\alpha}(I) = \alpha(I)$. The argument is similar if instead we have $a_r = 0$ and $b_1 \neq 0$ $\square$

Since $\alpha(I)$ is the degree of the smallest generator of I, the Waldschmidt constant can only take on a natural number in two variables. To find a monomial ideal with a non-integer Waldschmidt constant, we need to work in a ring with more variables. This is the focus of the next chapter.

Chapter 3

Waldschmidt constant for monomial ideals in $n > 2$ variables

In the previous chapter we saw how in two variables, the Waldschmidt constant of a monomial ideal is always equal to the initial degree $\alpha(I)$ (Theorem 2.5.3), which is a positive integer. In this chapter we will explore what values of the Waldschmidt constant can be obtained with a larger number of variables. The family of ideals we will investigate in Chapter 3 are in $R = K[x_1, \ldots, x_n]$ for $n \geq 3$ and have the following form

$$I = \langle x_2, \ldots, x_n \rangle^{e_1} \cap \langle x_1, x_3, \ldots, x_n \rangle^{e_2} \cap \cdots \cap \langle x_1, \ldots, x_{n-1} \rangle^{e_n} \tag{3.0.1}$$

with $e_1 \geq e_2 \geq \cdots \geq e_n > 0$. In other words, the intersection of powers of all height $n-1$ prime monomial ideals in $R = K[x_1, \ldots, x_n]$. Our main result is that by choosing the appropriate e_i's, we can attain a Waldschmidt constant equal to $\frac{q}{p}$ for almost all rational numbers greater than or equal to 1. By "almost all" we mean that for a given denominator p, there are only finitely many numerators q which are not attainable by this method (see Corollary 3.3.4).

3.1 Intersections of powers of height two prime monomial ideals in three variables

In this section we will compute the Waldschmidt constant for ideals in $R = K[x_1, x_2, x_3]$ of the form

$$I = \langle x_2, x_3\rangle^{e_1} \cap \langle x_1, x_3\rangle^{e_2} \cap \langle x_1, x_2\rangle^{e_3}$$

with $e_1 \geq e_2 \geq e_3 > 0$. The results of this section illustrate the ideas later in the chapter. We now compute the symbolic polyhedron of I. Label the associated primes of I by $P_1 = \langle x_2, x_3\rangle, P_2 = \langle x_1, x_3\rangle$, and $P_3 = \langle x_1, x_2\rangle$. Then Lemma 2.4.7 gives us the form of the Newton polyhedra of each primary component $Q_i = P_i^{e_i}$. Let $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]^T$ be a vector of real variables. Then we have

$$\begin{aligned} NP(Q_1 = {P_1}^{e_1}) &= \left\{\tfrac{1}{e_1}\mathbf{y}_2 + \tfrac{1}{e_1}\mathbf{y}_3 \geq 1 \mid \mathbf{y}_2, \mathbf{y}_3 \in \mathbb{R}_{\geq 0}\right\} \\ NP(Q_2 = {P_2}^{e_2}) &= \left\{\tfrac{1}{e_2}\mathbf{y}_1 + \tfrac{1}{e_2}\mathbf{y}_3 \geq 1 \mid \mathbf{y}_1, \mathbf{y}_3 \in \mathbb{R}_{\geq 0}\right\} \\ NP(Q_3 = {P_3}^{e_3}) &= \left\{\tfrac{1}{e_3}\mathbf{y}_1 + \tfrac{1}{e_3}\mathbf{y}_2 \geq 1 \mid \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}_{\geq 0}\right\}. \end{aligned}$$

Observe for I we have $\operatorname{Ass}(I) = \operatorname{maxAss}(I) = \{\sqrt{Q_1} = P_1, \sqrt{Q_2} = P_2, \sqrt{Q_3} = P_3\}$ so we have $Q_{\subseteq P_i} = Q_i$. Then by Definition 2.2.6 we compute $SP(I)$ by

$$SP(I) = \bigcap_{P \in \operatorname{maxAss}(I)} NP(Q_{\subseteq P}) = NP(Q_1) \cap NP(Q_2) \cap NP(Q_3)$$

which can be re-written as

$$SP(I) = \left\{ \mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0} \text{ with } A = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 \end{bmatrix} \right\}.$$

Now by Theorem 2.4.4, solving for the Waldschmidt constant $\widehat{\alpha}(I)$ amounts to solving the following linear optimization problem:

$$\text{Solve } \min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \text{ for} \tag{3.1.1}$$

$$A = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 \end{bmatrix},$$

where $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3]^T$ is a vector of variables. Define h_k to be the expression equal to the k-th coordinate of $A\mathbf{y}$:

$$\begin{aligned} h_1 = &\ \tfrac{1}{e_1}\mathbf{y}_2 + \tfrac{1}{e_1}\mathbf{y}_3 \\ h_2 = &\ \tfrac{1}{e_2}\mathbf{y}_1 + \tfrac{1}{e_2}\mathbf{y}_3 \\ h_3 = &\ \tfrac{1}{e_3}\mathbf{y}_1 + \tfrac{1}{e_3}\mathbf{y}_2. \end{aligned}$$

The symbolic polyhedron $SP(I)$ associated to the above linear optimization problem is defined by the intersection of six half-spaces. For $k = 1, 2, 3$ we have three half-spaces of defined by the inequality $h_k = \sum_{i=1, i\neq k}^{3} \frac{1}{e_k}\mathbf{y}_i \geq 1$. We also have three half-spaces coming from the non-negativity constraint: $\mathbf{y}_1 \geq 0, \mathbf{y}_2 \geq 0$, and $\mathbf{y}_3 \geq 0$. Changing the six inequalities to equations gives the six bounding hyperplanes of $SP(I)$. So for $k = 1, 2, 3$ we have three hyperplanes $h_k = \sum_{i=1, i\neq k}^{3} \frac{1}{e_k}\mathbf{y}_i = 1$ and 3 hyperplanes $\mathbf{y}_k = 0$.

A minimizer to the linear optimization problem (3.1.1) must be a vertex of $SP(I)$ by Theorem 2.1.13. The (potential) vertices of $SP(I)$ can be computed by intersecting 3 of the 6 bounding hyperplanes. We will denote such an intersection by $\mathbf{z} \in \mathbb{R}^3$. Among these $\binom{6}{3} = 20$ intersections, only those that exist and satisfy $A\mathbf{z} \geq \mathbf{1}$ and $\mathbf{z} \geq \mathbf{0}$ (i.e., those that are feasible) will be the vertices of $SP(I)$.

Let $H, Y \subseteq \{1, 2, 3\}$ with $|H| + |Y| = 3$. Then each choice of H and Y corresponds to an intersection of 3 hyperplanes. For example $H = \{1, 3\}$ and $V = \{2\}$ corresponds to the intersection of the hyperplanes $h_1 = 1, h_3 = 1$ and $\mathbf{y}_2 = 0$, i.e., the intersection of

$$\begin{aligned} h_1 &= \frac{1}{e_1}\mathbf{y}_2 + \frac{1}{e_1}\mathbf{y}_3 = 1 \\ h_3 &= \frac{1}{e_3}\mathbf{y}_1 + \frac{1}{e_3}\mathbf{y}_2 = 1 \\ &\qquad\qquad\quad \mathbf{y}_2 = 0. \end{aligned}$$

We will denote this intersection by $\mathbf{z}^{13|2}$, and similarly for other intersections. By analyzing the vertices of $SP(I)$ we will prove the following result on the Waldschmidt constant of I.

Theorem 3.1.1. *Let $R = K[x_1, x_2, x_3]$ and consider the following ideal*

$$I = \langle x_2, x_3\rangle^{e_1} \cap \langle x_1, x_3\rangle^{e_2} \cap \langle x_2, x_3\rangle^{e_3}$$

with $e_1 \geq e_2 \geq e_3 > 0$. If $e_1 \leq e_2 + e_3$, then

$$\widehat{\alpha}(I) = \frac{e_1 + e_2 + e_3}{2}.$$

Otherwise, $\widehat{\alpha}(I) = e_1$.

Proof. We will organize the intersections by $|H \cap Y|$.

Case 1: $|H \cap Y| = 0$.

Case 1.1: $|H \cap Y| = 0$ with $|H| = 3$ and $|Y| = 0$.

Here $H = \{1, 2, 3\}$. So we are computing the intersection of $h_1 = 1, h_2 = 1$ and $h_3 = 1$, i.e., the intersection of

$$\begin{aligned} h_1 = &\quad \tfrac{1}{e_1}\mathbf{y}_2 + \tfrac{1}{e_1}\mathbf{y}_3 = 1 \\ h_2 = &\quad \tfrac{1}{e_2}\mathbf{y}_1 + \tfrac{1}{e_2}\mathbf{y}_3 = 1 \\ h_3 = &\quad \tfrac{1}{e_3}\mathbf{y}_1 + \tfrac{1}{e_3}\mathbf{y}_2 = 1. \end{aligned}$$

This intersection is a distinguished intersection so we will label it $\mathbf{z}^*$ instead of $\mathbf{z}^{123|}$.

$$\text{Let } M = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & 1 \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & 1 \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & 1 \end{bmatrix}, \text{ then column 4 of rref}(M) = \mathbf{z}^* = \begin{bmatrix} \frac{-e_1+e_2+e_3}{2} \\ \frac{e_1-e_2+e_3}{2} \\ \frac{e_1+e_2-e_3}{2} \end{bmatrix}.$$

For $\mathbf{z}^*$ to be feasible it needs to satisfy the other three inequalities: $\mathbf{y}_1 \geq 0, \mathbf{y}_2 \geq 0$, and $\mathbf{y}_3 \geq 0$. In other words, each coordinate of $\mathbf{z}^*$ must be non-negative. Since we are assuming $e_1 \geq e_2 \geq e_3 > 0$, the first coordinate is minimal of the three, so $\mathbf{z}^*$ is feasible, and thus a vertex, when $\frac{-e_1+e_2+e_3}{2} \geq 0$, or $e_1 \leq e_2 + e_3$.

Case 1.2: $|H \cap Y| = 0$ with $|H| = 2$ and $|Y| = 1$.

There are three intersections in this case:

$$\begin{aligned} \mathbf{z}^{12|3} = &\quad (e_2, e_1, 0), \quad \text{feasible if } e_1 + e_2 \geq e_3 \quad \text{(always feasible)} \\ \mathbf{z}^{13|2} = &\quad (e_3, 0, e_1), \quad \text{feasible if } e_1 + e_3 \geq e_2 \quad \text{(always feasible)} \\ \mathbf{z}^{23|1} = &\quad (0, e_3, e_2), \quad \text{feasible if } e_2 + e_3 \geq e_1. \end{aligned}$$

Since we assume $e_i > 0$, these intersections are always non-negative. The feasibility conditions are attained by substituting the intersection point into $h_k \geq 1$ where k is the element missing from H. For example substituting $\mathbf{z}^{12|3} = (e_2, e_3, 0)$ into $h_3 = \frac{1}{e_3}\mathbf{y}_1 + \frac{1}{e_3}\mathbf{y}_2 \geq 1$ gives $e_1 + e_2 \geq e_3$ which is always true since $e_1 \geq e_3$. So we get that $\mathbf{z}^{12|3}$ and $\mathbf{z}^{13|2}$ will always be vertices of $SP(I)$, but $\mathbf{z}^{23|1}$ is a vertex only if $e_2 + e_3 \geq e_1$.

Case 1.3: $|H \cap Y| = 0$ with $|H| = 1$ and $|Y| = 2$.

Solving for $\mathbf{z}^{1|23}$, $\mathbf{z}^{2|13}$, and $\mathbf{z}^{3|12}$ gives an inconsistent system. So they are not intersection points. To see this, consider $\mathbf{z}^{1|23}$ which is the intersection of $h_1 = \frac{1}{e_1}\mathbf{y}_2 + \frac{1}{e_1}\mathbf{y}_3 = 1$, $\mathbf{y}_2 = 0$ and $\mathbf{y}_3 = 0$. Substituting $\mathbf{y}_2 = 0$ and $\mathbf{y}_3 = 0$ into $h_1 = 1$ gives $0 = 1$. Solving for $\mathbf{z}^{2|13}$, and $\mathbf{z}^{3|12}$ gives a similar contradiction.

Case 1.4: $|H \cap Y| = 0$ with $|H| = 0$ and $|Y| = 3$.

The intersection of $\mathbf{y}_1 = 0, \mathbf{y}_2 = 0$ and $\mathbf{y}_3 = 0$ is the zero vector. Note that $\mathbf{z}^{|123} = \mathbf{0}$ is never feasible since $A\mathbf{0} = \mathbf{0} \geq \mathbf{1}$, a contradiction.

Case 2: $|H \cap Y| = 1$.

Case 2.1: $|H \cap Y| = 1$ with $|H| = 2$ and $|Y| = 1$.

The intersections are:

$$\begin{aligned}
\mathbf{z}^{12|1} &= \quad (0, e_1 - e_2, e_2), \quad \text{feasible if } e_1 \geq e_2 + e_3 \\
\mathbf{z}^{12|2} &= \quad (-e_1 + e_2, 0, e_1), \quad \text{feasible if } e_2 \geq e_1 + e_3 \quad \text{(never feasible)} \\
\mathbf{z}^{13|1} &= \quad (0, e_3, e_1 - e_3), \quad \text{feasible if } e_1 \geq e_2 + e_3 \\
\mathbf{z}^{13|3} &= \quad (-e_1 + e_3, e_1, 0), \quad \text{feasible if } e_3 \geq e_1 + e_2 \quad \text{(never feasible)} \\
\mathbf{z}^{23|2} &= \quad (e_3, 0, e_2 - e_3), \quad \text{feasible if } e_2 \geq e_1 + e_3 \quad \text{(never feasible)} \\
\mathbf{z}^{23|3} &= \quad (e_2, -e_2 + e_3, 0), \quad \text{feasible if } e_3 \geq e_1 + e_2 \quad \text{(never feasible).}
\end{aligned}$$

The feasibility conditions are attained by substituting the intersection point into $h_k = 1$ where k is the element missing from H.

Case 2.2: $|H \cap Y| = 1$, $|H| = 1$ and $|Y| = 2$.

The six intersections are

$$\begin{aligned}
\mathbf{z}^{1|12} &= (0, 0, e_1), \quad \mathbf{z}^{1|13} = (0, e_1, 0) \\
\mathbf{z}^{2|13} &= (0, 0, e_2), \quad \mathbf{z}^{2|23} = (e_2, 0, 0), \\
\mathbf{z}^{3|12} &= (0, e_3, 0), \quad \mathbf{z}^{3|23} = (e_3, 0, 0).
\end{aligned}$$

These intersections are never feasible. If an intersection has only one non-zero entry, say in coordinate k, then substituting the intersection into $h_k \geq 1$ gives $0 \geq 1$.

This covers all 20 possible intersections. There are three types of $SP(I)$ depending on e_1 compared to $e_2 + e_3$:

Type 1 of $SP(I)$: The case that $e_1 < e_2 + e_3$.

There are four vertices: $\mathbf{z}^* = (\frac{-e_1+e_2+e_3}{2}, \frac{e_1-e_2+e_3}{2}, \frac{e_1+e_2-e_3}{2})$, $\mathbf{z}^{12|3} = (e_2, e_1, 0)$, $\mathbf{z}^{13|2} = (e_3, 0, e_1)$, and $\mathbf{z}^{23|1} = (0, e_3, e_2)$.

Recall a minimizer to the linear optimization problem (3.1.1) is a vertex of $SP(I)$ which minimizes $\mathbf{1}^T\mathbf{z}$. In other words, a vertex with minimal sum of coordinates. We compute the coordinate sum $\mathbf{1}^T\mathbf{z}$ for each of the above four vertices.

$$\mathbf{1}^T\mathbf{z}^* = \tfrac{-e_1+e_2+e_3}{2} + \tfrac{e_1-e_2+e_3}{2} + \tfrac{e_1+e_2-e_3}{2} = \tfrac{e_1+e_2+e_3}{2},$$
$$\mathbf{1}^T\mathbf{z}^{12|3} = e_2+e_3,\ \mathbf{1}^T\mathbf{z}^{13|2} = e_1+e_3,\ \text{and } \mathbf{1}^T\mathbf{z}^{23|1} = e_2+e_3.$$

We will now show $\mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+e_3}{2}$ is the minimum of the coordinate sums. Substituting the assumption $e_1 < e_2+e_3$ into $\frac{e_1+e_2+e_3}{2}$ gives:

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+e_3}{2} < \frac{e_2+e_3+e_2+e_3}{2} = e_2+e_3 = \mathbf{1}^T\mathbf{z}^{23|1} = \mathbf{1}^T\mathbf{z}^{12|3}.$$

Additionally, $e_2+e_3 \le e_1+e_3 = \mathbf{1}^T\mathbf{z}^{13|2}$ since $e_2 \le e_1$. So we get:

$$\mathbf{1}^T\mathbf{z}^* < \mathbf{1}^T\mathbf{z}^{23|1} = \mathbf{1}^T\mathbf{z}^{12|3} \le \mathbf{1}^T\mathbf{z}^{13|2}.$$

This shows $\mathbf{z}^*$ is the unique minimizer to the linear optimization problem (3.1.1). The optimal solution equals the Waldschmidt constant for I, so for Type 1 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3}{2}$.

Type 2 of $SP(I)$: The case that $e_1 = e_2+e_3$.

There are three vertices; the first two are $\mathbf{z}^{12|3} = (e_2, e_3, 0)$, and $\mathbf{z}^{13|2} = (e_3, 0, e_1)$. The third vertex is any of the following four intersections: $\mathbf{z}^* = (\frac{-e_1+e_2+e_3}{2}, \frac{e_1-e_2+e_3}{2}, \frac{e_1+e_2-e_3}{2})$, $\mathbf{z}^{23|1} = (0, e_3, e_2)$, $\mathbf{z}^{12|1} = (0, e_1-e_2, e_2)$, and $\mathbf{z}^{13|1} = (0, e_3, e_1-e_3)$. We observe that these four intersections are equal when $e_1 = e_2+e_3$. So for Type 2 we get $\widehat{\alpha}(I) = e_1$.

Type 3 of $SP(I)$: The case that $e_1 > e_2+e_3$.

There are four vertices: $\mathbf{z}^{12|3} = (e_2, e_1, 0)$, $\mathbf{z}^{13|2} = (e_3, 0, e_1)$, $\mathbf{z}^{12|1} = (0, e_1-e_2, e_2)$, and $\mathbf{z}^{13|1} = (0, e_3, e_1-e_3)$. So for Type 3 we get $\widehat{\alpha}(I) = e_1$. $\square$

The following corollary shows we can choose e_1, e_2, and e_3 to construct an ideal with Waldschmidt constant $\frac{q}{2} > 1$.

Corollary 3.1.2. *Fix a positive integer $q > 2$. Then write the integer partition $q =$*

$e_1 + e_2 + e_3$ *where*

$$\begin{cases} e_1 = \frac{q}{3}, e_2 = \frac{q}{3}, e_3 = \frac{q}{3} & \textit{if } q \equiv 0 \mod 3 \\ e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lfloor \frac{q}{3} \rfloor, e_3 = \lfloor \frac{q}{3} \rfloor & \textit{if } q \equiv 1 \mod 3 \\ e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lceil \frac{q}{3} \rceil, e_3 = \lfloor \frac{q}{3} \rfloor & \textit{if } q \equiv 2 \mod 3. \end{cases}$$

Then

$$I = \langle x_2, x_3 \rangle^{e_1} \cap \langle x_1, x_3 \rangle^{e_2} \cap \langle x_1, x_2 \rangle^{e_3}$$

is a monomial ideal in $R = K[x_1, x_2, x_3]$ *with* $\widehat{\alpha}(I) = \frac{q}{2}$.

Proof. We consider three cases that depend on the value of $q \mod 3$.

Case 1: $q \equiv 0 \mod 3$. Let $e_1 = \frac{q}{3}, e_2 = \frac{q}{3}, e_3 = \frac{q}{3}$. Since $q \equiv 0 \mod 3$, $\frac{q}{3}$ is an integer. We have $e_1 < e_2 + e_3$, so by Theorem 3.1.1 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3}{2} = \frac{q}{2}$.

Case 2: $q \equiv 1 \mod 3$. Let $e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lfloor \frac{q}{3} \rfloor, e_3 = \lfloor \frac{q}{3} \rfloor$. If $q = 4$, then $e_1 = 2$ and $e_2 = e_3 = 1$. Since $e_1 = e_2 + e_3$, by Theorem 3.1.1 we have $\widehat{\alpha}(I) = e_1 = 2$ or $\widehat{\alpha}(I) = \frac{q}{2}$. Otherwise if $q > 4$, observe that $q \equiv 1 \mod 3$ implies that $\lceil \frac{q}{3} \rceil = \frac{q+2}{3}$ and $\lfloor \frac{q}{3} \rfloor = \frac{q-1}{3}$. Then $e_1 < e_2 + e_3$ is true since $\frac{q+2}{3} < \frac{q-1}{3} + \frac{q-1}{3}$ holds for $q > 4$. So by Theorem 3.1.1 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3}{2} = \frac{\lceil \frac{q}{3} \rceil + \lfloor \frac{q}{3} \rfloor + \lfloor \frac{q}{3} \rfloor}{2} = \frac{\frac{q+2}{3} + \frac{q-1}{3} + \frac{q-1}{3}}{2} = \frac{q}{2}$.

Case 3: $q \equiv 2 \mod 3$. Let $e_1 = \lceil \frac{q}{3} \rceil, e_2 = \lceil \frac{q}{3} \rceil, e_3 = \lfloor \frac{q}{3} \rfloor$. Observe that $q \equiv 2 \mod 3$ implies that $\lceil \frac{q}{3} \rceil = \frac{q+1}{3}$ and $\lfloor \frac{q}{3} \rfloor = \frac{q-2}{3}$. Then $e_1 < e_2 + e_3$ is true since $\frac{q+1}{3} < \frac{q+1}{3} + \frac{q-2}{3}$ holds for $q \geq 5$. So by Theorem 3.1.1 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3}{2} = \frac{\lceil \frac{q}{3} \rceil + \lceil \frac{q}{3} \rceil + \lfloor \frac{q}{3} \rfloor}{2} = \frac{\frac{q+1}{3} + \frac{q+1}{3} + \frac{q-2}{3}}{2} = \frac{q}{2}$. □

The following two examples illustrate the above corollary.

Example 3.1.3. We will construct an ideal I with Waldschmidt constant equal to $\frac{17}{2}$. Since $17 \equiv 2 \mod 3$ we can set $e_1 = \lceil \frac{17}{3} \rceil = 6, e_2 = \lceil \frac{17}{3} \rceil = 6$, and $e_3 = \lfloor \frac{17}{3} \rfloor = 5$ so that

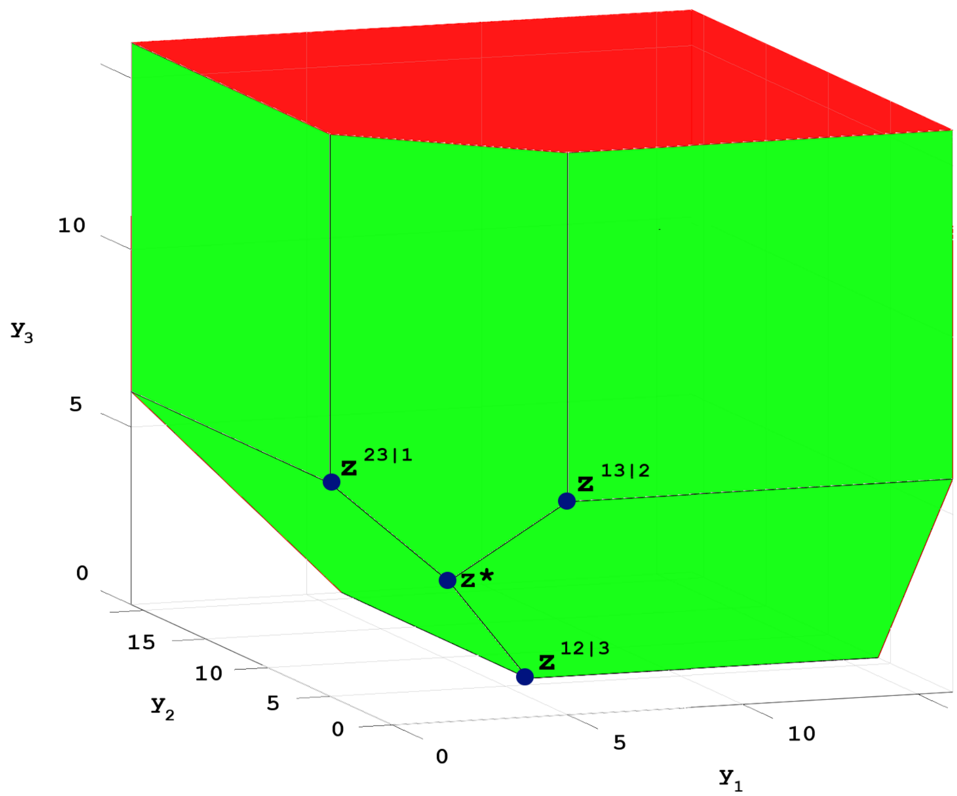
$$I = \langle x_2, x_3 \rangle^6 \cap \langle x_1, x_3 \rangle^6 \cap \langle x_1, x_2 \rangle^5$$

has $\widehat{\alpha}(I) = \frac{17}{2}$ by Corollary 3.1.2. Since $e_1 < e_2 + e_3$, we have $SP(I)$ of Type 1 (see proof of Theorem 3.1.1) and so it has four vertices:

$\mathbf{z}^* = (\frac{5}{2}, \frac{5}{2}, \frac{7}{2})$, $\mathbf{z}^{12|3} = (6, 6, 0), \mathbf{z}^{13|2} = (5, 0, 6)$, and $\mathbf{z}^{23|1} = (0, 5, 6)$.

See Figure 3.1 for an image of $SP(I)$.

Figure 3.1: Symbolic polyhedron of Example 3.1.3



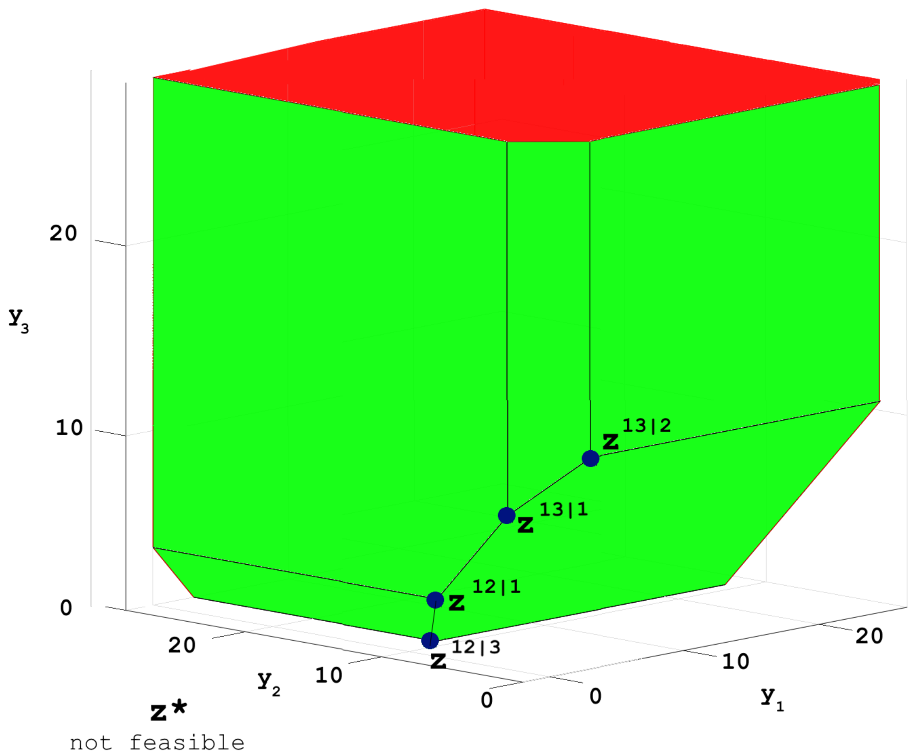
Example 3.1.4. Consider the ideal

$$I = \langle x_2, x_3 \rangle^{11} \cap \langle x_1, x_3 \rangle^3 \cap \langle x_1, x_2 \rangle^3.$$

Here we have $e_1 = 11, e_2 = 3$, and $e_3 = 3$. Since $e_1 > e_2 + e_3$, we know $\widehat{\alpha}(I) = e_1 = 11$ by Theorem 3.1.1.

We have $SP(I)$ of Type 3 (see proof of Theorem 3.1.1) which has four vertices: $\mathbf{z}^{12|3} = (3, 11, 0)$, $\mathbf{z}^{13|2} = (3, 0, 11), \mathbf{z}^{12|1} = (0, 8, 3)$, and $\mathbf{z}^{13|1} = (0, 3, 8)$. Observe the line segment joining $\mathbf{z}^{12|1} = (0, 8, 3)$ to $\mathbf{z}^{13|1} = (0, 3, 8)$ is a face of minimizers. Note we can compute $\mathbf{z}^* = (-\frac{5}{2}, \frac{11}{2}, \frac{11}{2})$ which lies outside of $SP(I)$. See Figure 3.2 for an image of $SP(I)$.

Figure 3.2: Symbolic polyhedron of Example 3.1.4



3.2 Intersections of powers of height three prime monomial ideals in four variables

In the previous section we showed that any rational Waldschmidt constant $\widehat{\alpha}(I) > 1$ with a denominator of two can be attained by intersecting powers of height two prime monomial ideals in $n = 3$ variables. In this section we will compute the Waldschmidt constant for ideals in $R = K[x_1, x_2, x_3, x_4]$ of the form

$$I = \langle x_2, x_3, x_4\rangle^{e_1} \cap \langle x_1, x_3, x_4\rangle^{e_2} \cap \langle x_1, x_2, x_4\rangle^{e_3} \cap \langle x_1, x_2, x_3\rangle^{e_4}$$

where $e_1 \geq e_2 \geq e_3 \geq e_4 > 0$. We will find that we can choose appropriate e_i's to attain any Waldschmidt constant $\widehat{\alpha}(I) = \frac{q}{3} > 1$ with the exception of $q = 5$.

In a similar fashion to the formulation of the linear optimization problem (3.1.1) in Section 3.1, we can use Lemma 2.4.7 and Theorem 2.4.4 to show that solving for the Waldschmidt constant $\widehat{\alpha}(I)$ amounts to solving the following linear optimization problem:

$$\text{Solve}\min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \text{ for} \tag{3.2.1}$$

$$A = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & \frac{1}{e_3} \\ \frac{1}{e_4} & \frac{1}{e_4} & \frac{1}{e_4} & 0 \end{bmatrix},$$

where $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3\ \mathbf{y}_4]^T$ is a vector of variables.

Define h_k to be the expression equal to the k-th coordinate of $A\mathbf{y}$:

$$\begin{aligned} h_1 &= \tfrac{1}{e_1}\mathbf{y}_2 + \tfrac{1}{e_1}\mathbf{y}_3 + \tfrac{1}{e_1}\mathbf{y}_4 \\ h_2 &= \tfrac{1}{e_2}\mathbf{y}_1 + \tfrac{1}{e_2}\mathbf{y}_3 + \tfrac{1}{e_2}\mathbf{y}_4 \\ h_3 &= \tfrac{1}{e_3}\mathbf{y}_1 + \tfrac{1}{e_3}\mathbf{y}_2 + \tfrac{1}{e_3}\mathbf{y}_4 \\ h_4 &= \tfrac{1}{e_4}\mathbf{y}_1 + \tfrac{1}{e_4}\mathbf{y}_2 + \tfrac{1}{e_4}\mathbf{y}_3. \end{aligned}$$

The symbolic polyhedron $SP(I)$ associated to the above linear optimization problem is defined by the intersection of 8 half-spaces. For $1 \leq k \leq 4$ we have 4 half-spaces defined by the inequality $h_k = \sum_{i=1,i\neq k}^{4} \frac{1}{e_k}\mathbf{y}_i \geq 1$. We also have 4 half-spaces coming from the non-negativity constraint: $\mathbf{y}_1 \geq 0, \mathbf{y}_2 \geq 0, \mathbf{y}_3 \geq 0$, and $\mathbf{y}_4 \geq 0$. Changing the 8 inequalities to equations gives the 8 bounding hyperplanes of $SP(I)$. So for $1 \leq k \leq 4$ we have 4 hyperplanes $h_k = \sum_{i=1,i\neq k}^{4} \frac{1}{e_k}\mathbf{y}_i = 1$ and 4 hyperplanes $\mathbf{y}_k = 0$.

A minimizer to the linear optimization problem is among the vertices of $SP(I)$. To compute the (possible) vertices of $SP(I)$ we intersect 4 of the 8 bounding hyperplanes. We will denote such an intersection by $\mathbf{z} \in \mathbb{R}^4$. Among these $\binom{8}{4} = 70$ intersections, those that exist and satisfy $A\mathbf{z} \geq \mathbf{1}$ and $\mathbf{z} \geq \mathbf{0}$ (i.e., those that are feasible) will be the vertices of $SP(I)$.

Denote $\mathbf{z}^*$ to be the intersection of the four $h_k = 1$ hyperplanes. The following lemma concerns this distinguished intersection.

Lemma 3.2.1. *If $\mathbf{z}^*$ is the intersection of the four hyperplanes $h_1 = 1, h_2 = 1, h_3 = 1$, and $h_4 = 1$, then it has the form*

$$\mathbf{z}^* = \begin{bmatrix} \frac{-2e_1+e_2+e_3+e_4}{3} \\ \frac{e_1-2e_2+e_3+e_4}{3} \\ \frac{e_1+e_2-2e_3+e_4}{3} \\ \frac{e_1+e_2+e_3-2e_4}{3} \end{bmatrix}.$$

Furthermore, $\mathbf{z}^*$ is a vertex of $SP(I)$ when $e_1 \leq \frac{e_2+e_3+e_4}{2}$.

Proof. We can compute $\mathbf{z}^*$ by reducing the augmented matrix $[A|\mathbf{1}]$.

$$[A|\mathbf{1}] = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & 1 \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \frac{1}{e_2} & 1 \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & \frac{1}{e_3} & 1 \\ \frac{1}{e_4} & \frac{1}{e_4} & \frac{1}{e_4} & 0 & 1 \end{bmatrix}, \text{ last column of rref}([A|\mathbf{1}]) = \mathbf{z}^* = \begin{bmatrix} \frac{-2e_1+e_2+e_3+e_4}{3} \\ \frac{e_1-2e_2+e_3+e_4}{3} \\ \frac{e_1+e_2-2e_3+e_4}{3} \\ \frac{e_1+e_2+e_3-2e_4}{3} \end{bmatrix}.$$

Feasibility of $\mathbf{z}^*$ requires its coordinates to be non-negative. Since we are assuming $e_1 \geq e_2 \geq e_3 \geq e_4 > 0$, the first coordinate of $\mathbf{z}^*$ is minimal, and so $\mathbf{z}^*$ will be a vertex of $SP(I)$ when the first coordinate is non-negative. So we get that $\mathbf{z}^*$ is a vertex when $e_1 \leq \frac{e_2+e_3+e_4}{2}$. $\square$

Our next theorem gives the Waldschmidt constant for our family of ideals.

Theorem 3.2.2. *Let $R = K[x_1, x_2, x_3, x_4]$ and consider the following ideal*

$$I = \langle x_2, x_3, x_4\rangle^{e_1} \cap \langle x_1, x_3, x_4\rangle^{e_2} \cap \langle x_1, x_2, x_4\rangle^{e_3} \cap \langle x_1, x_2, x_3\rangle^{e_4}.$$

Assume without loss of generality that $e_1 \geq e_2 \geq e_3 \geq e_4 > 0$. If $e_1 \leq \frac{e_2+e_3+e_4}{2}$, then

$$\widehat{\alpha}(I) = \frac{e_1 + e_2 + e_3 + e_4}{3}.$$

Otherwise, $\widehat{\alpha}(I) = e_1$.

Proof. To prove this we will compute the form of every intersection of 4 of the 8 hyperplanes, which exhausts every possible vertex of $SP(I)$. To specify the 4 hyperplanes, we consider $H, Y \subseteq \{1, 2, 3, 4\}$ with $|H| + |Y| = 4$. Each choice of H and Y will correspond to an intersection of 4 hyperplanes. For example, $H = \{1, 3\}$ and $Y = \{2, 4\}$ is the intersection of the hyperplanes $h_1 = 1, h_3 = 1, \mathbf{y}_2 = 0, \mathbf{y}_4 = 0$. We notate this intersection as $\mathbf{z}^{13|24}$ (and similarly for other choices of H and Y).

Associate to H the $|H|$-tuple $\mathbf{t}$ which records the elements of H in ascending order, e.g., the above H gives $\mathbf{t} = (1, 3)$. Similarly associate to Y the $|Y|$-tuple $\mathbf{s}$, e.g., the above Y gives $\mathbf{s} = (2, 4)$.

We will consider cases depending on $|H \cap Y|$. For each case we will (i) determine the form of the intersection $\mathbf{z}$ (if it exists), (ii) determine the constraints the intersection must satisfy to be a vertex of $SP(I)$, (iii) show that $e_1 \leq \mathbf{1}^T\mathbf{z}$ (for every $\mathbf{z}$ except $\mathbf{z}^*$), and (iv) show that $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$.

Case 1.1: $|H \cap Y| = 0$ with $|H| = 1$ and $|Y| = 3$.

The intersection in this case is always empty. If we choose $H = \{1\}$ and $Y = \{2, 3, 4\}$ then substituting $\mathbf{y}_2 = 0$, $\mathbf{y}_3 = 0$, and $\mathbf{y}_4 = 0$ into

$$h_1 = \frac{1}{e_1}\mathbf{y}_2 + \frac{1}{e_1}\mathbf{y}_3 + \frac{1}{e_1}\mathbf{y}_4 = 1$$

gives $0 = 1$. We get a similar contradiction for any other choice of H and Y.

Case 1.2: $|H \cap Y| = 0$ with $|H| = 2$ and $|Y| = 2$.

We compute the six intersections for this case:

$$\begin{aligned} \mathbf{z}^{12|34} &= [e_2, e_1, 0, 0]^T & \mathbf{z}^{13|24} &= [e_3, 0, e_1, 0]^T \\ \mathbf{z}^{14|23} &= [e_4, 0, 0, e_1]^T & \mathbf{z}^{23|14} &= [0, e_2, e_3, 0]^T \\ \mathbf{z}^{24|13} &= [0, e_4, 0, e_2]^T & \mathbf{z}^{34|12} &= [0, 0, e_4, e_3]^T. \end{aligned}$$

In general for this case we can write the intersection $\mathbf{z}$ as:

$$\mathbf{z}_i = \begin{cases} e_{\mathbf{t}_1} & \text{for } i = \mathbf{t}_2 \\ e_{\mathbf{t}_2} & \text{for } i = \mathbf{t}_1 \\ 0 & \text{for } i \in Y. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following two constraints are satisfied:

$$e_{\mathbf{s}_1} \leq e_{\mathbf{t}_1} + e_{\mathbf{t}_2} \tag{3.2.2}$$

$$e_{\mathbf{s}_2} \leq e_{\mathbf{t}_1} + e_{\mathbf{t}_2}. \tag{3.2.3}$$

To justify these inequalities, substituting $\mathbf{z}$ into $h_{\mathbf{s}_j} \geq 1$ gives the inequality $e_{\mathbf{t}_1} + e_{\mathbf{t}_2} \geq e_{\mathbf{s}_j}$ for $j = 1, 2$. Since each e_i is assumed to be non-negative, $\mathbf{z}$ is non-negative (i.e., every entry of $\mathbf{z}$ is non-negative). Observe we have coordinate sum $\mathbf{1}^T\mathbf{z} = e_{\mathbf{t}_1} + e_{\mathbf{t}_2}$. We will show that for any vertex $\mathbf{z}$ in this case we have $e_1 \leq \mathbf{1}^T\mathbf{z}$. If $1 \in H$, then we have: $e_1 = e_{\mathbf{t}_1} < e_{\mathbf{t}_1} + e_{\mathbf{t}_2} = \mathbf{1}^T\mathbf{z}$. If $1 \notin H$, then we have: $e_1 = e_{\mathbf{s}_1} \leq e_{\mathbf{t}_1} + e_{\mathbf{t}_2} = \mathbf{1}^T\mathbf{z}$, by applying inequality (3.2.2).

We will show that for any vertex $\mathbf{z}$ in this case we have $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. Indeed

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_{\mathbf{t}_1} + e_{\mathbf{t}_2} + e_{\mathbf{s}_1} + e_{\mathbf{s}_2}}{3} \leq \frac{3(e_{\mathbf{t}_1} + e_{\mathbf{t}_2})}{3} = e_{\mathbf{t}_1} + e_{\mathbf{t}_2} = \mathbf{1}^T\mathbf{z},$$

where we apply inequalities (3.2.2) and (3.2.3).

Case 1.3: $|H \cap Y| = 0$ with $|H| = 3$ and $|Y| = 1$.

First let $H = \{1, 2, 3\}$ and $Y = \{4\}$. Then the intersection is computed to be

$$\mathbf{z}^{123|4} = \left[\frac{-e_1 + e_2 + e_3}{2}, \frac{e_1 - e_2 + e_3}{2}, \frac{e_1 + e_2 - e_3}{2}, 0\right]^T.$$

Non-negativity of the first coordinate gives the constraint $e_1 \leq e_2 + e_3$. Substituting $\mathbf{z}^{123|4}$ into $h_4 = \frac{1}{e_4}\mathbf{y}_1 + \frac{1}{e_4}\mathbf{y}_2 + \frac{1}{e_4}\mathbf{y}_3 \geq 1$ gives $\frac{-e_1+e_2+e_3}{2e_4} + \frac{e_1-e_2+e_3}{2e_4} + \frac{e_1+e_2-e_3}{2e_4} \geq 1$ which simplifies to $e_4 \leq \frac{e_1+e_2+e_3}{2}$. Similarly for the other three intersections of this case we get:

$\mathbf{z}^{124|3} = [\frac{-e_1+e_2+e_4}{2}, \frac{e_1-e_2+e_4}{2}, 0, \frac{e_1+e_2-e_4}{2}]^T$ feasible if $e_1 \leq e_2 + e_4$ and $e_3 \leq \frac{e_1+e_2+e_4}{2}$.

$\mathbf{z}^{134|2} = [\frac{-e_1+e_3+e_4}{2}, 0, \frac{e_1-e_3+e_4}{2}, \frac{e_1+e_3-e_4}{2}]^T$ feasible if $e_1 \leq e_3 + e_4$ and $e_2 \leq \frac{e_1+e_3+e_4}{2}$.

$\mathbf{z}^{234|1} = [0, \frac{-e_2+e_3+e_4}{2}, \frac{e_2-e_3+e_4}{2}, \frac{e_2+e_3-e_4}{2}]^T$ feasible if $e_2 \leq e_3 + e_4$ and $e_1 \leq \frac{e_2+e_3+e_4}{2}$.

In general for this case we can write the intersection $\mathbf{z}$ as:

$$\mathbf{z}_i = \begin{cases} \frac{-e_i+\sum_{k=1,k\neq i}^{3} e_k}{2} & \text{for } i \in H \\ 0 & \text{for } i \in Y. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following two constraints are satisfied:

$$e_{\mathbf{t}_1} \leq e_{\mathbf{t}_2} + e_{\mathbf{t}_3} \tag{3.2.4}$$

$$e_{\mathbf{s}_1} \leq \frac{e_{\mathbf{t}_1} + e_{\mathbf{t}_2} + e_{\mathbf{t}_3}}{2}. \tag{3.2.5}$$

Constraint (3.2.4) is the non-negativity condition of the intersection, and (3.2.5) comes from substituting $\mathbf{z}$ into $h_{\mathbf{s}_1} \geq 1$ (recall $\mathbf{s}_1$ is the element missing from H). Observe we have coordinate sum $\mathbf{1}^T\mathbf{z} = \frac{e_{\mathbf{t}_1}+e_{\mathbf{t}_2}+e_{\mathbf{t}_3}}{2}$. We will show that for any vertex $\mathbf{z}$ in this case we have $e_1 \leq \mathbf{1}^T\mathbf{z}$. If $1 \in H$, then we have: $e_1 = e_{\mathbf{t}_1} = \frac{e_{\mathbf{t}_1}+e_{\mathbf{t}_1}}{2} \leq \frac{e_{\mathbf{t}_1}+e_{\mathbf{t}_2}+e_{\mathbf{t}_3}}{2} = \mathbf{1}^T\mathbf{z}$ (apply inequality (3.2.4)). If $1 \notin H$, then we have: $e_1 = e_{\mathbf{s}_1} \leq \frac{e_{\mathbf{t}_1}+e_{\mathbf{t}_2}+e_{\mathbf{t}_3}}{2} = \mathbf{1}^T\mathbf{z}$ (apply inequality (3.2.5)).

We will show that for any vertex $\mathbf{z}$ in this case we have $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$:

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_{\mathbf{t}_1} + e_{\mathbf{t}_2} + e_{\mathbf{t}_3} + e_{\mathbf{s}_1}}{3} \leq \frac{e_{\mathbf{t}_1} + e_{\mathbf{t}_2} + e_{\mathbf{t}_3} + \frac{e_{\mathbf{t}_1}+e_{\mathbf{t}_2}+e_{\mathbf{t}_3}}{2}}{3} = \frac{e_{\mathbf{t}_1} + e_{\mathbf{t}_2} + e_{\mathbf{t}_3}}{2} = \mathbf{1}^T\mathbf{z},$$

by applying inequality (3.2.5).

Case 1.4 $|H \cap Y| = 0$ and $|H| = 4$.

This the distinguished intersection $\mathbf{z}^*$ of Lemma 3.2.1.

Case 2: $|H \cap Y| = 1$. Note this requires $1 \leq |H|, |Y| < 4$.

Observe that since $|H| + |Y| = 4$ and $|H \cap Y| = 1$, there is one number which is common to H and Y and one number of $\{1, 2, 3, 4\}$ absent from H and Y. Let $H \cap Y = \{a\}$ and let $\{1, \ldots, 4\} \setminus (H \cup Y) = \{b\}$. Let l be the position of a in $\mathbf{t}$ and let p be the position of a in $\mathbf{s}$, i.e., $\mathbf{t}_l = \mathbf{s}_p = a$.

Case 2.1: $|H \cap Y| = 1$ with $|H| = 1$ and $|Y| = 3$.

The intersection of the corresponding hyperplanes in this case is always empty. If we choose $H = \{1\}$ and $Y = \{1, 2, 3\}$, then $a = 1$ and $b = 4$. The corresponding augmented matrix is

$$\begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ which row reduces to } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & e_1 \end{bmatrix}.$$

This matrix has full rank, but $[0, 0, 0, e_1]^T$ is not feasible since substituting it into $h_4 \geq 1$ gives $0 \geq 1$. In general, any intersection with only one non-zero entry in its j-th coordinate will not be a vertex since it will fail to satisfy $h_j \geq 1$.

Case 2.2: $|H \cap Y| = 1$ with $|H| = 2$ and $|Y| = 2$.

We will first display the interesections for $a = 1, 2, 3, 4$ and then show $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$ and $e_1 \leq \mathbf{1}^T\mathbf{z}$.

For this case let a' denote the element of H that is not a. Also let c denote the element of $\{1, 2, 3, 4\}$ that is not a, a' or b.

If $a = 1$, then we have the following six intersections:

$$\mathbf{z}^{12|13} = \begin{bmatrix} 0 \\ e_1 - e_2 \\ 0 \\ e_2 \end{bmatrix}, \mathbf{z}^{12|14} = \begin{bmatrix} 0 \\ e_1 - e_2 \\ e_2 \\ 0 \end{bmatrix}, \mathbf{z}^{13|12} = \begin{bmatrix} 0 \\ 0 \\ e_1 - e_3 \\ e_3 \end{bmatrix},$$

$$\mathbf{z}^{13|14} = \begin{bmatrix} 0 \\ e_3 \\ e_1-e_3 \\ 0 \end{bmatrix}, \mathbf{z}^{14|12} = \begin{bmatrix} 0 \\ 0 \\ e_4 \\ e_1-e_4 \end{bmatrix}, \mathbf{z}^{14|13} = \begin{bmatrix} 0 \\ e_4 \\ 0 \\ e_1-e_4 \end{bmatrix}.$$

If $a = 2$, then we have the following six intersections:

$$\mathbf{z}^{12|23} = \begin{bmatrix} e_2-e_1 \\ 0 \\ 0 \\ e_1 \end{bmatrix}, \mathbf{z}^{12|24} = \begin{bmatrix} e_2-e_1 \\ 0 \\ 0 \\ e_1 \end{bmatrix}, \mathbf{z}^{23|12} = \begin{bmatrix} 0 \\ 0 \\ e_2-e_3 \\ e_3 \end{bmatrix},$$

$$\mathbf{z}^{23|24} = \begin{bmatrix} e_2 \\ 0 \\ e_2-e_3 \\ 0 \end{bmatrix}, \mathbf{z}^{24|12} = \begin{bmatrix} 0 \\ 0 \\ e_2 \\ e_2-e_4 \end{bmatrix}, \mathbf{z}^{24|23} = \begin{bmatrix} e_4 \\ 0 \\ 0 \\ e_4-e_2 \end{bmatrix}.$$

If $a = 3$, then we have the following six intersections:

$$\mathbf{z}^{13|23} = \begin{bmatrix} e_3-e_1 \\ 0 \\ 0 \\ e_1 \end{bmatrix}, \mathbf{z}^{13|34} = \begin{bmatrix} e_3-e_1 \\ e_1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}^{23|13} = \begin{bmatrix} 0 \\ e_3-e_2 \\ 0 \\ e_2 \end{bmatrix},$$

$$\mathbf{z}^{23|34} = \begin{bmatrix} e_2 \\ e_3-e_2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}^{34|23} = \begin{bmatrix} e_4 \\ 0 \\ 0 \\ e_3-e_4 \end{bmatrix}, \mathbf{z}^{34|13} = \begin{bmatrix} 0 \\ e_4 \\ 0 \\ e_3-e_4 \end{bmatrix}.$$

If $a = 4$, then we have the following six intersections:

$$\mathbf{z}^{14|24} = \begin{bmatrix} e_4-e_1 \\ 0 \\ e_1 \\ 0 \end{bmatrix}, \mathbf{z}^{14|34} = \begin{bmatrix} e_4-e_1 \\ e_1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}^{24|14} = \begin{bmatrix} 0 \\ e_4-e_2 \\ e_2 \\ 0 \end{bmatrix},$$

$$\mathbf{z}^{24|34} = \begin{bmatrix} e_2 \\ e_4-e_2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}^{34|14} = \begin{bmatrix} 0 \\ e_3 \\ e_4-e_3 \\ 0 \end{bmatrix}, \mathbf{z}^{34|24} = \begin{bmatrix} e_3 \\ 0 \\ e_4-e_3 \\ 0 \end{bmatrix}.$$

For this case of we can write the intersection $\mathbf{z}$ as:

$$\begin{cases} \mathbf{z}_b = & e_{a'} \\ \mathbf{z}_{a'} = & e_a - e_{a'} \\ \mathbf{z}_{\mathbf{s}_j} = 0 & \text{for } j = 1, 2. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following constraints are satisfied:

$$e_a \geq e_{a'} \tag{3.2.6}$$

$$e_a \geq e_c \tag{3.2.7}$$

$$e_a \geq e_{a'} + e_b. \tag{3.2.8}$$

The constraint (3.2.6) is the non-negative requirement on the a' coordinate. The constraints (3.2.7) and (3.2.8) are obtainted by substituting the intersection $\mathbf{z}$ into $h_c \geq 1$ and $h_b \geq 1$, respectively.

The coordinate sum in this case is $\mathbf{1}^T\mathbf{z} = e_a$. We will now show that $e_1 \leq \mathbf{1}^T\mathbf{z}$ by showing that when $\mathbf{z}$ is a vertex we have $\mathbf{1}^T\mathbf{z} = e_a = e_1$. This is clear when $a = 1$. Observe that $\{1,2,3,4\} = \{a, a', c, b\}$. Now suppose $a \neq 1$. Then one of a', c, or b must equal 1. If $a' = 1$, then (3.2.6) gives $e_a \geq e_1$ and so $e_a = e_1$ (by assumed ordering of e_i's). Similarly if $c = 1$, then (3.2.7) gives $e_a = e_1$. If $b = 1$, then (3.2.8) gives $e_a \geq e_{a'} + e_1$. Since $e_1 \geq e_a$ and $e_{a'} > 0$, this condition is never satisfied. So when $\mathbf{z}$ is a vertex, $\mathbf{1}^T\mathbf{z} = e_1$ as needed.

We will now show that $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. Indeed

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1 + e_2 + e_3 + e_4}{3} = \frac{e_a + e_{a'} + e_c + e_b}{3} \leq \frac{3e_a}{3} = e_a = \mathbf{1}^T\mathbf{z},$$

by applying the inequalities (3.2.7) and (3.2.8).

Case 2.3: $|H \cap Y| = 1$ with $|H| = 3$ and $|Y| = 1$.

We will first display the vertices for $a = 1,2,3,4$ and then show $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$ and $e_1 \leq \mathbf{1}^T\mathbf{z}$.

For $a = 1$:

$\mathbf{z}^{123|1} = \begin{bmatrix} 0 \\ e_1-e_2 \\ e_1-e_3 \\ -e_1+e_2+e_3 \end{bmatrix}$ feasible if non-negative and $e_1 \geq \frac{e_2+e_3+e_4}{2}$.

$\mathbf{z}^{124|1} = \begin{bmatrix} 0 \\ e_1-e_2 \\ -e_1+e_2+e_4 \\ e_1-e_4 \end{bmatrix}$ feasible if non-negative and $e_1 \geq \frac{e_2+e_3+e_4}{2}$.

$\mathbf{z}^{134|1} = \begin{bmatrix} 0 \\ -e_1+e_3+e_4 \\ e_1-e_3 \\ e_1-e_4 \end{bmatrix}$ feasible if non-negative and $e_1 \geq \frac{e_2+e_3+e_4}{2}$.

For $a = 2$:

$\mathbf{z}^{123|2} = \begin{bmatrix} e_2-e_1 \\ 0 \\ e_2-e_3 \\ -e_2+e_1+e_3 \end{bmatrix}$ feasible if non-negative and $e_2 \geq \frac{e_1+e_3+e_4}{2}$.

$\mathbf{z}^{124|2} = \begin{bmatrix} e_2-e_1 \\ 0 \\ -e_2+e_1+e_4 \\ e_2-e_4 \end{bmatrix}$ feasible if non-negative and $e_2 \geq \frac{e_1+e_3+e_4}{2}$.

$\mathbf{z}^{234|2} = \begin{bmatrix} -e_2+e_3+e_4 \\ 0 \\ e_2-e_3 \\ e_2-e_4 \end{bmatrix}$ feasible if non-negative and $e_2 \geq \frac{e_1+e_3+e_4}{2}$.

For $a = 3$:

$\mathbf{z}^{123|3} = \begin{bmatrix} e_3-e_1 \\ e_3-e_2 \\ 0 \\ -e_3+e_1+e_2 \end{bmatrix}$ feasible if non-negative and $e_3 \geq \frac{e_1+e_2+e_4}{2}$.

$\mathbf{z}^{134|3} = \begin{bmatrix} e_3-e_1 \\ -e_3+e_1+e_4 \\ 0 \\ e_3-e_4 \end{bmatrix}$ feasible if non-negative and $e_3 \geq \frac{e_1+e_2+e_4}{2}$.

$\mathbf{z}^{234|3} = \begin{bmatrix} -e_3+e_2+e_4 \\ e_3-e_2 \\ 0 \\ e_3-e_4 \end{bmatrix}$ feasible if non-negative and $e_3 \geq \frac{e_1+e_2+e_4}{2}$.

For $a = 4$:

$\mathbf{z}^{124|4} = \begin{bmatrix} e_4-e_1 \\ e_4-e_2 \\ -e_4+e_1+e_2 \\ 0 \end{bmatrix}$ feasible if non-negative and $e_4 \geq \frac{e_1+e_2+e_3}{2}$.

$\mathbf{z}^{134|4} = \begin{bmatrix} e_4-e_1 \\ -e_4+e_1+e_3 \\ e_4-e_3 \\ 0 \end{bmatrix}$ feasible if non-negative and $e_4 \geq \frac{e_1+e_2+e_3}{2}$.

$\mathbf{z}^{234|4} = \begin{bmatrix} -e_4-e_2+e_3 \\ e_4-e_2 \\ e_4-e_3 \\ 0 \end{bmatrix}$ feasible if non-negative and $e_4 \geq \frac{e_1+e_2+e_3}{2}$.

In general for this case we can write the intersection $\mathbf{z}$ as:

$$\begin{cases} \mathbf{z}_b = -e_a + \sum_{k=1,k\neq l}^{3} e_{\mathbf{t}_k} & \\ \mathbf{z}_{\mathbf{t}_i} = e_a - e_{\mathbf{t}_i} & \text{for } 1 \leq i \leq 3. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following constraints are satisfied:

$$e_a \leq \sum_{k=1,k\neq l}^{3} e_{\mathbf{t}_k} \tag{3.2.9}$$

$$e_a \geq e_{\mathbf{t}_i} \text{ (for } 1 \leq i \leq 3) \tag{3.2.10}$$

$$e_a \geq \frac{e_b + \sum_{k=1,k\neq l}^{3} e_{\mathbf{t}_k}}{2}. \tag{3.2.11}$$

The constraints (3.2.9) and (3.2.10) are non-negativity requirements on the coordinates of $\mathbf{z}$. Substituting $\mathbf{z}$ into $h_b \geq 1$ gives the constraint (3.2.11). The coordinate sum in this case is $\mathbf{1}^T\mathbf{z} = e_a$. We will now show that $e_1 \leq \mathbf{1}^T\mathbf{z}$ by showing that when $\mathbf{z}$ is a vertex we have $\mathbf{1}^T\mathbf{z} = e_a = e_1$. This is clear when $a = 1$. Now suppose that $a > 1$ and $b \neq 1$. Then (3.2.10) for $i = 1$ gives $e_a \geq e_1$ and so $e_a = e_1$. Otherwise suppose that $a > 1$ and $b = 1$. Then (3.2.11) becomes

$$e_a \geq \frac{e_1 + \sum_{k=1,k\neq l}^{3} e_{\mathbf{t}_k}}{2}.$$

Applying (3.2.9) gives $e_a \geq \frac{e_1+e_a}{2}$ which requires $e_a = e_1$ since $e_1 \geq e_a$. So when $\mathbf{z}$ is a vertex the coordinate sum is $\mathbf{1}^T\mathbf{z} = e_1$.

We will now show that $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. Indeed

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+e_3+e_4}{3} = \frac{e_a + \sum_{k=1,k\neq a}^4 e_k}{3} \leq \frac{e_a+2e_a}{3} = e_a = \mathbf{1}^T\mathbf{z},$$

by applying constraint (3.2.11).

Case 3: $|H \cap Y| = 2$.

We will show this case is always infeasible. Suppose $H = \{1,2\}$ and $Y = \{1,2\}$. The corresponding augmented matrix is

$$\left[\begin{array}{ccccc} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & 1 \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \frac{1}{e_2} & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \text{ which row reduces to } \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & e_1 \\ 0 & 0 & 0 & 0 & 1-\frac{e_1}{e_2} \end{array}\right].$$

If $e_1 > e_2$, then the bottom row gives an inconsistent system. If $e_1 = e_2$, then A does not have full rank and $\mathbf{z}$ is not a vertex. We similarly do not get a vertex for other choices of H and Y in this case.

The above Cases 1, 2, and 3 exhaust all possible vertices.

We have shown that if $\mathbf{z}$ is any vertex of $SP(I)$, then $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. So if $\mathbf{z}^*$ is a vertex, then it is a minimizer to the linear optimization problem (3.2.1) and so $\widehat{\alpha}(I) = \mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+e_3+e_4}{3}$. We will now show that if $\mathbf{z}^*$ is not a vertex, then there is a vertex $\mathbf{z}$ such that $\mathbf{1}^T\mathbf{z} = e_1$ and thus $\widehat{\alpha}(I) = e_1$.

Now suppose the feasible condition of $\mathbf{z}^*$ is not satisfied so we have $e_1 > \frac{e_2+e_3+e_4}{2}$. We have also shown that if $\mathbf{z}$ is any vertex of $SP(I)$ (other than $\mathbf{z}^*$) , then $e_1 \leq \mathbf{1}^T\mathbf{z}$. So if $\mathbf{z}^*$ is not a vertex and if there is a vertex $\mathbf{z}$ with coordinate sum $\mathbf{1}^T\mathbf{z} = e_1$, then $\mathbf{z}$ must be a minimizer of the linear optimization problem 3.2.1.

First consider the intersection $\mathbf{z}^{123|1} = \begin{bmatrix} 0 \\ e_1-e_2 \\ e_1-e_3 \\ -e_1+e_2+e_3 \end{bmatrix}$ from Case 2.3.

It is a vertex when non-negative and the constraint $e_1 \geq \frac{e_2+e_3+e_4}{2}$ is satisfied. The constraint is always satisfied since we are now assuming $\mathbf{z}^*$ is not a vertex and so $e_1 > \frac{e_2+e_3+e_4}{2}$. We require $e_1 \leq e_2 + e_3$ and $e_1 \geq e_3$ for non-negativity. (Observe that $e_1 > \frac{e_2+e_3+e_4}{2}$ implies $e_1 > e_3$ so there will always be more than one non-zero coordinate.) So if $e_1 \leq e_2 + e_3$ then $\mathbf{z}^{123|1}$ is a vertex and a minimizer.

Suppose $\mathbf{z}^*$ and $\mathbf{z}^{123|1}$ are not vertices. Then we can consider the intersection $\mathbf{z}^{12|14} = \begin{bmatrix} 0 \\ e_1-e_2 \\ e_2 \\ 0 \end{bmatrix}$ with constraints $e_1 \geq e_2 + e_3$ and $e_1 \geq e_4$. If the intersection $\mathbf{z}^{123|1}$ failed to be vertex, then it must be that $e_1 > e_2 + e_3$ which means $\mathbf{z}^{12|14}$ is a vertex. (Observe that $e_1 > e_2 + e_3$ implies $e_1 > e_2$ so there will always be more than one non-zero coordinate.)

In summary, if $\mathbf{z}^*$ is a vertex of $SP(I)$, we have $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3+e_4}{3}$. Otherwise, at least one of $\mathbf{z}^{123|1}$ and $\mathbf{z}^{12|14}$ is a vertex and $\widehat{\alpha}(I) = e_1$. This concludes the proof of Theorem 3.2.2. □

Similar to Corollary 3.1.2 from Section 3.1, the following corollary allows us to control $\widehat{\alpha}(I)$ by choosing the appropriate e_i's.

Corollary 3.2.3. *Fix a positive integer $q > 3, q \neq 5$. Then write the integer partition $q = e_1 + e_2 + e_3 + e_4$ where*

$$\begin{cases} e_1 = \frac{q}{4}, e_2 = \frac{q}{4}, e_3 = \frac{q}{4}, e_4 = \frac{q}{4} & \text{if } q \equiv 0 \mod 4 \\ e_1 = \lceil\frac{q}{4}\rceil, e_2 = \lfloor\frac{q}{4}\rfloor, e_3 = \lfloor\frac{q}{4}\rfloor, e_4 = \lfloor\frac{q}{4}\rfloor & \text{if } q \equiv 1 \mod 4 \\ e_1 = \lceil\frac{q}{4}\rceil, e_2 = \lceil\frac{q}{4}\rceil, e_3 = \lfloor\frac{q}{4}\rfloor, e_4 = \lfloor\frac{q}{4}\rfloor & \text{if } q \equiv 2 \mod 4 \\ e_1 = \lceil\frac{q}{4}\rceil, e_2 = \lceil\frac{q}{4}\rceil, e_3 = \lceil\frac{q}{4}\rceil, e_4 = \lfloor\frac{q}{4}\rfloor & \text{if } q \equiv 3 \mod 4 \end{cases}$$

Then

$$I = \langle x_2, x_3, x_4 \rangle^{e_1} \cap \langle x_1, x_3, x_4 \rangle^{e_2} \cap \langle x_1, x_2, x_4 \rangle^{e_3} \cap \langle x_1, x_2, x_3 \rangle^{e_4}$$

is a monomial ideal in $R = K[x_1, x_2, x_3, x_4]$ with $\widehat{\alpha}(I) = \frac{q}{3}$.

Proof. We consider four cases that depend on the value of $q \mod 4$.

Case 1: $q \equiv 0 \mod 4$. Let $e_1 = e_2 = e_3 = e_4 = \frac{q}{4}$. Since $q \equiv 0 \mod 4$, $\frac{q}{4}$ is an integer.

We have $e_1 \leq \frac{e_2+e_3+e_4}{2}$, so by Theorem 3.2.2 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3+e_4}{3} = \frac{q}{3}$.

Case 2: $q \equiv 1 \mod 4$. Let $e_1 = \lceil \frac{q}{4} \rceil, e_2 = e_3 = e_4 = \lfloor \frac{q}{4} \rfloor$. If $q = 5$, then $e_1 = 2$ and $e_2 = e_3 = e_4 = 1$ which makes $e_1 \leq \frac{e_2+e_3+e_4}{2}$ false so we cannot apply Theorem 3.2.2. Otherwise if $q > 5$, then $q \equiv 1 \mod 4$ implies that $\lceil \frac{q}{4} \rceil = \frac{q+3}{4}$ and $\lfloor \frac{q}{4} \rfloor = \frac{q-1}{4}$. Then $e_1 \leq \frac{e_2+e_3+e_4}{2}$ is true since $\frac{q+3}{4} \leq (\frac{q-1}{4} + \frac{q-1}{4} + \frac{q-1}{4})/2$ holds for $q > 5$. So by Theorem 3.2.2 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3+e_4}{3} = \frac{\lceil \frac{q}{4} \rceil + \lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{4} \rfloor}{3} = \frac{\frac{q+3}{4} + \frac{q-1}{4} + \frac{q-1}{4} + \frac{q-1}{4}}{3} = \frac{q}{3}$.

Case 3: $q \equiv 2 \mod 4$. Let $e_1 = e_2 = \lceil \frac{q}{4} \rceil$ and $e_3 = e_4 = \lfloor \frac{q}{4} \rfloor$. Observe that $q \equiv 2 \mod 4$ implies that $\lceil \frac{q}{4} \rceil = \frac{q+2}{4}$ and $\lfloor \frac{q}{4} \rfloor = \frac{q-2}{4}$. Then $e_1 \leq \frac{e_2+e_3+e_4}{2}$ is true since $\frac{q+2}{4} \leq (\frac{q+2}{4} + \frac{q-2}{4} + \frac{q-2}{4})/2$ holds for $q > 5$. So by Theorem 3.2.2 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3+e_4}{3} = \frac{\lceil \frac{q}{4} \rceil + \lceil \frac{q}{4} \rceil + \lfloor \frac{q}{4} \rfloor + \lfloor \frac{q}{4} \rfloor}{3} = \frac{\frac{q+2}{4} + \frac{q+2}{4} + \frac{q-2}{4} + \frac{q-2}{4}}{3} = \frac{q}{3}$.

Case 4: $q \equiv 3 \mod 4$. Let $e_1 = e_2 = e_3 = \lceil \frac{q}{4} \rceil$ and $e_4 = \lfloor \frac{q}{4} \rfloor$. Observe that $q \equiv 2 \mod 4$ implies that $\lceil \frac{q}{4} \rceil = \frac{q+1}{4}$ and $\lfloor \frac{q}{4} \rfloor = \frac{q-3}{4}$. Then $e_1 \leq \frac{e_2+e_3+e_4}{2}$ is true since $\frac{q+1}{4} \leq (\frac{q+1}{4} + \frac{q+1}{4} + \frac{q-3}{4})/2$ holds for $q > 6$. So by Theorem 3.2.2 we get $\widehat{\alpha}(I) = \frac{e_1+e_2+e_3+e_4}{3} = \frac{\lceil \frac{q}{4} \rceil + \lceil \frac{q}{4} \rceil + \lceil \frac{q}{4} \rceil + \lfloor \frac{q}{4} \rfloor}{3} = \frac{\frac{q+1}{4} + \frac{q+1}{4} + \frac{q+1}{4} + \frac{q-3}{4}}{3} = \frac{q}{3}$. □

So for this family of ideals in $R = K[x_1, x_2, x_3, x_4]$ our method allows us to attain any rational Waldschmidt constant $\frac{q}{3} > 1$ except for $\frac{5}{3}$.

3.3 Intersections of powers of height $n-1$ prime monomial ideals in n variables

We will now generalize the techniques of Sections 3.1 and 3.2 to see what values of the Waldschmidt constant can be attained from ideals of the form

$$I = \langle x_2, \ldots, x_n \rangle^{e_1} \cap \langle x_1, x_3, \ldots, x_n \rangle^{e_2} \cap \cdots \cap \langle x_1, \ldots, x_{n-1} \rangle^{e_n},$$

where $e_1 \geq e_2 \geq \cdots \geq e_n > 0$. In a similar fashion to the formulation of the linear optimization problem (3.1.1) in Section 3.1, we can use Lemma 2.4.7 and Theorem 2.4.4 to show that solving for the Waldschmidt constant $\widehat{\alpha}(I)$ amounts to solving the following linear optimization problem:

$$\text{Solve} \min\{\mathbf{1}^T \mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \text{ for} \tag{3.3.1}$$

$$A_n = \begin{bmatrix} 0 & \frac{1}{e_1} & \cdots & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \cdots & \frac{1}{e_2} & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & \frac{1}{e_3} & \cdots & \frac{1}{e_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{e_{n-1}} & \cdots & \frac{1}{e_{n-1}} & \frac{1}{e_{n-1}} & 0 & \frac{1}{e_{n-1}} \\ \frac{1}{e_n} & \cdots & \frac{1}{e_n} & \frac{1}{e_n} & \frac{1}{e_n} & 0 \end{bmatrix},$$

where $\mathbf{y} = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n]^T$ is a vector of variables.

Lemma 3.3.1. *Consider the matrix A_n of the linear optimization problem* (3.3.1). *The* *inverse of A_n is:*

$$A_n^{-1} = \begin{bmatrix} -\frac{(n-2)e_1}{n-1} & \frac{e_2}{n-1} & \frac{e_3}{n-1} & \cdots & \frac{e_{n-1}}{n-1} & \frac{e_n}{e_1} \\ \frac{e_1}{n-1} & -\frac{(n-2)e_2}{n-1} & \frac{e_3}{n-1} & \cdots & \frac{e_{n-1}}{n-1} & \frac{e_n}{n-1} \\ \frac{e_1}{n-1} & \frac{e_2}{n-1} & -\frac{(n-2)e_3}{n-1} & \cdots & \frac{e_{n-1}}{n-1} & \frac{e_n}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{e_1}{n-1} & \frac{e_2}{n-1} & \frac{e_3}{n-1} & \cdots & -\frac{(n-2)e_{n-1}}{n-1} & \frac{e_n}{n-1} \\ \frac{e_1}{n-1} & \frac{e_2}{n-1} & \frac{e_3}{n-1} & \cdots & \frac{e_{n-1}}{n-1} & -\frac{(n-2)e_n}{n-1} \end{bmatrix}.$$

Proof. Recall that $(M)_{i,j}$ denotes the (i,j)-th entry of a matrix M. Then define

$$(A_n^{-1})_{i,j} = \begin{cases} -\frac{(n-2)e_i}{n-1} & \text{for } i = j, \\ \frac{e_i}{n-1} & \text{for } i > j, \\ \frac{e_j}{n-1} & \text{for } j > i. \end{cases}$$

For $i = j$ we get $(AA^{-1})_{i,i} = 0\frac{-(n-2)e_i}{n-1} + (n-1)\frac{e_i}{n-1}\frac{1}{e_1} = 1$. And for $i \neq j$ we get $(AA^{-1})_{i,j} = 0\frac{e_j}{n-1} - \frac{(n-2)e_j}{n-1}\frac{1}{e_i} + (n-2)\frac{e_j}{n-1}\frac{1}{e_i} = 0$. This shows $AA^{-1} = I_n$. □

Define h_k to be the expression equal to the k-th coordinate of the vector $A_n\mathbf{y}$:

$$h_k = \sum_{i=1, i\neq k}^{n} \frac{1}{e_k}\mathbf{y}_i.$$

The symbolic polyhedron $SP(I)$ associated to the above linear optimization problem is defined by the intersection of $2n$ half-spaces. For $1 \leq k \leq n$ we have n half-spaces defined by the inequality $h_k = \sum_{i=1,i\neq k}^{n} \frac{1}{e_k}\mathbf{y}_i \geq 1$. We also have n half-spaces coming from the non-negativity contraint: $\mathbf{y}_k \geq 0$ for $1 \leq k \leq n$. Changing the $2n$ inequalities to equations gives the $2n$ bounding hyperplanes of $SP(I)$. So for $1 \leq k \leq n$, we have n hyperplanes $h_k = \sum_{i=1,i\neq k}^{n} \frac{1}{e_k}\mathbf{y}_i = 1$ and n hyperplanes $\mathbf{y}_k = 0$.

A minimizer to the linear optimization problem is among the vertices of $SP(I)$. To compute the (possible) vertices of $SP(I)$ we intersect n of the $2n$ bounding hyperplanes.

We will denote such an intersection by $\mathbf{z} \in \mathbb{R}^n$. Among these $\binom{2n}{n}$ intersections, those that exist and satisfy $A_n\mathbf{z} \geq \mathbf{1}$ and $\mathbf{z} \geq \mathbf{0}$ will be the vertices of $SP(I)$.

Denote $\mathbf{z}^*$ to be the intersection of the $h_i = 1$ hyperplanes. The following lemma concerns this distinguished intersection.

Lemma 3.3.2. *If $\mathbf{z}^*$ is the intersection of the n hyperplanes $h_1 = 1, h_2 = 1, \dots, h_n = 1$, then it has the form*

$$\mathbf{z}^* = \begin{bmatrix} \frac{-(n-2)e_1+e_2+\cdots+e_n}{n-1} \\ \vdots \\ \frac{e_1+\cdots-(n-2)e_i+\cdots+e_n}{n-1} \\ \vdots \\ \frac{e_1+\cdots+e_{n+1}-(n-2)e_n}{n-1} \end{bmatrix}.$$

Furthermore, $\mathbf{z}^*$ is a vertex of $SP(I)$ when $e_1 \leq \frac{e_2+e_3+\cdots+e_n}{n-2}$.

Proof. Since h_k is the expression from k-th coordinate of $A_n\mathbf{y}$, $\mathbf{z}^*$ is the solution to $A_n\mathbf{y} = \mathbf{1}$. Using the inverse of A_n from Lemma 3.3.1 we get:

$$\mathbf{z}_i^* = (A_n^{-1}\mathbf{1})_i = \frac{-(n-2)e_i + \sum_{j=1,j\neq i}^{n} e_j}{n-1}.$$

Since A_n is an invertible matrix, $\mathbf{z}^*$ is the unique solution to $A_n\mathbf{y} = \mathbf{1}$. Since we are assuming $e_1 \geq \cdots \geq e_n$, the first coordinate of $\mathbf{z}^*$ is minimal. To satisfy $\mathbf{z}^* \geq \mathbf{0}$ it suffices to check that the first coordinate is non-zero. So the condition for $\mathbf{z}^*$ to be a vertex of $SP(I)$ is $\frac{-(n-2)e_1+e_2+\cdots+e_n}{n-1} \geq 0$ or $e_1 \leq \frac{e_2+e_3+\cdots+e_n}{n-2}$. □

We will prove the following theorem on the Waldschmidt constant for our family of ideals.

Theorem 3.3.3. *Let $R = K[x_1, \dots, x_n]$ and consider the following ideal*

$$I = \langle x_2, \dots, x_n\rangle^{e_1} \cap \langle x_1, x_3, \dots, x_n\rangle^{e_2} \cap \cdots \cap \langle x_1, \dots, x_{n-1}\rangle^{e_n}$$

with $e_1 \geq e_2 \geq \cdots \geq e_n > 0$. *If* $e_1 \leq \frac{e_2+e_3+\cdots+e_n}{n-2}$, *then*

$$\widehat{\alpha}(I) = \frac{e_1 + e_2 + \cdots + e_n}{n-1}.$$

Otherwise, $\widehat{\alpha}(I) = e_1$.

Proof. We will compute the form of every intersection of n of the $2n$ hyperplanes, which exhausts every possible vertex of $SP(I)$. To specify the n hyperplanes, we consider $H, Y \subseteq \{1, 2, \ldots, n\}$ with $|H| + |Y| = n$. Each choice of H and Y will correspond to an intersection of n hyperplanes. For example, if $n = 5$, then the choice $H = \{1, 5, 4\}$ and $Y = \{4, 3\}$ is the intersection of the hyperplanes $h_1 = 1, h_5 = 1, h_4 = 1, \mathbf{y}_4 = 0, \mathbf{y}_3 = 0$. Associate to H the $|H|$-tuple $\mathbf{t}$ which records the elements of H in ascending order, e.g., the above H gives $\mathbf{t} = (1, 4, 5)$. Similarly, associate to Y the $|Y|$-tuple $\mathbf{s}$, e.g., the above Y gives $\mathbf{s} = (3, 4)$.

We will consider cases depending on $|H \cap Y|$. For each case we will (i) determine the form of the intersection $\mathbf{z}$ (if it exists), (ii) determine the constraints the intersection must satisfy to be a vertex of $SP(I)$, (iii) show that $e_1 \leq \mathbf{1}^T\mathbf{z}$ (for every vector $\mathbf{z}$ except $\mathbf{z}^*$), and (iv) show that $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$.

Case 1: $|H \cap Y| = 0$.

Case 1.1: $|H \cap Y| = 0$ with $|H| = 0$ and $|Y| = n$.

If $|H| = 0$ and $|Y| = \{1, \ldots, n\}$, then we have $\mathbf{y}_i = 0$ for $1 \leq i \leq n$. So the intersection is $\mathbf{0}$, the zero vector. The zero vector is never a vertex since $A_n\mathbf{0} < \mathbf{1}$.

Case 1.2: $|H \cap Y| = 0$ with $|H| = 1$ with $|Y| = n - 1$.

Since $|H| = \{\mathbf{t}_1\}$, we have the equations $h_{\mathbf{t}_1} = \sum_{i=1, i\neq \mathbf{t}_1}^{n} \frac{1}{e_k}\mathbf{y}_i = 1$ and $\mathbf{y}_{\mathbf{s}_j} = 0$ for $1 \leq j \leq |Y|$. Since H and Y are disjoint, substituting each of the $n-1$ equations $\mathbf{y}_{\mathbf{s}_j} = 0$ for $1 \leq j \leq |Y|$ into $h_{\mathbf{t}_1}$ yields $0 = 1$, a contradiction.

Case 1.3: $|H \cap Y| = 0$ and $n > |H| \geq 2$.

After relabelling, we can suppose that $H = \{1, \ldots, |H|\}$ and $Y = \{|H|+1, \ldots, n\}$. Then solving for the intersection $\mathbf{z}$ is equivalent to solving $M\mathbf{y} = \mathbf{c}$ where the rows of M and $\mathbf{c}$ encode the hyperplane coefficients of $h_1 = 1, h_2 = 1 \ldots, h_{|H|} = 1, \mathbf{y}_{|H|+1} = 0, \ldots, \mathbf{y}_n = 0$, in that order.

We will show that M can be written as a block matrix. Let $A_{|H|}$ to be the same matrix as A_n from the linear optimization problem (3.3.1) with inverse A_n^{-1} from Lemma 3.3.1(ii), but now we will set $n = |H|$. Define L to be the $|H| \times |Y|$ matrix with every entry in row i equal to $\frac{1}{e_i}$. Define $\mathbf{0}_{|Y|\times|H|}$ to be the $|Y| \times |H|$ matrix with every entry equal to 0. Then we can write

$$M = \begin{bmatrix} A_{|H|} & L \\ \mathbf{0}_{|Y|\times|H|} & I_{|Y|} \end{bmatrix}.$$

Since M is a block upper triangular and $A_{|H|}$ and I_n are invertible, we have the form of the inverse of M:

$$M^{-1} = \begin{bmatrix} A_{|H|}^{-1} & -A_{|H|}^{-1}L \\ \mathbf{0}_{|Y|\times|H|} & I_{|Y|} \end{bmatrix}.$$

Note $-A_{|H|}^{-1}L$ simplifies to a $|H| \times |Y|$ matrix with $\frac{|H|-2}{|H|-1} - 1$ in every entry and that $\mathbf{c} = [1, \ldots, 1, 0, \ldots, 0]^T$ (i.e., $|H|$-many 1's followed by $|Y|$-many 0's). Solving for $\mathbf{z}$:

$$(M^{-1}\mathbf{c})_i = \mathbf{z}_i = \begin{cases} \frac{-(|H|-2)e_i + \sum_{k=1,k\neq i}^{|H|} e_k}{|H|-1} & \text{for } 1 \leq i \leq |H| \\ 0 & \text{for } |H| < i \leq n. \end{cases}$$

An example to illustrate this case: suppose $n = 5$ and $H = \{1, 2, 3\}$ and $Y = \{4, 5\}$. Then

$$M = \begin{bmatrix} A_3 & L \\ \mathbf{0}_{2\times3} & I_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \frac{1}{e_2} & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & \frac{1}{e_3} & \frac{1}{e_3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\mathbf{z} = M^{-1}\mathbf{c} = \begin{bmatrix} A_3^{-1} & -A_3^{-1}L \\ \mathbf{0}_{2\times 2} & I_2 \end{bmatrix} = \begin{bmatrix} -\frac{e_1}{2} & \frac{e_2}{2} & \frac{e_3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{e_1}{2} & -\frac{e_2}{2} & \frac{e_3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{e_1}{2} & \frac{e_2}{2} & -\frac{e_3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-e_1+e_2+e_3}{2} \\ \frac{e_1-e_2+e_3}{2} \\ \frac{e_1+e_2-e_3}{2} \\ 0 \\ 0 \end{bmatrix}.$$

The result generalizes to any disjoint partition of $\{1,\ldots,n\}$ into H and Y with $|H| \geq 2$. Suppose we have $H = \{\mathbf{t}_1,\ldots,\mathbf{t}_{|H|}\}$, and $Y = \{\mathbf{s}_1,\ldots,\mathbf{s}_{|Y|}\}$. Then M and M^{-1} have the same form as above but with permutated columns and permutated rows respectively. Note the general case of M and M^{-1} replace $e_1,\ldots,e_{|H|}$ with $e_{\mathbf{t}_1},\ldots,e_{\mathbf{t}_{|H|}}$. This is the form of $\mathbf{z}$ in general:

$$\begin{cases} \mathbf{z}_{\mathbf{t}_i} = \frac{-(|H|-2)e_{\mathbf{t}_i} + \sum_{k=1,k\neq \mathbf{t}_i}^{|H|} e_k}{|H|-1} & \text{for } 1 \leq i \leq |H| \\ \mathbf{z}_{\mathbf{s}_j} = 0 & \text{for } 1 \leq j \leq |Y|. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following constraints are satisfied:

$$e_{\mathbf{t}_1} \leq \frac{\sum_{i=1,i\neq 1}^{|H|} e_{\mathbf{t}_i}}{|H|-2} \tag{3.3.2}$$

$$e_{\mathbf{s}_j} \leq \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1} \text{ for } 1 \leq j \leq |Y|. \tag{3.3.3}$$

Non-negativity on $\mathbf{z}_{\mathbf{t}_1}$, the minimal coordinate, gives constraint (3.3.2). Substituting $\mathbf{z}$ into $h_{\mathbf{s}_j} \geq 1$ gives the constraint (3.3.3). The coordinate sum is $\mathbf{1}^T\mathbf{z} = \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1}$. We will show that for any vertex $\mathbf{z}$ in this case we have $e_1 \leq \mathbf{1}^T\mathbf{z}$. If $1 \in H$, then we have

$$e_1 = \frac{(|H|-2)e_1 + e_1}{|H|-1} \leq \frac{\sum_{i=1,i\neq 1}^{|H|} e_{\mathbf{t}_i} + e_1}{|H|-1} = \frac{\sum_{i=1,}^{|H|} e_{\mathbf{t}_i}}{|H|-1} = \mathbf{1}^T\mathbf{z}.$$

by applying the inequality (3.3.2). If $1 \in Y$, then (3.3.3) for $j = 1$ gives

$$e_1 = e_{\mathbf{s}_1} \leq \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1} = \mathbf{1}^T\mathbf{z}.$$

We will now show $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. Observe we can write

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1 + \cdots + e_n}{n-1} = \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i} + e_{\mathbf{s}_1} + \cdots + e_{\mathbf{s}_{|Y|}}}{n-1}.$$

Then repeatedly applying the inequality (3.3.3) for each $e_{\mathbf{s}_j}$ gives

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1 + \cdots + e_n}{n-1} \leq \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i} + \frac{|Y|\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1}}{n-1} = \frac{\frac{(|H|+|Y|-1)\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1}}{n-1} = \frac{\sum_{i=1}^{|H|} e_{\mathbf{t}_i}}{|H|-1} = \mathbf{1}^T\mathbf{z}.$$

Case 1.4: $|H \cap Y| = 0$ and $|H| = n$.

Here we have $H = \{1, \ldots, n\}, Y = \emptyset$. This is $\mathbf{z}^*$ of Lemma 3.3.2.

Case 2: $|H \cap Y| = 1$. Note this requires $1 \leq |H|, |Y| < n$.

Observe that since $|H| + |Y| = n$ and $|H \cap Y| = 1$, there is one number which is common to H and Y and one number of $\{1, \ldots, n\}$ absent from H and Y. Let $H \cap Y = \{a\}$ and let $\{1, \ldots, n\} \setminus (H \cup Y) = \{b\}$. Let l be the position of a in $\mathbf{t}$ and let p be the position of a in $\mathbf{s}$, i.e., $\mathbf{t}_l = \mathbf{s}_p = a$.

Case 2.1: $|H \cap Y| = 1$ and $|H| = 1$.

This case is never feasible. The intersection has the form $\mathbf{z}_b = e_a$, and for $1 \leq j \leq |Y|$ we have $\mathbf{z}_{\mathbf{s}_j} = 0$. Substituting $\mathbf{z}$ into $h_b \geq 1$ gives $0 = 1$, a contradiction.

Case 2.2: $|H \cap Y| = 1$ and $|H| \geq 2$.

We will begin with an example: suppose $n = 7$ and let $H = \{1, 2, 3, 4, 5\}$ and $Y = \{5, 6\}$. So $a = 5$ and $b = 7$. Then solving for the intersection $\mathbf{z}$ is equivalent to solving $M\mathbf{y} = \mathbf{c}$ where the rows of M and $\mathbf{c}$ encode the hyperplane coefficients of $h_1 = 1, h_2 = 1, h_3 = 1, h_4 = 1, h_5 = 1, \mathbf{y}_5 = 0, \mathbf{y}_6 = 0$. This is the matrix M and vector $\mathbf{c}$:

$$M = \begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} & \frac{1}{e_1} \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \frac{1}{e_2} & \frac{1}{e_2} & \frac{1}{e_2} & \frac{1}{e_2} \\ \frac{1}{e_3} & \frac{1}{e_3} & 0 & \frac{1}{e_3} & \frac{1}{e_3} & \frac{1}{e_3} & \frac{1}{e_3} \\ \frac{1}{e_4} & \frac{1}{e_4} & \frac{1}{e_4} & 0 & \frac{1}{e_4} & \frac{1}{e_4} & \frac{1}{e_4} \\ \frac{1}{e_5} & \frac{1}{e_5} & \frac{1}{e_5} & \frac{1}{e_5} & 0 & \frac{1}{e_5} & \frac{1}{e_5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The inverse of M is computed:

$$M^{-1} = \begin{bmatrix} -e_1 & 0 & 0 & 0 & e_5 & 1 & 0 \\ 0 & -e_2 & 0 & 0 & e_5 & 1 & 0 \\ 0 & 0 & -e_3 & 0 & e_5 & 1 & 0 \\ 0 & 0 & 0 & -e_4 & e_5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ e_1 & e_2 & e_3 & e_4 & -3e_5 & -4 & -1 \end{bmatrix}.$$

Then

$$\mathbf{z} = M^{-1}\mathbf{c} = \begin{bmatrix} -e_1 + e_5 \\ -e_2 + e_5 \\ -e_3 + e_5 \\ -e_4 + e_5 \\ 0 \\ 0 \\ e_1 + e_2 + e_3 + e_4 - 3e_5 \end{bmatrix}.$$

More generally let $H \cap Y = \{a\}$ and $|H| \geq 2$. Then solving for the intersection $\mathbf{z}$ is equivalent to solving $M\mathbf{y} = \mathbf{c}$ where the rows of M and $\mathbf{c}$ encode the hyperplane coefficients of $h_{\mathbf{t}_1} = 1, h_{\mathbf{t}_2} = 1, \ldots, h_{\mathbf{t}_{|H|}} = 1$, $\mathbf{y}_{\mathbf{s}_1} = 0, \ldots, \mathbf{y}_{\mathbf{s}_{|Y|}} = 0$, in that order. We will give the form of each column of M. There are four types of columns:

$(M)_{i,b} = \frac{1}{e_{\mathbf{t}_i}}$ for $1 \leq i \leq |H|,$ and 0 otherwise.

$(M)_{i,a} = \frac{1}{e_{\mathbf{t}_i}}$ for $1 \leq i \leq |H|, i \neq l,$ and 1 for $i = |H| + p,$ and 0 otherwise.

$(M)_{i,\mathbf{t}_k(k\neq l)} = \frac{1}{e_{\mathbf{t}_i}}$ for $1 \leq i \leq |H|, i \neq k,$ and 0 otherwise.

$(M)_{i,\mathbf{s}_j(j\neq p)} = 1$ for $i = |H| + j,$ and 0 otherwise.

Given the above M, we now give the form of each row of M^{-1}. There are three types of rows:

$$(M^{-1})_{b,i} = \begin{cases} e_{\mathbf{t}_i} & \text{for } 1 \leq i \leq |H|, i \neq l \\ -(|H|-2)e_{\mathbf{t}_l} & \text{for } i = l \\ -1 & \text{for } i = |H| + j, j \neq p \\ -(|H|-1) & \text{for } i = |H| + p. \end{cases}$$

$$(M^{-1})_{\mathbf{t}_k(k\neq l),i} = \begin{cases} -e_{\mathbf{t}_k} & \text{for } i = k \\ e_{\mathbf{t}_l} & \text{for } i = l \\ 1 & \text{for } i = |H| + p \\ 0 & \text{otherwise.} \end{cases}$$

$(M^{-1})_{\mathbf{s}_j,i} = 1$ for $i = |H| + j,$ and 0 otherwise.

We will now verify that this constructed M^{-1} is the inverse of M by showing the dot product $(M^{-1})_{i,*} \cdot (M)_{*,j} = 1$ if $i = j$, and $(M^{-1})_{i,*} \cdot (M)_{*,j} = 0$ if $i \neq j$.

$(M^{-1})_{b,*} \cdot (M)_{*,b} = \sum_{1\leq i\leq |H|, i\neq l} e_{\mathbf{t}_i} \frac{1}{e_{\mathbf{t}_i}} - (|H|-2)e_{\mathbf{t}_l} \frac{1}{e_{\mathbf{t}_l}} = (|H|-1) - (|H|-2) = 1.$

$(M^{-1})_{b,*} \cdot (M)_{*,a} = \sum_{1\leq i\leq |H|, i\neq l} e_{\mathbf{t}_i} \frac{1}{e_{\mathbf{t}_i}} + 1(-(|H|-1)) = (|H|-1) - (|H|-1) = 0.$

$(M^{-1})_{b,*} \cdot (M)_{*,\mathbf{t}_k(k\neq l)} = \sum_{1\leq i\leq |H|, i\neq k,l} e_{\mathbf{t}_i} \frac{1}{e_{\mathbf{t}_i}} - (|H|-2)e_{\mathbf{t}_l} \frac{1}{e_{\mathbf{t}_l}} = (|H|-2) - (|H|-2) = 0.$

$(M^{-1})_{b,*} \cdot (M)_{*,\mathbf{s}_j(j\neq p)} = \sum_{1\leq i\leq |H|, i\neq l} e_{\mathbf{t}_i} \frac{1}{e_{\mathbf{t}_i}} - (|H|-2)e_{\mathbf{t}_l} * \frac{1}{e_{\mathbf{t}_l}} + (-1)1 = (|H|-1) - (|H|-2) - 1 = 1 - 1 = 0.$

$(M^{-1})_{\mathbf{t}_k(k\neq l),*} \cdot (M)_{*,b} = -e_{\mathbf{t}_k} \frac{1}{e_{\mathbf{t}_k}} + e_{\mathbf{t}_l} \frac{1}{e_{\mathbf{t}_l}} = 0.$

$(M^{-1})_{\mathbf{t}_k(k\neq l),*} \cdot (M)_{*,a} = -e_{\mathbf{t}_k}\frac{1}{e_{\mathbf{t}_k}} + 1 \cdot 1 = -1 + 1 = 0.$

$(M^{-1})_{\mathbf{t}_k(k\neq l),*} \cdot (M)_{*,\mathbf{t}_k(k\neq l)} = e_{\mathbf{t}_l}\frac{1}{e_{\mathbf{t}_l}} = 1.$

$(M^{-1})_{\mathbf{t}_k(k\neq l),*} \cdot (M)_{*,\mathbf{s}_j(j\neq p)} = e_{\mathbf{t}_l}\frac{1}{e_{\mathbf{t}_l}} - e_{\mathbf{t}_k}\frac{1}{e_{\mathbf{t}_k}} = 0.$

$(M^{-1})_{\mathbf{s}_j,*} \cdot (M)_{*,b} = 0$ (no matching non-zero entries).

$(M^{-1})_{\mathbf{s}_j,*} \cdot (M)_{*,a}$:

Case 1: $j = p$, so this is $(M^{-1})_{a,*} \cdot (M)_{*,a} = 1 \cdot 1 = 1.$

Case 2: $j \neq p$, so this is $(M^{-1})_{\mathbf{s}_j,*} \cdot (M)_{*,a} = 0.$ (no matching non-zero entries)

$(M^{-1})_{\mathbf{s}_j,*} \cdot (M)_{*,\mathbf{t}_k(k\neq l)} = 0.$ (no matching non-zero entries)

$(M^{-1})_{\mathbf{s}_j,*} \cdot (M)_{*,\mathbf{s}_j(j\neq p)} = 1 \cdot 1 = 1.$

This verifies $M^{-1}M = I_n$. The intersection is computed by $\mathbf{z} = M^{-1}\mathbf{c}$ where $\mathbf{c} = [1, \ldots, 1, 0, \ldots, 0]^T$ (i.e., $|H|$-many 1's followed by $|Y|$-many 0's). The value of $\mathbf{z}_b$ will equal the sum of the first $|H|$ entries of the row $(M^{-1})_{b,*}$, and the value of $\mathbf{z}_{\mathbf{t}_k}$ is the sum of the first $|H|$ entries of the row $(M^{-1})_{\mathbf{t}_k(k\neq l),*}$. In summary the solution $\mathbf{z}$ has entries:

$$\begin{cases} \mathbf{z}_b = -(|H|-2)e_a + \sum_{i=1,i\neq l}^{|H|} e_{\mathbf{t}_i} & \\ \mathbf{z}_{\mathbf{t}_i} = e_a - e_{\mathbf{t}_i} & \text{for } 1 \leq i \leq |H|, \\ \mathbf{z}_{\mathbf{s}_j} = 0 & \text{for } 1 \leq j \leq |Y|. \end{cases}$$

This intersection $\mathbf{z}$ is a vertex if the following constraints are satisfied:

$$e_a \leq \frac{\sum_{i=1,i\neq l}^{|H|} e_{\mathbf{t}_i}}{|H|-2} \tag{3.3.4}$$

$$e_a \geq e_{\mathbf{t}_i} \text{ (for } 1 \leq i \leq |H|) \tag{3.3.5}$$

$$e_a \geq e_{\mathbf{s}_j} \text{ (for } 1 \leq j \leq |Y|) \tag{3.3.6}$$

$$e_a \geq \frac{e_b + \sum_{i=1,i\neq l}^{|H|} e_{\mathbf{t}_i}}{|H|-1} \tag{3.3.7}$$

The constraints (3.3.4) and (3.3.5) are non-negativity requirements on the coordinates

$\mathbf{z}_b$ and $\mathbf{z}_{\mathbf{t}_i}$, respectively . Substituting $\mathbf{z}$ into $h_{\mathbf{s}_j} \geq 1$ and $h_b \geq 1$ gives the constraints (3.3.6) and (3.3.7), respectively. The coordinate sum in this case is $\mathbf{1}^T\mathbf{z} = e_a$. We will now show that $e_1 \leq \mathbf{1}^T\mathbf{z}$ by showing that when $\mathbf{z}$ is a vertex we have $\mathbf{1}^T\mathbf{z} = e_a = e_1$. This is clear when $a = 1$. Now suppose that $a > 1, b \neq 1$ with $1 \in H$. Then (3.3.5) for $i = 1$ gives $e_a \geq e_1$ and so $e_a = e_1$. Similarly if $1 \in Y$, then (3.3.6) for $j = 1$ gives $e_a \geq e_1$ and so $e_a = e_1$. Otherwise suppose that $a > 1$ and $b = 1$. Then (3.3.7) becomes $e_a \geq \frac{e_1 + \sum_{i=1, i\neq l}^{|H|} e_{\mathbf{t}_i}}{|H|-1}$. Applying (3.3.4) gives $e_a \geq \frac{e_1 + (|H|-2)e_a}{|H|-1}$ which requires $e_a = e_1$ since $e_1 \geq e_a$. So when $\mathbf{z}$ is a vertex the coordinate sum is $\mathbf{1}^T\mathbf{z} = e_1$.

We will now show that $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$.

From the above we know that if $\mathbf{z}$ is a vertex we require $e_1 = e_a$, which implies $e_1 = e_2 = \cdots = e_a$ by the ordering of the e_i's. Recall that $\{1, \ldots, a\}$ contains l-many elements from H and p-many elements from Y with only $H \cap Y = \{a\}$. This means $\sum_{i=1, i\neq b}^{a} e_i = (l+p-1)e_a$. So we can write:

$$\mathbf{1}^T\mathbf{z}^* = \frac{e_1 + \cdots + e_n}{n-1} = \frac{(l+p-1)e_a + e_b + \sum\limits_{l<i\leq|H|} e_{\mathbf{t}_i} + \sum\limits_{p<j\leq|Y|} e_{\mathbf{s}_j}}{n-1}.$$

Observe the inequality (3.3.7) can be re-writtten as $e_b \leq (|H|-1)e_a - \sum\limits_{1\leq i<l} e_{\mathbf{t}_i} - \sum\limits_{l<i\leq|H|} e_{\mathbf{t}_i}$. Substituting this inequality for e_b into the above gives

$$\mathbf{1}^T\mathbf{z}^* \leq \frac{(l+p-1)e_a + (|H|-1)e_a - \sum\limits_{1\leq i<l} e_{\mathbf{t}_i} - \sum\limits_{l<i\leq|H|} e_{\mathbf{t}_i} + \sum\limits_{l<i\leq|H|} e_{\mathbf{t}_i} + \sum\limits_{p<j\leq|Y|} e_{\mathbf{s}_j}}{n-1}.$$

Now after simplifying and noting $\sum\limits_{1\leq i<l} e_{\mathbf{t}_i} = (l-1)e_a$ we get

$$\mathbf{1}^T\mathbf{z}^* \leq \frac{(|H|+p-1)e_a + \sum\limits_{p<j\leq|Y|} e_{\mathbf{s}_j}}{n-1}.$$

Since $\sum\limits_{p<j\leq|Y|} e_{\mathbf{s}_j}$ is $|Y|-p$ terms less than or equal to e_a, we see the numerator has $|H|+$

$p-1+|Y|-p=n-1$ terms less than or equal to e_a. This gives

$$\mathbf{1}^T\mathbf{z}^* \leq \frac{(|H|+p-1)e_a + \sum\limits_{p<j\leq|Y|} e_{\mathbf{s}_j}}{n-1} \leq \frac{(n-1)e_a}{n-1} = e_a = \mathbf{1}^T\mathbf{z}$$

as needed.

Case 3: $|H \cap Y| \geq 2$.

We will show this case is always infeasible. Start with $|H \cap Y| = 2$ and assume without loss of generality that $H \cap Y = \{1,2\}$. To compute the intersection $\mathbf{z}$ we solve the appropriate system $M\mathbf{y} = \mathbf{c}$ using the $n \times n+1$ augmented matrix $[M|\mathbf{c}]$. There will be a $4 \times (n+1)$ submatrix of $[M|\mathbf{c}]$ with rows corresponding to the hyperplanes $(h_1 = 1, h_2 = 1, \mathbf{y}_1 = 0, \mathbf{y}_2 = 0)$:

$$\begin{bmatrix} 0 & \frac{1}{e_1} & \frac{1}{e_1} & \cdots & \frac{1}{e_1} & 1 \\ \frac{1}{e_2} & 0 & \frac{1}{e_2} & \cdots & \frac{1}{e_2} & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \text{which row reduces to} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{e_1} & \cdots & \frac{1}{e_1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1-\frac{e_1}{e_2} \end{bmatrix}.$$

If $e_1 > e_2$ the bottom row gives an inconsistent system. If $e_1 = e_2$, then $[M|\mathbf{c}]$ does not have full rank and $\mathbf{z}$ is not a vertex. The same rows are present in $[M|\mathbf{c}]$ if $|H \cap Y| > 2$. So in all cases $\mathbf{z}$ fails to be a vertex.

The above Cases 1, 2, and 3 exhaust all possible vertices of $SP(I)$.

We have shown that if $\mathbf{z}$ is any vertex of $SP(I)$, then $\mathbf{1}^T\mathbf{z}^* \leq \mathbf{1}^T\mathbf{z}$. So if $\mathbf{z}^*$ is a vertex, then it is a minimizer to the linear optimization problem (3.3.1) and so $\widehat{\alpha}(I) = \mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+\cdots+e_n}{n-1}$. We will now show that if $\mathbf{z}^*$ is not a vertex, then there is a vertex $\mathbf{z}$ such that $\mathbf{1}^T\mathbf{z} = e_1$ and thus $\widehat{\alpha}(I) = e_1$.

Now suppose the feasibility condition of $\mathbf{z}^*$ (see Lemma 3.3.2) is not satisfied so we have $e_1 > \frac{e_2+e_3+\cdots+e_n}{n-2}$. We have also shown that if $\mathbf{z}$ is any vertex of $SP(I)$ (other than $\mathbf{z}^*$) , then $e_1 \leq \mathbf{1}^T\mathbf{z}$. So if $\mathbf{z}^*$ is not a vertex and if there is a vertex $\mathbf{z}$ with coordinate sum $\mathbf{1}^T\mathbf{z} = e_1$, then $\mathbf{z}$ must be a minimizer of the linear optimization problem (3.3.1) and so $\widehat{\alpha}(I) = e_1$.

Recall in the $n = 4$ case we examined following set of three intersections: $\mathbf{z}^* = \mathbf{z}^{1234|}, \mathbf{z}^{123|1}, \mathbf{z}^{12|14}$ and determined that if $\mathbf{z}^*$ was not a vertex, at least one of $\mathbf{z}^{123|1}, \mathbf{z}^{12|14}$ must be a vertex.

For the general $n \geq 5$ case, we will construct a similar set of of $n-1$ intersections:

H	Y	Constraint	Non-negativity
$\{1,\ldots,n\}$	$\emptyset$		$e_1 \leq \frac{\sum_{i=2}^{n} e_i}{n-2}$
$\{1,\ldots,n-1\}$	$\{1\}$	$e_1 \geq \frac{\sum_{i=2}^{n} e_i}{n-2}$	$e_1 \leq \frac{\sum_{i=2}^{n-1} e_i}{n-3}$
$\{1,\ldots,n-2\}$	$\{1,n\}$	$e_1 \geq \frac{\sum_{i=2}^{n-1} e_i}{n-3}$	$e_1 \leq \frac{\sum_{i=2}^{n-2} e_i}{n-4}$
$\{1,\ldots,n-3\}$	$\{1,n-1,n\}$	$e_1 \geq \frac{\sum_{i=2}^{n-2} e_i}{n-4}$	$e_1 \leq \frac{\sum_{i=2}^{n-3} e_i}{n-5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{1,2,3\}$	$\{1,5,\ldots,n\}$	$e_1 \geq \frac{e_2+e_3+e_4}{2}$	$e_1 \leq e_2+e_3$
$\{1,2\}$	$\{1,4,\ldots,n\}$	$e_1 \geq e_2+e_3$	

The first intersection listed above is $\mathbf{z}^*$. The $n-2$ intersections that follow $\mathbf{z}^*$ are from Case 2.2 with $a = 1$ and their "Constraint" and "Non-negativity" conditions obtained from (3.3.7) and (3.3.4), respectively. Each intersection from Case 2.2 has coordinate sum $\mathbf{1}^T\mathbf{z} = e_1$.

If the first intersection is not a vertex then, the failure of its non-negativity condition means the second intersection's constraint is satisfied. If the second intersection is not a vertex, then the failure of its non-negativity condition means the third intersection's constraint is satisfied, and so on. If the first $n-2$ intersections are not vertices, then we have $e_1 > e_2 + e_3$ which means the final intersection must be a vertex since $e_1 > e_2 + e_3$ satisfies its constraint and guarantees that it has more than one non-zero coordinate.

This shows that if $\mathbf{z}^*$ is a vertex we have $\widehat{\alpha}(I) = \mathbf{1}^T\mathbf{z}^* = \frac{e_1+e_2+\cdots+e_n}{n-1}$. Otherwise we have $\widehat{\alpha}(I) = e_1$. This concludes the proof of Theorem (3.3.3). $\square$

The technique of Corollary 3.2.3 for $n = 4$ misses only $\widehat{\alpha}(I) = \frac{5}{3}$. As n increases, more numbers are missed by the technique. The following corollary gives a condition on positive integers q, p which will guarantee the Waldschmidt constant $\frac{q}{p} \geq 1$ is attainable in $n = p+1$ variables using a technique generalizing Corollary 3.2.3.

Corollary 3.3.4. *Consider the fraction $\frac{q}{p} \geq 1$ for some positive integers q, p. Let $n = p+1$ and $q \equiv k \mod n$. Then write the integer partition $q = e_1 + e_2 + \cdots + e_n$ where we set*

$$e_i = \begin{cases} \lceil \frac{q}{n} \rceil & \textit{for } 1 \leq i \leq k \\ \lfloor \frac{q}{n} \rfloor & \textit{for } k+1 \leq i \leq n. \end{cases}$$

Now suppose $n^2 - (k+1)n + k \leq q$ is true. Then

$$I = \langle x_2, \ldots, x_n \rangle^{e_1} \cap \langle x_1, x_3, \ldots, x_n \rangle^{e_2} \cap \cdots \cap \langle x_1, \ldots, x_{n-1} \rangle^{e_n}$$

is a monomial ideal in $R = K[x_1, ..., x_n]$ with $\widehat{\alpha}(I) = \frac{q}{p}$.

Proof. By Theorem 3.3.3 we need $e_1 \leq \frac{e_2 + \cdots + e_n}{n-2}$ to get $\widehat{\alpha}(I) = \frac{q}{p}$.

Set

$$e_i = \begin{cases} \lceil \frac{q}{n} \rceil & \text{for } 1 \leq i \leq k \\ \lfloor \frac{q}{n} \rfloor & \text{for } k+1 \leq i \leq n. \end{cases}$$

Observe that if $q \equiv k \mod n$, then we have $\lceil \frac{q}{n} \rceil = \frac{q+n-k}{n}$ and $\lfloor \frac{q}{n} \rfloor = \frac{q-k}{n}$. We will now re-write the condition $e_1 \leq \frac{e_2 + \cdots + e_n}{n-2}$ in terms of q, k, and $n = p+1$. We start by re-writing the LHS and RHS of $e_1 \leq \frac{e_2 + \cdots + e_n}{n-2}$ in the following way:

$$LHS : e_1 = \frac{q+n-k}{n} = \frac{(n-2)(q+n-k)}{(n-2)n} = \frac{nq + n^2 - kn - 2q - 2n + 2k}{(n-2)n},$$

$$RHS : \frac{e_2 + \cdots + e_n}{n-2} = \frac{(k-1)\lceil \frac{q}{n} \rceil + (n-k)\lfloor \frac{q}{n} \rfloor}{n-2} = \frac{\frac{(k-1)q+n-k}{n} + \frac{(n-k)q-k}{n}}{n-2}$$

$$= \frac{\frac{kq+kn-k^2-q-n+k+nq-kn-kq+k^2}{n}}{n-2} = \frac{-q-n+k+nq}{(n-2)n}.$$

Then $LHS \leq RHS$ requires

$$\frac{nq + n^2 - kn - 2q - 2n + 2k}{(n-2)n} \leq \frac{-q - n + k + nq}{(n-2)n}.$$

Simplifying gives

$$nq + n^2 - kn - 2q - 2n + 2k \leq -q - n + k + nq$$

or equivalently,

$$n^2 - (k+1)n + k \leq q.$$

If this holds, then $e_1 \leq \frac{e_2 + \cdots + e_n}{n-2}$ holds. So by Theorem 3.3.3 we have

$$\widehat{\alpha}(I) = \frac{e_1 + e_2 + \cdots + e_n}{n-1} = \frac{q}{n-1} = \frac{q}{p}.$$

Note n here refers to the number of variables in $R = K[x_1, ..., x_n]$.

□

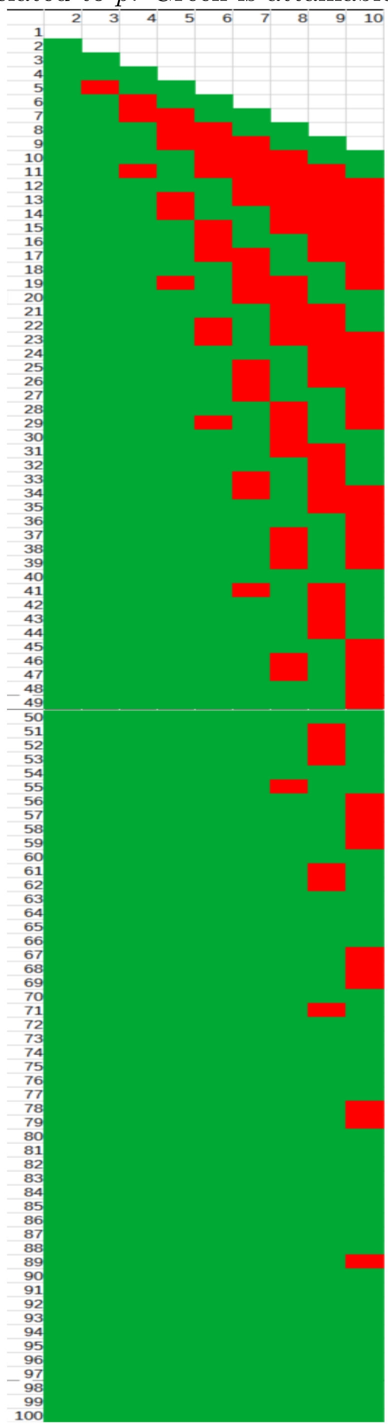
Corollary 3.3.5. *Consider the fraction* $\frac{q}{p} \geq 1$ *for some positive integers* q, p*. Let* $n = p+1$ *and* $q \equiv k \mod n$*. If* $n^2 \leq q$*, then we can find a monomial ideal* I *with* $\widehat{\alpha}(I) = \frac{q}{p}$*.*

Proof. Observe that $n^2 - (k+1)n + k = (n-1)(n-k)$. Since $k \in \{0, \ldots, n-1\}$ we have $(n-1)(n-k) \leq (n-1)(n) = n^2 - n \leq n^2$. So if $n^2 \leq q$ then the condition of Corollary 3.3.4 is satisfied. □

Observe that the above corollary implies that the technique of Corollary 3.3.4 only misses finitely many numerators q for each denominator $p = n - 1$. Figure 3.3 displays which values $\frac{q}{p} \geq 1$ are attainable by choosing e_i's by the technique of Corollary 3.3.4 for $1 \leq q \leq 100$ and $2 \leq p \leq 10$.

Figure 3.3: Waldschmidt constants attainable by Corollary 3.3.4

The Waldschmidt constants $\widehat{\alpha}(I) = \frac{q}{p} \geq 1$ attained by Corollary 3.3.4. Rows are associated to q and columns are associated to p. Green is attainable and red is not attainable.



Chapter 4

On the Waldschmidt constant of square-free principal Borel ideals

The content of Chapter 4 first appeared in the paper *On the Waldschmidt Constant of Square-free Principal Borel Ideals* [CMKSVT22] in collaboration with Camps Moreno, Sarmiento, and Van Tuyl.

The goal of Chapter 4 is to investigate the Waldschmidt constant of square-free principal Borel ideals. In Chapter 3 we saw that we could construct a monomial ideal that attains a Waldschmidt constant for "almost all" fractions $\frac{q}{p} \geq 1$. One motivation to study square-free principal Borel ideals is to fill in the "gaps" that are missed by the technique of Chapter 3 (e.g., those fractions indicated in red in Figure 3.3). In this chapter, we show that by allowing more variables we can attain any rational Waldschmidt constant $\frac{q}{p} \geq 1$ (see Corollary 4.3.2).

Given a monomial m, if $x_i|m$ and $j < i$, then we call $x_j \cdot \frac{m}{x_i}$ a **Borel move** of m. A monomial ideal is a **Borel ideal** (or a strongly stable ideal) if for every $m \in I$, all of the Borel moves of m are also in I. A monomial ideal I is a **principal Borel ideal** if there is a single monomial m such that every generator of I is obtained via a Borel move of m. The study of (principal) Borel ideals has a rich history; refer to [FMS11] and [Her02] for more on this topic.

4.1 Square-free principal Borel ideals

In this section we define square-free principal Borel ideals, introduce our notation, and then give the structure of their associated primes.

Definition 4.1.1. Let $X = \{m_1, \dots, m_t\}$ be a set of square-free monomials in $R = K[x_1, \dots, x_n]$. The **square-free Borel ideal generated by** X, denoted $\mathrm{sfBorel}(X)$, is the square-free monomial ideal generated by the square-free monomials that can be obtained via Borel moves from any monomial $m \in X$. If $X = \{m\}$, then we abuse notation and write $\mathrm{sfBorel}(m)$ for $\mathrm{sfBorel}(\{m\})$; furthermore, we call $\mathrm{sfBorel}(m)$ a **square-free principal Borel ideal**.

Example 4.1.2. Let $m = x_2x_5$ and let $I = \mathrm{sfBorel}(x_2x_5) \in K[x_1, \dots, x_5]$. The set of monomials attained via every Borel move on $m = x_2x_5$ is the following set:

$$B = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5\}.$$

The generators of $\mathrm{sfBorel}(x_2x_5)$ are the square-free monomials of B:

$$\mathrm{gens}(\mathrm{sfBorel}(x_2x_5)) = \{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_1x_5, x_2x_5\}.$$

The **support** of a square-free monomial $m = x_{i_1} \cdots x_{i_s}$ is the set $\mathrm{supp}(m) = \{i_1, \dots, i_s\}$. For our future arguments, we need two tuples that can be constructed from $\mathrm{supp}(m)$.

Definition 4.1.3. Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial. Let

$$T(m) = (t_0, t_1, \dots, t_k)$$

where $t_0 = s$ and $t_i = \max\{j < t_{i-1} \mid i_j < i_{j+1} - 1\}$. Furthermore, let

$$IT(m) = (i_{t_0}, i_{t_1}, \dots, i_{t_k}).$$

Remark 4.1.4. The following observations will hopefully help the reader with our notation. The t_i's are recording where the indices of $x_{i_1}x_{i_2}\cdots x_{i_s}$ are "jumping" by more than one. For example, if $m = x_2x_3x_5x_6x_8x_{10}$, then

$$T(m) = (t_0, t_1, t_2, t_3) = (6, 5, 4, 2)$$

records the positions where the indices increase by more than one. Note that we are recording this information from right-to-left. Equivalently, we can define the t_i's as follows. Consider the tuple $(i_1 - 1, i_2 - 2, \ldots, i_s - s)$. The t_i's are then the locations where $i_j - j < i_{j+1} - (j + 1)$, again reading right-to-left. In our example, $(2 - 1, 3 - 2, 5 - 3, 6 - 4, 8 - 5, 10 - 6) = (1, 1, 2, 2, 3, 4)$, so $t_0 = 6$, and $t_1 = 5$, $t_2 = 4$ and $t_3 = 2$ since these are the indices where $i_j - j < i_{j+1} - (j + 1)$. Continuing with this example, the tuple

$$IT(m) = (i_6, i_5, i_4, i_2) = (10, 8, 6, 3)$$

records the indices of the variables where the "jump" occurs.

The following lemma records some facts that follow immediately from the definitions.

Lemma 4.1.5. *Let m be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$. Then*

1. $s = t_0 > t_1 > \cdots > t_k \geq 1$,
2. $i_{t_0} > i_{t_1} > \cdots > i_{t_k}$, *and*
3. $i_{t_j} - t_j > i_{t_{j+1}} - t_{j+1}$ *for* $j = 0, \ldots, k - 1$.

Proof. (1) and (2) are immediate. For (3) we have

$$i_1 - 1 \leq i_2 - 2 \leq \cdots \leq i_j - j \leq \cdots \leq i_s - s \tag{4.1.1}$$

for all $j = 1, \ldots, s$. So

$$i_{t_{j+1}} - t_{j+1} < i_{t_{j+1}+1} - (t_{j+1} + 1) \leq i_{t_j} - t_j$$

since the t_i's are precisely the locations where the inequalities in (4.1.1) are strict. □

Recall that if I is an ideal, we let $\mathrm{Ass}(I)$ denote the *set of associated primes* of the ideal I. The following theorem characterizes the associated primes of $\mathrm{sfBorel}(I)$. This will allow us to construct the linear optimization problem required to compute the Waldschmidt constant. Note that the original statement of Theorem 4.1.6 involved the language of Alexander duals. We have given an equivalent expression for this statement.

Theorem 4.1.6 (Theorem 3.17, [FMS11])**.** *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and suppose $I = \mathrm{sfBorel}(m)$. Then*

$$\langle x_{j_1}, \ldots, x_{j_l} \rangle \in \mathrm{Ass}(I)$$

if and only if $x_{j_1} x_{j_2} \cdots x_{j_l}$ is a minimal generator of the square-free Borel ideal

$$\mathrm{sfBorel}(\{x_{t_k} x_{t_k+1} \cdots x_{i_{t_k}},\ x_{t_{k-1}} x_{t_{k-1}+1} \cdots x_{i_{t_{k-1}}},\ \ldots,\ x_{t_0} x_{t_0+1} \cdots x_{i_{t_0}}\}).$$

The monomials $x_{t_j} \cdots x_{i_{t_j}}$ for $j = 0, \ldots, k$ are minimal in the sense that if we remove any of them, we change the generators of the resulting square-free Borel ideal.

Example 4.1.7. In Remark 4.1.4 it was shown that the monomial $m = x_2 x_3 x_5 x_6 x_8 x_{10}$ has $T(m) = (6, 5, 4, 2)$ and $IT(m) = (10, 8, 6, 3)$. So the associated primes of $\mathrm{sfBorel}(m)$ are in one-to-one correspondence with the minimal generators of

$$\mathrm{sfBorel}(\{x_2x_3, x_4x_5x_6, x_5x_6x_7x_8, x_6x_7x_8x_9x_{10}\}).$$

Since we are dealing with square-free monomial ideals, Theorem 2.2.11 gives the procedure for computing the Waldschmidt constant for $\mathrm{sfBorel}(m)$. We will refer to the matrix A of Theorem 2.2.11 as the **matrix of associated primes** of I.

Example 4.1.8. Recall the monomial from the introduction:

$$m = x_{33215} x_{33216} \cdots x_{104348} \in K[x_1, \ldots, x_{104348}].$$

Here we have $T(m) = (71134)$ and $IT(m) = (104348)$. The associated primes of $\mathrm{sfBorel}(m)$ are in one-to-one correspondence with the generators of $\mathrm{sfBorel}(x_{71134} x_{71135} \cdots x_{104348})$.

But this is the ideal generated by all the square-free monomials of degree 33215 in 104348 variables, of which there are $\binom{104348}{33215}$. So the matrix of associated primes of sfBorel(m) will be a $\binom{104348}{33215} \times 104348$ matrix.

4.2 Upper bounds

In this section we give an upper bound on the Waldschmidt constant of a square-free principal Borel ideal. Our strategy is to show that there is enough structure in the linear optimization problem of Theorem 2.2.11 that we can bound the Waldschmidt constant.

We begin with a lemma which allows us to reduce to a smaller polynomial ring.

Lemma 4.2.1. *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial in $R = K[x_1, \ldots, x_n]$ with $I = \text{sfBorel}(m) \subseteq R$. Consider the same monomial m, but in the ring $S = K[x_1, \ldots, x_{i_s}]$, and let $J = \text{sfBorel}(m) \subseteq S$. Then $\widehat{\alpha}(I) = \widehat{\alpha}(J)$.*

Proof. By Theorem 4.1.6, the associated primes of I and J are the same (although viewed in different rings). So the matrix of associated primes of J in Theorem 2.2.11 is the same as the matrix of the associated primes of I, except that the columns in matrix of associated primes of I indexed by the variables $x_{i_s+1}, \ldots, x_n$ all contain zeroes. The result now follows from Theorem 2.2.11. $\square$

Before proceeding, we introduce additional notation. Given a square-free monomial m with $T(m) = (t_0, \ldots, t_k)$ and $IT(m) = (i_{t_0}, \ldots, i_{t_k})$, we have the inequalities

$$t_k < t_{k-1} < \cdots < t_0 = s \quad \text{and} \quad i_{t_k} < i_{t_{k-1}} < \cdots < i_{t_0}.$$

by Lemma 4.1.5. Let ℓ be smallest integer such that

$$i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}.$$

In particular, ℓ identifies where in the sequence of i_{t_j}'s we would place $t_0 = s$.

Let A be the matrix of associated primes of $I = \text{sfBorel}(m)$. We will let A_P denote the row associated to the associated prime P. The row A_P corresponds to a minimal generator m of the ideal in Theorem 4.1.6.

Example 4.2.2. For example, if $m = x_2x_3x_5x_6x_8x_{10}$, then we have $T(m) = (6, 5, 4, 2)$ and $IT(m) = (10, 8, 6, 3)$. For this monomial, $\ell = 2$ since $t_2 = 4$ and $t_0 = 6 \leq i_4 = i_{t_\ell} = 6$.

As the next lemma shows, we can bound the optimal solution of Theorem 2.2.11 by considering only a submatrix of the matrix of associated primes.

Lemma 4.2.3. *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \dots, t_k)$, $IT(m) = (i_{t_0}, i_{t_1}, \dots, i_{t_k})$, and ℓ as defined above. Let $I = \text{sfBorel}(m)$, and let A denote its matrix of associated primes. Let B be the submatrix of A where the j-th row of B corresponds to the associated prime $\langle x_{t_j}, \dots, x_{i_{t_j}} \rangle$ for $j = 0, \dots, k$. Suppose $\mathbf{x} \in \mathbb{R}^n$ is such that*

1. *$B\mathbf{x} \geq \mathbf{1}$,*

2. *$\mathbf{x}_j \geq \mathbf{x}_{j+1}$ for $1 \leq j \leq i_{t_\ell}$, and*

3. *$\mathbf{x}_{i_{t_\ell}} \geq \mathbf{x}_j$, for $i_{t_\ell} \leq j \leq n$.*

Then $A\mathbf{x} \geq \mathbf{1}$.

Proof. By Lemma 4.2.1, we can assume $n = i_s$. Consider any row A_P of A. By Theorem 4.1.6, P corresponds to a monomial m that is a Borel move of exactly one of

$$\{x_{t_k} \cdots x_{i_{t_k}}, \dots, x_{t_0} \cdots x_{i_{t_0}}\}.$$

Say m is a Borel move of $x_{t_j} \cdots x_{i_{t_j}}$. The j-th row of B (which corresponds to $x_{t_j} \cdots x_{i_{t_j}}$) is given by

$$B_j = (\underbrace{0, \dots, 0}_{t_j - 1}, \underbrace{1, \dots, 1}_{i_{t_j} - t_j + 1}, 0, \dots, 0).$$

The rows A_P and B_j have the same number of 1's. Additionally, A_P is formed from B_j by swapping some of the 1's with some of the 0's among the first $t_j - 1$ spots.

Since $B_j\mathbf{x} \geq 1$, we have $\mathbf{x}_{t_j} + \cdots + \mathbf{x}_{i_{t_j}} \geq 1$. Note that $A_P\mathbf{x}$ is formed from $B_j\mathbf{x}$ by subtracting some $\mathbf{x}_p$'s with $p \in \{t_j, \ldots, i_{t_j}\}$ and adding in some $\mathbf{x}_q$'s with $q \in \{1, \ldots, t_j - 1\}$.

If $p \in \{t_j, \ldots, i_{t_\ell}\}$, then the hypotheses imply that $\mathbf{x}_q \geq \mathbf{x}_p$ for all $q \in \{1, \ldots, t_j - 1\}$. If $p \in \{i_{t_\ell}, \ldots, i_{t_j}\}$, then $t_j < t_0 \leq i_{t_\ell} \leq p \leq i_{t_j}$. But then $\mathbf{x}_q \geq \mathbf{x}_{i_{t_\ell}} \geq \mathbf{x}_p$ for all $q \in \{1, \ldots, t_j - 1\}$. But this means

$$A_P\mathbf{x} \geq B_j\mathbf{x} \geq 1$$

because every time we subtract an $\mathbf{x}_p$ with $p \in \{t_j, \ldots, i_{t_j}\}$ we are replacing it with an $\mathbf{x}_q$ with $q \in \{1, \ldots, t_j - 1\}$ which is larger.

The result now follows since $A_P\mathbf{x} \geq 1$ for all rows of A. $\square$

We can now bound the Waldschmidt constant of a square-free principal Borel ideal in terms of $T(m), IT(m)$, and ℓ.

Theorem 4.2.4. *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and suppose $I = \mathrm{sfBorel}(m)$. If ℓ is the smallest integer such that $i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}$, then*

$$\begin{aligned} \widehat{\alpha}(I) \quad \leq \quad & (t_0 - t_\ell)\left(\frac{1}{i_{t_\ell} - t_\ell + 1}\right) + (i_{t_\ell} - i_{t_{\ell+1}})\left(\frac{1}{i_{t_\ell} - t_\ell + 1}\right) \\ & + \cdots + (i_{t_{k-1}} - i_{t_k})\left(\frac{1}{i_{t_{k-1}} - t_{k-1} + 1}\right) + i_{t_k}\left(\frac{1}{i_{t_k} - t_k + 1}\right). \end{aligned}$$

Proof. By Lemma 4.2.1, we can assume that the number of variables is $i_s = n$. Set $a =$

$i_{t_\ell} - t_\ell + 1$, and consider the vector $\mathbf{y} \in \mathbb{R}^{i_s = i_{t_0}}$ where

$$\begin{aligned} \mathbf{y}^T \quad = \quad & \Bigg(\underbrace{\frac{1}{i_{t_k} - t_k + 1}, \ldots, \frac{1}{i_{t_k} - t_k + 1}}_{i_{t_k}}, \underbrace{\frac{1}{i_{t_{k-1}} - t_{k-1} + 1}, \ldots, \frac{1}{i_{t_{k-1}} - t_{k-1} + 1}}_{i_{t_{k-1}} - i_{t_k}}, \\ & \ldots, \underbrace{\frac{1}{a}, \ldots, \frac{1}{a}}_{i_{t_\ell} - i_{t_\ell + 1}}, \underbrace{\frac{(t_{\ell-1} - t_\ell)}{(i_{t_{\ell-1}} - i_{t_\ell})a}, \ldots, \frac{(t_{\ell-1} - t_\ell)}{(i_{t_\ell - 1} - i_{t_\ell})a}}_{i_{t_{\ell-1}} - i_{t_\ell}}, \\ & \underbrace{\frac{(t_{\ell-2} - t_{\ell-1})}{(i_{t_{\ell-2}} - i_{t_{\ell-1}})a}, \ldots, \frac{(t_{\ell-2} - t_{\ell-1})}{(i_{t_{\ell-2}} - i_{t_{\ell-1}})a}}_{i_{t_{\ell-2}} - i_{t_{\ell-1}}}, \ldots, \underbrace{\frac{(t_0 - t_1)}{(i_{t_0} - i_{t_1})a}, \ldots, \frac{(t_0 - t_1)}{(i_{t_0} - i_{t_1})a}}_{i_{t_0} - i_{t_1}}\Bigg). \end{aligned}$$

Let A be the matrix of associated primes of I. We will use Lemma 4.2.3 to verify that $A\mathbf{y} \geq \mathbf{1}$. Theorem 2.2.11 then gives the required result after we sum all the entries of $\mathbf{y}$.

It follows from Lemma 4.1.5 (3) that

$$\frac{1}{i_{t_j} - t_j + 1} \geq \frac{1}{i_{t_{j-1}} - t_{j-1} + 1} \quad \text{for all } j = 1, \ldots, k.$$

These inequalities imply that the first i_{t_ℓ} entries of $\mathbf{y}$ form a non-increasing sequence. Thus, condition (2) of Lemma 4.2.3 holds for $\mathbf{y}$. In addition, it follows by Lemma 4.1.5 that

$$\frac{t_{j-1} - t_j}{i_{t_{j-1}} - i_{t_j}} < 1 \quad \text{for } j = 1, \ldots, \ell.$$

Hence $\frac{1}{a} \geq \mathbf{y}_r$ for all $r = i_{t_\ell}, \ldots, i_{t_0}$, and thus condition (3) of Lemma 4.2.3 also holds for $\mathbf{y}$.

Let B be the submatrix of A where the j-th row of B corresponds to the associated prime of I that is associated to $x_{t_j} \cdots x_{i_{t_j}}$. That is, written as a row vector:

$$B_j = (\underbrace{0, \ldots, 0}_{t_j - 1}, \underbrace{1, \ldots, 1}_{i_{t_j} - t_j + 1}, 0, \ldots, 0).$$

We now show that B and $\mathbf{y}$ satisfy condition (1) of Lemma 4.2.3, thus completing the proof.

Consider the j-th row of B, denoted B_j. If $j \geq \ell$, we then have

$$B_j \mathbf{y} \geq B_j \begin{pmatrix} \mathbf{0} \\ \frac{1}{i_{t_j}-t_j+1} \\ \vdots \\ \frac{1}{i_{t_j}-t_j+1} \\ \mathbf{y}_{i_{t_j}+1}^{i_{t_0}} \end{pmatrix} \geq 1$$

where $\mathbf{y}_{i_{t_j}+1}^{i_{t_0}}$ represents the last $i_{t_0} - i_{t_j}$ entries of $\mathbf{y}$, the fraction $\frac{1}{i_{t_j}-t_j+1}$ appears $i_{t_j} - t_j + 1$ times, and the $\mathbf{0}$ is the vector with $t_j - 1$ zeroes. Since every entry of this new vector is less than or equal to the corresponding entry in $\mathbf{y}$, the first inequality holds.

Now suppose that $j < \ell$. Consequently, note that $t_j < t_0 \leq i_{t_\ell} < i_{t_j}$. So we then have

$$\begin{aligned} B_j \mathbf{y} &= \sum_{r=1}^{i_s} (B_j)_r \mathbf{y}_r = \sum_{r=t_j}^{i_{t_\ell}} \mathbf{y}_r + \sum_{r=i_{t_\ell}+1}^{i_{t_j}} \mathbf{y}_r \\ &\geq \frac{i_{t_\ell} - t_j + 1}{i_{t_\ell} - t_\ell + 1} + \frac{t_{\ell-1} - t_\ell}{i_{t_\ell} - t_\ell + 1} + \cdots + \frac{t_j - t_{j+1}}{i_{t_\ell} - t_\ell + 1} \\ &= \frac{i_{t_\ell} - t_j + 1}{i_{t_\ell} - t_\ell + 1} + \frac{t_j - t_\ell}{i_{t_\ell} - t_\ell + 1} = 1. \end{aligned}$$

The inequality follows from the fact that $\mathbf{y}_r \geq \frac{1}{a}$ for all $1 \leq r \leq i_{t_\ell}$. Hence, $B_j \mathbf{y} \geq 1$ for all rows of j, so condition (1) of Lemma 4.2.3 also holds. $\square$

We derive the following corollary; in the next section we will show that this bound is exact under additional hypotheses.

Corollary 4.2.5. *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and suppose $I = \mathrm{sfBorel}(m)$. If ℓ is the smallest integer such that $i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}$, then*

$$\widehat{\alpha}(I) \leq \frac{t_0 - t_\ell + i_{t_\ell}}{i_{t_k} - t_k + 1}.$$

Proof. Recall that $t_0 = s$. Note that

$$\frac{1}{i_{t_k} - t_k + 1} \geq \frac{1}{i_{t_j} - t_j + 1} \quad \text{for } \ell \leq j \leq k.$$

The result now follows from Theorem 4.2.4 and this inequality. □

Example 4.2.6. Our bound in Theorem 4.2.4 is sharp. Continuing Example 4.2.2, if $m = x_2x_3x_5x_6x_8x_{10}$, then we have $T(m) = (6,5,4,2)$, $IT(m) = (10,8,6,3)$, and $\ell = 2$. Then

$$\hat{\alpha}(\text{sfBorel}(m)) \leq \frac{2}{3} + \frac{3}{3} + \frac{3}{2} = \frac{19}{6},$$

and this is the actual Waldschmidt constant.

4.3 Some exact values and lower bounds

In this section we compute the exact value of the Waldschmidt constant of principal square-free Borel ideals under some additional hypotheses. We then present a theorem that can be used to find lower bounds recursively.

Recall that ℓ is defined to be the smallest integer such that $i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}$, and consequently, $0 \leq \ell \leq k$. In the case that $t_0 = s \leq i_{t_k}$, that is, the case when $\ell = k$, we have the following exact formula:

Theorem 4.3.1. *Let $m = x_{i_1} \cdots x_{i_s}$ be a square-free monomial with $T(m) = (t_0, t_1, \ldots, t_k)$ and $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$, and suppose $I = \text{sfBorel}(m)$. If $t_0 = s \leq i_{t_k}$, then*

$$\widehat{\alpha}(I) = 1 + \frac{s-1}{i_{t_k} - t_k + 1}.$$

Proof. The hypotheses imply that $\ell = k$, with ℓ as in Corollary 4.2.5. Consequently, Corollary 4.2.5 then shows that

$$\widehat{\alpha}(I) \leq \frac{t_0 - t_k + i_{t_k}}{i_{t_k} - t_k + 1} = \frac{s - 1 + (i_{t_k} - t_k + 1)}{i_{t_k} - t_k + 1}$$

where we use the fact that $t_0 = s$.

For $j = 0, \ldots, k$, let P_j denote the associated prime of I that is associated with the monomial $x_{t_j} \cdots x_{i_{t_j}}$ using the correspondence of Theorem 4.1.6. For any other associated prime $P \in \text{Ass}(I)$, we will write $P \sim P_j$ if the prime P is associated to a monomial m that can be obtained from $x_{t_j} \cdots x_{i_{t_j}}$ via a Borel move. Note that each associated prime satisfies $P \sim P_j$ for exactly one $j \in \{0, \ldots, k\}$.

Let A be the matrix of associated primes of I. There are $|\mathrm{Ass}(I)|$ rows, and we write A_P for the row indexed by the associated prime P. Let $\mathbf{x} \in \mathbb{R}^{|\mathrm{Ass}(I)|}$. We will write $\mathbf{x}_P$ to denote the corresponding coordinate in $\mathbf{x}$. That is, if P indexes the i-th row of A, then $\mathbf{x}_P$ denotes the i-th coordinate of $\mathbf{x}$.

With the above notation, we now define the vector $\mathbf{y} \in \mathbb{R}^{|\mathrm{Ass}(I)|}$ as follows:

$$\mathbf{y}_P = \begin{cases} \frac{1}{\binom{i_{t_k}-1}{t_k-1}} \frac{s-1}{i_{t_k}} & P \sim P_k \\ \frac{1}{\binom{i_{t_k}}{s-1}} & P \sim P_0 \text{ and } \langle x_{i_{t_k}+1}, \ldots, x_{i_s} \rangle \subseteq P \\ 0 & \text{otherwise.} \end{cases}$$

Note that the second criterion means we are only interested in those prime ideals that arise from Borel moves of $x_{t_0} \cdots x_{i_{t_0}}$ that also contain the variables $\{x_{i_{t_k}+1}, \ldots, x_{i_s = i_{t_0}}\}$.

The r-th row of A^T is such that $(A^T)_{r,P} = 1$ if $x_r \in P$ and 0 otherwise. Then

$$(A^T \mathbf{y})_r = \sum_{x_r \in P} \mathbf{y}_P.$$

So, in order to compute $A^T\mathbf{y}$, we have to compute how many times x_r appears in some P such that $P \sim P_k$, or $P \sim P_0$ and $\langle x_{i_{t_k}+1}, \ldots, x_{i_s} \rangle \subseteq P$.

Observe that there are $\binom{i_{t_k}}{t_k-1}$ associated primes P such that $P \sim P_k$. This number is the number of Borel moves that can be made from $x_{t_k} \cdots x_{i_{t_k}}$. There are $\binom{i_{t_k}}{s-1} = \binom{i_{t_k}}{i_{t_k}+1-s}$ associated primes P such that $P \sim P_0$ and $\langle x_{i_{t_k}+1}, \ldots, x_{i_s} \rangle \subseteq P$. To see why this is true, suppose that we consider a Borel move of $x_{t_0} \cdots x_{i_{t_0}}$ that is also divisible by $x_{i_{t_k}+1} \cdots x_{i_{t_0}}$, i.e., the Borel move has the form $m'(x_{i_{t_k}+1} \cdots x_{i_{t_0}})$ where m' is a degree $i_{t_k} - t_0 + 1$ monomial in $\{x_1, \ldots, x_{i_{t_k}}\}$. Since $s = t_0 \leq i_{t_k}$, there are $\binom{i_{t_k}}{s-1} \geq 1$ possible m'.

For $i_{t_k} < r \leq i_{t_0}$ we have that x_r appears in every P such that $\mathbf{y}_P \neq 0$ and $P \sim P_0$.

Therefore

$$(A^T\mathbf{y})_r = \binom{i_{t_k}}{s-1}\frac{1}{\binom{i_{t_k}}{s-1}} = 1.$$

Now, for $1 \leq r \leq i_{t_k}$, x_r appears in $\binom{i_{t_k}-1}{t_k-1}$ elements P such that $P \sim P_k$, and in $\binom{i_{t_k}-1}{s-1}$ elements P such that $P \sim P_0$ and $\langle x_{i_{t_k}+1}, \ldots, x_{i_s}\rangle \subset P$. Therefore

$$\begin{aligned}(A^T\mathbf{y})_r &= \binom{i_{t_k}-1}{t_k-1}\left(\frac{1}{\binom{i_{t_k}-1}{t_k-1}}\frac{s-1}{i_{t_k}}\right) + \binom{i_{t_k}-1}{s-1}\frac{1}{\binom{i_{t_k}}{s-1}} \\ &= \frac{s-1}{i_{t_k}} + \frac{i_{t_k}-s+1}{i_{t_k}} = 1.\end{aligned}$$

This proves that $A^T\mathbf{y} = \mathbf{1}$. Finally

$$\begin{aligned}\mathbf{y}^T\mathbf{1} &= \binom{i_{t_k}}{t_k-1}\left(\frac{1}{\binom{i_{t_k}-1}{t_k-1}}\frac{s-1}{i_{t_k}}\right) + \binom{i_{t_k}}{s-1}\left(\frac{1}{\binom{i_{t_k}}{s-1}}\right) \\ &= \frac{i_{t_k}}{i_{t_k}-t_k+1}\frac{s-1}{i_{t_k}} + 1 = 1 + \frac{s-1}{i_{t_k}-t_k+1}.\end{aligned}$$

Due to the duality theorem (Theorem 2.1.16), we can conclude the result. $\square$

We arrive at the following corollary which was highlighted in the introduction.

Corollary 4.3.2. *Let* $I = \text{sfBorel}(x_i x_{i+1} \cdots x_{i+l})$. *Then*

$$\widehat{\alpha}(I) = \frac{i+l}{i}.$$

Consequently, for every rational number $\frac{a}{b} \geq 1$, *there exists a square-free principal Borel ideal* I *such that* $\widehat{\alpha}(I) = \frac{a}{b}$.

Proof. We have $T(m) = (l+1)$ and $IT(m) = (i+l)$. Now apply Theorem 4.3.1.

For the second statement, if $\frac{a}{b} = 1$, we can take $I = \text{sfBorel}(x_1) = \langle x_1\rangle$, from which it follows that $\widehat{\alpha}(I) = 1$. If $\frac{a}{b} > 1$, i.e., $a > b$, we have $\frac{a}{b} = \frac{b+(a-b)}{b}$. Then the result follows if we take $m = x_b x_{b+1} \cdots x_a = x_b x_{b+1} \cdots x_{b+(a-b)}$. $\square$

Remark 4.3.3. When $I = \text{sfBorel}(x_i x_{i+1} \cdots x_n)$, then I is generated by all the square-free monomials of degree $n-i+1$ in R. The Waldschmidt constants for these ideals were first computed in [BCG$^+$16, Theorem 7.5].

We now give a lower bound for the Waldschmidt constant of a square-free principal Borel ideal in terms of a smaller square-free principal Borel ideal.

Theorem 4.3.4. *Let* $m = x_{i_1} \cdots x_{i_s}$ *be a square-free monomial with* $T(m) = (t_0, t_1, \ldots, t_k)$ *and* $IT(m) = (i_{t_0}, i_{t_1}, \ldots, i_{t_k})$*, and suppose* $I = \mathrm{sfBorel}(m)$*. Suppose that* ℓ *is the smallest integer such that* $i_{t_{\ell+1}} < t_0 \leq i_{t_\ell}$*. Define* $\nu = i_{t_{\ell+1}} + 1$*. Then*

$$\widehat{\alpha}(I) \geq \widehat{\alpha}(\mathrm{sfBorel}(x_{i_1} \cdots x_{i_{t_{\ell+1}}})) + 1 + \frac{t_0 - \nu}{i_\nu - \nu + 1}.$$

Proof. By Lemma 4.2.1, we can assume we are working in the polynomial ring $K[x_1, \ldots, x_{i_{t_0}}]$. Consider the monomials

$$m_1 = x_{i_1} x_{i_2} \cdots x_{i_{t_{\ell+1}}} \in K[x_1, \ldots, x_{i_{t_{\ell+1}}}]$$

and

$$m_2 = x_{i_\nu} x_{i_\nu + 1} \cdots x_{i_{t_0}} \in K[x_\nu, \ldots, x_{i_{t_0}}].$$

Observe that while $m_1 m_2 | m$, m is not necessarily this product.

Let $I_1 = \mathrm{sfBorel}(m_1)$ and $I_2 = \mathrm{sfBorel}(m_2)$, in their respective rings, and furthermore, let $A(I_1)$ and $A(I_2)$ be the corresponding matrices of associated primes.

Take p to be the biggest integer such that $\nu \leq t_p$. Then

$$T(m_2) = (t_0 - \nu + 1, t_1 - \nu + 1, \ldots, t_p - \nu + 1)$$

and

$$IT(m_2) = (i_{t_0} - \nu + 1, i_{t_1} - \nu + 1, \ldots, i_{t_p} - \nu + 1).$$

We have $p \leq \ell$, then $i_{t_p} \geq t_0$. So by Theorem 4.3.1 we have

$$\widehat{\alpha}(I_2) = 1 + \frac{t_0 - \nu}{i_{t_p} - t_p + 1}.$$

Observe that $i_\nu - \nu + 1 = i_{t_p} - t_p + 1$, since $t_{p+1} < \nu \leq t_p$, meaning that $x_\nu \cdots x_{i_\nu}$ is a Borel movement of $x_{t_p} \cdots x_{i_{t_p}}$.

We claim that

$$\begin{bmatrix} A(I_1) & \mathbf{0} \\ \mathbf{0} & A(I_2) \end{bmatrix}$$

is a submatrix of $A(I)$, the matrix of associated primes of I.

First notice that any row of $A(I_1)$ is comes from a Borel movement of a corresponding $x_t x_{t+1} \cdots x_{i_t}$ for some $t \in T(m_1)$. By Theorem 4.1.6, these are also associated primes of I, implying that $[A(I_1)\ \mathbf{0}]$ is a submatrix of $A(I)$.

Now, for any $\ell + 1 > u > p$, $x_{t_u} \cdots x_{i_{t_u}}$ corresponds to a row of $A(I)$, and any of its Borel movements contain at least one x_j with $j < \nu$, otherwise $\nu \leq t_u$, contradicting the choice of p. For a row R, let $\mathrm{supp}(R)$ denote the set of indices of the non-zero entries of the row. Then for a row R in $[\mathbf{0}\ A(I_2)]$, there cannot exist a row R' of $A(I)$ with $\mathrm{supp}(R')$ $\subsetneq \mathrm{supp}(R)$. Thus any associated prime of I_2 can be viewed as an associated prime of I by Theorem 4.1.6. Thus $[\mathbf{0}\ A(I_2)]$ is a submatrix of $A(I)$ and by our choice of ν.

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be such that

$$A(I_1)^T \mathbf{y}_1 \leq \mathbf{1}, \quad \mathbf{1}^T \mathbf{y}_1 = \widehat{\alpha}(I_1) \ \text{ and } \ A(I_2)^T \mathbf{y}_2 \leq \mathbf{1}, \quad \mathbf{1}^T \mathbf{y}_2 = \widehat{\alpha}(I_2).$$

After permuting rows, we can assume that

$$A(I) = \begin{bmatrix} A(I_1) & \mathbf{0} \\ \mathbf{0} & A(I_2) \\ B_1 & B_2 \end{bmatrix}$$

where B_1, B_2 are some appropriately sized matrices. Set $\mathbf{z} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{0})$, where $\mathbf{0}$ is a vector of zeroes, where the number of zeroes in this vector are the same as the number of rows as B. Then $A(I)^T \mathbf{z} \leq \mathbf{1}$. Thus, by the dual version of Theorem 2.2.11, we have

$$\widehat{\alpha}(I) \geq \widehat{\alpha}(I_1) + \widehat{\alpha}(I_2) = \widehat{\alpha}(I_1) + 1 + \frac{t_0 - \nu}{i_\nu - \nu + 1}.$$

□

Theorem 4.3.4 reduces the problem of finding a lower bound on the principal square-free

Borel ideal $m = x_{i_1} \cdots x_{i_s}$ to finding a lower bound on the principal square-free Borel ideal of $x_{i_1} \cdots x_{i_{t_\ell+1}}$. Note one can now reapply Theorem 4.3.4 to this smaller ideal. At some point, the hypotheses of Theorem 4.3.1 will hold, which stops our recursive calculation.

This idea can be formally expressed as a formula, provided one is willing to introduce even further notation (involving further subscripts on our subscripts). Instead, we provide the following example in the hope of being more illuminating.

Example 4.3.5. Consider the monomial

$$m = x_3x_4x_5x_8x_9x_{10}x_{48}x_{49}x_{50}x_{98}x_{99}x_{100} \in K[x_1, \ldots, x_{100}]$$

and let $I = \text{sfBorel}(m)$. For this monomial $T(m) = (12, 9, 6, 3)$ and $IT(m) = (100, 50, 10, 5)$. Since $i_{t_2} = i_6 = 10 < t_0 = 12 < i_{t_1} = 50$, then $\nu = 11$ and Theorem 4.3.4 gives

$$\widehat{\alpha}(I) \geq \widehat{\alpha}(I_1) + 1 + \frac{12 - 11}{98 - 10 + 1} = \widehat{\alpha}(I_1) + \frac{90}{89}$$

where $I_1 = \text{sfBorel}(x_3x_4x_5x_8x_9x_{10}) = \text{sfBorel}(m_1)$. For this new monomial, we have $T(m_1) = (6, 3)$ and $IT(m_1) = (10, 5)$. Again using Theorem 4.3.4, we get

$$\widehat{\alpha}(I_1) \geq \widehat{\alpha}(I_2) + 1 + \frac{6 - 6}{10 - 6 + 1} = \widehat{\alpha}(I_2) + 1$$

where $I_2 = \text{sfBorel}(x_3x_4x_5)$. Then by Theorem 4.3.1 (or in this case, Corollary 4.3.2), we have $\widehat{\alpha}(I_2) = \frac{5}{3}$. Hence

$$\widehat{\alpha}(I) \geq \frac{5}{3} + 1 + \frac{90}{89} = \frac{982}{267}.$$

Note that if we apply the upper bound of Theorem 4.2.4 we get

$$3.6904 \approx \frac{155}{42} \geq \widehat{\alpha}(I) \geq \frac{982}{267} \approx 3.6779.$$

We finish this chapter with a result that allows us to make small changes to the generator of the square-free principal Borel without changing the Waldschmidt constant.

Theorem 4.3.6. *Let $I = \text{sfBorel}(x_{i_1} \cdots x_{i_{s-1}} x_{i_s})$ and $J = \text{sfBorel}(x_{i_1} \cdots x_{i_{s-1}} x_{i_s+r})$ for $r \in \mathbb{N}$. Then $\widehat{\alpha}(I) = \widehat{\alpha}(J)$.*

Proof. Let A be the matrix of associated primes of I and let $\mathbf{y} = (y_1, \ldots, y_{i_s})$ be an optimal solution to

$$\min\{\mathbf{1}^T\mathbf{x} \mid A\mathbf{x} \geq \mathbf{1}\}. \tag{4.3.1}$$

First consider $r = 1$ with $J = \mathrm{sfBorel}(x_{i_1} \cdots x_{i_{s-1}} x_{i_s+1})$. Let A' be the matrix of associated primes of J. Any element of $\mathrm{Ass}(J)$ not in $\mathrm{Ass}(I)$ includes both x_{i_s} and x_{i_s+1} as generators, so the columns of A' corresponding to x_{i_s} and x_{i_s+1} are identical. Let $\mathbf{y'} = (y'_1, \ldots, y'_{i_s+1})$ be an optimal solution to

$$\min\{\mathbf{1}^T\mathbf{x} \mid A'\mathbf{x} \geq \mathbf{1}\}. \tag{4.3.2}$$

and suppose for contradiction that $y'_1 + \cdots + y'_{i_s+1} = \widehat{\alpha}(J) < \widehat{\alpha}(I)$. But this means $(y'_1, \ldots, y'_{i_s} + y'_{i_s+1})$ is a feasible solution to (4.3.1), contradicting $\mathbf{y}$ being optimal and showing $\widehat{\alpha}(J) \geq \widehat{\alpha}(I)$. Observing that $(y_1, \ldots, y_{i_s}, 0)$ is a feasible solution to (4.3.2) gives $\widehat{\alpha}(I) \geq \widehat{\alpha}(J)$. This shows $\widehat{\alpha}(I) = \widehat{\alpha}(J)$, and an inductive argument gives the result for $r \in \mathbb{N}$. □

Chapter 5

Further Questions

We conclude this thesis with an example and questions for further investigation.

5.1 Waldschmidt constant in three variables with prescribed denominator

Example 5.1.1. Consider the following monomial ideal I in $R = K[x_1, x_2, x_3]$

$$I = \langle x_1^a, x_2^b \rangle \cap \langle x_1^c, x_3^d \rangle$$

and assume $a, b, c, d > 0$. Since I has no embedded primes and is presented in its irreducible form by Theorem 2.3.4, solving for the Waldschmidt constant $\widehat{\alpha}(I)$ amounts to solving the linear optimization problem:

$$\text{Solve } \min\{\mathbf{1}^T\mathbf{y} \mid A\mathbf{y} \geq \mathbf{1}, \mathbf{y} \geq \mathbf{0}\} \text{ for} \tag{5.1.1}$$

$$A = \begin{bmatrix} \frac{1}{a} & \frac{1}{b} & 0 \\ \frac{1}{c} & 0 & \frac{1}{d} \end{bmatrix},$$

where $\mathbf{y} = [\mathbf{y}_1 \ \mathbf{y}_2 \ \mathbf{y}_3]^T$ is a vector of variables.

The vertices of the symbolic polyhedron of I are the feasible intersections of the following

5 hyperplanes:

$$\begin{aligned} h_1 = &\ \tfrac{1}{a}\mathbf{y}_1 + \tfrac{1}{b}\mathbf{y}_2 = 1 \\ h_2 = &\ \tfrac{1}{c}\mathbf{y}_1 + \tfrac{1}{d}\mathbf{y}_3 = 1 \end{aligned}$$

$$\mathbf{y}_1 = 0, \mathbf{y}_2 = 0, \mathbf{y}_3 = 0.$$

Setting $H = \{1,2\}$ and $Y = \{1,2,3\}$ and following the procedure and notation of Section 3.1 (e.g., $\mathbf{z}^{12|1}$ is the intersection of the hyperplanes $h_1 = 1, h_2 = 1$ and $\mathbf{y}_1 = 0$). We get the following 10 intersections:

$$\begin{aligned}
\mathbf{z}^{12|1} &= (b,d,0), & \mathbf{z}^{12|2} &= (a, 0, \tfrac{d(c-a)}{c}), \\
\mathbf{z}^{12|3} &= (c, \tfrac{b(a-c)}{a}, 0), & \mathbf{z}^{1|12} &= \text{never feasible}, \\
\mathbf{z}^{1|13} &= \text{never feasible}, & \mathbf{z}^{1|23} &= (a,0,0), \\
\mathbf{z}^{2|12} &= \text{never feasible}, & \mathbf{z}^{2|13} &= \text{never feasible}, \\
\mathbf{z}^{2|23} &= (c,0,0), & \mathbf{z}^{|123} &= \text{never feasible}.
\end{aligned}$$

Now suppose $d > a > b = c = 1$. Observe then that $\mathbf{z}^{12|2}$ is infeasible since the second coordinate is negative. The intersection $\mathbf{z}^{2|23}$ is infeasible since it fails the inequality $h_1 \geq 1$.

Consider the intersection $\mathbf{z}^{12|3} = (1, \frac{(a-1)}{a}, 0)$ with coordinate sum $\mathbf{1}^T\mathbf{z}^{12|3} = 1 + \frac{(a-1)}{a} = \frac{a+(a-1)}{a}$. Observe that $\mathbf{1}^T\mathbf{z}^{12|3} = \frac{a+(a-1)}{a} < \mathbf{1}^T\mathbf{z}^{1|23} = a$ and $\mathbf{1}^T\mathbf{z}^{12|3} = \frac{a+(a-1)}{a} < \mathbf{1}^T\mathbf{z}^{12|1} = b+d$. This makes $\mathbf{z}^{12|3}$ the minimizer to the linear optimization problem (5.1.1). So we have $\widehat{\alpha}(I) = \frac{a+(a-1)}{a}$. This means we can always construct a monomial ideal I in three variables with a prescribed denominator a in its Waldschmidt constant.

The above example shows we can construct a monomial ideal I in $R = K[x_1, x_2, x_3]$ with Waldschmidt constant equal to any prescribed denominator, but with limitations on the numerator. By Corollary 3.1.2 we can construct a monomial ideal I in $R = K[x_1, x_2, x_3]$ with Waldschmidt constant equal to $\frac{q}{2} > 1$ for prescribed $q > 2$. This leads to the following question:

Question 5.1.2. Is it possible to construct a monomial ideal I in $R = K[x_1, x_2, x_3]$ such that $\widehat{\alpha}(I) = \frac{q}{p}$ for any $\frac{q}{p} \geq 1$? If not, which values are forbidden?

5.2 Efficiently constructing a monomial ideal with a prescribed Waldschmidt constant

We have developed two techniques for creating monomial ideals with a prescribed Waldschmidt constant. The Section 3.3 technique (Corollary 3.3.4) works for fractions $\frac{q}{p}$ that satisfy certain conditions on the relative size of q and p, and the value of $q \mod p+1$. The Chapter 4 technique (Corollary 4.3.2) of using square-free Borel principal ideal allows us to construct a monomial ideal with any rational prescribed Waldschmidt constant $\frac{q}{p} \geq 1$. The example of the square-free Borel ideal which approximates π in the introduction (see Equation 1.2.2) would not work with Chapter 3's technique.

Suppose we want to construct a monomial ideal I with $\widehat{\alpha}(I) = \frac{101}{2}$.

By Corollary 3.1.2 the following ideal in $R = K[x_1, x_2, x_3]$

$$I = \langle x_2, x_3 \rangle^{34} \cap \langle x_1, x_3 \rangle^{34} \cap \langle x_1, x_2 \rangle^{33}$$

has $\widehat{\alpha}(I) = \frac{101}{2}$.

Instead, if we construct a square-free principal Borel ideal I with using (Corollary 4.3.2), we get the ideal $I = \text{sfBorel}(x_2 \cdots x_{101})$ which also has $\widehat{\alpha}(I) = \frac{101}{2}$. But notice this ideal is in $R = K[x_1, \ldots, x_{101}]$. So the technique of Section 3.3 is more efficient in that it requires only 3 variables instead of 101.

We conclude this thesis with the following question:

Question 5.2.1. For a prescribed fraction $\frac{q}{p} \geq 1$, what is the monomial ideal I that most "efficiently" (e.g., in terms of minimal number of variables or associated primes) attains the Waldschmidt constant $\widehat{\alpha}(I) = \frac{q}{p}$?

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