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EXTENSION SPACES

FUNCTIONS, UNIFORMITIES AND EXTENSION SPACES

By

JOHN CHRISTOPHER TAYLOR, M.A.

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AUTHOR: John Christopher Taylor, B.Sc. (Acadia University)

M.A. (Queen's University)

SUPERVISOR: Professor B. Banaschewski

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SCOPE AND CONTENTS: This thesis is concerned with the extensions of a completely regular space and in particular with those that can be obtained by means of continuous real-valued functions or compatible uniformities. The main innovation is the concept of a process which systematizes the procedures for extending topological spaces and makes the proofs more formal. In addition processes are shown to be of interest in their own right.

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INTRODUCTION

The specific aim of this thesis is to consider the totality of extensions of a completely regular space E, especially those that can be defined by means of continuous real-valued functions on E and by means of the uniform structures of E. In addition, on a more abstract level, it is concerned with general methods or processes for the construction of topological spaces. These processes are used to construct extensions.

Let ξ be a particular kind of structure that can be attached to any set E (e.g. a topology for E) and let Ξ be a category with objects the pairs (E,X), where E is a set and X is a ξ-structure for E, and a suitable class of functions $\alpha: E \longrightarrow E'$ that are (X,X') ξ-homorphisms (e.g. the continuous functions or the open functions). If Σ is the category of topological spaces (i.e. objects topological spaces $(E,\underline{O})$ and maps the continuous functions) <u>a ξ-process $\mathcal{P}$ on a subcategory Ξ_p of Ξ consists of a covariant functor $P: \Xi_o \longrightarrow \Xi$ and a family $(p_X)_{(E,X) \text{ in } \Xi}$ of functions $p_X: E \longrightarrow P(E,X)$ such that:</u>

<u>(ξP_1) $p_X E$ is dense in $P(E,X)$; and</u>

<u>(ξP_2) if α is a mapping of Ξ_o in $\mathrm{Hom}((E,X),(E',X'))$ then</u>

$$\underline{P(\alpha) \circ p_X = p_{X'} \circ \alpha .}$$

Three kinds of process are used in this thesis: function processes, where X is a collection S of real-valued functions on E; uniform processes, where X is a uniformity for E; and topological processes, where X is a topology for E. The specific applications of each of these kinds of processes are briefly discussed in the following chapter by chapter summary of the thesis.

Chapter one is devoted to a discussion of our function processes that correspond to four descriptions of Hewitt's υ-extension of a completely regular space E, and to an application of the results of this discussion to the problem <u>(EA): characterize the subalgebras S of C_E for which $S = C_X|E$, where X is an extension of E</u> (C_E denotes the algebra of continuous real-valued functions on E).

In section one function processes are defined using the specific category Φ of objects (E,S), where S is a collection of real-valued functions on E, and with $\mathrm{Hom}((E,S),(E',S'))$ the set of all functions $\alpha: E \longrightarrow E'$ such that $S' \circ \alpha$ is contained in S. The concepts of a homomorphism and of an isomorphism of one function process into another are introduced as a restricted type of natural equivalence and three general kinds of invariant properties discussed. The first type is a topological property. If (t) is a topological property a function process $\mathcal{P}$ on a subcategory Ψ is said to be a (t)-process when each space $P(E,S)$, (E,S) in Ψ has property (t). The second kind deals with the 'extension' of the functions f in S as continuous functions f_P to the space $P(E,S)$ and with properties of the resulting collection S_P of continuous functions on $P(E,S)$. The third and last type of invariant property relates properties of the maps α in $\mathfrak{X}$ to properties of the continuous functions $P(\alpha)$. In addition another property of function processes is introduced which is not, strictly speaking, an invariant. It is the property of idempotence which is possessed by a process $\mathcal{P}$ when it can be iterated to produce a new process $\mathcal{P} \circ \mathcal{P}$ isomorphic to $\mathcal{P}$.

Section two is devoted to a discussion of the idempotent Tychonoff process $\mathfrak{J}$. It associates with any collection S of real-valued functions on a set E the subspace obtained by taking the closure of the natural image of E in $\prod_{f \text{ in } S} R_f$ under the evaluation mapping. It is shown to be characterized (up to isomorphism) by four invariant properties of the first type (theorem 5).

The third section discusses S-completely regular filters on a set E, where S is a collection of real-valued functions on E. The results of this section are used in section four to define the compact process $\mathfrak{M}$, which associates with each (E,S) the space of maximal S-completely regular filters on E. The subprocess $\mathcal{F}$ of $\mathfrak{M}$ is defined by considering for each (E,S) the subspace of $M(E,S)$ consisting of the $\underline{U}(S)$-Cauchy filters. It is shown to be isomorphic to $\mathfrak{J}$ (theorem 10).

In section five the first of two 'algebraic' processes is defined. This is the process $\mathcal{H}$ which associates with any real unitary function algebra the set of real-valued unitary algebra homomorphisms together with the Zariski topology. Given additional conditions on the function algebra it is shown (theorem 12) that $\mathcal{H}$ and $\mathfrak{J}$ are isomorphic processes on a subcategory of $\mathfrak{E}$.

Section six is devoted to a consideration of the process $\mathcal{L}$. It associates with each translation lattice of functions that contains the constants and is closed under multiplication by (-1) the space of translation lattice homomorphisms which map the zero function to zero and which commute with (-1). This process is isomorphic to $\mathfrak{J}$ on a subcategory of $\mathfrak{E}$ (theorem 13). The section concludes with the result (theorem 14) that on a suitable subcategory of $\mathfrak{E}$ the four processes $\mathfrak{J}$, $\mathfrak{F}$, $\mathcal{H}$ and $\mathcal{L}$ are isomorphic and satisfy all the invariant properties introduced in section one.

In section seven problem (EA) is discussed. It is clear, in view of Hewitt's υ-extension of completely regular spaces, that it is sufficient to consider the problem for the Q-extensions of E. As an application of theorem 14 it is shown that (up to isomorphism) any one of the processes $\mathfrak{J}$, $\mathfrak{F}$, $\mathcal{H}$ and $\mathcal{L}$ provides all the Q-extensions of E. A preliminary solution to (EA) is then stated (theorem 18). It shows that (EA) is equivalent to the problem .

(AC): characterize C_E as an algebra of continuous real-valued functions on E.

In the last section of this chapter it is shown that problem (AC) may be solved by a solution to a third problem, problem

(BC): characterize C_E^* as an algebra of continuous real-valued functions on E. Theorem 20 states a solution to (BC) which is then used to solve in turn (AC) and (EA) (theorems 21 and 22)

The second chapter discusses the extensions of a completely regular space E by making use of the well known fact that the quotient space of a compact space defined by a closed equivalence relation is also compact.

Section one shows that there is a Galois connection between the set of closed equivalence relations $\underline{r}$ on a compact space K and the uniformly closed unitary subalgebras $\underline{a}$ of C_K (theorem 1). It is shown that every continuous image of a compact space is essentially a quotient space of K defined by a closed equivalence relation on K.

In section two these results are applied to the Stone-Cech compactification $(\beta E, i)$ of E. The uniformly closed unitary subalgebras $\underline{a}$ of C_E^* define the compact spaces K into which E may be continuously mapped with dense image as the quotient spaces $\beta E/\underline{r}(\underline{a}) = K(\underline{a})$, where $\underline{r}(\underline{a})$ is the closed equivalence relation on βE associated with $\underline{a}$. When $\underline{a}$ contains $\underline{a}'$ it is shown that there is a canonical map $\pi(\underline{a}',\underline{a}):K(\underline{a}) \longrightarrow K(\underline{a}')$ such that $\pi(\underline{a}',\underline{a}) \circ \pi(\underline{a}) = \pi(\underline{a}')$, where $\pi(\underline{a}):\beta E \longrightarrow K(\underline{a})$ is the natural mapping. The pair $(K(\underline{a}), \pi(\underline{a}) \circ i)$ is an extension of E iff the weak topology $\underline{O}(E,\underline{a})$ is the topology of E . Such algebras are said to be characteristic and are denoted by $\underline{c}$. The extensions $(K(\underline{c}), \pi(\underline{c}) \circ i)$ are a representative collection of compact extensions of E in 1 - 1 correspondence with the characteristic algebras $\underline{c}$. It is shown that E is locally compact iff E has a smallest characteristic algebra (theorem 5) . In addition if $\beta E \cap \complement iE$ contains two points the characteristic algebras of E determine E (to within homeomorphism) (theorem 6) .

Section three considers the translation lattices $\mathcal{L}(\underline{a}) = [f \mid \text{for all } \lambda \geqslant 0 \ (f \cap \lambda) \cup (-\lambda) \text{ is in } \underline{a}]$ associated with each uniformly closed unitary subalgebra $\underline{a}$ of C^*_E . They are characterized internally by theorem 8 and by means of the realtions $\underline{r}(\underline{a})$ in theorem 10 .

The fourth section considers the set of extensions of E obtained taking the subspaces X of $K(\underline{c})$ containing $(\pi(\underline{c}) \circ i)E$. Every extension is isomorphic to one of them (theorem 11) and a non-redundant subset may be chosen. Another method of obtaining extensions by means of continuous functions is defined using the compact extensions $(K(\underline{c}), \pi(\underline{c}) \circ i)$, characteristic. If S is a subset of C_E then $\mathcal{E}_{\underline{c}}(S)$ is the largest subset of $K(\underline{c})$ to which each function in S has a continuous real-valued extension. After a discussion of some obvious properties of the operators $\mathcal{E}_{\underline{c}}$ it is shown (theorem 16) that if $\underline{c} \subseteq S \subseteq \mathcal{L}(\underline{c})$ then the extensions $(\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i)$ and $(T(E,S), t_S)$ are isomorphic. As a result every Q-extension of E may be obtained using the operators $\mathcal{E}_{\underline{c}}$. In particular Hewitt's -extension of E is $(\mathcal{E}_{\underline{C}_E^*}(C_E), i) = (\upsilon E, i)$.

In section five two quasi-orders for extensions of E are defined. If $X_1 = (X_1, j_1)$ and $X_2 = (X_2, j_2)$ are extensions of E(i.e. $j_i : E \longrightarrow X_i$ is an embedding in the completely regular space X_i) then $\leq$ and $\trianglelefteq$ are defined by setting $X_1 \quad X_2$ if there is a continuous function $\gamma_{21} : X_1 \longrightarrow X_2$ such that $\gamma_{21} \circ j_1 = j_2$ and $X_1 \leq X_2$ if $X_1 \trianglelefteq X_2$ and γ_{21} is an embedding. Necessary and sufficient conditions are obtained in theorem 18 (theorem 20) on a Q-extension (Y,k) of E in order that it be $\leq$ ($\trianglelefteq$) a given extension (X,j) of E. These results are extended in section six to arbitrary extensions (theorems 22 and 23) by making use of the canonical mappings $\pi(\underline{c}', \underline{c})$.

The seventh section discusses two Q-extensions of E associated with a given characteristic algebra $\underline{c}$ of E. The first one may be constructed by process $\mathcal{L}$ applied to $(E, \mathcal{L}(\underline{c}))$ (theorem 24), whereas the second is defined to be the intersection of all the Q-subspaces of $K(\underline{c})$ containing $(\pi(\underline{c}) \circ i)E$.

Section eight returns to problem (EA) of chapter one and provides another solution by giving (in theorem 27) an explicit algebraic construction of the extension algebras of E in terms of the characteristic algebras $\underline{c}$ of E and the extension algebras that contain C_E^*.

The chapter concludes with a discussion of additional properties of the lattices $\mathcal{L}(\underline{a})$ and a conjecture concerning them which, if correct, solves (AC) for Q-spaces.*) It is shown (theorem 29) that a lattice $\mathcal{L}(\underline{a})$ is closed under addition iff it is closed under multiplication which is the case iff it is closed under continuous composition. The conjecture states that, for Q-spaces, C_E is the only collection of continuous functions on E satisfying the conditions of theorem 8 which is closed under addition and that contains unbounded functions when E is not compact. In the case of locally compact spaces countable at infinity the conjecture holds (lemma 13).

Chapter three is analoguous to the first four sections of chapter one. The analogy is obtained by considering uniformities $\underline{U}$ in place of collections S of real-valued functions and compatible uniformities $\underline{U}$ on topological spaces (i.e. the $\underline{U}$-uniform topology is coarser than the topology of the space) in place of collections of continuous functions.

The first section is a brief discussion of some results for compatible uniformities that are used later in this chapter

*) See Erratum p. 260.

and in chapter four. The analogue of the fact that a continuous real-valued function on a topological space is determined by its value on a dense subset states (theorem 3) that a compatible uniformity is determined by its restriction to a dense subset.

Section two is the analogue of section one in chapter one with the category Φ replaced by the category Υ . This category has the pairs $(E,\underline{U})$, where $\underline{U}$ is a uniformity for E, as objects and $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ the $(\underline{U},\underline{U}')$-uniformly continuous functions $\alpha:E \longrightarrow E'$. In addition to the definition of isomorphism and the discussion of invariants, this section also introduces a natural (covariant) functor $U:\Phi \longrightarrow \Upsilon$ If (E,S) is in Φ then $U(E,S) = (E,\underline{U}(S))$ and if α is a map of Φ $U(\alpha) = \alpha$. By means of this functor every uniform process $\mathcal{P}$ on Υ induces a function process $\mathcal{P}_U$ on Φ .

In section three the well known construction of the separated space associated with a uniform space $(E,\underline{U})$ is shown to provide a uniform process $\mathcal{S}$. This process is defined (up to isomorphism) by a 'universal' property (theorem 6) . It is also shown that the separated space associated with $(E,\underline{U})$ is complete iff $(E,\underline{U})$ is complete.

The fourth section discusses the space of Cauchy filters associated with a uniform space and shows that it too defines a uniform process on Υ .

To correspond to the section on S-completely regular, the fifth section deals with $\underline{U}$-completely regular filters on a set E and obtains analogous results.

The analogue of process $\mathfrak{M}$ is the uniform process $\mathfrak{B}$ which is defined in section six by associating with each $(E,\underline{U})$ the space of maximal $\underline{U}$-completely regular filters on E . It is shown that this process associates with each separated uniform space its Samuel compactification and that it is defined (up to isomorphism) as a compact process with a 'universal' property (theorem 12) . The induced function process $\mathfrak{B}_J$ is not isomorphic to $\mathfrak{M}$.

In section seven process $\mathfrak{G}$ is defined as a subprocess of $\mathfrak{B}$ by associating with each $(E,\underline{U})$ the subspace of $B(E,\underline{U})$ consisting of the $\underline{U}$-Cauchy filters in $B(E,\underline{U})$. It is shown to have a universal property and (analogous to theorem 5 in chapter one) to be defined up to isomorphism by four invariant properties of uniform processes (theorem 16) . One immediate consequence of this theorem is the well known result that the separated space associated with the Cauchy filter space defined by $(E,\underline{U})$ is isomorphic to the separated space associated with the Cauchy filter space of the separated space defined by $(E,\underline{U})$. In addition the well known theorem on the extension of uniformly continuous functions (theorem 16) is proved in a manner which is formally identical with the proofs of theorem 13 and 20 in chapter two. This suggests a basic 'extension' theorem exists for every process.

The fourth chapter is a brief discussion of the extensions of a completely regular space E that may be obtained by completing E in its various structures. It is roughly analogous to chapter two.

The first section shows that all the topologically complete extensions (X,j) of E (i.e. X is complete in its finest structure) may be obtained by applying $\mathcal{G}$ to the objects $(E,\underline{U})$ where $\underline{U}$ is a structure of E. The totally bounded structures of E define its compact extensions and are in 1 - 1 correspondence with characteristic algebras of E (theorem 2). An equivalence relation is defined on the set of structures of E by identifying $\underline{U}_1$ and $\underline{U}_2$ if $G(E,\underline{U}_1) = G(E,\underline{U}_2)$. While it is shown that each equivalence class has a finest member the problem of characterizing such extremal structures is left open.

In the second section the quasi-orders of chapter two are considered when restricted to the topologically complete extensions of E. Necessary and sufficient conditions are obtained in order that two of these extensions be suitably related by these orders (theorems 4 and 5). These results suggest two basic types of problems for topologically complete extensions. An example of the second type is stated and solved (theorem 6). It states that a subspace X of $K(\underline{c})$ may be obtained by completing E in $\underline{U}(S)$, where $\underline{c} \subseteq S \subseteq \mathcal{X}(\underline{c})$, iff X is a union of G_δ sets in $K(\underline{c})$.

Section three is an exposition of well known results on the collection of $\underline{U}$-uniformly continuous functions on a set E.

The fourth section discusses very briefly, in terms of process $\mathcal{G}$, three uniformities associated with a given uniformity $\underline{U}$ on a set E. They are respectively the finest totally bounded, σ-bounded, and function uniformities coarser than $\underline{U}$.

The chapter concludes with some results which characterize those collections $C(\underline{U})$ of all $\underline{U}$-uniformly continuous real-valued functions on a completely regular space E, $\underline{U}$ a structure of E, which are extension algebras of E. A connection with the conjecture of chapter two is established.

The last chapter, chapter five, discusses some known constructions of compact spaces in the framework of topological processes.

Section one introduces and defines the concept of a topological process on a subcategory Σ_0 of the category Σ of topological spaces. Unlike the sections on function and uniform processes. this section contains no discussion of invariants. This is because they are not used to discuss the construction of compact spaces and because it is more or less obvious how an analogous discussion should proceed. Continuous topological processes are defined and a compactification on Σ_0 is defined to be a continuous compact topological process on the subcategory Σ_0. Natural functors $A: \Sigma_0 \longrightarrow \Phi$ are defined and used to obtain topological processes from function processes. It is proved that every continuous C-process $\mathcal{P}$ on a subcategory Σ_0 is isomorphic to one obtained by means of a natural functor A and the function process $\mathcal{H}$ (theorem 1). As a corollary to this theorem it is shown that when $\mathcal{P}$ is a compactification the associated natural functor A is uniquely defined by requiring $A(\underline{O})$ to be a uniformly closed unitary subalgebra of $C^{*}_{(E,\underline{O})}$. This raises the basic problem: given a compactification on Σ_0 determine the associated natural functor.

The remainder of the chapter is devoted to a discussion of this problem for the following compactifications: the Stone-Cech compactification; the Alexandroff one-point compactification; Banaschewski's zero-dimensional compactification; Freudenthal's rim-compact process; and Freudenthal's $\mathcal{L}$-compactification.

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CHAPTER ONE

THE CONSTRUCTION OF TOPOLOGICAL SPACES FROM COLLECTIONS OF FUNCTIONS

§1. Function processes. Consider the category Φ with objects the pairs (E,S) where E is a non-void set and S is a non-void collection of real-valued functions on E, and with $\mathrm{Hom}((E,S),(E',S'))$ the set of all functions $\alpha:E\longrightarrow E'$ such that $S' \circ \alpha = [f' \circ \alpha | f' \text{ in } S']$ is contained in S. Let Σ denote the category with the topological spaces $X,Y,\ldots$ as objects and $\mathrm{Hom}(X,Y)$ the set of all continuous functions $\gamma:X\longrightarrow Y$.

Definition 1. A function process $\mathcal{P}$ on a subcategory Ψ of Φ consists of a covariant functor $P:\Psi\longrightarrow\Sigma$ and a family $(p_S)_{(E,S) \text{ in } \Psi}$ of functions $p_S:E\longrightarrow P(E,S)$ such that:

(FP_1) $p_SE = [p_Sx | x \text{ in } E]$ is dense in the space $P(E,S)$; and

(FP_2) if α is a mapping of Ψ in $\mathrm{Hom}((E,S),(E',S'))$ then

$P(\alpha) \circ p_S = p_{S'} \circ \alpha$.

Remarks. 1. Categories and functors are considered from the 'naive' standpoint of Godement [1]. A subcategory is obtained by restricting both objects and mappings, although in this thesis only subcategories obtained by restricting objects are considered.

2. The following commutative diagram illustrates (FP_2):

$$\begin{tikzcd} (E,S) \arrow[r, "\alpha"] \arrow[d, "p_S"'] & (E',S') \arrow[d, "p_{S'}"] \\ P(E,S) \arrow[r, "P(\alpha)"] & P(E',S') \end{tikzcd}$$

3. The type of function process defined here could more properly be called a real function process since the functions involved are always real-valued. By changing the range of the functions different types of function processes could be considered.

Examples of function processes on $\mathfrak{E}$

1. Define $P(E,S)$ to be the discrete space $[E]$ with $P(\alpha)E = E$ and $p_S x = E$ for all x in E.

2. Define $P(E,S)$ to be the discrete space E with $P(\alpha) = \alpha$ and $p_S x = x$ for all x in E.

3. Define $P(E,S)$ to be the space $(E, \underline{O}(E,S))$ where $\underline{O}(E,S)$ is the weak topology defined by S i.e. the coarsest topology for E with respect to which all the functions in S are continuous. Define $P(\alpha) = \alpha$ and $p_S x = x$ for all x in E.

4. Let $\underline{r}(S)$ be the equivalence relation defined on E by setting $x\,\underline{r}(S)y$ if $fx = fy$ for all f in S. Define $P(E,S)$ to be the quotient space of $(E,\underline{O}(E,S))$ by $\underline{r}(S)$. Define $P(\alpha)$ to be the unique continuous function $\bar{\alpha}$ such that $\bar{\alpha} \circ \pi(S) = \pi(S') \circ \alpha$, where $\pi(S): E \longrightarrow E/\underline{r}(S)$ is the natural mapping. Define p_S to be $\pi(S)$.

In general if $\mathcal{P}$ is a function process on a subcategory $\mathfrak{X}$ of $\mathfrak{T}$ and if $\mathfrak{X}_o$ is a subcategory of $\mathfrak{X}$ then $\mathcal{P}$ defines a process $\mathcal{P}|\mathfrak{X}_o$ on $\mathfrak{X}_o$ – the restriction of $\mathcal{P}$ to $\mathfrak{X}_o$. If (E,S) is in $\mathfrak{X}_o$ then $(P|\mathfrak{X}_o)(E,S) = P(E,S)$ and $(p|\mathfrak{X}_o)_S = p_S$. If α is a mapping of $\mathfrak{X}_o$ then $(P|\mathfrak{X}_o)(\alpha) = P(\alpha)$.

<u>General Remark</u>. In algebra there exist many constructions which are closely related to the construction of topological spaces by means of processes. Consider for example the constructions of free (left) R-modules, R a given ring. One method of construction corresponds to each set E the (left) module $M(E)$ of R-valued functions for E which take on the value 0 except at a finite number of points of E. The set E may be mapped into $M(E)$ by the function m_E defined by setting $m_E x = f_x$, where $f_x x = 1$ and $f_x y = 0$ if $y \neq x$. The pair $(M(E),m_E)$ has the property that if M is any R-module and $m:E \longrightarrow M$ is a function then there exists a unique R-linear function $\gamma:M(E) \longrightarrow M$ such that $\gamma \circ m_E = m$ as shown in the diagram

$$\begin{tikzcd} E \arrow[r, "m_E"] \arrow[dr, "m"'] & M(E) \arrow[d, "\gamma"] \\ & M \end{tikzcd}$$

This 'universal' property implies that if $\alpha:E \longrightarrow E'$ is a function then there is a unique R-linear function $M(\alpha):M(E) \longrightarrow M(E')$ such that $M(\alpha) \circ m_E = m_{E'} \circ \alpha$. The uniqueness part of the 'universal' property is equivalent to the fact that $m_E E$ is a set of module generators for $M(E)$. It follows that this method of constructing free R-modules may be considered to consist of a covariant functor M (easily established from the uniqueness condition) on the category of sets valued in the category of R-modules together with a family $(m_E)_E$ of functions $m_E : E \longrightarrow M(E)$ such that:

(1) $m_E E$ is a set of module generators for $M(E)$ (which corresponds to (FP_1)); and

(2) if $\alpha : E \longrightarrow E'$ is a function then $M(\alpha) \circ m_E = m_{E'} \circ \alpha$ (this corresponds to (FP_2)).

There are many other algebraic constructions which can be presented in this setting, including all the constructions of algebraic entities defined by 'universal' properties such as the tensor product of a right R-module with a left R-module, the tensor algebra on an R-module (R-commutative) and so on. For other examples see Chevalley [2].

To return to function processes, consider two processes $\mathcal{P}$ and $\mathcal{Q}$ on a subcategory $\mathfrak{X}$ of $\mathfrak{E}$. They may be compared by means of the restricted type of natural transformation defined in

Definition 2. A homomorphism γ of the process $\mathcal{P}$ on $\mathfrak{X}$ into the process $\mathcal{Q}$ on $\mathfrak{X}$ consists of a family $(\gamma(E,S))_{(E,S) \text{ in } \mathfrak{X}}$ of continuous functions $\gamma(E,S) : P(E,S) \longrightarrow Q(E,S)$ such that:

(1) if α is a mapping of $\mathfrak{X}$ in $\mathrm{Hom}((E,S), (E',S'))$

then

$\gamma(E',S') \circ P(\alpha) = Q(\alpha) \circ \gamma(E,S)$, and

(2) for each (E,S) in Ψ $\gamma(E,S) \circ p_S = q_S$.

A homomorphism γ is called an isomorphism if each $\gamma(E,S)$ is a homeomorphism. When an isomorphism γ of $\mathcal{P}$ into $\mathcal{Q}$ exists they are said to be isomorphic processes on Ψ .

Remarks. The first condition shows that a homomorphism is a natural transformation. The second imposes an additional condition on the natural transformations. These two conditions together with (FP_2) state that the following diagram is commutative:

$$\begin{tikzcd}
(E,S) \arrow[rr, "p_S"] \arrow[dr, "q_S"'] \arrow[dd, "\alpha"'] & & P(E,S) \arrow[dl, "\gamma(E,S)"'] \arrow[dd, "P(\alpha)"] \\
& Q(E,S) \arrow[dd, "Q(\alpha)"] & \\
(E',S') \arrow[rr, "p_{S'}"] \arrow[dr, "q_{S'}"'] & & P(E',S') \arrow[dl, "\gamma(E',S')"] \\
& Q(E',S') &
\end{tikzcd}$$

The relation of isomorphism between processes on a fixed subcategory Ψ of Φ is an equivalence relation. It is clearly reflexive and symmetric. Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be three processes on Ψ and let γ and δ be homomorphisms of $\mathcal{P}$ into $\mathcal{Q}$ and of $\mathcal{Q}$ into $\mathcal{R}$. Then if (E,S) and (E',S') are two objects of Ψ and α is a mapping of Ψ in Hom $((E,S), (E',S'))$ it follows that:

(1) $\delta(E,S) \circ \gamma(E,S) : P(E,S) \longrightarrow R(E,S)$ is a continuous function;

(2) $\delta(E,S) \circ \gamma(E,S) \circ p_S = \delta(E,S) \circ q_S = r_S$; and

$$(3)\ \delta(E',S') \circ \gamma(E',S') \circ P(\alpha) = \delta(E',S') \circ Q(\alpha) \circ \gamma(E,S)$$
$$= R(\alpha) \circ \delta(E,S) \circ \gamma(E,S).$$

This shows that δ and γ define a homomorphism of $\mathcal{P}$ into $\mathcal{R}$ which is an isomorphism if both δ and γ are isomorphisms. In particular the relation of isomorphism is transitive.

The introduction of the concept of isomorphism for processes on a fixed subcategory Σ leads to an investigation of those properties of processes that are invariant under isomorphism i.e., those properties possessed by both or neither of a pair of isomorphic processes. Three basic types of invariant properties are of interest.

The first and simplest type of invariant property is a topological property. <u>If (t) is a topological property and $\mathcal{P}$ is a process on Σ such that all the spaces $P(E,S)$, (E,S) in Σ, have property (t) then $\mathcal{P}$ is said to be a (t)-process on Σ or to satisfy property (t) on Σ</u>. Since topological properties are precisely those invariant under homeomorphisms it is clear that any topological property of a process is invariant under isomorphism. For example a process $\mathcal{P}$ on Σ is a Hausdorff process on Σ if each space $P(E,S)$, (E,S) in Σ is a Hausdorff space. In the case of Hausdorff processes condition (1) in definition 2 is a consequence of condition (2) as stated in

<u>Lemma 1. Let $\mathcal{P}$ and $\mathcal{Q}$ be processes on Σ and let $(\gamma(E,S))_{(E,S)\text{ in }\Sigma}$ be a family of continuous functions $\gamma(E,S):P(E,S) \longrightarrow Q(E,S)$</u>

<u>such that $\gamma(E,S) \circ p_S = q_S$. If $\mathcal{Q}$ is a Hausdorff process this family defines a homomorphism γ of $\mathcal{P}$ into $\mathcal{Q}$</u>.

Proof: From (FP_2) and the hypothesis it follows that

$$\gamma(E',S') \circ P(\alpha) \circ p_S = \gamma(E',S') \circ p_{S'} \circ \alpha = q_{S'} \circ \alpha$$

$= Q(\alpha) \circ q_S = Q(\alpha) \circ \gamma(E,S) \circ p_S$. Since (FP_1) states that p_SE is dense in $P(E,S)$ it follows that $\gamma(E',S') \circ P(\alpha) = Q(\alpha) \circ \gamma(E,S)$ when $Q(E',S')$ is Hausdorff.

The second type of invariant property connects the functions in S with continuous functions on $P(E,S)$. Let $\mathcal{P}$ be a process on $\mathfrak{X}$ and if (E,S) is an object of $\mathfrak{X}$ let $C_{P(E,S)}$ denote the algebra of continuous real-valued functions on the space $P(E,S)$. Since p_SE is dense in $P(E,S)$ the correspondence $g \longrightarrow g \circ p_S$ of $C_{P(E,S)} \longrightarrow C_{P(E,S)} \circ p_S$ is 1 - 1. It is even an algebra homomorphism when the operations in $C_{P(E,S)} \circ p_S$ are defined pointwise. In general, as for example in example 1 there is no connection between S and $C_{P(E,S)} \circ p_S$. In the case however of examples 2, 3 and 4 it can be seen that S is contained in $C_{P(E,S)} \circ p_S$ for each object in $\mathfrak{X}$.

It is clear that a process $\mathcal{P}$ on $\mathfrak{X}$ is such that S is contained in $C_{P(E,S)} \circ p_S$ for each object of $\mathfrak{X}$ iff $\mathcal{P}$ satisfies condition

(FP_3) <u>for each object (E,S) of $\mathfrak{X}$ every function f in S defines a continuous function f_P on $P(E,S)$ with $f_P \circ p_S = f$.</u>

<u>Property (FP_3) is invariant under isomorphism</u>. Let $\mathcal{P}$ and $\mathcal{Q}$ be two isomorphic processes on the subcategory $\mathfrak{X}$. If (E,S) is an object of $\mathfrak{X}$ let $\gamma(E,S) : P(E,S) \longrightarrow Q(E,S)$ be a homeomorphism such that $\gamma(E,S) \circ p_S = q_S$. Assume $\mathcal{Q}$ satisfies (FP_3)

and let $S_Q = [f_Q | f \text{ in } S]$. If f_Q is in S let $f_P = f_Q \circ \gamma(E,S)$. The functions f_P are continuous and $f_P \circ p_S = f_Q \circ \gamma(E,S) \circ p_S = f_Q \circ q_S = f$. Hence $\mathcal{P}$ satisfies (FP_3) and so this property is invariant.

Since the functions f_Q and f_P are uniquely defined by being continuous and by satisfying $f_Q \circ q_S = f_P \circ p_S = f$ it follows that the correspondence $f_Q \longrightarrow f_P$ of S_Q with $S_P = [f_P | f \text{ in } S]$ is 1 - 1 and onto. This result has two immediate consequences. First, S_Q separates the points of $Q(E,S)$ iff S_P separates the points of $P(E,S)$. Second, $\gamma(E,S)$ is a homeomorphism with respect to the weak topologies defined by S_P and S_Q. This is because if f is in S, $\varepsilon > 0$ and u_o, v_o are such that $\gamma(E,S)u_o = v_o$, then $\gamma(E,S)[u \in P(E,S) | \; |f_P u - f_P u_o| < \varepsilon] = [v \in Q(E,S) | \; |f_Q v - f_Q v_o| < \varepsilon]$ and $[u \in P(E,S) | \; |f_P u - f_P u_o| < \varepsilon] = \gamma^{-1}(E,S)[v \in Q(E,S) | \; |f_Q v - f_Q v_o| < \varepsilon]$. This proves that the following two properties of processes are also invariant under isomorphism:

(FP_4) <u>$\mathcal{P}$ satisfies (FP_3) and for each object (E,S) of $\mathfrak{X}$ S_P separates the points of $P(E,S)$</u>;

<u>and</u>

(FP_5) <u>$\mathcal{P}$ satisfies (FP_3) and for each object (E,S) of $\mathfrak{X}$ the topology of $P(E,S)$ is $\underline{O}(P(E,S), S_P)$.</u>

Using the notation of the previous two paragraphs let $\mathcal{P}$ and $\mathcal{Q}$ be isomorphic processes on $\mathfrak{X}$ that satisfy (FP_3).

The coarsest uniformity on $P(E,S)$ such that each f_P in S_P is uniformly continuous is generated by the surroundings $V(f_P,\varepsilon) = [(u,w)\ u,w \text{ in } P(E,S) \mid |f_P u - f_P w| < \varepsilon]$. Denote this uniformity by $\underline{U}(S_P)$. Let $\underline{U}(S_Q)$ denote the corresponding uniformity on $Q(E,S)$. It is clear that $(\gamma(E,S) \times \gamma(E,S))\, V(f_P,\varepsilon) = V(f_Q,\varepsilon)$ and that $V(f_P,\varepsilon) = (\gamma(E,S) \times \gamma(E,S))^{-1} V(f_Q,\varepsilon)$. Consequently $\gamma(E,S)$ is an isomorphism of the uniform space $(P(E,S), \underline{U}(S_P))$ with the uniform space $(Q(E,S), \underline{U}(S_Q))$. This proves the invariance of property

(FP_6) <u>$\mathcal{P}$ satisfies (FP_3) and for each object (E,S) of Ψ the set $P(E,S)$ is complete in the uniformity $\underline{U}(S_P)$.</u>

The third general type of invariant property of a process $\mathcal{P}$ on Ψ relates properties of the mappings α of Ψ to properties of the corresponding continuous functions $P(\alpha)$. Two examples are of interest for the purposes of this thesis. The first is

(FP_7) <u>if (E,S), $(E'S')$ are two objects of Ψ and if α is a mapping of Ψ in $\mathrm{Hom}((E,S),(E',S'))$ such that $S = S' \circ \alpha$, then $P(\alpha)$ embeds $P(E,S)$ on a closed subspace of $P(E',S')$.</u>

<u>This is an invariant property.</u> Let $\mathcal{Q}$ be a second process on Ψ which is isomorphic to $\mathcal{P}$ and let $\gamma(E,S)$ and $\gamma(E',S')$ be homeomorphisms such that $\gamma(E',S') \circ P(\alpha) = \alpha(\alpha) \circ \gamma(E,S)$. Since $\gamma^{-1}(E,S)$ is also a homeomorphism it follows that

$Q(\alpha)$ embeds $Q(E,S)$ on a closed subspace of $Q(E',S')$ when $P(\alpha)$ embeds $P(E,S)$ on a closed subspace of $P(E',S')$. The second example which is obviously an invariant property is

(FP$_3$) <u>if (E,S), (E',S') are two objects of $\mathfrak{T}$ and if α is a mapping of $\mathfrak{T}$ in $\mathrm{Hom}((E,S),(E',S'))$ such that $S = S' \circ \alpha$ and the correspondence $f' \longrightarrow f' \circ \alpha$ is 1 - 1, then $P(\alpha)$ is a homeomorphism.</u>

Let $\mathfrak{T}$ be a subcategory of $\mathfrak{F}$ and denote by $\mathcal{P}$ a process on $\mathfrak{T}$ that satisfies (FP$_3$). It induces a covariant functor $P_* : \mathfrak{T} \longrightarrow \mathfrak{F}$ defined as follows: if (E,S) is an object of $\mathfrak{T}$ define $P_*(E,S)$ to be $(P(E,S), S_P)$ - considering $P(E,S)$ as a set - and if α is a mapping of $\mathfrak{T}$ in $\mathrm{Hom}((E,S),(E',S'))$ define $P_*(\alpha)$ to be $P(\alpha)$. To show that P_* is a covariant functor it is sufficient to show that $P_*(\alpha)$ is in $\mathrm{Hom}(P_*(E,S), P_*(E',S'))$. Let f' be in S'. The function $f'_P \circ P(\alpha) = (f' \circ \alpha)_P$ because $f'_P \circ P(\alpha) \circ p_S = f'_P \circ p_{S'} \circ \alpha = f' \circ \alpha$. Consequently $S'_P \circ P(\alpha)$ is contained in S_P. Let $P_*\mathfrak{T}$ be the image of $\mathfrak{T}$ under P_*.

If $\mathcal{Q}$ is a process defined on a subcategory containing $P_*\mathfrak{T}$ then $Q \circ P_* : \mathfrak{T} \longrightarrow \Sigma$ is a covariant functor. Define $\mathcal{Q} \,\square\, \mathcal{P}$ to be the functor $Q \,\square\, P = Q \circ P_*$ and the family $((q \,\square\, P)_S)_{(E,S) \text{ in } \mathfrak{T}}$ of functions $(q \,\square\, P)_S = q_{S_P} \circ p_S : E \longrightarrow (Q \,\square\, P)(E,S) = Q(P(E,S), S_P)$. Then $\mathcal{Q} \,\square\, \mathcal{P}$ satisfies

(FP_2) in definition 1. Let α be a mapping of $\mathcal{T}$ in $\mathrm{Hom}((E,S),(E',S'))$. Then $(Q \square P)(\alpha) \circ (q \square p)_S = Q(P(\alpha)) \circ q_{S_P} \circ p_S = q_{S'_P} \circ P(\alpha) \circ p_S = q_{S'_P} \circ p_{S'} \circ \alpha = (q \square p)_{S'} \circ \alpha$. In general, however, does not satisfy (FP_1) as the following example shows.

Example. Assume that $\mathcal{P}$ is a process on $\mathcal{T}$ that satisfies (FP_3). Let $\mathcal{Q}$ be the process defined in example 2. If (E,S) is an object of $\mathcal{T}$ then $(q \square p)_S E = q_{S_P}(p_S E) = p_S E$, which is dense in $(Q \square P)(E,S) = P(E,S)$ - as a discrete space - iff $p_S E = P(E,S)$. This condition is not necessarily satisfied if $\mathcal{P}$ has (FP_3) - see process $\mathcal{J}$ in the next section - and so in general for this $\mathcal{Q}$, $\mathcal{Q} \square \mathcal{P}$ is not a process on $\mathcal{T}$.

The reason why $(q \square p)_S E$ is not necessarily dense in $(Q \square P)(E,S)$ is due to the fact that in general q_{S_P} is not related to the topology of $P(E,S)$ defined by $\mathcal{P}$. Specifically it is not continuous since <u>$(q \square p)_S$ E is dense in $(Q \square P)(E,S)$ if q_{S_P} is continuous</u>. Let U be a non-void open subset of $(Q \square P)(E,S) = Q(P(E,S), S_P)$. Since q_{S_P} $P(E,S)$ is dense in this set $q_{S_P}^{-1}(U)$ is non-void. When q_{S_P} is continuous it is an open subset of $P(E,S)$. The fact that $p_S E$ is dense in $P(E,S)$ then implies that $p_S^{-1}(q_{S_P}^{-1}(U))$ is non-void. In other words $(q \square p)_S E \cap U \neq \emptyset$ i.e. $(q \square p)_S E$ is dense in $(Q \square P)(E,S)$.

This introduces the following problem: if $\mathcal{P}$ is a process on $\mathcal{T}$ and (E,S) is an object of $\mathcal{T}$ when is

$p_S : E \longrightarrow P(E,S)$ continuous with respect to a topology $\underline{O}_E$ for E? One answer to this question is stated as

Theorem 1. Let $\mathcal{P}$ be a process on $\mathfrak{T}$ that satisfies (FP_5) and let (E,S) be an object of $\mathfrak{T}$. If O_E is a topology for E then $p_S : E \longrightarrow P(E,S)$ is continuous with respect to $\underline{O}_E$ iff each f in S is continuous with respect to $\underline{O}_E$.

Proof: Since $\mathcal{P}$ satisfies (FP_3) $S = S_P \circ p_S$ and so when p_S is continuous every function f in S is also continuous.

On the other hand $S = S_P \circ p_S$ also implies that $p_S^{-1}[\underline{O}(P(E,S), S_P)] = \underline{O}(E,S)$. If $\underline{O} = \underline{O}(P(E,S), S_P)$ then p_S is continuous with respect to $p_S^{-1}\,\underline{O}$ and $\underline{O}$. Since the functions in S_P are $\underline{O}$ - continuous it follows that $p_S^{-1}\,\underline{O}$ contains $\underline{O}(E,S)$ or equivalently that the functions in S are $p_S^{-1}\,\underline{O}$ - continuous. Let f be a function in S and choose $\varepsilon > 0$ and v_0 in $P(E,S)$. Then $p_S^{-1}[v \text{ in } P(E,S) \mid |f_P v - f_P v_0| < \varepsilon]$ is open in $\underline{O}(E,S)$. If x_1 is in E such that $|f_P(p_S x_1) - f_P v_0| < \varepsilon$ and $\varepsilon' < \varepsilon - |f_P(p_S x_1) - f_P v_0|$, then $p_S[x \text{ in } E \mid |fx - fx_1| < \varepsilon']$ is contained in $[v \text{ in } P(E,S) \mid |f_P v - f_P v_0| < \varepsilon] = U$. Consequently $p_S^{-1}\,U$ is open in $\underline{O}(E,S)$ and so $p_S^{-1}\underline{O}$ is contained in $\underline{O}(E,S)$.

If $\mathcal{P}$ satisfies (FP_5) it then follows that p_S is continuous when $\underline{O}(E,S)$ is coarser than $\underline{O}_E$ i.e. when each function in S is $\underline{O}_E$ - continuous.

<u>Corollary. Let $\mathcal{P}$ be a process on $\mathfrak{T}$ that satisfies (FP_3) and let $\mathcal{Q}$ be a process on a subcategory containing $P_* \mathfrak{T}$. If $\mathcal{Q}$ satisfies (FP_5) then $\mathcal{Q} \square \mathcal{P}$ is a process on $\mathfrak{T}$.</u>

Proof: The functions f_p in S_p are all continuous functions on $P(E,S)$. Since $\mathcal{Q}$ satisfies (FP_5) this implies that each q_{S_p} is continuous. Hence $(q \square p)_S\ E$ is dense in $(Q\ \ P)(E,S)$ for each (E,S) in $\mathfrak{T}$.

Let $\mathcal{P}$ be a process on $\mathfrak{T}$ for which $\mathcal{P} \square \mathcal{P}$ is defined i.e. which satisfies (FP_3) and is such that $P_* \mathfrak{T}$ is a subcategory of $\mathfrak{T}$. If $\mathcal{P}$ satisfies (FP_5) then by the corollary to theorem 1. $\mathcal{P} \square \mathcal{P}$ is a process on $\mathfrak{T}$ (when it is defined). In many cases there is no essential difference between $\mathcal{P}$ and $\mathcal{P} \square \mathcal{P}$ - they are isomorphic processes on $\mathfrak{T}$. A process $\mathcal{P}$ on $\mathfrak{T}$ is said to be <u>idempotent</u> if $\mathcal{P} \square \mathcal{P}$ is an isomorphic process on $\mathfrak{T}$.

<u>Remarks.</u> Two questions that occur in this context are the following:

(1) if $\mathcal{P} \square \mathcal{P}$ is defined is it necessarily a process; and (2) if $\mathcal{P} \square \mathcal{P}$ is a process is it necessarily idempotent?

Let $\mathcal{P}$ be a process on $\mathfrak{T}$ such that $\mathcal{P} \square \mathcal{P}$ is defined. If $\mathfrak{T}_0$ is a subcategory of $\mathfrak{T}$ then $\mathcal{P} | \mathfrak{T}_0$ and $(\mathcal{P} \square \mathcal{P}) | \mathfrak{T}_0$ are defined. This does not necessarily imply that $\mathcal{P}|\mathfrak{T}_0 \square \mathcal{P}|\mathfrak{T}_0$ is defined since this is the case iff $P_* \mathfrak{T}_0$ is contained in $\mathfrak{T}_0$. However when $\mathcal{P}|\mathfrak{T}_0 \square \mathcal{P}|\mathfrak{T}_0$ is defined it is clear that $\mathcal{P}|\mathfrak{T}_0 \square \mathcal{P}|\mathfrak{T}_0 = (\mathcal{P} \square \mathcal{P})|\mathfrak{T}_0$.

The property of being an idempotent process is not independent of the other properties of processes as shown by

Theorem 2. Let Ψ be a subcategory of Φ obtained by restricting the objects and let $\mathcal{P}$ be a process on Ψ such that $P_*\Psi$ is a subcategory of Ψ. If $\mathcal{P}$ satisfies (FP_5) and (FP_7) or (FP_8) then $\mathcal{P}$ is idempotent.

Proof: Since $\mathcal{P}$ satisfies (FP_5) $\mathcal{P} \square \mathcal{P}$ is a process on Ψ. Consider the following commutative diagram, where (E,S) is an object of Ψ:

$$\begin{tikzcd}
(E,S) \arrow[r, "p_S"] \arrow[d, "p_S"'] & P(E,S) \arrow[d, "P(p_S)"] \\
(P(E,S), S_P) \arrow[r, "p_{S_P}"] & P(P(E,S), S_P)
\end{tikzcd}$$

The continuous function $P(p_S)$ exists because $(P(E,S),\ S_P)$ is an object of Ψ and p_S is a mapping of Φ, hence a mapping of Ψ.

When $\mathcal{P}$ satisfies (FP_7) or (FP_8) $P(p_S)$ is a homeomorphism. The function p_S has the property that $S = S_P \circ p_S$ and that the correspondence $f_P \longrightarrow f_P \circ p_S = f$ is $1 - 1$. This shows that $P(p_S)$ is a homeomorphism if $\mathcal{P}$ satisfies (FP_8). In addition if $\mathcal{P}$ satisfies (FP_7) it follows that $P(p_S)$ embeds $P(E,S)$ on a closed subspace of $P(P(E,S),\ S_P)$. It is clear that $P(p_S)P(E,S)$ contains $(p_{S_P} \circ p_S)E$ which is dense in $P(P(E,S),\ S_P)$. Consequently $P(p_S)$ is a homeomorphism.

The family $(P(p_S))_{(E,S) \text{ in } \mathfrak{T}}$ of homeomorphisms defines an isomorphism of $\mathcal{P}$ into $\mathcal{P} \,\square\, \mathcal{P}$. Since $(p \,\square\, p)_S = p_{S_p} \circ p_S$ the homeomorphisms $P(p_S)$ satisfy condition (2) of definition 2 by virtue of the commutativity of the diagram. Let α be a mapping of $\mathfrak{T}$ in Hom((E,S), (E',S')). Then $P(p_{S'}) \circ P(\alpha) = P(p_{S'} \circ \alpha) = P(P(\alpha) \circ p_S)$ $= P(P(\alpha)) \circ P(p_S) = (P \,\square\, P)(\alpha) \circ P(p_S)$. This shows that the homeomorphisms $P(p_S)$ define a natural transformation of $\mathcal{P}$ into $\mathcal{P} \,\square\, \mathcal{P}$. This proves that $\mathcal{P}$ is idempotent.

Remarks. This proof is valid for any subcategory $\mathfrak{T}$ for which $p_S : E \longrightarrow P(E,S)$ is a mapping of $\mathfrak{T}$. In all the applications of this theorem the categories are obtained by merely restricting the objects of $\mathfrak{T}$, and so they are of this type. The argument used to show that the $P(p_S)$ define a natural transformation is the first explicit use of the fact that P is a covariant functor.

The property of being idempotent is not an invariant property because if $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic processes on a subcategory $\mathfrak{T}$ it is possible that $\mathcal{P} \,\square\, \mathcal{P}$ be defined and that $\mathcal{Q} \,\square\, \mathcal{Q}$ be undefined. Consider the following almost trivial

Example. Let E and E' be two distinct sets with $\alpha : E \longrightarrow E'$ 1 - 1 and onto. Let $\mathfrak{T}$ be the subcategory consisting of the single object (E, F_E) - where F_E is the collection of all real-valued functions on E - and the single mapping being the identity mapping i of E into itself. Define $\mathcal{P}$ as follows: let

$P(E,F_E)$ be the discrete space E ; let $P(i)$ be the identity mapping of E into itself; and let $p_{F_E} = P(i)$. Define $\mathcal{Q}$ in a similar way using the discrete space E' and $q_{F_E} = \alpha$. It is clear that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic processes on $\mathfrak{X}$ for which $\mathcal{P}$ is idempotent and $\mathcal{Q} \square \mathcal{Q}$ is not defined.

Assume that $\mathfrak{X}$ is a subcategory of $\overline{\mathfrak{X}}$ obtained by restricting the objects. Let $\mathcal{P}$ and $\mathcal{Q}$ be two processes on $\mathfrak{X}$ that satisfy (FP_3) and are such that $P_* \mathfrak{X}$ and $Q_* \mathfrak{X}$ are subcategories of $\mathfrak{X}$. Then four 'potential' processes are defined: $\mathcal{P} \square \mathcal{P}$, $\mathcal{Q} \square \mathcal{P}$, $\mathcal{P} \square \mathcal{Q}$ and $\mathcal{Q} \square \mathcal{Q}$. Let γ be a homomorphism of $\mathcal{P}$ into $\mathcal{Q}$. For each object (E,S) in $\mathfrak{X}$ the homomorphism γ and the four 'potential' processes define the following diagram, where S_P and S_Q are the collections of continuous functions defined by $S_P \circ p_S = S_Q \circ q_S = S$: (note that $\gamma(E,S)$ is in $\mathrm{Hom}((P(E,S),S_P),(Q(E,S),S_Q))$ and hence is a mapping of $\mathfrak{X}$)

$$\begin{tikzcd}
 & & (E,S) \arrow[dl, "p_S"'] \arrow[drr, "q_S"] & & & \\
 & (P(E,S),S_P) \arrow[rrr, "\gamma(E,S)"] \arrow[dl, "p_{S_P}"'] \arrow[ddr, "q_{S_P}"] & & & Q(E,S),S_Q) \arrow[dl, "p_{S_Q}"'] \arrow[ddr, "q_{S_Q}"] & \\
P(P(E,S),S_P) \arrow[rrr, "P(\gamma(E,S))"'] \arrow[drr, "\gamma(P(E,S),S_P)"'] & & & P(Q(E,S),S_Q) \arrow[drr, "\gamma(Q(E,S),S_Q)"] & & \\
 & & Q(P(E,S),S_P) \arrow[rrr, "Q(\gamma(E,S)"'] & & & Q(Q(E,S),S_Q)
\end{tikzcd}$$

This diagram is commutative with the possible exception of the bottom 'face' . Let X and Y be topological spaces and

$\psi : X \longrightarrow Y$ a continuous function. Since for any subset A of X $\psi \bar{A}$ is contained in $\overline{\psi A}$ it follows that ψ maps dense subsets on dense subsets iff ψX is dense in Y. The continuous functions $\gamma(E,S)$, $\gamma(P(E,S), S_P)$, and $\gamma(Q(E,S), S_Q)$ all map their domains on dense subsets of the spaces in which they are valued. The commutativity of the end 'faces' of the diagram has the following consequences: (1) $\mathcal{Q} \square \mathcal{P}$ is a process if $\mathcal{P} \square \mathcal{P}$ is a process; and (2) $\mathcal{Q} \square \mathcal{Q}$ is a process if $\mathcal{P} \square \mathcal{Q}$ is a process. In addition the homomorphism γ defines a homomorphism $\gamma(\mathcal{P})$ of $\mathcal{P} \square \mathcal{P}$ into $\mathcal{Q} \square \mathcal{P}$ and a homomorphism $\gamma(\mathcal{Q})$ of $\mathcal{P} \square \mathcal{Q}$ into $\mathcal{Q} \square \mathcal{Q}$. The homomorphisms $\gamma(\mathcal{P})$ and $\gamma(\mathcal{Q})$ are defined by setting $\gamma(\mathcal{P})(E,S) = \gamma(P(E,S), S_P)$ and $\gamma(\mathcal{Q})(E,S) = \gamma(Q)E,S), S_Q)$. To show, for example that $\gamma(\mathcal{P})$ is a homomorphism of $\mathcal{P} \square \mathcal{P}$ into $\mathcal{Q} \square \mathcal{P}$ it is sufficient to prove that $\gamma(\mathcal{P})(E',S') \circ (P \square P)(\alpha) = (Q \square P)(\alpha) \circ \gamma(\mathcal{P})(E,S)$, since γ already satisfies condition (1) of definition 2. It is clear that $\gamma(\mathcal{P})(E',S') \circ (P \square P)(\alpha) = \gamma(P(E',S')\ S'_P) \circ P(P(\alpha)) = Q(P(\alpha)) \circ \gamma(P(E,S), S_P)$ since $P(\alpha)$ is a mapping of in $\mathrm{Hom}((P(E,S), S_P), (P(E',S'), S'_P))$. Because $Q(P(\alpha)) \circ \gamma(P(E,S),S_P) = (Q \square P)(\alpha) \circ \gamma(\mathcal{P})(E,S)$ it then follows that $\gamma(\mathcal{P})$ is a homomorphism. From the definition of $\gamma(\mathcal{P})$ and $\gamma(\mathcal{Q})$ it is clear that they are isomorphisms if γ is an isomorphism.

Consider the identity $P(\gamma(E,S)) \circ p_{S_P} = p_{S_Q} \circ \gamma(E,S)$. The continuous function $P(\gamma(E,S))$ maps $P(P(E,S), S_P)$ on a dense subset of $P(Q(E,S)$ iff $p_{S_Q} \circ \gamma(E,S)$ maps $P(E,S)$ on a dense subset of this space. This is the case if $\gamma(E,S)$ is onto. In particular, *if γ is an isomorphism of $\mathcal{P}$ into $\mathcal{Q}$ then $\mathcal{P} \square \mathcal{Q}$ is a process when $\mathcal{P} \square \mathcal{P}$ is a process.* This is because $P(\gamma(E,S))$ maps its domain on a dense subset of its value space when γ is an isomorphism. *The function $P(\gamma(E,S))$ is a homeomorphism when $\gamma(E,S)$ is a homeomorphism* because P is a functor.

If $\mathcal{P} \square \mathcal{P}$ and $\mathcal{P} \square \mathcal{Q}$ are processes on $\mathcal{T}$ then the functions $P(\gamma(E,S))$ define a homomorphism $\mathcal{P}(\gamma)$ of $\mathcal{P} \square \mathcal{P}$ into $\mathcal{P} \square \mathcal{Q}$. From the diagram it follows that $P(\gamma(E,S)) \circ (p \square p)_S = (p \square q)_S$. Let (E',S') be a second object of $\mathcal{T}$ and let α be a map of $\mathcal{T}$ in $\mathrm{Hom}((E,S),(E',S'))$. Then $P(\gamma(E',S')) \circ (P \square P)(\alpha) = P(\gamma(E',S') \circ (P(\alpha)) = P(Q(\alpha) \circ \gamma(E,S)) = P(Q(\alpha)) \circ P(\gamma(E,S)) = (P \square Q)(\alpha) \circ P(\gamma(E,S))$. This shows that the functions $P(\gamma(E,S))$ define a homomorphism of $\mathcal{P} \square \mathcal{P}$ into $\mathcal{P} \square \mathcal{Q}$. *This homomorphism $\mathcal{P}(\gamma)$ is an isomorphism if γ is an isomorphism.*

Similarly if $\mathcal{Q} \square \mathcal{P}$ and $\mathcal{Q} \square \mathcal{Q}$ are processes on $\mathcal{T}$ *the functions $Q(\gamma(E,S))$ define a homomorphism $\mathcal{Q}(\gamma)$ of $\mathcal{Q} \square \mathcal{P}$ into $\mathcal{Q} \square \mathcal{Q}$ which is an isomorphism if γ is an isomorphism.*

With the aid of these results it is easy to prove

Theorem 3. Let $\mathcal{J}$ be a subcategory of $\mathcal{T}$ obtained by restricting the objects. Let $\mathcal{P}$ and $\mathcal{Q}$ be isomorphic processes on $\mathcal{J}$ such that $\mathcal{P} \square \mathcal{P}$ and $\mathcal{Q} \square \mathcal{Q}$ are defined. If $\mathcal{P} \square \mathcal{P}$ is a process on $\mathcal{J}$ then so is $\mathcal{Q} \square \mathcal{Q}$. When this is the case they are isomorphic processes on $\mathcal{J}$. If $\mathcal{P}$ is idempotent so is $\mathcal{Q}$.

Proof: Let γ be an isomorphism of $\mathcal{P}$ into $\mathcal{Q}$. If $\mathcal{P}\,\square\,\mathcal{P}$ is a process on $\mathfrak{X}$ so is $\mathcal{P}\,\square\,\mathcal{Q}$. Since $\mathcal{P}\,\square\,\mathcal{Q}$ is a process on $\mathfrak{X}$ it follows that $\mathcal{Q}\,\square\,\mathcal{Q}$ is a process on $\mathfrak{X}$.

The homomorphisms $\mathcal{P}(\gamma)$ and $\gamma(\mathcal{Q})$ are isomorphisms. Consequently $\gamma(\mathcal{Q}) \circ \mathcal{P}(\gamma)$ is an isomorphism of $\mathcal{P}\,\square\,\mathcal{P}$ into $\mathcal{Q}\,\square\,\mathcal{Q}$.

If $\mathcal{P}$ is idempotent then $\mathcal{P}\,\square\,\mathcal{P}$ is a process on $\mathfrak{X}$ isomorphic to $\mathcal{P}$. This implies that $\mathcal{Q}\,\square\,\mathcal{Q}$ is a process on $\mathfrak{X}$ which is isomorphic to $\mathcal{P}\,\square\,\mathcal{P}$ and hence to $\mathcal{P}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic this implies $\mathcal{Q}$ and $\mathcal{Q}\,\square\,\mathcal{Q}$ are isomorphic i.e. $\mathcal{Q}$ is idempotent.

Remarks. The proof of this theorem will apply to a pair of isomorphic processes $\mathcal{P}$ and $\mathcal{Q}$ on $\mathfrak{X}$ isomorphic under γ such that $\gamma(E,S)$ is a mapping of $\mathfrak{X}$ in $\mathrm{Hom}((P(E,S),\ S_P)\ (Q(E,S),\ S_Q))$. This leaves the question open when γ does not have this property. Another question left open is the problem of the connection between $\mathcal{P}\,\square\,\mathcal{P}$ and $\mathcal{P}\,\square\,\mathcal{Q}$: is $\mathcal{P}\,\square\,\mathcal{Q}$ necessarily a process if $\mathcal{P}\,\square\,\mathcal{P}$ is a process?

§2 Process $\mathfrak{J}$. Let I be an index set and for each i in I let R_i be the set of real numbers together with the usual uniformity which is defined by the metric $d(x,y) = |x - y|$. Let $\prod_{i \text{ in } I} R_i$ be the cartesian product of the spaces R_i together with the product uniformity. It is a Hausdorff uniform space which is complete because each of the coordinate spaces R_i is complete. If i is in I define $\pi_i : \prod_{i \text{ in } I} R_i \longrightarrow R_i$ by setting $\pi_i z = z_i$ when $z = (z_i)_{i \text{ in } I}$. The product uniformity is then the uniformity $\underline{U}((\pi_i)_{i \text{ in } I})$ i.e. the coarsest one for which each real-valued function π_i is uniformly continuous. Consequently the uniform topology of $\prod_{i \text{ in } I} R_i$ is the weak topology $\underline{O}(\prod_{i \text{ in } I} R_i, (\pi_i)_{i \text{ in } I})$.

With these observations on the product of a family of real lines it is relatively easy to define and discuss a process on Φ which will be denoted by $\mathfrak{J}$.

Let (E,S) be an object of Φ and consider S as an index set. Form $\prod_{f \text{ in } S} R_f$ and define $t_S: E \longrightarrow \prod R_f$ by setting $t_S x = (fx)_{f \text{ in } S}$. Let $T(E,S)$ be the topological space obtained by taking the closure of $t_S E$ in $\prod_{f \text{ in } S} R_f$ and the subspace topology for this closed set.

If f is in S defined f_T to be $\pi_f | T(E,S)$. The functions f_T are continuous and satisfy $f_T \circ p_S = f$. The topology of $T(E,S)$ is the weak topology $\underline{O}(T(E,S), S_T)$, where $S_T = [f_T | f \text{ in } S]$. Since $T(E,S)$ is a closed subset of $\prod_{f \text{ in } S} R_f$ it is complete in the induced uniformity $\underline{U}((\pi_f)_{f \text{ in } S}) | T(E,S)$ which is clearly $\underline{U}(S_T)$.

Let (E',S') be a second object of Φ and let α be in $\mathrm{Hom}((E,S),(E',S'))$. Since $S' \circ \alpha$ is contained in S it is possible to define a function $\alpha_{S',S} : t_S E \quad T(E',S')$ by setting $\alpha_{S',S} \circ t_S = t_{S'} \circ \alpha$. This function is $(\underline{U}(S_T) | t_S E, \underline{U}(S'_T))$ - uniformly continuous and since $(T(E',S'), \underline{U}(S'_T))$ is a complete separated uniform space the fact that $t_S E$ is dense in $T(E,S)$ implies that $\alpha_{S',S}$ has a unique $(\underline{U}(S_T), \underline{U}(S'_T))$ -

uniformly continuous extension $T(\alpha):T(E,S) \longrightarrow T(E',S')$ (see Bourbaki [3] p. 151).

Let (E'',S'') be a third object of $\mathfrak{E}$ and let $\alpha':E' \longrightarrow E''$ be a second mapping of $\mathfrak{E}$. It is clear that $\alpha'_{S'',S'} \circ \alpha_{S',S} = (\alpha' \circ \alpha)_{S'',S}$ and hence from the uniqueness of the extension that $T(\alpha') \circ T(\alpha) = T(\alpha' \circ \alpha)$. Consequently $T: \mathfrak{E} \longrightarrow \Sigma$ is a covariant functor.

Since for each object (E,S) in $\mathfrak{E}$ t_S maps E on the dense subset $t_S E$ of $T(E,S)$ and since for α in $\mathrm{Hom}((E,S),(E',S'))$ $T(\alpha) \circ t_S = \alpha_{S',S} \circ t_S = t_{S'} \circ \alpha$, it follows that the functor T and the family $(t_S)_{(E,S)}$ in $\mathfrak{E}$ of functions t_S defines a process on $\mathfrak{E}$. This process will be denoted by $\mathfrak{J}$.

Remark. Tychonoff [4] made use of this method of construction to show that every completely regular space may be embedded in a compact space. Instead of using the whole line R he used the unit interval $[0,1]$ and functions with values in that range. For this reason process $\mathfrak{J}$ will be called the Tychonoff process.

The process $\mathfrak{J}$ has all of the specific process properties introduced in § 1 with the exception of (FP_8). This result is stated as

Theorem 4. The process $\mathfrak{J}$ on $\mathfrak{E}$ satisfies (FP_3),(FP_4),(FP_5),(FP_6) and (FP_7). Consequently it is completely regular and idempotent. It does not satisfy (FP_8) on $\mathfrak{E}$.

Proof: In the course of the definition of $\mathfrak{J}$ it was shown that $\mathfrak{J}$ satisfies (FP_3), (FP_5) and (FP_6). Since for any index set I the projections π_i, i in I, separate the points of $\prod_{i \text{ in } I} R_i$ it is clear that $\mathfrak{J}$ satisfies (FP_4).

Let α be in $\mathrm{Hom}((E,S),(E',S'))$ and such that $S = S' \circ \alpha$. Define the function $\theta: \prod_{f \text{ in } S} R_f \longrightarrow \prod_{f' \text{ in } S'} R_{f'}$, by setting $\theta((Y_f)_{f \text{ in } S}) = (Z_{f'})_{f' \text{ in } S}$ where $f' \circ \alpha = f$ implies $Z_{f'} = Y_f$. This function embeds $\prod_{f \text{ in } S} R_f$ in $\prod_{f' \text{ in } S'} R_{f'}$.

In addition $\theta \circ t_S = t_{S'} \circ \alpha$ since if x is in E, $(\theta \circ t_S)x = \theta((fx)_{f \text{ in } S}) = ((f' \circ \alpha)x)_{f' \text{ in } S'} = (t_{S'} \circ \alpha)x$. It follows that $T(\alpha) = \theta | T(E,S)$ since $t_S E$ is dense in $T(E,S)$. This proves that $T(\alpha)$ maps $T(E,S)$ homeomorphically onto the closure in $T(E',S')$ of $(t_{S'} \circ \alpha)E$. Hence $\mathfrak{J}$ satisfies (FP_7).

Process $\mathfrak{J}$ is completely regular because it satisfies (FP_4) and (FP_5). It is idempotent in view of theorem 2 since it satisfies (FP_5) and (FP_7).

To prove that $\mathfrak{J}$ does not satisfy (FP_8) on Φ it is sufficient to find a pair of objects in Φ and a mapping α such that the conditions of (FP_8) are satisfied but for which $T(\alpha)$ is not a homeomorphism. Consider the following

Example. Let E' be the space R of real numbers and let S' be the collection of polynomial functions f' on R. If $f'(X)$ is in $R[X]$ the corresponding polynomial function f' is defined by setting $f'x$ equal to the value of $f'(X)$ at x. Let

E be any closed interval of R, say $[-1,1]$ and let $\alpha: E \longrightarrow E'$ be the natural injection. Define S to be $S' \circ \alpha$. Since S' consists of polynomial functions $f' \longmapsto f' \circ \alpha$ is $1-1$. The continuous function $T(\alpha)$ is not a homeomorphism. This is because $T(E,S)$ is compact and $T(E',S')$ is not (for any object (E,S) of $\mathfrak{F}$ $T(E,S)$ is compact iff all the functions in S are bounded).

An interesting property of the Tychonoff process $\mathcal{T}$ is that the first four properties of theorem 4 characterize it. In other words the following 'converse' of theorem 4 holds

Theorem 5. Let $\mathcal{P}$ be a process on $\mathfrak{F}$ that satisfies (FP_3), (FP_4), (FP_5) and (FP_6). Then $\mathcal{P}$ is isomorphic to $\mathcal{T} \mid \mathfrak{F}$.

Proof: Let (E,S) be an object of $\mathfrak{F}$. Since $\mathcal{P}$ satisfies (FP_3) and (FP_4) it follows that $p_S x = p_S y$ iff $f_P(p_S x) = f_P(p_S x)$ for each f in S i.e. iff $fx = fy$ for each f in S. This means that $t_S x = t_S y$ iff $fx = fy$ for each f in S. Therefore there is a $1-1$ function $\gamma_1(E,S): p_S E \longrightarrow T(E,S)$ such that $\gamma_1(E,S) \circ p_S = t_S$.

If f is in S let f_T denote the continuous function on $T(E,S)$ with $f_T \circ t_S = f$. Then $f_T \circ \gamma_1(E,S) = f_P \mid p_S E$. This means that $\gamma_1(E,S)$ is an isomorphism of the uniform space $(p_S E, \underline{U}(S_P) \mid p_S E)$ with the uniform space $(t_S E, \underline{U}(S_T) \mid t_S E)$. The argument used to establish the invariance of (FP_6) also applies here to prove the assertion.

If $\gamma_1(E,S)$ is considered as a function $: p_S E \longrightarrow T(E,S)$ it follows from considerations of uniform continuity that it has a unique $(\underline{U}(s_P), \underline{U}(s_T))$ - uniformly continuous extension $\gamma(E,S): P(E,S) \longrightarrow T(E,S)$ (see Bourbaki [3] p 151). Similarly $\gamma_1^{-1}(E,S): t_S E \longrightarrow P(E,S)$ has a unique $(\underline{U}(s_T), \underline{U}(S_P))$ - uniformly continuous extension to $T(E,S)$. This proves that $\gamma(E,S)$ is an isomorphism of $(P(E,S), \underline{U}(S_P))$ with $(T(E,S), \underline{U}(S_T))$.

Since $\mathcal{P}$ and $\mathcal{J}$ satisfy (FP_5) it follows that $\gamma(E,S)$ is a homeomorphism of $P(E,S)$ with $T(E,S)$.

The functions $\gamma(E,S)$ are such that $\gamma(E,S) \circ p_S = \gamma_1(E,S) \circ p_S = t_S$ and since $\mathcal{J}$ is a Hausdorff process lemma 1 shows that the family $(\gamma(E,S))_{(E,S) \text{ in } \Psi}$ defines an isomorphism of $\mathcal{P}$ into $\mathcal{J} \mid \Psi$.

<u>Corollary.</u> <u>Let Ψ be a subcategory of Φ obtained by restricting the objects. If $\mathcal{P}$ is a process on Ψ that satisfies (FP_3), (FP_4), (FP_5) and (FP_6) then $\mathcal{P}$ satisfies (FP_7). Consequently if $\mathcal{P} \square \mathcal{P}$ is defined $\mathcal{P}$ is idempotent.</u>

<u>Proof:</u> $\mathcal{P}$ is isomorphic to $\mathcal{J} \mid \Psi$ by the theorem. Since $\mathcal{J}$ satisfies (FP_7) by theorem 4 it follows from the invariance of (FP_7) that $\mathcal{P}$ satisfies this condition. If $\mathcal{P} \square \mathcal{P}$ is defined then by theorem 2 $\mathcal{P}$ is idempotent.

A process on Φ, which is isomorphic to $\mathcal{J}$, can be defined by associating with each object (E,S) of Φ a space of filters. These filters are defined by means of the functions in S and are loosely speaking the trace filters on E of the neighbourhood filters of $T(E,S)$. They form the subject of the next section.

§3. S-completely regular filters on E. Let E be a set and let S denote a collection of real-valued functions on E. The functions of S may be used to separate the subsets of E in the following way.

Definition 3. Let F_1 containing F_2 be two subsets of E. S is said to completely separate F_2 from $\complement F_1$ if there exist two integers $k, n > 0$ with $0 \leq k \leq n$, n functions $f_1, \ldots, f_n$ in S and $2n$ real numbers $\lambda_1(e), \ldots, \lambda_k(e), \mu_1(e), \ldots \mu_{n-k}(e)$, $e = 1$ or 2 such that:

(1) $\lambda_i(1) > \lambda_i(2)$, $i = 1, \ldots, k$ and $\mu_j(1) < \mu_j(2)$ $j = 1, \ldots, n-k$; and

(2) $$F_2 \subseteq \left(\bigcap_{i=1}^{k} [x | f_i x < \lambda_i(2)]\right) \cap \left(\bigcap_{j=1}^{n-k} [x | f_{k+j} x > \mu_j(2)]\right)$$
$$\subseteq \left(\bigcap_{i=1}^{k} [x | f_i x < \lambda_i(1)]\right) \cap \left(\bigcap_{j=1}^{n-k} [x | f_{k+j} x > \mu_j(1)]\right) \subseteq F_1 .$$

Remarks. This definition is cumbersome because S is not assumed to have any particular structure. When it has a sufficiently elaborate structure this definition is equivalent to: F_2 and $\complement F_1$ are completely separated if there exists a function g in S with $0 \leq g \leq 1$ and $F_2 \subseteq [x | gx = 0] \subseteq [x | gx < 1] \subseteq F_1$. For example this is the case if S is a lattice that contains the constant functions and is closed under the addition of and multiplication by the constant functions (all operations being defined pointwise in the usual way). A special case of this example is obtained when S is the collection of real-valued functions on E continuous with respect to some topology $\underline{0}$ for E. It is in this context that the concept of complete separation usually arises particularly when $\underline{0}$ is a completely regular topology. Then F_2 and $\complement F_1$ are defined to be completely separated if there is an $\underline{0}$ - continuous function g with $0 \leq g \leq 1$ and $F_2 \subseteq [x|gx = 0]$ and $\complement F_1 \subseteq [x|gx = 1]$.

The filters $\underline{F}$ on E that are of interest in this section are those that satisfy the condition of

Definition 4. A filter $\underline{F}$ on E is said to be S-completely regular if F_1 in $\underline{F}$ implies there exists F_2 in $\underline{F}$ such that F_2 and $\complement F_1$ are completely separated by S.

Remarks. In the case where E is a completely regular space and S is the collection of continuous real-valued functions on E this definition of S-completely regular filters coincides with that of Banaschewski [5]. The notion of a completely regular filter is implicit in the work of Alexandroff [6].

Examples of S-completely regular filters.

1. Let f be a function in S and assume λ is in $\mathbb{R}E$. Then the countable collection of sets $[x|fx < \lambda + 1/n] \cap [x|fx > \lambda - 1/n]$ for $n > 0$ form a filter basis on E. The resulting filter is clearly S-completely regular since if $n_1 < n_2$ and $F_i = [x|fx < \lambda + 1/n_i] \cap [x|fx > \lambda - 1/n_i]$ $i = 1, 2$ then F_2 and $\complement F_1$ are completely separated by S.

2. Let f be a function in S which is not bounded above (below) by a constant function. Then the countable collection of sets $[x|fx > n]$ $([x|fx < -n])$ for $n > 0$ form a filter basis on E. The filter defined in either case is S-completely regular.

3. If x is a point of E let $\underline{M}_x$ be the filter generated by the collection of sets $[u|fu < fx + 1/n] \cap [u|fu > fx-1/n]$ where f is in S and $n > 0$. This filter is S-completely regular (it is here that all the details of definition 3 are necessary when no structure for S is assumed).

The third example $\underline{M}_x$ is an S-completely regular filter along which each f in S converges to fx. The function f converges to λ along the first filter and to $+\infty(-\infty)$ along the filter of example 2. This suggeststhat the convergence of the functions in S along the S-completely regular filters should be considered.

Since infinite limits are to be admitted it is convenient to describe briefly the two point compactification $\overline{R}$ of the real numbers R obtained by adjoining $+\infty$ and $-\infty$. Let $\overline{R}$ be the set $R \cup [+\infty] \cup [-\infty]$ and define a topology on $\overline{R}$ by defining neighbourhood filters for each point. If x is in R and x is in a subset V of $\overline{R}$ then V is an $\overline{R}$ - neighbourhood of x iff $V \cap R$ is on R-neighbourhood of x (in the usual topology for the real numbers R). A fundamental system of neighbourhoods for $+\infty(-\infty)$ is defined to be the countable filter basis of sets $[x\epsilon R|x > n] \cup [+\infty]$ $([x\epsilon R|x < -n] \cup [-\infty])$. It is clear that $\overline{R}$ is compact in this topology and that it contains R as a dense subspace.

The first result on the convergence of functions in S along arbitrary S-completely regular filters is

Theorem 6. If $\underline{F}$ is an S-completely regular filter and if f is in S, then there exists an S-completely regular filter $\underline{F}'$ containing $\underline{F}$ such that $\lim_{\underline{F}'} f$ exists in $\bar{R}$.

Proof: Assume that $\lim_{\underline{F}} f$ does not exist in $\bar{R}$ (otherwise $\underline{F}$ itself is a filter that will do). Then there are two possibilities. Either, for some A in $\underline{F}$ fA is a bounded set of real numbers or, for each A in $\underline{F}$ fA is unbounded.

In the first case the closure $\overline{fA}$ of fA in R is compact and so $\bigcap_{A\in\underline{F}} \overline{fA} \neq \phi$. If λ is in $\overline{fA}$, then $\underline{F}$ and the sets $[x|\lambda - 1/n < fx < \lambda + 1/n]$, $n > 0$ generate a filter $\underline{F}'$ along which f converges to λ.

In the second case either $\underline{F}$ and the sets $[x|fx > n]$, $n > 0$, or $\underline{F}$ and the sets $[x|fx < -n]$, $n > 0$ generate a filter on E. Assume that the first of these possibilities holds. Let $\underline{F}'$ be the filter generated by $\underline{F}$ and the sets $[x|fx > n]$, $n > 0$. Clearly $\lim_{\underline{F}'} f = +\infty$, which is in $\bar{R}$.

The filter $\underline{F}'$ is S-completely regular because of examples 1 and 2 and the following

Lemma 2. Let $\underline{F}$ and $\underline{F}'$ be two S-completely regular filters on E that generate a filter $\underline{F}$ on E. Then $\underline{F}$ is S-completely regular.

Proof: The sets $F \cap F'$, F in $\underline{F}$ and F' in $\underline{F}'$ form a filter basis for $\underline{F}''$. Let F_1'' be in $\underline{F}''$ and choose F_1 in $\underline{F}$, F_1' in $\underline{F}'$ with $F_1 \cap F_1'$ contained in F_1''. Let F_2, F_2' be in $\underline{F}$ and $\underline{F}'$ respectively and such that F_2 and $\complement F_1$, F_2' and $\complement F_1'$ are completely separated by S. Then $F_2 \cap F_2'$ and $\complement(F_1 \cap F_1')$ are completely separated by S (an almost immediate consequence of definition 3). Hence $F_2 \cap F_2'$ and $\complement F_1''$ are completely separated by S. Since $F_2 \cap F_2'$ is in $\underline{F}''$ it follows that $\underline{F}''$ is S-completely regular.

To complete the proof of theorem 6 it remains to consider the situation where $\underline{F}$ and the sets $[x \mid fx > n]$, $n > 0$ do not generate a filter. In this instance $\underline{F}$ and the sets $[x \mid fx < -n]$, $n > 0$ generate a filter $\underline{F}'$ on E. In view of example 2 and the above lemma $\underline{F}'$ is an S-completely regular filter. Since f converges along $\underline{F}'$ to $-\infty$ the theorem follows.

The S-completely regular filters $\underline{F}$ on E may be partially ordered by inclusion. This order is inductive and so by Zorn's lemma maximal S-completely regular filters $\underline{H}$ exist above each filter $\underline{F}$. It follows from theorem 6 that if $\underline{H}$ is a maximal S-completely regular filter on E then $\lim_{\underline{H}} f$ exists in $\overline{E}$ for all f in S. The converse of this statement is a corollary to the following theorem,

<u>Theorem 7. If $\underline{F}$ is an S-completely regular filter and if $\lim\limits_{\underline{F}} f$ exists in $\bar{R}$ for all f in S, then $\underline{F}$ is generated by the family of sets f^{-1} where V is any $\bar{R}$ neighbourhood of $\lim\limits_{\underline{F}} f$.</u>

Proof: Let F_1 be a set in $\underline{F}$ and let F_2 be a subset of F_1 be a set in $\underline{F}$ with F_2 and $\complement F_1$ completely separated by S. This means that there exist two integers n, k with $0 \leq k \leq n$, n functions $f_1, \ldots, f_n$ in S, and $2n$ real numbers $\lambda_1(e), \ldots, \lambda_k(e)$, $\mu_1(e), \ldots, \mu_{n-k}(e)$, $e = 1$ or 2 such that:

(1) $\lambda_i(1) > \lambda_i(2)$, $i = 1, \ldots, k$ and $\mu_j(1) < \mu_j(2)$, $i = 1, \ldots, n-k$; and

(2)
$$F_2 \subseteq \left(\bigcap_{i=1}^{k} [x \mid f_i x < \lambda_i(2)]\right) \cap \left(\bigcap_{j=1}^{n-k} [x \mid f_{k+j} x > \mu_j(2)]\right)$$
$$\subseteq \left(\bigcap_{i=1}^{k} [x \mid f_i x < \lambda_i(1)]\right) \cap \left(\bigcap_{j=1}^{n-k} [x \mid f_{k+j} x > \mu_j(1)]\right) \subseteq F_1 .$$

For any f in S let $\hat{f}(\underline{F})$ denote $\lim\limits_{\underline{F}} f$. Since F_2 is in $\underline{F}$ it follows from (2) that $\hat{f}_i(\underline{F}) \leq \lambda_i(2)$, $i = 1, \ldots, k$ and $\hat{f}_{k+j}(\underline{F}) \geq \mu_j(2)$, $j = 1, \ldots, n-k$. As a result $[u \in R \mid u < 1/2(\lambda_i(1) + \lambda_i(2))] \cup [-\infty]$ is an $\bar{R}$ - neighbourhood V_i of $\hat{f}_i(\underline{F})$ $i = 1, \ldots, k$ and $[u \in R \mid u > 1/2(\mu_j(1) + \mu_j(2))] \cup [+\infty]$ is an $\bar{R}$ - neighbourhood V_{k+j} of $\hat{f}_j(\underline{F})$ $j = 1, \ldots, n-k$.

Since $f_i^{-1} V_i = [x \mid f_i x < 1/2(\lambda_i(1) + \lambda_i(2))]$ and $f_{k+j}^{-1} V_{k+j} = [x \mid f_{k+j} x > 1/2(\mu_j(1) + \mu_j(2))]$ it follows from (2) that

$$F_2 \subseteq (\bigcap_{i=1}^{k} f_i^{-1} V_i) \cap (\bigcap_{j=1}^{n-k} f_{k+j}^{-1} V_{k+j}) \subseteq F_1.$$

The set F_1 was chosen as an arbitrary set in $\underline{F}$ and as a result the theorem holds.

<u>Corollary 1. Let $\underline{F}$ be an S-completely regular filter on E. Then $\lim_{\underline{F}} f$ exists in $\overline{R}$ for all f in S iff $\underline{F}$ is a maximal S-completely regular filter.</u>

Proof: It has been shown that every f in S converges along a maximal S-completely regular filter.

On the other hand if $\lim_{\underline{F}} f$ exists in $\overline{R}$ for all f in S let $\underline{M}$ containing $\underline{F}$ be a maximal S-completely regular filter. It is clear that $\lim_{\underline{F}} f = \lim_{\underline{M}} f$ for each f in S because $\underline{M}$ is finer than $\underline{F}$. From the theorem it then follows that $\underline{F}$ and $\underline{M}$ are generated by the same family of sets. Consequently $\underline{F} = \underline{M}$.

<u>Corollary 2. Let $\underline{M}_1$ and $\underline{M}_2$ be two maximal S-completely regular filters on E. Then $\underline{M}_1 = \underline{M}_2$ iff $\lim_{\underline{M}_1} f = \lim_{\underline{M}_2} f$ for each f in S.</u>

Proof: If $\lim_{\underline{M}_1} f = \lim_{\underline{M}_2} f$ for each f in S then both filters are generated by the same family of sets and so are identical.

<u>Corollary 3. If x is in E let $\underline{M}_x$ be the filter defined in example 3. It is a maximal S-completely regular filter on E.</u>

Proof: It is S-completely regular and as observed in example 3 each f in S converges along $\underline{M}_x$ to fx. By corollary 1 it is therefore a maximal S-completely regular filter.

The maximal S-completely regular filters $\underline{M}_x$, x in E, all have the property that for each f in S $\lim_{\underline{M}_x} f$ is finite or equivalently $\lim_{\underline{M}_x} f$ is in R. The maximal S-completely regular filters with this property are characterized by

<u>Theorem 8.</u> Let $\underline{F}$ <u>be an S-completely regular filter on E. Then for all f in S $\lim_{\underline{F}} f$ exists in R iff</u> $\underline{F}$ <u>is a</u> $\underline{U}(S)$ <u>- Cauchy filter on E.</u>

Proof: The uniformity $\underline{U}(S)$ is defined in § 2 to be the uniformity on E generated by the entourages $V(f,\varepsilon)$ where f is in S and $\varepsilon > 0$ and $V(f,\varepsilon) = [(x,y) \mid |fx - fy| < \varepsilon]$.

A filter $\underline{G}$ on E is $\underline{U}(S)$ - Cauchy iff it contains $V(f,\varepsilon)$ - small sets for any f in S and $\varepsilon > 0$. Hence if $\underline{G}$ is a $\underline{U}(S)$ - Cauchy filter on E $\lim_{\underline{G}} f$ exists in R for each f in S, regardless of whether or not $\underline{G}$ is S-completely regular.

Conversely let $\underline{F}$ be an S-completely regular filter on E such that $\lim_{\underline{F}} f$ exists in R for each f in S. From theorem 7 it follows that $\underline{F}$ contains $V(f,\varepsilon)$ - small sets for any f in S and $\varepsilon > 0$. Consequently $\underline{F}$ is a $\underline{U}(S)$ - Cauchy filter on E.

<u>Remark</u>. The $\underline{U}(S)$ - Cauchy filters on E that are S-completely regular are precisely the minimal $\underline{U}(S)$ - Cauchy filters. This can be seen from the next theorem.

The relation of maximal S-completely regular filters on E to filters on E along which all the functions in S converge is stated as

Theorem 9. Let $\underline{G}$ be a filter on E such that $\lim_{\underline{G}} f$ exists in $\overline{R}$ for each f in S. Then $\underline{G}$ contains a unique maximal S-completely regular filter $\underline{M}$. The filter $\underline{M}$ is generated by the sets $f^{-1}V$ where V is any $\overline{R}$ - neighbourhood of $\lim_{\underline{G}} f$. If $\underline{G}$ is $\underline{U}(S)$ - Cauchy so is $\underline{M}$.

Proof: Assume that $\underline{M}_1$ and $\underline{M}_2$ are maximal S-completely regular filters contained in $\underline{G}$. Then $\lim_{\underline{M}_1} f = \lim_{\underline{G}} f$ for each f in S and so by corollary 2 to theorem 7 $\underline{M}_1 = \underline{M}_2$.

The filter $\underline{F}$ generated by the sets $f^{-1}V$ where V is any $\overline{R}$ - neighbourhood of $\lim_{\underline{G}} f$ is S-completely regular. This is a routine consequence of definition 3. Since $\lim_{\underline{F}} f$ exists for each f in S it follows from corollary 1 to theorem 7 that $\underline{F}$ is maximal. The filter $\underline{F}$ is contained in $\underline{G}$ and so must be the unique maximal S-completely regular filter $\underline{M}$ contained in $\underline{G}$.

If $\underline{G}$ is $\underline{U}(S)$ - Cauchy then $\lim_{\underline{G}} f$ is in R for each f in S. This is true for $\underline{M}$ as well and so by theorem 8 $\underline{M}$ is $\underline{U}(S)$ - Cauchy.

§4. Processes $\mathfrak{M}$ and $\mathfrak{F}$. If (E,S) is an object of $\mathfrak{E}$ consider the collection of maximal S-completely regular filters $\underline{M}$ on E. This set may be given a topology as follows: if a subset U of E is in the weak topology $\underline{O}(E,S)$ let $U^* = [\underline{M} | U$ is in $\underline{M}]$; the sets U^* form a base for a topology on this set since $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$. Let $M(E,S)$ denote the resulting topological space.

Let α be a mapping in Hom$((E,S), (E',S'))$. If $\underline{M}$ is in $M(E,S)$ let $\alpha\underline{M}$ be the filter generated on E' by the sets αF, F in $\underline{M}$. Since S contains $S' \circ \alpha$ it follows from corollary 1 to theorem 7 that $\lim\limits_{\underline{M}} f' \circ \alpha$ $= \lim\limits_{\alpha\underline{M}} f'$ exists in $\overline{R}$ for each f' in S'. By theorem 9 there is a unique filter $\underline{M}'$ in $M(E',S')$ contained in $\alpha\underline{M}'$. Define $M(\alpha)$ by setting $M(\alpha)\,\underline{M} = \underline{M}'$.

The function $M(\alpha)$ is continuous. Let U' be an $\underline{O}(E',S')$ open set in $M(\alpha)\underline{M}_o$. Theorem 7 shows that there are m functions f_i' in S'. and m $\overline{R}$ - neighbourhoods V_i of $\lambda_i = \lim\limits_{\alpha\underline{M}_o} f_i = \lim\limits_{M(\alpha)\underline{M}_o} f_i$ such that U' contains $\bigcap\limits_{i=1}^{m} f_i'^{-1} V_i$. Let W_i be an open $\overline{R}$ - neighbourhood of λ_i such that V_i contains $\overline{W}_i$. The set $W = \bigcap\limits_{i=1}^{m} (f_i'^{-1} \circ \alpha) W_i$ is in $\underline{O}(E,S)$ and belongs to $\underline{M}_o$ by theorem 7. Furthermore if W is in $\underline{M}$ then $\lim\limits_{\alpha\underline{M}} f_i'$ is in $\overline{W}_i$ and hence in V_i for $i = 1, \ldots, m$. Therefore $M^{-1}(\alpha)\,(U')^*$ contains $\underline{M}_o$ and the neighbourhood W^* of $\underline{M}_o$. Consequently $M(\alpha)$ is continuous at $\underline{M}_o$ and hence is continuous on $M(E,S)$ since $\underline{M}_o$ is arbitrary.

Let α' be in Hom $((E',S'), (E'',S''))$. Then $M(\alpha' \circ \alpha) =$

$M(\alpha') \circ M(\alpha)$. Let $\underline{M}$ be in $M(E,S)$. The filters $(\alpha' \circ \alpha)\underline{M}$ and $\alpha'(\alpha\underline{M})$ are identical. Since $\alpha(\alpha\underline{M})$ contains $\alpha'(M(\alpha)\underline{M})$ it follows that $(\alpha' \circ \alpha)\underline{M}$ contains $(M(\alpha') \circ M(\alpha))\underline{M}$. Therefore $M(\alpha' \circ \alpha)\underline{M} = (M(\alpha') \circ M(\alpha))\underline{M}$. Consequently $M: \Phi \longrightarrow \Sigma$ is a covariant functor.

Let (E,S) be an object of Φ. Corollary 3 of theorem 7 states that $\underline{M}_x$ - the filter on E generated by the sets $[u|fx - 1/n < fu < fx + 1/n]$, f in S and $n > 0$ - is a maximal S-completely regular filter on E. Define $m_S: E \longrightarrow M(E,S)$ by setting $m_S x = \underline{M}_x$.

The set $m_S E$ is a dense subset of $M(E,S)$. To prove this it is sufficient to show that when U is in the weak topology $\underline{O}(E,S)$ then $U^M \cap m_S E = m_S U$. Assuming this to be so the fact that the sets U^M form a base for the topology implies $m_S E$ is dense in $M(E,S)$ as $m_S U = \phi$ iff $U = \phi$ and $\phi^M = \phi$.

Let U be in $\underline{O}(E,S)$ and let x be a point of U. Then there exist m functions $f_1, \ldots, f_m$ in S and $n > 0$ such that $\bigcap_{i=1}^{m} [u|f_i x - 1/n < f_i u < f_i x + 1/n]$ is contained in U . From the definition of $\underline{M}_x$ this shows that U is in $\underline{M}_x$ iff x is in U i.e. $U^M \cap m_S E = m_S U$.

Let (E',S') be a second object of Φ and let α be in $\mathrm{Hom}\,((E,S), (E',S'))$. If x is in E then $\alpha\underline{M}_x$ contains $\underline{M}'_{\alpha x}$ and so $M(\alpha)\underline{M}_x = \underline{M}'_{\alpha x}$ or in other words $M(\alpha) \circ m_S = m_{S'} \circ \alpha$.

Consequently <u>the functor M and the family $(m_S)_{(E,S)}$ in Φ of functions m_S define a process on Φ which will be denoted by $\mathfrak{M}$.</u>

If (E,S) is any object of Φ the functions f in S define $\bar{R}$ - valued functions F on $M(E,S)$ such that $F \circ m_S = f$ by setting $F\underline{U} = \lim_{\underline{U}} f$ for each $\underline{U}$ in $M(E,S)$. <u>The functions F are all continuous.</u> Let V be an open subset of $\bar{R}$. Theorem 5 and its first corollary show that $\lim_{\underline{U}} f$ is in V iff $f^{-1}V$ is in $\underline{U}$. Since R is a subspace of $\bar{R}$, $f^{-1}V$ is in $\underline{O}(E,S)$. This means that $(f^{-1}V)^*$ is defined and that $F^{-1}V = (f^{-1}V)^* = [\underline{U} | f^{-1}V \text{ is in } \underline{U}]$. Since this is an open subset of $M(E,S)$ it follows that F is continuous. <u>The topology of $M(E,S)$ is the weak topology $O(M(E,S), \bar{R})$.</u> Since this topology is coarser than the topology of $M(E,S)$ it is sufficient to show that the sets U^* are open in this weak topology. Let U be in $\underline{O}(E,S)$ and assume that $\underline{U}$ is in U^* i.e. U is in $\underline{U}$. Theorem 7 shows that there are n functions $f_1,\dots,f_n$ and n $\bar{R}$-neighbourhoods $V_1,\dots,V_n$ of $F_1\underline{U},\dots, F_n\underline{U}$ such that $\bigcap_{i=1}^{n} f_i^{-1}V_i$ is contained in U. The sets $f_i^{-1}V_i$ are in $\underline{O}(E,S)$ and $F_i^{-1}V_i = (f_i^{-1}V_i)^*$. This implies that

$\bigcap_{i=1}^{n} F_i^{-1}V_i)^*$ is contained in U^* and since $\bigcap_{i=1}^{n} f_i^{-1} V_i$ is in $\underline{U}$

(theorem 7) it follows that U^* is an $\underline{O}(M(E,S),\bar{S})$ - neighbourhood of every point it contains i.e. U^* is in $\underline{O}(M(E,S)\bar{S})$.

One property of $\mathfrak{M}$ that distinguishes it from $\mathfrak{J}$ is the fact that each $M(E,S)$ is a compact space i.e. $\mathfrak{M}$ is a compact process. To prove this $M(E,S)$ may be mapped into the compact space $\prod_{f \text{ in } S} \bar{R}_f$ by the function δ where $\delta\underline{M}$ = $(\bar{F}(\underline{M}))_{f \text{ in } S} = (\lim_{\underline{M}} f)_{f \text{ in } S}$. Corollary 2 to theorem 7 shows that δ is 1 - 1 . It is also an embedding since the topology of $M(E,S)$ is the weak topology $\underline{O}(M(E,S),\bar{S})$. Hence $M(E,S)$ is compact iff $\delta(M(E,S))$ is closed.

Let Z be in the closure of $\delta(M(E,S))$. Then Z is also in the closure of $\delta(m_S E)$ since $m_S E$ is dense in $M(E,S)$. Let $Z = (Z_f)_{f \text{ in } S}$ and let V_f denote an arbitrary $\bar{R}$ - neighbourhood of Z_f . The family of sets $f^{-1}(V_f)$ generates a filter $\underline{F}$ on E . To prove this take n functions f_i and n corresponding neighbourhoods V_i . Since Z is in the closure of $\delta(m_S E)$ there is a point x of E with $((\delta \circ m_S)x)_{fi}$ in V_i . The point x is in $\bigcap_{i=1}^{n} f_i^{-1} V_i$ because $((\delta \circ m_S(x)f_i = f_i x$. The filter $\underline{F}$ is S-completely regular and $\lim_{\underline{F}} f = Z_f$ for each f in S . This shows that $\underline{F}$ is in $M(E,S)$ and that $\delta\,\underline{F} = Z$. Consequently $M(E,S)$ is compact.

Process $\mathfrak{J}$ is not a compact process because as observed in the example of theorem 4 $T(E,S)$ is compact iff every function in S is bounded. As a result $\mathfrak{J}$ and $\mathfrak{M}$ are not isomorphic processes on Φ.

Remarks. From the proof of the compactness of $\mathfrak{M}$ it can be seen that $\mathfrak{M}$ is essentially a modified version of $\mathfrak{I}$. The modification consists of replacing the real line by its two-point compactification $\bar{R}$. In general each method of extending the real line will produce a corresponding modification of $\mathfrak{I}$. In each case the modified process will satisfy suitably modified versions of $(FP_3)(FP_4)(FP_5)$ and (FP_6) - assuming only completely regular extensions are considered.

The second process of this section - process $\mathfrak{F}$ - is defined by means of $\mathfrak{M}$. If (E,S) is an object of Φ define $F(E,S)$ to be the subspace of $M(E,S)$ consisting of all the $\underline{U}(S)$ - Cauchy filters in that set. If α is in $Hom((E,S),(E',S'))$ define $F(\alpha)$ to be $M(\alpha)|M(E,S)$. To show that $F: \Phi \longrightarrow \Sigma$ is a covariant functor it is sufficient to prove that $F(\alpha):F(E,S) \longrightarrow F(E',S')$ and that $F(\alpha' \circ \alpha) = F(\alpha') \circ F(\alpha)$. Both of these assertions are consequences of theorem 9 and the definition of $M(\alpha)$. If $\underline{M}$ is in $M(E,S)$ $M(\alpha)\underline{M}$ is the unique maximal S'-completely regular filter $\underline{M}'$ contained in $\alpha\underline{M}$. Since α is $(\underline{U}(S), \underline{U}(S'))$ - uniformly continuous $\alpha\underline{M}$ is $\underline{U}(S')$ - Cauchy if $\underline{M}$ is $\underline{U}(S)$ - Cauchy. Theorem 9 then shows that $M(\alpha)\underline{M}$ is $\underline{U}(S')$ - Cauchy and so $F(\alpha)$ maps $F(E,S)$ into $F(E',S')$. It also proves that $F(\alpha' \circ \alpha) = F(\alpha') \circ F(\alpha)$ since when $\underline{M}$ is a $\underline{U}(S)$ - Cauchy filter in $M(E,S)$, the $\underline{U}(S'')$ - Cauchy filter $M(\alpha' \circ \alpha)\underline{M} = M(\alpha')\underline{M}'$ where $\underline{M}' = M(\alpha)\underline{M}$ is $\underline{U}(S')$ -

Cauchy.

Theorem 8 shows that the filters $\underline{M}_x$, x in E, are all $\underline{U}(S)$ - Cauchy. Consequently $m_S E$ is contained in $F(E,S)$. Let $f_S = m_S$. Then $f_S E$ is dense in $F(E,S)$ and $F(\alpha) \circ f_S = f_{S'} \circ \alpha$.

<u>The process $\mathfrak{F}$ is defined to be the functor F and the family $(f_S)_{(E,S) \text{ in } \mathfrak{E}}$ of functions f_S.</u>

From the definition of $\mathfrak{F}$ as a 'subprocess' of $\mathfrak{M}$ it is clear that the natural injections $i(E,S):F(E,S) \longrightarrow M(E,S)$ define a homomorphism of $\mathfrak{F}$ into $\mathfrak{M}$.

The all inclusive property of this process is stated as

<u>Theorem 10.</u> <u>$\mathfrak{F}$ and $\mathfrak{T}$ are isomorphic processes on $\mathfrak{E}$. Hence $\mathfrak{F}$ is idempotent.</u>

Proof: Let (E,S) be an object of $\mathfrak{E}$. Then $\prod_{f \text{ in } S} R_f$ is a subspace of $\prod \bar{R}_f$. Theorem 8 shows that δ maps $F(E,S)$ onto $\delta(M(E,S)) \cap \prod_{f \text{ in } S} R_f$. Let $\gamma(E,S) = \delta | F(E,S)$. Then $\gamma(E,S)$ is a homeomorphism of $F(E,S)$ with a closed subset of $\prod_{f \text{ in } S} R_f$ such that $\gamma(E,S) \circ f_S = t_S$. This implies that $\gamma(E,S)F(E,S) = T(E,S)$. Since $\mathfrak{T}$ is a Hausdorff process these homeomorphisms define an isomorphism of $\mathfrak{F}$ into $\mathfrak{T}$ (lemma 1).

The 'potential' process $\mathfrak{F} \square \mathfrak{F}$ is defined because $\mathfrak{F}$ is defined on $\mathfrak{E}$ and satisfies (FP_3). $\mathfrak{F}$ is therefore idempotent (by the corollary to theorem 5 or by either of theorems 2 and 3).

Remarks. The process $\mathcal{F}$ can be defined without explicit use of process $\mathcal{M}$. When this is done it is possible to prove this theorem by means of an argument similar to that used to show that $\mathcal{M}$ is compact.

One property of process $\mathcal{F}$ which will be useful later in the chapter is stated as

Lemma 3. Let (E,S) be an object of Φ. Let g denote a real-valued function on E. Then g is in $C_{F(E,S)} \circ f_S$ iff $\lim_{\underline{M}} g$ exists in R for each $\underline{M}$ in $F(E,S)$.

Proof: Let h be a continuous real-valued function on $F(E,S)$. Let $\underline{M}$ be a point of $F(E,S)$ and let $\mathcal{V}_F(\underline{M})$ denote the neighbourhood filter of $\underline{M}$. It has a basis of sets $U^* \cap F(E,S)$ where U is in $\underline{O}(E,S)$ and belongs to $\underline{M}$. Consequently $f_S^{-1}\mathcal{V}_F(\underline{M}) = \underline{M}$. Since h is continuous $\lim_{\mathcal{V}_F(\underline{M})} h$ exists in R for each $\underline{M}$. This shows that $\lim_{\underline{M}}(h \circ f_S)$ exists in R for each $\underline{M}$ since $f_S\underline{M}$ contains $\mathcal{V}_F(\underline{M})$.

Assume that g is a real-valued function on E such that $\lim_{\underline{M}} g$ exists in R for each $\underline{M}$ in $F(E,S)$. Then g is $\underline{O}(E,S)$ continuous since the filters $\underline{M}_x$ are the neighbourhood filters of the points of E.

Define $\bar{g}$ on $F(E,S)$ by setting $\bar{g}\,\underline{M} = \lim_{\underline{M}} g$. Let V and W be open neighbourhoods of $\bar{g}\,\underline{M}_o$ such that $\overline{W}$ is contained in V. Then $\underline{M}_o$ is in $(g^{-1}W)^*$ which is contained in V

since if $g^{-1}W$ is in $\underline{H}$ $\lim_{\underline{H}} g$ is in $\overline{W}$. This proves that $\overline{g}$ is continuous. Since $\overline{g}$ of $f_S = g$ the lemma is proved.

Processes $\mathfrak{F}$ and $\mathfrak{M}$ coincide on the subcategory consisting of those objects (E,S) for which every maximal S-completely regular filter is $\underline{U}(S)$ - Cauchy. <u>This is the case iff each function in S is bounded.</u> When every function f in S is bounded it is clear that $\lim_{\underline{H}} f$ is real for each f in S and $\underline{H}$ in $M(E,S)$. This shows that the sets $F(E,S)$ and $M(E,S)$ coincide when every function in S is bounded. Assume that S contains an unbounded function f_o. Then one of the two filters of example 2 in § 3 is defined. Since there is a maximal S-completely regular filter $\underline{H}$ finer than it, it follows that $\lim_{\underline{H}} f_o = +\infty$ (or $-\infty$). The filter $\underline{H}$ is not $\underline{U}(S)$ - Cauchy.

Let Φ^* be the subcategory of Φ obtained by considering only those objects (E,S) for which S consists entirely of bounded functions. <u>Then $\mathfrak{F} \mid \Phi^* = \mathfrak{M} \mid \Phi^*$ and so $\mathfrak{J} \mid \Phi^*$ is isomorphic to $\mathfrak{M} \mid \Phi^*$.</u>

<u>Remark.</u> Alexandroff [6] essentially used process $\mathfrak{M}$ to construct βE. Banaschewski [5] used this construction of βE to construct υE in a way analogous to the construction of $\mathfrak{F}$ as a "subprocess" of $\mathfrak{M}$. Since $\mathfrak{F}$ is isomorphic to $\mathfrak{J}$ it satisfies (FP_3). If (E,S) is an object of Φ it is clear that for f in S f_F is the restriction of $\overline{f}$ to $F(E,S)$ i.e., $f_F\underline{H} = \lim_{\underline{H}} f$ for each $\underline{H}$ in $F(E,S)$.

§ 5. Process $\mathcal{H}$. Let (E,S) be an object of Φ. The collection S will be said to be a unitary real algebra on E if S has the following properties: when f and g are in S the functions $f + g$, fg are in S where $(f + g)x = fx + gx$ and $(fg)x = (fx)(gx)$ for each x in E; the function 1 is in S where $1x = 1$ for each x in E; and if λ is any real number and f is in S then λf is in S, where $(\lambda f)x = \lambda(fx)$ for each x in E.

Let A be the subcategory of Φ obtained by restricting the objects to be those pairs (E,S) such that S is a unitary real algebra on E.

If (E,S) is an object of A let $H(E,S)$ be the set of real-valued unitary algebra homomorphisms $h:S \longrightarrow R$ (where unitary means that $h(1) = 1$) together with the topology generated by the sets V_f, f in S, where $V_f = [h \text{ in } H(E,S) | h(f) \neq 0]$. This topology is called Zariski topology. The sets V_f, f in S, form a base for the Zariski topology since for f,g in S $V_f \cap V_g = V_{fg}$.

Let (E',S') be a second object of A and let α be in $\mathrm{Hom}((E,S),(E',S'))$. The correspondence $f' \longrightarrow f' \circ \alpha$ is a unitary algebra homomorphism of S' into S. Consequently it induces a function $H(\alpha):H(E,S) \longrightarrow H(E',S')$ - the transpose of this correspondence - which is defined by setting $H(\alpha)h = h'$ where $h'(f') = h(f' \circ \alpha)$. The function h' is a real-valued algebra homomorphism and it is unitary because $1' \circ \alpha = 1$ and so $h'(1') = h(1' \circ \alpha) = h(1) = 1$.

The function $H(\alpha)$ is continuous. If f' is in S' then $H^{-1}(\alpha)V'_{f'} = [h \text{ in } H(E,S) \mid (H(\alpha)h)(f') \neq 0] = [h \text{ in } H(E,S) \mid h(f' \circ \alpha) \neq 0] = V_{f' \circ \alpha}$. Since the sets $V'_{f'}$, f' in S', generate the topology of $H(E',S')$ this shows that $H(\alpha)$ is continuous.

Let (E'',S'') be a third object of A and let α' be in $\mathrm{Hom}((E',S'),(E'',S''))$. Then $H(\alpha' \circ \alpha) = H(\alpha') \circ H(\alpha)$. Let h be in $H(E,S)$ and pick f'' in S''. $(H(\alpha')[H(\alpha)h])(f'') = [H(\alpha)h](f'' \circ \alpha') = h(f'' \circ \alpha' \circ \alpha) = (H(\alpha' \circ \alpha)h)(f'')$. This completes the proof of the fact that $H: A \longrightarrow \Sigma$ is a covariant functor.

Let (E,S) be an object of A. Define $h_S: E \quad H(E,S)$ by setting $h_S x = h_x$, where $h_x f = fx$ for any function f in S. Since S is a unitary real algebra on E h_x is a real-valued unitary algebra homomorphism of S for each x in E.

The set $h_S E$ is dense in $H(E,S)$. If $V_f \cap h_S E = \phi$ $h_x f = fx = 0$ for all x in E. This means that $f = 0$ and so $V_f = V_0 = \phi$. The fact that the sets V_f, f in S, form a base for the topology of $H(E,S)$ implies that $h_S E$ is dense in $H(E,S)$.

If x is in E then $(H(\alpha)h_x)(f') = h_x(f' \circ \alpha) = (f' \circ \alpha)x = f'(\alpha x) = h'_{\alpha x} f$. This shows that $H(\alpha) \circ h_S = h_{S'} \circ \alpha$.

Consequently <u>the functor H on A and the family $(h_S)_{(E,S) \text{ in } A}$ of functions h_S define a process on A which will be denoted by $\mathcal{H}$.</u>

<u>$\mathcal{H}$ is a T_1-process on A.</u> Let (E,S) be an object of A and let $h_1 \neq h_2$ be two points in $H(E,S)$. Since they are distinct there exists at least one function f in S with $h_1(f) \neq h_2(f)$. Let $g = f - h_2(f)\cdot 1$. This function is in S and $h_1(g) \neq 0$, $h_1(g) = 0$. Consequently h_1 is in V_g and h_2 is in V_g. This shows that $\mathbb{C}[h_2]$ is open and hence that $[h_2]$ is closed.

<u>$\mathcal{H}$ satisfies (FP_4) on A.</u> Let (E,S) and (E',S') be two objects of A and let α be in $\mathrm{Hom}((E,S),(E',S'))$. Assume that $S = S' \circ \alpha$ and that $f' \longrightarrow f' \circ \alpha$ is $1-1$. The correspondence $f' \longrightarrow f' \circ \alpha$ is then a unitary isomorphism of the algebra S' with the algebra S. As a result $H(\alpha)$ is $1-1$ and every h' in $H(E',S')$ is $H(\alpha)\,h$ for some h in $H(E,S)$. If f is in S and equals $f' \circ \alpha$ then $H(\alpha)V_f = [H(\alpha)h \mid h(f) \neq 0] = [H(\alpha)h \mid h(f'\circ\alpha) \neq 0] = [H(\alpha)h \mid (H(\alpha)h)(f') \neq 0] = V'_{f'}$. This shows that $H(\alpha)$ is an open mapping and as a result that $H(\alpha)$ is a homeomorphism.

<u>Process $\mathcal{H}$ is not Hausdorff on A and it does not satisfy (FP_3) on A.</u> Consider the following

<u>Example.</u> Let (E,S) be the object of A where E is the space of real numbers R and S is the collection of polynomial functions on R. If $f(X)$ is in $R[X]$ it defines the polynomial function f on R by setting fx equal to the value of $f(X)$ at x. The correspondence $f(X) \longrightarrow f$ is an algebra isomorphism of $R[X]$ with S. Hilbert's Nullstellensatz shows that each

maximal ideal of $R[X]$ consists of all polynomials with a common real root. This implies that $h_S E = H(E,S)$ in this case. Since a polynomial $f(X)$ has only a finite number of real roots it follows that for every f in S $\complement V_f$ is finite. The fact that $h_S E$ is not finite shows that $H(E,S) = h_S E$ is not a Hausdorff space. In addition the functions cannot all define continuous functions f_H on $H(E,S)$ such that $f_H \circ h_S = f$. This is because they would then separate the points of $H(E,S)$ which would imply that the topology is Hausdorff.

Remark. This example shows that $\mathfrak{H}$ is not isomorphic to $\mathfrak{J}|A$.

Let (E,S) be an object of A. The functions f in S do define real-valued functions $\bar{f}$ on $H(E,S)$ in a "natural" way by setting $\bar{f}h = h(f)$ for each h in $H(E,S)$. The functions $\bar{f}$ are such that $\bar{f} \circ h_S = f$ for each f in S. In addition $\bar{S} = [\bar{f} \mid f \text{ in } S]$ is a unitary real algebra on $H(E,S)$. If f and g are in S then $\overline{f+g}\, h = h(f + g) = h(f) + h(g) = \bar{f}h + \bar{g}h$ and similarly $\overline{fg}h = (\bar{f}h)(\bar{g}h)$, $\bar{1}h = 1$, and $(\lambda\bar{f})h = \lambda(\bar{f}h)$ for each h in $H(E,S)$.

In general, as the previous example shows, $\bar{S}$ contains functions that are not continuous. Since $V_f = \bar{f}^{-1}(R - [0])$ it follows that each $\bar{f}$ in $\bar{S}$ is continuous iff $\underline{C}(H(E,S),\bar{S})$ is the topology of $H(E,S)$. The main purpose of the remainder of this section is to obtain sufficient conditions on S for $\bar{S}$ to consist of continuous functions. These conditions determine a subcategory of A on which it will be shown that the restrictions of $\mathcal{H}$ and $\mathcal{J}$ are isomorphic processes.

Remarks. Suppose that (E,S) is an object in A such that each function f in S does define a continuous function f_H on $H(E,S)$ for which $f_H \circ h_S = f$. What is the connection between f_H and $\bar{f}$? If $H(E,S)$ is a Hausdorff space then $f_H = \bar{f}$ for each f in S. This can be proved by showing that S_H is a unitary real algebra on $H(E,S)$ and defining $\delta: H(E,S) \to H(E,S)$ by setting $(\delta h)(f) = f_H h$. Since the functions in S_H are continuous δ is continuous. It is clear that $\delta \circ h_S = h_S$ and so when $H(E,S)$ is a Hausdorff space $\delta h = h$ for each h in $H(E,S)$. This implies that $f_H h = (\delta h)(f) = h(f) = \bar{f}h$ for each h in $H(E,S)$ and so $f_H = \bar{f}$. The question prompted by this result is whether the functions f_H exist when $H(E,S)$ is Hausdorff i.e. if $H(E,S)$ is Hausdorff are the functions in $\bar{S}$ all continuous?

Let (E,S) be an object of A and define $\gamma(E,S)$: $H(E,S) \to \prod_{f \text{ in } S} R_f$ by setting $\gamma(E,S)h = (\bar{f}h)_{f \text{ in } S} = (h(f))_{f \text{ in } S}$.

The function $\gamma(E,S)$ is an embedding iff each $\bar{f}$ in $\bar{S}$ is continuous. This is because $\underline{O}(H(E,S),\bar{S})$ is the topology of $H(E,S)$ iff each $\bar{f}$ in $\bar{S}$ is continuous.

The image set $\gamma(E,S)\,H(E,S)$ is a closed subset of $\prod_{f \text{ in } S} R_f$. Let $Z = (Z_f)_{f \text{ in } S}$ be in the closure of the image.

It follows from the definition of the product topology that $Z_{f+g} = Z_f + Z_g$, $Z_{fg} = Z_f Z_g$, $Z_1 = 1$ and $Z_{\lambda f} = \lambda Z_f$. This shows that $h_Z : S \longrightarrow R$, where $h_Z(f) = Z_f$ is a unitary algebra homomorphism. Since $\gamma(E,S)h_Z = Z$ it follows that the image is closed.

From the definition of $\gamma(E,S)$ it follows that $\gamma(E,S) \circ h_S = t_S$. As a result <u>$T(E,S)$ is contained in $\gamma(E,S)\ H(E,S)$</u>. In particular if $\gamma(E,S)$ is an embedding it is a homeomorphism of $T(E,S)$ with $H(E,S)$.

If f,g are in S set $f \leqslant g$ if $fx \leqslant gx$ for each x in E. It is clear that $\leqslant$ defines a partial order on S which is preserved under addition by arbitrary functions and under multiplication by positive ones i.e. $f \leqslant g$ implies $f + k \leqslant g + k$ for all k in S and $fk \leqslant gk$ if $k \leqslant 0$. <u>A unitary algebra homomorphism $h : S \longrightarrow R$ is said to be order preserving if $f \leqslant g$ implies $h(f) \leqslant h(g)$ or equivalently if $f \geqslant 0$ implies that $h(f) \geqslant 0$.</u>

Let h be in $H(E,S)$. <u>Then $\gamma(E,S)h$ is in $T(E,S)$ iff h is order preserving.</u> Let $\gamma(E,S)h$ be in $T(E,S)$. If f in S is $\geqslant 0$ then $(t_S x)_f = fx \geqslant 0$. Since $T(E,S)$ is the closure in $\prod_{f \text{ in } S} E_f$ of $t_S E$ it follows that $h(f) \geqslant 0$ i.e. h is order preserving. Assume $\gamma(E,S)h$ is not in $T(E,S)$. Then there exist m functions $f_1, \ldots, f_m$ in S and some $\varepsilon > 0$ such that for all x in E at least one of the $|h(f_i) - f_i x| \geqslant \varepsilon$. Let $g_i = h(f_i) \cdot 1 - f_i$ and let $g = \sum_{i=1}^{m} g_i^2$. This function is in S and if x is in E $gx \geqslant \varepsilon^2 > 0$.

Since $\varepsilon^2.1$ is in S it follows that $g - \varepsilon^2.1 \geqslant 0$. The homomorphism h does not preserve order because $h(g - \varepsilon^2.1) = h(g) - \varepsilon^2.1 = -\varepsilon^2.1 < 0$.

Note. This assertion and its proof are due to Isbell [7].

Remark. Since the aim of the rest of this section is to obtain conditions on S in order that the functions of $\overline{S}$ are all continuous it is natural to wonder why the process was not defined using the weak topology $\sigma(H(E,S),\overline{S})$. The reason is that *$h_E E$ is dense in $H(E,S)$ with respect to $\sigma(H(E,S),\overline{S})$ iff every h in $H(E,S)$ is order preserving.* This is an immediate consequence of the previous assertion.

The connection between the continuity of the functions in $\overline{S}$ and the order preserving properties of the homomorphisms in $H(E,S)$ is stated as

Theorem 11. *Let (E,S) be an object of A. If each $\overline{f}$ in $\overline{S}$ is continuous then every h in $H(E,S)$ is order preserving. Conversely if each h in $H(E,S)$ is order preserving and S satisfies condition $(*)$*

– *for real $\lambda \geqslant 0$ and f in S the functions $f \cap \lambda$ and $f \cup (-\lambda)$ are in S, where $(f \cap \lambda)x = \min(fx, \lambda)$ and $(f \cup (-\lambda))x = \max(fx, -\lambda)$* – *then each $\overline{f}$ in $\overline{S}$ is continuous.*

Proof: If each $\overline{f}$ in $\overline{S}$ is continuous then $\gamma(E,S)\, H(E,S) = T(E,S)$ and so every real-valued unitary algebra homomorphism is order preserving.

To prove the second half of the theorem it is sufficient to find a function g in S given h_0 in $H(E,S)$, f in S and $\varepsilon > 0$ such that $V_g = [h \text{ in } H(E,S) \mid |\bar{f}h - \bar{f}h_0| < \varepsilon] = [h \text{ in } H(E,S) \mid |h(f) - h_0(f)| < \varepsilon]$. Let $g_1 = f - h_0(f)$ and let $g = (g_1 \cup 0 - g_1 \cap 0) \cap \varepsilon - \varepsilon.1$. This function is in S and has the desired property.

If f is in S and the real number λ is ≥ 0 then for each x in E $h_x(f \cap \lambda) = (f \cap \lambda)x = fx \cap \lambda = h_x(f) \cap \lambda$ and $h_x(f \cup (-\lambda)) = h_x(f) \cup (-\lambda)$. From the characterization of the order preserving algebra homomorphisms h of $H(E,S)$ it follows that for such a homomorphism h, $h(f \cap \lambda) = h(f) \cap \lambda$ and $h(f \cup (-\lambda)) = h(f) \cup (-\lambda)$. Furthermore if $|f| = f \cup 0 - f \cap 0$ then $h(|f|) = h(f) \cup 0 - h(f) \cap 0 = |h(f)|$ for any order preserving homomorphism h.

Consequently when each h in $H(E,S)$ is order preserving it follows that $h(g) = h(|g_1| \cap \varepsilon - \varepsilon.1) = h(|g_1| \cap \varepsilon) - \varepsilon = h(|g_1|) \cap \varepsilon - \varepsilon = |h(g_1)| \cap \varepsilon - \varepsilon = |h(f) - h_0(f)| \cap \varepsilon - \varepsilon$. This shows that $h(g) = 0$ iff $|h(f) - h_0(f)| \geq \varepsilon$.

To complete this section it is sufficient to obtain a condition on S which implies that every h in $H(E,S)$ is order preserving. Isbell [7] defined a unitary real algebra S on E to be <u>closed under bounded inversion if $1/f$ is in S when $f \geq 1$</u>, where $(1/f)x = 1/fx$. He showed that <u>if S is closed under bounded inversion then each h in $H(E,S)$ is order preserving.</u> Let h be in $H(E,S)$ and such that for some $f \geqslant 0$ $h(f) = \lambda < 0$. The function $f - \lambda \geqslant -\lambda > 0$ and so has an inverse g in S. This implies that $1 = h(1) = h(f - \lambda)\cdot h(g) = 0$ which is a contradiction.

Let A' be the subcategory of A obtained by restricting the objects to be those objects (E,S) of A for which S satisfies condition (*) of theorem 10 and is closed under bounded inversion. On this category the restrictions of $\mathcal{H}$ and $\mathcal{T}$ are isomorphic as stated in

Theorem 12. $\mathcal{T}|A'$ and $\mathcal{H}|A'$ are isomorphic processes on A'. $\mathcal{H}|A'$ is idempotent.

Proof: If (E,S) is an object of A' it follows from the previous assertion and theorem 10 that $\bar{S}$ consists entirely of continuous functions. As a result $\gamma(E,S):H(E,S) \longrightarrow T(E,S)$ is a homeomorphism. Since $\gamma(E,S) \circ h_S = t_S$ and $\mathcal{T}$ is Hausdorff it follows from lemma 1 that these homeomorphisms define an isomorphism of $\mathcal{H}|A'$ into $\mathcal{T}|A'$.

To prove $\mathcal{H}|A'$ is idempotent it is sufficient to show that $\mathcal{H}|A' \,\square\, \mathcal{H}|A'$ is defined (see corollary to theorem 5). Now A' contains $H_* A'$ because if (E,S) is in A' it follows that $\bar{S}$ is closed under bounded inversion and satisfies condition (*) of theorem 10. Consequently $\mathcal{H}|A'$ is idempotent.

Remarks. 1. Theorem 12 could have been obtained by a subprocess $\mathcal{H}_o$ of $\mathcal{H}$ and establishing an isomorphism theorem with $\mathcal{T}$. For each object (E,S) in A define $H_o(E,S)$ to be the subspace of $H(E,S)$ consisting of all the order preserving homomorphisms in $H(E,S)$. Define $H_o(\alpha)$ to be $H(\alpha)|H(E,S)$ and let $(h_o)_S = h_S$.

It is not hard to see that this defines a subprocess $\mathcal{H}_o$ of $\mathcal{H}$. In addition $\mathcal{H}_o$ is isomorphic to $\mathcal{J}$ when both are restricted to the subcategory of A obtained by considering those objects (E,S) for which S satisfies $(*)$. The desired homeomorphisms are the functions $\gamma(E,S)|H_o(E,S) = \gamma_o(E,S)$. Since their inverses are always continuous this raises the question as to whether they define a homomorphism of $\mathcal{J}|A$ into $\mathcal{H}$. This is so iff $\gamma_o^{-1}(E',S') \circ T(\alpha) = H(\alpha) \circ \gamma_o^{-1}(E,S)$ when α is in $\mathrm{Hom}((E,S),(E',S'))$. This does not appear to be the case in general since if Z is in $T(E,S)$ and $(H(\alpha) \circ \gamma_o^{-1}(E,S))Z = h'$ then $h'(f') = Z_{f' \circ \alpha}$ but $(\gamma_o^{-1}(E',S') \circ T(\alpha)Z = h$ where $h''(f') = (T(\alpha)Z)_{f'}$. In general $(T(\alpha)Z)_{f'} \neq Z_{f' \circ \alpha}$. The equality holds if $f' \longrightarrow f' \circ \alpha$ is an injection of S' into S because $T(\alpha)$ is then the restriction to $T(E,S)$ of the projection of $\prod_{f \text{ in } S} R_f$ on $\prod_{f' \text{ in } S'} R_{f'}$, which maps $(Z_f)_{f \text{ in } S} \longrightarrow (Z_{f' \circ \alpha})_{f' \text{ in } S}$.

2. The proof of the idempotence of $\mathcal{H}|A'$ could have been achieved by defined a modified form of composition of with itself utilizing the functions $\bar{F}$ in a more or less obvious way to define a covariant functor $H_+ : A \longrightarrow A$. Since $\mathcal{H}$ satisfies (FP_6) it is not hard to see that $\mathcal{H}$ is idempotent with respect to this modified type of composition. In addition this new type of composition coincides with the original one on A'.

3. The origins of this process go back to Stone's work on Boolean rings where he introduces a topology into the space of maximal ideals of a Boolean ring which is the analogue of the Zariski-topology. In the case of a Boolean algebra it is this topology.

§6. Translation lattices of functions and the processes $\mathcal{L}_+, \mathcal{L}_-$ and $\mathcal{L}$. Let E be a set and denote by S a collection of real-valued functions $f, g \dots$ on E. The collection S may be partially ordered by setting $f \leq g$ if $fx \leq gx$ for each x in E. S becomes a lattice in this order if f and g in S imply that $f \cup g$ and $f \cap g$ are in S where $(f \cup g)x = \max[fx, gx]$ and $(f \cap g)x = \min[fx, gx]$ for each x in E. If in addition S contains with each f and real number λ the function $f + \lambda$ defined by setting $(f + \lambda)x = fx + \lambda$ for each x in E then S will be called a translation lattice of functions on E.

Remarks. Translation lattices were first defined by Kaplansky [8]. He defined a translation lattice to be a distributive lattice L such that for x in L and λ real a sum $x + \lambda$ is defined in L which satisfies:

(TL_1) $x + 0 = x$ for each x in L;

(TL_2) $(x + \lambda) + \mu = x + (\lambda + \mu)$ for each x in L and λ, μ real;

(TL_3) $\lambda > 0$ implies $x + \lambda > x$ for each x in L;

(TL_4) $x > y$ implies $x + \lambda > y + \lambda$ for each real λ; and

(TL_5) for any pair x, y in L there exists λ such that $x + \lambda > y$; and

(TL_6) $x < y + \lambda$ for each $\lambda > 0$ implies $x \leq y$.

Condition (TL_5) is not in general satisfied by a translation lattice of functions on a set. It was used by Kaplansky to obtain a representation of a translation lattice as a translation lattice of continuous functions on a compact space. Shirota [9] defined a translation lattice to be a lattice L with an external operation + that satisfies (TL_1) to (TL_4) inclusive. This definition of Shirota's is equivalent to saying that a translation lattice consists of a lattice L and an order preserving monoid homomorphism θ of the totally ordered abelian group of real numbers under addition into the lattice ordered monoid of order preserving functions that map L into itself. This last order is defined by setting $\delta_1 \leq \delta_2$, where each $\delta_i : L \longrightarrow L$ is order preserving, if $\delta_1 x \leq \delta_2 x$ for each x in L. In other words a translation lattice L is a lattice that admits the real numbers as lattice automorphisms. This suggests a possible representation theory for ordered groups - admitting the group as lattice automorphisms of a given lattice L.

A translation lattice homomorphism l of S into the real numbers is a lattice homomorphism $l : S \longrightarrow R$ such that $l(f + \lambda) = l(f) + \lambda$. From the definition of a translation lattice of functions on a set E it is clear that each x in E

defines a translation lattice homomorphism $\ell_x : S \longrightarrow R$, where $\ell_x(f) = fx$.

The real numbers may be considered as the translation lattice of constant functions on a one-point set. In this case addition has the usual meaning and the lattice order is the usual total order of the real numbers.

Let $\ell : S \longrightarrow R$ be a translation lattice homomorphism. Define $\ell + \lambda$ on S by setting $(\ell + \lambda)f = \ell(f) + \lambda$. Then $\ell + \lambda$ is also a translation lattice homomorphism of S into the real numbers. From the standpoint of the lattice S these homomorphisms are essentially the same.

Pick f_o in S. Then if $\ell : S \longrightarrow R$ is a translation lattice homomorphism there is a unique translation lattice homomorphism $\ell_o : S \longrightarrow R$ such that $\ell_o(f_o) = 0$ and $\ell = \ell_o + \lambda_o$. If $\ell = \ell_o + \lambda$ and $\ell_o(f_o) = 0$ then $\ell(f_o) = (\ell_o + \lambda)f_o = \ell_o(f_o) + \lambda = \lambda$. This shows that $\ell_o = \ell - \ell(f_o)$ is the unique translation lattice homomorphism with $\ell_o(f_o) = 0$ and $\ell = \ell_o + \lambda_o$.

Assume that S contains the zero function 0 defined by setting $0x = 0$ for each x in S. Then S contains the constant functions on E. Such a translation lattice of functions on E will be said to be <u>a translation lattice that contains the constants.</u>

Let Λ be the subcategory of $\mathfrak{L}$ obtained by restricting the objects to be those pairs (E,S) for which S is a translation lattice that contains the constants. If (E,S) is an object of Λ define $L_o(E,S)$ to be <u>the set</u> of all translation lattice homomorphisms $\iota : S \longrightarrow R$ which map the zero function on 0. If (E',S') is a second object of Λ and if α is in $\mathrm{Hom}((E',S),(E',S'))$ define $L_o(\alpha): L_o(E,S) \longrightarrow L_o(E',S')$ by setting $(L_o(\alpha)\iota)(f') = \iota(f' \circ \alpha)$. $L_o(\alpha)\iota$ is a translation homomorphism because the correspondence $f' \longrightarrow f' \circ \alpha$ is a lattice homomorphism such that $f' + \lambda \longrightarrow (f' + \lambda) \circ \alpha$ $= f' \circ \alpha + \lambda$. If (E'',S'') is a third object and α' is in $\mathrm{Hom}((E',S'),(E'',S''))$ then $L_o(\alpha' \circ \alpha) = L_o(\alpha') \circ L_o(\alpha)$ since $(L_o(\alpha' \circ \alpha)\iota)(f'') = \iota(f'' \circ \alpha' \circ \alpha) = (L_o(\alpha)\iota)(f'' \circ \alpha') =$ $(L_o(\alpha')[L_o(\alpha)\iota])(f'')$, for each f'' in S''.

If (E,S) is an object of Λ define $\iota_S : E \longrightarrow L_o(E,S)$ by setting $\iota_S x = \iota_x$ where $\iota_x(f) = fx$ for each f in S. Then if α is in $\mathrm{Hom}((E,S),(E',S'))$ $L_o(\alpha) \circ \iota_S = \iota_{S'} \circ \alpha$ since $(L_o(\alpha)\,\iota_x)(f') = \iota_x(f' \circ \alpha) = (f' \circ \alpha)x = f'(\alpha x) = \iota'_{\alpha x}(f')$.

It follows that a process on Λ may be defined by giving topologies for the sets $L_o(E,S)$ such that the subsets $\iota_S E$ are dense and the functions $L_o(\alpha)$ are continuous. Since the development so far is parallel to that of process $\mathcal{H}$ on A it is natural to consider the topology generated by the sets W_f, f in S, where $W_f = [\iota \in L_o(E,S) \mid \iota(f) \neq 0]$. Unfortunately these sets do not appear to form a base for this topology and so the fact that $W_f \cap \iota_S E \neq \phi$ implies $W_f = W_o = \phi$ cannot be used to prove that $\iota_S E$ is dense in the resulting space.

However if $\underline{O_+}$ is the topology generated by the sets W_f, f in S with $f \geq 0$ they do form a base since $W_f \cap W_g =$

$W_{f \cap g}$ when $f,g \geqslant 0$. Similarly if $\underline{O}_-$ is the topology generated by the sets W_f, f in S and $f \leqslant 0$ this topology has these sets for a base since $W_f \cap W_g = W_{f \cup g}$ if $f, g \leqslant 0$.

Define $L_+(E,S)$ to be $L_o(E,S)$ with the "positive" topology $\underline{O}_+$. Then $\mathcal{l}_S E$ is dense in this space because the sets W_f, $f \geqslant 0$ form a base for O_+ . If α is in $\mathrm{Hom}((E,S),(E',S'))$ and $f' \geqslant 0$ then $L_o^{-1}(\alpha)W_{f'} = [\mathcal{l} \,\epsilon L_o(E,S) \mid (L(\alpha)\mathcal{l})(f') = \mathcal{l}(f' \circ \alpha) \neq 0] = W_{f' \circ \alpha}$ and since $f' \circ \alpha \geqslant 0$ $L_o(\alpha)$ is a continuous function: $L_+(E,S) \longrightarrow L_+(E',S')$. Consequently <u>if $L_+ : \Lambda \longrightarrow \Sigma$ is the covariant functor with $L_+(E,S) = (L_o(E,S), \underline{O}_+)$ and $L_+(\alpha) = L_o(\alpha)$ then together with the family $(\mathcal{l}_S)_{(E,S) \text{ in } \Lambda}$ of functions $\mathcal{l}_S$ it defines a process $\mathcal{L}_+$ on Λ.</u>

Similarly define $L_-(E,S)$ to be the set $L_o(E,S)$ together with the negative topology $\underline{O}_-$. As before each $\mathcal{l}_S E$ is dense and each $L_o(\alpha)$ is continuous. Let $L_-(\alpha) = L_o(\alpha)$. <u>Then $L_- : \Lambda \longrightarrow \Sigma$ is a covariant functor which together with the family $(\mathcal{l}_S)_{(E,S) \text{ in } \Lambda}$ of functions $\mathcal{l}_S$ defines a process $\mathcal{L}_-$ on Λ.</u>

As in the case of process $\mathcal{H}$ if (E,S) is an object of Λ the functions f in S define real-valued functions $\bar{f}$ on $L_o(E,S)$ in a "natural" way by setting $\bar{f}\mathcal{l} = \mathcal{l}(f)$ for each $\mathcal{l}$ in $L_o(E,S)$. The functions $\bar{f}$ are such that $\bar{f} \circ \mathcal{l}_S = f$ and $\bar{S} = [\bar{f} \mid f \epsilon S]$ is a translation lattice of functions on $L_o(E,S)$. If f and g are in S then $\overline{f \cup g}\,\mathcal{l} = \mathcal{l}(f \cup g) =$

$\ell(f) \cup \ell(g) = \bar{f}\ell \cup \bar{g}\ell$ and similarly $\overline{f \cap g}\,\ell = \bar{f}\ell \cup \bar{g}\ell$ and $\overline{f+\lambda}\,\ell = \bar{f}\ell + \lambda$. This shows <u>$\bar{S}$ is a translation lattice which contains the constants</u> since in addition $\bar{0}\ell = 0$ for each ℓ in $L_o(E,S)$.

These functions $\bar{f}$ connect the positive and negative topologies since $\underline{O}_+ \cup \underline{O}_-$ (the smallest topology finer than both) $= \underline{O}(L_o(E,S),\bar{S})$. To prove this it is sufficient to show that for f_o in S, $\varepsilon > 0$ and ℓ_o in $L_o(E,S)$ $[\ell \varepsilon L_o(E,S) \mid |\bar{f}_o\ell - \bar{f}_o\ell_o| < \varepsilon]$ is the intersection of an O_+ -open set with an O_- -open set.

Let $g = f_o - \ell_o(f_o)$. This function is in S and so are $g_1 = (g \cap \varepsilon) - \varepsilon$ and $g_2 = (g \cup (-\varepsilon)) + \varepsilon$. If ℓ is in $L_o(E,S)$ $\ell(g_1) = \ell(g \cap \varepsilon) - \varepsilon = \ell(g) \cap \ell(\varepsilon) - \varepsilon =$ $(\ell(g) \cap \varepsilon) - \varepsilon$ since $\ell(\varepsilon) = \ell(0+\varepsilon) = \ell(0) + \varepsilon = \varepsilon$ and similarly $\ell(g_2) = (\ell(g) \cup (-\varepsilon)) + \varepsilon$. The functions g_1 and g_2 are respectively ≤ 0 and ≥ 0. Furthermore $\ell(g_1) \neq 0$ iff $\ell(g) < \varepsilon$ and $\ell(g_2) \neq 0$ iff $\ell(g) > -\varepsilon$. Therefore $W_{g_1} \cap W_{g_2} = [\ell \varepsilon L_o(E,S) \mid |\ell(f_o) - \ell_o(f_o)| < \varepsilon]$ $= [\ell \varepsilon L_o(E,S) \mid |\bar{f}_o\ell - \bar{f}_o\ell_o| < \varepsilon]$. This proves that $\underline{O}_+ \cup \underline{O}_-$ $= \underline{O}(L_o(E,S),\bar{S})$.

The functions $\bar{f}$, f in S, define a function $\delta(E,S):L_o(E,S) \longrightarrow \prod_{f \varepsilon S} R_f$ by setting $\delta(E,S)\,\iota = (\bar{f}\iota)_{f\varepsilon S} = (\iota(f))_{f\varepsilon S}$. The image of $L_o(E,S)$ under $\delta(E,S)$ is closed. Let Z be in the closure of $\delta(E,S)L_o(E,S)$. If $Z = (Z_f)_{f\varepsilon S}$ then $Z_{f\cup g} = Z_f \cup Z_g$, $Z_{f\cap g} = Z_f \cap Z_g$ and $Z_{f+\lambda} = Z_f + \lambda$. Consequently Z defines a translation lattice homomorphism $\iota_Z:S \longrightarrow R$ by setting $\iota_Z(f) = Z_f$. Since $Z_o = 0$ it is in $L_o(E,S)$ and $\delta(E,S)\,\iota_Z = Z$. This shows that $\delta(E,S)L_o(E,S)$ is a closed subset of $\prod_{f\varepsilon S} R_f$.

It is clear that $\delta(E,S) \circ \iota_S = t_S$ and hence as a result that $\delta(E,S)L_o(E,S)$ contains $T(E,S)$. From this it follows that <u>$\iota_S E$ is dense in the topology generated by the sets W_f, f in S, iff $\delta(E,S)L_o(E,S) = T(E,S)$.</u> This is because this topology is $\underline{O}_+ \cup \underline{O}_- = \underline{O}(L_o(E,S),\bar{S})$ and so $\delta(E,S)$ embeds $(L_o(E,S), \underline{O}_+ \cup \underline{O}_-)$ in $\prod_{f\varepsilon S} R_f$.

Assume that S is a translation lattice that contains the constants which is also closed under multiplication by (-1) i.e. if f is in S, -f is in S where $(-f)x = -fx$. <u>Then $\delta(E,S)$ is in $T(E,S)$ iff $\iota(-f) = -\iota(f)$ for all f in S.</u>

Since the homorphisms ℓ_x are such that $\ell_x(-f) = -\ell_x(f)$ for all f in S it is clear that if $\delta(E,S)\ell$ is in $T(E,S)$ $\ell(-f) = -\ell(f)$ for all f in S. Suppose $\ell_o(-f) = -\ell_o(f)$ for each f in S but that $\delta(E,S)\ell_o$ is not in $T(E,S)$. Then there exist n functions $f_1,\ldots,f_n$ in S and $\varepsilon > 0$ such that if x is in E at least one of $|f_i x - \ell_o(f_i)| \geqslant \varepsilon$ $i = 1,\ldots,n$. Let $g_i = f_i - \ell_o(f_i)$ and define $g_i^1 = -([g_i \cap \varepsilon] - \varepsilon)$ and $g_i^2 = [g_i \cup (-\varepsilon)] + \varepsilon$. Since $\ell_o(-f) = -\ell_o(f)$ it follows that $\ell_o(g_i^1) = \ell_o(g_i^2) = \varepsilon$, $i = 1,\ldots,n$. Let $g = \bigcap_{i=1}^{n} [g_i^1 \cap g_i^2]$. Then $\ell_o(g) = \varepsilon$. However $g = 0$. If x is in E $gx = \bigcap_{i=1}^{n} [g_i^1 x \cap g_i^2 x]$ and at least one of the $g_i^1 x \cap g_i^2 x = 0$, $i = 1,\ldots,n$. Since $g_i^1 \cap g_i^2 \geqslant 0$ $i = 1,\ldots,n$ it follows that $gx = 0$. Since $g = 0$ $\ell_o(g) = 0$. This is a contradiction. Hence $\delta(E,S)\ell_o$ is in $T(E,S)$ if $\ell_o(-f) = -\ell_o(f)$ for each f in S.

<u>Remark.</u> If S is a unitary real algebra on E which is also closed under $\cup$ and $\cap$ then it follows that <u>if $\ell : S \quad R$ is a translation lattice homomorphism such that $\ell(0) = 0$ and $\ell(-f) = -\ell(f)$ for each f in S then ℓ is also an algebra homomorphism.</u>

Let Λ' be the subcategory of Λ obtained by restricting the objects (E,S) to those pairs for which S is closed under multiplication by (-1) . If (E,S) is an object of Λ' define $L(E,S)$ to be the subspace of $L_+(E,S)$ consisting of all ℓ in $L_0(E,S)$ for which $\ell(-f) = -\ell(f)$ for all f in S. If (E',S') is a second object and α is in $\mathrm{Hom}((E,S),(E',S'))$ let $L(\alpha) = L_0(\alpha)|L(E,S)$. This function maps $L(E,S)$ into $L(E',S')$. Since $(-f') \circ \alpha = -(f' \circ \alpha)$ if f' is in S', this fact implies that $(L(\alpha)\ell)(-f') =$ $\ell((-f') \circ \alpha) = \ell(-(f' \circ \alpha)) = -\ell(f' \circ \alpha) = -(L(\alpha)\ell)(f')$. It is clear that $L(\alpha' \circ \alpha) = L(\alpha') \circ L(\alpha)$ if α' is in $\mathrm{Hom}((E,S),(E',S'))$.

Then $L: \Lambda' \longrightarrow \Sigma$ is a covariant functor. Since for each (E,S) in Λ', $\ell_S E$ is contained in $L(E,S)$ it follows that L together with the family $(\ell_S)_{(E,S) \text{ in } \Lambda'}$ of functions ℓ_S defines a process on Λ' . It will be denoted by $\mathcal{L}$.

This process $\mathcal{L}$ is a 'subprocess' of $\mathcal{L}_+ | \Lambda'$. It is also a 'subprocess' of $\mathcal{L}_- | \Lambda'$. This is because $\underline{Q}_+ | L(E,S) = \underline{Q}_- | L(E,S)$ for any object (E,S) of Λ' . If (E,S) is in Λ' and $f \geqslant 0$ is in S then $W_f \cap L(E,S) =$ $W_{-f} \cap L(E,S)$ and hence the two topologies coincide on this set.

Since $\underline{O}_+ \cup \underline{O}_- = \underline{O}(L_o(E,S),\bar{S})$ it follows that the topology of $L(E,S)$ is the restriction of this weak topology to $L(E,S)$. One consequence of this result is

<u>Theorem 13. $\mathcal{L}$ and $\mathfrak{J}\,|\Lambda'$ are isomorphic processes on Λ' . $\mathcal{L}$ is idempotent.</u>

Proof: If (E,S) is in Λ' $\delta(E,S)|L(E,S)$ is an embedding in $\prod\limits_{f \text{ in } S} R_f$. It is a homeomorphism $\gamma(E,S):L(E,S) \longrightarrow T(E,S)$ which satisfies $\gamma(E,S) \circ \mathfrak{l}_S = t_S$. Since $\mathfrak{J}$ is Hausdorff lemma 1 shows that these homeomorphisms define an isomorphism of $\mathcal{L}$ into $\mathfrak{J}\,|\,\Lambda'$.

As in the proof of theorem 12 it follows that $\mathcal{L}$ is idempotent if $\mathcal{L} \square \mathcal{L}$ is defined. If (E,S) is in Λ' and $S_L = \bar{S}|L(E,S)$ then $(L(E,S),S_L)$ is also in Λ' . This is because $(-f_L)\mathfrak{l} = -(f_L\mathfrak{l}) = -(\mathfrak{l}(f)) = \mathfrak{l}(-f) = -f_L\mathfrak{l}$ and so S_L is closed under multiplication by -1 . Since $\bar{S}$ is a translation lattice of functions on $L_o(E,S)$ which contains the constants it follows that $(L(E,S),S_L)$ is in Λ' . Consequently $\mathcal{L} \square \mathcal{L}$ is defined and so $\mathcal{L}$ is idempotent (corollary to theorem 5).

<u>Remarks.</u> It is not hard to see that $\mathcal{L}$ satisfies (FP_δ) and hence as a result $\mathfrak{J}\,|\,\Lambda'$ satisfies (FP_δ) . Process $\mathcal{L}$ was suggested by a result of Shirota's [9] (his theorem 9) . The proof given by Shirota does not appear to be correct.

To conclude this section on translation lattice processes it will be shown that $\underline{\mathcal{L}_+|\,\Lambda'}$ and $\underline{\mathcal{L}_-|\,\Lambda'}$ are

<u>isomorphic processes on Λ'.</u> Let (E,S) be an object of Λ' and denote by $J:S \longrightarrow S$ the mapping defined by setting $Jf = -f$. Then $(J \circ J)f = f$ for all f in S and $J(f \cup g) = Jf \cap Jg$, $J(f \cap g) = Jf \cup Jg$, $J(f + \lambda) = Jf - \lambda$. Let $K:R \longrightarrow R$ be the corresponding antiautomorphism of the lattice R i.e. $Kx = -x$ if x is in R.

Define $\gamma(E,S):L_o(E,S) \longrightarrow L_o(E,S)$ by setting $\gamma(E,S)\ell = K \circ \ell \circ J$. It is a routine calculation to show that $K \circ \ell \circ J$ is in $L_o(E,S)$ when ℓ is in $L_o(E,S)$. The function $\gamma(E,S)$ is $1-1$ and onto since $(\gamma(E,S) \circ \gamma(E,S))$ $= K \circ (K \circ \ell \circ J) \circ J = (K \circ K) \circ \ell \circ (J \circ J) = \ell$ for each in $L_o(E,S)$. The function $\gamma(E,S):L_+(E,S) \longrightarrow L_-(E,S)$ is a homeomorphism. If $f \geq 0$ is in S then $Jf \leq 0$ and $\gamma(E,S)W_f = W_{Jf}$. This is because $\ell(f) = 0$ iff $(K \circ \ell \circ J)(Jf)$ $= -\ell(f) = 0$. Similarly if $f \leq 0$ is in S then $Jf \geq 0$ and $\gamma(E,S)W_f = W_{Jf}$. This shows that $\gamma(E,S)$ is an $(\underline{O}_+, \underline{O}_-)$ - homeomorphism. The family $(\gamma(E,S))_{(E,S) \text{ in } \Lambda'}$ of homeomorphisms $\gamma(E,S)$ defines an isomorphism γ of $\mathcal{L}_+ | \Lambda'$ into $\mathcal{L}_- | \Lambda'$. If (E,S) is in Λ' then $\gamma(E,S) \circ \ell_S = \ell_S$ because $K \circ \ell_x \circ J = \ell_x$, for each x in E. Let f be in S then $(K \circ \ell_x \circ J)(f) = (K \circ \ell_x)(-f) = K((-f)x) = K(-fx) =$ $fx = \ell_x(f)$. If (E',S') is a second object of Λ' and α is in $\mathrm{Hom}((E,S),(E',S'))$ let ℓ be in $L_o(E,S)$. Then
$(K \circ (L_o(\alpha)\ell) \circ J')(f') = (K \circ (L_o(\alpha)\ell))(-f') = K(\ell((-f') \circ \alpha))$
$= K(\ell(-(f' \circ \alpha))) = (K \circ \ell \circ J)(f' \circ \alpha) = (L_o(\alpha)(K \circ \ell \circ J))(f')$.

This shows that $\gamma(E',S') \circ L_+(\alpha) = L_-(\alpha) \circ \gamma(E,S)$, and hence that γ is an isomorphism of $\mathcal{L}_+|\Lambda'$ into $\mathcal{L}_-|\Lambda'$.

Remark. For any (E,S) in Λ' the set $L(E,S) = [\,l \text{ in } L_0(E,S) | K \circ l \circ J = l\,]$. If either $L_+(E,S)$ or $L_-(E,S)$ is a Hausdorff space it follows that $L(E,S) = L_0(E,S)$. Hence if $\mathcal{L}_+|\Lambda'$ or $\mathcal{L}_-|\Lambda'$ is a Hausdorff process they coincide and are both equal to $\mathcal{L}$. Is $\mathcal{L}_+|\Lambda'$ a Hausdorff process on Λ'?

Note. Processes $\mathcal{H}$ and $\mathcal{L}$ are examples of 'algebraic' function processes. Any suitable structure defined for collections of functions on a set provides a corresponding process. As with $\mathcal{H}$ and $\mathcal{L}$ algebraic properties may be deduced by comparing the process with $\mathcal{J}$.

To conclude this discussion of specific processes it is convenient to combine theorems 10, 12 and 13. Let $A' \cap \Lambda'$ denote the subcategory of $\mathfrak{E}$ obtained by restricting the objects to be those common to A' and Λ' i.e. those pairs (E,S) where S is a unitary real algebra on E closed under bounded inversion and the binary operations $\cup$ and $\cap$. The result of combining these three theorems is stated as

Theorem 14. <u>$\mathcal{J}|A' \cap \Lambda'$, $\mathcal{F}|A' \cap \Lambda'$, $\mathcal{H}|A' \cap \Lambda'$ and $\mathcal{L}|A' \cap \Lambda'$ are isomorphic processes on $A' \cap \Lambda'$. They all satisfy (FP_3), (FP_4), (FP_5) (FP_6), (FP_7) and (FP_8) . In addition they are all idempotent (as processes on $A' \cap \Lambda'$) .</u>

Proof: The first two assertions are an immediate consequence of theorems 10, 12 and 13 and the properties of these processes.

To prove that they are all idempotent it is sufficient to show that $\mathcal{J}|A'\cap\Lambda' \,\square\, \mathcal{J}|A'\cap\Lambda'$ and so on are all defined. If (E,S) is $A'\cap\Lambda'$ then $(T(E,S),\ S_T)$ is also in $A'\cap\Lambda'$. This is a consequence of the properties of the product topology and the fact that $t_S E$ is dense in $T(E,S)$. $(F(E,S),S_F)$, $(H(E,S),S_H)$ and $(L(E,S),S_L)$ are also in $A'\cap\Lambda'$. This can be established in each case by computation or by noting that the homeomorphisms from $T(E,S)$ to $F(E,S)$, $H(E,S)$ and $L(E,S)$ induce correspondences of S_T with S_F, S_H and S_L which preserve the properties of closure under bounded inversion and $\cup$ and $\cap$.

§7. <u>The extension algebras of E.</u> Let E denote a completely regular space and let C_E denote the algebra of all real-valued continuous functions on E. In the introduction the original problem was stated as

(EA): <u>characterise the subalgebras S of C_E for which $S = C_X|E$, where X is an extension of E.</u>

Two definitions are introduced to make this problem more precise.

<u>Definition 5.</u> <u>An extension of a completely regular space E is a pair (X, j) where X is a completely regular space and $j: E \longrightarrow X$ is a homeomorphic embedding with jE dense in X.</u>

<u>Definition 6.</u> <u>If (X, j) is an extension of E the subalgebra $C_X \circ j = [g \circ j \mid g \text{ in } C_X]$ of C_E is called an extension algebra of E.</u>

When no confusion is likely X is used to denote (X,j) and $C_{X|E}$ to denote $C_X \circ j$.

Two extensions (X_1,j_1) and (X_2,j_2) are said to be isomorphic if there is a homeomorphism $j_{21} : X_1 \longrightarrow X_2$ such that $j_{21} \circ j_1 = j_2$. For the purposes of the original problem isomorphic extensions are identical since $C_{X_1} \circ j_1 = C_{X_2} \circ j_{21} \circ j_1 = C_{X_2} \circ j_2$.

Let $\mathcal{P}$ be a process defined on a subcategory Ψ of Φ. If an object (E,S) of Φ is in Ψ then $\mathcal{P}$ will be said to be defined at (E,S). Let (E,S) be an object for which $\mathcal{P}$ is defined. Then it associates with (E,S) the pair $(P(E,S),p_S)$ where $p_S : E \longrightarrow P(E,S)$ is a function with dense image. Theorem 1 gives necessary and sufficient conditions on a topology $\underline{O}_E$ for E in order that p_S be $\underline{O}_E$ - continuous, assuming that $\mathcal{P}$ satisfies (FP_5) at (E,S) (from the statement of properties (FP_3),$(FP_4)(FP_5)$ and (FP_6) it is clear what is meant by saying that a process satisfies one of them at a specific object). If the process $\mathcal{P}$ satisfies (FP_4) and (FP_5) at (E,S) the space $P(E,S)$ is completely regular. This leads to the consideration of those completely regular topologies of E for which $(P(E,S),p_S)$ is an extension of E. The following theorem shows that under these conditions there is a unique completely regular topology with this property.

<u>Theorem 15. Let (E,S) be an object of Φ and let $\mathcal{P}$ be a process defined at (E,S) which satisfies (FP_4) and (FP_5) at (E,S). Let O_E be a topology for E. Then $p_S: E \longrightarrow P(E,S)$ is an O_E - embedding iff $O_E = O(E,S)$ and S separates the points of E.</u>

Proof: Theorem 1 states that p_S is O_E - continuous iff $O(E,S)$ is coarser than O_E. On the other hand p_S is O_E - open iff $O(E,S)$ contains O_E, since $S_P \circ p_S = S$ and $\mathcal{P}$ satisfies (FP_5) at (E,S).

Since $\mathcal{P}$ satisfies (FP_4) at (E,S) p_S is 1 - 1 iff S separates the point of E.

Combining these results the theorem follows.

<u>Examples of extensions of E obtained from processes.</u>

1. Let E be a completely regular space. The object (E,C_E) is in $A' \cap \Lambda'$ and so $\mathcal{J}$, $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{L}$ may be applied to it. In each case an extension of E is obtained since by theorem 14 each of these processes satisfies (FP_4) and (FP_5) at (E,C_E). Since the processes $\mathcal{J}|A' \cap \Lambda'$, $\mathcal{F}|A' \cap \Lambda'$, $\mathcal{H}|A' \cap \Lambda'$ and $\mathcal{L}|A' \cap \Lambda'$ are isomorphic the extensions $(T(E,C_E), t_{C_E})$, $(F(E,C_E)f_{C_E})$, $(H(E,C_E),h_{C_E})$ and $(L(E,C_E), l_{C_E})$ are all isomorphic. If X is any of them it is clear that $C_X|E = C_E$.

2. Let E be a completely regular space. The object (E,C_E^{*}) is in $A' \cap \Lambda'$ where C_E^{*} is the collection of all bounded continuous real-valued functions on E. As before each of the

processes $\mathcal{T}$, $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{L}$ are defined at (E,C_E^*) and they yield the isomorphic extensions $(T(E,C_E^*),t_{C_E^*})$, $(F(E,C_E^*),f_{C_E^*}) = (M(E,C_E^*),m_{C_E^*})$, $(H(E,C_E^*),h_{C_E^*})$ and $(L(E,C_E^*),\ell_{C_E^*})$. If X is any of these extensions of E then $C_X|E = C_E^*$. All these extensions are compact. In addition the extensions of example 1 can all be embedded in these extensions. For example consider the extensions defined by $\mathcal{H}$. Let $\overline{C}_E^*$ be the bounded functions in $\overline{C}_E$. Then $\overline{C}_E^* \circ h_{C_E} = C_E^*$ and $\overline{f} \longrightarrow \overline{f} \circ h_{C_E}$ is 1 - 1 . Since $\mathcal{H}$ satisfies (FP_8) $H(h_{C_E}):H(E,C_E) \longrightarrow H(H(E,C_E),\overline{C}_E^*)$ is a homeomorphism and the following diagram is commutative

$$\begin{tikzcd}
E \arrow[r, "h_{C_E}"] \arrow[d, "h_{C_E^*}"'] & H(E, C_E) \arrow[d, "h_{\overline{C}_E^*}"] \\
H(E, C_E^*) \arrow[r, "H(h_{C_E})"] & H(H(E, C_E), \overline{C}_E^*)
\end{tikzcd}$$

The function $H^{-1}(h_{C_E}) \circ h_{\overline{C}_E^*}$ embeds $H(E,C_E)$ as a dense subspace of $H(E,C_E^*)$.

The extensions (X,j) in these two examples have one property in common: namely, all the real-valued algebra homomorphisms of C_X are of the form h_x, x in X, where

$h_x(f) = fx$. This is because theorem 14 states that $\mathcal{H}|A' \cap \Lambda'$ is idempotent and so $h_{C_X} : X \longrightarrow H(X, C_X)$ is a homeomorphism since $\mathcal{H}|A' \cap \Lambda'$ is Hausdorff (which shows that the homeomorphism: $H(E,C_E) \longrightarrow H(H(E,C_E), C_{H(E,C_E)})$ is $h_{C_{H(E,C_E)}}$).

Hewitt [10] first considered topological spaces with this property. He called them Q-spaces and defined them in terms of the maximal ideals of the algebra of all continuous real-valued functions. If E is a set and S is a unitary real algebra on E then the kernel of h_x is $M_x = [f \text{ in } S | fx = 0]$. A maximal ideal M of S is said to be fixed if $M = M_x$ for some x in E, otherwise it is said to be free. M is said to be real if S/M is isomorphic to the field R of real numbers as an R-algebra. Otherwise M is said to be hyperreal. Using this terminology Hewitt defined a Q-space to be a completely regular space for which every free maximal ideal of C_E is by hyperreal or equivalently for which every real maximal of C_E is fixed. This definition is equivalent to

Defintion 7. A completely regular space E is said to be a Q-space if $h_{C_E} : E \longrightarrow H(E, C_E)$ is a homeomorphism.

The equivalence follows from the fact that h_{C_E} is a homeomorphism iff it is onto i.e. iff every real maximal ideal is fixed.

Since the space $H(E,C_E)$ is homeomorphic to the space $H(F,C_F)$ when C_E and C_F are isomorphic real algebras it follows that two Q-spaces E and F are homeomorphic iff C_E and C_F are isomorphic real algebras.

If E is a Q-space and F is homeomorphic to E under γ then F is a Q-space. This is because $H(\gamma)$ is a homeomorphism and $h_{C_F} = H(\gamma) \circ h_{C_E} \circ \gamma^{-1}$.

Examples of Q-spaces.

1. A compact space is a Q-space. Let K be a compact space. It is completely regular and so $h_{C_K} : K \longrightarrow H(K,C_K)$ is an embedding. Since K is compact h_{C_K} is a homeomorphism.

2. A locally compact space E which is countable at infinity is a Q-space. Consider the diagram in the second example of extensions defined by processes. Identify E with $h_{C_E} E$ and $H(E,C_E)$ with its homeomorphic image under $H^{-1}(h_{C_E}) \circ h_{C_E^*}$.

Then $E \subseteq H(E,C_E) \subseteq H(E,C_E^*)$.

Lemma 3. A completely regular space E is locally compact and countable at infinity iff there is a function f_o in C_E with $0 \leq f_o \leq 1$ such that:

(1) f_o has no zeros in E i.e. $1/f_o$ is in C_E ; and

(2) if h in $H(E,C_E^*)$ is not in $h_{C_E^*}E$ then $h(f_o) = 0$.

Assuming this lemma it is easy to prove the assertion. If h in $H(E,C_E)$ is not in $E = h_{C_E} E$ then $h(f_o) = 0$. $1/f_o$ has a finite value at h i.e. $h(1/f_o)$ is a real number. Therefore $0 = h(f_o). h(1/f_o) = h(1) = 1$ which is a contradiction. Consequently h_{C_E} is a homeomorphism and so E is a Q-space.

Proof of the lemma: If such a function f_o exists then E is locally compact and countable at infinity. Consider E as a subspace of $H(E,C_E^*)$ and $\overline{F}_o$ as the extension of f_o to this extension, where $\overline{F}_o h = h(f_o)$. Since $\overline{F}_o$ is continuous $Z(\overline{F}_o) = [h \text{ in } H(E,C_E^*) | h(f_o) = 0]$ is open and coincides with E. The fact that $H(E,C_E^*)$ is compact implies E is locally compact. Also $[h \text{ in } H(E,C_E^*) | h(f_o) \geqslant 1/n] = E_n$ is compact and $E = \bigcup_{n \geqslant 1} E_n$. This shows that E is countable at infinity.

If E is locally compact it is an open subset of $H(E,C_E^*)$. If E is countable at infinity then there is an increasing sequence $(E_n)_{n \geqslant 1}$ of compact subsets such that $\bigcup_{n \geqslant 1} E_n = E$. A compact space is normal and so for each n there is a continuous function f_n on $H(E,C_E^*)$ with $0 \leq f_n \leq 1$ and $f_n | E_n = 1$, $f_n | \complement E = 0$. Let $f = \sum_{n \geqslant 1} 2^{-n}.f_n$. Then f is a continuous function on $H(E,C_E)$ such that $0 \quad f \quad 1$ and $fh = 0$ iff h is in $\complement E$. The function $f_o = f|E$ is the desired function.

3. <u>For any integer $n > 0$ R^n is a non-compact Q-space.</u>
This is a special case of example 2 since R^n is a locally compact space countable at infinity for each $n > 0$.

Q-spaces are characterized by the following theorem which was essentially proved by Hewitt [10].

<u>Theorem 16.</u> <u>A completely regular space E is a Q-space iff it is complete in the structure $\underline{U}(C_E)$.</u>

Proof: Since $\mathcal{H}\,|A' \cap \Lambda'$ and $\mathfrak{J}\,|A' \cap \Lambda'$ are isomorphic by theorem 14 it is clear that E is a Q-space iff $t_{C_E}: E \longrightarrow T(E, C_E)$ is a homeomorphism. The space $T(E, C_E)$ is complete in the structure $\underline{U}((C_E)_T)$ since $\mathfrak{J}$ satisfies (FP_6). Identifying E with its image under t_{C_E} it follows that $\underline{U}(C_E) = \underline{U}((C_E)_T)|E$.

Consequently if t_{C_E} is a homeomorphism E is complete in $\underline{U}(C_E)$. Conversely if E is complete in $\underline{U}(C_E)$ it is a closed subset of $T(E, C_E)$ and therefore coincides with this set. Hence t_{C_E} is a homeomorphism.

<u>Corollary 1.</u> <u>If $\mathcal{P}$ is a process on a subcategory Ψ that satisfies $(FP_3), (FP_4), (FP_5)$ and (FP_6) then $\mathcal{P}$ is a Q-process on Ψ. Hence $\mathfrak{J}$ is a Q-process on Φ.</u>

Proof: Since for any object (E,S) of Ψ S_P is contained in $C_{P(E,S)}$ it follows that $\underline{U}(S_P)$ is contained in $\underline{U}(C_{P(E,S)})$. When $P(E,S)$ is complete in $\underline{U}(S_P)$ then it is complete in $\underline{U}(C_{P(E,S)})$ and hence is a Q-space. The next corollary is due to Shirota [9].

<u>Corollary 2. A completely regular space E is a Q-space iff every closed subspace is a Q-space.</u>

Since $C_E|F$ is contained in C_F it follows that F is complete in $\underline{U}(C_F)$ and so by the theorem is a Q-space.

An extension (X,j) of E will be said to be a Q-extension of E if X is a Q-space. Since $\mathcal{H}|A'\cap\Lambda'$ is isomorphic to $\mathcal{J}|A'\cap\Lambda'$ it follows that $(H(E,C_E),h_{C_E})$ is a Q-extension of E. It has the following 'universal' property noted by Mrowka [11] which is stated as

<u>Theorem 17. Let E be a completely regular space and let (Y,g) be a pair consisting of a Q-space Y and a continuous function $g:E\longrightarrow Y$. Then there exists a unique continuous function $w:H(E,C_E)\longrightarrow Y$ such that $w \circ h_{C_E} = g$.</u>

Proof: Since $C_Y \circ g$ is contained in C_E and H is a functor on Λ, the following commutative diagram exists

$$\begin{tikzcd}
E \arrow[r, "h_{C_E}"] \arrow[d, "g"'] & H(E,C_E) \arrow[d, "H(g)"] \\
Y \arrow[r, "h_{C_Y}"] & H(Y,C_Y)
\end{tikzcd}$$

Since Y is a Q-space h_{C_Y} is a homeomorphism and $w = h_{C_Y}^{-1} \circ H(g)$ is the required function. It is unique because $h_{C_E}E$ is dense in $H(E,C_E)$ and Y is Hausdorff.

This theorem shows that $(H(E,C_E)$ satisfies the conditions of

Definition 8. Let E be a completely regular space and let (Y,g) be a pair consisting of a Q-space Y and a continuous function $g:E \longrightarrow Y$. An ν extension of E is a pair (Y_1,g_1) such that for any other pair (Y,g) there is a unique continuous function $w:Y_1 \longrightarrow Y$ with $w \circ g_1 = g$.

There are three immediate formal consequences of this definition. First, if $(Y_1,g_1$ and (Y_2,g_2) are ν-extensions of E then there is a unique homeomorphism $w_{21}:Y_1 \longrightarrow Y_2$ such that $w_{21} \circ g_1 = g_2$. From the definition there exist unique continuous functions w_{ij} $i \neq j$ equal to 1 or 2 with $w_{ij} \circ g_j = g_i$. This shows that $w_{ij} \circ w_{ji} \circ g_i = g_i$. The uniqueness condition in the definition shows that $w_{ij} \circ w_{ji} : Y_i \longrightarrow Y_i$ is the identity mapping and hence that $w_{ij} = w_{ji}^{-1}$. Therefore w_{ij} is a homeomorphism, in particular w_{21} is a homeomorphism. Second, if (Y_1,g_1) is an ν-extension of E Y_1 is the only Q-subspace of Y_1 containing g_1E. Let V be a Q-space of Y_1 containing g_1E. The pair (V_1g_1) is admissible and so there is a unique continuous function $w:Y_1 \longrightarrow V$ such that $w \circ g_1 = g_1$. If $n:V \longrightarrow W$ is the natural injection $n \circ g_1 = g_1$ and so $n \circ w \circ g_1 = g_1$. The uniqueness condition implies that $n \circ w:Y_1 \longrightarrow Y_1$ is the identity mapping and so $V = V_1$. Third, if E' is a completely regular space and $f:E \longrightarrow E'$ is continuous then given ν-extensions (Y_1,g_1) and (Y_1',g_1')

<u>of E and E' respectively there is a unique continuous function $f_1 : Y_1 \longrightarrow Y_1'$ such that $f_1 \circ g_1 = g_1' \circ f$.</u> This is an automatic consequence of the definition since $(Y_1', g_1' \circ f)$ is an admissible pair for E .

Corollary 2 of theorem 16 and the second property of υ-extensions show that <u>$g_1 E$ is dense in Y_1</u> when (Y_1, g_1) is an υ-extension of E .

So far no use has been made of the complete regularity of the space E . Even the pair $(H(E,C_E), h_{C_E})$ exists independently of the complete regularity of E . To say that E is completely regular is equivalent to asserting that h_{C_E} is an embedding. Consequently <u>if E is completely regular then an υ-extension of E is also an extension of E .</u>

<u>Remarks.</u> Hewitt [10] introduced the concept of an υ-extension of E by showing that (up to isomorphism) there exists a unique Q-extension υE of the completely regular space E such that $C_{\upsilon E}|E = C_E$. He gave two constructions of υE . One was by means of the real maximal ideals of C_E and was essentially $(H(E,C_E), h_{C_E})$. The other construction was by means of the Tychonoff process $\mathfrak{I}$ and yielded the extension $(T(E,C_E), t_{C_E})$. The υ-extensions defined in definition 8 coincide with those of Hewitt since $(H(E,C_E), h_{C_E})$ satisfies both definitions. Other constructions of Hewitt's space υE were provided by Shirota [9] . He gave two, the most important of which will be considered in chapter three section one. He also considered the construction of

υE from the translation lattice C_E but his work does not appear to be correct. Banaschewski [5] showed that $(F(E,C_E),f_{C_E})$ is also an υ-extension of E.

One result of Hewitt's work is that (EA) can be completely solved by considering just the Q-extensions of E. This is because for any extension (X,j) of E the extension $(H(X,C_X),h_{C_X} \circ j)$ is a Q-extension of E and $C_X|E =$

$C_{H(X,C_X)}|E$.

Let (Y,k) be a Q-extension of E. It is isomorphic to the extension $(H(E,C_Y \circ k),h_{C_Y} \circ k)$. Consider the commutative diagram

$$\begin{tikzcd} E \arrow[r, "h_{C_Y \circ k}"] \arrow[d, "k"'] & H(E, C_Y \circ k) \arrow[d, "H(k)"] \\ Y \arrow[r, "h_{C_Y}"] & H(Y,C_Y) \end{tikzcd}$$

Since the correspondence $g \longrightarrow g \circ k$ is 1 - 1 and onto it follows that $H(k)$ is a homeomorphism. Y is a Q-space and so h_{C_Y} is a homeomorphism. The homeomorphism $h_{C_Y}^{-1} \circ H(k):H(E,C_Y \circ k) \longrightarrow Y$ satisfies $h_{C_Y}^{-1} \circ H(k) \circ h_{C_Y} \circ k = k$, and so the two extensions are isomorphic.

Consequently the construction process $\mathcal{H}$ applied to suitable objects (E,S) yields 'all' the Q-extensions of E. The collections S are restricted to be unitary subalgebras of C_E that are:

(1) closed under bounded inversion;

(2) closed under $\cup$ and $\cap$ i.e. S is a sublattice of C_E ; and such that

(3) $\underline{O}(E,S)$ is the topology of E . In other words those collections S for which (E,S) is in $A' \cap \Lambda'$ and $\underline{O}(E,S)$ is the topology of E. Since the extension algebras of S satisfy these conditions it follows that process $\mathcal{H}$ applied to these objects yields all the Q-extensions of E (see theorems 14, 15 and corollory 1 to theorem 16) .

To conclude this section, the following theorem reduces (EA) to a new problem.

Theorem 18. Let S be a unitary subalgebra of C_E . S is an extension algebra of E iff

(1) S is a sublattice closed under bounded inversion;

(2) $O(E,S)$ is the topology of E ; and

(3) $S_H = C_{H(E,S)}$.

Proof: If $S = C_Y|E$ where (Y,k) is a Q-extension of E, S satisfies (1) and (2) . It also satisfies (3) since (Y,k) is isomorphic to $(H(E,C_Y \circ k),h_{C_Y \circ k}) = (H(E,S),h_S)$.

Conversely if S satisfies (1) and (2) then by theorems 14 and 15 $(H(E,S),h_S)$ is an extension of E . If S satisfies (3) then $S = S_H|E = C_{H(E,S)}|E$ is an extension algebra.

Condition (3) of this theorem shows that (EA) could be solved by providing a suitable solution to problem

(AC): <u>if E is a completely regular space characterize the collection C_E as an algebra of continuous functions on E.</u>

A satisfactory solution to (EA) would also provide a solution for (AC) since C_E is equal to its largest extension algebra. In this sense the problems (EA) and (AC) are equivalent.

<u>Concluding remarks.</u> The argument following the remarks on Hewitt's υ-extension could (in view of theorem 14) equally well have been carried out by applying any one of the processes $\mathfrak{J}$, $\mathfrak{F}$ or $\mathcal{L}$ to obtain the extensions. In addition there would be no advantage in using any particular one of these processes since in the last analysis it will be as difficult to prove $S_H = C_{H(E,S)}$ as to show, for example, that $S_T = C_{T(E,S)}$. When the investigation of (EA) started the various constructions of υE were known. The argument following the preceding remarks was developed in four ways, one to correspond to each of the constructions of a Hewitt υ-extension. It became evident that there was no advantage in using one way rather than another, and in attempt to show this clearly it was found convenient to develop the idea of a function process and to carry the discussion through in the above manner.

§8. <u>A characterization of C_E.</u> Let F_E denote the collection of all real-valued functions on E. It may be given a topology by defining the fundamental neighbourhoods of a function f to be the sets $V_{f,1/n} = [g \mid |fx - gx| < 1/n$ for each x in $E]$. This topology is called *the topology of uniform convergence* on E. A subcollection S of F_E is said to be *uniformly closed* if it is a closed subset of F_E. When E is given a topology C_E is a uniformly closed subcollection of F_E.

The collection F_E is a unitary real algebra on E. It is a lattice with respect to the partial order $\leq$ defined by setting $f \leq g$ if $fx \leq gx$ for each x in E, since F_E is closed under $\cup$ and $\cap$. In addition if f in F_E has no zeros in E then $1/f$ is in F_E. This function *$1/f$ is called the inverse of f.* The collection C_E is a sub-lattice of C_E that is closed under inversion i.e. if f in C_E has an inverse $1/f$ then $1/f$ is in C_E. A subcollection S of F_E is said to be *closed under positive inversion* if when f in S is such that $fx > 0$ for each x in E then $1/f$ is in S. A subalgebra S of F_E is closed under inversion when it is closed under positive inversion if f in S implies $f \cup 0$ and $f \cap 0$ are in S. This is because when f in S has no zeros in E neither has $|f| = f \cup 0 - f \cap 0$, and so $1/|f|$ is in S. The inverse of f is $f \cdot 1/|f|^2$ which is in S.

With the aid of this terminology a preliminary characterization of C_E may be stated as

<u>Theorem 19. Let E be a completely regular space and assume that S is a subalgebra of C_E which contains C_E^M. Then $S = C_E$ if S is uniformly closed and is closed under positive inversion.</u>

Proof: If f is in C_E and $f \cup 0$, $f \cap 0$ are in S then f is in S. Hence it is sufficient to show that S contains all positive continuous functions on E.

Let $f \geqslant 0$ be in C_E and set $g = f + 1$. The function $g \wedge n$ is in C_E^M and so in S for each $n > 0$. Since $g \wedge n \geqslant 1 > 0$ the fact that S is closed under positive inversion implies that $1/g \wedge n = 1/g \cap 1/n$ is in S. The functions $1/g \cap 1/n$ tend uniformly to $1/g$ as n tends to infinity and as S is uniformly closed $1/g$ is in S. The function $1/g$ has no zeros in E and is positive. Therefore g is in S, which implies that $f = g - 1$ is in S.

For any unitary subalgebra S of F_E the collection S^M of bounded functions that belong to S is a subalgebra of F_E^M. Theorem 19 shows that to obtain a solution to (AC) it is sufficient to characterize those uniformly closed subalgebras S of C_E which are closed under positive inversion and for which $S = C_E^M$. In other words a solution to (AC) is provided by a solution to problem

(BC): <u>for a completely regular space E characterize C_E^M as an algebra of bounded continuous functions on E.</u>

Remark. On the face of it (BC) appears to be a harder problem than (AC) . In chapter two section three it will be shown to be an equivalent problem in a non-trivial sense.

An almost trivial solution for (BC) is provided by

Theorem 20. _Let S^M be a unitary subalgebra of C_E^M . $S^M = C_E^M$ iff S^M satisfies_

(F^M): when f in S^M factors as $f = gh$ then g and h are in S^M if they are bounded and continuous.

Proof: C_E^M satisfies (F^M) by definition. Assume that $S^M \neq C_E^M$ and let f in C_E^M be a function that does not belong to S^M . Since S^M contains the constants it may be assumed that $f \geqslant 1$. The function $1/f$ belongs to C_E^M and $1 = f \cdot 1/f$. The unit of C_E^M is in S^M and so S^M does not satisfy (F^M) .

This theorem provides a similar solution to (AC) which may be stated as

Theorem 21. _Let S be a unitary subalgebra of C_E that is uniformly closed and closed under positive inversion. Then $S = C_E$ iff S satisfies_

(F): when f in S factors as $f = gh$ then g and h are in S if they are continuous.

Proof: When S satisfies (F) it follows that S^M satisfies (F^M) . Theorem 20 shows that $S^M = C_E^M$. Under these circumstances theorem 19 applies and so $S = C_E$.

To conclude this chapter consider the application of theorem 21 to (EA) by means of theorem 18. Let S be a unitary subalgebra of C_E that satisfies conditions (1) and (2) of theorem 18. It is necessary and sufficient to give conditions on S in order that S_H satisfy the conditions of theorem 21.

The algebra S_H is uniformly closed iff S is uniformly closed. As S is a sublattice of C_E it follows that S_H is a sublattice of $C_{H(E,S)}$. Consequently S_H is closed under positive inversion iff it is closed under inversion. This is the case iff $1/f$ is in S whenever f in S is such that $h(f) \neq 0$ for all h in $H(E,S)$.

The algebra S_H satisfies (F) on $H(E,S)$ iff the algebra S satisfies

<u>(F_S)</u>: <u>when f in S factors as $f = gh$ then g and h are in S if they converge to finite limits along the filters in the set $F(E,S)$ i.e. the minimal $U(S)$ - Cauchy filters on E.</u>

Since the extensions $(F(E,S),f_S)$ and $(H(E,S),h_S)$ are isomorphic this assertion is an immediate consequence of lemma 3 in section four.

This completes the proof of the following strong version of theorem 18.

<u>Theorem 22.</u> <u>Let S be a unitary subalgebra of C_E. S is an extension algebra of C_E iff:</u>

(1) <u>S is a sublattice closed under bounded inversion;</u>

(2) <u>$O(E,S)$ is the topology of E;</u>

(3) <u>S is uniformly closed;</u>

(4) <u>$1/f$ is in S when $h(f) \neq 0$ for each h in $H(E,S)$; and</u>

(5) <u>S satisfies (F_S) .</u>

CHAPTER TWO

THE EXTENSIONS OF A COMPLETELY REGULAR SPACE

§1. Subalgebras and equivalence relations. Let A denote an arbitrary real associative algebra with unit 1 and let $H(A)$ be the set of unitary real-valued algebra homomorphisms h of A. The Zariski topology for $H(A)$ is the topology generated by the sets V_a, a in A, where $V_a = [h \text{ in } H(A) \mid h(a) \neq 0]$. Let $H(A)$ also denote the resulting topological space.

Denote by $\mathcal{R}$ the set of equivalence relations r on $H(A)$ and by $\mathcal{A}$ the set of unitary subalgebras B of A. Both sets may be partially ordered: set $r_1 \leq r_2$ if $r_1[h] \subseteq r_2[h]$ for all h in $H(A)$ ($r[h]$ is the r-equivalence class of h); set $B_1 \leq B_2$ if B_1 is a subalgebra of B_2.

Every equivalence relation r defines a unitary subalgebra $\alpha(r)$ of A and every unitary subalgebra B of A defines an equivalence relation $\rho(B)$ on $H(A)$ in the following way: $\alpha(r) = [a \text{ in } A \mid h_1 r\, h_2 \text{ implies } h_1(a) = h_2(a)]$; and $h_1\, \rho(B) h_2$ if $h_1|B = h_2|B$.

This defines two functions $\alpha: \mathcal{R} \longrightarrow \mathcal{A}$ and $\rho: \mathcal{A} \longrightarrow \mathcal{R}$. If $r_1 \leq r_2$ then $\alpha(r_1) \geq \alpha(r_2)$ and similarly if $B_1 \leq B_2$ it follows that $\rho(B_1) \geq \rho(B_2)$. As a result α and ρ

define a Galois connection between $\mathfrak{a}$ and $\mathfrak{R}$ Birkhoff [12]. The immediate problem that arises is the problem of characterising the 'closed' subalgebras and equivalence relations, i.e. those subalgebras B for which $B = (\alpha \circ \rho)(B)$ and those equivalence relations $\underline{r}$ for which $\underline{r} = (\rho \circ \alpha)(\underline{r})$.

Consider the special case of this problem that arises when $A = C_K$, K a compact space. Since a compact space is a Q-space $H(K,C_K) = H(C_K)$ is homeomorphic to K under h_{C_K}. If K and $H(K,C_K)$ are identified by means of h_{C_K} then the problem can be stated in terms of equivalence relations $\underline{r}$ on K. From now on let $\underline{r}$ denote an equivalence relation on K and let $\underline{a}$ denote a unitary subalgebra of C_K. Let $\rho(\underline{a})$ be the equivalence relation on K defined by $x\,\rho(\underline{a})\,y$ if $fx = fy$ for each f in A. With these conventions a complete solution to the problem is stated as

<u>Theorem 1. Let K be a compact space and let $A = C_K$. Let $\underline{r}$ be an equivalence relation on K and let $\underline{a}$ be a unitary subalgebra of C_K. Then $\underline{r} = (\rho \circ \alpha)(\underline{r})$ iff $\underline{r}$ is closed and $\underline{a} = (\alpha \circ \rho)(\underline{a})$ iff $\underline{a}$ is uniformly closed.</u>

Proof: First, in both assertions the conditions are necessary.

Let $\underline{r}$ be an equivalence relation on K and let $\underline{a} = \alpha(\underline{r})$. Then $\underline{a}$ is uniformly closed. If f is in the uniform closure of $\underline{a}$ and $x\,\underline{r}\,y$, then $gx = gy$ for all g in $\underline{a}$ and so $fx = fy$. In other words f is in $\underline{a}$.

Let $\underline{a}$ be a unitary subalgebra of C_K and let $\underline{r} = \rho(\underline{a})$. Consider the pair $(T(K,\underline{a}), t_{\underline{a}})$ corresponding to the object $(K,\underline{a})$ under process $\mathcal{J}$. Since $\mathcal{J}$ satisfies (FP_3) and (FP_4) it follows that $x\, \underline{r}\, y$ iff $t_{\underline{a}}x = t_{\underline{a}}y$. If $x\, \underline{r}\, y$ then $fx = fy$ for all f in $\underline{a}$ and so $(f_T \circ t_{\underline{a}}$ and so $(f_T \circ t_{\underline{a}})y$ for all f in $\underline{a}$, which implies that $t_{\underline{a}}x = t_{\underline{a}}y$. Conversely if $t_{\underline{a}}x = t_{\underline{a}}y$ then $fx = (f_T \circ t_{\underline{a}})x = (f_T \circ t_{\underline{a}})y = fy$ for all f in $\underline{a}$ and so $x\, \underline{r}\, y$. The space $T(K,\underline{a})$ is Hausdorff and $t_{\underline{a}}$ is a continuous function by theorem 1. This means that the subset C of $K \times K$ consisting of the pairs (x,y) for which $t_{\underline{a}}x = t_{\underline{a}}y$ is a closed set. Then because K is compact this implies that $\underline{r}$ is closed (see Bourbaki [3] p.97).

Consider the sufficiency of the condition in each case.

Let $\underline{r}$ be a closed equivalence relation on K. Then $K/\underline{r}$ is a compact space Samuel [13] (also Bourbaki [3]. p 97). Let $\pi(\underline{r}) : K \longrightarrow K/\underline{r}$ be the natural mapping. It follows that f is in $\alpha(\underline{r})$ iff $f = f_1 \circ \pi(\underline{r})$ where f_1 is a continuous real-valued function on $K/\underline{r}$ i.e. $\alpha(\underline{r}) = C_{K/\underline{r}} \circ \pi(\underline{r})$. It is clear that $\alpha(\underline{r})$ contains $C_{K/\underline{r}} \circ \pi(\underline{r})$. Let f be in $\alpha(\underline{r})$. There is a unique real-valued function f_1 on $K/\underline{r}$ such that $f = f_1 \circ \pi(\underline{r})$. This function f_1 is continuous. Let O be an open subset of the real numbers. Then $f^{-1} O$ is an open $\underline{r}$-saturated subset of K. Since $\pi(\underline{r})(f^{-1} O) = f_1^{-1} O$ it follows that this set is open and hence that f_1 is continuous. Consequently $\alpha(\underline{r}) = C_{K/\underline{r}} \circ \pi(\underline{r})$. Since a compact space is completely regular, the functions in $C_{K/\underline{r}}$ separate the points and so if $fx = fy$ for each f in $\alpha(\underline{r})$, $\pi(\underline{r})x = \pi(\underline{r})y$ i.e. $x \, \underline{r} \, y$. This proves that $\underline{r} = (\rho \circ \alpha)(\underline{r})$ when $\underline{r}$ is closed.

Let $\underline{a}$ be a uniformly closed unitary subalgebra of C_K. Then $\rho(\underline{a})$ is a closed equivalence relation on K and so $K/\rho(\underline{a})$ is a compact space. Since $\underline{a}$ is contained in $(\alpha \circ \rho)(\underline{a})$ it follows that every f in $\underline{a}$ may be written as $f = f_1 \circ \pi(\rho(\underline{a}))$ where f_1 is a (uniquely defined) continuous function on $K/\rho(\underline{a})$. Let $\underline{a}_1 = [f_1 | f \text{ in } \underline{a}]$. Then $\underline{a}_1$ is a uniformly closed unitary subalgebra of $C_{K/\rho(\underline{a})}$. By the Stone-Weierstrass theorem it coincides with $C_{K/\rho(\underline{a})}$ iff it separates the points. This is certainly the case as $\pi(\rho(\underline{a})x = \pi(\rho(\underline{a}))y$ iff $fx = fy$ for each f in $\underline{a}$. Therefore $\underline{a} = \underline{a}_1 \circ \pi(\rho(\underline{a})) = C_{K/\rho(\underline{a})} \circ \pi(\rho(\underline{a})) = \alpha(\rho(\underline{a})) = (\alpha \circ \rho)(\underline{a})$ when $\underline{a}$ is a uniformly closed unitary subalgebra of C_K. .

Remarks. Having solved the general problem for the special case $A = C_K$, K compact, it is natural to consider how the proof applies to case $A = C_E$, E a Q-space. Besides needing a characterization of those equivalence relations $\underline{r}$ for which $E/\underline{r}$ is a Q-space (or just completely regular) it is necessary to have some analogue of the Stone-Weierstrass theorem. This would be a solution to (AC) by an internal condition. Although the solution in chapter one is not of this type it could be applied here. This theorem also justifies the following change in notation:

<u>If $\mathfrak{a}$ is a uniformly closed subalgebra of C_K let $\rho(\mathfrak{a})$ be denoted by $\pi(\mathfrak{a})$.</u>

<u>Examples of uniformly closed unitary subalgebras $\mathfrak{a}$ of C_K.</u>

1. If D is a closed subset of K let $\mathfrak{a}_D = [f \text{ in } C_K | f$ is constant on $D]$. This is a subalgebra of this type. For example $\mathfrak{a}_K$ equals the collection of constant functions and $\mathfrak{a}_\phi = C_K$. The corresponding equivalence relations identify all points and no points respectively. Other examples of closed subsets of K are the finite sets and in particular the two points sets $[x,y]$. The algebras $\mathfrak{a}_{[x,y]}$ x,y in K are the maximal algebras of this type that are distinct from C_K.

2. Let $D_1,\dots,D_m$ be m mutually disjoint closed subsets of K. The algebra $\bigcap_{i=1}^{m} \mathfrak{a}_{D_i}$ is a uniformly closed unitary subalgebra of C_K which contains $\mathfrak{a}_{\bigcap_{i=1}^{m} D_i}$. The algebra $\bigcap_{i=1}^{m} \mathfrak{a}_{D_i}$ could be denoted $\mathfrak{a}_{D_1,\dots,D_m}$. It is distinct from C_K iff at least one of the sets D_i contains two or more points.

3. Let $\underline{a}$ be a proper uniformly closed unitary subalgebra of C_K. Then $\underline{a}$ is an intersection of maximal subalgebras distinct from C_K since $\underline{a} = \bigcap_{x\, \underline{r}(\underline{a})y} \underline{a}[x,y]$ - the intersection over all pairs $[x,y]$ such that $x\, \underline{r}(\underline{a})y$. In addition $\underline{a} = \underline{a}_{D_1,\ldots,D_m}$ with each D_i finite iff ther exist only a finite number of pairs $[x,y]$ with $x \neq y$ and $x\, \underline{r}(\underline{a})y$.

4. Another description of $\underline{a}$ may be given in terms of the non-trivial $\underline{r}(\underline{a})$ - equivalence classes $(D_\alpha)_{\alpha \text{ in } \Omega}$ i.e. those classes that contain at least two points. It is clear that $\underline{a} = \bigcap_{\alpha \text{ in } \Omega} \underline{a}_{D_\alpha}$. In this connection the following kind of problems arise: can any of the classes D_α be omitted without changing $\bigcap_{\alpha \text{ in } \Omega} \underline{a}_{D_\alpha}$? what kind of separation statements can be made about the D_α? for example are there mutually disjoin open sets U_α with D_α contained in U_α for each α in Ω? and so on. These questions are of importance because as will be seen later in this chapter they have a definite bearing on (AC). This is in section nine, where the concept of a set determining an algebra $\underline{a}$ is introduced.

Let (F,f) be a pair consisting of a Hausdorff space F and a continuous function $f: K \longrightarrow F$ with $fK = F$. For convenience such a pair will be called a continuous image of K. If (F_1,f_1) and (F_2,f_2) are continuous images of

K they are said to be isomorphic if there is a homeomorphism $f_{21}:F_1 \longrightarrow F_2$ such that $f_{21} \circ f_1 = f_2$. If (F,f) is a continuous image of K then it is obvious that F is compact space.

If $\underline{r}$ is a closed equivalence relation on K then $(K/\underline{r}, \pi(\underline{r}))$ is a continuous image of K. Furthermore $(K/\underline{r}_1, \pi(\underline{r}_1))$ and $(K/\underline{r}_2, \pi(\underline{r}_2))$ are isomorphic continuous images of K iff $\underline{r}_1 = \underline{r}_2$ i.e. iff they are equal. This is because for any closed equivalence relation $\underline{r}$ on K $\underline{r} = (\rho \circ \alpha)(\underline{r})$ by theorem 1 and $\alpha(\underline{r}) = C_{K/\underline{r}} \circ \pi(\underline{r})$.

Let (F,f) be a continuous image of K. Then since F is Hausdorff the relation $x\,\underline{r}\,y$ if $fx = fy$ is a closed equivalence relation on K (see the proof of theorem 1). Let $\gamma:K/\underline{r} \longrightarrow F$ be the 1 - 1 continuous function such that $\gamma \circ \pi(\underline{r}) = f$. Since $K/\underline{r}$ is compact γ is a homeomorphism and so (F,f) is isomorphic to $(K/\underline{r}, \pi(\underline{r}))$. Consequently <u>every continuous image of K is essentially a quotient space of K by a closed equivalence relation $\underline{r}$ on K.</u>

§2. <u>The compact extensions of a completely regular space.</u>

Let E denote a completely regular space. Stone [14] and Čech [15] showed that a compact extension (B,b) of E is defined up to isomorphism by requiring that $C_B \circ b = C_E^M$. Any compact extension with this property is called a Stone-Čech compactification of E.

The extension $(H(E,C^*_E),h_{C_E})$ is a Stone-Čech compactification of E. It will be denoted by $(\beta E,i)$.

Let K be a compact space and let $k:E \longrightarrow K$ be a continuous function. Then there is a unique continuous function $f:\beta E \longrightarrow K$ such that $f \circ i = k$. The uniqueness follows from the fact that $iE = h_{C_E^*}E$ is dense in $\beta E = H(E,C^*_E)$. Consider the commutative diagram

$$\begin{tikzcd} E \arrow[r, "h_{C_E^*}"] \arrow[d, "k"'] & H(E, C_E^*) \arrow[d, "H(k)"] \\ K \arrow[r, "h_{C_K}"] & H(K, C_K) \end{tikzcd}$$

Since a compact space is a Q-space h_{C_K} is a homeomorphism and $f = h^{-1}_{C_K} \circ H(k)$ is the desired function.

This shows that $(\beta E,i)$ satisfies the condition of Definition 1. A β-extension of a completely regular space E is a pair (B,b), where B is a compact space and $b:E \longrightarrow B$ is a continuous function, such that when (K,k) is any other pair of this type there exists a unique continuous function $f:B \longrightarrow K$ with $f \circ b = k$.

There are three immediate formal consequences of this definition which preserve the analogy between it and definition 8 of chapter one. First, any two β-extensions are isomorphic, where two admissible pairs (K_1,k_1) and (K_2,k_2) are said to be isomorphic if there is a homeomorphism $k_{21}: K_1 \longrightarrow K_2$ with $k_{21} \circ h_1 = h_2$. Second, *if (B,b) is a β-extension and $bE \subseteq B_1 \subseteq B$ is such that B_1 is a compact subspace then $B_1 = B$.* Third, *if $g: E \longrightarrow E'$ is continuous and (B,b), (B',b') are β-extensions of E and E' respectively, then there is a unique continuous function $g_1: B \longrightarrow B'$ such that $g_1 \circ b = b' \circ g$.* The proofs of these assertions are formally the same as those of the corresponding assertions following definition 8 of chapter one.

The second consequence implies that *if (B,b) is a β-extension of E then bE is dense in B* because a closed subspace of a compact space is also compact.

As in the case of the definition of an υ-extension of E the fact that $(\beta E, i)$ is an extension of E and a β-extension of E implies that every β-extension (B,b) of E is also an extension of E. Furthermore it implies that β-extensions of E and Stone-Čech compactifications of E are one and the same thing.

Remarks. Čech's original construction of a β-extension of E is as follows. Let I denote the closed unit interval $[0,1]$ and let $S_0 = [f \text{ in } C_E^R \mid 0 \leq f \leq 1]$. The extension $(T(E,S_0), t_{S_0})$ of E is a β-extension of E. It is an extension of E by theorem 15 of chapter one. It is a compact extension since S_0 consists entirely of bounded functions. It is a Stone-Čech compactification of E since every g in C_E^* may be expressed as $g = \lambda g_o + \mu$ where g_o is in S_o. Hewitt [10] showed that $(H(E,C_E^*),h_{C_E^*})$ is a Stone-Cech compactification of E. Theorem 14 of chapter one shows that $(L(E,C_E^*), \iota_{C_E^*})$ and $(F(E,C_E^*),f_{C_E^*}) = (M(E,C_E^*),m_{C_E^*})$ are also β-extensions of E. This last construction is due to Alexandroff [6].

<u>Convention.</u> If f is in C_E^* it will be convenient to let f also denote the function $\bar{f}$ on βE such that $\bar{f} \circ i = f$. With this identification $C_E^* = C_{\beta E}$ and the closed equivalence relations $\underline{r}$ on βE are all of the form $\underline{r} = \underline{r}(\underline{a})$, where $\underline{a}$ is a uniformly closed unitary subalgebra of C_E^*. The natural mappings $\pi(\underline{r}(\underline{a}))$ will be denoted by $\pi(\underline{a})$.

<u>Theorem 2.</u> <u>Let K be a compact space and let $k:E \longrightarrow K$ be a continuous function. The following statements are equivalent when $\underline{a} = C_K \circ k$:</u>

(1) <u>kE is dense in K;</u>

(2) <u>there is a homeomorphism (unique) $g:H(E,\underline{a}) \longrightarrow K$ such that $g \circ h_{\underline{a}} = k$; and</u>

(3) <u>if $f:\beta E \longrightarrow K$ is the continuous function with $f \circ i = k$ then $f(\beta E) = K$.</u>

Proof. Consider the commutative diagram

$$\begin{tikzcd} E \arrow[r, "h_{\underline{a}}"] \arrow[d, "k"'] & H(E,\underline{a}) \arrow[d, "H(k)"] \\ K \arrow[r, "h_{C_K}"] & H(K,C_K) \end{tikzcd}$$

If kE is dense in K the correspondence $f \longrightarrow f \circ k$ of C_K with $\underline{a} = C_K \circ k$ is 1 - 1 and since $\mathcal{H}$ satisfies $(FP_{\underline{a}})$ $H(k)$ is a homeomorphism. The function $g = h_{C_K}^{-1} \circ H(k)$ is the desired homeomorphism when K is compact.

In the following diagram

$$\begin{tikzcd} E \arrow[r, "h_{C_E^*}"] \arrow[d, "e"'] & H(E,C_E^*) \arrow[d, "H(e)"] \\ E \arrow[r, "h_{\underline{a}}"] & H(E,\underline{a}) \end{tikzcd}$$

where $ex = x$ for all x in E, the continuous function $H(e)$ is onto. This is because it maps $H(E,C_E^*)$ on a compact, hence closed, subset of $H(E,\underline{a})$ which contains the dense set $h_{\underline{a}}E$. If there is a homeomorphism $g:H(E,\underline{a}) \longrightarrow K$ such that $g \circ h_{\underline{a}} = k$ then $f = g \circ H(e)$. This follows from the uniqueness of f and the fact that $g \circ H(e) \circ i = g \circ H(e) \circ h_{C_E} = g \circ h_{\underline{a}} \circ e = g \circ h_{\underline{a}} = k$.

Consequently $f(\beta E) = K$ if statement (2) holds.

Assume that $f(\beta E) = K$. Then $kE = (f \circ i)E$ is a dense subset of K because iE is a dense subset of βE.

Corollary. Let K be a compact space and let $k:E \longrightarrow K$ be a continuous function with kE dense in K. The following statements are equivalent when $\underline{a} = C_K \circ k$:

(1) <u>(K,k) is an extension of E</u> ;

(2) <u>$(H(E,\underline{a}),h_{\underline{a}})$ is an extension of E</u> ;

(3) <u>$\mathcal{O}(E,\underline{a})$ is the topology of E; and</u>

(4) <u>$\pi(\underline{a}) \circ i$ is an embedding.</u>

Proof: The equivalence of (1) and (2) is an immediate consequence of the theorem. Statements (2) and (3) are equivalent by theorems 14 and 15 since $(E,\underline{a})$ is an object of $A' \cap \Lambda'$ (note that $\underline{a} = C_K \circ k$) .

Since kE is dense in K the theorem shows that (K,k) is a continuous image of βE . There is a homeomorphism $n: \beta E/\underline{r}(\underline{a}) \longrightarrow K$ such that $n \circ \pi(\underline{a}) = f$. Therefore $n \circ \pi(\underline{a}) \circ i = f \circ i = k$ and so $\pi(\underline{a}) \circ i$ is an embedding iff k is an embedding.

<u>Remarks.</u> Banaschewski [16] essentially showed the equivalence of (1) and (3) . $(H(E,\underline{a}),h_{\underline{a}})$ may of course be an extension of E even if kE is not dense in K . It will then be isomorphic to $(\overline{kE},k)$.

If E is a completely regular space let $\underline{\mathfrak{a}} = \underline{\mathfrak{a}}(E)$ denote the collection of uniformly closed unitary subalgebras $\underline{a}$ of C_E^* . Let $\underline{\mathfrak{C}} = \underline{\mathfrak{C}}(E)$ be the subcollection of consisting of those algebras $\underline{a}$ in $\underline{\mathfrak{a}}$ for which $\mathcal{O}(E,\underline{a})$ is the topology of E . The algebras in $\underline{\mathfrak{C}}$ will be called the <u>characteristic algebras of E</u> . When $\underline{\mathfrak{a}}$ is ordered by inclusion it is a complete lattice and $\underline{\mathfrak{C}}$ is a conditionally complete sublattice. In addition if $\underline{c}$ is in $\underline{\mathfrak{C}}$ and $\underline{c}$ is contained in $\underline{a}$ then $\underline{a}$ itself is in $\underline{\mathfrak{a}}$. In general

$\mathfrak{C}$ is not a complete lattice, as it will be shown later that $\mathfrak{C}$ is a complete lattice iff E is locally compact (theorem 5) .

The relations $\underline{r}(\underline{c})$ for $\underline{c}$ in may be distinguished among the $\underline{r}(\underline{a})$ for $\underline{a}$ in $\mathfrak{a}$ and as a result provide another method of characterizing the characteristic algebras of E . This characterization is stated as

Theorem 3. For an algebra $\underline{a}$ in $\mathfrak{a}$ the following conditions are equivalent:

(1) $\pi(\underline{a}) \circ i$ is an embedding ;

(2) if x is in iE and $y \neq x$ is in βE there exists f in $\underline{a}$ with $fx \neq fy$; and

(3) for all x in iE $\underline{r}(\underline{a})[x] = [x]$.

Proof: From the definition of $\underline{r}(\underline{a})$ (2) and (3) are clearly equivalent. The equivalence of (1) and (2) is an immediate consequence of the following lemma since iE is dense in βE.

Lemma 1. Let F be a Hausdorff space and let $\underline{r}$ be an open (closed) equivalence relation on F . Let E be a subset of F such that E is dense in $\underline{r}[E]$. If $\pi : F \longrightarrow F/\underline{r}$ denotes the natural mapping then $\pi|E$ is an embedding iff for all x in E, $\underline{r}[x] = [x]$.

Proof: Assume that for all x in E $\underline{r}[x] = [x]$. Then $E = \underline{r}[x]$ and since $\underline{r}$ is open (closed) $E/\underline{r}|E$ may be homeomorphically embedded in $F/\underline{r}$ by a function j such that $\pi \circ n = j \circ \pi_E$, where $n : E \longrightarrow F$ is the natural injection and

$\pi_E : E \longrightarrow E/\underline{r}|E$ is the natural mapping (see Bourbaki [3] p82 and p85). Since π_E is obviously a homeomorphism in this case it follows that $\pi \circ n = \pi|E$ is an embedding.

Suppose $\pi|E$ is an embedding. Let $G = \pi E$ and denote by $f:F \longrightarrow F$ the identity function where $fx = x$ for all x in F. Let $\varphi : G \longrightarrow E$ be the continuous function such that $\varphi \circ \pi|E = f|E$. If Θ is the function $\varphi \circ \pi|\underline{r}[E] : \underline{r}[E] \longrightarrow E$ contained in $\underline{r}[E]$, then $\Theta|E = f|E$. Since F is Hausdorff and E is dense in $\underline{r}[E]$ it follows that $\Theta = f|\underline{r}[x]$. Consequently $E = \underline{r}[E]$ and so for all x in E $\underline{r}[x] = [x]$ since $\pi|E$ is $1 - 1$.

An immediate consequence of this theorem is the following

<u>Corollary. If (K,k) is a compact extension of E and $f:\beta E \longrightarrow K$ is the continuous function with $f \circ i = k$, then $f(\beta E \cap \complement iE)$ is contained in $K \cap \complement kE$.</u>

Proof: If $\underline{a} = C_K \circ k$ then $\pi(\underline{a}) \circ i$ is an embedding by the corollary to theorem 2. The proof of this corollary shows that there is a homeomorphism $n : \beta E/\underline{r}(\underline{a}) \longrightarrow K$ such that $n \circ \pi(\underline{a}) = f$. This theorem shows that if y is in $\beta E \cap \complement iE$ then $\pi(\underline{a})y$ is not in $\pi(\underline{a})(iE)$, because $\pi(\underline{a})y = \pi(\underline{a})x$ implies x is not in iE by (2). Since $n \circ \pi(\underline{a}) \circ i = f \circ i = k$ it follows that n maps $\complement(\pi(\underline{a})(iE))$ on kE, i.e. $f(\beta E \cap \complement iE) = K \cap \complement KE$.

<u>Remark.</u> More generally if (K,k) is a pair where K is a compact space and $k:E \longrightarrow K$ is continuous, then $f(\beta E \cap \complement iE)$ is contained in $K \cap \complement kE$ iff iE is $\underline{r}(\underline{a})$ - saturated where $\underline{a} = C_K \circ k$.

To each $\underline{c}$ in corresponds the collection of compact extensions (K,k) of E with $C_K \circ k = \underline{c}$. One such extension is $(\beta E/\underline{r}(c), \pi(\underline{c}) \circ i)$. Let $K(\underline{c}) = (K(\underline{c}), \pi(\underline{c}) \circ i)$ denote this extension of E. Since $\underline{r}(C_E^*)$ is the identity equivalence relation, $K(C_E^*)$ and $(\beta E,i)$ may be identified. Although the definition of $(\beta E,i)$ as $(H(E,C_E^*),h_{C_E^*})$ suggests that the compact extension corresponding to $\underline{c}$ should be taken to be $(H(E,\underline{c}),h_{\underline{c}})$ it will be more convenient to use the quotient construction $K(\underline{c})$.

The correspondence $\underline{c} \longrightarrow K(\underline{c})$ defined for $\underline{c}$ in $\underline{\mathfrak{C}}$ may be extended to $\underline{\mathfrak{A}}$ by defining $K(\underline{a}) = (\beta E/\underline{r}(\underline{a}), \pi(\underline{a}))$.

Let $\underline{a}'$ and $\underline{a}$ be two elements of with $\underline{a}'$ contained in $\underline{a}$. When $\underline{a}$ is identified with $C_{K(\underline{a})}$ then $\underline{a}'$ is identified with a uniformly closed unitary subalgebra of $C_{K(\underline{a})}$. Let $\underline{r}(\underline{a}'|\underline{a})$ denote the closed equivalence relation on $K(\underline{a})$ defined by $\underline{a}'$ considered as a subalgebra of $C_{K(\underline{a})}$. In other words for x,y in $K(\underline{a})$ $x\,\underline{r}(\underline{a}'|\underline{a})y$ iff $f'x = f'y$ for all f' in $\underline{a}'$, where f' denotes both a function on E and a function on $K(\underline{a})$ such that $f' \circ \pi(\underline{a}) \circ i = f'$.

Let $\pi_1(\underline{a}',\underline{a}):K(\underline{a}) \longrightarrow K(\underline{a})/\underline{r}(\underline{a}',\underline{a})$ be the natural mapping. It is clear that $C_{K(\underline{a})/\underline{r}(\underline{a}',\underline{a})} \circ \pi_1(\underline{a}',\underline{a}) \circ \pi(\underline{a}) = \underline{a}'$ and hence there is a homeomorphism $\delta:K(\underline{a})/\underline{r}(\underline{a}',\underline{a}) \longrightarrow K(\underline{a}')$ such that $\delta \circ \pi_1(\underline{a}',\underline{a}) \circ \pi(\underline{a}) = \pi(\underline{a}')$ (see the discussion of continuous images in section 1). Let $\pi(\underline{a}',\underline{a}) = \delta \circ \pi_1(\underline{a}',\underline{a})$. Then $\pi(\underline{a}',\underline{a}):K(\underline{a}) \longrightarrow K(\underline{a}')$ is the function (continuous) such that $\pi(\underline{a}',\underline{a}) \circ \pi(\underline{a}) = \pi(\underline{a}')$.

Let $\underline{a}$, $\underline{a}'$ and $\underline{a}$ be three elements of $\mathcal{a}$ such that $\underline{a}'$ contains $\underline{a}$ and is contained in $\underline{a}$. Then $\pi(\underline{a},\underline{a}') \circ \pi(\underline{a}',\underline{a}) = \pi(\underline{a},\underline{a})$ since $\pi(\underline{a},\underline{a}') \circ \pi(\underline{a}',\underline{a}) \circ \pi(\underline{a}) = \pi(\underline{a},\underline{a}') \circ \pi(\underline{a}') = \pi(\underline{a})$.

Using this notation $\pi(\underline{a}) = \pi(\underline{a},C_E^M)$ since $\pi(\underline{a},C_E^M) \circ \pi(C_E^M) = \pi(\underline{a})$. This fact suggests the following analogue of theorem 3.

<u>Theorem 4. Let $\underline{a}',\underline{a}$ be two elements of $\mathcal{a}$ with $\underline{a}'$ contained in $\underline{a}$ and let X be a subset of $K(\underline{a})$ that contains $(\pi(\underline{a}) \circ i)E$. The following conditions are equivalent:</u>

(1) $\pi(\underline{a}',\underline{a})|X:X \longrightarrow K(\underline{a}')$ <u>is an embedding;</u>

(2) <u>if x is in X and $y \neq x$ is in $K(\underline{a})$ there exists f' in $\underline{a}'$ with $f'x \neq f'y$;</u>

(3) <u>for all x in X $\underline{r}(\underline{a}',\underline{a})[x] = [x]$; and</u>

(4) <u>$\underline{a}'$ when extended to X is a characteristic algebra of X.</u>

Proof: Since $r(\underline{a}',\underline{a})$ is closed and X is dense in $K(\underline{a})$ (1) and (3) are equivalent by lemma 1. From the definition of $r(\underline{a}',\underline{a})$ it is clear that (2) and (3) are equivalent.

If $\underline{a}'$ is identified with $C_{K(\underline{a}')}$ it follows that (1) implies (4), provided the restriction to X of the extension to $K(\underline{a}')$ of a function f' in $\underline{a}'$ is its extension to X. Let f' be in $\underline{a}'$ and let $\overline{F'}$ be its extension to $K(\underline{a}')$ i.e. $\overline{F'} \circ \pi(\underline{a}') \circ i = f'$. If $f' = \overline{F'} \circ \pi(\underline{a}',\underline{a})|X$ then $f' \circ \pi(\underline{a}) \circ i = \overline{F'} \circ \pi(\underline{a}',\underline{a}) \circ \pi(\underline{a}) \circ i =$ [because $X \supseteq (\pi(\underline{a}) \circ i)E)$] $= \overline{F'} \circ \pi(\underline{a}') \circ i = f'$. This shows that f' is the extension of f' to X.

Assume that $\underline{a}'$ satisfies (4) and let $\underline{a}'$ also denote its extension to X. Then $(H(X,\underline{a}'),h_{\underline{a}'})$ is a compact extension of X. The function $\theta = h_{\underline{a}'} \circ \pi(\underline{a}) \circ i : E \longrightarrow H(X,\underline{a}')$ is continuous with dense image. Consequently there is a homeomorphism $\delta : H(X,\underline{a}') \longrightarrow K(\underline{a}')$ such that $\delta \circ g = \pi(\underline{a}')$ where $g : \beta E \longrightarrow H(X,\underline{a}')$ satisfies $g \circ i = \theta$. Let $\gamma = \delta \circ h_{\underline{a}'} : X \longrightarrow K(\underline{a}')$. It is a homeomorphic embedding and $\gamma \circ \pi(\underline{a}) \circ i = \delta \circ h_{\underline{a}'} \circ \pi(\underline{a}) \circ i = \delta \circ g \circ i = \pi(\underline{a}') \circ i$. Consequently $\gamma = \pi(\underline{a}',\underline{a})|X$ and so (1) implies (4).

Remark. The relations $r(\underline{a}',\underline{a})$ and the mappings $(\underline{a}',\underline{a})$ for $\underline{a}' \subseteq \underline{a}$ are used in section five to investigate the corrections between the various extensions of E. For example this theorem shows when a subset X of $K(\underline{c})$ that contains $(\pi(\underline{c}) \circ i)E$ may be embedded in the compact extension $K(\underline{c}')$ for when $\underline{c}$ contains $\underline{c}'$, i.e. when the extension X is 'contained in' the extension $K(\underline{c}')$.

This section concludes with a discussion of some

Examples of characteristic algebras of E.

1. Let D be a closed subset of βE. The algebra $\underline{a}_D = [f \text{ in } C_E^* \mid f|D \text{ is constant}]$ is a characteristic algebra if $D \cap iE = \emptyset$ since βE is normal. When $D \cap iE = \emptyset$ let $\underline{c}_D$ also denote the algebra $\underline{a}_D$. Specific examples of such closed sets are the finite subsets disjoint from iE in particular the two point subsets of $\beta E \cap \complement\, iE$. If $x \neq y$ are two points of $\beta E \cap \complement\, iE$ $\underline{c}_{[x,y]}$ is a maximal characteristic algebra of E distinct from C_E^*. When $\underline{c} \neq C_E^*$ is a characteristic algebra of E $\underline{r}(\underline{c})$ identifies at least two points and so $\underline{c} \subseteq \underline{c}_{[x,y]}$ for some pair of points identified by $\underline{r}(\underline{c})$. This shows that $\underline{c}$ is defined by the maximal characteristic algebras that contain it.

For any point x in $\beta E \cap \complement\, iE$ $\underline{c}_{[x]} = \underline{c}_\emptyset = C_E^*$. If $\beta E \cap \complement\, iE$ is a closed subset of βE then $\underline{c}_{\beta E \cap \complement iE}$ is the smallest characteristic algebra of E. This is because $\underline{c}$ is characteristic iff the non-trivial $\underline{r}(\underline{c})$-equivalence classes all lie in $\beta E \cap \complement\, iE$ (theorem 3). The set $\beta E \cap \complement\, iE$ is closed iff E is locally compact, from which follows the first half of

Theorem 5. The conditionally complete lattice $\mathfrak{C}(E)$ is a complete lattice iff E is locally compact.

Proof: Consider the algebra $\underline{a} = \bigcap_{\underline{c} \text{ in } \mathfrak{C}(E)} \underline{c}$ in $\mathfrak{A}(E)$. If f is in $\underline{a}$ then f is constant on $\beta E \cap \complement\, iE$ because for $x \neq y$ in

$\beta E \cap \complement iE, c_{[x,y]}$ is a characteristic algebra and so $fx = fy$. For each f in $\underline{a}$ let $\lambda_f = f[\beta E \cap \complement iE]$. Then $\bigcap_{f \text{ in } \underline{a}} f^{-1}([\lambda_f])$ is a closed set D containing $\beta E \cap \complement iE$ and if x is in $\beta E \cap \complement iE$ $\underline{r}(\underline{a})[x] = D$. $\underline{\mathfrak{C}}(E)$ is a complete lattice iff $\underline{a}$ is a characteristic algebra. This is the case iff $D \cap iE = \emptyset$ or in other words iff $D = \beta E \cap \complement iE$. Consequently if $\underline{\mathfrak{C}}(E)$ is a complete lattice $\beta E \cap \complement iE$ is closed and so E is locally compact. The converse has been proved.

Remark. This theorem is equivalent to Samuel's characterization of locally compact spaces [13]. The equivalence is discussed in chapter four.

The maximal characteristic algebras of E (distinct from C_E^*) characterize E as a completely regular space when $\beta E \cap \complement iE$ contains two points. This result known to Banaschewski [16] is stated as

Theorem 6. Let E and E' be two completely regular spaces and let $h: C_E^* \longrightarrow C_{E'}^*$ be a unitary algebra isomorphism. The following statements are equivalent when $\beta E \cap \complement iE$ contains two points;

(1) there is a homeomorphism $\theta: E \longrightarrow E'$ such that ${}^t h \circ h_{C_{E'}} \circ \theta = h_{C_E}$;

(2) $h(\underline{a})$ is a maximal characteristic algebra of E' iff $\underline{a}$ is a maximal characteristic algebra of E; and

(3) $h(\underline{a})$ is a characteristic algebra of E' iff $\underline{a}$ is a characteristic algebra of E.

Proof: it is clear that ${}^{t}h : H(E', C^{*}_{E'}) \longrightarrow H(E, C^{*}_{E})$ is a homeomorphism. Consider the commutative diagram

$$\begin{tikzcd}
E \arrow[rr, "h_{C^*_E}"] \arrow[d, "\theta"'] & & H(E, C^*_E) \arrow[d, shift right, "H(\theta)"'] \\
E' \arrow[rr, "h_{C^*_{E'}}"] & & H(E', C^*_{E'}) \arrow[u, shift right, "{}^{t}h"']
\end{tikzcd}$$

which exists when (1) holds. $H(\theta) = {}^{t^{-1}}h$ because the diagram is commutative. Since $H(\theta)$ maps $h_{C^*_E}E$ on $h_{C^*_{E'}}E'$ and its complement on $\complement h_{C^*_{E'}}E'$ it follows that $h(\underline{a}_{[x,y]}) = \underline{a}_{[H(\theta)x,\, H(\theta)y]}$ which is then characteristic. If $h(\underline{a}_{[x,y]}) = \underline{a}_{[H(\theta)x, H(\theta)y]}$ is characteristic then x and y do not belong to $h_{C^*_E}E$ and so $\underline{a}_{[x,y]}$ is characteristic. This shows that (1) implies (2).

Every algebra $\underline{a}$ in $\underline{\mathfrak{A}}(E)$ is defined by the maximal algebras $\underline{a}_{[x,y]}$ containing it. It is characteristic iff all the $\underline{a}_{[x,y]}$ containing $\underline{a}$ are characteristic. Since $h(\underline{a}_1)$ lies in $h(\underline{a}_2)$ iff $\underline{a}_1$ is contained in $\underline{a}_2$ it follows that (2) implies by (3).

Assume that (3) holds. Let $E_1' = ({}^{t^{-1}}h \circ h_{C^*_E})E$ and let $E_2' = h_{C^*_{E'}}E'$. They are subsets of $\beta E'$ such that if $\underline{a}'$ is in $\underline{\mathfrak{A}}(E')$ then $\underline{a}'|E_1'$ is a characteristic algebra of the subspace E_1' iff $a'|E_2'$ is a characteristic algebra of the subspace E_2'. To show that (1) holds it is necessary and sufficient to prove that $E_1' = E_2'$. Since $\beta E'$ and the natural injections n_j' of E_j' are β-extensions of the E_j' this follows from the following lemma as $\beta E' \cap \complement iE_j'$ contains two points when (3) holds and $\beta E \cap \complement iE$ contains two points.

Lemma 2. Let K be a compact space and let E_1, E_2 be two subspaces with E_1 and E_2 each containing at least two points. Then $E_1 = E_2$ if these subspaces satisfy the following condition: when a is a uniformly closed unitary subalgebra of C_K $a|E_1$ is a characteristic algebra of E_1 iff $a|E_2$ is a characteristic algebra of E_2.

Proof: Assume that $E_1 \neq E_2$ and that x_1 is in $E_1 \cap \complement E_2$. Let $x_2 \neq x_1$ be a second point in $\complement E_2$. The algebra $a_{[x_1,x_2]}$ has the property that $a_{[x_1,x_2]}|E_2$ is a characteristic algebra of E_2 but $a_{[x_1,x_2]}|E_1$ is not a characteristic algebra of E_1 (see theorem 4). This is a contradiction.

Therefore $E_1 \cap \complement E_2 = \phi$ i.e. $E_1 \subseteq E_2$. From the symmetry of the argument this implies that $E_1 = E_2$.

Remarks. Theorem 2 is the best result that can be obtained in view of the fact that there are completely regular spaces E such that $\beta E \cap \complement iE$ consists of one point (e.g. the space of ordinals less than the first uncountable ordinal). For such a space E the restriction mapping of $C_{\beta E} \longrightarrow C_E^M$ is an algebra isomorphism for which (3) holds. Obviously (1) is false in this case.

2. Let $D_1,\dots,D_m$ be m closed mutually disjoint subsets of $\beta E \cap \complement E$. If $D = \bigcup_{i=1}^{m} D_i$ then $\underline{c}_D$ is defined and $\underline{c}_D = \underline{a}_D$ is contained in $\underline{a}_{D_1,\dots,D_m} = \bigcap_{i=1}^{m} \underline{c}_{D_i}$. This shows that $\underline{a}_{D_1,\dots,D_m}$ is characteristic and justifies setting $\underline{a}_{D_1,\dots,D_m} = \underline{c}_{D_1,\dots,D_m}$.

The characteristic algebras $\underline{c}_{D_1,\dots,D_m}$ have the property that the intersection of any two is characteristic and is of this type.

This raises the question of whether $\mathcal{C}$ is a lattice. If it is not in general, what are the topological consequences for E when $\mathcal{C}(E)$ is a lattice? In view of theorem 1 these questions be considered in terms of closed equivalence relations on βE. Let $\underline{a}_1$ and $\underline{a}_2$ be two elements of $\mathfrak{A}(E)$ and let $\underline{a} = \underline{a}_1 \cap \underline{a}_2$. Then $\underline{r}(a)$ is the coarsest closed equivalence relation finer than $\underline{r}(\underline{a}_1)$ and $\underline{r}(\underline{a}_2)$. If $\underline{r}(\underline{a}_1)$ and $\underline{r}(\underline{a}_2)$ satisfy condition (3) of theorem (3) does $\underline{r}(\underline{a})$ satisfy this condition? It is not hard to see that the coarsest equivalence relation finer than $\underline{r}(\underline{a}_1)$ and $\underline{r}(\underline{a}_2)$ satisfies this condition. Is it a closed equivalence relation?

§3. Translation lattices associated with the algebras $\underline{a}$ in $\mathfrak{A}(E)$. Let E be any completely regular space and let C_E and C_E^* denote respectively the collection of continuous real-valued functions on E and the collection of bounded continuous real-valued functions on E. Then C_E^* determines C_E as shown by . . .

<u>Lemma 3.</u> $C_E = [f \text{ in } F_E \mid \text{for } \lambda \geq 0 \ (f \cap \lambda) \cup (-\lambda) \text{ is in } C_E^*]$

Proof: It is clear that C_E is contained in this set. Let f be a real-valued function on E such that $\lambda \geq 0$ implies $(f \cap \lambda) \cup (-\lambda)$ is in C_E^*. Pick x_0 in E and $\varepsilon > 0$ and let $U_0 = [x \text{ in } E \mid |fx_0 - fx| < \varepsilon]$. Let λ_0 be greater than $|fx_0 - \varepsilon|$ and $|fx_0 + \varepsilon|$ and let $f_0 = (f \cap \lambda_0) \cup (-\lambda_0)$. Then $U_0 = [x \text{ in } E \mid |f_0x_0 - f_0x| < \varepsilon]$ and so is open. This is because $f_0x_0 = fx_0$ and so by the choice of λ_0 $|f_0x_0 - f_0x| < \varepsilon$ iff $|fx_0 - fx| < \varepsilon$.

This connection between C_E^* and C_E suggests the following definition of a function $\mathcal{L}$ on $\mathfrak{a} = \mathfrak{a}(E)$: if $\underline{a}$ is in $\mathfrak{a}$ let $\mathcal{L}(\underline{a}) = [f \text{ in } F_E \mid \text{ for } \lambda \geq 0 \ (f \cap \lambda) \cup (-\lambda) \text{ is in } \underline{a}]$. Some obvious properties of $\mathcal{L}$ are listed in

<u>Theorem 7.</u> <u>The function $\mathcal{L}$ is such that:</u>

(1) <u>if a_1 is contained in a_2 then $\mathcal{L}(a_1)$ is contained in $\mathcal{L}(a_2)$, in particular $\mathcal{L}(a)$ is contained in C_E for all a;</u>

(2) <u>$\mathcal{L}(a_1 \cap a_2) = \mathcal{L}(a_1) \cap \mathcal{L}(a_2)$ and $\mathcal{L}(a_1 \cup a_2)$ contains $\mathcal{L}(a_1) \cup \mathcal{L}(a_2)$;</u>

(3) <u>for all a, a is contained in $\mathcal{L}(a)$; and</u>

(4) <u>$\mathcal{L}(a_1) \cap a_2 = a_1 \cap a_2$, in particular $\mathcal{L}(a) \cap C_E^* = a$ for all a and so $\mathcal{L}$ is 1 - 1.</u>

Proof: The first two properties are immediate consequences of the definition of $\mathcal{L}$ (note that $\underline{a}_1 \cup \underline{a}_2$ is the smallest algebra in $\underline{\mathcal{A}}$ containing $\underline{a}_1$ and $\underline{a}_2$) and lemma 3 .

Since each $\underline{a}$ in $\underline{\mathcal{A}}$ is isomorphic to C_K for some compact space K (for example $K = H(E,\underline{a})$ or $\beta E/\underline{r}(\underline{a})$) it follows that $\underline{a}$ is a lattice that contains the constants and so (3) is satisfied. Property (4) holds because of (3) and the obvious fact that f in $\mathcal{L}(\underline{a})$ is bounded iff f is in $\underline{a}$.

The collections $\mathcal{L}(\underline{a})$ of continuous functions on E are characterized by

<u>Theorem 8. Let S be a collection of continuous functions on E . Then there exists an algebra $\underline{a}$ in $\underline{\mathcal{A}}(E)$ with $S = \mathcal{L}(\underline{a})$ iff S satisfies the following conditions:</u>

($\mathcal{L}_1$) <u>S is a translation sublattice of C_E that contains the constants and is closed under multiplication by real numbers;</u>

($\mathcal{L}_2$) <u>S is uniformly closed;</u>

($\mathcal{L}_3$) <u>S is closed under positive inversion ; and</u>

($\mathcal{L}_4$) <u>f is in S iff $f \cup 0$ and $f \cap 0$ are both in S .</u>

<u>When S satisfies these conditions $S^* = S \cap C_E^*$ is in $\underline{\mathcal{A}}(E)$ and $S = \mathcal{L}(S^*)$.</u>

Proof: First consider the necessity of these conditions. Let $\underline{a}$ be in $\underline{\mathcal{A}}$ and consider $\mathcal{L}(\underline{a})$.

Let f and g be in $\mathcal{L}(\underline{a})$ and let λ be ≥ 0. The distributivity of the lattice of real numbers implies that $[(f \cap g) \cap \lambda] \cup (-\lambda) = [(f \cap \lambda) \cap (g \cap \lambda)] \cup (-\lambda) = [(f \cap \lambda) \cup (-\lambda)] \cap [(g \cap \lambda) \cup (-\lambda)]$. Since $\underline{a}$ is a lattice $f \cap g$ is in $(\underline{a})$. Similarly $f \cup g$ is in $\mathcal{L}(\underline{a})$. $\mathcal{L}(\underline{a})$ contains the constants because $\underline{a}$ is a unitary subalgebra and $\underline{a}$ is contained in $\mathcal{L}(\underline{a})$. If α is > 0 then αf is in $\mathcal{L}(\underline{a})$ because for $\lambda \geq 0$ $(\alpha f \cap \lambda) \cup (-\lambda) = \alpha[(f \cap \lambda/\alpha) \cup (-\lambda/\alpha)]$. Since $[(-1)f \cap \lambda] \cup (-\lambda) = (-1)[(f \cap \lambda) \cup (-\lambda)]$ it follows that $\mathcal{L}(\underline{a})$ is closed under multiplication by real numbers.

To show that $\mathcal{L}(\underline{a})$ satisfies $(\mathcal{L}_1)$ it remains to show that $\mathcal{L}(\underline{a})$ is a translation lattice i.e. if f is in $\mathcal{L}(\underline{a})$ and α denotes the function $\alpha x = \alpha$ for all x in E then $f + \alpha$ is in $\mathcal{L}(\underline{a})$. Let λ be ≥ 0 and consider $[(f + \alpha) \cap \lambda] \cup (-\lambda) = [f \cap (\lambda - \alpha) + \alpha] \cup (-\lambda) = [f \cap (\lambda - \alpha)] \cup (-\lambda - \alpha) + \alpha$. Since $\underline{a}$ is a translation lattice it is sufficient to show that when f is in $\mathcal{L}(\underline{a})$ and $\lambda > \mu$ then $(f \cap \lambda) \cup \mu)$ is in $\underline{a}$.

Let $\lambda_0 = \max[|\lambda|, |\mu|]$. The function $f_0 = (f \cap \lambda_0) \cup (-\lambda_0)$ is in $\underline{a}$ and so also are the functions $f_0 \cap \lambda$ and $(f_0 \cap \lambda) \cup \mu$. From the definition of λ_0 it follows that $(f \cap \lambda) \cup \mu = (f_0 \cap \lambda) \cup \mu$ and so $(f \cap \lambda) \cup \mu$ is in $\underline{a}$.

This shows that $\mathcal{L}(\underline{a})$ satisfies $(\mathcal{L}_1)$.

Let $(f_n)_n$ be a sequence of functions in $\mathcal{L}(\underline{a})$ that converge uniformly to some function f . If λ is $\geqslant 0$ then the functions $(f_n \cap \lambda) \cup (-\lambda)$ converge uniformly to $(f \cap \lambda) \cup (-\lambda)$. Since $\underline{a}$ is uniformly closed $(f \cap \lambda) \cup (-\lambda)$ is in $\underline{a}$ and so f itself is in $\mathcal{L}(\underline{a})$. Consequently $\mathcal{L}(\underline{a})$ is uniformly closed.

Let f in $\mathcal{L}(\underline{a})$ be $\geqslant 0$ and assume that for each x in E $fx \neq 0$. The function $1/f$ is continuous and if $\varepsilon > 0$ $1/f \cup 1/\varepsilon$ is in $\mathcal{L}(\underline{a})$. Let δ be >0 and consider $(f \cup \delta) \cap \varepsilon$ which is in $\underline{a}$ since f is in $\mathcal{L}(\underline{a})$. This function is $\geqslant \delta \cap \varepsilon$ and since $\underline{a}$ is closed under bounded inversion (being isomorphic to C_K for some compact space K) $1/(f \cup \delta) \cap \varepsilon = 1/(f \cup \delta) \cup 1/\varepsilon = (1/f \cap 1/\delta) \cup 1/\varepsilon$ is in $\underline{a}$. Using the distributivity of the lattice of real numbers it follows that $(1/f \cap 1/\delta) \cup 1/\varepsilon = (1/f \cup 1/\varepsilon) \cap (1/\delta \cup 1/\varepsilon) = (1/f \cup 1/\varepsilon) \cap 1/\delta$. Let $\lambda = 1/\delta$. Then $(1/f \cup 1/\varepsilon) \cap \lambda$ is in $\underline{a}$ for all $\lambda \geqslant 0$ and so $1/f \cup 1/\varepsilon$ is in $\mathcal{L}(\underline{a})$ for all $\varepsilon > 0$. The functions $1/f \cup 1/n$ are in $\mathcal{L}(\underline{a})$ for each $n > 0$ and as they converge uniformly to $1/f$ it follows that $1/f$ is in $\mathcal{L}(\underline{a})$ because $\mathcal{L}(\underline{a})$ is uniformly closed. This shows that $\mathcal{L}(\underline{a})$ is closed under positive inversion.

If f is in $\mathcal{L}(\underline{a})$ then since $\mathcal{L}(\underline{a})$ satisfies $(\mathcal{L}_1)$ $f \cup 0$ and $f \cap 0$ are in $\mathcal{L}(\underline{a})$. Assume that f is a function such that $f \cup 0$ and $f \cap 0$ are in $\mathcal{L}(\underline{a})$. Then f is in $\mathcal{L}(\underline{a})$. When λ is $\geqslant 0$, $(f \cap \lambda) \cup (-\lambda) = (f \cup 0) \cap \lambda +$

$(f \cap 0) \cup (-\lambda)$ and since $\underline{a}$ is closed under addition $(f \cap \lambda) \cup (-\lambda)$ is in $\underline{a}$ if λ is ≥ 0.

This completes the proof of the necessity of these four conditions.

Assume that S satisfies $(\mathcal{L}_1)$, $(\mathcal{L}_2)$, $(\mathcal{L}_3)$ and $(\mathcal{L}_4)$. Conditions $(\mathcal{L}_1)$ and $(\mathcal{L}_2)$ show that S^* is a uniformly closed translation sublattice of C_E^* which contains the constants and is closed under multiplication by real numbers. Assume that S^* is an algebra. Then S^* is in $\underline{\mathcal{A}}(E)$.

If f is in S and λ is ≥ 0 then $(f \cap \lambda) \cup (-\lambda)$ is in S^* and so S is contained in $\mathcal{L}(S^*)$.

Since S satisfies $(\mathcal{L}_4)$ and is closed under multiplication by (-1) it follows that $S = \mathcal{L}(S^*)$ if S contains all the positive functions in $\mathcal{L}(S^*)$. Let $f \geq 0$ be in $\mathcal{L}(S^*)$. From the first part of the theorem $f \cup 1/n$ is in $\mathcal{L}(S^*)$ for each $n > 0$. Also $1/f \cup 1/n = 1/f \cap n$ is in $\mathcal{L}(S^*)$ for all $n > 0$. If $g_n = 1/f \cap n$, it is in S^* and hence in S. For each x in E $g_n x > 0$ and since S satisfies $(\mathcal{L}_3)$ $1/g_n = f \cup 1/n$ is in S for each $n > 0$. The functions $f \cup 1/n$ converge uniformly to f and since S satisfies $(\mathcal{L}_2)$ it follows that f is in S. Consequently $S = \mathcal{L}(S^*)$.

This completes the proof of the theorem assuming that S^* is an algebra.

The following lemma completes the theorem by showing that S^* is an algebra if S satisfies $(\mathcal{L}_1)$ and $(\mathcal{L}_2)$.

Lemma 4. Let E be any set and let S^* be a uniformly closed translation sublattice of F_E^* that contains the constant functions. Then S^* is a subalgebra iff it is closed under multiplication by the real numbers.

Proof: The object (E,S^*) is in Λ' and so process $\mathcal{L}$ is defined for (E,S^*). The space $L(E,S^*)$ is compact since it is homeomorphic to $T(E,S^*)$. The collection $(S^*)_L$ of functions f_L on $L(E,S^*)$ such that $f_L \circ \mathfrak{l}_S = f$ is a uniformly closed translation lattice which contains the constants and separates the points of $L(E,S^*)$.

If S^* is closed under multiplication by real numbers the fact that $\mathcal{L}$ is isomorphic to $\mathfrak{J}\,|\,\Lambda'$ implies that $(S^*)_L$ is also closed under multiplication by real numbers. When $(S^*)_L$ is closed under multiplication by real numbers it has the two-point property i.e. if $x \neq y$ are two points of $L(E,S^*)$ and if λ, μ are two real numbers there exists a function f_L in $(S^*)_L$ with $f_L x = \lambda$ and $f_L y = \mu$. Assume that g_L is a function in $(S^*)_L$ such that $\lambda_1 = g_L x \neq g_L y = \mu_1$. Let $\varepsilon = \dfrac{\lambda - \mu}{\lambda_1 - \mu_1}$. Then εg_L is in $(S^*)_L$ and $(\varepsilon g_L)x - (\varepsilon g_L)y = \lambda - \mu$. If $\delta = \mu - \varepsilon\mu_1$ then $f_L = \varepsilon g_L + \delta$ is a function with the desired property since $f_L x - f_L y = \lambda - \mu$ and $f_L y = \mu$.

Stone's proof of the Stone-Weierstrass theorem (see Bourbaki [17]) shows that a lattice of continuous real-valued functions on a compact space K is uniformly dense in C_K if it has the two-point property. Since $(S^*)_L$ is uniformly closed it follows that $(S^*)_L = C_{L(E,S^*)}$. As a result S^* is an algebra since $S^* = C_{L(E,S^*)} \circ \iota_{S^*}$.

Note. Theorems 7 and 8 show that a solution to (AC) provides a solution to (BC) since $\mathcal{L}$ is 1 - 1 and a solution to (AC) characterises $\mathcal{L}(C^*_E)$ among the lattices satisfying $(\mathcal{L}_1)$, $(\mathcal{L}_2)$, $(\mathcal{L}_3)$ and $(\mathcal{L}_4)$.

The characterization of the lattices $\mathcal{L}(\underline{a})$ given by theorem 8 is an internal one. For some purposes it is more useful to relate $\mathcal{L}(\underline{a})$ to the equivalence relation $\underline{r}(\underline{a})$ on βE. In order to do this it is necessary to show that every function f in C_E defines an $\bar{R}$ - valued continuous function $\underline{f}$ on βE with $\underline{f} \circ i = f$.

First consider the following lemma,

<u>Lemma 5. If f is in C_E let $f_n = (f \cap n) \cup (-n)$ where $n \geq 0$ is an integer. Then if x is in βE the sequence $(f_n x)_n$ is monotone.</u>

Proof: To simplify the notation identify E with iE and any g in C^*_E with its extension to βE.

For all x in βE $f_0 x = 0$. Assume $f_n x > 0$ for some $n > 0$. Consider $f_{n+1} x$. Since $(f_{n+1} \cap n) \cup (-n) = f_n$

Similarly if $\underline{f}x_0 = -\infty$ the function is continuous at x_0.

Since x_0 is arbitrary it follows that $\underline{f}$ is continuous.

Remark. If $\underline{C_E}$ denotes $[\underline{f} \mid f \text{ in } C_E]$ it coincides with $C_{\beta E}$ iff $C_E = C_E^M$ i.e. iff E is pseudocompact. Also $\underline{C_E}$ may be made into an algebra by requiring the correspondence $\underline{f} \longrightarrow f$ to be an algebra isomorphism. It will not in general be true that $\underline{C_E}$ is an algebra of functions.

Theorem 9 permits a second characterisation of the lattices $\mathcal{L}(\underline{a})$. It is stated as

Theorem 10. Let $\underline{a}$ be in $\mathcal{A}(E)$. Then f is in $\mathcal{L}(\underline{a})$ iff for x, y in βE, $x r(\underline{a}) y$ implies $\underline{f}x = \underline{f}y$.

Proof: If f is in $\mathcal{L}(\underline{a})$ then for each $n \geq 0$ f_n is in $\underline{a}$. Hence if $x r(\underline{a}) y$, $f_n x = f_n y$ for each $n \geq 0$. This shows that $\underline{f}x = \underline{f}y$.

Assume that $\underline{f}x = \underline{f}y$ when $x r(\underline{a}) y$. The function $(\underline{f} \cap n) \cup (-n)$ is a continuous function on βE which coincides on iE with $f_n = (f \cap n) \cup (-n)$. Therefore $(\underline{f} \cap n) \cup (-n) = f_n$. Consequently if $x r(\underline{a}) y$, $f_n x = f_n y$ and so the functions f_n are in $\underline{a}$ for each $n \geq 0$.

Let λ be ≥ 0 and let $n > \lambda$ be an integer. Since $(f \cap \lambda) \cup (-\lambda) = (f_n \cap \lambda) \cup (-\lambda)$ it follows that $(f \cap \lambda) \cup (-\lambda)$ is in $\underline{a}$ and hence that f is in $\mathcal{L}(\underline{a})$.

Corollary 1. For any a in $\mathcal{A}(E)$ if f, g in $\mathcal{L}(\underline{a})$ are ≥ 0 then $f + g$ is in $\mathcal{L}(\underline{a})$. If f is in $\mathcal{L}(\underline{a})$ and g is in a then $f + g$ is in $\mathcal{L}(\underline{a})$.

If f and g are positive continuous functions then $\underline{f}$ and $\underline{g}$ are positive. Therefore $\underline{f}x + \underline{g}x = (\underline{f + g})x$ for each x in βE. The first assertion follows immediately from the theorem.

If g is in C_E^* then $g = \underline{g}$ and so $\underline{g}$ is bounded. Consequently if f is any continuous function $\underline{f}x + \underline{g}x = (\underline{f + g})x$ for each x in βE. The rest of the corollary follows from the theorem. Another almost immediate corollary is

Corollary 2. Let a be in $\mathcal{A}(E)$ and let $C^\infty_{K(\underline{a})}$ be the collection of continuous $\bar{R}$ - valued functions g on $K(\underline{a})$ such that $g(\pi(\underline{a}) \circ i)E$ is real-valued. Then $\mathcal{L}(\underline{a}) = C^\infty_{K(\underline{a})} \circ \pi(\underline{a}) \circ i$.

Proof: If $\underline{g}$ is in $C^\infty_{K(\underline{a})}$ then $\underline{g} \circ \pi(\underline{a})$ is in $C_{\beta E} = \underline{C}_E$. Since it is compatible with $r(\underline{a})$ $\underline{g} \circ \pi(\underline{a}) \circ i$ is in $\mathcal{L}(\underline{a})$.

Conversely if f is in $\mathcal{L}(\underline{a})$ the function $\underline{f}$ is compatible with $r(\underline{a})$. Therefore there is an $\bar{R}$ - valued function $\underline{g}$ on $K(\underline{a})$ such that $\underline{g} \circ \pi(\underline{a}) = \underline{f}$. If O is an open subset of $\bar{R}$ then $\underline{f}^{-1}O$ is an open $r(\underline{a})$ - saturated subset of βE. Since $\pi(\underline{a})\,\underline{f}^{-1}O = \underline{g}^{-1}O$ it follows that $\underline{g}$ is continuous.

§4. <u>The construction of extensions of E.</u> Let E be a completely regular space and let (Y,j) denote an extension of E. It will be considered to have been constructed if any isomorphic extension has been constructed. In this sense to construct all the extensions of E it is sufficient to construct a collection of extensions of E such that any extension (Y,j) is isomorphic to one of them. Such a collection of extensions is said to be <u>representative.</u> One of the purposes of this section is to construct a representative <u>set</u> of extensions of E which is <u>non-redundant</u> in the sense that no two distinct extensions in the set are isomorphic.

From the definition of a characteristic algebra and theorem 3 it follows that if $\underline{c}$ is in $\subseteq(E)$ then $(K(\underline{c}), \pi(\underline{c}) \circ i)$ is a compact extension of E. Consequently if X is a subset of $K(\underline{c})$ that contains $(\pi(\underline{c}) \circ i)E$ then as a subspace of $K(\underline{c})$ $(X, \pi(\underline{c}) \circ i)$ is an extension of E. Let $\Xi(E)$ be the collection of extensions of this type. It is a set and it is a representative collection as stated in

<u>Theorem 11. If (Y,j) is an extension of E there exists an extension $(X, \pi(\underline{c}) \circ i)$ in $\Xi(E)$ isomorphic to (Y,j). The algebra $\underline{c}$ may be taken to be $C_Y^* \circ j$.</u>

Proof: First, assume Y is a compact space. Then $C_Y^* \circ j = C_Y \circ j$ is a characteristic algebra $\underline{c}$ of E. By the definition of a β-extension j has an extension f to βE with $f \circ i = j$. Since jE is dense in Y theorem 2 shows that

(Y,f) is a continuous image of βE. Therefore there is a homeomorphism $\gamma: K(\underline{c}) \longrightarrow Y$ such that $\gamma \circ \pi(\underline{c}) = f$. This means that $\gamma \circ \pi(\underline{c}) \circ i = j$ and so (Y,j) is isomorphic to $(K(\underline{c}), \pi(\underline{c}) \circ i)$.

Second, assume Y is not compact. Since Y is completely regular it has a β-extension $(\beta Y, i_Y)$. The extension $(\beta Y, i_Y \circ j)$ of E is compact and isomorphic to $(K(\underline{c}), \pi(\underline{c}) \circ i)$ where $\underline{c} = C_Y^* \circ j$. Let $\gamma: K(\underline{c}) \longrightarrow \beta Y$ be the homeomorphism such that $\gamma \circ \pi(\underline{c}) \circ i = i_Y \circ j$. The space Y is mapped homeomorphically by $\gamma^{-1} \circ i_Y$ on $(\gamma^{-1} \circ i_Y)Y = X$ which is contained in $K(\underline{c})$. Clearly X contains $(\pi(\underline{c}) \circ i)E$ and so (Y,j) is isomorphic to $(X, \pi(\underline{c}) \circ i)$.

Remark. The set $\Xi(E)$ is defined by means of the specific compact extensions $K(\underline{c})$, $\underline{c}$ in $\sqsubseteq(E)$, of E. Any other specific choice of compact extensions of E defines a corresponding representative of set of extensions of E.

The extensions $(X, \pi(\underline{c}) \circ i)$ in $\Xi(E)$ may be constructed as quotients of extensions in βE in view of

Theorem 12. If Z is a subset of βE that contains iE and $\underline{c}$ is in $\sqsubseteq(E)$ then $(Z/\underline{r}(\underline{c})|Z, \pi_Z(\underline{c}) \circ i)$ is an extension of E (where $\pi_Z(\underline{c})$ is the natural mapping of Z onto $Z/\underline{r}(\underline{c})|Z$). If Z is $\underline{r}(\underline{c})$-saturated this extension is isomorphic to the extension $(\pi(\underline{c})Z, \pi(\underline{c}) \circ i)$ of E. If X is a subset of $K(\underline{c})$ that contains $(\pi(\underline{c}) \circ i)E$ the extension $(X, \pi(\underline{c}) \circ i)$ is isomorphic to the extension $(Z/\underline{r}(\underline{c})|Z, \pi_Z(\underline{c}) \circ i)$ where $Z = \pi^{-1}(\underline{c})X$.

Proof: Consider the diagram

$$\begin{tikzcd}
iE \arrow[r, "n"] \arrow[d, "\pi_{iE}(\underline{c})"] & Z \arrow[r, "m"] \arrow[d, "\pi_Z(\underline{c})"] & \beta E \arrow[d, "\pi(\underline{c})"] \\
iE/\underline{r}(\underline{c})|iE \arrow[r, "\underline{n}"] & Z/\underline{r}(\underline{c})|Z \arrow[r, "\underline{m}"] & \beta E/\underline{r}(\underline{c})
\end{tikzcd}$$

where $n(ix) = ix$ for all x in E, $m(y) = y$ for all y in Z, and $\underline{n}, \underline{m}$ are the unique continuous 1 - 1 functions which are defined by requiring the diagram to be commutative.

Since $\underline{c}$ is a characteristic algebra $\pi(\underline{c}) \circ m \circ n = \pi(\underline{c})|iE$ is an embedding and $\pi_{iE}(\underline{c})$ is a homeomorphism. Consequently $\underline{m} \circ \underline{n}$ is an embedding. This implies that $\underline{n}$ is an embedding. Let U be an open subset of $iE/\underline{r}(\underline{c})|iE$ and let O be an open subset of $\beta E/\underline{r}(c)$ such that $(\underline{m} \circ \underline{n})U = O \cap (\underline{m} \circ \underline{n})(iE/\underline{r}(c)|ie)$. If $P = \underline{m}^{-1}O$ then $\underline{n}U = P \cap n(iE/\underline{r}(\underline{c})|iE)$ and since $\underline{m}$ is continuous it follows that $\underline{n}$ is an embedding.

Therefore $\pi_Z(\underline{c}) \circ n \circ i = \pi_Z(\underline{c}) \circ i$ is an embedding and so $(Z/\underline{r}(\underline{c})|Z, \pi_Z(\underline{c}) \circ i)$ is an extension of Ej.

The second assertion is a particular case of a well known result for closed equivalence relations and saturated subsets (see Bourbaki [3] p65) which shows that $\underline{m}$ is an embedding if Z is $\underline{r}(\underline{c})$-saturated.

The third assertion follows from the first two since $\pi^{-1}(\underline{c})X$ is $\underline{r}(\underline{c})$-saturated.

Remark. The 1 - 1 correspondence between the $\underline{r}(\underline{c})$-saturated subsets Z of βE that contain iE and the subsets X of $K(\underline{c})$ that contain $(\pi(\underline{c}) \circ i)E$ $(Z \longrightarrow \pi(\underline{c})Z$ and $X \longrightarrow \pi^{-1}(\underline{c})X)$ sets up a 1 - 1 correspondence between the corresponding extensions of E. This shows that $\Xi(E)$ may be constructed without duplication by means of the relations $\underline{r}(\underline{c})$ and these $\underline{r}(c)$-saturated subsets of βE.

The relation '$(X,\pi(c) \circ i)$ is isomorphic to $(X',\pi(c') \circ i)$' is an equivalence relation on $\Xi(E)$. Let $(I_\alpha)_{\alpha \text{ in } \Omega}$ be the corresponding partition of $\Xi(E)$ into isomorphism classes I_α. It is clearly sufficient for the purposes of constructing a representative non-redundant set of extensions of E to choose precisely one extension from each I_α.

Let $(X,\pi(\underline{c}) \circ i)$ and $(Y,\pi(\underline{c}) \circ i)$ be isomorphic extensions of E and let $j:X \longrightarrow Y$ be the homeomorphism such that $j \circ \pi(\underline{c}) \circ i = \pi(\underline{c}) \circ i$. Since $(\pi(\underline{c}) \circ i)E$ is dense in X and Y is Hausdorff it follows that $jx = x$ for each x in X and so $X = Y$. Therefore for each $\underline{c}$ in $\underline{C}$ there is at most one subset X of $K(\underline{c})$ with $(X,\pi(\underline{c}) \circ i)$ in a given isomorphism class I_α.

Let $(X,\pi(\underline{c}) \circ i)$ be in I_α and let $\underline{c}_\alpha$ be the characteristic algebra $C^*_X \circ \pi(\underline{c}) \circ i$. The algebra $\underline{c}_\alpha$ is independent of the representative of I_α since if (Z,k) and

(Y,j) are isomorphic extensions of E $C_Z^* \circ k = C_Y^* \circ j$.
Theorem 11 shows that there exists a (necessarily unique) subset X_α of $K(\underline{c}_\alpha)$ for which $(X_c, \pi(\underline{c}_\alpha) \circ i)$ is in I_α. Consequently the family $((X_\alpha, \pi(\underline{c}_\alpha) \circ i))_{\alpha \text{ in } \Omega}$ of extensions $(X_\alpha, \pi(\underline{c}_\alpha) \circ i)$ is a representative non-redundant set of extensions of E.

Remark. This shows that for a given construction of the compact extensions of E a representative non-redundant set of extensions of E may be defined by identifying each extension (Y,j) with a subspace of that compact extension of E which loosely speaking is a β-extension of (Y,j).

As shown in section seven of chapter one the processes $\mathcal{J}$, $\mathcal{I}$, $\mathcal{H}$ and $\mathcal{L}$ provide methods of constructing a representative collection of Q-extensions of E. The remainder of this section is concerned with another method of obtaining a representative collection of Q-extensions of E. It consists of defining subsets of the compact extensions $K(\underline{c})$ to correspond to collections S of continuous real-valued functions on E.

Let $\underline{c}$ be in $\mathfrak{C}$ and let E also denote $(\pi(\underline{c}) \circ i)E$. Then E is a subset of $K(\underline{c})$. Let f be in C_E and denote by $\underline{F}_x$ the trace filter on E of $\mathcal{V}(x)$ - the neighbourhood filter of x in $K(\underline{c})$. If x is in E $\lim\limits_{\underline{F}_x} f$ exists in R and is fx. If x is in E and $\lim\limits_{\underline{F}_x} f$ exists in R

then f has a continuous real-valued extension f to the subspace $E \cup [x]$ with $fx = \lim_{F_x} f$ (see Bourbaki [3] p 54). Let $\mathcal{E}_{\underline{c}}([f]) = [x \text{ in } K(\underline{c}) \mid \lim_{F_x} f \text{ exists in } R]$ considered as a subspace of $K(\underline{c})$. Since it contains $E = (\pi(\underline{c}) \circ i)E$, $\mathcal{E}_{\underline{c}}([f]) = (\mathcal{E}_{\underline{c}}([f]), \pi(\underline{c}) \circ i)$ is an extension of E. The space $\mathcal{E}_{\underline{c}}([f])$ is the largest subspace of $K(\underline{c})$ to which f has a continuous finite-valued extension.

The operator $\mathcal{E}_{\underline{c}}$ may be defined for any collection S of continuous real valued functions on E by setting $\mathcal{E}_{\underline{c}}(S) = \bigcap_{f \text{ in } S} \mathcal{E}_{\underline{c}}([f])$. Some elementary properties of it are listed in

<u>Theorem 13. Let c be in $\mathfrak{C}$ and let S_1, S_2 be two subsets of C_E. Then it follows that:</u>

(1) <u>$\mathcal{E}_{\underline{c}}(S_1 \cup S_2) = \mathcal{E}_{\underline{c}}(S_1) \cap \mathcal{E}_{\underline{c}}(S_2)$;</u>

(2) <u>$\mathcal{E}_{\underline{c}}(S_1 \cap S_2)$ contains $\mathcal{E}_{\underline{c}}(S_1) \cup \mathcal{E}_{\underline{c}}(S_2)$; and</u>

(3) <u>if S_1 is contained in S_2 then $\mathcal{E}_{\underline{c}}(S_1)$ contains $\mathcal{E}_{\underline{c}}(S_2)$.</u>

Proof: (1) implies (2) and (2) implies (3). Property (1) holds because $\mathcal{E}_{\underline{c}}(S_1) \cap \mathcal{E}_{\underline{c}}(S_2) = \left(\bigcap_{f \text{ in } S_1} \mathcal{E}_{\underline{c}}([f])\right) \cap$

$$\left(\bigcap_{f \text{ in } S_2} \mathcal{E}_{\underline{c}}([f])\right) = \bigcap_{f \text{ in } S_1 \cup S_2} \mathcal{E}_{\underline{c}}([f]) = \mathcal{E}_{\underline{c}}(S_1 \cup S_2) .$$

Instead of fixing $\underline{c}$ and varying S as in theorem 13 consider a fixed S and the relations between the members of the family $((\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i))_{\underline{c} \text{ in } \underline{\mathcal{C}}}$ of extensions of E. The first and more or less obvious result is stated as

<u>Theorem 14.</u> <u>Let c_1, c_2 be two elements of $\underline{\mathcal{C}}$ with c_1 contained in c_2. The set $\pi(\underline{c}_1^{-1}\underline{c}_2)\mathcal{E}_{\underline{c}_1}(S)$ is the largest subset X of $\mathcal{E}_{\underline{c}_2}(S)$ such that:</u>

(1) <u>X contains $(\pi(\underline{c}_2) \circ i) E$;</u>

(2) <u>X is $\underline{r}(\underline{c}_1,\underline{c}_2)$ - saturated ; and</u>

(3) <u>if $x \; \underline{r}(\underline{c}_1,\underline{c}_2) y$ for x in X then $\tilde{f}x = \tilde{f}y$ for each f in S</u> (where $\tilde{f}$ is the extension of f to $\mathcal{E}_{\underline{c}_2}(S)$) .

Proof: Since $(\pi(\underline{c}_1) \circ i)E$ lies in $\mathcal{E}_{\underline{c}_1}(S)$ and $\pi(\underline{c}_1,\underline{c}_2) \circ \pi(\underline{c}_2) = \pi(\underline{c}_1)$ it follows that $\pi^{-1}(\underline{c}_1\underline{c}_2)\, \mathcal{E}_{\underline{c}_1}(S)$ contains $(\pi(\underline{c}_2) \circ i)E$. The set $\pi(\underline{c}_1^{-1},\underline{c}_2)\, \mathcal{E}_{\underline{c}_1}(S)$ is obviously $\underline{r}(\underline{c}_1,\underline{c}_2)$ - saturated and is a subset of $\mathcal{E}_{\underline{c}_2}(S)$. It also satisfies (3) .

Assume that X is contained in $\mathcal{E}_{\underline{c}_2}(S)$ and satisfies these three conditions and let $X_1 = \pi(\underline{c}_1,\underline{c}_2)X$.

It is clear that X_1 is a subset of $\mathcal{E}_{\underline{c}_1}(S)$ containing $(\pi(\underline{c}_1) \circ i)E$ and so $X = \pi(\underline{c}_1^{-1}, \underline{c}_2)X_1$ is contained in $\pi(\underline{c}_1^{-1}, \underline{c}_2)\ \mathcal{E}_{\underline{c}_1}(S)$.

An almost immediate consequence of this theorem is

<u>Theorem 15. Let $\underline{c}_1, \underline{c}_2$ be two elements of $\mathcal{C}$ with $\underline{c}_1$ contained in $\underline{c}_2$. The following statements are equivalent:</u>

(1) <u>$\pi(\underline{c}_1, \underline{c}_2)\ \mathcal{E}_{\underline{c}_2}(S) = \mathcal{E}_{\underline{c}_1}(S)$;</u>

(2) <u>$\pi(\underline{c}_1^{-1}, \underline{c}_2)\ \mathcal{E}_{\underline{c}_1}(S) = \mathcal{E}_{\underline{c}_2}(S)$; and</u>

(3) <u>$\mathcal{E}_{\underline{c}_2}(S)$ is $\underline{r}(\underline{c}_1, \underline{c}_2)$ - saturated and for x in this set $x\ \underline{r}(\underline{c}_1, \underline{c}_2)y$ implies $\tilde{f}x = \tilde{f}y$ for each f in S .</u>

<u>In particular if S is contained in $\mathcal{L}(\underline{c}_1)$ then $\pi(\underline{c}_1, \underline{c}_2)\ \mathcal{E}_{\underline{c}_2}(S) = \mathcal{E}_{\underline{c}_1}(S)$ for every $\underline{c}_2$ in $\mathcal{C}$ that contains $\underline{c}_1$.</u>

Proof: Theorem 14 shows that (2) and (3) are equivalent. Obviously (2) implies (1) . On the other hand if (1) holds then $\pi^{-1}(\underline{c}_1, \underline{c}_2)\ \mathcal{E}_{\underline{c}_1}(S)$ contains $\mathcal{E}_{\underline{c}_2}(S)$ which shows by theorem 14 that (2) is true.

Assume that S is contained in $\mathcal{L}(\underline{c}_1)$. Using the notation of theorem 10 it follows that every f defines a continuous $\overline{K}$ - valued function $\underline{f}$ in $C^{\infty}_{K(\underline{c}_2)}$ since $\mathcal{L}(\underline{c}_1)$ is contained in $\mathcal{L}(\underline{c}_2)$. Since S lies in $\mathcal{L}(\underline{c}_1)$ these functions $\underline{f}$, $\tilde{f}$ in S, are compatible with $\underline{r}(\underline{c}_1,\underline{c}_2)$.

The set $\mathcal{E}_{\underline{c}_2}(S) = [x \text{ in } K(\underline{c}_2) \mid |\underline{f}x| < \infty$ for each f in S] . This set is $\underline{r}(\underline{c}_1,\underline{c}_2)$ - saturated (since the functions $\underline{f}$ are compatible with $\underline{r}(\underline{c}_1,\underline{c}_2)$ and since $\tilde{f} = \underline{f} \mid \mathcal{E}_{\underline{c}_2}(S)$ it follows that statement (3) is satisfied. Consequently all three statements hold in this case.

To show that the operators $\mathcal{E}_{\underline{c}}$, $\underline{c}$ in $\underline{C}$, define a representative collection of Q-extensions of E it is sufficient to establish the following connection between these operators and process $\mathcal{J}$.

<u>Theorem 16. If S is a subset of $\mathcal{L}(\underline{c})$ that contains $\underline{c}$ the extensions $(T(E,S),t_S)$ and $(\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i)$ are isomorphic.</u>

Proof: Since $\mathcal{J}$ satisfies (FP_4) and (FP_5) theorem 15 of chapter one shows that $(T(E,S),t_S)$ is an extension of E when S contains a characteristic algebra of E .

Let $\tilde{S} = [\tilde{f} \mid f \text{ in } S]$ where $\tilde{f}$ is the extension of f to $\mathcal{E}_{\underline{c}}(S)$. Consider the objects (E,S) and $(\mathcal{E}_{\underline{c}}(S), \tilde{S})$ of Φ and the mapping $\pi(\underline{c}) \circ i$ in $\mathrm{Hom}((E,S)(\mathcal{E}_{\underline{c}}(S),\tilde{S}))$. Since $\tilde{S} \circ \pi(\underline{c}) \circ i = S$ and the correspondence $\tilde{f}$ $\quad \tilde{f} \circ \pi(\underline{c}) \circ i = f$ is 1 - 1 the fact that $\mathcal{J}$ satisfies (FP_7) implies that $T(\pi(\underline{c}) \circ i)$ embeds $T(E,S)$ on a closed subset of $T(\mathcal{E}_{\underline{c}}(S),\tilde{S})$.

Consider the commutative diagram

$$\begin{tikzcd}
(E,S) \arrow[r, "t_S"] \arrow[d, "\pi(\underline{c}) \circ i"'] & T(E,S) \arrow[d, "T(\pi(\underline{c}) \circ i)"] \\
(\mathcal{E}_{\underline{c}}(S), \tilde{S}) \arrow[r, "t_{\tilde{S}}"] & T(\mathcal{E}_{\underline{c}}(S), \tilde{S}) .
\end{tikzcd}$$

The function $t_{\tilde{S}}$ is continuous since the topology of $\mathcal{E}_{\underline{c}}(S)$ is the weak topology $\underline{\sigma}(\mathcal{E}_{\underline{c}}(S), \tilde{S})$ (see theorem 1 of chapter one). Consequently $(t_{\tilde{S}} \circ \pi(\underline{c}) \circ i)E$ is dense in $T(\mathcal{E}_{\underline{c}}(S), \tilde{S})$. This implies that $T(\pi(\underline{c}) \circ i)$ is a homeomorphism.

To prove the theorem it is sufficient to show that $t_{\tilde{S}}$ is a homeomorphism. Theorem 15 of chapter one shows that $t_{\tilde{S}}$ is a homeomorphic embedding and so it is sufficient to show that $t_{\tilde{S}}\, \mathcal{E}_{\underline{c}}(S) = T(\mathcal{E}_{\underline{c}}(S), \tilde{S})$.

Since S is a subset of $\mathcal{X}(\underline{c})$ containing $\underline{c}$ it follows that $S^{*} = \underline{c}$. Let $\underline{c}_T = (S_T)^{*} = (S^{*})_T$. Then $\underline{c}_T$ is a characteristic algebra of $T(E,S)$ and $(H(T(E,S),\underline{c}_T), h_{\underline{c}_T})$ is a compact extension of $T(E,S)$. It is clear by the Stone-Weierstrass theorem that $C_{H(T(E,S),\underline{c}_T)} \circ h_{\underline{c}_T} = \underline{c}_T$. This shows that the compact extension $(H(T(E,S),\underline{c}_T),\ h_{\underline{c}_T} \circ t_S)$ of E is isomorphic to the extension $(K(\underline{c}), \pi(\underline{c}) \circ i)$. Let $\gamma : K(\underline{c}) \longrightarrow H(T(E,S),\underline{c}_T)$ be the homeomorphism that makes the following diagram commutative

$$\begin{tikzcd}
E \arrow[r, "t_S"] \arrow[d, "\pi(\underline{c})\circ i"'] & T(E,S) \arrow[d, "h_{\underline{c}_T}"] \\
K(\underline{c}) \arrow[r, "\gamma"] & H(T(E,S),\underline{c}_T)
\end{tikzcd}$$

Define the homeomorphic embedding $\theta : T(\mathcal{E}_{\underline{c}}(S), \tilde{S})$ by setting $\theta = \gamma^{-1} \circ h_{\underline{c}_T} \circ T(\pi^{-1}(\underline{c}) \circ i)$. Then $\theta \circ t_S \circ \pi(\underline{c}) \circ i =$

$\gamma^{-1} \circ h_{\underline{c}_T} \circ T(\pi(\underline{c}) \circ i) \circ t_{\tilde{S}} \circ \pi(\underline{c}) \circ i = \gamma^{-1} \circ h_{\underline{c}_T} \circ t_S =$ $\pi(\underline{c}) \circ i$. Therefore $\theta \circ t_{\tilde{S}}$ is the identity mapping of $\mathcal{E}_{\underline{c}}(S)$ into itself.

It is clear that $\theta\, T(\mathcal{E}_{\underline{c}}(S), \tilde{S})$ is contained in $\mathcal{E}_{\underline{c}}(S)$ and therefore since θ is 1 - 1 that $t_S\, \mathcal{E}_{\underline{c}}(S) = T(\mathcal{E}_{\underline{c}}(S), \tilde{S})$. This completes the proof of the theorem.

This theorem has a number of corollaries that are all consequences of properties of process $\mathfrak{I}$.

<u>Corollary 1. If S is a subset of $\mathcal{L}(\underline{c})$ then $\mathcal{E}_{\underline{c}}(S)$ is a Q-space.</u>

Proof: $\mathcal{E}_{\underline{c}}(S) = \mathcal{E}_{\underline{c}}(S \cup \underline{c})$ which is homeomorphic to the space $T(E, S \cup \underline{c})$. Since $\mathfrak{I}$ is a Q-process the corollary follows.

<u>Corollary 2. If X is a subset of $K(\underline{c})$ containing $(\pi(\underline{c}) \circ i)E$ and if $K(\underline{c}) \cap \complement X$ is a union of G_δ - sets then as a subspace X is a Q-space?</u>

Proof: Let x_0 be in $\complement X$. Then there is a sequence $(U_n)_{n \geq 0}$ of open sets in $K(\underline{c})$ such that $U_n \supseteq U_{n+1}$ and such that x_0 is in $\bigcap_{n \geq 0} U_n$ which lies in $\complement X$. For each n let g_n be a function $C_{K(\underline{c})}$ such that $g_n x_0 = 0$, $g_n | \complement U_n = 1$, and $0 \leq g_n \leq 1$. The function $g_{x_0} = \sum_{n \geq 0} 2^{-n} g_n$ is in $C_{K(\underline{c})}$ and $Z(g_{x_0}) = [x \text{ in } K(\underline{c}) | gx = 0]$ contains x_0 and is disjoint

from X. This is because $Z(g)$ lies in $\bigcap_{n \geq 0} U_n$.
Let $g_{x_0} = g_{x_0} \circ \pi(\underline{c}) \circ i$. Then g_{x_0} is in $\underline{c}$ and since $g_{x_0} x > 0$ for each x in E it follows that $1/g_{x_0}$ is in $\mathcal{L}(\underline{c})$. Clearly $\mathcal{E}_{\underline{c}}([1/g_{x_0}])$ contains X and omits x_0. Therefore $X = \mathcal{E}_{\underline{c}}(S)$ where $S = [1/g_{x_0} | x_0 \text{ in } K(\underline{c}) \cap \complement X]$ which is a subset of $\mathcal{L}(\underline{c})$. This corollary follows from corollary 1.

<u>Corollary 3. Let S be a translation sublattice of continuous real-valued functions on E that contains the constants and is closed under multiplication by (-1). If $S^* = S \quad C_E^*$ is a characteristic algebra $\underline{c}$ of E the extensions $(L(E,S), 1_S)$ and $(\mathcal{E}_{\underline{c}})(S), \pi(\underline{c}) \circ i)$ are isomorphic.</u>

Proof: It is clear that S is a subset of $\mathcal{L}(\underline{c})$ containing $\underline{c}$. The result follows from the theorem and theorem 13 of chapter one which states that $J \mid \Lambda'$ and $\mathcal{L}$ are isomorphic processes.

<u>Corollary 4. Let S be a unitary subalgebra of continuous real-valued functions on E which is closed under bounded inversion and which is also a sublattice of C_E. If $S^* = S \cap C_E^*$ is a characteristic algebra $\underline{c}$ of E the extensions $(H(E,S),h_S)$ and $(\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i)$ are isomorphic.</u>

Proof: Analogous to the proof of corollary 3 (with 14 replaced by 12).

When $S = C_E$ corollary 4 shows that the extensions $(\mathcal{E}_{C_E^*}(C_E), i)$ and $(H(E,C_E),h_{C_E})$ are isomorphic. Consequently

$(\mathcal{E}_{C^*_E}(C_E), i)$ is an υ-extension of E. Let $\upsilon E = \mathcal{E}_{C^*_E}(C_E)$.

As shown in section seven of chapter one a representative collection of Q-extensions of E may be obtained by applying any one of the processes $\mathfrak{J}$, $\mathfrak{F}$, $\mathfrak{H}$ or $\mathcal{L}$ to the objects (E,S) where S is an extension algebra. If S is an extension algebra then S^* is a characteristic and S is a subset of $\mathcal{L}(S^*)$ containing S^*. Consequently the family $((\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i))_{\underline{c} \text{ in } \underline{C}}$ of Q-extensions $(\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i)$, S a subset of $\mathcal{L}(\underline{c})$ is a representative collection of Q-extensions of E.

§5. <u>Two quasi-orders on the collection of extensions of E.</u>
Let $(X, \pi(\underline{c}) \circ i)$ and $(Y, \pi(\underline{c}) \circ i)$ be two extensions of E in $\Xi(E)$ that both 'lie in' $K(\underline{c})$. A natural way to compare these extensions is to compare the subsets X and Y of $K(\underline{c})$ and to set $(X, \pi(\underline{c}) \circ i) \leq (Y, \pi(\underline{c}) \circ i)$ iff X is contained in Y. The relation thus defined has the following obvious extension to the collection of all extensions of E.

<u>Definition 2.</u> <u>Let $X_1, j_1)$ and (X_2, j_2) be two extensions of E. Set $(X_1, j_1) \leq (X_2, j_2)$ if there exists a homeomorphic embedding $j_{21}: X_1 \longrightarrow X_2$ such that $j_{21} \circ j_1 = j_2$.</u>

This definition extends $\leq$ since in the previous situation the natural injection of X into Y is the desired embedding.

The relation $\leq$ is obviously reflexive. If $(X_1, j_1) \leq (X_2, j_2) \leq (X_3, j_3)$ and j_{21}, j_{32} are the corresponding embeddings, then $j_{32} \circ j_{21} : X_1 \longrightarrow X_3$ is a homeomorphic embedding such that $j_{32} \circ j_{21} \circ j_1 = j_{32} \circ j_2 = j_3$. Consequently the relation $\leq$ is transitive.

If (X_1, j_1) and (X_2, j_2) are isomorphic extensions then it is clear that $(X_1, j_1) \leq (X_2, j_2) \leq (X_1, j_1)$. Conversely assume that this is the case and let j_{21} and j_{12} be suitable embeddings. Then $(j_{12} \circ j_{21}) \circ j_1 = j_{12} \circ j_2 = j_1$ and $(j_{21} \circ j_{12}) \circ j_2 = j_{21} \circ j_1 = j_2$. Since X_1 and X_2 are Hausdorff and $j_1 E$ is dense in X_1 it followstthat j_{12} and j_{21} are homeomorphisms. Consequently $(X_1, j_1) \leq (X_2, j_2) \leq (X_1, j_1)$ iff the extensions are isomorphic. This shows that $\leq$ induces a partial order on the set of isomorphism classes of the collection of extensions of E.

Assume that $(X_1, j_1) \leq (X_2, j_2)$ and let $j_{21} : X_1 \longrightarrow X_2$ be an embedding such that $j_{21} \circ j_1 = j_2$. The following lemma shows that j_{21} is unique not only as an embedding but also as a continuous function.

Lemma 6. Let (X_1, j_1) and (X_2, j_2) be two extensions of E. Then there exists at most one continuous function $\gamma_{21} : X_1 \longrightarrow X_2$ such that $\gamma_{21} \circ j_1 = j_2$.

Proof: Let γ_{21} and γ'_{21} be two continuous functions with this property. Then $\gamma_{21}|j_1E = \gamma'_{21}|j_1E$. Since j_1E is dense in X_1 and X_2 is Hausdorff it follows that $\gamma_{21} = \gamma'_{21}$.

Remark. In view of this lemma it makes sense to speak of the existence of the function γ_{21} for a pair of extensions (X_1,j_1) and (X_2,j_2) of E. Consider three extensions (X_i,j_i) $i = 1,2,3$ of E. If γ_{21} and γ_{32} exist so does γ_{31} and $\gamma_{31} = \gamma_{32} \circ \gamma_{21}$. This is because $\gamma_{32} \circ \gamma_{21} \circ j_1 = \gamma_{32} \circ j_2 = j_3$.

Lemma 6 suggests the definition of a second quasi-order for extensions of E.

Definition 3. Let (X_1,j_1) and (X_2,j_2) be two extensions of E. Set $(X_1,j_1) \precsim (X_2,j_2)$ if the continuous function $\gamma_{21}: X_1 \longrightarrow X_2$ exists (such that $\gamma_{21} \circ j_1 = j_2$).

The arguments used when dealing with the relation may be repeated to show: $\precsim$ is a quasi-order which induces a partial order on the set of isomorphism classes of the collection of extensions of E.

It is clear that $(X_1,j_1) \precsim (X_2,j_2)$ if $(X_1,j_1) \leq (X_2,j_2)$ i.e. $\precsim$ is a 'coarser' relation than $\leq$.

Let (X,j) be an extension of E. An examination of the proof of theorem 11 shows that $(X,j) \leq (K(C_X^M \circ j),\pi(C_X^M \circ j) \circ i)$. Hence this theorem may be restated as

<u>Theorem 17. If (X,j) is an extension of E there exists an element $\underline{c}$ of $\mathfrak{C}$ such that $(X,j) \leq (K(\underline{c}), \pi(\underline{c}) \circ i)$.</u>

This raises the question as to what are necessary and sufficient conditions on $\underline{c}$ in order that (Y,j) be $\leq (K(\underline{c}), \pi(\underline{c}) \circ i)$. An answer to this question is contained in

<u>Theorem 18. Let (X,j) be an extension of E and let (Y,k) be a Q-extension of E . Then $(X,j) \leq (Y,k)$ iff $C_Y \circ k$ is contained in $C_X \circ j$ and $C_Y^{*} \circ k$, when extended to X, is a characteristic algebra of X .</u>

Proof: Assume $(X,j) \leq (Y,k)$ and let $\iota : X \longrightarrow Y$ be the embedding such that $\iota \circ j = k$. Obviously $C_Y \circ \iota \subseteq C_X$ and $C_Y^{*} \circ \iota$ is a characteristic algebra of X . This shows that $C_Y \circ k = C_Y \circ \iota \circ j$ is contained in $C_X \circ j$ and that $C_Y^{*} \circ k$, when extended to X, is a characteristic algebra of X (since it is $C_Y^{*} \circ$) .

Conversely assume that $C_Y \circ k$ is contained in $C_X \circ j$ and that $C_Y^{*} \circ k$, when extended to X, is a characteristic algebra of X . If S is the algebra $C_Y \circ k$ extended to X then S^{*} is the extension of $C_Y^{*} \circ k$. Since S is a sublattice of C_X that contains the constants, the topology $\underline{O}(X,S)$ is also the topology $\underline{O}(X,S^{*})$, which is the topology of X since S^{*} is a characteristic algebra. Furthermore the object (X,S) is in $A' \cap \Lambda'$ since $C_Y \circ k$ is closed under bounded inversion.

Consider the following commutative diagram

$$\begin{tikzcd}
Y \arrow[d, "h_{C_Y}"'] & E \arrow[l, "k"'] \arrow[r, "j"] \arrow[d, "h_{C_Y \circ k}"] & X \arrow[d, "h_S"] \\
H(Y, C_Y) & H(E, C_Y \circ k) \arrow[l, "H(k)"'] \arrow[r, "H(j)"] & H(X, S)
\end{tikzcd}$$

Since $\mathcal{H}$ satisfies (FP_δ) $H(j)$ and $H(k)$ are homeomorphisms. The fact that Y is a C-space means that h_{C_Y} is a homeomorphism. Since (X,S) is in $A' \cap \Lambda'$ and $\underline{O}(X,S)$ is the topology of X it follows from theorems 14 and 15 of chapter one that h_S is an embedding. Therefore the mapping $\iota = h_{C_U}^{-1} \circ H(k) \circ H^{-1}(j) \circ h_S : X \longrightarrow Y$ is an embedding such that $\iota \circ j = k$. Hence $(X,j) \leq (Y,k)$.

Let (X,j) be an extension of E. It is said to be <u>maximal with respect to $\leq$</u> if $(X,j) \leq (X',j')$ implies that $(X',j') \leq (X,j)$. Theorem 17 shows that such a maximal extension is compact. Since compact spaces are absolutely closed the converse holds as stated in

<u>Theorem 19.</u> <u>(X,j) is maximal with respect to $\leq$ iff X is compact.</u>

Proof: It is sufficient to show that (X,j) is maximal if X is compact. Assume that $(X,j) \leq (X',j')$ and that $\iota : X \longrightarrow X'$ is the embedding with $\iota \circ j = j'$. The subspace ιX of X' is compact and since X' is Hausdorff it is also closed. Since $\iota \circ j = j'$ it follows that ιX contains $j'E$ which is dense in X. Consequently $\iota X = X'$ and ι is a homeomorphism. This shows that $(X',j') \leq (X,j)$ and hence that (X,j) is maximal.

Theorems 18 and 19 are concerned with the quasi-order $\leqslant$. There are analogous theorems for the quasi-order $\preceq$. The first of these is

<u>Theorem 20.</u> <u>Let (X,j) be an extension of E and let (Y,k) be a Q-extension of E. Then (X,j) (Y,k) iff $C_Y \circ k$ is contained in $C_X \circ j$.</u>

Proof: Assume that $\gamma : X \longrightarrow Y$ is the continuous function such that $\gamma \circ j = k$. Since $C_Y \circ \gamma$ is contained in C_X it follows that $C_Y \circ k = C_Y \circ \gamma \circ j$ is contained in $C_X \circ j$.

Conversely assume that $C_Y \circ k$ is contained in $C_X \circ j$. Then the following diagram is commutative

$$\begin{tikzcd}
Y \arrow[d, "h_{C_Y}"'] & E \arrow[l, "k"'] \arrow[r, "j"] \arrow[d, "h_{C_Y \circ k}"] & X \arrow[d, "h_{C_X}"] \\
H(Y, C_Y) & H(E, C_Y \circ k) \arrow[l, "H(k)"'] \arrow[r, "H(j)"] & H(X, C_X)
\end{tikzcd}$$

The functions $H(j)$ and $H(k)$ are homeomorphisms since satisfies (FP_8), and h_{C_Y} is a homeomorphism since Y is a Q-space. Let $\gamma = h_{C_Y}^{-1} \circ H(k) \circ H^{-1}(j) \circ h_{C_X}$. Then $\gamma : X \longrightarrow Y$ is continuous and $\gamma \circ j = k$. Consequently $(X,j) \preceq (Y,k)$.

An extension (X,j) is said to be <u>maximal with respect to $\preceq$</u> if $(X,j) \preceq (X',j')$ implies $(X',j') \preceq (X,j)$. These maximal extensions are characterized by

<u>Theorem 21. (X,j) is maximal with respect to $\preceq$ iff X is compact and $C_X \circ j$ is a minimal characteristic algebra of E.</u>

Proof: If (X,j) is maximal with respect to $\preceq$ it is maximal with respect to $\leq$. Theorem 19 shows that X is compact. Let $\underline{c}$ in $\mathfrak{C}$ be such that $\underline{c}$ is contained in $C_X \circ j$. Theorem 20 shows that $(X,j) \preceq (K(\underline{c}),\pi(\underline{c}) \circ i)$. Since (X,j) is maximal the continuous function $\gamma:X \longrightarrow K(c)$ such that $\gamma \circ j = \pi(\underline{c}) \circ i$ is a homeomorphism. Consequently $C_{K(\underline{c})} \circ \gamma = C_X$ and so $\underline{c} = C_{K(\underline{c})} \circ \pi(\underline{c}) \circ i = C_{K(\underline{c})} \circ \gamma \circ j = C_X \circ j$. This shows that $C_X \circ j$ is a minimal characteristic algebra of E.

Conversely assume that (X,j) satisfies these conditions. Let (X',j') be an extension of E and let $\gamma:X \longrightarrow X'$ be a continuous function such that $\gamma \circ j = j'$. Then, as in the proof of theorem 19, $\gamma X = X'$ and X' is compact. Since $C_{X'} \circ j' = C_{X'} \circ \gamma \circ j'$ it follows that $C_{X'} \circ j' = C_X \circ j$. This shows that $C_X = C_{X'} \circ \gamma$. The fact that C_X separates the points of X implies that γ is 1 - 1. Since γ is 1 - 1 onto and X is compact γ is a homeomorphism. Therefore $(X',j') \preceq (X,j)$ and so (X,j) is maximal with respect to $\preceq$.

<u>Remark.</u> Theorem 20 shows that $(\beta E,i)$ is $\preceq$ all compact extensions of E. Usually $(\beta E,i)$ is considered to be the largest compact extension of E. This is because the usual quasi-ordering of the compact extensions of E is the opposite order to $\preceq$. If E is locally compact theorem 21 shows that there is (up to isomorphism) a unique extension of E maximal with respect

to $\preceq$. This extension is clearly the Alexandroff one-point compactification of E and is obtained by adding the smallest number of points to E'. Intuitively this makes $\preceq$ less attractive than its opposite order. The reason for its use in this section is that it is an obvious 'extension' of the 'natural' quasi-order $\leq$.

§6. The quasi-orders restricted to $\Xi(E)$. Theorems 18 and 20 give necessary and sufficient conditions on an extension (X,j) in order that it be $\leq$ and $\preceq$ a specific Q-extension (Y,k) . What can be said if instead of a specific Q-extension any particular extension (Z,l) is considered? Since $\Xi(E)$ is a representative collection of extensions of E it is sufficient to consider these two problems when the extensions belong to $\Xi(E)$. In what follows, the answers are given in terms of the relations $\underline{r}(\underline{c}',\underline{c})$ and the functions $\pi(\underline{c}',\underline{c})$ where $\underline{c}'$ contained in $\underline{c}$ are two elements of $\mathfrak{C} = \mathfrak{C}(E)$.

Let X_1 and X_2 denote the extensions $(X_1,\pi(\underline{c}_1) \circ i)$ and $(X_2,\pi(\underline{c}_2) \circ i)$ of E corresponding to the algebras $\underline{c}_1$ and $\underline{c}_2$. First consider the case where $\underline{c}_1$ is contained in $\underline{c}_2$. The following lemmas solve the problems.

Lemma 7. If $\underline{c}_1$ is contained in $\underline{c}_2$ then $X_2 \preceq X_1$ iff $\pi(\underline{c}_1,\underline{c}_2)X_2$ is a subset of X_1 or equivalently iff $\pi^{-1}(\underline{c}_1,\underline{c}_2)X_1$ contains X_2 .

Lemma 8. If $\underline{c}_1$ is contained in $\underline{c}_2$ then $X_2 \leq X_1$ iff (1) $\pi(\underline{c}_1,\underline{c}_2)X_2$ is a subset of X_1 and (2) $\underline{r}(\underline{c}_1,\underline{c}_2)[x] = [x]$ for each x in X_2, i.e. $\underline{c}_1$ when extended to X_2 is a characteristic algebra of X_2.

Proofs: Consider lemma 7. If $\pi(\underline{c}_1,\underline{c}_2)X_2$ is a subset of X_1 then $X_2 \trianglelefteq X_1$ because $\pi(\underline{c}_1,\underline{c}_2) \circ \pi(\underline{c}_2) = \pi(\underline{c}_1)$. Conversely assume $X_2 \trianglelefteq X_1$. Then $X_2 \trianglelefteq K(\underline{c}_1)$. Clearly $\pi(\underline{c}_1,\underline{c}_2)|X_2$: $X_2 \longrightarrow K(\underline{c}_1)$ is the continuous function γ on X_2 with $\gamma \circ \pi(\underline{c}_2) \circ i = \pi(\underline{c}_1) \circ i$. Since $\gamma: X_2 \longrightarrow K(\underline{c}_1)$ it follows from the fact that $X_2 \trianglelefteq X_1$ that γX_2 is a subset of X_1: i.e. $\pi(\underline{c}_1,\underline{c}_2)X_2$ is contained in X_1.

Lemma 8 is an immediate consequence of lemma 7 and theorem 4.

The second simple situation occurs when $\underline{c}_2$ is contained in $\underline{c}_1$. The solutions are stated in the next two lemmas.

Lemma 9. If $\underline{c}_2$ is contained in $\underline{c}_1$ then $X_2 \trianglelefteq X_1$ iff $X_2 \leq X_1$.

Lemma 10. If $\underline{c}_2$ is contained in $\underline{c}_1$ then $X_2 \leq X_1$ iff (1) $X_1' = \pi^{-1}(\underline{c}_2,\underline{c}_1)X_2$ is a subset of X_1 and (2) $\underline{r}(\underline{c}_2,\underline{c}_1)[x] = [x]$ for each x in X_1'.

Proofs: Assume that $\gamma: X_2 \longrightarrow X_1$ is a continuous function such that $\gamma \circ \pi(\underline{c}_2) \circ i = \pi(\underline{c}_1) \circ i$. Then $\pi(\underline{c}_2,\underline{c}_1) \circ \gamma \circ \pi(\underline{c}_2) \circ i = \pi(\underline{c}_2) \circ i$ and so $\pi(\underline{c}_2,\underline{c}_1) \circ \gamma: X_2 \longrightarrow K(\underline{c}_2)$ is the identity mapping of X_2 on itself. Also $\gamma \circ \pi(\underline{c}_2,\underline{c}_1)|\gamma X_2 \circ \pi(\underline{c}_1) \circ i = \gamma \circ \pi(\underline{c}_2) \circ i = \pi(\underline{c}_1) \circ i$ and so $\gamma \circ \pi(\underline{c}_2,\underline{c}_1)|\gamma X_2: \gamma X_2 \rightarrow \gamma X_2$ is the identity mapping of γX_2 on itself. Therefore γ is an embedding and so $X_2 \leq X_1$ if $X_2 \trianglelefteq X_1$. This proves lemma 9.

Assume that $X_2 \leq X_1$ and that $j_{12}: X_2 \longrightarrow X_1$ is the embedding with $j_{12} \circ \pi(\underline{c}_2) \circ i = \pi(\underline{c}_1) \circ i$. Consider the diagram (commutative)

$$\begin{tikzcd}
& X_2 \arrow[r] \arrow[dd, "j_{12}"] & K(\underline{c}_2) \arrow[dd, "\pi(\underline{c}_2, \underline{c}_1)"] \\
E \arrow[ur, "\pi(\underline{c}_2)\circ i"] \arrow[dr, "\pi(\underline{c}_1)\circ i"'] & & \\
& X_1 \arrow[r] & K(\underline{c}_1)
\end{tikzcd}$$

where the unlabelled maps are the natural injections. Since $j_{12}X_2$ is homeomorphic to X_2 it follows that $\underline{c}_2$ when extended to $j_{12}X_2$ is a characteristic algebra of this space. Theorem 4 shows that for each x in $j_{12}X_2$ $\underline{r}(\underline{c}_2, \underline{c}_1)[x] = [x]$ and consequently that $j_{12}X_2$ is $\underline{r}(\underline{c}_2, \underline{c}_1)$ - saturated. Since the diagram is commutative $(\pi(\underline{c}_2, \underline{c}_1) \circ j_{12})x = x$ for each x in X_2 and so $j_{12}X_2 = \pi^{-1}(\underline{c}_2, \underline{c}_1)X_2 = X'$. This shows that if $X_2 \leq X_1$ the conditions of the lemma are satisfied.

Conversely if X_2 and X_1 are such that (1) and (2) hold then by theorem 4 $\pi(\underline{c}_2, \underline{c}_1) | X_1' : X_1' \longrightarrow X_2$ is a homeomorphism and so X_1' and X_2 are isomorphic extensions. Since $X_1' \leq X_1$ it follows that $X_2 \leq X_1$.

These solutions of the two problems for the special cases $\underline{c}_1$ contained in $\underline{c}_2$ and $\underline{c}_2$ contained in $\underline{c}_1$ can be used to obtain solutions for the general case. In the case of the relation the solution is stated as

<u>Theorem 22. Let $X_1 = (X_1, \pi(\underline{c}_1) \circ i)$ and $X_2 = (X_2, \pi(\underline{c}_2) \circ i)$ be two extensions in $\Xi(E)$ associated with the algebras $\underline{c}_1$ and $\underline{c}_2$. The following conditions are equivalent:</u>

(1) $X_2 \leq X_1$; and

(2) if $c = c_1 \cup c_2$ and $X = \pi^{-1}(c_2,c)X_2$, then

(i) X is isomorphic to X_2 ,

(ii) $\pi(c_1,c)X$ is a subset of X_1, and

(iii) c_1 when extended to X is a characteristic algebra of X or equivalently for each x in X $r(c_1,c)[x] = [x]$.

Proof: If (2) (ii) and (iii) are satisfied then by lemma 8 $X \leq X_1$. Therefore condition (2) implies (1) .

Assume that $X_2 \leq X_1$. Let $c = c_1 \cup c_2$ and let $X = \pi^{-1}(c_2,c)X$. The extensions X and X_2 are isomorphic by lemmas 10 and 7 if for each x in X $r(c_2,c)[x] = [x]$. This is the case because c is a subalgebra of $C^*_{X_2}|E$. Therefore $X \leq X_1$ and so by lemma 8, since c_1 is contained in c , conditions (2)(ii) and (iii) are satisfied. Therefore (1) implies (2) .

An answer to the second problem is given as

Theorem 23. Let $X_1 = (X_1, \pi(c_1) \circ i)$ and $X_2 = (X_2, \pi(c_2) \circ i)$ be two extensions in $\Xi(E)$ associated with the algebras c_1 and c_2 . The following conditions are equivalent:

(1) $X_2 \trianglelefteq X_1$; and

(2) if $c = c_1 \cup c_2$ and $X = \pi^{-1}(c_2,c)X_2$, then

(i) X is a subset of $\pi^{-1}(c_1,c)X_1$, and

(ii) for x in X if y is in $r(c_2,c)[x]$ then it is in $r(c_1,c)[x]$.

Proof: Lemma 7 shows that $X \trianglelefteq X_2$ and so if $X_2 \trianglelefteq X_1$ it not in $\Xi(E)$. The same is true of lemmas 7,8 and 10. On the other hand lemma 9 has the following consequence: <u>let (Y,j) and (Z,ι) be two extensions of E such that $C_Y^* \circ j$ is a subalgebra of $C_Z^* \circ \iota$; then $(Y,j) \preceq (Z,\iota)$ iff $(Y,j) \leq (Z,\iota)$.</u> This assertion follows from theorem 11 and lemma 9. It is clear that the same result holds if 'a characteristic algebra of Z is a characteristic algebra of Y'. More precisely, if there are characteristic algebras $\underline{c}^Z$ and $\underline{c}^Y$ of Z and Y respectively such that $\underline{c}^Z|E$ contains $\underline{c}^Y|E$.

§7. <u>Two Q-extensions of E associated with a characteristic algebra $\underline{c}$</u>. In chapter one section seven an υ-extension of E is shown to be a Q-extension of $E \preceq$ every other Q-extension of E. Various examples of υ-extensions are given in that chapter and a further one, namely $(\upsilon E,i)$, is given in section four following corollary 4 of theorem 16. The extension $(\upsilon E,i)$ is defined to be the extension $(\mathcal{E}_{C_E^*}(C_E),i)$.

Two other descriptions of the subset υE of βE can be given. First, <u>it is the intersection of all the subsets X of βE that contain iE and as subspaces of βE are Q-spaces.</u> Let X be such a subset of βE. Since $C_X \circ i$ is contained in C_E it follows from theorem 13 that $\upsilon E = \mathcal{E}_{C_E^*}(C_E)$ is a subset of $\mathcal{E}_{C_E^*}(C_X \circ i)$. It is clear that X is contained in $\mathcal{E}_{C_E^*}(C_X \circ i)$. As an extension of E

$\mathcal{E}_{C_E^*}(C_X \circ i)$ is isomorphic to $(H(E, C_X \circ i), h_{C_X} \circ i)$ by corollary 4 of theorem 16. Since $\mathfrak{F}$ satisfies (FP_δ) this extension is isomorphic to the extension $(H(X, C_X), h_{C_X} \circ i)$. Consider the following commutative diagram

$$\begin{tikzcd}
 & E \arrow[ld, "i"'] \arrow[r, "i"] \arrow[d, "h_{C_X \circ i}"] & X \arrow[d, "h_{C_X}"] \\
\mathcal{E}_{C_E^*}(C_X \circ i) \arrow[r, "j"'] & H(E, C_X \circ i) \arrow[r, "H(i)"] & H(X, C_X).
\end{tikzcd}$$

Since X is a Q-space h_{C_X} is a homeomorphism and therefore $\theta = h_{C_X}^{-1} \circ H(i) \circ j : \mathcal{E}_{C_E^*}(C_X \circ i) \longrightarrow X$ is a homeomorphism. Since $\theta \circ i = i$ it follows that θ is the identity mapping and so $\mathcal{E}_{C_E^*}(C_X \circ i) = X$. Therefore υE is a subset of X and since υE is a Q-space the result follows.

Second, *$\mathbb{C}\upsilon E$ is the union of all the G_δ-subsets Z of βE disjoint from βE .* Corollary 2 of theorem 16 shows that the union of all the G_δ-subsets of βE is a Q-subspace and so by the previous result it contains υE . It coincides with υE as a special case of

Lemma 11. *Let S be contained in $\mathcal{L}(\underline{c})$. Then $\mathbb{C}\mathcal{E}_{\underline{c}}(S)$ is a union of G_δ-sets disjoint from $(\pi(\underline{c}) \circ i)E$.*

Proof: If f is in S let $\underline{f}$ be the function in $C^\infty_{X(\underline{c})}$ such that $\underline{f} \circ \pi(\underline{c}) \circ i$ (see corollary 2 of theorem 10). Let x_0 be in $\mathbb{C}\mathcal{E}_{\underline{c}}(S)$. Then there is a function f in S with $\underline{f}x_0$

$= +\infty$ say. The function $\underline{f} \cup 1$ is in $C^{\infty}_{K(\underline{c})}$ since $f \cup 1$ is in $\mathcal{L}(c)$. It is infinite at those points y where $\underline{f}y = +\infty$. The function $g = 1/f \cup 1$ is in $\mathcal{L}(\underline{c})$ and since it is bounded is in $\underline{c}$. Furthermore if x is in E $gx \neq 0$. Let g also denote its extension to $K(\underline{c})$. Then $gx_0 = 0$ and so x_0 is in the G_δ-set $Z(g) = [y \text{ in } K(\underline{c}) \mid gy = 0]$. Since $gy = 0$ iff $\underline{f} \cup 1$ is infinite, i.e. iff $fy = +\infty$ it follows that $Z(g) \cap \mathcal{E}_{\underline{c}}(S) = \phi$. Similarly this argument may be applied to the case when $\underline{f}x_0 = -\infty$ by considering $\underline{f} \cap (-1)$.

Corollary. If S is a subset of $\mathcal{L}(c)$ that contains c and with any function f the functions $f \cap 1$ and $f \cup (-1)$, then $\complement\, \mathcal{E}_{\underline{c}}(S) = \bigcup_{f \text{ in } \underline{c}, 1/f \text{ in } S} Z(f)$.

Proof: It is an immediate consequence of the proof of the lemma.

These two descriptions of υE suggest the following definition.

Definition 4. Let $\underline{c}$ be a characteristic algebra of E. Define $\upsilon(\underline{c})E$ to be the intersection of all the Q-subspaces X of $K(\underline{c})$ that contain $(\pi(\underline{c}) \circ i)E$. Define $\upsilon_\delta(\underline{c})E$ to be the complement of the union of G_δ-subsets of $K(\underline{c})$ disjoint from $(\pi(\underline{c}) \circ i)E$.

Using this notation the above descriptions of υE show that $\upsilon E = \upsilon(C^*_E)E = \upsilon_\delta(C^*_E)E$.

Definition 4 defines for each characteristic algebra of E the two extensions $(\upsilon(\underline{c})E, \pi(\underline{c}) \circ i)$ and $(\upsilon_\delta(\underline{c})E, \pi(\underline{c}) \circ i)$. The next two theorems state that these are both Q-extensions.

Consider first the space $\upsilon_\delta(\underline{c})E$. Not only is it a Q-space but it can be constructed from the lattice $\mathcal{L}(\underline{c})$ as stated in

Theorem 24. If $\underline{c}$ is a characteristic algebra then $\upsilon_\delta(\underline{c})E = \mathcal{E}_{\underline{c}}(\mathcal{L}(\underline{c}))$. Consequently $\upsilon_\delta(\underline{c})E$ is a Q-space and the extensions $(\upsilon_\delta(\underline{c})E, \pi(\underline{c}) \circ i)$ and $(L(E, \mathcal{L}(\underline{c})), \iota_{\mathcal{L}(\underline{c})})$ are isomorphic.

Proof: Lemma 11 shows that $\mathcal{E}_{\underline{c}}(\mathcal{L}(\underline{c}))$ contains $\upsilon_\delta(\underline{c})E$. The proof of corollary 2 of theorem 16 shows that $\upsilon_\delta(\underline{c})E = \mathcal{E}_{\underline{c}}(S)$ where S is a subset of $\mathcal{L}(\underline{c})$. Since $\mathcal{E}_{\underline{c}}(\mathcal{L}(\underline{c}))$ is contained in $\mathcal{E}_{\underline{c}}(S)$ by theorem 13, it follows that $\upsilon_\delta(\underline{c})E = \mathcal{E}_{\underline{c}}(\mathcal{L}(\underline{c}))$.

Corollary 1 (or corollary 2) of theorem 16 shows that $\upsilon_\delta(\underline{c})E$ is a Q-subspace. The third corollary of the same theorem shows that the extensions $(\upsilon_\delta(\underline{c})E, \pi(\underline{c}) \circ i) = (\mathcal{E}_{\underline{c}}(\mathcal{L}(\underline{c})), \pi(\underline{c}) \circ i)$ and $(L(E, \mathcal{L}(\underline{c})), \iota_{\mathcal{L}(\underline{c})})$ are isomorphic.

The space $\upsilon(\underline{c})E$ is also a Q-space as stated in

Theorem 25. If $\underline{c}$ is a characteristic algebra then $\upsilon(\underline{c})E$ as a subspace of $K(\underline{c})$ is a Q-space.

Proof: It is sufficient to show that when $(X_\alpha)_{\alpha \text{ in } \Omega}$ is a family of subsets X_α of $K(\underline{c})$ each containing $(\pi(\underline{c}) \circ i)E$, then $X = \bigcap_{\alpha \text{ in } \Omega} X_\alpha$ is a Q-space if each X_α is a Q-space.

This may be proved by means of

<u>Lemma 12. Let X be a subset of $K(\underline{c})$ containing $(\pi(\underline{c}) \circ i)E$. As a subspace of $K(\underline{c})$ X is a Q-space iff there exists a collection S of continuous real-valued functions such that:</u>

(1) $\pi^{-1}(\underline{c})X = Z = \mathcal{E}_{C_E^*}(S)$; and

(2) on Z the extensions of the functions in S are compatible with $\underline{r}(\underline{c})|Z$.

<u>When X is a Q-space the set S may be taken to be $C_X \circ \pi(\underline{c}) \circ i$.</u>

Given the lemma, the proof of this theorem is as follows. Let $Z_\alpha = \pi^{-1}(\underline{c})X_\alpha$. Then $Z_\alpha = \mathcal{E}_{C_E^*}(S_\alpha)$ where $S_\alpha = C_{X_\alpha} \circ \pi(\underline{c}) \circ i$. Let $Z = \pi^{-1}(c)X = \pi^{-1}(c)(\bigcap_{\alpha \text{ in } \Omega} X_\alpha) = \bigcap_{\alpha \text{ in } \Omega} (\pi^{-1}(c)X_\alpha) = \bigcap_{\alpha \text{ in } \Omega} Z_\alpha$. It is clear that $Z = \mathcal{E}_{C_E^*}(\bigcup_{\alpha \text{ in } \Omega} S_\alpha)$ and that on Z the extensions of the functions in $\bigcup_{\alpha \text{ in } \Omega} S_\alpha$ are compatible with $\underline{r}(\underline{c})$. The lemma shows that $X = \pi(\underline{c})Z$ is a Q-space.

Proof of lemma 12: Assume that X is a subset of $K(\underline{c})$ containing $(\pi(\underline{c}) \circ i)E$ and let $\underline{c}_1 = C_X^* \circ \pi(\underline{c}) \circ i = C_X^*|E$.

Consider the following diagram

$$\begin{tikzcd}
 & E \arrow[dl, "\pi(\underline{c}_1)\circ i"'] \arrow[r, "\pi(\underline{c})\circ i"] \arrow[d, "h_{C_X|E}"] & X \arrow[d, "h_{C_X}"] \\
\mathcal{E}_{\underline{c}_1}(C_X|E) \arrow[r, "j"'] & H(E, C_X|E) \arrow[r, "H(\pi(\underline{c})\circ i)"] & H(X, C_X)
\end{tikzcd}$$

where j is the homeomorphism that exists by corollary 4 of theorem 16. The function $H(\pi(\underline{c}) \circ i)$ is a homeomorphism since $\mathcal{H}$ satisfies (FP_δ) and when X is a Q-space h_{C_X} is also a homeomorphism. Then $h_{C_X}^{-1} \circ H(\pi(\underline{c}) \circ i) \circ j$: $\mathcal{E}_{\underline{c}_1}(C_X|E) \longrightarrow X$ is a homeomorphism θ such that $\theta \circ \pi(\underline{c}_1) \circ i = \pi(\underline{c}) \circ i$. Therefore $\theta = \pi(\underline{c}, \underline{c}_1) \mid \mathcal{E}_{\underline{c}_1}(C_X|E)$.

Since $\mathcal{E}_{\underline{c}}(C_X|E)$ contains X it follows from theorem 14 that $\pi(\underline{c}, \underline{c}_1)\, \mathcal{E}_{\underline{c}_1}(C_X|E) = \mathcal{E}_{\underline{c}}(C_X|E) = X$. Theorem 15 shows that $\mathcal{E}_{\underline{c}_1}(C_X|E) = \pi^{-1}(\underline{c}, \underline{c}_1)\, \mathcal{E}_{\underline{c}}(C_X|E)$ and also that $\mathcal{E}_{C_E^*}(C_X|E) = \pi^{-1}(\underline{c}_1)\, \mathcal{E}_{\underline{c}_1}(C_X|E)$.

Consequently $Z = \pi^{-1}(\underline{c})X = \pi^{-1}(\underline{c})\, \mathcal{E}_{\underline{c}}(C_X|E) = \pi^{-1}(\underline{c}_1)\, [\pi^{-1}(\underline{c}, \underline{c}_1)\, \mathcal{E}_{\underline{c}}(C_X|E)] = \pi^{-1}(\underline{c}_1)\, \mathcal{E}_{\underline{c}_1}(C_X|E) = \mathcal{E}_{C_E^*}(C_X|E)$.

Clearly $C_X|E$ when extended to Z is compatible with $\underline{r}(\underline{c})$ on Z and so conditions (1) and (2) are necessary.

Assume that Z is an $\underline{r}(\underline{c})$-saturated subset of βE containing iE and such that $Z = \mathcal{E}_{C_E^*}(S)$ where S is a subset of C_E which when extended to Z is compatible with $\underline{r}(\underline{c})$ on Z. If $X = \pi(\underline{c})Z$ then S is contained in $C_X|E$

and so by theorem 13 $Z = \mathcal{E}_{C_E^*}(C_X|E)$.

Let $\underline{c}_1 = C_X^*|E$. Theorem 15 shows that $\pi^{-1}(\underline{c}_1)\,\mathcal{E}_{\underline{c}_1}(C_X|E)$ $= Z$ and therefore $X = \pi(\underline{c})Z = (\pi(\underline{c},\underline{c}_1) \circ \pi(\underline{c}_1))\pi^{-1}(\underline{c}_1)\,\mathcal{E}_{\underline{c}_1}(C_X|E)$ $= \pi(\underline{c},\underline{c}_1)\,\mathcal{E}_{\underline{c}_1}(C_X|E)$. Corollary 1 of theorem 16 shows that $\mathcal{E}_{\underline{c}_1}(C_X|E)$ is a Q-space. To complete the proof it is sufficient to show that $\pi(\underline{c},\underline{c}_1)\,|\,\mathcal{E}_{\underline{c}_1}(C_X|E) : \mathcal{E}_{\underline{c}_1}(C_X|E) \longrightarrow X$ is a homeomorphism. Let x be in $\mathcal{E}_{\underline{c}_1}(C_X|E)$ and assume $y\ \underline{r}(\underline{c},\underline{c}_1)x$. Then since $\underline{c}_1 = C_X^*|E$ y is in $\mathcal{E}_{\underline{c}_1}(C_X|E)$. This shows that $y = x$ because $\underline{c}_1$ separates the points of $K(\underline{c}_1)$. By theorem 4 $\pi(\underline{c},\underline{c}_1)\,|\,\mathcal{E}_{\underline{c}_1}(C_X|E)$ is a homeomorphism and so X is a Q-space.

<u>Remark.</u> Theorem 25 does not provide another explicit construction of $\upsilon(\underline{c})E$. Does such a construction exist that uses one of the function processes or the operator $\mathcal{E}_{\underline{c}}$? Another problem of interest is the determination of those characteristic algebras $\underline{c}$ of E for which $\upsilon(\underline{c})E = \upsilon_\delta(\underline{c})E$.

§8. <u>A construction of the extension algebras of E .</u> Let E denote a completely regular space. The purpose of this section is to show that all the extension algebras of E may be constructed from the characteristic algebras $\underline{c}$ and the extension algebras of E that contain C_E^* . These particular algebras are characterized by

Theorem 26. Let S be a subalgebra of C_E that contains C_E^*. Then S is an extension algebra iff

(1) S is a sublattice of C_E closed under bounded inversion;

(2) S is uniformly closed; and

(3) $1/f$ is in S when f in S is such that $h(f) \neq 0$ for all h in $H(E,S)$.

Proof: The necessity of these conditions is clear (see theorem 18 of chapter one).

Since S contains C_E^* $\underline{O}(E,S)$ is the topology of E. The object (E,S) is in $\Lambda' \cap \Lambda'$ and so by theorems 14 and 15 of chapter one $(H(E,S),h_S)$ is an extension of E and the algebra $\bar{S}$ consists of continuous functions. Since S satisfies (2) and (3) $\bar{S}$ is uniformly closed and closed under positive inversion. Also $C_E^* = \bar{S}^*|E$ is contained in $C_{H(E,S)}^*|E$ which is also contained in C_E^* and so $\bar{S}^* = C_{H(E,S)}^*$. Therefore $\bar{S} = C_{H(E,S)}$ by theorem 19 of chapter one. Consequently $S = C_{H(E,S)}|E$ and so S is an extension algebra.

If S is an extension algebra containing C_E^* then every characteristic algebra $\underline{c}$ of E is a subalgebra of S. Therefore each $\underline{c}$ defines an equivalence relation on $H(E,S)$ (see section one) which in turn defines a new subalgebra of S.

Let $S(\underline{c}) = [f \text{ in } S \mid \text{for } h_1, h_2 \text{ in } H(E,S),\ h_1|\underline{c} = h_2|\underline{c}$ implies $h_1(f) = h_2(f)]$. Algebras like this are extension algebras and every extension algebra is of this form as stated in

<u>Theorem 27.</u> <u>Let S be an extension algebra of E that contains C_E^* and let $\underline{c}$ be a characteristic algebra. Then the algebra $S(\underline{c})$ is an extension algebra. Conversely if S_1 is an extension algebra there exists an extension algebra S containing C_E^* and a characteristic algebra $\underline{c}$ such that $S_1 = S(\underline{c})$. In this case $\underline{c}$ may be taken to be S_1^*.</u>

Proof: Let $Z = \mathcal{E}_{C_E^*}(S)$. Since C_E^* is contained in S it follows from corollary 4 of theorem 16 that $S = C_Z|E$ since (Z,i) and $(H(E,S),h_S)$ are isomorphic extensions of E (see theorem 18 of chapter one). Furthermore Z is a Q-space and consequently $S(\underline{c})$ is the collection of functions in S whose extensions to Z are compatible with $\underline{r}(\underline{c})|Z$. Therefore $S(\underline{c}) = C_X \circ j$ where $X = Z/\underline{r}(\underline{c})|Z$ and $j = \pi_Z(\underline{c}) \circ i$. Since (X,j) is an extension of E by theorem 12 it follows that $S(\underline{c})$ is an extension algebra.

Let S_1 be an extension algebra and let $\underline{c} = S_1^*$. It is a characteristic algebra of E. Let $Z = \mathcal{E}_{C_E^*}(S_1)$ and let $X = \mathcal{E}_{\underline{c}}(S_1)$. Since S_1 is contained in $\mathcal{L}(\underline{c})$ they are both Q-spaces and by theorem 15 $\pi^{-1}(\underline{c})X = Z$. If $S =$

$C_Z|E$ then S is an extension algebra containing C_E^*. The first part of the theorem shows that $S_1 = S(\underline{c})$.

Remarks. The correspondence $(S,\underline{c}) \longrightarrow S(\underline{c})$ is not 1 - 1. For example if $S = C_E$ and $x \neq y$ are two points in βE that do not lie in E then $C_E = C_E(C_E^*) = C_E(\underline{c}_{[x,y]})$. However for any extension algebra S_1 the algebra $C_{C_E}(S_1)|E$ is uniquely defined among those extension algebras S containing C_E^* for which $S(S_1^*) = S_1$ by the following property: if h_1 is in $H(E,S_1)$ then there exists h in $H(E,S)$ such that $h|S_1 = h_1$. The characteristic algebras are extension algebras and so the theorem suggests the following problem. If S_2 is contained in S_1 are a pair of extension algebras of E is $S_1(S_2) = [f \text{ in } S_1 |$ for h_1,h_2 in $H(E,S_1)$, $h_1|S_2 = h_2|S_2$ implies $h_1(f) = h_2(f)]$ also an extension algebra? This is so since $S_1(S_2) = S_1(S_2^*)$ and the argument of the theorem applies when $\mathcal{E}_{C_E^*}$ is replaced by $\mathcal{E}_{S_1^*}$ and $\pi(\underline{c})$ by $\pi(S_2^*,S_1^*)$.

§ 9. A conjecture concerning the lattices $\mathcal{L}(\underline{a})$. Let E be a completely regular space and let $\underline{a}$ be in $\mathcal{A}(E)$. The lattices $\mathcal{L}(\underline{a}) = [f \text{ in } F_E |$ for $\lambda \geq 0$ $(f \cap \lambda) \cup (-\lambda)$ is in $\underline{a}]$ are characterized as collections S of continuous functions on

E with the following properties:

($\mathcal{L}_1$) S is a translation sublattice of C_E that contains the constants and is closed under multiplication by real numbers ;

($\mathcal{L}_2$) S is uniformly closed ;

($\mathcal{L}_3$) S is closed under positive inversion ; and

($\mathcal{L}_4$) f is in S iff $f \cup 0$ and $f \cap 0$ are in S .

The lattice $\mathcal{L}(C_E^{*}) = C_E$ has additional properties. For example it is a vector lattice and also satisfies the conditions of

Definition 5. A collection S of real-valued functions on a set E is said to be closed under continuous composition or to be composition closed if for $f_1,\ldots,f_n$ in S and g a continuous function on R^n, $n \geq 1$, the function $k = g(f_1,\ldots,f_n)$ defined by $kx = g(f_1x,\ldots,f_nx)$ is in S .

Remarks. This definition is due to Isbell [7] . A collection S is certainly composition closed if for $f_1,\ldots,f_n$ in S and g a continuous function on the closure in R^n of $[(f_1x,\ldots,f_nx) \mid x \text{ in } E]$ the function $g(f_1,\ldots,f_n)$ is in S . The operations of addition, multiplication, inversion, and taking the maximum or minimum of a pair of functions may be defined by means of continuous functions g . Consequently if S is composition closed, it is closed under all these operations.

Do these additional properties of $\mathcal{L}(C_E^*)$ distinguish it from the other lattices that satisfy $(\mathcal{L}_1),(\mathcal{L}_2),(\mathcal{L}_3)$ and $(\mathcal{L}_4)$? Since every $\underline{a}$ in $\underline{\mathfrak{a}}$ is equal to $C_{K(\underline{a})} \circ \pi(\underline{a}) \circ i$ it follows that none of these additional properties distinguish $\mathcal{L}(\underline{a})$ from $\mathcal{L}(C_E^*)$ if $\mathcal{L}(\underline{a}) = \underline{a}$. The algebras $\underline{a}$ for which this happens are characterised by

<u>Theorem 28. $\mathcal{L}(\underline{a}) = \underline{a}$ iff when $\underline{a}$ is extended to βE $Z(f) \neq \phi$ for f in $\underline{a}$ implies $Z(f) \cap iE \neq \phi$.</u>

Proof: Assume that f is an unbounded function in $\mathcal{L}(\underline{a})$. Then $f \cup 0$ or $f \cap 0$ is unbounded and so f may be assumed to be a positive function belonging to $\mathcal{L}(\underline{a})$ but not $\underline{a}$. When extended to βE $\underline{f}$ is necessarily infinite at some point. If $g = f \cup 1$ the same is true of $\underline{g}$. The function g does not vanish at any point of E and so the bounded function $k = 1/g$ is in $\mathcal{L}(\underline{a})$ and hence in $\underline{a}$. Furthermore $kx = 0$ iff $gx = +\infty$ and therefore $Z(k) \cap iE = \phi$. From the choice of g $Z(k) \neq \phi$.

Let k be a function in $\underline{a}$ such that $Z(k) \neq \phi$ and $Z(k) \cap iE \neq \phi$. The function k may be assumed to be ≥ 0 since otherwise k^2 could be used. The function $g = 1/k$ is in $\mathcal{L}(\underline{a})$ by $(\mathcal{L}_3)$ and it is clear that $\underline{g}x = +\infty$ iff $kx = 0$. Therefore g is unbounded.

<u>Remark.</u> <u>This theorem provides a characterisation of pseudo-compact spaces: E is pseudocompact iff for f in C_E^* $Z(f) \neq \phi$ implies $Z(f) \cap iE \neq \phi$.</u>

There are, however, many examples of algebras $\underline{a}$ for which $\mathcal{L}(\underline{a}) \neq \underline{a}$ when E is not pseudocompact. Among these there are a lot of characteristic algebras as shown by the following

Example. Assume E is not pseudocompact and let $x \neq y$ be two points in βE that are not in υE. Then there is a positive function $f_{x,y}$ in $C_{\beta E} = C_E^*$ which vanishes at x and y and such that $Z(f_{x,y}) \cap iE = \phi$ (this is because $\complement \upsilon E$ is a union of G_δ sets of βE that are disjoint from iE). Consequently $\mathcal{L}(\underline{c}_{[x,y]}) \neq \underline{c}_{[x,y]}$ by theorem 28. Let $g = 1/f_{x,y}$. It is in $\mathcal{L}(\underline{c}_{[x,y]})$ and $\underline{g}x = \underline{g}y = +\infty$. If k is any function in $C_{\beta E} = C_E^*$ then $(k + \underline{g})x = (k + \underline{g})y = +\infty$ and so by theorem 10 $k + g$ is in $\mathcal{L}(\underline{c}_{[x,y]})$. This shows that $\mathcal{L}(\underline{c}_{[x,y]})$ is not closed under addition. Assume the opposite. Then $k = k + g + (-1)g$ is in $\mathcal{L}(\underline{c}_{[x,y]})$ and so C_E^* is contained in $\mathcal{L}(\underline{c}_{[x,y]})$. Therefore $C_E^* = \underline{c}_{[x,y]}$ and $x = y$. This is a contradiction. Consequently if m is the cardinal number of $\complement \upsilon E$ there are at least m examples of characteristic algebras $\underline{c}$ of E such that $\mathcal{L}(\underline{c})$ is not a vector lattice.

These examples of algebras $\underline{a}$ for which $\mathcal{L}(\underline{a})$ is not closed under addition lead to the consideration of those algebras $\underline{a}$ such that $\mathcal{L}(\underline{a})$ is not closed under multiplication. The following theorem shows that these two kinds of algebras $\underline{a}$ coincide.

<u>Theorem 29. Let $\underline{a}$ be a uniformly closed unitary subalgebra of C_E^*. The following statements are equivalent:</u>

(1) $\mathcal{L}(\underline{a})$ <u>is closed under addition;</u>

(2) $\mathcal{L}(\underline{a})$ <u>is closed under multiplication; and</u>

(3) $\mathcal{L}(\underline{a})$ <u>is closed under continuous composition.</u>

Proof: Obviously (3) implies both (1) and (2). The proof of the converses depends on the following lemma, a modification of a result due to Isbell [7].

<u>Lemma 13. Let E be locally compact and countable at infinity. Let $\underline{c}$ be a characteristic algebra of E. The following statements are equivalent:</u>

(1) $\mathcal{L}(\underline{c})$ <u>is closed under addition;</u>

(2) $\mathcal{L}(\underline{c})$ <u>is closed under multiplication; and</u>

(3) $\underline{c} = C_E^*$; <u>i.e. $\mathcal{L}(\underline{c}) = C_E$.</u>

Proof: Since (3) implies (1) and (2) it is sufficient to consider the converses.

The space E is locally compact and therefore the functions in C_E^* constant on $\complement iE$ define a characteristic algebra which by theorem 5 is contained in $\underline{c}$. Lemma 4 of chapter one shows that $\underline{c}$ contains a positive function f_o in C_E^* for which $Z(f_o) = \complement iE$, since E is also countable at infinity. The function $1/f_o$ is in $\mathcal{L}(\underline{c})$ by $(\mathcal{L}_3)$.

Assume that $\mathcal{L}(\underline{c})$ is closed under addition (multiplication) and let g be in C_E^*. The function $1/f_o + g$ $(f_o \cdot g)$ is infinite on $\complement iE$ (vanishes on $\complement iE$) and so $1/f_o + g$ $(f_o \cdot g)$ is in $\mathcal{L}(\underline{c})$ by theorem 10. Since $\mathcal{L}(\underline{c})$ is closed under addition (multiplication) $g = 1/f_o + g + (-1)1/f_o$ ($g = 1/f_o \cdot f_o \cdot g$) is in $\mathcal{L}(\underline{c})$. Therefore $\underline{c} = C_E^*$ and (1) implies (3) ((2) implies (3)).

To return to the proof of the theorem let $f_1, \ldots, f_n$ be n functions in $\mathcal{L}(\underline{a})$ and let Z be the closure in R^n of $[(f_1x, \ldots, f_nx) | x \text{ in } E]$. Consider the set C of continuous real-valued functions g on Z such that $g(f_1, \ldots, f_n)$ is in $\mathcal{L}(\underline{a})$. The set C inherits properties from the translation lattice $\mathcal{L}(\underline{a})$. For example if g_1, g_2 are in C then $g_1 \cup g_2$ and $g_1 \cap g_2$ are in C. This is because if x is in E $(g_1 \cup g_2)(f_1, \ldots, f_n)x = g_1(f_1, \ldots, f_n)x \cup g_2(f_1, \ldots, f_n)x$ and similarly for $g_1 \cap g_2$. In a similar manner it follows that C satisfies $(\mathcal{L}_1), (\mathcal{L}_2), (\mathcal{L}_3)$ and $(\mathcal{L}_4)$. The projections $\pi_j | Z$ are in C where $\pi_j(f_1, \ldots, f_n) = f_j$ and so the topology of Z is the weak topology $\underline{C}(Z, C^*)$ since C satisfies $(\mathcal{L}_1)$. In other words C^* is a characteristic algebra of Z. Since R^n is locally compact and countable at infinity Z as a closed subspace also satisfies these properties.

Lemma 13 applies to C and Z. Since C is closed under addition (multiplication) when $\mathcal{L}(a)$ is closed under addition (multiplication) it follows from the lemma that $C = C_Z$ if $\mathcal{L}(\underline{a})$ satisfies (1) or (2). Consequently $\mathcal{L}(a)$ is composition closed if $\mathcal{L}(\underline{a})$ satisfies (1) or (2).

Remark. The proofs of this theorem and lemma are essentially the proofs given by Isbell [7] for his theorem 1.13 and corollary 1.14.

The algebras $\underline{a}$ in $\mathfrak{a}$ determine the closed equivalence relations $\underline{r}(\underline{a})$ on βE and are in turn defined by these relations (see sections one and two). A subset X of βE will be said to determine $\underline{a}$ if the functions f in $C_{\beta E} = C_E^*$ such that $f|X$ is compatible with $r(\underline{a})|X$ are in $\underline{a}$.

The algebras $\underline{a}$ for which $\mathcal{L}(\underline{a})$ is closed under continuous composition are characterized by

Theorem 30. *$\mathcal{L}(\underline{a})$ is closed under continuous composition iff for any f in $\underline{a}$ $Z(f) \cap iE = \phi$ implies that $\complement Z(f)$ determines $\underline{a}$.*

Proof: Let f in $\underline{a}$ be such that $Z(f) \cap iE = \phi$ and assume that $Z(f)$ does not determine $\underline{a}$. The function f may be assumed to be positive since f^2 is in $\underline{a}$ and has the same two properties. Let g be a function in C_E^* such that $g|\complement Z(f)$ is compatible with $\underline{r}(\underline{a})|\complement Z(f)$ but such that g is not in $\underline{a}$. Since $k = 1/f$ is in $\mathcal{L}(\underline{a})$ and $\underline{k}$ is infinite on $Z(f)$ the function $\underline{k} + g$ is infinite on $Z(f)$. Theorem 10 shows that $k + g$ is in $\mathcal{L}(\underline{a})$. Since g is not in $\underline{a}$, $\mathcal{L}(\underline{a})$ is not closed under addition and hence is not closed under continuous composition.

Assume that $\mathcal{L}(\underline{a})$ is not composition closed or equivalently is not closed under addition. Then there exist two functions f, g in $\mathcal{L}(\underline{a})$ such that $f + g$ is not in $\mathcal{L}(\underline{a})$. Corollary 1 of theorem 10 shows that f and g are both unbounded. Since by $(\mathcal{L}_4)$ $f \cup 0$, $f \cap 0$, $g \cup 0$ and $g \cap 0$ are in $\mathcal{L}(\underline{a})$ the functions $k = f \cup 0 + g \cup 0$ and $l = f \cap 0 + g \cap 0$ are in $\mathcal{L}(\underline{a})$ by corollary 1 of theorem 10. Also $k + l = f + g$ and so f and g may be assumed to be respectively positive and negative.

Since $\mathcal{L}(\underline{a})$ is a translation lattice it contains $f_1 = f + 1$ and $g_1 = g - 1$. As $f \geq 0$ and $g \leq 0$, $f_1 \geq 1$ and $g_1 \leq -1$. By $(\mathcal{L}_3)$ and $(\mathcal{L}_1)$ the functions $1/f_1$ and $1/g_1$ are in $\underline{a}$ and so also is $k = 1/f_1 . 1/g_1$. It is clear that $Z(k) = Z(1/f_1) \cup Z(1/g_1)$ and since $Z(1/f_1) = \complement \mathcal{E}_{C^*_E}([f_1])$ and $Z(1/g_1) = \complement \mathcal{E}_{C^*_E}([g_1])$ it follows that $\complement Z(k) = \mathcal{E}_{C^*_E}([f_1]) \cap \mathcal{E}_{C^*_E}([g_1]) = \mathcal{E}_{C^*_E}([f_1, g_1])$. Consequently on $\complement Z(k)$ $\underline{f}_1 + \underline{g}_1 = \underline{f} + \underline{g}$ is compatible with $\underline{r}(\underline{a})$ as both functions are finite on that set. This means that $(f + g)_n = [(f + g) \cap n] \cup (-n)$ is a bounded continuous function such that $(f + g)_n | \complement Z(k)$ is compatible with $\underline{r}(\underline{a}) | \complement Z(k)$. However $f + g$ is not in $\mathcal{L}(\underline{a})$ and therefore for some n_o $(f + g)_{n_o}$ is not in $\underline{a}$. This shows that $\complement Z(k)$ does not determine $\underline{a}$. It is clear that $Z(k) \cap iE = \phi$.

<u>Remark.</u> This theorem may be stated in terms of the compact space $K(\underline{a})$. If f is in $\underline{a}$ let it also denote its 'extension' to $K(\underline{a})$ and let $Z_{\underline{a}}(f) = [x \text{ in } K(a) | fx = 0, \ f \text{ in } \underline{a}]$. Then $\mathcal{L}(\underline{a})$ is composition closed iff $(\pi(\underline{a}) \circ i)E \cap Z_{\underline{a}}(f) = \phi$

implies $C_{K(\underline{a})} | \mathbf{C} Z_{\underline{a}}(f) = C^{*}_{\mathbf{C} Z_{\underline{a}}(f)}$. The concept of a set X determining $\underline{a}$ is essentially the same as an equivalence relation from which non-trivial classes may be omitted (see example 4 in section one) .

The results of this section have been obtained in an attempt to make the following conjecture plausible.*)

Conjecture. Let E be a Q-space and let S be a subset of C_E that satisfies $(\mathcal{L}_1)$,$(\mathcal{L}_2)$,$(\mathcal{L}_3)$ and $(\mathcal{L}_4)$. Then $S = C_E$ iff S satisfies

(1) $O(E,S)$ is the topology of E ;

(2) $S = S^{*}$ iff E is compact ; and

(3) S is closed under addition (or equivalently composition or multiplication closed).

The conjecture is restricted to Q-spaces because if E is pseudocompact $\mathcal{L}(\underline{a}) = \underline{a}$ for every $\underline{a}$ in $\mathcal{Q}(E)$ and there seems to be no satisfactory 'algebraic' way to distinguish C^{*}_{E} from the other characteristic algebras of E (other than the factoring condition of theorem 20 in chapter one).

When E is compact the conjecture holds by virtue of the Stone-Weierstrass theorem. In fact for compact spaces it is equivalent to this theorem because a compact space K has exactly one characteristic algebra and it is the sole algebra in $\mathcal{Q}(K)$ that separates the points of K .

Lemma 13 shows that the conjecture holds for all locally compact spaces that are countable at infinity (they are Q-spaces,

*) See Erratum p. 260.

see section seven of chapter one) .

In addition the example of this section shows that if E is not a compact space then when E is a Q-space there are many characteristic algebras $\underline{c} \neq C_E^*$ for which $\mathcal{L}(\underline{c})$ satisfies conditions (1) and (2) of the conjecture but not condition (3) .

The alternative form of theorem 30 (in the previous remark) shows that to obtain a counterexample for the conjecture it is necessary and sufficient to exhibit a compact space K and a dense subset E such that:

(1) <u>as a subspace E is a Q-space</u> ;

(2) <u>$C_E^* \neq C_K|E = \underline{c}$</u> ;

(3) <u>there is a function f in C_K with $Z(f) \cap E = \emptyset$ and $Z(f) \neq \emptyset$</u> ; <u>and</u>

(4) <u>if g is in C_K and $Z(g) \cap E = \emptyset$ then $C^*_{\complement Z(g)} = C_K | \complement Z(g)$.</u>

When this is the case $\mathcal{L}(\underline{c})$ satisfies the conditions of the conjecture.

An example of this type of situation is provided by a Q-space E with the following property: <u>there is a closed subset D of βE distinct and disjoint from iE that lies in no G_δ-set disjoint from iE .</u> Let $\underline{c}_D$ be the characteristic algebra of functions constant on E . Then if f is in $\underline{c}_D$ and $Z(f) \cap iE = \emptyset$ it follows that $Z(f) \cap D = \emptyset$. By theorem 30 $\mathcal{L}(\underline{c}_D)$ is composition closed. Furthermore $\mathcal{L}(\underline{c}_D) \neq \underline{c}_D$. Let

x_o be in βE and in neither iE nor D. Since E is a C-space there is a function g in $C_{\beta E}$ for which $gx_o = 0$, $g|D = 1$ and $Z(g) \cap iE = \emptyset$. By theorem 28 $\mathcal{L}(\underline{c}_D) \neq \underline{c}_D$.

The importance of this conjecture is due to the fact that it extends the Stone-Weierstrass theorem to a class of completely regular spaces which contains those that are locally compact and countable at infinity. For which spaces is it correct?*)

*) See Erratum p. 260.

CHAPTER THREE

THE CONSTRUCTION OF TOPOLOGICAL SPACES FROM UNIFORMITIES

§1. Preliminaries. Let E denote a completely regular space and let $\underline{U}$ be a uniformity for E. The uniformity $\underline{U}$ is said to be compatible (with the topology of E) if the $\underline{U}$-uniform topology is coarser than the topology of the space E. A uniformity $\underline{U}$ for E is compatible iff it has a filter basis of open surroundings (entourages). Every uniformity has a base consisting of surroundings open with respect to the product of the uniform topology. Hence if it is compatible it has a base of open surroundings. Conversely if a uniformity has a base of open surroundings then for any point the neighbourhood filter contains the filter defined by the uniformity. Consequently it is compatible. A compatible uniformity $\underline{U}$ for E is said to be a structure of E when the $\underline{U}$-uniform topology coincides with that of the space E. If $\underline{U}$ is a compatible uniformity for E, E is said to be complete in $\underline{U}$ when every $\underline{U}$-Cauchy filter converges in the $\underline{U}$-uniform topology.

Examples of uniformities for E:

1. Let S be a collection of real-valued functions on E. Let $\underline{U}(S)$ denote the uniformity for E generated by the sets $V(f,\varepsilon) = [(x,y)\,|\,|fx - fy| < \varepsilon]$ where f is in S and $\varepsilon > 0$. The $\underline{U}(S)$-uniform topology is the weak topology $\underline{O}(\mathcal{E},S)$. Hence $\underline{U}(S)$ is a compatible uniformity iff the functions in S are all continuous. This suggests that there is a possible parallel

between compatible uniformities and continuous functions. A uniformity of this type is called a <u>function uniformity</u>.

2. Let $\underline{a}$ be a uniformly closed unitary subalgebra of C_E^H. Then $\underline{U}(\underline{a})$ is a compatible uniformity for E which is a structure of E iff $\underline{a}$ is a characteristic algebra of E. In particular $\underline{U}(C_E^H)$ is a structure of E.

3. Let $\underline{c}$ be a characteristic algebra and let S be a collection of functions containing $\underline{c}$ and contained in $\mathcal{L}(\underline{c})$. Then $\underline{U}(S)$ is a structure of E. In particular $\underline{U}(C_E)$ is a structure of E. It is the finest compatible function uniformity for E.

If $(\underline{U}_i)_{i \text{ in } I}$ is any family of compatible uniformities for E then the uniformity $\underline{U}$ generated by this family is also compatible. This is because the $\underline{U}$-uniform topology is the supremum of the $\underline{U}_i$-uniform topologies. If one of the $\underline{U}_i$ is a structure of E then $\underline{U}$ itself is a structure of E. Since E has at least one structure, for example $\underline{U}(C_E)$, it follows that <u>there exists a finest structure of E</u> which will be denoted by $\underline{U}^E$. It is clear that a uniformity $\underline{U}$ for E is compatible iff $\underline{U}^E$ contains $\underline{U}$.

While there is a coarsest compatible uniformity for E - namely the uniformity consisting of the single set $E \times E$ - in general there is no coarsest structure. Samuel [13] showed that the existence of a coarsest structure characterizes the locally compact spaces.

A useful lemma due to Weil [18] is the following well known variant of Urysohn's lemma. It is stated without proof as

Lemma 1. Let U be a uniformity for E and let F be a subset of E. If V is a surrounding of U there is a uniformly continuous function f on E such that $0 \leq f \leq 1$, $fx = 0$ for x in F and $fx = 1$ for x in $\complement(V[F])$.

The examples of compatible uniformities in example 2 all satisfy the following equivalent conditions:

(1) for any surrounding V in U there exists a finite number of points $x_1, \ldots, x_n$ in E with $E = \bigcup_{i=1}^{n} V[x_i]$; and

(2) for any surrounding V in U there exists a finite number of V-small sets $F_1, \ldots, F_n$ such that $E = \bigcup_{i=1}^{n} F_i$.

Since $V[x]$ is V^2-small for any x in E, it is clear that (1) implies (2). On the other hand if x_i is in F_i then $V[x_i]$ contains F_i and so (2) implies (1). The uniformities U for E that satisfy either (1) or (2) are said to be totally bounded. Condition (2) implies that the uniformity generated by any family of totally bounded uniformities is itself totally bounded. Since $U(C_E^*)$ is a totally bounded structure of E a finest totally bounded structure U^{E*} exists. The totally bounded uniformities are of interest because of the close connection between them and compact spaces. This is stated as the following well known theorem,

Theorem 1. *Let E be a completely regular space. The following statements are equivalent:*

(1) *E is compact ;*

(2) *E is complete in $\underline{U}(C_E^*)$;*

(3) *E is complete in $\underline{U}^E$; and*

(4) *E is complete in a totally bounded structure.*

Proof: A compact space is a Q-space and since by theorem 16 of chapter one a completely regular space is a Q-space iff it is complete in $\underline{U}(C_E)$ it follows that (1) implies (2) .

Since $\underline{U}^E$ contains $\underline{U}(C_E^*)$ it is clear that (2) implies (3) and since $\underline{U}^E$ is totally bounded (3) implies (4) .

Assume that E satisfies (4) and that $\underline{U}$ is a totally bounded structure of E in which it is complete. Let $\underline{F}_o$ be an ultrafilter on E and if V is in $\underline{U}$ let $F_1,\ldots,F_n$ be n V-small sets such that $E = \bigcup_{i=1}^{n} F_i$. Since $\underline{F}_o$ is an ultrafilter it contains of the sets F_i . Therefore every ultrafilter is $\underline{U}$ - Cauchy and hence converges. This shows that E is compact and so (4) implies (1) .

Remark. It is known (see Banaschewski [16]) that for any completely regular space $\underline{U}^{E*} = \underline{U}(C_E^*)$ and so (2) and (3) are actually identical. This result is established in the first section of chapter four as theorem 2 .

The compatible uniformities in examples 1 and 2 are totally bounded iff the functions in S are all bounded. However, all these uniformities $\underline{U}$ do satisfy the following weaker equivalent conditions:

(1) for any surrounding V in $\underline{U}$ there exists a sequence $(x_n)_n$ of points x_n in E such that $E = \bigcup_{n=1}^{\infty} V[x_n]$; and

(2) for any surrounding V in $\underline{U}$ there exists a sequence $(F_n)_n$ of V-small sets F_n such that $E = \bigcup_{n=1}^{\infty} F_n$.

Since $V[x]$ is V^2-small for any x in E, (1) implies (2). Choose x_n in F_n for each n. Then $V[x_n]$ contains F_n and so (2) implies (1). A uniformity $\underline{U}$ is said to be σ-bounded if it satisfies either (1) or (2). Condition (2) implies that a uniformity generated by a family of σ-bounded uniformities is also σ-bounded. Since $\underline{U}(C_E)$ is a σ-bounded structure of E it follows that E has a finest σ-bounded structure which will be denoted by $\underline{U}^{E\sigma}$.

The σ-bounded compatible uniformities of a completely regular space E may be used to distinguish the Q-spaces as shown by the following theorem, due to Shirota [9].

Theorem 2. Let E be a completely regular space. The following statements are equivalent:

(1) E is a Q-space;

(2) E is complete in $\underline{U}(C_E)$;

(3) E is complete in $\underline{U}^{E\sigma}$; and

(4) E is complete in a σ-bounded structure of E.

Proof: Theorem 16 of chapter one states that (1) and (2) are equivalent. It is clear that (3) and (4) are equivalent and also that (2) implies (3).

Assume that E is complete in $\underline{U}^{E\sigma}$ and let $\underline{F}$ be any $\underline{U}(C_E)$ - Cauchy filter. Then since every continuous real-valued function converges along $\underline{F}$ it follows from theorem 9 of chapter one that there is a unique maximal C_E - completely regular filter $\underline{M}$ contained in $\underline{F}$. The remark following definition 3 of chapter one shows that $\underline{M}$ has a basis of zero sets Z i.e. sets $Z = Z(g) = [x \text{ in } E \mid gx = 0]$ where g is in C_E^{*}. To prove that E is complete in $\underline{U}(C_E)$ it is sufficient to find a $\underline{U}^{E\sigma}$- Cauchy filter containing $\underline{M}$ as this will imply that both $\underline{M}$ and $\underline{F}$ converge.

Consider filters on E that have a basis of zero sets. If any two generate a filter it too has a basis of zero sets, because for any two functions g_1 and g_2 $Z(g_1) \cap Z(g_2) = Z(g_1 g_2)$. If Z is a zero set then the filter of sets containing Z obviously has a basis of zero sets. Consequently if Z is a zero set which together with a filter generates a new filter, the new filter has a basis of zero sets if the original filter had such a basis. Filters of this type, i.e. with a basis of zero sets, when ordered by inclusion form an inductive set and so by Zorn's lemma any filter of this type is contained in a maximal one. Among these maximal filters there are $\underline{U}^{E}$ -

Cauchy filters as shown by

<u>Lemma 2. Let $\underline{F}$ a filter on E maximal among those with a basis of zero sets. Then $\underline{F}$ is a $\underline{U}^{E\sigma}$ - Cauchy filter if for any sequence $(F_n)_n$ of sets F_n in $\underline{F}$ $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.</u>

Proof: Let V and W be surroundings of $\underline{U}^E$ such that W^3 is contained in V. Let $(D_n)_n$ be a sequence of W-small sets such that $E = \bigcup_{n=1}^{\infty} D_n$. For each n let f_n be a continuous function on E such that $Z(f_n)$ contains D_n and is contained in $W[D_n]$. Such functions exist by lemma 1. The zero sets $Z(f_n)$ cover E and each of them is W^3-small and hence V-small.

Since $\underline{F}$ is a maximal for each n $Z(f_n)$ is in $\underline{F}$ or else there exists a set F_n in $\underline{F}$ with $F_n \cap Z(f_n) = \emptyset$. Assume that none of the $Z(f_n)$ belong to $\underline{F}$ and choose F_n disjoint from $Z(f_n)$ for each n. Then $\bigcap_{n=1}^{\infty} F_n = (\bigcap_{n=1}^{\infty} F_n) \cap (\bigcup_{K=1}^{\infty} Z(f_K)) = \bigcup_{K=1}^{\infty}((\bigcap_{n=1}^{\infty} F_n) \cap Z(f_K)) = \emptyset$. This is a contradiction. Therefore $\underline{F}$ contains one of the $Z(f_n)$ and as a result is a $\underline{U}^{E\sigma}$- Cauchy filter.

To complete the proof of this theorem it is sufficient in view of lemma 2 to show that any filter $\underline{F}$ with a basis of zero sets which contains $\underline{M}$ has the property that any countable intersection of sets in $\underline{F}$ is non-void. Assume this is not the case. Then there is a filter $\underline{F}$ on E with a basis of zero sets that contains $\underline{M}$ and a sequence $(g_n)_n$ of functions in C_E^* with $0 \leq g_n \leq 1$ such that for each n $Z(g_n)$ is in $\underline{F}$

and such that $\bigcap_{n=1}^{\infty} Z(g_n) = \phi$. Let $g = \sum_{n=1}^{\infty} 2^{-n} g_n$. It is in C_E^* and converges along $\underline{F}$ to zero. Therefore g converges along $\underline{M}$ to zero. Furthermore if x is in E $gx \neq 0$ and so $1/g$ is in C_E. It therefore converges along $\underline{M}$ to a finite limit. This is a contradiction. Therefore $\underline{M}$ is a convergent filter and so (3) implies (2).

Remarks. The statements of theorems 1 and 2 are parallel. This raises the following problems. Are the uniformities $\underline{U}(C_E)$ and $\underline{U}^{E\sigma}$ the same for any completely regular space E? If not, for which spaces E do they coincide? The proof that (3) implies (2) is essentially the same as that given by Shirota [9].

This preliminary section concludes with a theorem about the uniqueness of compatible uniformities. Its corollary will be used later in this chapter to identify particular uniformities.

Theorem 3. Let X be a topological space and let F be a dense subset. Let $\underline{U}_1$ and $\underline{U}_2$ be two compatible uniformities on X. If $\underline{U}_1 | F = \underline{U}_2 | F$ then $\underline{U}_1 = \underline{U}_2$.

Proof: Since F is a dense subset of X $F \times F$ is a dense subset of $X \times X$. Let V_1 be a surrounding of $\underline{U}_1$ that is open. Then $\overline{V}_1$ contains $\overline{V_1 \cap F \times F}$ which in turn contains $\overline{V_1 \cap F \times F} = \overline{V}_1$. Since $\underline{U}_1$ is compatible it has a base of open surroundings V_1. Their closures $\overline{V}_1$ also form a base for $\underline{U}_1$. Therefore the sets $\overline{V_1 \cap F \times F}$ form a base for $\underline{U}_1$.

Consequehtly if $\underline{U}_1|F = \underline{U}_2|F$ $\underline{U}_1$ and $\underline{U}_2$ have a common base and so are identical.

Corollary. Let X be a topological space and let $p:E \longrightarrow X$ be a function on the set E with pE dense in X. Let $\underline{U}_1$ and $\underline{U}_2$ be compatible uniformities on X. Then $\underline{U}_1 = \underline{U}_2$ if their inverse images under p coincide.

Proof: For any uniformity $\underline{U}$ on X $(p \times p)(p \times p)^{-1}\underline{U} = \underline{U}|pE$. This is because if V is in $\underline{U}$ (x,y) is in $(p \times p)^{-1}V$ iff (px,py) is in V. Consequently when the inverse images under p coincide $\underline{U}_1|pE = \underline{U}_2|pE$. Since $pE = F$ is a dense subset of X the theorem implies that $\underline{U}_1 = \underline{U}_2$.

Remarks. While compatible uniformities were not defined for arbitrary topological spaces it is clear what they are: uniformities whose uniform topology is coarser than the topology of the space. The space is completely regular iff a separated structure exists. This theorem is analogous to the theorem which asserts that two continuous functions valued in a Hausdorff space coincide if they agree on a dense subset of their domain.

§2. Uniform processes. Consider the category Υ with objects the pairs $(E,\underline{U})$, where E is a non-void set and $\underline{U}$ is a uniformity for E, and with $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ the set of all $(\underline{U},\underline{U}')$ - uniformly continuous functions $\alpha:E \longrightarrow E'$. A general method of associating with each object of Υ a topological space is said to be a uniform process if it satisfies the conditions of

Definition 1. *A uniform process $\mathcal{P}$ on Υ consists of a covariant functor $P: \Upsilon \longrightarrow \Sigma$ and a family $(p_{\underline{U}})_{(E,\underline{U})\text{ in }\Upsilon}$ of functions $p_{\underline{U}}: E \longrightarrow P(E,\underline{U})$ such that:*

(UP_1) $p_{\underline{U}}E$ is dense in $P(E,\underline{U})$; and

(UP_2) if α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ then

$$P(\alpha) \circ p_{\underline{U}} = p_{\underline{U}'} \circ \alpha .$$

Remarks. 1. Condition (UP_2) states that the following diagram is commutative

$$\begin{tikzcd}
(E,\underline{U}) \arrow[r, "p_{\underline{U}}"] \arrow[d, "\alpha"'] & P(E,\underline{U}) \arrow[d, "P(\alpha)"] \\
(E',\underline{U}') \arrow[r, "p_{\underline{U}'}"] & P(E',\underline{U}').
\end{tikzcd}$$

2. This definition differs from definition 1 of chapter one in two respects. In the first place it defines processes on Υ not on subcategories of Υ . This is because for the purposes of this thesis there is no immediate point in considering processes on subcategories of Υ . The second difference is much more basic. In the case of a uniform process it is nonsensical to speak of a 'real' uniform process because no analogue to the value space exists.

Examples of uniform processes on Υ

1. The first two examples of function process on Φ suggest correspondingly 'trivial' examples of uniform processes.

2. To correspond to the third example of a function process on Φ there is the following process. Let $P(E,\underline{U})$ be the set E

together with the $\underline{U}$-uniform topology. Define $P(\alpha)$ to α and $p_{\underline{U}}x = x$ for all x in E.

The close analogy between the definition of a function process and the definition of a uniform process suggests that it is possible to develop the discussion of uniform processes in a manner parallel to the development in chapter one of the 'theory of function processes'.

Consider two processes $\mathcal{P}$ and $\mathcal{Q}$ on $\mathfrak{T}$. They may be compared by means of the restricted type of natural homomorphism defined in

Definition 2. <u>A homomorphism γ of the process $\mathcal{P}$ into the process $\mathcal{Q}$ consists of a family $(\gamma(E,\underline{U}))_{(E,\underline{U})}$ in $\mathfrak{T}$ of continuous functions $\gamma(E,\underline{U}): P(E,\underline{U}) \longrightarrow Q(E,\underline{U})$ such that:</u>

(1) <u>if α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ then $\gamma(E',\underline{U}') \circ P(\alpha) = Q(\alpha) \circ \gamma(E,\underline{U})$; and</u>

(2) <u>for each $(E,\underline{U})$ in $\mathfrak{T}$ $\gamma(E,\underline{U}) \circ p_{\underline{U}} = q_{\underline{U}}$.</u>

A homomorphism γ is called an <u>isomorphism</u> if each $\gamma(E,\underline{U})$ is a homeomorphism. When an isomorphism γ of $\mathcal{P}$ into $\mathcal{Q}$ exists they are said to be <u>isomorphic processes.</u>

As in the case of function processes it is easy to show that the relation of isomorphism between uniform processes is an equivalence relation.

The introduction of the concept of isomorphism for uniform processes leads to a consideration of those properties of uniform processes that are invariant under isomorphism. The simplest kind of invariant property is a topological property. If (t) is a topological property a uniform process $\mathcal{P}$ is said to be a (t)-process if for each $(E,\underline{U})$ in Υ $P(E,\underline{U})$ satisfies (t) .

As in the case of function processes when   is a Hausdorff process condition (2) of definition 2 implies condition (1) . This result is stated as

Lemma 3. Let $\mathcal{P}$ and $\mathcal{Q}$ be two uniform processes and let $(\gamma(E,\underline{U}))_{(E,\underline{U}) \text{ in } \Upsilon}$ be a family of continuous functions $\gamma(E,\underline{U}) : P(E,\underline{U}) \longrightarrow Q(E,\underline{U})$ such that $\gamma(E,\underline{U}) \circ p_{\underline{U}} = q_{\underline{U}}$. If $\mathcal{Q}$ is a Hausdorff process this family defines a homomorphism of $\mathcal{P}$ into $\mathcal{Q}$.

Proof: The same as that of lemma 1 in chapter one.

In the case of function processes properties (FP_3),(FP_4), (FP_5) and (FP_6) deal with the 'extension' of the functions in S as continuous functions on the space $P(E,S)$. The analogous properties for uniform processes deal with the 'extension' of the uniformity $\underline{U}$ to the space $P(E,\underline{U})$ as a compatible uniformity.

Let $\mathcal{P}$ be a uniform process. If $(E,\underline{U})$ is an object of Υ then there is a finest compatible uniformity $\underline{U}_P$ for $P(E,\underline{U})$ such that $p_{\underline{U}}$ is $(\underline{U}, \underline{U}_P)$-uniformly continuous. The properties (UP_3),(UP_4),(UP_5) and (UP_6) are defined as follows:

(UP_3) for each $(E,\underline{U})$ in Υ , $\underline{U} = (p_{\underline{U}} \times p_{\underline{U}})^{-1}\underline{U}_P$;

(UP_4) for each $(E,\underline{U})$ in Υ , $\underline{U}_P$ is a separated uniformity ;

(UP_5) *for each* $(E,\underline{U})$ *in* Υ, $\underline{U}_P$ *is a structure of* $P(E,\underline{U})$ i.e. the $\underline{U}_P$-uniform topology is the topology of $P(E,\underline{U})$; and

(UP_6) *for each* $(E,\underline{U})$ *in* Υ, $P(E,\underline{U})$ *is complete in* $\underline{U}_P$.

All four of these properties are invariant under isomorphism. Let $\mathcal{P}$ and $\mathcal{Q}$ be isomorphic processes on Υ and let $(\gamma(E,\underline{U}))_{(E,\underline{U}) \text{ in } \Upsilon}$ be a family of homeomorphisms that satisfies the conditions of definition 2. Let $(E,\underline{U})$ be an object of Υ . Then $\gamma(E,\underline{U})$ is an isomorphism of the uniform spaces $(P(E,\underline{U}),\underline{U}_P)$ and $(Q(E,\underline{U}),\underline{U}_Q)$. The inverse image of $\underline{U}_Q$ under $\gamma(E,\underline{U})$ is a compatible uniformity for $P(E,\underline{U})$ whose inverse image under $p_{\underline{U}}$ is the inverse image of $\underline{U}_Q$ under $q_{\underline{U}}$. This shows that $\gamma(E,\underline{U})$ is $(\underline{U}_P,\underline{U}_Q)$ - uniformly continuous. Similarly $\gamma^{-1}(E,\underline{U})$ is $(\underline{U}_Q,\underline{U}_P)$ - uniformly continuous. This shows that $\gamma(E,\underline{U})$ is an isomorphism of these two uniform spaces. From this fact it follows that the above properties are invariant.

The argument used here suggests that if $\mathcal{P}$ is a uniform process and *if* α *is in* $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ *then* $P(\alpha)$ *is a* $(\underline{U}_P,\underline{U}'_P)$*-uniformly continuous function.* This is an almost immediate consequence of the fact that $P(\alpha) \circ p_{\underline{U}} = p_{\underline{U}'} \circ \alpha$. The function $p_{\underline{U}'} \circ \alpha$ is $(\underline{U},\underline{U}'_P)$-uniformly continuous and so $p_{\underline{U}}$ is uniformly continuous with respect to the inverse image under $P(\alpha)$ of $\underline{U}'_P$. Since $P(\alpha)$ is continuous this uniformity has a basis of open surroundings and is therefore compatible.

The definition of $\underline{U}_P$ then implies that $P(\alpha)$ is $(\underline{U}_P,\underline{U}'_P)$-uniformly continuous.

__Remark.__ If a process satisfies (UP_3) then the corollary to theorem 3 shows that for any object $(E,\underline{U})$ the uniformity $\underline{U}_P$ is the only compatible uniformity of $P(E,\underline{U})$ whose inverse image under $p_{\underline{U}}$ is $\underline{U}$. For some processes $\mathcal{P}$ this property makes it easy to give a specific description of the uniformity $\underline{U}_P$.

One application of these properties is stated as the corollary to

__Lemma 4.__ __Let $\mathcal{P}$ be a uniform process that satisfies (UP_3). Then for an object $(E,\underline{U})$ in $\mathcal{T}$, $\underline{U}_P$ is totally bounded iff $\underline{U}$ is totally bounded.__

Proof: Since $\underline{U} = (p_{\underline{U}} \times p_{\underline{U}})^{-1}\underline{U}_P$ it follows that $\underline{U}$ is totally bounded when $\underline{U}_P$ is totally bounded.

Assume that $\underline{U}$ is totally bounded and let V_P be a closed surrounding in $\underline{U}_P$. Let $V = (p_{\underline{U}} \times p_{\underline{U}})^{-1}V_P$ and let $F_1,\ldots,F_n$ be n V-small sets such that $\bigcup_{i=1}^{n} F_i = E$. The sets $p_{\underline{U}}F_i$ are V_P-small and since V_P is closed so are their closures $\overline{p_{\underline{U}}F_i}$ in $P(E,\underline{U})$. Since $p_{\underline{U}}E$ is dense in $P(E,\underline{U})$ it follows that the sets $\overline{p_{\underline{U}}F_i}$, $i = 1,\ldots,n$ cover $P(E,\underline{U})$.

The uniformity $\underline{U}_P$ has a base of closed surroundings and as a result is totally bounded.

__Corollary.__ __Let $\mathcal{P}$ be a uniform process that satisfies (UP_3), (UP_4), (UP_5) and (UP_6). If $(E,\underline{U})$ is an object of $\mathcal{T}$ then__

$\underline{P(E,U)}$ is compact if $\underline{U}$ is totally bounded.

Proof: The lemma shows that $\underline{U}_p$ is totally bounded. The result follows from theorem 1.

Remark. It will be shown in section seven that any two uniform processes that satisfy these four properties are isomorphic. This is the analogue of theorem 5 in chapter one.

The third general type of invariant property of a process $\mathcal{P}$ relates properties of the mappings α to properties of the continuous functions $P(\alpha)$. An example of this type analogous to (FP_7) is

(UP_7) if $(E,\underline{U}),(E',\underline{U}')$ are two objects of Υ and if α is in $\mathrm{Hom}((E,\underline{U})(E',\underline{U}'))$ such that $\underline{U} = (\alpha \times \alpha)^{-1}\underline{U}'$ then $P(\alpha)$ embeds $P(E,\underline{U})$ on a closed subspace of $P(E',\underline{U}')$.

This property is invariant. The proof of this assertion is the same as that of (FP_7). Since property (FP_8) is satisfied by 'algebraic' function processes there does not appear to be a useful analogue for uniform processes.

Another difference between uniform and function processes is that if $\mathcal{P}$ is a uniform process it (always) induces a covariant functor $P_*: \Upsilon \longrightarrow \Upsilon$ which is defined as follows: if $(E,\underline{U})$ is an object of Υ let $P_*(E,\underline{U}) = (P(E,\underline{U}),\underline{U}_p)$ and if α is a mapping of Υ let $P_*(\alpha) = P(\alpha)$. This defines a covariant functor because if $\alpha: E \longrightarrow E'$ is $(\underline{U},\underline{U}')$-uniformly continuous $P(\alpha)$ is $(\underline{U}_p,\underline{U}'_p)$-uniformly continuous.

If $\mathcal{Q}$ is a process on Υ then $\mathcal{Q} \square \mathcal{P}$ is defined to be the covariant functor $Q \square P = Q \circ P_{*}$ and the family $((q \square p)_{\underline{U}})_{(E,\underline{U}) \text{ in } \Upsilon}$ of functions $(q \square p)_{\underline{U}} = q_{\underline{U}_P} \circ p_{\underline{U}}$.

As in the case of function processes $\mathcal{Q} \square \mathcal{P}$ satisfies the second condition of definition 1 since when α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$, $(Q \square P)(\alpha) \circ (q \square p)_{\underline{U}} = Q(P(\alpha)) \circ q_{\underline{U}_P} \circ p_{\underline{U}}$ $= q_{\underline{U}'_P} \circ P(\alpha) \circ p_{\underline{U}} = q_{\underline{U}'_P} \circ p_{\underline{U}'} \circ \alpha = (q \square p)_{\underline{U}} \circ \alpha$.

In general $\mathcal{Q} \square \mathcal{P}$ is not a uniform process as the following example shows.

Example. As in the analogous example in chapter one if $\mathcal{Q}$ is the process on Υ such that $Q(E,\underline{U})$ is the discrete space E , $q_{\underline{U}}x = x$ for each x in E and $Q(\alpha) = \alpha$, then $\mathcal{Q} \square \mathcal{P}$ is a process iff $p_{\underline{U}}E = P(E,\underline{U})$ for every object in Υ . As will be seen later this is not the case for every uniform process $\mathcal{P}$.

If the functions $q_{\underline{U}_P}$ are all continuous then $\mathcal{Q} \square \mathcal{P}$ is a process. This introduces the following problem: if $\mathcal{P}$ is a uniform process and $(E,\underline{U})$ is an object of Υ when is $p_{\underline{U}}: E \longrightarrow P(E,\underline{U})$ continuous with respect to a topology $\underline{O}_E$ for E? One answer to this question is stated as

Theorem 4. *Let $\mathcal{P}$ be a uniform process that satisfies (UP_5) and let $(E,\underline{U})$ be an object of Υ . If $\underline{O}_E$ is a topology for E then $p_{\underline{U}}E \longrightarrow P(E,\underline{U})$ is continuous with respect to $\underline{O}_E$ when $\underline{O}_E$ is finer than the $\underline{U}$-uniform topology of E . In addition if $\mathcal{P}$ satisfies (UP_3) the converse holds.*

Proof: Since $p_{\underline{U}}$ is $(\underline{U},\underline{U}_p)$-uniformly continuous it is continuous with respect to the $\underline{U}$-uniform topology and the $\underline{U}_p$-uniform topology, which is the topology of $P(E,\underline{U})$. Furthermore when $(p_{\underline{U}} \times p_{\underline{U}})^{-1}\underline{U}_p = \underline{U}$ it follows that the inverse image of the topology of $P(E,\underline{U})$ under $p_{\underline{U}}$ is the $\underline{U}$-uniform topology. This proves the theorem.

Corollary. Let $\mathcal{P}$ and $\mathcal{Q}$ be two uniform processes. Then $\mathcal{Q}\square\mathcal{P}$ is a process when $\mathcal{Q}$ satisfies (UP_5) .

Proof: Since the uniformities $\underline{U}_p$ are compatible the $\underline{U}_p$-uniform topology is coarser than the topology of $P(E,\underline{U})$. The theorem shows that all the functions $q_{\underline{U}_p}$ are continuous and so $\mathcal{Q}\square\mathcal{P}$ is a process.

If $\mathcal{P}$ is a uniform process then $\mathcal{P}\square\mathcal{P}$ is defined. If it is a process isomorphic to $\mathcal{P}$ then $\mathcal{P}$ is said to be idempotent. As in the case of function processes two obvious questions arise. First, is $\mathcal{P}\square\mathcal{P}$ necessarily a process? Second, when $\mathcal{P}\square\mathcal{P}$ is a process is $\mathcal{P}$ idempotent?

For the purposes of this thesis the following theorem on idempotence of uniform processes will be sufficient.

Theorem 5. Let $\mathcal{P}$ be a uniform process that satisfies (UP_5), and (UP_7) . Then $\mathcal{P}$ is idempotent.

Proof: Since $\mathcal{P}$ satisfies (UP_5) $\mathcal{P}\square\mathcal{P}$ is a uniform process. Consider the following commutative diagram

$$\begin{tikzcd}
(E,\underline{U}) \arrow[r, "p_{\underline{U}}"] \arrow[d, "p_{\underline{U}}"'] & P(E,\underline{U}) \arrow[d, "P(p_{\underline{U}})"] \\
(P(E,\underline{U}),\underline{U}_p) \arrow[r, "p_{\underline{U}_p}"] & P(P(E,\underline{U}),\underline{U}_p).
\end{tikzcd}$$

Since $\mathcal{P}$ satisfies (UP_7) $P(p_{\underline{U}})$ is a homeomorphism. This is because $(p_{\underline{U}_P} \circ p_{\underline{U}})E$ is dense in $P(P(E,\underline{U}),\underline{U}_P)$.

The family $(P(p_{\underline{U}}))_{(E,\underline{U}) \text{ in } \Upsilon}$ of homeomorphisms defines an isomorphism of $\mathcal{P}$ into $\mathcal{P}\square\mathcal{P}$. Since $(p \square p)_{\underline{U}} = p_{\underline{U}_P} \circ p_{\underline{U}}$ the homeomorphisms $P(p_{\underline{U}})$ satisfy condition (2) of definition since the diagram is commutative. Let α be in $\mathrm{Hom}((E,\underline{U})(E',\underline{U}'))$. Then $P(p_{\underline{U}'}) \circ P(\alpha) = P(p_{\underline{U}'} \circ \alpha) = P(P(\alpha) \circ p_{\underline{U}}) = (P\square P)(\alpha) \circ P(p_{\underline{U}})$. Consequently the family of homeomorphisms defines an isomorphism of $\mathcal{P}$ into $\mathcal{P}\square\mathcal{P}$ and so $\mathcal{P}$ is idempotent.

Uniform processes may be related to function processes in the following natural way. Define the covariant functor $U: \Phi \longrightarrow \Upsilon$ by setting $U(E,S) = (E,\underline{U}(S))$ and $U(\alpha) = \alpha$ (this defines a covariant functor because if α is in $\mathrm{Hom}((E,S),(E',S))$ it is $(\underline{U}(S),\underline{U}(S'))$-uniformly continuous. Let $\mathcal{P}$ be a uniform process on Υ and consider the covariant functor $P \circ U: \Phi \longrightarrow \Sigma$ and the family $(p_{\underline{U}(S)})_{(E,S) \text{ in } \Phi}$ of functions $p_{\underline{U}(S)}: E \longrightarrow (P \circ U)(E,S) = P(E,\underline{U}(s))$. It is clear that they constitute a process on Φ which will be denoted by $\mathcal{P}_U$. <u>The function process $\mathcal{P}_U$ is said to be induced by the natural functor U and the uniform process $\mathcal{P}$</u>. In the case of the uniform processes in examples 1 and 2 the corresponding induced function processes are the examples 1,2 and 3 of section one in chapter one.

§3. <u>Process $\mathcal{S}$</u> . This process is the analogue of example 4 of a function process in chapter one. Let $(E,\underline{U})$ be an object of

$\mathcal{T}$ and let $C = \bigcap_{V \text{ in } \underline{U}} V$. The set C defines an equivalence relation on E which will also be denoted by C. Let $E_1 = E/C$ and let $\pi_1 : E \longrightarrow E_1$ be the natural mapping which associates with each x in E its C-equivalence class. Let $S(E,\underline{U})$ be the set E_1 and the quotient topology corresponding to the $\underline{U}$-uniform topology on E.

Assume that $(E',\underline{U}')$ is a second object and that α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$. Then since α is $(\underline{U},\underline{U}')$-uniformly continuous $(\alpha \times \alpha)^{-1}C'$ contains C and so (α_x, α_y) is in C' when (x,y) is in C. Let $S(\alpha) : E_1 \longrightarrow E_1'$ be the unique function for which $\pi_1' \circ \alpha = S(\alpha) \circ \pi_1$. (Since α is continuous with respect to the uniform topologies $S(\alpha)$ is a continuous function, i.e. $S(\alpha)$ is in $\mathrm{Hom}(S(E,\underline{U}), S(E',\underline{U}'))$.

When α' is in $\mathrm{Hom}((E',\underline{U}'),(E'',\underline{U}''))$ then $S(\alpha' \circ \alpha) = S(\alpha') \circ S(\alpha)$ because $S(\alpha') \circ S(\alpha) \circ \pi_1 = S(\alpha') \circ \pi_1' \circ \alpha = \pi_1'' \circ \alpha' \circ \alpha$. Consequently $S : \mathcal{T} \longrightarrow \Sigma$ is a covariant functor.

For each object $(E,\underline{U})$ in $\mathcal{T}$ let $s_{\underline{U}}$ be the natural mapping π_1. Then by definition $S(\alpha) \circ s_{\underline{U}} = s_{\underline{U}'} \circ \alpha$ and $s_{\underline{U}}E = S(E,\underline{U})$. This shows that <u>the functor S and the family $(s_{\underline{U}})_{(E,\underline{U}) \text{ in } \mathcal{T}}$ of functions $s_{\underline{U}}$ defines a uniform process $\mathcal{S}$.</u>

<u>Theorem 6.</u> <u>Process $\mathcal{S}$ satisfies (UP_3), (UP_4) and (UP_5). If $\mathcal{P}$ is any process on $\mathcal{T}$ that satisfies these three properties then there is a unique homomorphism γ of $\mathcal{S}$ into $\mathcal{P}$.</u>

Proof: Let $(E,\underline{U})$ be an object of $\mathfrak{T}$ and consider $s_{\underline{U}}:E \longrightarrow S(E,\underline{U})$. If V_S is in $\underline{U}_S$ then $(s_{\underline{U}} \times s_{\underline{U}})^{-1}V_S = V$ is in $\underline{U}$ and from the definition of $s_{\underline{U}}$, $C \circ V \circ C = V$ and $(s_{\underline{U}} \times s_{\underline{U}})V = V_S$.

The collection of surroundings V in $\underline{U}$ for which $C \circ V \circ C = V$ is a base for the uniformity $\underline{U}$ since $C \circ V \circ C$ contains V and is contained in V^3. If V is such a surrounding of $\underline{U}$ let $V_1 = (\pi_1 \times \pi_1)V$. The sets V_1 are a base for a uniformity on E_1 which is $\underline{U}_S$.

First, note that $V = (\pi_1 \times \pi_1)^{-1}V_1$. If (x,y) is such that $(\pi_1 x, \pi_1 y)$ is in V_1 there are points u and v in E such that (u,v) is in V and $\pi_1 u = \pi_1 x$, $\pi_1 v = \pi_1 y$. Therefore (x,y) is in $C \circ V \circ C = V$. Since $(\pi_1 \times \pi_1)^{-1}V_1$ contains V this proves the equality.

Second, $(V_1)^{-1} = (V^{-1})_1$, V_1 contains W_1 if V contains W and $V_1 \circ V_1 = (V \circ V)_1$. The first two assertions are clear and so is the fact that $(V \circ V)_1$ is contained in $V_1 \circ V_1$. Let (x_1,z_1) be in $V_1 \circ V_1$. Then there is a point y_1 in E_1 with (x_1,y_1) and (y_1,z_1) in V_1. Let $\pi_1 x = x_1$, $\pi_1 y = y_1$ and $\pi_1 z = z_1$. Since $V = (\pi_1 \times \pi_1)^{-1}V_1$ (x,y) and (y,z) are in V and so (x,z) is in $V \circ V$. Therefore $(V \circ V)_1$ contains $V_1 \circ V_1$ and the equality follows. This proves that the sets V_1 form a base for a uniformity on E which has $\underline{U}$ for its inverse image under $\pi_1 = s_{\underline{U}}$.

The natural mapping is uniformly continuous with respect to this new uniformity and as a result is continuous with respect to the corresponding uniform topology. Consequently this new uniformity is compatible. It follows that it is $\underline{U}_S$ by the remark following (UP_3).

The uniformity $\underline{U}_S$ is separated. If (x_1, y_1) is in each V_1 then (x,y) is in each V of $\underline{U}$ if $\pi_1 x = x_1$ and $\pi_1 y = y_1$. Consequently (x,y) is in C and so $\pi_1 x = \pi_1 y$ i.e. $x_1 = y_1$.

The uniformity $\underline{U}_S$ is a structure of $S(E,\underline{U})$. Let O_1 be an open set of the quotient topology for E_1 and let $O = \pi_1^{-1} O_1$. Then O is a C-saturated open set. Let x be in O and let V be a surrounding in $\underline{U}$ such that $C \circ V \circ C = V$ and $V[x]$ is contained in O. Then $\pi_1(V[x]) = V_1[\pi_1 x]$ which is contained in $\pi_1 O = O_1$. If (x,y) is in V then $(\pi_1 x, \pi_1 y)$ is in V_1 and so $V_1[\pi_1 x]$ contains $\pi_1(V[x])$. The opposite inclusion holds because $C \circ V \circ C = C$. This shows that the quotient topology for E_1 is the $\underline{U}_S$-uniform topology.

This completes the proof of the first assertion.

Let $\mathcal{P}$ be a uniform process on $\mathfrak{T}$. If there is a homomorphism γ of $\mathcal{S}$ into $\mathcal{P}$ it is unique as for each $(E,\underline{U})$ in $\mathfrak{T}$ $s_{\underline{U}}E = S(E,\underline{U})$. Furthermore for this same reason the argument of lemma 3 applies to show that a homomorphism γ of $\mathcal{S}$ into $\mathcal{P}$ is defined by a family of continuous functions that satisfy condition (2) of definition 2.

Assume $\mathcal{P}$ satisfies (UP_3), (UP_4) and (UP_5). Let $(E,\underline{U})$ be an object of $\mathfrak{T}$. Since $\mathcal{P}$ satisfies the first two of these properties $p_{\underline{U}}x = p_{\underline{U}}y$ iff (x,y) is in each surrounding of $\underline{U}$. Therefore there is a 1 - 1 function $\gamma(E,\underline{U}):S(E,\underline{U}) \longrightarrow P(E,\underline{U})$ such that $\gamma(E,\underline{U}) \circ s_{\underline{U}} = p_{\underline{U}}$ (since $s_{\underline{U}}E = S(E,\underline{U})$). Theorem 4 shows that $p_{\underline{U}}$ is continuous with respect to the $\underline{U}$-uniform topology. Since $\gamma(E,\underline{U}) \circ s_{\underline{U}} = p_{\underline{U}}$ it follows from the definition of a quotient topology that $\gamma(E,\underline{U})$ is continuous.

Remark. The second assertion of this theorem states that $\mathcal{S}$ satisfies a 'universal' property with respect to (UP_3), (UP_4), and (UP_5). It is clear that this 'universal' property defines the separation process $\mathcal{S}$ up to isomorphism. The uniform space $(S(E,\underline{U}),\underline{U}_S)$ is called the separated space associated with $(E,\underline{U})$.

An important property of this process is that it preserves completeness as stated in

Theorem 7. $(S(E,\underline{U}),\underline{U}_S)$ is a complete uniform space iff $(E,\underline{U})$ is complete.

Proof: Let $\underline{F}$ denote a $\underline{U}$-Cauchy filter on E and let $\underline{F}_1$ denote a $\underline{U}_S$-Cauchy filter on E_1. Since $s_{\underline{U}}$ is $(\underline{U},\underline{U}_S)$-uniformly continuous $s_{\underline{U}}\underline{F}$ is a $\underline{U}_S$-Cauchy filter on E_1 which converges if $\underline{F}$ converges in the $\underline{U}$-uniform topology.

On the other hand $s_{\underline{U}}^{-1}\underline{F}_1$ is a $\underline{U}$-Cauchy filter which converges if $\underline{F}_1$ converges. Since $\mathcal{S}$ satisfies (UP_3) $s_{\underline{U}}^{-1}\underline{F}_1$ is a $\underline{U}$-Cauchy filter. If $\underline{F}_1$ converges to a point of $S(E,\underline{U})$, say $s_{\underline{U}}x$, then $\underline{F}_1$ is finer than the neighbourhood filter $\mathcal{V}_1(s_{\underline{U}}x)$. This filter consists of sets $V_S[s_{\underline{U}}x]$, V_S in $\underline{U}_S$ and so $s_{\underline{U}}^{-1}\mathcal{V}_1(s_{\underline{U}}x) = \mathcal{V}(x)$ - the $\underline{U}$-uniform neighbourhood filter of x. This shows that $s_{\underline{U}}^{-1}\underline{F}_1$ converges if $\underline{F}_1$ converges.

Assume that $(E,\underline{U})$ is complete. Then $s_{\underline{U}}^{-1}\underline{F}_1$ converges in the $\underline{U}$-uniform topology and so does $s_{\underline{U}}(s_{\underline{U}}^{-1}\underline{F}_1)$ which contains $\underline{F}_1$. Hence $\underline{F}_1$ converges and so $(S(E,\underline{U}),\underline{U}_S)$ is a complete uniform space.

Assume that $(S(E,\underline{U}),\underline{U}_S)$ is complete then $s_{\underline{U}}\underline{F}$ converges and so also does $s_{\underline{U}}^{-1}(s_{\underline{U}}\underline{F})$ which is contained in $\underline{F}$. Therefore $\underline{F}$ converges and so $(E,\underline{U})$ is a complete space.

This section concludes with the following obvious result. <u>If $(E,\underline{U})$ is an object in $\mathfrak{T}$ then $s_{\underline{U}}$ is a homeomorphism iff $\underline{U}$ is a separated uniformity. Consequently $\mathfrak{S}$ is idempotent.</u> The first assertion since $s_{\underline{U}}$ is 1 - 1 iff it is a homeomorphism. The process $\mathfrak{S}$ is idempotent because $\mathfrak{S}\circ\mathfrak{S}$ is a process and each $s_{\underline{U}_S}$ is a homeomorphism.

§4. <u>The Cauchy filter process $\mathfrak{N}$.</u> If $(E,\underline{U})$ is an object of $\mathfrak{T}$ let $N(E,\underline{U})$ be the set of $\underline{U}$(Cauchy filters $\underline{F}$ on E together with the following topology for E: if F is any subset of E let $F^* = [\underline{F}$ in $N(E,\underline{U}) | F$ is in $\underline{F}]$; the sets F^* form a base for a topology for $N(E,\underline{U})$ since $(F_1 \cap F_2)^* = F_1^* \cap F_2^*$.

Let $(E',\underline{U}')$ be a second object of $\mathfrak{T}$ and let α be in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$. Define $N(\alpha)$ by setting $N(\alpha)\underline{F} = \alpha\underline{F}$. This is a $\underline{U}'$-Cauchy filter because α is $(\underline{U},\underline{U}')$-uniformly continuous. The function $N(\alpha)$ is continuous since $N^{-1}(\alpha)(F')^* = (\alpha^{-1}F')^*$ for any subset F' of E'.

If α' is in $\mathrm{Hom}((E',\underline{U}'),(E,\underline{U}))$ then $N(\alpha' \circ \alpha) = N(\alpha') \circ N(\alpha)$ since $(\alpha' \circ \alpha)\underline{F} = \alpha'(\alpha\underline{F})$ as both filters have a common base. Consequently $N: \Upsilon \longrightarrow \Sigma$ is a covariant functor.

When $(E,\underline{U})$ is an object of Υ define $n_{\underline{U}}: E \longrightarrow N(E,\underline{U})$ by setting $n_{\underline{U}}x$ equal to the filter $\dot{x}$ of sets that contain x. It is $\underline{U}$-Cauchy and if F is a subset of E $n_{\underline{U}}x$ is in F^{*} iff x is in F i.e. $F^{*} \cap n_{\underline{U}}E = n_{\underline{U}}F$.

Let α be in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$. Then $N(\alpha) \circ n_{\underline{U}} = n_{\underline{U}'} \circ \alpha$ because $\alpha(\dot{x}) = (\dot{\alpha x})$. This shows that <u>the functor N and the family $(n_{\underline{U}})_{(E,\underline{U}) \text{ in } \Upsilon}$ of functions $n_{\underline{U}}$ define a process which will be denoted by $\mathfrak{n}$.</u>

<u>Remarks.</u> For any object $(E,\underline{U})$ of Υ there are two obvious mappings of E into the set $N(E,\underline{U})$. One is the function $n_{\underline{U}}$ and the other is the mapping which corresponds the $\underline{U}$-uniform neighbourhood filter of x to each point x. If the second one is used the topology of $N(E,\underline{U})$ has to be defined in terms of the sets F^{*} where F is in the $\underline{U}$-uniform topology of E. Given this modified version of the functor N the second type of mappings fail to satisfy condition (UP_2). For this reason the functions $n_{\underline{U}}$ are the ones used to define $\mathfrak{n}$. These functions and the modified version of N also define a process.

For the purposes of this thesis, the properties of $\mathfrak{n}$ that are of interest are stated in

Theorem 8. $\mathfrak{n}$ is a T_0-process which satisfies (UP_3) and (UP_6). It is not idempotent.

Proof: Let $(E,\underline{U})$ be an object of $\mathcal{T}$ and let $\underline{F},\underline{F}'$ be two distinct $\underline{U}$-Cauchy filters on E. Then either $\underline{F}$ contains a set F not in $\underline{F}'$ or $\underline{F}'$ contains a set F' not in $\underline{F}$ and so there is a neighbourhood of one of these two filters that does not contain the other.

To prove that $\mathfrak{n}$ satisfies (UP_3) it is necessary and sufficient to find a compatible uniformity for $N(E,\underline{U})$ whose inverse image under $n_{\underline{U}}$ is $\underline{U}$.

If V is a symmetric surrounding of $\underline{U}$ let V_2 be the set of pairs of $\underline{U}$-Cauchy filters that share a V-small set. The family $(V_2)_{V \text{ symmetric in } \underline{U}}$ is a base for a uniformity for $N(E,\underline{U})$. The sets V_2 are symmetric and if V contains W then V_2 contains W_2. Also $V_2 \circ V_2$ is contained in $(V \circ V)_2$. Let $\underline{F}$, $\underline{F}'$ and $\underline{F}$ be three $\underline{U}$-Cauchy filters such that $\underline{F}$ and $\underline{F}'$ share the V-small set F and $\underline{F}'$ and $\underline{F}$ share the V-small set F'. Then $F \cap F' \neq \emptyset$ and if x is in $F \cap F'$ then $V[x]$ contains both F and F'. It is a $V \circ V$-small set common to $\underline{F}$ and $\underline{F}$ and so $V_2 \circ V_2$ is contained in $(V \circ V)_2$. This shows that the sets V_2 are a base for a uniformity for $N(E,\underline{U})$. It is clearly compatible because for any filter $\underline{F}$ in $N(E,\underline{U})$ and surrounding V_2 if F is a V-small set in $\underline{F}$ then F^* is contained in $V_2[\underline{F}]$.

The inverse image of this uniformity under $n_{\underline{U}}$ is $\underline{U}$. This is because $(n_{\underline{U}} \times n_{\underline{U}})^{-1} V_2 = V$. If $\dot{x}$ and $\dot{y}$ share a V-small set then (x,y) is in V. If (x,y) are in the symmetric surrounding V so are (x,x), (y,y) and (y,x). The set $[x,y]$ is therefore V-small and common to $\dot{x}$ and $\dot{y}$.

This shows that $\underline{U}_N$ is generated by the sets V_2, or more accurately, they form a base for $\underline{U}_N$ and also that $\mathfrak{n}$ satisfies (UP_3).

The uniform space $(N(E,\underline{U}),\underline{U}_N)$ is complete. Let $\underline{F}_2$ be a $\underline{U}_N$-Cauchy filter on $N(E,\underline{U})$ and let $\underline{F}'_2$ be the filter generated by the sets $V_2[F_2]$ where F_2 is in $\underline{F}_2$. It is a $\underline{U}_N$-Cauchy filter contained in $\underline{F}'_2$. Furthermore since $n_{\underline{U}}E$ is dense in $N(E,\underline{U})$ and $\underline{U}_N$ is compatible $\underline{F} = n_{\underline{U}}^{-1}\underline{F}'_2$ is a proper filter on E. Since $\mathfrak{n}$ satisfies (UP_3) it is a $\underline{U}$-Cauchy filter. The filter $n_{\underline{U}}\underline{F}$ on $N(E,\underline{U})$ converges to $\underline{F}$ since for any set F in $\underline{F}$ $n_{\underline{U}}F = n_{\underline{U}}E \cap F^{*}$ is a subset of F^{*}. This proves that $\underline{F}'_2$, which is contained in $n_{\underline{U}}\underline{F}$, also converges to $\underline{F}$ and hence that $\underline{F}_2$ converges to $\underline{F}$. Note that in the case of $\underline{F}'_2$ and $\underline{F}_2$ the convergence is with respect to the $\underline{U}_N$-uniform topology.

A cardinality argument may be used to prove that is not idempotent. Assume that $\mathfrak{n} \square \mathfrak{n}$ is a process and let $(E,\underline{U})$ be an object of $\mathfrak{T}$ such that E is an infinite set and $\underline{U}$ is totally bounded. Then every ultra filter on E is a $\underline{U}$-Cauchy filter. If $\underline{m}$ is the cardinal of E then there are $2^{2^{\underline{m}}}$ distinct ultrafilters on E (see Pospíšil [19]) and so the cardinal of $N(E,\underline{U})$ is greater than or equal to $2^{2^{m}}$. Lemma 4 shows that $\underline{U}_N$ is totally bounded and so this argument applies to show that $N(E,\underline{U})$ and $N(N(E,\underline{U}),\underline{U}_N)$ are not even equipotent, let alone homeomorphic.

Remarks. If the modified version of the functor N is used the theorem holds with the exception that $\mathfrak{n}$ is no longer a T_0-process. The process $\mathfrak{n}$ is of especial interest because it is a non-trivial example of a process that is not idempotent. The proof of this fact does not show that $\mathfrak{n} \square \mathfrak{n}$ is a process and so it could be simplified if this were not the case. Process $\mathfrak{n}$ suggests that another non-trivial process could be defined in the following way (either as a function or a uniform process): correspond to (E,S) or $(E,\underline{U})$ the set of ultrafilters on E with the topology analogous to that of $N(E,\underline{U})$ and map E into this set by corresponding to x the ultrafilter $\dot{x}$. If $\alpha: E \longrightarrow E'$ is any function it maps ultrafilters on E onto ultrafilters on E' since ultrafilters are characterized by the property that they contain a subset or its complement. It is not hard to see that this defines a Hausdorff (function or uniform) process which is again not idempotent. Both these examples are in a sense 'extravagant' processes. Are there non-idempotent processes which do not appear to be so 'extravagant'?

Corollary. $\mathfrak{S} \square \mathfrak{n}$ and $\mathfrak{S} \square \mathfrak{n} \square \mathfrak{S}$ are two uniform processes that satisfy (UP_3), (UP_4), (UP_5) and (UP_6).

Proof: Since $\mathfrak{S}$ satisfies (UP_3) and (UP_5) it follows that $\mathfrak{S} \square \mathfrak{N}$ is a process and that $\mathfrak{S} \square \mathfrak{N} \square \mathfrak{S}$ is a process if $\mathfrak{N} \square \mathfrak{S}$ is a process (see the corollary to theorem 4) . $\mathfrak{N} \square \mathfrak{S}$ is a process because for each object $(E,\underline{U})$ in $\mathfrak{T}$, $s_{\underline{U}}E = S(E,\underline{U})$ and so $(n_{\underline{U}_S} \circ s_{\underline{U}})E = n_{\underline{U}_S} S(E,\underline{U})$ which is dense in $N(S(E,\underline{U}),\underline{U}_S)$.

Both processes satisfy the first three properties because $\mathfrak{S}$ satisfies them (theorem 6) . They satisfy (UP_6) since $\mathfrak{N}$ does and because $\mathfrak{S}$ preserves completion (theorem 7) .

Remark. Exercise 5 on page 156 of Bourbaki [3] states that these two processes are isomorphic. This will be proved in section seven by showing that the four properties of this corollary characterize a class of isomorphic processes.

§5. U-completely regular filters. Let E be a set and let $\underline{U}$ be a uniformity for E . The filters on E that are of interest in this section are those that satisfy the condition of

Definition 3. A filter $\underline{F}$ on E is said to be $\underline{U}$-completely regular if F in $\underline{F}$ implies there is a set D in $\underline{F}$ and a surrounding V of $\underline{U}$ with $V[D]$ contained in F .

When the uniformity is defined by a collection S of functions on E an S-completely regular filter is a $\underline{U}(S)$-completely regular filter. This is an immediate consequence of definitions 3 and 4 of chapter one and the fact that if $\varepsilon = \lambda_1 - \lambda_2$ is positive $V(f,\varepsilon)[F_2]$ is contained in $\underline{F}_1$

when F_2 is contained in $[x|fx < \lambda_2]$ and F_1 contains $[x|fx < \lambda_1]$.

Examples of U-completely regular filters

1. If F is any subset of E the family $(V[F])_{V \text{ in } \underline{U}}$ of sets V[F] is a U-completely regular filter . In particular the U-uniform neighbourhood filters of the points of E are U-completely regular.

2. If U contains $\underline{U}_1$ then the $\underline{U}_1$-completely regular filters are U-completely regular. If S is the set of U-uniformly continuous real-valued functions then U(S) is contained in U and so every U(S)-completely regular filter (in particular every S-completely regular filter) is also a U-completely regular filter.

When ordered by inclusion it is clear that the U-completely regular filters form an inductive set and so by Zorn's lemma every U-completely regular filter is contained in a maximal U-completely regular filter. These maximal filters M have the property that for any U-completely regular filter F either M contains F or F and M do not generate a proper filter. This is a consequence of

Lemma 5. Let F and F' be two U-completely regular filters on E that generate a filter F on E . Then F is U-completely regular.

Proof: Let F be a set in F . Then there are sets F and F' in F and F' respectively such that $F \cap F'$ is in F and is contained in F . Let D and D' be subsets of F and F' which belong to $\underline{F}$ and $\underline{F}'$ and are such that there is a surrounding V in $\underline{U}$ with $V[D]$ contained in F and $V[D']$ contained in F'. The set $D \cap D'$ is in $\underline{F}$ and since $V[D] \cap V[D']$ contains $V[D \cap D']$ it follows that F contains $V[D \cap D']$. Hence $\underline{F}$ is $\underline{U}$-completely regular.

Corollary 1. If f is a $\underline{U}$-uniformly continuous real-valued function on E and $\underline{F}$ is a $\underline{U}$-completely regular filter then there exists a $\underline{U}$-completely regular filter $\underline{F}'$ containing $\underline{F}$ such that $\lim_{\underline{F}'} f$ exists in $\bar{R}$.

Proof: From example 2 it follows that if S is the collection of $\underline{U}$-uniformly continuous real-valued functions on E then the S-completely regular filters are $\underline{U}$-completely regular filters. Consequently the argument of theorem 6 in chapter one applies to prove the existence of such an $\underline{F}'$.

Corollary 2. If f is a $\underline{U}$-uniformly continuous real-valued function on E and $\underline{M}$ is a maximal $\underline{U}$-completely regular filter then $\lim_{\underline{M}} f$ exists in $\bar{R}$.

Proof: It is an immediate consequence of corollary 1.

This second corollary may be used to show that there are $\underline{U}(S)$-completely regular filters that are not S-completely regular. Consider the following

Example. Let E be the set R of real numbers and let $S = [e]$ where $ex = x$ for each x in E. The uniformity $\underline{U}(S)$ is the usual uniformity for R. The maximal S-completely regular filters are the neighbourhood filters of the point with respect to the usual topology and the two filters at 'infinity' generated by the families $((\leftarrow, -n))_n$ and $((n, \rightarrow))_n$. The function sin x is a $\underline{U}(S)$-uniformly continuous function which does not converge along the filters at 'infinity' . Consequently corollary 2 to lemma 5 shows that there are maximal $\underline{U}(S)$-completely regular filters which are not S-completely regular.

Given any filter $\underline{F}$ on E the sets $V[F]$, F in $\underline{F}$ and V in $\underline{U}$ generate a filter $\underline{U}[\underline{F}]$ which is contained in $\underline{F}$. This is obvious since $(V_1 \cap V_2)[F_1 \cap F_2]$ is contained in $V_1[F_1] \cap V_2[F_2]$. <u>The filter $\underline{U}[\underline{F}]$ is $\underline{U}$-completely regular and $\underline{F}$ is $\underline{U}$-completely regular iff $\underline{U}[\underline{F}] = \underline{F}$</u> . Let F be in $\underline{F}$ and V,W be surroundings of $\underline{U}$ such that V contains W^2 . Then $W[W[F]]$ is contained in $V[F]$ and so $\underline{U}[\underline{F}]$ is $\underline{U}$-completely regular. If $\underline{F}$ is $\underline{U}$-completely regular the sets $V[F]$, F in $\underline{F}$ and V in $\underline{U}$ form a base for $\underline{F}$ and so $\underline{F} = \underline{U}[\underline{F}]$. The converse is obvious. <u>If $\underline{F}'$ is $\underline{U}$-completely regular and $\underline{F}$ contains $\underline{F}'$ then $\underline{U}[\underline{F}]$ contains $\underline{F}'$</u> . It is clear that $\underline{U}[\underline{F}]$ contains $\underline{U}[\underline{F}']$, but this filter equals $\underline{F}'$.

<u>If $\underline{F}$ is a $\underline{U}$-Cauchy filter so is $\underline{U}[\underline{F}]$</u> . Let V and W be surroundings of $\underline{U}$ such that V contains W^3 . If F is a W-small set in $\underline{F}$ then $W[F]$ is a W^3-small set (and hence V-small) in $\underline{U}[\underline{F}]$. <u>If $\underline{F}'$ contains $\underline{F}$ and $\underline{F}$ is a $\underline{U}$-Cauchy filter then $\underline{U}[\underline{F}'] = \underline{U}[\underline{F}]$.</u> It is sufficient to prove that $\underline{U}[\underline{F}]$ contains $\underline{U}[\underline{F}']$. Let V and W be surroundings in $\underline{U}$ with W^2 contained in V and let F, F'

be W-small sets in $\underline{F}$ and $\underline{F}'$ respectively. If x is in $F \cap F'$ then $W^2[x]$ contains $W[F]$ and is contained in $V[F']$. Consequently $\underline{U}[\underline{F}]$ contains $\underline{U}[\underline{F}']$.

There are two immediate consequences of this last assertion. The first is that *a $\underline{U}$-Cauchy $\underline{U}$-completely regular filter is a maximal $\underline{U}$-completely regular filter.* The second is a characterization of the minimal $\underline{U}$-Cauchy filters: *a $\underline{U}$-Cauchy filter is a minimal $\underline{U}$-Cauchy filter iff it is $\underline{U}$-completely regular.* If $\underline{F}$ is a minimal $\underline{U}$-Cauchy filter then $\underline{F} = \underline{U}[\underline{F}]$ and so it is $\underline{U}$-completely regular. If $\underline{F}$ is a $\underline{U}$-Cauchy $\underline{U}$-completely regular filter and $\underline{F}$ contains a $\underline{U}$-Cauchy filter $\underline{F}'$ then $\underline{F} = \underline{U}[\underline{F}] = \underline{U}[\underline{F}']$ which is contained in $\underline{F}$. Hence $\underline{F} = \underline{F}'$ and so $\underline{F}$ is minimal.

Remark. This shows that if S is any collection of real-valued functions on E the maximal S-completely regular filters that are $\underline{U}(S)$-Cauchy are the maximal $\underline{U}(S)$-completely regular filters that are $\underline{U}(S)$-Cauchy.

The maximal $\underline{U}$-completely regular filters are characterized as follows: *a $\underline{U}$-completely regular filter $\underline{F}$ is maximal iff there exists an ultrafilter $\underline{F}_o$ containing $\underline{F}$ with $\underline{F} = \underline{U}[\underline{F}_o]$.* If $\underline{M}$ is a maximal $\underline{U}$-completely regular filter and $\underline{F}_o$ is any ultrafilter containing $\underline{M}$ then $\underline{U}[\underline{F}_o]$ is clearly equal to $\underline{M}$. Let $\underline{F}_o$ be an ultrafilter and assume that $\underline{F}$ is a $\underline{U}$-completely regular filter containing $\underline{U}[\underline{F}_o]$. Let F be a set in $\underline{F}$ that is not in $\underline{U}[\underline{F}_o]$. Then there is a set D in $\underline{F}$ and a symmetric surrounding V in $\underline{U}$ such that F

contains $V^3[D]$. The sets $V[D]$ and $V[\complement V^2[D]]$ are disjoint. Since F is not in $\underline{U}[F_o]$ it follows that $V^2[D]$ is not in F_o . Therefore $\complement V^2[D]$ is in F_o and so $V[\complement V^2[D]]$ is in $\underline{U}[F_o]$ and hence is in F . This is a contradiction. Consequently $F = \underline{U}[F_o]$ i.e. $\underline{U}[F_o]$ is a maximal $\underline{U}$-completely regular filter.

One consequence of this result is that <u>if $\underline{U}$ contains $\underline{U}_1$ and if M is a maximal $\underline{U}$-completely regular filter then $\underline{U}_1[M]$ is a maximal $\underline{U}_1$-completely regular filter.</u> Let F_o be an ultrafilter containing M . Then $\underline{U}_1[F_o]$ contains $\underline{U}_1[M]$ and since it is $\underline{U}$-completely regular is contained in M . Therefore $\underline{U}_1[M] = \underline{U}_1[F_o]$ and the assertion follows.

<u>A $\underline{U}$-completely regular filter has a basis of sets in the $\underline{U}$-uniform topology.</u> This is because with respect to the $\underline{U}$-uniform topology $\underline{U}$ has a base of open surroundings W and for any such surroundings the sets $W[F]$ are in the $\underline{U}$-uniform topology when F is a subset of E .

Let E' be a second set and let $\underline{U}'$ be a uniformity for E' . Denote by $\alpha: E \longrightarrow E'$ a $(\underline{U},\underline{U}')$-uniformly continuous function. In this situation the following lemma will be of use.

<u>Lemma 6. Assume that $(\alpha \times \alpha)^{-1}\underline{U}' = \underline{U}$. Let F be a filter on E . The following statements hold:</u>

(1) $\underline{U}[\underline{F}]$ is contained in $\alpha^{-1}(\alpha\underline{F})$;

(2) $\underline{U}[\alpha^{-1}(\alpha\underline{F})] = \alpha^{-1}(\underline{U}'[\alpha\underline{F}])$;

(3) $\underline{U}[\underline{F}] = \alpha^{-1}(\underline{U}'[\alpha\underline{F}])$.

Proof: Obviously (1) and (2) imply (3) since $\underline{U}[\underline{U}[\underline{F}]] = \underline{U}[\underline{F}]$.

Let F be a subset of E and assume αy is in αF . Then there is a point x in F with $\alpha x = \alpha y$. Therefore (x,y) belongs to every surrounding V of $\underline{U}$ and so V[F] contains $\alpha^{-1}(\alpha F)$. This proves (1) .

Let V be a surrounding of $\underline{U}$ and let V' be a surrounding of $\underline{U}'$ such that $(\alpha \times \alpha)^{-1}V' = V$. Then for any subset F of E, $\alpha^{-1}(V'[\alpha F]) = V[\alpha^{-1}(\alpha F)]$. If (x,y) is in V and x is in $\alpha^{-1}(\alpha F)$ then $(\alpha x, \alpha y)$ is in V' and αx is in αF . Consequently y is in $\alpha^{-1}(V'[\alpha F])$. Conversely if $(\alpha x, \alpha y)$ is in V' and αx is in αF then (x,y) is in V and x is in $\alpha^{-1}(\alpha F)$. This proves the equality from which (2) follows.

Corollary 1. *If $\underline{F}$ is $\underline{U}$-completely regular then $\underline{F} = \alpha^{-1}(\underline{U}'[\alpha\underline{F}]) = \alpha^{-1}(\alpha\underline{F})$.*

Proof: $\underline{U}[\underline{F}]$ is contained in $\alpha^{-1}(\alpha\underline{F})$ which in turn is contained in $\underline{F}$. If $\underline{F}$ is $\underline{U}$-completely regular $\underline{F} = \underline{U}[\underline{F}]$ and the corollary follows from (3) .

Corollary 2. *If $\underline{M}$ is a maximal $\underline{U}$-completely regular filter then $\underline{U}'[\alpha\underline{M}]$ is a maximal $\underline{U}'$-completely regular filter.*

Proof: Let $\underline{F}'$ be a $\underline{U}'$-completely regular filter containing $U'[\alpha\underline{M}]$. Since α is $(\underline{U},\underline{U}')$-uniformly continuous $\alpha^{-1}\underline{F}'$ is a $\underline{U}$-completely regular filter. Corollary 1 shows that it contains $\underline{M}$ and hence is equal to $\underline{M}$. Since $\alpha\underline{M} = \alpha(\alpha^{-1}\underline{F}')$ contains $\underline{F}'$ it follows that $\underline{F}' = \underline{U}'[\alpha\underline{M}]$ and so $\underline{U}'[\alpha\underline{M}]$ is a maximal $\underline{U}'$-completely regular filter.

The corollaries to this lemma are used to prove the following theorem which has an application in the next section analogous to the use of theorem 9 of chapter one in the definition of process $\mathfrak{M}$.

Theorem 9. Let $\alpha: E \longrightarrow E'$ be a $(\underline{U},\underline{U}')$-uniformly continuous function. If $\underline{M}$ is a maximal $\underline{U}$-completely regular filter on E then $\underline{U}'[\alpha\underline{M}]$ is a maximal $\underline{U}'$-completely regular filter on E'. When $\underline{M}$ is $\underline{U}$-Cauchy, $\underline{U}'[\alpha\underline{M}]$ is $\underline{U}'$-Cauchy. If in addition $(\alpha \times \alpha)^{-1}\underline{U}' = \underline{U}$ the correspondence $\underline{M} \longrightarrow \underline{U}'(\alpha\underline{M}]$ is $1 - 1$.

Proof: Let $\underline{U}_1 = (\alpha \times \alpha)^{-1}\underline{U}'$. Let $\underline{M}_1 = \underline{U}_1[\underline{M}]$. It is a maximal $\underline{U}_1$-completely regular filter as $\underline{U}_1$ is contained in $\underline{U}$.

The filter $\underline{U}'[\alpha\underline{M}_1]$ is a maximal $\underline{U}'$-completely regular filter by corollary 2. Since $\alpha\underline{M}$ contains $\alpha\underline{M}_1$ it follows that $\underline{U}'[\alpha\underline{M}_1] = \underline{U}'[\alpha\underline{M}]$ which proves the first assertion.

If $\underline{M}$ is $\underline{U}$-Cauchy then $\alpha\underline{M}$ is $\underline{U}'$-Cauchy as α is $(\underline{U},\underline{U}')$-uniformly continuous. Consequently $U'[\alpha\underline{M}]$ is $\underline{U}'$-Cauchy.

The last assertion is an immediate consequence of corollary 1 since under the additional assumption $\underline{M} = \alpha^{-1}(\underline{U}'[\alpha\underline{M}])$.

§6. Process $\mathcal{B}$. Let $(E,\underline{U})$ be an object of $\mathcal{T}$. Define $B(E,\underline{U})$ to be the set of maximal $\underline{U}$-completely regular filters $\underline{M}$ together with the topology generated by the sets O^*, O a subset of E in the $\underline{U}$-uniform topology, where $O^* = [\underline{M} \text{ in } B(E,\underline{U}) | O \text{ is in } \underline{M}]$. The sets O^* form a base for this topology since $O_1^* \cap O_2^* = (O_1 \cap O_2)^*$.

If $(E',\underline{U}')$ is a second object of $\mathcal{T}$ and α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ define $V(\alpha)$ by setting $B(\alpha)\underline{M} = \underline{U}'[\alpha\underline{M}]$ which by theorem 9 is in $B(E',\underline{U}')$. The function $B(\alpha)$: $B(E,\underline{U}) \longrightarrow B(E',\underline{U}')$ is continuous. Let O' be a subset of E' in the $\underline{U}'$-uniform topology and assume $B(\alpha)\underline{M}_0$ is in $(O')^*$, i.e. O' is in $B(\alpha)\underline{M}_0$. Let O_1' be an open set in $B(\alpha)\underline{M}_0$ and W' a surrounding of $\underline{U}'$ with O' containing $W'[O_1']$. If $\alpha^{-1}(O_1')$ is in $\underline{M}$ then O_1' is in $\alpha\underline{M}$ and so $W'[O_1']$ is in $\underline{U}'[\alpha\underline{M}] = B(\alpha)\underline{M}$. Therefore the set $B^{-1}(\alpha)(O')^*$ contains the neighbourhood $(\alpha^{-1}(O_1'))^*$ of $\underline{M}_0$ since α is continuous with respect to the uniform topologies. Hence $B(\alpha)$ is continuous.

Let α' be in $\mathrm{Hom}((E',\underline{U}'),(E'',\underline{U}''))$. Then $B(\alpha' \circ \alpha) = B(\alpha') \circ B(\alpha)$. If $\underline{M}$ is in $B(E,\underline{U})$ then $B(\alpha' \circ \alpha)\underline{M} = \underline{U}''[\alpha'(\alpha\underline{M})]$ which contains $\underline{U}''[\alpha'(\underline{U}'[\alpha\underline{M}])] = (B(\alpha') \circ B(\alpha))\underline{M}$. Since these filters are maximal they coincide and the identity holds. Consequently $B: \Upsilon \longrightarrow \Sigma$ is a covariant functor.

When $(E;\underline{U})$ is an object of Υ define $b_{\underline{U}}x$ to be the $\underline{U}$-uniform neighbourhood filter $\underline{M}_x$ of the point x. It is a $\underline{U}$-Cauchy filter and since it is also a $\underline{U}$-completely regular filter it is in $B(E,\underline{U})$. The set $b_{\underline{U}}E$ is dense in $B(E,\underline{U})$ because for any set O in the $\underline{U}$-uniform topology $O^* \cap b_{\underline{U}}E = b_{\underline{U}}O$.

If α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ then $\underline{U}'[\alpha\underline{M}_x] = \underline{M}'_{\alpha x}$. This is because $\alpha\underline{M}_x$ is a filter which generates a filter with $\underline{M}'_{\alpha x}$ and so $\underline{U}'[\alpha\underline{M}_x]$ and $\underline{M}'_{\alpha x}$ generate a proper filter, i.e. these two maximal filters coincide.

This shows that <u>the functor B and the family $(b_{\underline{U}})_{(E,\underline{U})\text{ in }\Upsilon}$ of functions $b_{\underline{U}}$ define a uniform process $\mathcal{B}$</u>.

The basic properties of this process are stated as

<u>Theorem 10.</u> <u>$\mathcal{B}$ is a compact process that satisfies (UP_4), (UP_5) (UP_6) and (UP_7). It is idempotent.</u>

Proof: It is idempotent by virtue of theorem 5 if it satisfies the first assertion.

Let $(E,\underline{U})$ be an object of Υ. The space $B(E,\underline{U})$ is Hausdorff. If $\underline{M}_1$ and $\underline{M}_2$ are two distinct maximal $\underline{U}$-completely regular filters then there are two disjoint sets O_1 and O_2 in the $\underline{U}$-uniform topology such that O_i is in $\underline{M}_i$. Since $O_1^* \cap O_2^* = (O_1 \cap O_2)^* = \phi^*$ which is void, it follows that $\underline{M}_1$ and $\underline{M}_2$ have disjoint neighbourhoods.

To show that every open cover has a finite subcover it is sufficient to consider open covers $\mathfrak{X}^*$ of $B(E,\underline{U})$ by sets O^* from the base of the topology.

Let $\mathfrak{X}^*$ be an open cover of $B(E,U)$ by sets O^*, where O is a subset of E in the $\underline{U}$-uniform topology. Assume that $\mathfrak{X}^*$ has no finite subcover. This implies that the cover $\mathfrak{X}$ of E by the sets O has no finite subcover. Therefore the sets $\complement O$, O in $\mathfrak{X}$ form a filter basis for a filter $\underline{F}$ on E.

This filter lies in no $\underline{U}$-completely regular filter on E since $\mathfrak{X}^*$ is a cover of $B(E,\underline{U})$. Let $\underline{F}_o$ be an ultrafilter containing $\underline{F}$. Then $\underline{U}[\underline{F}_o]$ does not contain $\underline{F}$. Consequently there are n sets $O_1,\ldots,O_n$ in $\mathfrak{X}$ such that $\bigcap_{i=1}^{n} \complement O_i$ is not in $\underline{U}[\underline{F}_o]$, i.e. if F is in $\underline{F}_o$ and V is in $\underline{U}$ then $V[F] \cap (\bigcup_{i=1}^{n} O_i) \neq \emptyset$. Since $\underline{U}$ has a basis of symmetric surroundings this is equivalent to $F \cap V[\bigcup_{i=1}^{n} O_i] \neq \emptyset$ i.e. $V[\bigcup_{i=1}^{n} O_i] = \bigcup_{i=1}^{n} V[O_i]$ is in $\underline{F}_o$. This leads to a contradiction in the following way.

If O^* is in $\mathfrak{X}^*$ and $\underline{M}$ is in O^* there is an open set O_1 in $\underline{M}$ such that for some V in $\underline{U}$ $V[O_1]$ is contained in O. Choose subsets O_1 of O in this way to obtain a refinement $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$ i.e. $\mathfrak{X}_1^*$ consists of sets O_1^* such that each O_1 is not only contained in some O in $\mathfrak{X}$ but there is a surrounding V of $\underline{U}$ such that $V[O_1]$ lies in some O in $\mathfrak{X}$. Assume $\mathfrak{X}_1^*$ has no finite subcover.

Let F_1 be the filter defined by $\mathfrak{X}_1$. It is contained in $\underline{F}$ and for the same reason as before $\underline{F}_1$ does not lie in $\underline{U}[\underline{F}_o]$. Therefore there are m sets $O_{11},\ldots,O_{1m}$ in $\mathfrak{X}_1$ such that $\bigcup_{i=1}^{m} V[O_{1j}]$ is in $\underline{F}_o$ for each V in $\underline{U}$. However there are sets O_j in and surroundings V_j in $\underline{U}$ such that O_j contains $V_j[O_{1j}]$ for $j = 1,\ldots,m$. Let $V = \bigcap_{j=1}^{m} V_j$. Then $\bigcup_{j=1}^{m} O_j$ contains $\bigcup_{j=1}^{m} V[O_{1j}]$ which is in $\underline{F}_o$. This is a contradiction since $\bigcap_{j=1}^{n} \complement O_j$ is in $\underline{F}_o$. Therefore $B(E,\underline{U})$ is compact.

The unique structure of $B(E,\underline{U})$ has as a basis the graphs $\bigcup_{i=1}^{n} (O_i^* \times O_i^*)$ of the finite open covers $O_1^*,\ldots,O_n^*$ of $B(E,\underline{U})$, where the sets O_i are in the $\underline{U}$-uniform topology. The inverse image under $b_{\underline{U}}$ of such a graph is the graph $\bigcup_{i=1}^{n} (O_i \times O_i)$ which is a subset of $E \times E$ containing the diagonal. The argument used to show that $B(E,\underline{U})$ is compact also proves that there is a refinement $O_{11}^*,\ldots,O_{1m}^*$ of this cover and a surrounding V of $\underline{U}$ such that each $V[O_{1j}]$ lies in some O_i. Since $O_{11},\ldots,O_{1m}$ is a cover of E this proves that $\bigcup_{i=1}^{n} (O_i \times O_i)$ contains V. Consequently $\underline{U}_B$ is the unique structure of $B(E,\underline{U})$ and so $\mathcal{B}$ automatically satisfies (UP_4), (UP_5) and (UP_6).

Let α be in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ and assume that $(\alpha \times \alpha)^{-1}\underline{U}' = \underline{U}$. Theorem 9 shows that $B(\alpha)$ is $1 - 1$. Let

O be a subset of E in the $\underline{U}$-uniform topology and assume that O is in the maximal $\underline{U}$-completely regular filter $\underline{M}_o$. Pick V' in $\underline{U}'$ an open surrounding such that if $V = (\alpha \times \alpha)^{-1}V'$ there is a set F in $\underline{M}_o$ with $V^2[F]$ contained in O. The set αF is in $\alpha\underline{M}_o$ and the open set $V'[\alpha F]$ is in $B(\alpha)\underline{M}_o$. Furthermore $\alpha^{-1}(V'[\alpha F]) = V[\alpha^{-1}(\alpha F)]$ which is contained in $V^2[F]$ (see the proof of lemma 6). Therefore $\underline{M}_o$ is in $B(\alpha)^{-1}(V'[\alpha F])^*$ which is contained in O^*. This shows that $B(\alpha)(O^*)$ is an open subset of the subspace $B(\alpha)B(E,\underline{U})$. Consequently $B(\alpha)$ is an embedding and since $\underline{B}$ is compact it follows that $\mathcal{B}$ satisfies (UP_7).

Remark. In general if $\mathcal{P}$ is a compact process that satisfies (UP_5) it also satisfies (UP_4) and (UP_6).

The 'universal' property that characterizes $\mathcal{B}$ is an immediate consequence of a corollary to

Theorem 11. Let E be a set and let $\underline{U}$ be a separated uniformity for E. Let $\underline{O}$ be the uniform topology and let E also denote the topological space $(E,\underline{O})$. Then $b_{\underline{U}}: E \longrightarrow B(E,\underline{U})$ is an embedding.

Proof: If O is in $\underline{O}$ then $O^* \cap b_{\underline{U}}E = b_{\underline{U}}O$ and so $b_{\underline{U}}$ is an embedding iff it is 1 - 1. Since $\underline{U}$ is a separated structure $b_{\underline{U}}$ is 1 - 1 and is therefore an embedding.

Corollary 1. $\underline{O}$ is a completely regular topology.

Proof: A compact space is completely regular and any subspace of a completely regular space is also completely regular.

Corollary 2. Let K be a compact space and let $\underline{U}^K$ be its unique structure. Then $b_{\underline{U}}K : K \longrightarrow B(K,\underline{U}^K)$ is a homeomorphism.

Proof: $b_{\underline{U}}K$ embeds K on a dense subset of the Hausdorff space $B(K,\underline{U}^K)$. Since K is compact it follows that $b_{\underline{U}}K$ maps K onto $B(K,\underline{U}^K)$.

An immediate consequence of corollary 2 is that $\mathcal{B}$ has the following 'universal' property for each object $(E,\underline{U})$ of $\mathcal{T}$: let K be a compact space and let $k : E \longrightarrow K$ be a $(\underline{U},\underline{U}^K)$-uniformly continuous function; then there exists a unique continuous function $k_B : B(E,\underline{U}) \longrightarrow K$ such that $k_B \circ b_{\underline{U}} = k$. Consider the following commutative diagram

$$\begin{tikzcd}
E \arrow[r, "b_{\underline{U}}"] \arrow[d, "k"'] & B(E,\underline{U}) \arrow[d, "B(k)"'] \\
K \arrow[r, "b_{\underline{U}}k"] & B(K,\underline{U}^k)
\end{tikzcd}$$

By corollary 2 $b_{\underline{U}}K$ is a homeomorphism and so $k_B = b_{\underline{U}}^{-1}K \circ B(k)$ is the desired function. It is unique because $b_{\underline{U}}E$ is dense in $B(E,\underline{U})$ and K is Hausdorff.

One application of this 'universal' property is the following result. If f is a bounded $\underline{U}$-uniformly continuous real-valued function on E there is a unique continuous real-valued function f_B on $B(E,\underline{U})$ such that $f_B \circ b_{\underline{U}} = f$. Consider the closure of fE. It is a compact space K with respect to which f plays the role of k. The result follows immediately.

This 'universal' property also shows that there is a homomorphism of the function process $\mathcal{B}_U$ into $\mathfrak{M}$. This is because for each object (E,S) in Φ the function $m_S : E \longrightarrow M(E,S)$ is $(\underline{U}(S), \underline{U}^{M(E,S)})$-uniformly continuous and so the continuous function $(m_S)_B : B(E,\underline{U}(S)) \longrightarrow M(E,S)$ is defined such that $(m_S)_B \circ b_{\underline{U}(S)} = m_S$. Since $\mathfrak{M}$ is a Hausdorff process lemma 1 of chapter one shows that the family $((m_S)_B)_{(E,S) \text{ in } \Phi}$ of functions $(m_S)_B$ defines a homomorphism of $\mathcal{B}_U$ into $\mathfrak{M}$. This homomorphism is not an isomorphism because of the example following corollary 2 of lemma 5 in section five.

The 'universal' property characterizes $\mathcal{B}$ as a compact process that satisfies (UP_5) as shown by

Theorem 12. Let $\mathcal{P}$ be a uniform process that is compact and satisfies (UP_5). Then there is a unique homomorphism γ of $\mathcal{B}$ into $\mathcal{P}$.

Proof: For each object $(E,\underline{U})$ in Υ the function $p_{\underline{U}} : E \longrightarrow P(E,\underline{U})$ is $(\underline{U},\underline{U}_P)$-uniformly continuous. Since $\mathcal{P}$ satisfiess (UP_5) and is compact $\underline{U}_P$ is the unique structure of $P(E,\underline{U})$. The 'universal' property defines a continuous function $(p_{\underline{U}})_B : B(E,\underline{U}) \longrightarrow P(E,\underline{U})$ with $(p_{\underline{U}})_B \circ b_{\underline{U}} = p_{\underline{U}}$. Define $\gamma(E,\underline{U})$ to be $(p_{\underline{U}})_B$. Then since $\mathcal{P}$ is a Hausdorff process lemma 3 shows that the family $(\gamma(E,\underline{U}))_{(E,\underline{U}) \text{ in } \Upsilon}$ defines a homomorphism γ of $\mathcal{B}$ into $\mathcal{P}$.

A compact process is a Hausdorff process and so the homomorphism γ is unique (in view of condition (2) of definition 2).

Concluding remarks. Samuel [13] was the first to demonstrate a compact extension of a separated uniform space $(E,\underline{U})$ with the 'universal' property. His construction was by means of a quotient of the compact space of ultrafilters on E. Two ultrafilters $\underline{F}_0$ and $\underline{F}'_0$ are identified if $\underline{U}[\underline{F}_0] = \underline{U}[\underline{F}'_0]$. The construction of this space by means of the filters $\underline{U}[\underline{F}_0]$ i.e. the construction of $B(E,\underline{U})$, is due to Banaschewski [20] (although he defined the topology in another way). It is perhaps worth noting that three compact extensions of the space R of real numbers may now be associated with the identity function e. They are $(M(R,[e]), m_{[e]})$, $(B(R,\underline{U}([e])), b_{\underline{U}([e])})$ and the Stone-Cech compactification $(\beta R,i)$. This last extension is associated with e since it is the Wallman extension of the normal space R which is characterized by the fact that sets with disjoint closures in R (i.e. $\overline{eF_1}$ and $\overline{eF_2}$ are disjoint) have disjoint closures in the compact extension. The Samuel compactification of R associated with $\underline{U}([e])$ is not a β-extension because there are bounded continuous real-valued functions that are not uniformly continuous (and hence have no extension to $B(E,\underline{U}([e]))$. These three distinct extensions correspond to three different ways of using functions to distinguish between sets. Let S be a collection of real-valued functions on a set E and let F_1 and F_2 be two subsets of E with

F_1 containing F_2. Then F_2 and $\complement F_1$ are said to be:

(1) <u>S-completely separated</u> if the conditions of definition 3 in chapter one are satisfied;

(2) <u>$\underline{U}$(S)-completely separated</u> if there is a surrounding V in $\underline{U}(S)$ such that F_1 contains $V[F_2]$; and

(3) <u>S-closure separated</u> if there is a finite number of functions, say $f_1,\dots,f_n$ in S such that $\bigcap_{i=1}^{n} \overline{f_i F_2}$ is contained in the interior of $\bigcap_{i=1}^{n} f_i F_1$ i.e. $\bigcap_{i=1}^{n} \overline{f_i F_2}$ and $\bigcap_{i=1}^{n} \overline{f_i \complement F_1}$ are disjoint.

These three types of separation are progressively stronger and when $E = R$ and $S = [e]$ they are mutually distinct since they define the above three distinct compact extensions of R.

§7. <u>Process $\mathfrak{G}$.</u> This process is defined by means of $\mathfrak{B}$ in a manner analogous to the way in which in chapter one $\mathfrak{F}$ is defined as a subprocess of $\mathfrak{M}$.

Let $(E,\underline{U}($ be an object of $\mathfrak{T}$. Define $G(E,\underline{U})$ to be the subspace of $B(E,\underline{U})$ consisting of the maximal $\underline{U}$-completely regular filters that are $\underline{U}$-Cauchy. If α is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ define $G(\alpha)$ to be $B(\alpha)|G(E,\underline{U})$. It maps $G(E,\underline{U})$ into $G(E',\underline{U}')$ in view of theorem 9. Furthermore if α' is in $\mathrm{Hom}((E,\underline{U}),(E',\underline{U}'))$ then $G(\alpha' \circ \alpha) = G(\alpha') \circ G(\alpha)$ by direct computation. Therefore $G\colon \mathfrak{T} \longrightarrow \Sigma$ is a covariant functor.

If $(E,\underline{U})$ is an object of $\mathfrak{T}$ then $b_{\underline{U}}E$ is contained in the set $G(E,\underline{U})$. Define $g_{\underline{U}}$ to be $b_{\underline{U}}$. It follows immediately that <u>the functor G and the family $(g_{\underline{U}})_{(E,\underline{U}) \text{ in } \mathfrak{T}}$ of functions $g_{\underline{U}}$ define a uniform process. Let $\mathfrak{G}$ denote this process.</u>

Since $\mathfrak{G}$ is a subprocess of $\mathfrak{B}$ it follows that <u>$\mathfrak{G}$ satisfies (UP_4), (UP_5) and (UP_7) .</u> The first two properties belong to $\mathfrak{G}$ since for any object $(E,\underline{U})$ in $\mathfrak{T}$ $\underline{U}_G$ obviously contains $\underline{U}_B | G(E,\underline{U})$. $\mathfrak{G}$ satisfies (UP_7) because it is a subprocess of a process with this property.

<u>Process $\mathfrak{G}$ satisfies (UP_3) and (UP_6) .</u> To prove this assertion an explicit description of $\underline{U}_G$ is obtained by essentially the same argument that is used to describe $\underline{U}_M$ in theorem 8.

Let $(E;\underline{U})$ be an object of $\mathfrak{T}$. If V is a symmetric surrounding of $\underline{U}$ let V_3 be the set of pairs of filters $\underline{M}$ in $G(E,\underline{U})$ which share a V-small set. The argument of theorem 8 shows that the family $(V_3)_{V \text{ symmetric in } \underline{U}}$ of sets V_3 is a base of a uniformity for $G(E,\underline{U})$.

This uniformity is compatible. Let V be a symmetric surrounding of $\underline{U}$ and let $\underline{M}$ be a point of $G(E,\underline{U})$. Then $\underline{M}$ contains a V-small set O in the $\underline{U}$-uniform topology and so $O^{*} \cap G(E,\underline{U})$ is contained in $V_3[\underline{M}]$. Therefore the corresponding uniform topology is coarser than the topology of $G(E,\underline{U})$.

This uniformity is $\underline{U}_G$ because its inverse image under $g_{\underline{U}}$ is $\underline{U}$. Let V and W be symmetric surroundings of $\underline{U}$

such that V contains W^2 and such that W is open with respect to the $\underline{U}$-uniform topology. If $(g_{\underline{U}}x, g_{\underline{U}}y)$ is in V_3 the neighbourhood filters share a V-small set and so (x,y) is in V. On the other hand if (x,y) is in W the V-small open set $W[x]$ contains x and y and so $(g_{\underline{U}}x, g_{\underline{U}}y)$ is in V_3. Therefore $(g_{\underline{U}} \times g_{\underline{U}})^{-1}V_3$ contains W and is contained in V. This proves that the inverse image under $g_{\underline{U}}$ of the uniformity generated by the sets V_3 is equal to $\underline{U}$.

The uniform space $(G(E,\underline{U}),\underline{U}_G)$ is complete. Let $\underline{F}_3$ be a $\underline{U}$-Cauchy filter on $G(E,\underline{U})$ and let $\underline{F}_3'$ be the filter generated by the sets $V_3[F_3]$ where F_3 is in $\underline{F}_3$ and V_3 is the surrounding corresponding to a symmetric V in $\underline{U}$. It is a $\underline{U}_G$-Cauchy filter and $\underline{F} = g_{\underline{U}}^{-1} \underline{F}_3'$ is a $\underline{U}$-Cauchy filter. If the filter $g_{\underline{U}}\underline{F}$ converges to $\underline{U}[\underline{F}]$ then since it is finer than $\underline{F}_3'$ both $\underline{F}_3'$ and $\underline{F}_3$ converge.

The filter $g_{\underline{U}}\underline{F}$ is finer than $g_{\underline{U}}(\underline{U}[\underline{F}])$. If O is an open set in $\underline{U}[\underline{F}]$ (with respect to the $\underline{U}$-uniform topology) then $g_{\underline{U}}O = O^{*} \cap g_{\underline{U}}E$ and so $g_{\underline{U}}(\underline{U}[\underline{F}])$ converges to $\underline{U}[\underline{F}]$. This proves that $g_{\underline{U}}\underline{F}$ converges. Consequently $\mathcal{G}$ satisfies (UP_6) and so the following theorem has been proved

<u>Theorem 13. Process $\mathcal{G}$ satisfies (UP_3), (UP_4), (UP_5), (UP_6) and (UP_7). Consequently it is idempotent.</u>

<u>Corollary 1. If $(E,\underline{U})$ is an object of $\mathcal{T}$ $G(E,\underline{U}) = B(E,\underline{U})$ iff $\underline{U}$ is totally bounded.</u>

Proof: If $G(E,\underline{U}) = B(E,\underline{U})$ then $\underline{U}_G = \underline{U}_B$ which is totally bounded. Then by lemma 4 $\underline{U}$ is totally bounded.

If $\underline{U}$ is totally bounded lemma 4 shows that $\underline{U}_G$ is totally bounded. Then by theorem 1 $G(E,\underline{U})$ is compact and as it is a dense subset of $B(E,\underline{U})$ it follows that $G(E,\underline{U})$ $= B(E,\underline{U})$.

<u>Remark.</u> This corollary is analogous to the result in chapter which shows that $\mathfrak{J}$ and $\mathfrak{M}$ coincide on Φ^* .

<u>Corollary 2. If $(E,\underline{U})$ is an object of $\mathcal{T}$ then $\underline{U}_G$ is σ-bounded iff $\underline{U}$ is σ-bounded. Consequently $G(E,\underline{U})$ is a Q-space if $\underline{U}$ is σ-bounded.</u>

Proof: If $\underline{U}_G$ is σ-bounded it is obvious that $\underline{U}$ is σ-bounded. Assume that $\underline{U}$ is σ-bounded and let V and W be two symmetric surroundings of $\underline{U}$ such that V contains W^3 . Assume also that W is open with respect to the $\underline{U}$-uniform topology. Let $(F_n)_n$ be a sequence of W-small sets that covers E . If $\underline{M}$ is in $G(E,\underline{U})$ let F be a W-small set in $\underline{M}$. Then $F \cap F_{n_o} \neq \phi$ for some n_o and so $\underline{M}$ contains the open set $O_{n_o} = W[F_{n_o}]$. These sets are V-small and the sequence $(O_n^* \cap G(E,\underline{U}))_n$ is a cover of $G(E,\underline{U})$ by V_3-small sets. Hence $\underline{U}_G$ is σ-bounded.

The second assertion is an immediate consequence of Shirota's theorem (theorem 2) .

Another property of $\mathfrak{G}$ which is a consequence of the fact that it is a subprocess of $\mathfrak{B}$ is stated as

Theorem 14. Let E be a completely regular space and let $\underline{U}$ be a compatible uniformity for E. Then $g_{\underline{U}}: E \longrightarrow G(E,\underline{U})$ is an embedding iff $\underline{U}$ is a structure of E.

Proof: Theorem 11 shows that $g_{\underline{U}}$ is an embedding with respect to the $\underline{U}$-uniform topology iff $\underline{U}$ is a separated uniformity. It is therefore an embedding of the space E if $\underline{U}$ is a structure of E. On the other hand if $g_{\underline{U}}$ is an embedding then $\underline{U}$ is a separated uniformity and so the $\underline{U}$-uniform topology is the topology of E i.e. $\underline{U}$ is a structure of E.

Corollary. Let E be a completely regular space and let $\underline{U}$ be a structure of E. Then $g_{\underline{U}}: E \longrightarrow G(E,\underline{U})$ is a homeomorphism iff E is complete in $\underline{U}$.

Proof: Since E is a structure of E the theorem shows that $g_{\underline{U}}$ is an embedding. It is also an isomorphism of the uniform space $(E,\underline{U})$ with the space $(g_{\underline{U}}E, \underline{U}_G | g_{\underline{U}}E)$ since it is 1 - 1. If $g_{\underline{U}}$ is a homeomorphism theorem 13 shows that E is complete in $\underline{U}$. Conversely if E is complete in $\underline{U}$ then $g_{\underline{U}}E$ is a closed set as it is complete in the restriction of $\underline{U}_G$. Hence $g_{\underline{U}}E = G(E,\underline{U})$ and so $g_{\underline{U}}$ is a homeomorphism.

An immediate consequence of this consequence of this corollary is that $\mathfrak{G}$ has the following 'universal' property for each object $(E,\underline{U})$ in $\mathfrak{T}$: let $(F,\underline{V})$ be a complete separated uniform space and let $\alpha: E \longrightarrow F$ be a $(\underline{U},\underline{V})$-uniformly continuous function; then there exists a unique continuous function $\alpha_G: G(E,\underline{U}) \longrightarrow \underline{V}$ such that $\alpha_G \circ g_{\underline{U}} = \alpha$. Consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "g_{\underline{U}}"] \arrow[d, "\alpha"'] & G(E,\underline{U}) \arrow[d, "G(\alpha)"] \\ F \arrow[r, "g_{\underline{V}}"] & G(F,\underline{V}). \end{tikzcd}$$

The corollary shows that $g_{\underline{V}}$ is a homeomorphism and hence $\alpha_G = g_{\underline{V}}^{-1} \circ G(\alpha)$ is the desired function. It is unique because F is a Hausdorff space. *The function α_G is $(\underline{U}_G,\underline{V})$-uniformly continuous* because $G(\alpha)$ is $(\underline{U}_G,\underline{V}_G)$-uniformly continuous and $g_{\underline{V}}$ is an isomorphism of $(F,\underline{V})$ with $(G(F,\underline{V}),\underline{V}_G)$.

As in the case of process B this 'universal' property may be applied to extend uniformly continuous real-valued functions. *If f is a $\underline{U}$-uniformly continuous real-valued function on E there is a unique continuous real-valued function f_G on $G(E,\underline{U})$ such that $f_G \circ g_{\underline{U}} = f$. This function is $\underline{U}_G$-uniformly continuous.* These assertions follow immediately from the 'universal' property since the set of real numbers is complete in its usual uniformity which is separated.

From this it follows that for any object $(E,\underline{U})$ of *$\underline{U}_G$ is a function uniformity iff $\underline{U}$ is a function uniformity.* Assume that $\underline{U}_G$ is a function uniformity. Then since $\underline{U}$ is the inverse image of $\underline{U}_G$ under $g_{\underline{U}}$ the 'restrictions' of the functions that define $\underline{U}_G$ (by means of $g_{\underline{U}}$) is a collection S of functions such that $\underline{U} = \underline{U}(S)$. Conversely assume that

$\underline{U} = \underline{U}(S)$. Then each f in S has a $\underline{U}_G$-uniformly continuous 'extension' f_G to $G(E,\underline{U})$. If $S_G = [f_G | f \text{ in } S]$ then $\underline{U}_G$ contains the compatible uniformity $\underline{U}(S_G)$. These two uniformities coincide since the inverse image of $\underline{U}(S_G)$ under $g_{\underline{U}}$ is $\underline{U}(S_G \circ g_{\underline{U}}) = \underline{U}(S) = \underline{U}$.

As an immediate consequence $\mathcal{G}_U$ is isomorphic to $\mathcal{J}$. Since $(\underline{U}(S))_G = \underline{U}(S_G)$ it follows from theorem 13 that $\mathcal{G}_U$ satisfies $(FP_3),(FP_4),(FP_5)$ and (FP_6) . Therefore by theorem 5 of chapter one $\mathcal{G}_U$ and $\mathcal{J}$ are isomorphic.

To return to the 'universal' property, consider the following theorem which characterizes $\mathcal{G}$.

Theorem 15. Let $\mathcal{P}$ be a uniform process that satisfies (UP_4), (UP_5), and (UP_6) . Then there is a unique homomorphism γ of $\mathcal{G}$ into $\mathcal{P}$. If $\mathcal{P}$ satisfies (UP_3) this homomorphism is an isomorphism.

Proof: Since $\mathcal{P}$ satisfies UP_4 and UP_6 for any object $(E,\underline{U})$ of $\mathcal{T}$ $(P(E,\underline{U}),\underline{U}_P)$ is a complete separated uniform space. The 'universal' property defines the $(\underline{U}_G,\underline{U}_P)$-uniformly continuous function $(p_{\underline{U}})_G : G(E,\underline{U}) \longrightarrow P(E,\underline{U})$. Since $\underline{U}_P$ is a structure of $P(E,\underline{U})$ this function is continuous. Let $\gamma(E,\underline{U}) = (p_{\underline{U}})_G$. Then as $\gamma(E,\underline{U}) \circ g_{\underline{U}} = p_{\underline{U}}$ and $\mathcal{P}$ is Hausdorff, lemma 3 shows that the family $(\gamma(E,\underline{U}))_{(E,\underline{U}) \text{ in } \mathcal{T}}$ of functions $\gamma(E,\underline{U})$ defines a homomorphism γ of $\mathcal{G}$ into $\mathcal{P}$. It is unique because $\mathcal{P}$ is Hausdorff.

Assume that $\mathcal{P}$ also satisfies (UP_3) . Consider the following commutative diagram for any object $(E,\underline{U})$ of $\mathfrak{T}$

$$\begin{tikzcd}
(E,\underline{U}) \arrow[r, "g_{\underline{U}}"] \arrow[d, "p_{\underline{U}}"'] & \mathcal{G}(E,\underline{U}) \arrow[d, "\mathcal{G}(p_{\underline{U}})"] \\
(P(E,\underline{U}),\underline{U}_P) \arrow[r, "g_{\underline{U}_P}"] & \mathcal{G}(P(E,\underline{U}),\underline{U}_P).
\end{tikzcd}$$

The corollary to theorem 14 shows that $g_{\underline{U}_P}$ is a homeomorphism and so $\gamma(E,\underline{U}) = g_{\underline{U}_P}^{-1} \circ G(p_{\underline{U}})$. Since the inverse image of U_P under $p_{\underline{U}}$ is $\underline{U}$ the fact that $\mathcal{G}$ satisfies (UP_7) shows that $G(p_{\underline{U}})$ is an embedding. It is also a homeomorphism because the closed image of $G(E,\underline{U})$ under the embedding $G(p_{\underline{U}})$ contains the dense set $(g_{\underline{U}_P} \circ p_{\underline{U}})E$.

Consequently $\gamma(E,\underline{U})$ is a homeomorphism and so γ is an isomorphism.

<u>Corollary.</u> <u>$\mathcal{G}$ is isomorphic to $\mathcal{S} \square \mathcal{N}$ and $\mathcal{S} \square \mathcal{N} \square \mathcal{S}$.</u>

Proof: It is an immediate consequence of this theorem and the corollary to theorem 8.

<u>Remarks.</u> When $(E,\underline{U})$ is a separated uniform space the uniform space $(G(E,\underline{U}),\underline{U}_G)$ and the embedding $g_{\underline{U}}$ constitute a completion of $(E,\underline{U})$. A completion of $(E,\underline{U})$ is a complete separated uniform space $(X,\underline{V})$ and a function $\alpha:E \longrightarrow X$ which is a dense embedding with respect to the uniform topologies and such that $(\alpha \times \alpha)^{-1}\underline{V} = \underline{U}$; i.e. such that α is an isomorphism of $(E,\underline{U})$ with $(\alpha E,\underline{V}|\alpha E)$. The argument used in the last part of theorem 15 may be used to show that given a pair of completions $((X_i,\underline{V}_i),\alpha_i)$ $i = 1,2$ of $(E,\underline{U})$ there is a homeomorphism $\gamma_{21}:X_1 \longrightarrow X_2$ which is a $(\underline{V}_1,\underline{V}_2)$-isomor-

phism. This uniqueness is usually proved by means of the following well known theorem on the extension of uniformly continuous functions,

Theorem 16. Let $(X,\underline{V})$, $(Y,\underline{U})$ and $(E,\underline{W})$ be three uniform spaces and let $\alpha : E \longrightarrow Y$ be such that $(\alpha \times \alpha)^{-1}\underline{U} = \underline{W}$ and αE is dense in Y with respect to the $\underline{U}$-uniform topology. If $\gamma : E \longrightarrow X$ is a $(\underline{W},\underline{V})$-uniformly continuous function then there is a unique continuous function $\gamma : Y \longrightarrow X$ such that $\gamma_\alpha \circ \alpha = \gamma$ when $(X,\underline{V})$ is a complete separated uniform space. The function γ_α is $(\underline{U}\underline{V})$-uniformly continuous.

Proof: Consider the following commutative diagram

$$\begin{tikzcd}
X & E \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & Y \\
G(X,\underline{V}) \arrow[from=u, "g_{\underline{V}}"'] & G(E,\underline{W}) \arrow[l, "G(\gamma)"'] \arrow[r, "G(\alpha)"] \arrow[from=u, "g_{\underline{W}}"'] & G(Y,\underline{U}) \arrow[from=u, "g_{\underline{U}}"']
\end{tikzcd}$$

The corollary to theorem 14 shows that $g_{\underline{V}}$ is a homeomorphism (and hence a $(\underline{V},\underline{V}_G)$-isomorphism). Since $g_{\underline{U}}$ is continuous with respect the $\underline{U}$-uniform topology by theorems 4 and 13 it follows that $(g_{\underline{U}} \circ \alpha)E$ is dense in $G(Y,\underline{U})$. This shows that $G(\alpha)$ is a homeomorphism since $\mathcal{G}$ satisfies (UP_7). The function $g_{\underline{V}}^{-1} \circ G(\gamma) \circ G(\alpha)^{-1} \circ g_{\underline{U}}$ is the desired continuous function γ_α. It is unique because X is a Hausdorff space.

The function γ_α is $(\underline{U},\underline{W})$-uniformly continuous because $G(\alpha)$ is a $(\underline{W}_G,\underline{U}_G)$-isomorphism and $G(\gamma)$ is $(\underline{W}_G,\underline{V}_G)$-uniformly continuous.

<u>Corollary. If a uniform process $\mathcal{P}$ satisfies (UP_3) the function process $\mathcal{P}_U$ induced by U satisfies (FP_3) .</u>

Proof: Let (E,S) be an object of Φ . Let $\underline{W} = \underline{U}(S)$ and let $(Y,\underline{U}) = (P(E,\underline{U}(S)),(\underline{U}(S))_P)$ and $\alpha = p_{\underline{U}}(S)$. In place of X and γ consider the real numbers R and a function in S . The corollary follows immediately.

To continue these extended remarks it is worth noticing the close similarity of this theorem to theorem 20 of chapter two. The statement and proof in each case are essentially the same. This suggests that to every process $\mathcal{P}$ (function or uniform) there corresponds an analogous extension which may be proved in an analogous way. The proof may be broken in two - each half referring to each 'cell' of the diagram. In other words the extension theorem is an immediate consequence of an analogue to the corollary of theorem 14 and a condition on the mappings α which ensures that $P(\alpha)$ is a homeomorphism.

This suggestion raises the following problem. Is there a process $\mathcal{P}$ (function, uniform or perhaps of a third type) whose extension theorem is the well known extension theorem for continuous functions valued in a regular space (see Bourbaki [3] p54)?

In the case of process $\mathcal{B}$ the extension theorem is the theorem obtained from theorem 16 by assuming in addition that X is compact.

CHAPTER FOUR

THE TOPOLOGICALLY COMPLETE EXTENSIONS OF A COMPLETELY REGULAR SPACE

§1. The construction of topologically complete extensions. Let E be a completely regular space and let (X,i) be an extension of E. It will be said to be a topologically complete extension of E if X is complete in some structure or equivalently if X is complete in $\underline{U}^X$ -its finest structure.

The corollary to theorem 14 in chapter three shows that process $\mathcal{G}$ applied to the objects $(E,\underline{U})$, where $\underline{U}$ is a structure of E, yields the topologically complete extensions $(G(E,\underline{U}),g_{\underline{U}})$ of E. This collection of extensions is a representative collection of topologically complete extensions in view of

Theorem 1. Let (X,i) be a topologically complete extension of E and let $\underline{W}$ be a structure of X in which it is complete. If $\underline{W}|E = (i \times i)^{-1}\underline{W}$ it is a structure of E and the extensions (X,i) and $(G(E,\underline{W}|E),g_{\underline{W}|E})$ are isomorphic.

Proof: $\underline{W}|E$ is a structure of E because i is a homeomorphic embedding. Consequently $(G(E,\underline{W}|E),g_{\underline{W}|E})$ is an extension of E. The function i is $(\underline{W}|E,\underline{W})$-uniformly continuous and so by the universal property of $\mathcal{G}$ has an extension as a continuous function $i_G : G(E,\underline{W}|E) \longrightarrow X$ such that $i_G \circ g_{\underline{W}|E} = i$.

On the other hand theorem 16 of chapter three shows that there is a continuous function $(g_{\underline{W}|E})_i : X \longrightarrow G(E,\underline{W}|E)$ such that $(g_{\underline{W}|E})_i \circ i = g_{\underline{W}|E}$. Consequently the extensions are isomorphic.

Remark. The proof of this theorem shows that the homeomorphism: $X \longrightarrow G(E,\underline{W}|E)$ which extends i is also an isomorphism of $(X,\underline{W})$ with $(G(E,\underline{W}|E),(\underline{W}|E)_G)$. If the extension $(G(E,\underline{U}),g_{\underline{U}})$ is said to be obtained by completing E in the structure $\underline{U}$, then this theorem states that every topologically complete extension of E may be obtained by completing E in some structure.

The compact extensions, and more generally the Q-extensions, of E are particular types of topologically complete extensions of E. Let $\underline{U}$ be a totally bounded structure of E. Then corollary 1 of theorem 13 in chapter three states that $(G(E,\underline{U}),g_{\underline{U}})$ is a compact extension of E. Process $\mathfrak{G}$ applied to the objects $(E,\underline{U})$, where $\underline{U}$ is a totally bounded structure of E, yields 'all' the compact extensions of E. This is a consequence of theorem 1 and the fact that if X is compact and $\underline{W}$ is its unique structure, then $\underline{W}|E$ is totally bounded.

Similarly process $\mathfrak{G}$ applied to the objects $(E,\underline{U})$, where $\underline{U}$ is a σ-bounded structure of E, yields 'all' the Q-extensions of E. Corollary 2 of theorem 13 in chapter three shows that $(G(E,\underline{U}),g_{\underline{U}})$ is a Q-extension if $\underline{U}$ is σ-bounded. Since X is a Q-space iff it is complete in some σ-bounded structure $\underline{W}$ (theorem 2 of chapter three) and since $\underline{W}|E$ is then σ-bounded the result follows.

The totally bounded structures of E may be described in terms of the characteristic algebras $\underline{c}$ of E. If $\underline{c}$ is a characteristic algebra of E then $\underline{U}(\underline{c})$ is a totally bounded structure of E. The compact extensions $(G(E,\underline{U}(\underline{c}), g_{\underline{U}(\underline{c})})$ and $(T(E,\underline{c}), t_{\underline{c}})$ are isomorphic since $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{J}$ are isomorphic function processes. The extension $(T(E,\underline{c}), t_{\underline{c}})$ is isomorphic to the extension $(K(\underline{c}), \pi(\underline{c}) \circ i)$ since $\underline{c}_T = C_{T(E,\underline{c})}$.

If $\underline{U}$ is a totally bounded structure of E then $G(E,\underline{U}) = B(E,\underline{U})$. Consequently $\underline{U} = (b_{\underline{U}} \times b_{\underline{U}})^{-1}\underline{U}(C_{B(E,\underline{U})})$ $= \underline{U}(C_{B(E,\underline{U})} \circ b_{\underline{U}})$. If $\underline{c} = C_{B(E,\underline{U})} \circ b_{\underline{U}}$ it is a characteristic algebra of E since $(B(E,\underline{U}), b_{\underline{U}})$ is a compact extension of E(theorem 11 of chapter three). This completes the proof of

Theorem 2. A structure $\underline{U}$ of E is totally bounded iff there exists a characteristic algebra $\underline{c}$ of E such that $\underline{U} = \underline{U}(\underline{c})$. The extensions $(G(E,\underline{U}(\underline{c})), g_{\underline{U}(\underline{c})})$ and $(K(\underline{c}), \pi(\underline{c}) \circ i)$ are isomorphic.

Remark. The proof of this theorem shows that a uniformity $\underline{U}$ for a set E is totally bounded iff there is a uniformly closed unitary subalgebra $\underline{a}$ of F_E^* such that $\underline{U} = \underline{U}(\underline{a})$. Furthermore $\underline{a}$ is uniquely determined by $\underline{U}$.

This theorem shows that the correspondence $\underline{U} \longrightarrow (G(E,\underline{U}), g_{\underline{U}})$-which associates with each structure $\underline{U}$ of E the completion of E in $\underline{U}$ - is 1 - 1 when restricted to the totally bounded structures of E . This is not the case for the correspondence itself as shown by the following

Example. Let R be the space of real numbers. It is a Q-space and therefore complete in $\underline{U}(C_R)$. It is also complete in its usual uniformity $\underline{U}([e])$ where $e:R \longrightarrow R$ is the identity function. Therefore $G(R,\underline{U}(C_R)) = G(R,\underline{U}([e]))$ which equals the set of neighbourhood filters of the points of R . Consequently the two extensions are identical. However $\underline{U}(C_R) \neq \underline{U}([e])$ because there are continuous functions, for example $e^x \sin x$, which are not $\underline{U}([e])$-uniformly continuous.

This correspondence therefore defines an equivalence relation $\underline{e}$ on the set of structures of E . If $\underline{U}_1$ and $\underline{U}_2$ are two structures set $\underline{U}_1 \,\underline{e}\, \underline{U}_2$ if the sets $G(E,\underline{U}_1)$ and $G(E,\underline{U}_2)$ are identical i.e. if the minimal $\underline{U}_1$-Cauchy filters are the minimal $\underline{U}_2$-Cauchy filters. It is clear that $\underline{e}$ is an equivalence relation and that $\underline{U}_1 \,\underline{e}\, \underline{U}_2$ iff the corresponding completions of E are identical.

This equivalence relation is of especial interest because of

Lemma 2. Let $\underline{U}_1$ and $\underline{U}_2$ be two structures of E such that the extensions $(G(E,\underline{U}_1), g_{\underline{U}_1})$ and $(G(E,\underline{U}_2), g_{\underline{U}_2})$ are isomorphic. Then the sets $G(E,\underline{U}_1)$ and $G(E,\underline{U}_2)$ are identical and so the extensions are identical.

Proof: If $\underline{U}$ is a structure of E and if $\underline{M}$ is a point in $G(E,\underline{U})$ let ${}_G(\underline{M})$ be the neighbourhood filter of $\underline{M}$. It has as a basis the sets $O^* \cap G(E,\underline{U})$ where O is an open subset of E contained in $\underline{M}$. Since $g_{\underline{U}}^{-1}(O^* \cap G(E,\underline{U})) = O$ it follows that $\underline{M} = g_{\underline{U}}^{-1}(\mathcal{V}_G(\underline{M}))$.

Let $\gamma: G(E,\underline{U}_1) \longrightarrow G(E,\underline{U}_2)$ be a homeomorphism such that $\gamma \circ g_{\underline{U}_1} = g_{\underline{U}_2}$. Let $\underline{M}_1$ and $\underline{M}_2$ be two points such that $\gamma \underline{M}_1 = \underline{M}_2$. Then $\gamma^{-1}(\mathcal{V}_G(\underline{M}_2)) = \mathcal{V}_G(\underline{M}_1)$ and so $\underline{M}_1 = g_{\underline{U}_1}^{-1}(\mathcal{V}_G(\underline{M}_1)) = g_{\underline{U}_2}^{-1}(\mathcal{V}_G(\underline{M}_2)) = \underline{M}_2$. Therefore $G(E,\underline{U}_1) = G(E,\underline{U}_2)$ and so the extensions are identical.

This lemma shows that to obtain a non-redundant representative collection of topologically complete extensions of E, it is necessary and sufficient to choose one structure from each of the $\underline{e}$-equivalence classes. One way to do this is to show that each $\underline{e}$-equivalence class contains a finest member. This can easily be done by first proving two elementary properties of the relation $\underline{e}$.

Let $\underline{U}$ be a structure of E and let $(X,i) = (G(E,\underline{U}), g_{\underline{U}})$. If $\underline{U}\ \underline{e}\ \underline{V}$ then (X,i) is the completion of E in $\underline{V}$ and so X is complete in the structure $\underline{U}_G$ and $\underline{V}_G$. On the other hand if $\underline{M}_1$ is a structure of X in which it is complete and if $\underline{V} = \underline{M}_1|E$ theorem 1 and lemma 1 show that $\underline{V}\ \underline{e}\ \underline{U}$. In view of theorem 3 of chapter three it follows that there is a 1 - 1 correspondence between the elements of $\underline{e}[\underline{U}]$-the $\underline{e}$-equivalence class of $\underline{U}$- and the structures of X in which it is complete. The correspondence is $\underline{V} \longrightarrow \underline{V}_G$ which preserves order. Since X is topologically complete it is complete in its finest structure $\underline{U}^X$ whose restriction $\underline{U}^X|E$ is the finest structure of E $\underline{e}$-equivalent to $\underline{U}$. This completes the proof of

<u>Theorem 3. Let $\underline{U}$ be a structure of E. Then the e-equivalence class $\underline{e}[\underline{U}]$ of $\underline{U}$ contains a finest structure.</u>

<u>Remarks.</u> The existence of the finest structure in $\underline{e}[\underline{U}]$ was pointed out to the author by Banaschewski in a written communication. He also gave a preliminary characterization of these 'extremal' uniformities: the finest structure in $\underline{e}[\underline{U}]$ is the finest uniformity coarser than the intersection of the filters $(\pi \times \pi)\underline{M}$ on $E \times E$, where for each $\underline{M}$ in $G(E,\underline{U})$, $(\pi \times \pi)\underline{M}$ is the filter generated by the sets $F \times F$, F in $\underline{M}$.

It follows from Shirota's theorem (theorem 2 of chapter three) that for a structure $\underline{U}$ of E, $G(E,\underline{U})$ is a Q-space iff $\underline{U}$ is $\underline{e}$-equivalent to some σ-bounded structure of E. Furthermore, since any completely regular space X has a finest σ-bounded structure $\underline{U}^{X\sigma}$ and a finest function structure $\underline{U}^{X\varphi} = \underline{U}(C_X)$, when $G(E,\underline{U})$ is a Q-space the above argument shows that the class $\underline{e}[\underline{U}]$ contains a finest σ-bounded structure and a finest function structure. The fact that the argument applies depends on the result: $\underline{U}_G$ is σ-bounded (a function structure) iff $\underline{U}$ is σ-bounded (a function structure).

The main problem that arises with all these 'extremal' structures is their characterization. The 'extremal' function structures are clearly in 1 - 1 correspondence with the extension algebras of E . Since the internal characterization of extension algebras seems to be complicated it is likely that the problem of characterising these extremal structures of E is also difficult.

§2. The quasi-orders restricted to the topologically complete extensions. Since for a given completely regular space E the completions in its structures form a representative collection of topologically complete extensions it is sufficient to consider $\leq$ and $\preceq$ restricted to the family $((G(E,\underline{U}),g_{\underline{U}}))_{\underline{U} \text{ a structure of } E}$ of extensions $(G(E,\underline{U}),g_{\underline{U}})$ of E .

The first result is a consequence of the fact that $\mathcal{G}$ is a uniform process. It is stated as

Theorem 4. Let $\underline{U}$ and $\underline{W}$ be two structures of E . The following statements are equivalent:

(1) $\underline{W}$ contains $\underline{U}$; and

(2) $(G(E,\underline{W}),g_{\underline{W}}) \preceq (G(E,\underline{U}),g_{\underline{U}})$ and the continuous function is $(\underline{W}_G,\underline{U}_G)$-uniformly continuous.

Proof: Since $\underline{W}$ contains $\underline{U}$ the identity function $e:E \longrightarrow E$ is $(\underline{W},\underline{U})$-uniformly continuous. Consequently the continuous function $G(e):G(E,\underline{W}) \longrightarrow G(E,\underline{U})$ such that $G(e) \circ g_{\underline{W}} = g_{\underline{U}} \circ e = g_{\underline{U}}$ is defined. The function $G(e)$ is $(\underline{W}_G,\underline{U}_G)$-uniformly continuous (see section two of chapter three) . This shows that (1) implies (2) .

On the other hand if $\gamma : G(E,\underline{W}) \longrightarrow G(E,\underline{U})$ is a continuous $(\underline{W}_G, \underline{U}_G)$-uniformly continuous function such that $\gamma \circ g_{\underline{W}} = g_{\underline{U}}$, then $\underline{U} = (g_{\underline{U}} \times g_{\underline{U}})^{-1} \underline{U}_G$ is contained in $(g_{\underline{W}} \times g_{\underline{W}})^{-1} \underline{W}_G = \underline{W}$. Consequently (2) implies (1).

Remark. If $\gamma : X \longrightarrow Y$ is a continuous function and X, Y are completely regular spaces then γ is $(\underline{U}^X, \underline{U}^Y)$ - and $(\underline{U}((X), \underline{U}(C_Y))$-uniformly continuous. Consequently if (X,i) and (Y,j) are topologically complete extensions and $X \preceq Y$ the continuous function $\gamma : X \longrightarrow Y$ such that $\gamma \circ i = j$ is uniformly continuous with respect to structures in which X and Y are complete (with both of them function structures if X and Y are Q-spaces).

If $\underline{U}$ is a structure of E contained in the structure $\underline{W}$ it defines a subuniformity $\underline{U}_{(G,\underline{W})}$ of $\underline{W}_G$. The uniformity $\underline{U}_{(G,\underline{W})}$ is the compatible uniformity for $G(E,\underline{W})$ whose inverse image under $g_{\underline{W}}$ is $\underline{U}$. In this notation $\underline{W}_G = \underline{W}_{(G,\underline{W})}$. The notation is introduced in order to state the analogue for $\leq$ of theorem 4 which is

Theorem 5. Let $\underline{U}$ and $\underline{W}$ be two structures of E such that $\underline{W}$ contains $\underline{U}$. The following statements are equivalent:

(1) $\underline{U}_{(G,\underline{W})}$ is a structure of $G(E,\underline{W})$;

(2) $(G(E,\underline{W}), g_{\underline{W}}) \leq (G(E,\underline{U}), g_{\underline{U}})$; and

(3) $G(E,\underline{W})$ is contained in $G(E,\underline{U})$ i.e. every minimal $\underline{W}$-Cauchy filter is a minimal $\underline{U}$-Cauchy filter.

Proof: Consider the commutative diagram

$$\begin{tikzcd}
(E,\underline{U}) \arrow[r, "g_{\underline{W}}"] \arrow[d, "g_{\underline{U}}"'] & (G(E,\underline{W}),\underline{U}_{(G,\underline{W})}) \arrow[d, "g_{\underline{U}_{(G,\underline{W})}}"] \\
G(E,\underline{U}) \arrow[r, "G(g_{\underline{W}})"] & G(G(E,\underline{W}),\underline{U}_{(G,\underline{W})})
\end{tikzcd}$$

where $G(g_{\underline{W}})$ is defined because $g_{\underline{W}}$ is $(\underline{U},\underline{U}_{(G,\underline{W})})$-uniformly continuous. It is a homeomorphism because $\mathcal{G}$ satisfies (UP_7), the inverse image of $\underline{U}_{(G,\underline{W})}$ under $g_{\underline{W}}$ is $\underline{U}$, and $(g_{\underline{U}_{(G,\underline{W})}} \circ g_{\underline{W}})E$ is a dense subset of $G(G(E,\underline{W}),\underline{U}_{(G,\underline{W})})$.

Consequently the continuous function $G(g_{\underline{W}})^{-1} \circ g_{\underline{U}_{(G,\underline{W})}}$ $: G(E,\underline{W}) \longrightarrow G(E,\underline{U})$ is an embedding iff $g_{\underline{U}_{(G,\underline{W})}}$ is an embedding i.e. iff $\underline{U}_{(G,\underline{W})}$ is a structure of $G(E,\underline{W})$. This shows that (1) implies (2).

Assume that $\gamma : G(E,\underline{W}) \longrightarrow G(E,\underline{U})$ is an embedding with $\gamma \circ g_{\underline{W}} = g_{\underline{U}}$. If $\underline{M}$ is in $G(E,\underline{W})$ then $\underline{M} = g_{\underline{W}}^{-1}(\mathcal{V}_G(\underline{M})) = g_{\underline{U}}^{-1}(\mathcal{V}_G(\gamma\, \underline{M})) = \gamma\, \underline{M}$. Consequently $G(E,\underline{W})$ is contained in $G(E,\underline{U})$.

If $G(E,\underline{W})$ is contained in $G(E,\underline{U})$ then the restriction of $\underline{U}_G$ to $G(E,\underline{W})$ is a compatible uniformity whose inverse image under $g_{\underline{W}}$ is $\underline{U}$. Therefore $\underline{U}_{(G,\underline{W})} = \underline{U}_G | G(E,\underline{W})$. Since $\underline{U}_G$ is a structure of $G(E,\underline{U})$ this implies that $\underline{U}_{(G,\underline{W})}$ is a structure of $G(E,\underline{W})$.

Remark. In this theorem it is only of interest to consider the case where $\underline{W}$ contains $\underline{U}$ since theorem 4 shows that this implies that the continuous function on $G(E,\underline{W})$ that extends $g_{\underline{U}}$ is $(\underline{W}_G,\underline{U}_G)$-uniformly continuous.

These two theorems suggest the following two types of general problem for each of the quasi-orders $\leq$ and $\preceq$. Let $X = (X,i)$ and $Y = (Y,i)$ be two topologically complete extensions of E and assume that $X \leq Y$ or $X \preceq Y$. Then two basic types of questions occur:

(1) if X has been obtained by completing E in $\underline{W}_o$ when can Y be obtained by completing E in a coarser structure? and

(2) if Y has been obtained by completing E in $\underline{U}_o$ when can X be obtained by completing E in a finer structure?

When $\underline{W}_o$ and $\underline{U}_o$ are extremal structures of E these questions are trivial unless they are modified by putting additional conditions on the respectively coarser and finer structures of E .

An example of a modification of the second kind of question for $\leq$ is the following problem. Let $\underline{c}$ be a characteristic algebra of E and let $X = (X, \pi(\underline{c}) \circ i)$ be an extension of E with $X \leq K(\underline{c})$. Characterize those X which may be obtained by completing E in a function structure $\underline{U} = \underline{U}(S)$ where S contains $\underline{c}$ and is contained in $\mathcal{L}(\underline{c})$

(note that this implies $\underline{U}(S)$ contains $U(\underline{c})$ and that $K(\underline{c})$ may be obtained by completing E in $U(c)$). The solution is stated as

Theorem 6. $(X, \pi(\underline{c}) \circ i)$ is isomorphic to $(G(E,\underline{U}(S)), \varepsilon_{\underline{U}(S)})$ where S contains $\underline{c}$ and is contained in $\mathcal{L}(\underline{c})$ iff $\complement X$ is a union of G_δ-sets of $K(\underline{c})$.

Proof: $(G(E,\underline{U}(S)), \varepsilon_{\underline{U}(S)})$ is isomorphic to $(T(E,S), t_S)$ as $\mathfrak{G}_{\mathfrak{U}}$ and $\mathfrak{J}$ are isomorphic processes. The extension $(T(E,S), t_S)$ is also isomorphic to $(\mathcal{E}_{\underline{c}}(S), \pi(\underline{c}) \circ i)$ by theorem 16 of chapter two. To prove the theorem is sufficient to prove that $X = \mathcal{E}_{\underline{c}}(S)$ where S contains $\underline{c}$ and is contained in $\mathcal{L}(\underline{c})$ iff $\complement X$ is a union of G_δ-sets of $K(\underline{c})$. Lemma 11 of chapter two shows that this condition is necessary. This condition is sufficient in view of the proof of corollary 2 to theorem 16 in chapter two.

§3. Functions defined by uniformities. Let E be a set and let $\underline{U}$ be a uniformity for E. Denote by $C(\underline{U})$ the collection of $\underline{U}$-uniformly continuous real-valued functions on E and let $C^{*}(\underline{U})$ be the collection of bounded functions in $C(\underline{U})$. The following theorem about $C(\underline{U})$ is well known.

Theorem 7. $C(\underline{U})$ is a uniformly closed vector sublattice of F_E which contains the constants and is closed under bounded inversion.

Proof: It is a routine application of the fact that f is in $C(\underline{U})$ iff for every $\varepsilon > 0$ $[(x,y) \mid |fx - fy| < \varepsilon]$ is in $\underline{U}$.

Remark. Since a vector lattice of functions contains f iff it contains $f \cup 0$ and $f \cap 0$ it follows that $C(\underline{U})$ satisfies conditions $(\mathcal{L}_1)(\mathcal{L}_2)$ and $(\mathcal{L}_4)$ of theorem 8 in chapter two. As it is a vector lattice theorem 29 in this same chapter shows that the extra condition $(\mathcal{L}_3)$ is a very strong one to be satisfied by $C(\underline{U})$.

Corollary. $C^*(\underline{U})$ is a uniformly closed unitary subalgebra of F_E^*.

Proof: From the theorem it follows that $C^*(\underline{U})$ satisfies the conditions of lemma 4 in chapter two. Since it is closed under multiplication by real numbers the corollary follows.

Remark. When E is a completely regular space and $\underline{U}$ is a compatible uniformity then $C^*(\underline{U})$ is an algebra $\underline{a}$ and $\mathcal{L}(\underline{a})$ contains $C(\underline{U})$.

The collections $C(\underline{U})$ and $C^*(\underline{U})$ are connected with processes $\mathfrak{G}$ and $\mathfrak{B}$ as shown by

Theorem 8. If $(E,\underline{U})$ is an object of $\mathcal{T}$ then the following statements hold:

(1) $C(\underline{U}) = C(\underline{U}_G) \circ g_{\underline{U}}$;

(2) $C^*(\underline{U}) = C^*(\underline{U}_G) \circ g_{\underline{U}}$; and

(3) $C^*(\underline{U}) = C_B(E,\underline{U}) \circ b_{\underline{U}}$.

Proof: Since $g_{\underline{U}}$ is $(\underline{U},\underline{U}_G)$-uniformly continuous and $b_{\underline{U}}$ is $(\underline{U},\underline{U}_B)$-uniformly continuous these results are immediate consequences of the 'universal' properties of $\mathfrak{G}$ and $\mathfrak{B}$ as applied to the extension of real-valued functions.

Corollary 1. $C^*(\underline{U})$ when extended to $G(E,\underline{U})$ is a characteristic algebra.

Proof: Since $G(E,\underline{U})$ is a subspace of $B(E,\underline{U})$ this is obvious.

Corollary 2. The $\underline{U}$-uniform topology is the weak topology $\underline{O}(E,C^*(\underline{U}))$.

Proof: If O_G is the topology of $G(E,\underline{U})$ its inverse image under $g_{\underline{U}}$ is the $\underline{U}$-uniform topology. This corollary follows from the previous one.

One immediate consequence of corollary 2 is the following obvious characterization of compatible uniformities, which is stated as

Theorem 9. Let E be a completely regular space and let $\underline{U}$ be a uniformity for E. Then $\underline{U}$ is compatible (is a structure) iff $C^*(U)$ is contained in C^*_E (is a characteristic algebra of E).

Proof: It is a consequence of corollary 2 to theorem 8 and the definition of a compatible uniformity (structure) of E in chapter three.

§4. Three uniformities associated with given uniformity. Let E be a set and let $\underline{U}$ be a uniformity for E. The uniformity consisting of the single set $E \times E$ is certainly σ-bounded, a function structure, and totally bounded as it is the uniformity generated by the constant functions. Therefore there exists a finest uniformity of each of these types coarser than $\underline{U}$. Let these uniformities be denoted respectively by $\underline{U}^{\sigma}$, $\underline{U}^{\varphi}$ and $\underline{U}^*$.

The uniformity $\underline{U}^{*} = \underline{U}(C^{*}(\underline{U}))$. This is because, in view of the remark following theorem 2, $\underline{U}^{*} = \underline{U}(\underline{a})$ where $\underline{a}$ is a uniquely defined uniformly closed unitary subalgebra of F_{E}^{*}. Hence $C^{*}(\underline{U})$ contains $\underline{a}$ but since $\underline{U}(C^{*}(\underline{U}))$ is totally bounded and contained in $\underline{U}$ the converse holds i.e. $\underline{a} = C^{*}(\underline{U})$.

The uniformity $\underline{U}^{\varphi} = \underline{U}(C(\underline{U}))$ since $\underline{U}$ contains $\underline{U}(S)$ iff $C(\underline{U})$ contains S.

In addition $\underline{U}^{*} = (\underline{U}^{\sigma})^{*} = (\underline{U}^{\varphi})^{*}$ and $(\underline{U}^{\sigma})^{\varphi} = \underline{U}^{\varphi}$. One consequence of these results is

Theorem 10. Let E be a completely regular space and let $\underline{U}$ be a uniformity for E. Then all four uniformities $\underline{U}^{*}$, $\underline{U}$, $\underline{U}$ and $\underline{U}$ are compatible (structures of E) when one of them is compatible (a structure of E).

Proof: Since for any uniformity $\underline{U}$, $\underline{U}^{*} = \underline{U}(C^{*}(\underline{U}))$ the result follows from theorem 9.

Corollary. Let E be a completely regular space. The following statements are equivalent:

(1) E has a coarsest structure ;

(2) E has a coarsest totally bounded structure ; and

(3) E is locally compact.

Proof: The theorem shows that (1) and (2) are equivalent. Since the totally bounded structures are all of the form $\underline{U}(\underline{c})$ by theorem 2, where $\underline{c}$ is a characteristic algebra of E. Therefore there exists a coarsest totally bounded structure of E iff there is a smallest characteristic algebra i.e. iff $\underline{\mathcal{C}}(E)$ is a complete lattice. By theorem 15 of chapter two this is equivalent to (3) .

Remark. This result is due to Samuel [13] .

Another consequence of these preliminary results is

Theorem 11. Let E be a set and let $\underline{U}$ be a uniformity for E . Then $G(E,\underline{U}) \subseteq G(E,\underline{U}^{\sigma}) \subseteq G(E,\underline{U}^{\varphi}) \subseteq G(E,\underline{U}^{*}) = B(E,\underline{U})$.

Proof: If $G(E,\underline{U}^{*}) = B(E,\underline{U})$ then $B(E,\underline{U}) = B(E,\underline{U}^{\sigma}) = B(E,\underline{U}^{\varphi})$. Since a $\underline{U}$-Cauchy filter is $\underline{U}^{\sigma}$-Cauchy and a $\underline{U}^{\sigma}$-Cauchy filter is $\underline{U}^{\varphi}$-Cauchy the result follows.

Let $(E,\underline{U})$ be any object of $\mathcal{T}$. The function $b_{\underline{U}}$ is $(\underline{U}^{*}, \underline{U}_B)$-uniformly continuous since the inverse image of $\underline{U}_B$ under $b_{\underline{U}}$ is totally bounded. Consider the following commutative diagram

$$\begin{tikzcd}
(E,\underline{U}^{*}) \arrow[r, "b_{\underline{U}^{*}}"] \arrow[d, "b_{\underline{U}}"'] & B(E,\underline{U}^{*}) \arrow[d, "B(b_{\underline{U}})"'] \\
(B(E,\underline{U}),\underline{U}_B) \arrow[r, "b_{\underline{U}_B}"] & B(B(E,\underline{U}),\underline{U}_B).
\end{tikzcd}$$

The function $b_{\underline{U}_B}$ is a homeomorphism by corollary 2 of theorem 11 in chapter three. Since process $\mathcal{B}$ satisfies (UP_7) and (UP_5) it follows that $B(b_{\underline{U}})$ is also homeomorphism. The 'trace filter' argument in lemma 1 may be applied here to show that $B(E,\underline{U}^{*}) = B(E,\underline{U})$ because there is a homeomorphism $\gamma : B(E,\underline{U}^{*}) \longrightarrow B(E,\underline{U})$ such that $\gamma \circ b_{\underline{U}^{*}} = b_{\underline{U}}$. Since $G(E,\underline{U}^{*}) = B(E,\underline{U}^{*})$, by corollary 1 of theorem 13 in chapter three, it follows that $G(E,\underline{U}^{*}) = B(E,\underline{U})$.

<u>Corollary. Let E be a completely regular space and let $\underline{U}$ be a structure of E. Then $(G(E,\underline{U}), g_{\underline{U}}) \leq (G(E,\underline{U}^{\sigma}), g_{\underline{U}^{\sigma}}) \leq (G(E,\underline{U}^{\varphi}), g_{\underline{U}^{\varphi}}) \leq (G(E,\underline{U}^{*}), g_{\underline{U}}) = (B(E,\underline{U}), g_{\underline{U}})$.</u>

Proof: It is an immediate consequence of the definition of and the theorem.

The subset $G(E,\underline{U})$ of $B(E,\underline{U})$ is described by means of the uniformity $\underline{U}$. Are descriptions of this type possible for $G(E,\underline{U}^{\sigma})$ and $G(E,\underline{U}^{\varphi})$? The following theorem characterizes $G(E,\underline{U}^{\varphi})$ when the functions f in $C^{*}(\underline{U})$ are considered as functions on $B(E,\underline{U})$.

<u>Theorem 12. $\underline{M}$ in $B(E,\underline{U})$ is in $G(E,\underline{U}^{\varphi})$ iff there is a function f in $C^{*}(\underline{U})$ such that $f\,\underline{M} \neq 0$ and $1/f$ is in $\underline{C}(\underline{U})$.</u>

Proof: Let $S = C(\underline{U})$ and let X be the subspace $b_{\underline{U}}E$ of $B(E,\underline{U})$. Let $\underline{c}$ be the characteristic algebra $C_{B(E,\underline{U})}|X = C^{*}(\underline{U})|X$. Then $G(E,\underline{U}^{\varphi}) = \mathcal{E}_{\underline{c}}(S_G|X)$, where S_G is the extension of S to $G(E,\underline{U})$ i.e. $S_G = C(\underline{U}_G)$. Assuming this to be the case the theorem follows from the corollary to lemma 11 of chapter two.

$(G(E,\underline{U}^{\varphi}), g_{\underline{U}^{\varphi}})$ is 'isomorphic' to $(T(E,S), t_S)$ since $\underline{U}^{\varphi} = \underline{U}(S)$ and $\mathcal{G}_{\underline{U}}$ is isomorphic to $\mathfrak{J}$. Furthermore $(T(E,S), t_S)$ is 'isomorphic' to $(T(X,S_G|X), t_{S_G|X} \circ g_{\underline{U}})$ since $X = g_{\underline{U}}E$ and $\mathfrak{J}$ satisfies (FP_7). Theorem 16 of chapter two shows that $(T(X,S_G|X), t_{S_G|X})$ is 'isomorphic' to $\mathcal{E}_{\underline{c}}(S_G|X)$ and the natural injection. Therefore there is a homeomorphism

$\gamma : G(E,\underline{U}^{\varphi}) \longrightarrow \mathcal{E}_{\underline{c}}(S_G|X)$ such that $\gamma \circ g_{\underline{U}}E = \gamma \circ h_{\underline{U}}E$. Hence $G(E,\underline{U}^{\varphi}) = \mathcal{E}_{\underline{c}}(S_G|X)$.

Remarks. The problem of describing $G(E,\underline{U})$ by means of $\underline{U}$ seems to be much more difficult - presumably because of the somewhat inaccessible nature of the uniformity $\underline{U}$. In the case of a completely regular space E, Shirota's Theorem (theorem 2 in chapter three) shows that $G(E,\underline{U}^{E\sigma}) = G(E,\underline{U}^{E\varphi})$ since $\underline{U}^{E\varphi} = \underline{U}(C_E)$. An examination of the proof that (3) implies (2) in this theorem shows that - by using Weil's lemma (lemma 1 of chapter three) - if $\underline{M}$ is a filter in $B(E,\underline{U})$ which is $\underline{U}^{\varphi}$-Cauchy but not $\underline{U}^{\sigma}$-Cauchy then there is a function f in $C^{*}(\underline{U})$ with $Z(f)$ disjoint from $G(E,\underline{U}^{\sigma})$ and $f\,\underline{M} = 0$. Therefore $\complement G(E,\underline{U}^{\sigma})$ is a union of G_δ-subsets of $B(E,\underline{U})$. This suggests the following problem: is $\complement G(E,\underline{U}^{\sigma})$ the union of the G_δ-subsets of $B(E,\underline{U})$ disjoint from $G(E,\underline{U})$?

§5. Extension algebras and structures. Let E be a completely regular space and let $\underline{U}$ be a structure of E. This section considers the question of when $C(\underline{U})$ is an extension algebra of E. One answer is given as

Theorem 13. $C(\underline{U})$ is an extension algebra iff $C(\underline{U}) = \underline{C}_{L(E,C(\underline{U}))} \circ \iota_{C(\underline{U})}$.

Proof: Since $\underline{U}$ is a structure of E the weak topology $\underline{O}(E,C(\underline{U}))$ is the topology of E (see corollary 2 of theorem 8) and so $(L(E,C(\underline{U})), \iota_{C(\underline{U})})$ is an extension of E. Therefore the condition is sufficient.

Conversely if $C(\underline{U})$ is an extension algebra of E the object $(E, C(\underline{U}))$ is in the category $A' \cap \Lambda'$. Then since $\mathcal{L}|A' \cap \Lambda'$ and $\mathcal{H}|A' \cap \Lambda'$ are isomorphic processes it follows from theorem 18 of chapter one that the condition is necessary.

The extension $(L(E,C(\underline{U})), \iota_{C(\underline{U})})$ is isomorphic to the extension $(T(E,C(\underline{U})), t_{C(\underline{U})})$ since $\mathcal{L}$ and $\mathfrak{J}|\Lambda'$ are isomorphic processes on Λ'. As $\mathcal{G}_U$ and $\mathfrak{J}$ are isomorphic processes, it follows that both extensions are isomorphic to the extension $(G(E,\underline{U}^{\varphi}), g_{\underline{U}^{\varphi}})$ because $\underline{U}^{\varphi} = \underline{U}(C(\underline{U}))$.

Let S be a collection of real-valued functions on a set E and let $\underline{V}$ be a uniformity for E. The collection S is said to be $\underline{V}$-inversion closed if $1/f$ is in S when $\lim_{\underline{M}} f$ exists and is different from zero for each minimal $\underline{V}$-Cauchy filter $\underline{M}$.

With the aid of this definition and the fact that $(L(E,C(\underline{U})), \iota_{C(\underline{U})})$ and $(G(E,\underline{U}^{\varphi}), g_{\underline{U}^{\varphi}})$ are isomorphic extensions of E it is not hard to obtain the following theorem,

<u>Theorem 14. $C(\underline{U})$ is an extension algebra iff $C(\underline{U})$ is $\underline{U}^{\varphi}$-inversion closed and $C^*(\underline{U}) = C^*_{G(E,\underline{U}^{\varphi})}|E$.</u>

Proof: When $C(\underline{U})$ satisfies these conditions its extension to $G(E,\underline{U}^{\varphi})$ satisfies the conditions of theorem 19 in chapter one. Therefore $C(\underline{U}) = C_{G(E,\underline{U}^{\varphi})}|E$.

If $C(\underline{U})$ is an extension algebra then $C(\underline{U}) = C_{G(E,\underline{U}^{\varphi})}|E$ as a result of theorem 13. Therefore $C(\underline{U})$ satisfies the conditions of the theorem.

Remarks. This theorem is quite similar to theorem 22 in chapter one since the conditions are stated partly in terms of the $C(\underline{U})$-completely regular filters that are $\underline{U}(C(\underline{U})) = \underline{U}^{\varphi}$-Cauchy. There is a connection between this theorem and the conjecture of chapter two. $G(E,\underline{U}^{\varphi})$ is a Q-space and the extension of $C(\underline{U})$ to $G(E,\underline{U}^{\varphi})$ satisfies $(\mathcal{L}_1),(\mathcal{L}_2),(\mathcal{L}_3)$ and $(\mathcal{L}_4)$ of theorem 8 in chapter two when $C(\underline{U})$ is $\underline{U}^{\varphi}$-inversion closed. The conjecture asserts that as a result $C^*(\underline{U}) = C^*_{G(E,\underline{U}^{\varphi})}|E$. This provides a setting in which to look for a counterexample to the conjecture.*)

If S is an extension algebra of E then there exists a Q-extension (Y,g) of E such that $S = C_Y \circ g = C_Y|E$. Since Y is complete in $\underline{U}(C_Y)$ and the extension $(G(E,\underline{U}(S)),g_{\underline{U}(S)})$ is isomorphic to (Y,g) by theorem 1 it follows that:

(1) $S = C(\underline{U}(S))$;

(2) S is $(\underline{U}(S))^{\varphi} = \underline{U}(S)$-inversion closed; and

(3) $S^* = C^*_{G(E,(\underline{U}(S))^{\varphi})}|E$. Combined with theorem 14 this completes the proof of

Theorem 15. S is an extension algebra of E iff there is a uniformity U for E such that:

(1) $S = C(\underline{U})$;

(2) S is $\underline{U}^{\varphi}$-inversion closed; and

(3) $S^* = C^*_{G(E,\underline{U}^{\varphi})}|E$.

*) See Erratum p. 260.

Remarks. The unsatisfactory part of theorems 14 and 15 is the requirement that $C^*(\underline{U}) = C^*_{G(E,\underline{U}^\varphi)}|E$. If the conjecture of chapter two is valid it can be omitted. On the other hand failing that, it is still of interest to attempt to characterize internally those uniformities $\underline{U}$ for which $C^*(\underline{U}) = C^*(\underline{U}) = C^*_{G(E,\underline{U}^\varphi)}|E$ or for which $C^*(\underline{U}) = C^*_{G(E,\underline{U})}|E$. It is not hard to see that the extremal structures of section one satisfy the second more restrictive proporty.

CHAPTER FIVE

THE CONSTRUCTION OF COMPACT SPACES

§1. Topological processes. The objects of the category Σ of topological spaces may be written as pairs $(E,\underline{O})$ where E is a set and $\underline{O}$ is a topology for E. In this notation $\mathrm{Hom}((E,\underline{O}),(E',\underline{O}'))$ is the collection of $(\underline{O},\underline{O}')$-continuous functions $\alpha: E \longrightarrow E'$. This formal similarity of Σ to Φ and $\mathcal{T}$ suggests the following definition,

Definition 1. If Σ_o is a subcategory of Σ a topological process $\mathcal{P}$ on Σ_o consists of a covariant functor $P: \Sigma_o \longrightarrow \Sigma$ and a family $(p_{\underline{O}})_{(E,\underline{O})\text{ in }\Sigma_o}$ of functions $p_{\underline{O}}: E \longrightarrow P(E,\underline{O})$ such that:

(TP$_1$) $p_{\underline{O}}E$ is dense in $P(E,\underline{O})$; and

(TP$_2$) if α is a mapping of Σ_o in $\mathrm{Hom}((E,\underline{O}),(E',\underline{O}'))$ then

$$P(\alpha) \circ p_{\underline{O}} = p_{\underline{O}'} \circ \alpha .$$

Remark. 1. As preliminary examples of topological processes on Σ the first three examples of function processes in chapter one suggest corresponding examples of topological process. An analogue of the fourth example is the process which associates with each topological space a completely regular space by identifying points that are not distinguished by the continuous real-valued functions.

2. By now it is apparent that to each kind of structure that can be attached to a set there is a corresponding category and a corresponding type of process. For each type of process a corresponding theory can be built up. Given two processes of the same type it is obvious what is meant by a homomorphism of one into the other. The simplest kind of invariants are always the topological properties and in each case the analogue of lemma 1 in chapter one may be proved. In the case of topological processes the second type of invariant is similar to that for uniform processes. If $\mathcal{P}$ is a topological process on Σ_0 and $(E,\underline{O})$ is an object of this category (in place of $\underline{U}_P$) the topology $\underline{O}_P$ is defined as the finest topology for $P(E,\underline{O})$ which is coarser than the topology of this space and for which $p_{\underline{O}}$ is $(\underline{O},\underline{O}_P)$-continuous.

Given a topological process $\mathcal{P}$ on Σ_0 the fact that the functions $p_{\underline{O}}:E \longrightarrow P(E,\underline{O}),(E,\underline{O})$ in Σ_0, are defined on and valued in a topological space suggests a class of problems of the following type: characterize those objects $(E,\underline{O})$ of Σ_0 for which $p_{\underline{O}}$ is an embedding (a continuous, open, or closed function). If every function $p_{\underline{O}}$ is continuous the process is said to be a <u>continuous process</u> (note that this is equivalent to asserting that $\underline{O}_P$ is the topology of $P(E,\underline{O})$ for each $(E,\underline{O})$ in Σ_0 and that this property is the analogue of (UP_3) and hence should be denoted by (TP_3)). It is clear that this is an invariant property of topological processes. A continuous compact topological process on Σ_0 is also called <u>a compactification on Σ_0</u>.

Topological processes may be defined by function processes and the following special kind of functor.

Definition 2. A functor A on a subcategory Σ_0 of valued in Φ is said to be a natural functor if it satisfies:

(1) if $(E,\underline{O})$ is in Σ_0 then $A(E,\underline{O}) = (E,\underline{A}(\underline{O}))$, where $A(\underline{O})$ is a collection of $\underline{O}$-continuous real-valued functions on E; and

(2) if α is a mapping of Σ_0 then $A(\alpha) = \alpha$.

Remarks. Roughly speaking, a natural functor on Σ_0 chooses a collection of continuous functions for each space in Σ_0 such that the choice is compatible with the mappings of Σ_0. Natural functors on subcategories of Φ and Υ valued in any of the three categories Φ, Υ or Σ are similarly defined. Essentially a natural functor replaces compatible structures on the sets by other compatible structures of the same or another type.

Examples of natural functors

1. For each $(E,\underline{O})$ in Σ let $C(\underline{O}) = C_{(E,\underline{O})}$-the algebra of $\underline{O}$-continuous real-valued functions on E.

2. For each $(E,\underline{O})$ in Σ let $C^*(\underline{O}) = C^*_{(E,\underline{O})}$, where $C^*_{(E,\underline{O})}$ is the collection of bounded functions in $C_{(E,\underline{O})}$.

Remarks. Other natural functors associate with each (E,S) in Φ the topological space $(E,\underline{O}(E,S))$ or the uniform space $(E,\underline{U}(S))$ and with each $(E,\underline{U})$ in Υ the space $(E,\underline{O}(E,\underline{U}))$ or the object $(E,C(\underline{U}))$ in Φ (the topology $\underline{O}(E,\underline{U})$ denotes the $\underline{U}$-uniform topology on E). Another natural functor on Σ associates with each $(E,\underline{O})$ the uniform space $(E,\underline{U}^{(E,\underline{O})})$, where $\underline{U}^{(E,\underline{O})}$ is the finest $\underline{O}$-compatible uniformity for E.

Let A be a natural functor on a subcategory Σ_o and let $\mathcal{P}$ be a function process defined on the subcategory $A\Sigma_o$ of Φ. Define $\mathcal{P}_A$ to be the topological process on Σ_o consisting of the covariant functor $P \circ A$ and the family $(p_{A(\underline{O})})_{(E,\underline{O}) \text{ in } \Sigma_o}$ of functions $p_{A(\underline{O})}: E \longrightarrow P(E, A(\underline{O})) = (P \circ A)(E,\underline{O})$. Although the definition of $\mathcal{P}_A$ presupposes that a process is defined it is in any case trivial to verify this, keeping in mind the definition of a natural functor. The topological process $\mathcal{P}_A$ on Σ_o is said to be induced by the natural functor A and the function process $\mathcal{P}$.

Examples of induced topological processes on Σ

1. For any space $(E,\underline{O})$ the object $(E, C_{(E,\underline{O})})$ is in $A' \cap \Lambda'$ and so the induced processes $\mathcal{J}_C$, $\mathcal{F}_C$, $\mathcal{H}_C$ and $\mathcal{L}_C$ are all defined on Σ. Since the original function processes are all isomorphic on $A' \cap \Lambda'$ (theorem 14 of chapter one) it follows that the induced processes are all isomorphic. Another process that is induced is $\mathcal{M}_C$.
2. For the same reason the natural functor C^* induces the topological processes $\mathcal{J}_{C^*}$, $\mathcal{F}_{C^*} = \mathcal{M}_{C^*}$, $\mathcal{H}_{C^*}$ and $\mathcal{L}_{C^*}$ which are all isomorphic processes on Σ.

The fact that there are topological processes induced by function processes raises the following question. Which topological processes on a subcategory Σ_o of Σ can (up to isomorphism) be 'factored' as a 'product' of a function process and a natural functor? A result in this direction is stated as

Theorem 1. Let $\mathcal{P}$ be a continuous Q-topological process on a subcategory Σ_o of Σ. Then there exists a natural functor $A: \Sigma_o \longrightarrow \Phi$ such that:

(1) $A\Sigma_o$ is contained in $A' \cap \Lambda'$; and

(2) $\mathcal{H}_A$ is isomorphic to $\mathcal{P}$.

Proof: If $(E,\underline{O})$ is in Σ_o let $A(\underline{O}) = c_{P(E,\underline{O})} \circ p_{\underline{O}}$. Since $\mathcal{P}$ is a continuous topological process on Σ_o this defines a natural functor A which satisfies (1).

For any object $(E,\underline{O})$ in Σ_o the following commutative diagram is defined.

$$\begin{tikzcd} E \arrow[r, "h_{A(\underline{O})}"] \arrow[d, "p_{\underline{O}}"'] & H(E, A(\underline{O})) \arrow[d, "H(p_{\underline{O}})"] \\ P(E,\underline{O}) \arrow[r, "h_{c_{P(E,\underline{O})}}"] & H(P(E,\underline{O}), c_{P(E,\underline{O})}). \end{tikzcd}$$

Since $\mathcal{H}$ satisfies (FP_8) $H(p_{\underline{O}})$ is a homeomorphism. When $\mathcal{P}$ is a Q-process $h_{c_{P(E,\underline{O})}}$ is a homeomorphism too. The function $\gamma(E,\underline{O}): H(E,A(\underline{O})) \longrightarrow P(E,\underline{O})$ defined as $\gamma(E,\underline{O}) = h^{-1}_{c_{P(E,\underline{O})}} \circ H(p_{\underline{O}})$, is also a homeomorphism.

Since $\mathcal{P}$ is a Hausdorff process the family $(\gamma(E,\underline{O}))_{(E,\underline{O}) \text{ in } \Sigma_o}$ of homeomorphisms $\gamma(E,\underline{O})$ defines an isomorphism of $\mathcal{H}_\Lambda$ into $\mathcal{P}$.

Corollary. Let $\mathcal{P}$ be a compactification on Σ_0. Then there exists a unique natural functor $A: \Sigma_0 \longrightarrow \Phi$ such that:

(1) for each $(E,\underline{O})$ in Σ_0, $A(\underline{O})$ is a uniformly closed unitary subalgebra of $C^*_{(E,\underline{O})}$; and

(2) $\mathcal{H}_A$ is isomorphic to $\mathcal{P}$.

Furthermore if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two isomorphic processes of this type on Σ_0, the corresponding natural functors A_1 and A_2 are identical.

Proof: If A is a natural functor on Σ_0 that satisfies (1) then $A(\underline{O}) = C_{H(E,A(\underline{O}))} \circ h_{A(\underline{O})}$. When $\mathcal{H}_A$ is isomorphic to $\mathcal{P}$ this means that $A(\underline{O}) = C_{P(E,\underline{O})} \circ p_{\underline{O}}$ and so A is uniquely defined by $\mathcal{P}$. The second assertion is now obvious.

Remarks. This corollary shows that $\mathcal{H} | A' \cap \Lambda' \cap \Phi^*$ plus the family of natural functors A on Σ_0 valued in $A' \cap \Lambda' \cap \Phi^*$ yields (up to isomorphism) all the compactifications on Σ_0. In other words $\mathcal{H} | A' \cap \Lambda' \cap \Phi^*$ satisfies the conditions of the following definition. Given a kind of structure that can be attached to a set, a compact process $\mathcal{P}$ on a subcategory of the corresponding category is called a general method of compactification if, for any natural functor A on a subcategory Σ_0 of Σ with valued in the subcategory, the induced process $\mathcal{P}_A$ is a compactification on Σ_0.

Other examples of a general method of compactification are provided by the restrictions of $\mathcal{J}$, $\mathcal{F}$ and $\mathcal{L}$ to $A' \cap \Lambda' \cap \Phi^*$. The process $\mathcal{M}$ on Φ is a general method of compactification as also is the process $\mathcal{B}$ on Υ . Since $\mathcal{B}|\Upsilon^* = \mathcal{G}|\Upsilon^*$, where Υ^* is the subcategory of Υ obtained by restricting the objects to be the pairs $(E,\underline{U})$ where $\underline{U}$ is totally bounded, it follows that $\mathcal{G}|\Upsilon^*$ is another general method of compactification. General methods of compactification are also of interest because Freudenthal's results on the relation $\Subset$ and compactification (see Freudenthal [21]), and the work of Fan and Gottesman [22] on normal bases can, by suitable choice of structure, be shown to define two more examples.

The corollary to theorem 1 raises the following problem: given a specific compactification on a subcategory Σ_0 find the associated natural functor $\Lambda: \Sigma_0 \longrightarrow \Phi$. In the following sections of this chapter this problem is considered for five known compactifications.

Remark. Given any general method of compactification a corresponding problem exists providing the compactification can be suitably 'factored' .

To conclude this section consider the following subprocess of a compactification $\mathcal{P}$ on Σ_0: define $\mathcal{P}_\delta$ to be the functor P_δ and the family $(p_{\underline{O}})_{(E,\underline{O}) \text{ in } \Sigma_0}$, where $P_\delta(E,\underline{O})$ is the complement of the union of G_δ sets in $P(E,\underline{O})$ disjoint from $p_{\underline{O}}E$ and $P_\delta(\alpha) = P(\alpha)$ if α is a mapping of Σ_0. Let Λ be the natural functor on Σ_0 defined by $\mathcal{P}$ and define Λ_δ by setting $\Lambda_\delta(E,\underline{O}) = (E, \mathcal{L}(\Lambda(\underline{O})))$ and $\Lambda_\delta(\alpha) = \alpha$. The argument of theorem 24 in chapter two may be applied to show that $\mathcal{P}_\delta$ and $\mathcal{L}_{\Lambda_\delta}$ are isomorphic processes on Σ_o. This shows that $\mathcal{P}_\delta$ is a Q-process. Hence every compactification $\mathcal{P}$ on Σ_o has associated with it the continuous Q-process $\mathcal{P}_\delta$, which can be obtained by means of the natural functor Λ of and the lattice function $\mathcal{L}$.

§ 2. The Stone-Čech compactification on Σ. As noted in chapter two Stone and Čech defined a compact extension for a completely regular space E by requiring that every function in C^*_E have an extension. One construction of such a space is $(\beta E, i) = (H(E, C^*_E), h_{C_E})$. This topological process on the sub-category of completely regular spaces may be extended to Σ as the process $\mathcal{H}_{C^*}$ defined in the previous section. The natural functor is of course the functor C^* also defined in section one. It was observed in chapter two that h_{C_E} is an embedding iff the space E is completely regular. Consequently the compactification $\mathcal{H}_{C^*}$ on Σ provides an extension of an object $(E, \underline{O})$ in Σ iff $\underline{O}$ is a completely regular topology.

§3. The Alexandroff compactification on Σ_p. Alexandroff [23] described a 'one-point' extension of a locally compact space as follows. A point p_∞ (could be [E]) is adjoined to E and the topology of $E \cup [p_\infty]$ is defined by setting a subset O open if it is an open subset of E or if it contains p_∞ and its complement is a compact subset of E. Since E is locally compact this defines a topology for $E \cup [p_\infty]$ with respect to which it is a compact extension of E.

A completely regular E is locally compact iff it is an open subset of βE (identifying E and iE) and so it is clear that $[p_\infty]$ corresponds to the closed set $\beta E \cap \complement E$. In other words the corresponding equivalence relation identifies all the points of $\beta E \cap \complement E$. This means that the characteristic algebra associated with the 'one-point' compact extension $E \cup [p_\infty]$ is the algebra of functions in C_E^M whose extensions to βE are constant on $\beta E \cap \complement E$. This algebra is the uniform closure of the subalgebra of C_E^M of functions g that are constant outside some compact subset of E (depending on g). To prove this it is clearly sufficient to show that any function g in C_E^M which is constant on $\beta E \cap \complement E$ may be uniformly approximated by functions in C_E^M which are constant outside some compact subset of E. It is not hard to see that this is the case.

If E and E' are locally compact and $\alpha : E \longrightarrow E'$ is a continuous function it has an extension to the 'one-point' extensions iff α is proper i.e. iff the inverse image of a compact set under α is compact (see Bourbaki [3] p103).

Define Σ_p to be the subcategory of Σ obtained by restricting the maps to be proper. Define the natural functor A_∞ on Σ_p by setting $A_\infty(\underline{O})$ equal to the uniform closure of the algebra of $\underline{O}$-continuous bounded real-valued functions g on E that are constant outside some compact subset (depending on g). This defines a natural functor because if α is a proper function in $\mathrm{Hom}((E,\underline{O}),(E',\underline{O}'))$ and g' is constant outside the compact set D' then $g' \circ \alpha$ is constant outside the compact set $\alpha^{-1}D'$.

The Alexandroff compactification on Σ_p is the process $\mathcal{H}_{A_\infty}$ which is isomorphic on the subcategory of locally compact spaces to Alexandroff's original process. Hence A_∞ is the associated natural functor. Furthermore $A_\infty(\underline{O})$ is a characteristic algebra of $(E,\underline{O})$ iff $\underline{O}$ is locally compact in view of

Lemma 1. Let S be a collection of real valued functions on a set E and let S_c be its uniform closure in F_E. Then $\underline{O}(E,S) = \underline{O}(E,S_c)$.

Proof: Let x_o be a point in E and let f be in S_c. Pick $\varepsilon > 0$ and an integer n such that $\varepsilon/2 > 1/n$. Let f_n be in S and such that $|f_nx - fx| < 1/n$ for all x in E. Then $[x \text{ in } E \mid |f_nx_o - f_nx| < \varepsilon/2]$ is contained in $[x \text{ in } E \mid |fx_o - fx| < \varepsilon]$. This shows that $\underline{O}(E,S) = \underline{O}(E,S_c)$.

Remark. The definition of A_∞ suggests the definition of two more natural functors on Σ_p. Replace the phrase 'are constant' by 'assume only a finite number of values' or by 'assume only a countable number of values'. The resulting algebras are characteristic when $A_\infty(\underline{O})$ is characteristic. Is the converse true? What are the resulting compact spaces like and how can they be described without the use of functions?

§4. Banaschewski's zero-dimensional compactification on Σ.

Banaschewski [24] described a zero-dimensional compact extension

(K_z, i_z) of a zero-dimensional space E which is defined up to isomorphism by the usual 'universal' property: if (K,k) is a compact extension of E and K is zero-dimensional then there is a unique continuous function $k_z : K_z \longrightarrow K$ such that $k_z \circ i_z = k$.

To describe K_z it is necessary and sufficient to characterise $C_{K_z} \circ i_z$. Let K be a compact zero-dimensional space. Then C_K is the uniform closure of the algebra generated by the continuous characteristic functions on E (this is a consequence of the Stone-Weierstrass theorem). If (K,k) is an extension of E then $C_K \circ k$ is the uniform closure of an algebra generated by continuous characteristic functions on E . Consequently $C_{K_z} \circ i_z$ is the uniform closure of the algebra generated by all the continuous characteristic functions on E . Obviously this algebra contains $C_{K_z} \circ i_z$ and to prove the converse it is sufficient to observe that its maximal ideal space is zero-dimensional.

Define the natural functor A_z on Σ by setting $A_z(\underline{O})$ equal to the uniform closure of the algebra generated by the $\underline{O}$-continuous characteristic functions on E . The compactification $\mathcal{H}_{A_z}$ is the zero-dimensional process on Σ . In view of lemma 1 it is clear that $\mathcal{H}_{A_z}$ provides an extension of $(E,\underline{O})$ iff $\underline{O}$ is a zero-dimensional topology for E . Restricted to the zero-dimensional spaces $\mathcal{H}_{A_z}$ defines Banaschewski's maximal zero-dimensional space (K_z, i_z).

Remark. The fact that a compactification can be associated with dimension zero raises the problem as to whether this is possible for any finite dimension n. The argument used to define A_z may be applied when the algebra C_K is suitably characterized for a compact n-dimensional space K.

§5. Freudenthal's rim-compact process on Σ_p. An open subset 0 of a topological space E is said to be rim-compact if the boundary $\mathbf{B}(0)$ of 0 is compact. If 0 is rim-compact so is $\complement\overline{0}$ and if $0'$ is rim-compact so is $0 \cup 0'$ since $\mathbf{B}(0 \cup 0')$ is a closed subset of $\mathbf{B}(0) \cup \mathbf{B}(0')$.

A filter $\underline{F}$ on E is said to be rim-compact if for each set F in $\underline{F}$ there is a rim-compact set 0 in $\underline{F}$ such that F contains $\overline{0}$. A topological space E is said to be rim-compact if all the neighbourhood filters are rim-compact and E is Hausdorff. This is equivalent to asserting that E is Hausdorff and has a basis of rim-compact sets.

A continuous function f on a topological space E is said to be rim-compact if f and $(-1)f$ satisfy the condition (r.c.): when $\lambda < \mu$ there is a rim-compact set 0 with $[x \text{ in } E \mid fx < \lambda] \subseteq 0 \subseteq \overline{0} \subseteq [x \text{ in } E \mid fx < \mu]$.

These concepts are linked by the following lemma,

Lemma 2. Let E be a rim-compact space and let U be an open set containing the closure of the rim-compact set 0. Then there exists a rim-compact function f with $0 \leq f \leq 1$ such that $f|0 = 0$ and $f|\complement U = 1$.

Proof: Since the boundary $\mathcal{B}(O)$ is rim-compact and E is rim-compact there exists a finite collection $O_1, \ldots, O_n$ of rim-compact sets O_i such that $\bigcup_{i=1}^{n} O_i$ contains $\mathcal{B}(O)$ and $\overline{O}_i$ is contained in U for each i . Let $W = O \cup (\bigcup_{i=1}^{n} O_i)$. Then $O \subseteq \overline{O} \subseteq W \subseteq \overline{W} \subseteq U$. Repeating this argument the proof of Urysohn's lemma shows that there exists a bounded continuous function f with $0 \leq f \leq 1$ such that $f|O = 0$ and $f|\complement U = 1$. The definition of this function shows that it is rim-compact.

Corollary 1. A rim-compact space is completely regular.

Proof: Obvious.

If S denotes the collection of bounded rim-compact functions on E then two other immediate corollaries to this lemma are

Corollary 2. If E is rim-compact $O(E,S)$ is the topology of E , and

Corollary 3. If E is rim-compact a filter F on E is rim-compact iff it is S-completely regular.

The collection S has a definite structure as stated in

Theorem 2. The collection S is a uniformly closed unitary subalgebra of C_E^* . Hence it is a characteristic algebra of the rim-compact space E .

Proof: It is sufficient, in view of lemma 4 in chapter two, to prove that S is a uniformly closed translation lattice of functions on E that contains the constants and is closed under multiplication by the real numbers.

The set S obviously contains the constants and is closed under multiplication by real numbers. If f and g are in S and λ is any number then $[x|(f \cap g)x < \lambda] = [x|fx < \lambda] \cup [x|gx < \lambda]$ and $[x|(f \cup g)x < \lambda] = [x|fx < \lambda] \cap [x|gx < \lambda]$. Consequently since the intersection and union of two rim-compact sets are both rim-compact, S contains $f \cap g$ and $f \cup g$. The set S is obviously closed under the addition of constants. Furthermore it is easily seen to be uniformly closed.

Hence S is a uniformly closed unitary subalgebra of C_E^* and in view of corollary 2 to lemma 2 it is a characteristic algebra of E.

Freudenthal [21] showed that if E is rim-compact the space of maximal rim-compact filters on E i.e. the space $F(E,S)$ by corollary 3 to lemma 2, and the embedding f_S constitute a compact extension of E. Since $F(E,S)$ is compact it follows from the Stone-Weierstrass theorem and properties of the process $\mathfrak{F}$ that $C_{F(E,S)} \circ f_S = S$.

Define the natural functor A_r on Σ_p by setting $A_r(\underline{O})$ equal to the collection of bounded $\underline{O}$-continuous rim-compact functions on E. This defines a natural functor because if α is a proper continuous function the inverse image of a rim-compact set is a rim-compact set.

<u>Freudenthal's rim-compact process on Σ_5 is the compactification</u> $\mathcal{H}_{\Lambda_r}$ (or the isomorphic process $\mathcal{F}_{\Lambda_r}$) . It agrees with his original method of associating a compact space with a rim-compact space and clearly provides an extension of $(E,\underline{O})$ iff $\underline{O}$ is a rim-compact topology.

<u>Remarks.</u> From the point of view of the natural functor Λ_r, theorem 2 is the important result. It is an immediate formal consequence of the condition (r.c) and the fact that the intersection and union of two rim-compact sets are both rim-compact. Hence any other general class of open sets that is closed under finite union and intersection define a corresponding natural functor and consequently a compactification. For example consider the open-closed sets or the open sets with sequentially compact boundaries. In the first case the resulting compactification is Banaschewski's process. Since a compact spaceiis sequentially compact it is not hard to see that there is a homomorphism of the compactification defined by the second example into Freudenthal's rim-compact process.

§6. <u>Freudenthal's $\perp$ -compactification on Σ_5 .</u> Freudenthal [25] defined a compact space that can be associated with a given topological space $(E,\underline{O})$ by means of a relation $\perp$ defined for open subsets of $(E,\underline{O})$. This relation is defined with the aid of the following concept.

In a topological space E <u>a subset D is said to connect the subsets F_1 and F_2 of E</u> if, in the subspace $F_1 \cup D \cup F_2$, there is no decomposition into two open sets which separate F_1 and F_2. It is clear that if D connects F_1 and F_2 then D' connects F_1 and F_2 when D' contains D. Also if D connects F_1 and F_2 it connects F_1' and F_2 when F_1' contains F_1. A more important property is that if D connects $F_1 \cup F_2$ and F then $D \cap \complement F_2$ connects F_1 and F or $D \cap \complement F_1$ connects F_2 and F.

<u>The relation δ is defined on a space $(E,\underline{O})$ as follows:</u> if O_1 and O_2 are in $\underline{O}$(i.e. are open) set $O_1 \delta O_2$ when for any decreasing (i.e. D_n contains D_{n+1}) sequence $(D_n)_n$ of closed sets D_n that connect O_1 and O_2 $\bigcap_n D_n$ does not lie in $O_1 \delta O_2$.

<u>Lemma 3.</u> <u>The relation δ is clearly symmetric and has the following properties:</u>

(δ_1) <u>if $O_1 \delta O_2$ and O_1 contains O_1' then $O_1' \delta O_2$; and</u>

(δ_2) <u>if $O_1 \delta O_2$ and $O_1' \delta O_2$ then $(O_1 \cup O_1') \delta O_2$.</u>

Proof: The first property is an immediate consequence of the fact that if D connects O_1' and O_2 it also connects O_1 and O_2. The second property follows from the third property of the relation 'one set connects two others'. Let $(D_n)_n$ be a decreasing sequence of closed sets D_n that connect $(O_1 \cup O_1')$ and O_2. If $D_n \cap \complement O_1'$ does not connect O_1 and O_2 for all integers n there is an integer n_0 such that $D_{n_0} \cap \complement O_1'$ does not connect O_1 and O_2. Consequently if $n \geq n_0$

$D_n \cap \complement O_1'$ does not connect O_1 and O_2 and so for $n \geq n_o$ $D_n \cap \complement O_1$ connects O_1 and O_2. Therefore $D_n \cap \complement O_1$ connects O_1 and O_2 for all integers n.

Consequently either the closed sets $D_n \cap \complement O_1'$ connect O_1 and O_2 for all integers n or the sets $D_n \cap \complement O_1$ connect O_1' and O_2 for all n. Assume that one of these situations, say the first, holds. If $\bigcap_n D_n$ lies in $O_1 \cup O_1' \cup O_2$ then $\bigcap_n (D_n \cap \complement O_1') = (\bigcap_n D_n) \cap \complement O_1'$ lies in $O_1 \cup O_2$. This contradicts the fact that $O_1 \delta O_2$. Consequently, since the same argument applies when the second situation holds, it follows that $(O_1 \cup O_1') \delta O_2$.

<u>Corollary. Let O_1, O_2 and O_1', O_2' be four open sets. The following statements hold:</u>

(1) <u>if O_i contains O_i', $i = 1,2$, and $O_1 \delta O_2$ then $O_1' \delta O_2'$; and</u>

(2) <u>if $O_1 \delta O_2$ and $O_1' \delta O_2'$ then $(O_1 \cap O_1') \delta (O_2 \cup O_2')$ and $(O_1 \cup O_1') \delta (O_2 \cap O_2')$.</u>

Proof: The first assertion is an immediate consequence of (δ_1) and the fact that δ is symmetric.

If $O_1 \delta O_2$ and $O_1' \delta O_2'$ then (δ_1) shows that $(O_1 \cap O_1') \delta O_2$ and $(O_1 \cap O_1') \delta O_2'$. The second statement follows from this, property (δ_2) and the symmetry of δ.

Freudenthal showed that there exists for every topological space $(E,\underline{O})$ a compact space K_δ and an $\underline{O}$-continuous function $i_\delta : E \longrightarrow K_\delta$ such that:

(1) $i_\delta E$ is dense in K_δ ;

(2) if O_1 and O_2 are open subsets of K_δ with disjoint closure then $(i_\delta^{-1} O_1)\, \delta\, (i_\delta^{-1} O_2)$; and

(3) if (K,k) is any pair that satisfies the previous conditions then there exists a (unique) continuous function $k_\delta : K_\delta \longrightarrow K$ such that $k_\delta \circ i_\delta = k$.

The pair (K_δ, i_δ) is clearly defined up to isomorphism by these three conditions. Freudenthal called such a pair <u>a maximal δ-compactification of $(E,\underline{O})$.</u>

The construction of (K_δ, i_δ) given by Freudenthal seems to be rather involved and as a result it will not be discussed. Instead the algebra $C_{K_\delta} \circ i_\delta$ will be described and the compactification obtained as its homorphism space.

Let $(E,\underline{O})$ be a topological space and let $k : E \longrightarrow K$ be a continuous function with kE a dense subset of the compact space K . Then (K,k) satisfies (2) iff for every g in C_K and $\lambda < \mu$ $[x \text{ in } E \mid (g \circ k)x < \lambda]\, \delta\, [x \text{ in } E \mid (g \circ k)x > \mu]$ (this is because a compact space is normal). In other words every function f in $C_K \circ k$ satisfies the following condition:

<u>(δ) if $\lambda < \mu$ then $[x \text{ in } E \mid fx < \lambda]\, \delta\, [x \text{ in } E \mid fx > \mu]$.</u>

Let $A_\delta(\underline{O})$ be the collection of functions in $C^{*}_{(E,\underline{O})}$ that satisfy (δ) . This collection has a definite structure as stated in

<u>Theorem 3. $A_{\mathcal{U}}(\underline{O})$ is a uniformly closed unitary subalgebra of $C^*(E,\underline{O})$.</u>

Proof: It is sufficient, in view of lemma 4 in chapter two, to prove that $A_{\mathcal{U}}(\underline{O})$ is a uniformly closed translation lattice of functions on E that contains the constants and is closed under multiplication by the real numbers.

Since $\phi\,\mathcal{U}\,E$ it is clear that $A_{\mathcal{U}}(\underline{O})$ contains the constants. It is clear that $A_{\mathcal{U}}(\underline{O})$ is closed under multiplication by positive numbers and also, since $\mathcal{U}$ is symmetric, by (-1). Statement (1) of the corollary to lemma 3 shows that $A_{\mathcal{U}}(\underline{O})$ is uniformly closed and statement (2) shows that $A_{\mathcal{U}}(\underline{O})$ is closed under $\wedge$ and $\vee$. Since $A_{\mathcal{U}}(\underline{O})$ is obviously closed under the addition of constants, the theorem follows.

<u>Corollary. $(H(E,A_{\mathcal{U}}(\underline{O})),\ h_{A_{\mathcal{U}}(\underline{O})})$ is a maximal $\mathcal{U}$-compactification of $(E,\underline{O})$.</u>

Proof: This pair obviously satisfies the preliminary conditions and condition (1). Since $A_{\mathcal{U}}(\underline{O}) = C_{H(E,\ A_{\mathcal{U}}(\underline{O}))}|E$ it satisfies (2). From the definition of $A_{\mathcal{U}}(\underline{O})$ it follows that if (K,k) satisfies conditions (1) and (2) then $A_{\mathcal{U}}(\underline{O})$ contains $C_K \circ k = C_K|E$. Consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "h_{A_{\mathcal{U}}(\underline{O})}"] \arrow[d, "k"'] & H(E,A_{\mathcal{U}}(\underline{O})) \arrow[d, "H(k)"] \\ K \arrow[r, "h_{C_K}"] & H(K,C_K). \end{tikzcd}$$

As a compact space is a Q-space, h_{C_K} is a homeomorphism and so $h_{C_K}^{-1} \circ H(k)$ is the desired continuous function k .

Remarks. The proof of the theorem and the corollary depend solely on the symmetry of δ , the fact that $\phi \delta E$ and the corollary to lemma 3 . Any other general relation on $\underline{O}$ with these formal properties defines a corresponding algebra, by means of which a corresponding maximal compactification is defined. One example of this type of relation is obtained by relating O_1 and O_2 if there is a rim-compact set O with $O_1 \subseteq O \subseteq \overline{O} \subseteq \complement O_2$. The resulting compactification is the rim-compact process of the previous section. Another example is obtained by modifying the definition of δ itself. Replace the phrases 'decreasing sequence $(D_n)_n$ of closed sets D_n' and '$\cap_n D_n$' by 'filter $\underline{F}$ with a basis of closed sets D' and '$\cap F$ f in $\underline{F}$' . It is not hard to see that the proof of lemma 3 applies to this modification of δ and consequently that the corollary to the lemma holds.

The algebra $A_\delta(\underline{O})$ is defined for any topological space $(E,\underline{O})$. Hence to obtain a natural functor it is sufficient to define a class of $(\underline{O},\underline{O}')$-continuous functions $\alpha: E \longrightarrow E'$ such that $A_\delta(\underline{O})$ contains $A_\delta(\underline{O}') \circ \alpha$. This is the case if $O_1' \delta O_2'$ implies $(\alpha^{-1}O_1') \delta (\alpha^{-1}O_2')$. Let $O_i = \alpha^{-1}O_i'$ and assume that $(D_n)_n$ is a decreasing sequence of closed sets D_n such that each D_n connects O_1 and O_2 .

If α is closed then the sets αD_n connect αO_1 and αO_2. Since αO_i lies in O_i' it follows that $(\alpha O_1)\,\mathcal{U}\,(\alpha O_2)$ and so $\bigcap_n(\alpha D_n)$ is not contained in $(\alpha O_1)\cup(\alpha O_2)$. This implies $\bigcap_n D_n$ does not lie in $O_1 \cup O_2$ if in addition α preserves countable intersections.

Let Σ_δ be the subcategory of Σ obtained by restricting the maps to those that are closed and which preserve countable intersections. Then $A_\mathcal{U}$ is defined as a natural functor on Σ_δ by setting $A_\mathcal{U}(E,\underline{O}) = (E,A_\mathcal{U}(\underline{O}))$ and $A_\mathcal{U}(\alpha) = \alpha$ if α is a map of Σ_δ. Freudenthal's $\mathcal{U}$-compactification on Σ_δ is the compact process $\mathcal{H}_{A_\mathcal{U}}$.

Freudenthal was not interested in compact extensions provided by this compactification. The ostensible motivation for this process is the fact that it can be used to obtain Mazurkiewicz' 'topological frontier' of a manifold (see Mazurkiewicz [25]). Freudenthal proved that if E is a manifold and d^* is the natural metric for E (i.e. $d^*(x,y) = \inf_{x,y \text{ in } C \text{ a continuum}} \operatorname{diam}(C)$) then Mazurkiewicz space is obtained by applying this $\mathcal{U}$-compactification to the topological space that results from completing E in d^*.

While the problem of characterizing those topological spaces $(E,\underline{O})$ for which $\mathcal{H}_{A_\mathcal{U}}$ provides an extension remains open, a preliminary result is stated as

<u>Theorem 4. Let $(E, \underline{O})$ be a locally compact space and assume that if $x \neq y$ there are disjoint open sets O_x and O_y containing x and y respectively with $O_x \mathfrak{d} O_y$. Then $A_{\mathfrak{d}}(\underline{O})$ is a characteristic algebra.</u>

Proof: Pick O_x and O_y with compact closures that lie disjoint open sets O'_x and O'_y. Let f_x be a continuous function with $0 \leq f_x \leq 1$, $f_x x = 0$ and $f_x | \complement O_x = 1$. Let f_y be a continuous function with $0 \leq f_y \leq 1$, $f_y y = 1$ and $f_y | \complement O_y = 0$. Define f to be $f_x + f_y$. Then $0 \leq f \leq 2$, $fx = 0$, $fy = 2$ and $f | \complement O_x \cap \complement O_y = 1$.

The function f is in $A_{\mathfrak{d}}(\underline{O})$. If $\lambda < \mu \leq 1$ and D connects $[x|fx < \lambda]$ and $[x|fx > \mu]$ then $D \cap \overline{O}_x \cap [x|fx \geq \lambda] \cap [x|fx \leq \mu]$ is non-void. Since $\overline{O}_x$ is compact this implies that $[x|fx < \lambda] \mathfrak{d} [x|fx > \mu]$. If $1 \leq \lambda < \mu$ a similar argument applies with O_y instead of O_x. If $\lambda < 1 < \mu$ then $[x|fx < \lambda]$ lies in O_x and $[x|fx > \mu]$ lies in O_y and so $[x|fx < \lambda] \mathfrak{d} [x|fx > \mu]$. This shows that f satisfies $(\mathfrak{d})$ and hence is in $A_{\mathfrak{d}}(\underline{O})$. From the definition of f and the fact that x and y are arbitrary points it follows that $A_{\mathfrak{d}}(\underline{O})$ is a characteristic algebra.

<u>Remark.</u> The proof of this theorem applies to the modification of $\mathfrak{d}$ defined in the remarks following the corollary to theorem 3.

CONCLUDING REMARKS

To conclude this thesis it seems fitting to make some general remarks about the main innovation that has been introduced, namely the concept of a process. From the use that has been made of processes it is clear that they have a significant role to play in the theory of extensions, if for no other reason than the fact that they introduce order into what was a relatively chaotic situation. This order is apparent in that it is now possible to compare and classify the various methods of constructing topological spaces and in addition to make more formal the proofs of theorems about the extensions (for example theorem 16 in chapter three). It also appears likely that most results on extension procedures (for example Stone's work on Boolean maps) can be discussed in the framework of topological processes.

While processes may be applied with good effect to the problem of constructing extensions they are also of interest in themselves. With the introduction of the concept of isomorphism the way is open to the consideration of the various invariant properties. Of the three general types considered the third seems to be the most interesting one from the point of view of additional results. However, it is perhaps worth noting that all the theorems which characterize processes by means of invariant or 'universal' properties involve only properties of the second type.

In general topology topological processes are likely to be the most important. The function processes and the uniform processes are of interest in that they can be used to define topological properties by means of suitable natural functors. From the point of view of topology they appear as a means to an end i.e. the definition of topological processes.

In the introduction a more or less intuitive definition of an ξ-process was given. The intuitive aspect of the definition being found in the phrases 'kind of structure' and 'ξ-homomorphism' which may be given an explicit formal meaning (see Bourbaki [27]) . While a process as defined in the introduction always constructs topological spaces there is no obvious reason why a process should be restricted in this way. In other words if ξ_1 and ξ_2 are two kinds of structures a (ξ_2, ξ_1)-process could be defined in the analogous way to consist of a functor which 'turns' a ξ_1-structure on a set into a ξ_2-structure on another set together with a family of functions that is related to the functor in the analogous way. If τ denotes the kind of structure that is called a topology then, in this notation, a ξ-process is a (τ, ξ)-process. It appears possible to introduce natural functors in this general setting and in short to carry most of the details over to the general setting. In an analogous fashion the (ξ, ξ)-processes will be of basic interest in the study of ξ-structures and the

$(\mathfrak{S}, \mathfrak{S}_1)$-processes of interest as a means of defining $(\mathfrak{S}, \mathfrak{S})$-processes with the aid of $(\mathfrak{S}_1, \mathfrak{S})$-natural functors.

In this setting many of the well known algebraic constructions can be considered. For example for a commutative ring R the passage from an R-module M to a specific construction of a tensor algebra T over M is essentially a process which replaces module structure by algebra structure.

Consequently the concept of a process seems to present a number of interesting possibilities for further investigation.

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Erratum

I. R. Isbell informs me that "The important conjecture of §9, Chapter Two, is false, and is refuted by a complicated example in the literature you cite and by a simpler example to appear soon. (1) See Example 1.22 of your reference [7]. (2) The Baire functions on the real line also give a counter-example. This is M. Henriksen and D. G. Johnson, "On the structure of a class of archimedean lattice ordered algebras", Fund. Math., to appear. Further, Theorem 5.2 of that paper will show that your conjecture is valid for Lindeloff spaces (extending Isbell's result for locally compact spaces countable at infinity)."

Another counterexample to the conjecture, provided by B. Banaschewski, may be described as follows: Let E be the discrete space of cardinality $\aleph_1$ and $\geqslant$ a well-ordering of E such that each proper segment of $(E, \leqslant)$ is countable. Then consider the set A of all real functions f on E such that $f(x) = f(a)$ for all $x \geqslant a$ with some suitable a (depending on f). Clearly, A is distinct from C_E, contains unbounded functions and the characteristic functions of all subsets $\{x\}$ of E. Moreover, A is closed under any finitary and countable operations which are defined "pointwise", in view of the choice of the well-ordering. Hence, A is a unitary algebra of functions, sublattice of C_E and closed under uniform convergence, i.e., satisfies all conditions of the conjecture. Since E is a Q-space (which follows from Satz 2 of M. Landsberg, Der Durchschnittsgrad hypercharakteristischer Filter, Math. Annalen 131 (1956), 429 - 434) this disproves the conjecture.