A Fuzzy-Smooth Variable Structure Filtering Strategy

for State and Parameter Estimation

S. Andrew Gadsden
Department of Mechanical Engineering
McMaster University
Hamilton, Ontario, Canada
gadsden@mcmaster.ca

Mohammed Al-Shabi
Department of Mechatronics Engineering
Philadelphia University
Jerash, Jordan
mshabi@philadelphia.edu.jo

Saeid R. Habibi
Department of Mechanical Engineering
McMaster University
Hamilton, Ontario, Canada
habibi@mcmaster.ca

***Abstract*—This article introduces a new filtering strategy based on combining elements of fuzzy logic and the smooth variable structure filter (SVSF). A revised formulation of the SVSF is presented in an effort to combine it with fuzzy logic, and is referred to as the RSVSF. Computer simulations are used to compare the new strategy, referred to as the Fuzzy-SVSF, with other popular Kalman-based estimation methods. Preliminary results indicate that combining fuzzy logic with the SVSF yields an improved estimation result and improved stability to system and modeling changes and uncertainties.**

Keywords—fuzzy logic; smooth variable structure filter; kalman filter; estimation

I. INTRODUCTION

Knowledge of the model and external environment, and an understanding of the uncertainties, is important for the safe and reliable control of any mechanical or electrical system. Quite often, exact knowledge of the system and its surroundings is not possible. In these cases, good approximations may yield satisfactory results.

Fuzzy logic, sets, and theory were first proposed in [*1*,*2*]. It has since gained considerable attention in the area of intelligent and adaptive control, as well as artificial intelligence [*3*,*4*,*5*]. As the name suggests, fuzzy logic deals with reasoning conditions that may be approximated rather than using exact or fixed values. Consider the area of control, where instead of using digital values of 0 or 1 to describe a condition, fuzzy logic considers continuous values between 0 and 1 (inclusive). For example, if a system behaves according to one of two models, it may be possible to represent the system by using a blend or combination of the two models; as opposed to only one or the other. Fuzzy logic was first successfully applied on the control of a train, where the comfort and reliability was improved [*5*]. It has since been applied to a variety of commercial and industrial applications, including: elevators, vehicles, dishwashers, cameras, and even software [*5*].

According to [*6*], the fuzzy control concept employs a type of linguistic approach, such that the controlled variable values depends on inferences from IF-THEN-ELSE rules. Linear systems may be represented in a piecewise fashion, such that rules may be drawn. These rules can be used to approximate the system. Note that fuzzy logic may be applied to both linear and nonlinear systems, and it has been shown to improve overall control performance and reliability [*5*].

This paper studies and compares the application of six different estimation strategies. Popular Kalman filter (KF)-based methods are studied, as well as the relatively new smooth variable structure filter (SVSF). The purpose of this paper is to introduce a new type of estimation strategy based on fuzzy logic and the SVSF, referred to as the Fuzzy-SVSF. Conceptually, this method is based on the Fuzzy-KF introduced in [*6*]. These estimation strategies are applied on an aerospace system for comparison purposes. The paper is organized as follows. The KF and SVSF estimation strategies are summarized, followed by the fuzzy-based methods. The simulation setup is described, and the six estimation strategies are applied to the system. The results are shown and discussed, followed by the main conclusions of the paper.

II. ESTIMATION STRATEGIES

A. Kalman Filter

The most well-known and extensively studied estimation strategy is referred to as the Kalman filter (KF). It was introduced in the 1960s in an effort to solve a linear stochastic estimation problem [*7*,*8*]. By making use of a statistically optimal gain equation, the KF yields the most accurate state estimates under some strict assumptions [*9*]. For example, the system under study must be well-defined (known) and linear. Furthermore, the noise must be Gaussian and white. These conditions lead to a statistically optimal solution to a linear stochastic estimation problem [*10*]. The following equations, used recursively, make up the KF process.

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \tag{2.1}$$

$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{2.2}$$

$$K_{k+1} = P_{k+1|k}H^T\left(HP_{k+1|k}H^T + R_k\right)^{-1} \tag{2.3}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(z_{k+1} - H\hat{x}_{k+1|k}\right) \tag{2.4}$$

$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{2.5}$$

The first two equations comprise the prediction stage, and are used to predict the state estimate and state error covariance, respectively. The following equations comprise the update stage. The KF gain is used to update the state estimates based on the measurement error, as well as update the state error covariance matrix. These values are used in the next time step, in a recursive process. Refer to the appendix for a list of nomenclature and their corresponding definition.

B. Revised KF (RKF)

In an effort to combine the KF with fuzzy logic, a revised KF was created [*6*]. In the revised KF strategy, the state estimate is corrected by a gain that is lagged by one time step. According to [*6*], the current state estimate is revised based on the previous a posteriori output error rather than the current a priori output estimate. The RKF is shown as follows [*6*].

$$K_{k+1} = AP_k H^T \left(HP_{k|k}H^T + R_k\right)^{-1} \tag{2.5}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + K_{k+1}\left(z_k - H\hat{x}_{k|k}\right) \tag{2.6}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + Bu_k \tag{2.7}$$

$$P_{k+1} = (A - K_{k+1}H)P_k A^T + Q_k \tag{2.8}$$

First the revised Kalman gain is calculated, and used to find an a priori state estimate. This estimate is then revised further based on the input to the system. The updated state error covariance matrix is used again in the following time step. The process is repeated iteratively.

C. Smooth Variable Structure Filter (SVSF)

The smooth variable structure filter (SVSF) was first introduced in 2007 [*11*]. It was based on an earlier, more complicated derivation [*12*]. The SVSF is based on sliding mode estimation and is formulated in a predictor-corrector fashion. The sliding mode strategy allows the estimation strategy to be very robust to modeling uncertainties and errors [*10*]. Since its introduction, the SVSF has been significantly developed and expanded [*13*,*14*,*15*,*16*]. The SVSF estimation process may be summarized by the following sets of equations [*10*].

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \tag{2.9}$$

$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{2.10}$$

$$e_{z,k+1|k} = z_{k+1} - H\hat{x}_{k+1|k} \tag{2.11}$$

$$\begin{aligned} K_{k+1} &= H^+ \left[\left(\left|e_{z,k+1|k}\right| + \gamma\left|e_{z,k|k}\right|\right) \right. \\ &\left. \cdot\, sat\left(\frac{e_{z,k+1|k}}{\psi}\right)\right]\left[diag\left(e_{z,k+1|k}\right)\right]^{-1} \end{aligned} \tag{2.12}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z,k+1|k} \tag{2.13}$$

$$\begin{aligned} P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T \\ + K_{k+1}R_{k+1}K_{k+1}^T \end{aligned} \tag{2.14}$$

$$e_{z,k+1|k+1} = z_{k+1} - H\hat{x}_{k+1|k+1} \tag{2.15}$$

The SVSF gain is a function of the measurement error e_z, a smoothing boundary layer ψ, and a convergence rate γ. The boundary layer width is defined as a measurement of the uncertainty in the estimation process. In particular, the estimation process may be illustrated in the following figure.

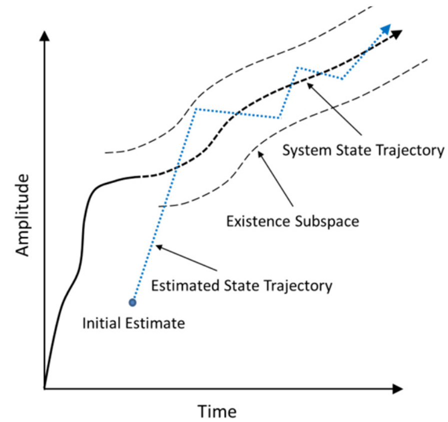

Fig. 1. The SVSF estimation process is illustrated here. An initial estimate is forced towards the true system state trajectory, and is bounded to it within a region referred to as the existence subspace.

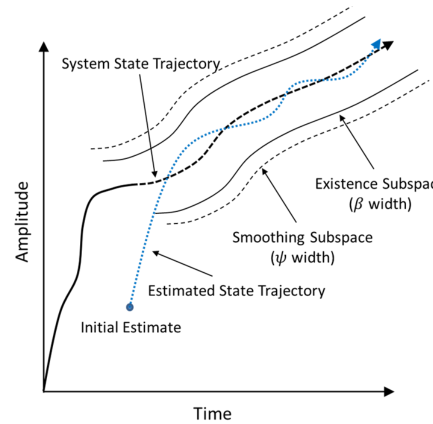

Fig. 2a. The SVSF estimated trajectory is smoothed when the smoothing subspace encompasses the existence subspace [*10*]. In other words, the uncertainties are being suppressed in the estimation process.

The SVSF gain yields a robust estimation strategy, due to the inherent switching effect. It is bounded-input, bounded-output (BIBO) stable and is based on a decreasing estimation error [*11*]. Note that the smoothing boundary layer is described in more detail in Figs. 2a and 2b. When the existence subspace boundary is defined smaller than the smoothing boundary layer, the estimated state trajectory is smoothed. However, chattering remains when the smoothing term is too small, as uncertainties have been underestimated. The magnitude of chattering may be used to determine the amount of uncertainty present in the estimation process [*13*].

D. Revised Smooth Variable Structure Filter (RSVSF)

This paper introduces a revised SVSF formulation in order to combine the SVSF with fuzzy logic, similar to [*6*]. The strategy is similar to (2.5) through (2.8). The SVSF gain (2.12) is revised accordingly to formulate the RSVSF gain:

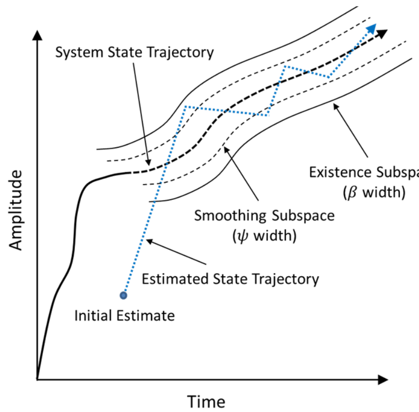

Fig. 2b. Chattering is present during the SVSF estimation process wen the smoothing subspace is defined too small [*10*]. In this case, not all of the uncertainties have been considered.

$$K_{k+1} = AH^{+}\left[\left(\left|e_{z,k+1|k}\right| + \gamma\left|e_{z,k|k}\right|\right) \cdot sat\left(\frac{e_{z,k+1|k}}{\psi}\right)\right]\left[diag\left(e_{z,k+1|k}\right)\right]^{-1} \quad (2.16)$$

Note that the initial measurement errors are set to zero. The state estimates and state error covariance are updated as per (2.6) through (2.8), while also calculating the a priori and a posteriori measurement errors. The process is repeated iteratively.

III. Fuzzy-Based Methods

A. Fuzzy-KF

The fuzzy-based methods used in this paper are based on [*6*]. The first step is to calculate the Fuzzy-KF gain by using the measurement or initial condition, as follows.

$$\kappa_{k+1} = AP_{2,k}H^{T}\left(HP_{2,k}H^{T} + \varphi_{k}\right)^{-1} \quad (3.1)$$

The a priori estimate is then calculated by the following:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + \kappa_{k+1}e_{z,k+1|k} \quad (3.2)$$

The state estimate is then calculated and used again in the next iteration.

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + Bu_{k+1} \quad (3.3)$$

The next step is to calculate the state error covariance in order to obtain the measurement estimate for the next time step.

$$\omega_{k+1} = \kappa_{k+1}P_{1,k}\kappa_{k+1}^{T} \quad (3.4)$$

$$P_{2,k+1} = (A - \kappa_{k+1}H)P_{2,k}A^{T} + \omega_{k+1} \quad (3.5)$$

Finally, the covariance is updated using the measurement estimation error covariance, as follows:

$$P_{1,k+1} = HP_{2,k+1}H^{T} + \varphi_{k+1} \quad (3.6)$$

The above process is repeated iteratively. More detail on the Fuzzy-KF may be found in [*6*].

B. Fuzzy-SVSF

Conceptually, the fuzzy-based SVSF estimation process is very similar to the Fuzzy-KF, except that the gain is formulated differently. The equations are listed here for completeness.

$$\kappa_{k+1} = AK_{svsf} \quad (3.7)$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + \kappa_{k+1}e_{z,k+1|k} \quad (3.8)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + Bu_{k+1} \quad (3.9)$$

$$\omega_{k+1} = \kappa_{k+1}P_{1,k}\kappa_{k+1}^{T} \quad (3.10)$$

$$P_{2,k+1} = (A - \kappa_{k+1}H)P_{2,k}A^{T} + \omega_{k+1} \quad (3.11)$$

$$P_{1,k+1} = HP_{2,k+1}H^{T} + \varphi_{k+1} \quad (3.12)$$

The above equations represent the Fuzzy-SVSF process, which is repeated iteratively. The initial state estimations and measurement errors are typically set to zero. The SVSF gain defined by (2.12) is used in (3.7).

IV. Simulation Setup and Results

The estimation strategies are applied and simulated on an electrohydrostatic actuator (EHA) for comparison purposes. An EHA is a commonly used hydraulic circuit used in aerospace systems, and has been described in detailed in [*6*,*10*,*17*,*15*]. The state space equation of the EHA is described mathematically as follows:

$$x_{k+1} = Ax_{k} + Bu_{k} + w_{k} \quad (4.1)$$

The state vector corresponds to the EHA position (cm), velocity (cm/s), and acceleration (cm/s^2). The initial state values are set to zero. The system matrix and input matrix are defined respectively by:

$$A = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -522.02 & -28.616 & 0.9418 \end{bmatrix} \quad (4.2)$$

$$B = [0 \quad 0 \quad 542.02]^{T} \quad (4.3)$$

where T is the sample rate defined as 0.001 seconds. The system noise w is random with absolute maximum amplitude 0.001 for all three states. A random signal with maximum amplitude of 1 is used as the system input u. The measurement equation is defined as follows:

$$z_{k} = Hx_{k} + v_{k} \quad (4.4)$$

where the measurement matrix is defined by:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (4.5)$$

The measurement noise v is random with absolute maximum amplitude 0.1 for both states.

The initial filter estimates are set to 0, and the initial state error covariance matrix is defined as a diagonal matrix of 10. The system and measurement noise covariances are defined respectively as follows:

$$Q = \begin{bmatrix} 1\times10^{-6} & 0 & 0 \\ 0 & 1\times10^{-6} & 0 \\ 0 & 0 & 1\times10^{-6} \end{bmatrix} \quad (4.6)$$

$$R = \begin{bmatrix} 1 \times 10^{-2} & 0 \\ 0 & 1 \times 10^{-2} \end{bmatrix} \quad (4.7)$$

The SVSF 'memory' of convergence rate is set as $\gamma = 0.1$ and the smoothing boundary layer widths is set, by tuning based on minimizing state estimation error, as $\psi = [5 \quad 50 \quad 500]^T$. Two cases were studied. The first case involves normal conditions, and the second case involves the presence of modeling uncertainty half-way through the simulation. The system model is changed half-way to the following:

$$A = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ -250 & -15 & 0.85 \end{bmatrix} \quad (4.8)$$

A. Normal Case

The following figure shows the estimation results for the normal case. All six strategies are able to follow the true position trajectory well. The KF and RKF methods converge the slowest to the true position trajectory; however, once on the trajectory the methods perform very well. In fact, the KF-based methods performed the best for this case. This was to be expected given that the system is linear, well-defined, and in the presence of Gaussian system and measurement noise.

The root mean square error (RMSE) results are calculated for this case, and the results are shown in the following table.

Table 1. RMSE Results for Normal Case

	Position (cm)	Velocity (cm/s)	Acceleration (cm/s^2)
KF	1.48×10^{-3}	1.75×10^{-2}	3.85
RKF	1.47×10^{-3}	1.99×10^{-2}	4.02
Fuzzy-KF	1.63×10^{-3}	4.65×10^{-2}	6.37
SVSF	1.66×10^{-3}	3.95×10^{-2}	5.07
RSVSF	1.65×10^{-3}	4.77×10^{-2}	6.49
Fuzzy-SVSF	1.64×10^{-3}	4.78×10^{-2}	6.42

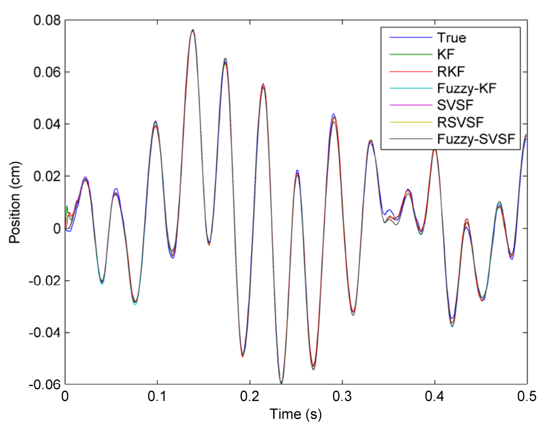

Fig. 2. EHA estimation results for all six filtering strategies, for the normal case. All filters are able to follow the trajectory well.

B. Fault Case

The following figure shows the results for the case when modeling uncertainty is injected half-way through the simulation.

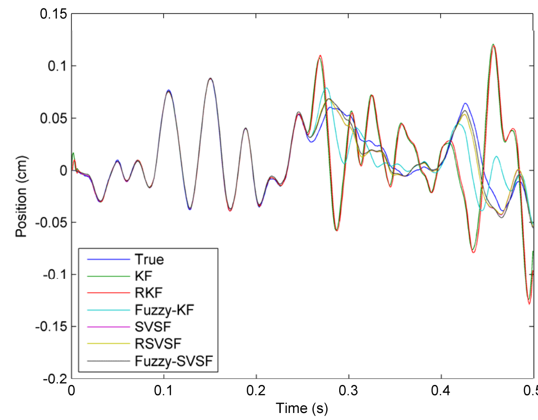

Fig. 3. EHA estimation results for all six filtering strategies, for the fault case. The modeling uncertainty is injected half-way through the simulation. The SVSF-based methods yield the most accurate results in this case.

The RMSE results are calculated for this case, and the results are shown in the following table.

Table 2. RMSE Results for Fault Case

	Position (cm)	Velocity (cm/s)	Acceleration (cm/s^2)
KF	4.15×10^{-2}	0.46	156
RKF	4.15×10^{-2}	0.57	164
Fuzzy-KF	1.56×10^{-2}	1.78	270
SVSF	7.05×10^{-3}	0.48	178
RSVSF	7.02×10^{-3}	0.63	179
Fuzzy-SVSF	5.42×10^{-3}	0.68	181

As the SVSF is more robust to modeling uncertainties and errors, it was expected to outperform the KF in this case. The position estimation results demonstrate the expectations. In fact, the SVSF-based strategies outperformed the KF-based strategies by about an order of magnitude. The fuzzy-based methods yielded even more accurate results. For example, the Fuzzy-KF outperformed the other KF-based methods; likewise, the Fuzzy-SVSF outperformed the other SVSF-based methods. However, all six methods yielded about the same velocity and acceleration results. For control in an uncertain system environment, the Fuzzy-SVSF method should be considered.

V. Conclusions

This paper introduced a new estimation strategy which combined elements of fuzzy logic with the smooth variable structure filter (SVSF), and is referred to as the Fuzzy-SVSF. Preliminary results indicate an improvement to position estimation accuracy as well as stability to modeling changes and instability. Future research will involve implementing the Fuzzy-SVSF on a physical setup, built for experimentation, as well as provide a detailed proof of stability.

APPENDIX

The list of nomenclature is shown as follows.

Table 3. List of Nomenclature

A	System matrix
B	Input gain or matrix
C or H	Measurement matrix
I	Identity matrix ('ones' along the diagonal)
K	Gain (KF or SVSF)
P	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix
S	Measurement error covariance matrix
T	Time step or sample rate
e	Measurement error
x	State(s)
z	Measurement(s)
w	System noise
v	Measurement noise
γ	SVSF 'memory' or convergence rate
ψ	SVSF smoothing boundary layer
κ	Fuzzy-based gain
φ	Probability density function of measurement error
ω	Fuzzy function of a residual
^	Superscript refers to an estimate
$k\|k$	Subscript refers to a posteriori ('after the fact')
$k+1\|k$	Subscript refers to a priori value ('before the fact')

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