A Square-Root Formulation of the Sliding Innovation Filter for Target Tracking

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ABSTRACT

The sliding innovation filter (SIF) is a state and parameter estimation strategy based on sliding mode concepts. It has seen significant development and research activity in recent years. In an effort to improve upon the numerical stability of the SIF, a square-root formulation is derived. The square-root SIF is based on Potter's algorithm. The proposed formulation is computationally more efficient and reduces the risks of failure due to numerical instability. The new strategy is applied on target tracking scenarios for the purposes of state estimation. The results are compared with the popular Kalman filter.

Keywords: Square-root, Kalman filter, smooth variable structure filter, target tracking

1. INTRODUCTION

Estimation theory is a field of study considered to be a branch of statistics and signal processing [1]. True states of a system are typically plagued by noisy measurements or observations made of that system, and consequently, estimation theory deals with extracting knowledge of these true states from such measurements. Since the purpose is to remove unwanted noise from a signal, the term 'filter' is appropriate and commonly used for estimation theory techniques. Many estimation strategies rely on the update or refinement of estimates based on some gain [2]. The Kalman filter (KF) is considered to be one of the most population estimation theory techniques and is formulated in a predictor-corrector fashion [3]. The KF is known to yield an optimal solution for estimation problem linear in nature, and its gain optimization is based on minimizing the state error covariance matrix [4]. It has been demonstrated in the literature that a symmetric and positive-definite error covariance matrix is necessary to properly represent the state vector components' statistics [5]. By definition, a symmetric matrix is defined as any square matrix with its corresponding transpose being exactly equal. Consider a symmetric matrix P, then $P = P^T$ is satisfied, and it can subsequently be considered positive-definite if $b^T P b > 0$ is also satisfied, where b is a non-zero vector with real entries. These two definitions mentioned basically ensure that the state error covariance matrix's off-diagonal elements are equal to each other (i.e., $p_{ij} = p_{ji}$), as well as ensuring elements along the diagonal are real and positive values. This means that the square of each estimation error is guaranteed to be positive.

Square-root, or factored-form filters as described in [6], make use of three powerful linear algebra techniques to help ensure numerical stability [7, 8, 9]. These three techniques include QR decomposition, Cholesky factor updating, and efficient least squares [10, 11]. Breaking up the covariance matrix into factored terms is involved, which are then propagated forward and updated at each measurement [5]. Then, these factors are multiplied together to form the covariance matrix once again, ensuring its positive-definiteness. Potter's square-root filter and Bierman-Thornton's UD filters are two of the most popular square-root filters [12]. Although the accuracy of Potter's square-root filter is similar to that of the UD filter, it is computationally more expensive [13].

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UD filtering was introduced in the late 1970s and is based on transformation methods which involve an upper triangle covariance factorization [14, 15]. Despite being considered a square-root filter variant, UD filtering does not involve computing any square roots. Instead, the covariance P is defined by:

$$P = UDU^T \tag{1.1}$$

where U is an upper triangular matrix with diagonal elements that are unity (all 1), and $D = diag(d_1, \dots, d_n)$. The matrices U and D are referred to as the UD factors of the covariance matrix P. Several different strategies exist to perform UD decomposition, creating U and D matrices [16]. Furthermore, the numerical stability of filtering strategies can be improved by factoring the covariance matrix into Cholesky factors [17]. This improvement in stability was first encountered through attempts to improve the stability of the KF with finite-precision arithmetic [16]. In essence, the nature of the KF is unchanged, and instead an equivalent statistical parameter is used which has been found to be more resistant to round-off errors [18]. An increase in the arithmetic precision results in a reduction of the effects of round-errors, improving the overall stability of the filter.

This paper is organized as follows: a background on the Kalman filter (KF) and sliding innovation filter (SIF) and their equations are summarized in section 2. In section 3, the square-root formulations of the KF are introduced, followed by an exposition of the new square-root SIF. Section 4 presents and describes the target tracking scenario. The results of implementing the square-root KF and square-root SIF are then outlined, and a comparison of the different methods is provided. Finally, the paper is concluded and future work is described in section 5.

2. ESTIMATION STRATEGIES

The Kalman Filter

The core of the Kalman filter (KF) algorithm are shown in the following equations, and are used in an iterative fashion. In equations (2.1) and (2.2), the a priori state estimate $\hat{x}_{k+1|k}$ is defined based on knowledge of the system F and previous state estimate $\hat{x}_{k|k}$, and the corresponding state error covariance matrix $P_{k+1|k}$, respectively.

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Gu_k \tag{2.1}$$

$$P_{k+1|k} = FP_{k|k}F^T + Q_k \tag{2.2}$$

The Kalman gain K_{k+1} is defined by (2.3), and is used to update the state estimate $\hat{x}_{k+1|k+1}$ as shown in (2.4). The gain makes use of an innovation covariance S_{k+1}, which is defined as the inverse term found in (2.3).

$$K_{k+1} = P_{k+1|k}H^T\left(HP_{k+1|k}H^T + R_{k+1}\right)^{-1} \tag{2.3}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(z_{k+1} - H\hat{x}_{k+1|k}\right) \tag{2.4}$$

The a posteriori state error covariance matrix $P_{k+1|k+1}$ is then calculated by (2.5), and is used iteratively, as per (2.2).

$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{2.5}$$

The derivation of the KF is well documented, with details available in [19, 3, 20]. The KF gain is unique in its ability to yield an optimal solution to the linear estimation problem, however, this comes at a cost of stability and robustness. Assumptions used in the derivation include the system model being known and linear, that the system and measurement noises are white, and that the states have initial conditions with known means and variances [21, 13]. However, the previous assumptions often do not hold in a number of applications, and if violated, the KF yields suboptimal results and can become unstable [22]. In addition, the KF is sensitive to computer precision and the complexity associated with computations of matrix inversions [16]. However, with modern computing power and capabilities, this drawback has been significantly reduced. The extended Kalman filter (EKF) is a natural extension of the KF method for nonlinear systems and measurements. A nonlinear system or measurement equations can be linearized according to their Jacobian matrix. The partial derivatives are used to compute linearized system and measurement matrices F and H, respectively found as follows [23]:

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k|k}, u_k} \tag{2.6}$$

$$H_{k+1} = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k+1|k}} \tag{2.7}$$

The equations (2.6) and (2.7) linearize the nonlinear system or measurement functions around the current estimate of the state [3]. These values can then be used as per equations (2.1) through (2.5). This comes the tradeoff of optimality; as such, the EKF yields a suboptimal solution to the nonlinear estimation problem [20]. Other Kalman-based methods exist beyond the EKF, and include the unscented Kalman filter (UKF) and the cubature Kalman filter (CKF) [24]. Although these methods yield improvements on the EKF, a number of strict assumptions still apply. Furthermore, the presence of modeling errors, uncertainties, and disturbances can still lead to unstable estimates.

The Sliding Innovation Filter

Similar to the Kalman filter (KF), the SIF is formulated as a predictor-corrector estimation method [25]. The state estimates and state error covariances are first predicted using values obtained at the previous time step (or initialization), and then the state estimates and state error covariance are updated based on the measurements and correction term at the current time step. The correction term in this case is referred to as the SIF gain.

The prediction stage includes calculating the predicted or *a priori* ('before the fact') state estimates $\hat{x}_{k+1|k}$, the predicted state error covariance $P_{k+1|k}$, and the predicted innovation $\tilde{z}_{k+1|k}$ as per the following three equations, respectively. Note that the nonlinear SIF process is the same as the nonlinear KF process, except that system and measurements equations are no longer linear.

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k \tag{2.8}$$

$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{2.9}$$

$$\tilde{z}_{k+1|k} = z_{k+1} - C\hat{x}_{k+1|k} \tag{2.10}$$

The update stage includes calculating the SIF gain K_{k+1}, the updated or *a posteriori* ('after the fact') state estimates $\hat{x}_{k+1|k+1}$, and the updated state error covariance $P_{k+1|k+1}$ as per the following three equations, respectively:

$$K_{k+1} = C^+\overline{sat}\left(\left|\tilde{z}_{k+1|k}\right|/\delta\right) \tag{2.11}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\tilde{z}_{k+1|k} \tag{2.12}$$

$$P_{k+1|k+1} = (I - K_{k+1}C)P_{k+1|k}(I - K_{k+1}C)^T \dots$$
$$\dots + K_{k+1}R_{k+1}K_{k+1}^T \tag{2.13}$$

Note that C^+ refers to the pseudoinverse of the measurement matrix, $\overline{sat}$ refers to the diagonal of the saturation term, sat refers to the saturation of a value (yields a result between -1 and +1), $\left|\tilde{z}_{k+1|k}\right|$ refers to the absolute value of the innovation, δ refers to the sliding boundary layer width, and I refers to the identity matrix (of dimension n-by-n where n is the number of states). Equations (2.8) through (2.13) represent the SIF estimation process for linear systems and measurements defined by (2.1) and (2.2), respectively.

The main difference between the KF and SIF strategies is in the structure of the gain. For the KF, the gain is derived as a function of the state error covariance, which offers optimality [26, 27]. However, for the SIF, the gain is based on the measurement matrix, the innovation, and a sliding boundary layer term. Although the state error covariance is not used to calculate the SIF gain, it still provides useful information as it represents the amount of estimation error in the filtering process. Figure 1 provides an overview of the SIF estimation concept. An initial estimate is pushed towards the sliding boundary layer which is defined based on the amount of uncertainties in the estimation process. Once inside the sliding boundary layer, the estimates are forced to switch about the true state trajectory by the SIF gain.

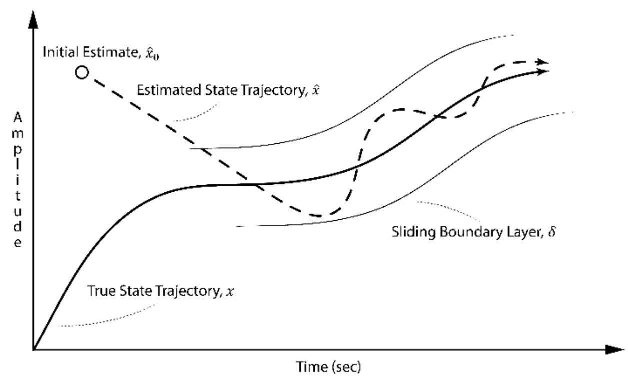

Figure 1. The above figure illustrates the SIF estimation concept.

The state estimates are updated with their corresponding innovation and sliding boundary layer term. The SIF gain effectively acts as a switching term, which forces the measurement errors to be bounded towards the true state trajectory. The sliding boundary layer δ is defined as a function of the modeling uncertainty and noise present in the estimation process. The width can be tuned to obtain the desired estimation result. Another starting point for tuning is to use the values of the measurement noise covariance. For example, $\delta = 10diag(R)$. The values can then be tuned by trial-and-error, grid search methods, or optimization techniques to reduce the estimation error.

For the cases when there are fewer measurements than states $(m < n)$, artificial measurements can be created based on existing measurements to create a full measurement matrix. The structure could also be modified as per a Luenberger observer or other strategies as per [28, 29]. This process would be required to estimate parameters of the system matrix using the SIF.

3. SQUARE-ROOT FORMULATIONS

The Square-Root Kalman Filter

James Potter and Angus Andrews developed the square-root formulation of the KF, and the method described in this section is often referred to as Potter's algorithm [13, 16]. As per Cholesky factorization, suppose that the square root of the state error covariance matrix P is available, such that $P = SS^T$. Modifying (2.2) yields:

$$P_{k+1|k} = S_{k+1|k}S_{k+1|k}^{\mathrm{T}} = FS_{k|k}S_{k|k}^{T}F^{T} + Q_k^{1/2}Q_k^{T/2} \tag{3.1}$$

Equation (3.1) is essentially (2.2). Modifying (2.3) yields:

$$K_{k+1} = S_{k+1|k}S_{k+1|k}^T H^T \left(HS_{k+1|k}S_{k+1|k}^T H^T + R_{k+1}\right)^{-1} \tag{3.2}$$

The updated state error covariance (2.5) then becomes:

$$P_{k+1|k+1} = (I - K_{k+1}H)S_{k+1|k}S_{k+1|k}^T(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{3.3}$$

Alternatively, this can be written as follows [13]:

$$P_{k+1|k+1} = S_{k+1|k}(I - a\phi\phi^T)S_{k+1|k}^T \tag{3.4}$$

where a and ϕ are defined as:

$$\begin{gathered} a = \left(\phi^T\phi + R_{i,k+1}\right)^{-1} \\ \phi = S_{k+1|k}^T H^T \end{gathered} \tag{3.5}$$

Note that i refers to the i^{th} element of the corresponding matrix or vector. As per [13], the a posteriori square-root covariance matrix can be calculated as follows:

$$S_{k+1|k+1} = S_{k+1|k}(I - a\gamma\phi\phi^T) \tag{3.6}$$

where γ is given as [13]:

$$\gamma = \left(1 + \sqrt{aR_{i,k+1}}\right) \tag{3.7}$$

Equations (3.1) through (3.7) can be used in conjunction with the standard KF estimation process. The main difference is that the update equation is used to update S instead of P, and the process is repeatedly iteratively [13].

The Square-Root SIF

This paper introduces the square-root formulation of the SIF, hereafter referred as to SR-SIF. It is based on the same approach as the square-root KF. For linear systems and measurements, the SR-SIF estimation processed is summarized by the following set of equations. For nonlinear systems and measurements, the nonlinearities may be linearized as per the EKF methodology. The state estimates $\hat{x}_{k+1|k}$ and square-root covariance $S_{k+1|k}$ are first calculated (noting the change in notation), as follows:

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Gu_k \tag{3.8}$$

$$S_{k+1|k}S_{k+1|k}^T = FS_{k|k}S_{k|k}^T F^T + Q_k^{1/2}Q_k^{T/2} \tag{3.9}$$

The predicted measurement $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ are calculated next.

$$\hat{z}_{k+1|k} = H\hat{x}_{k+1|k} \tag{3.10}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{3.11}$$

Next, the gain K_{k+1} is calculated as per (3.12).

$$K_{k+1} = C^{+}\overline{sat}\left(\left|e_{z,k+1|k}\right|/\delta\right) \tag{3.12}$$

The a posteriori square-root covariance matrix $S_{k+1|k+1}$ is calculated next as follows:

$$S_{k+1|k+1} = S_{k+1|k}(I - a\gamma\phi\phi^T) \tag{3.13}$$

where $a = \left(\phi^T\phi + R_{i,k+1}\right)^{-1}$, $\phi = S_{k+1|k}^T H^T$, and $\gamma = \left(1 + \sqrt{aR_{i,k+1}}\right)$.

The SR-SIF estimation process is summarized by (3.8) through (3.13). It is important to note that in this case, the gain is not affected by the square-root covariance calculation. However, the SR-SIF formulation sets the framework for future work and implementation in other types of SIF that rely on the covariance [4].

4. COMPUTER EXPERIMENTS

Target Tracking Problem Setup

The target tracking problem involved in this study is based on a generic air traffic control (ATC) scenario found in [21] and is as described in [4]. A radar stationed at the origin provides direct position-only measurements, with a standard deviation of $50\ m$ in each coordinate. The average motion of the target of interest is illustrated in Figure 2.

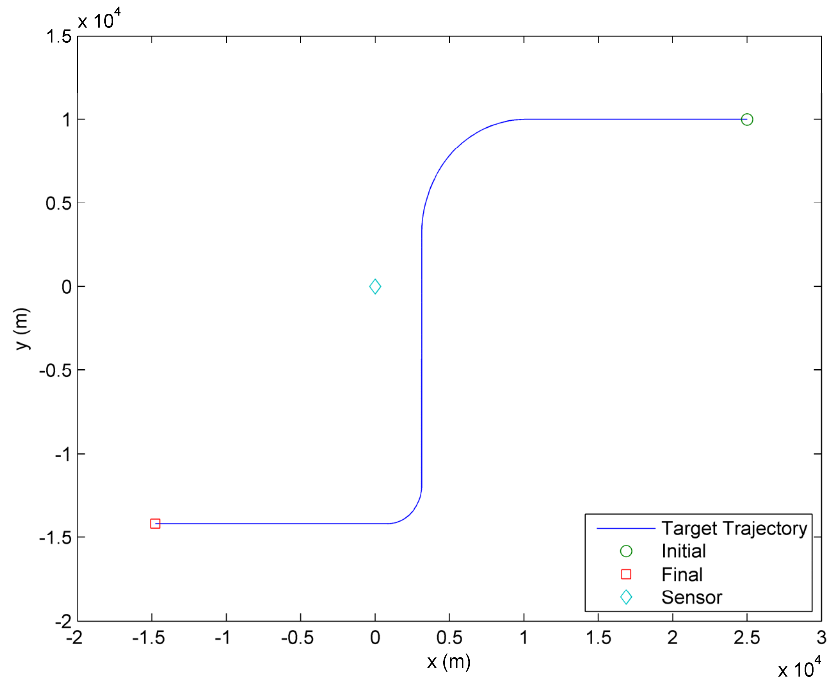

Figure 2. True target trajectory for the nonlinear estimation problem.

As shown in Figure 2, an aircraft starts from an initial position of $[25{,}000\ m, 10{,}000\ m]$ at time $t = 0\ s$, and flies westward at $120\ m/s$ for $125\ s$. A coordinated turn is then performed by the aircraft for a period of $90\ s$ at a rate of $1^{\circ}/s$. Then, the aircraft proceeds to fly southward at $120\ m/s$ for $125\ s$, followed by another coordinated turn for $30\ s$ at $3^{\circ}/s$. Finally, the aircraft continues to fly westward until it reaches the ultimate points of its trajectory, indicating the end of the simulation.

The behaviour of civilian aircraft in ATC scenarios may be modeled by two different modes of operation: uniform motion (UM) which involves a straight flight path with a constant speed and course, and maneuvering which includes turning or climbing and descending [21]. In the case of this study, maneuvering refers to a coordinated turn (CT) model, where a turn is made at a constant turn rate and speed. The uniform motion model used for this target tracking problem is given by (4.1) [21, 30].

$$x_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ 0 & \frac{1}{2}T^2 \\ T & 0 \\ 0 & T \end{bmatrix} w_k \tag{4.1}$$

The state vector of the aircraft may be defined as follows:

$$x_k = [\xi_k \quad \eta_k \quad \dot{\xi}_k \quad \dot{\eta}_k]^T \tag{4.2}$$

The first two states refer to the position along the x-axis and y-axis, respectively, and the last two states refer to the velocity along the x-axis and y-axis, respectively. The sampling time used in this simulation was 5 seconds. When using the CT model, the state vector needs to be augmented to include the turn rate, as shown in (4.3) [21]. The CT model may be considered nonlinear if the turn rate of the aircraft is not known. Note that a left turn corresponds to a positive turn rate, and a right turn has a negative turn rate. This sign convention follows the commonly used trigonometric convention (the opposite is true for navigation convention) [21]. As per [21, 30], the CT model is given by (4.4).

$$x_k = [\xi_k \quad \eta_k \quad \dot{\xi}_k \quad \dot{\eta}_k \quad \omega_k]^T \tag{4.3}$$

$$x_{k+1} = \begin{bmatrix} 1 & 0 & \frac{sin\omega_k T}{\omega_k} & -\frac{1-cos\omega_k T}{\omega_k} & 0 \\ 0 & 1 & \frac{1-cos\omega_k T}{\omega_k} & \frac{sin\omega_k T}{\omega_k} & 0 \\ 0 & 0 & cos\omega_k T & -sin\omega_k T & 0 \\ 0 & 0 & sin\omega_k T & cos\omega_k T & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2}T^2 & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 \\ T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} w_k \tag{4.4}$$

Since the radar stationed at the origin provides direct position measurements only, the measurement equation may be linearly formed as follows:

$$z_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} x_k + v_k \tag{4.5}$$

Equations (4.1) through (4.5) were used to generate the true state values of the trajectory and the radar measurements for this target tracking scenario. As previously mentioned, the EKF uses a linearized form of the system and measurement matrices. In this case, the system defined by (4.4) is nonlinear, such that the Jacobian of it yields a linearized form as shown in (4.6). The terms in the last column of (4.6) are correspondingly defined in (4.7) [21].

$$\left[\nabla_x F_{k,x}^T\right]^T\Big|_{x_k=\hat{x}_k} = \begin{bmatrix} 1 & 0 & \frac{sin\widehat{\omega}_k T}{\widehat{\omega}_k} & -\frac{1-cos\widehat{\omega}_k T}{\widehat{\omega}_k} & F_{\widehat{\omega}1} \\ 0 & 1 & \frac{1-cos\widehat{\omega}_k T}{\widehat{\omega}_k} & \frac{sin\widehat{\omega}_k T}{\widehat{\omega}_k} & F_{\widehat{\omega}2} \\ 0 & 0 & cos\widehat{\omega}_k T & -sin\widehat{\omega}_k T & F_{\widehat{\omega}3} \\ 0 & 0 & sin\widehat{\omega}_k T & cos\widehat{\omega}_k T & F_{\widehat{\omega}4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

$$\begin{bmatrix} F_{\widehat{\omega}1} \\ F_{\widehat{\omega}2} \\ F_{\widehat{\omega}3} \\ F_{\widehat{\omega}4} \end{bmatrix} = \begin{bmatrix} \frac{(cos\widehat{\omega}_k T)T}{\widehat{\omega}_k}\hat{\dot{\xi}}_k - \frac{(sin\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\hat{\dot{\xi}}_k - \frac{(sin\widehat{\omega}_k T)T}{\widehat{\omega}_k}\widehat{\dot{\eta}_k} - \frac{(-1+cos\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\widehat{\dot{\eta}_k} \\ \frac{(sin\widehat{\omega}_k T)T}{\widehat{\omega}_k}\hat{\dot{\xi}}_k - \frac{(1-cos\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\hat{\dot{\xi}}_k - \frac{(cos\widehat{\omega}_k T)T}{\widehat{\omega}_k}\widehat{\dot{\eta}_k} - \frac{(sin\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\widehat{\dot{\eta}_k} \\ -(sin\widehat{\omega}_k T)T\hat{\dot{\xi}}_k - (cos\widehat{\omega}_k T)T\widehat{\dot{\eta}_k} \\ (cos\widehat{\omega}_k T)T\hat{\dot{\xi}}_k - (sin\widehat{\omega}_k T)T\widehat{\dot{\eta}_k} \end{bmatrix} \tag{4.7}$$

To generate the results for this section, the following values were used for the initial state error covariance matrix $P_{0|0}$, the system noise matrix Q, and the measurement noise matrix R.

$$P_{0|0} = \begin{bmatrix} R_{11} & 0 & 0 & 0 & 0 \\ 0 & R_{22} & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

$$Q = L_1 \begin{bmatrix} \frac{T^3}{3} & 0 & \frac{T^2}{2} & 0 & 0 \\ 0 & \frac{T^3}{3} & 0 & \frac{T^2}{2} & 0 \\ \frac{T^2}{2} & 0 & T & 0 & 0 \\ 0 & \frac{T^2}{2} & 0 & T & 0 \\ 0 & 0 & 0 & 0 & \frac{L_2}{L_1}T \end{bmatrix} \tag{4.9}$$

$$R = 50^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.10}$$

Note that L_1 and L_2 are referred to as power spectral densities, and were defined as 0.16 and 0.01, respectively [30]. The system and measurement noise (w_k and v_k) were generated using their respective covariance values (Q and R). Also, when using the UM model, the fifth row and column of (4.8) and (4.9) were truncated. For the standalone SVSF estimation process, the limit on the sliding boundary layer widths were defined as $\delta = [500 \quad 1{,}000 \quad 500 \quad 1{,}000 \quad 1]^T$. These parameters were tuned based on some knowledge of the uncertainties (i.e., magnitude of noise) and with the goal of decreasing the estimation error. It is required to transform the measurement matrix into a square matrix (i.e., identity), such that an 'artificial' measurement is created. It is possible to derive 'artificial' velocity measurements based on the available position measurements. For example, consider the following artificial measurement vector y_k for the SIF:

$$y_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ (z_{1,k+1} - z_{1,k})/T \\ (z_{2,k+1} - z_{2,k})/T \\ 0 \end{bmatrix} \tag{4.11}$$

The accuracy of (4.11) depends on the sampling rate T. Applying the above type of transformation to non-measured states allows a measurement matrix equivalent to the identity matrix. The estimation process would continue as in the previous section, where $H = I$. Note however that the artificial velocity measurements would be delayed one time step. Furthermore, it is assumed that the artificial turn rate measurement is set to 0, since no artificial measurement could be created based on the available measurements. A total of 500 Monte Carlo runs were performed, and the results were averaged.

Simulation Results

In this study's experimentation, both the square-root KF and the proposed SR-SIF were implemented and applied to the target tracking problem and setup described in earlier sections. Results of the target tracking are shown below in Figure 3.

The square-root based SIF was able to accurately follow the target trajectory, regardless of which flight model was in operation. The square-root based EKF, however, had trouble at the occurrence of the coordinated aircraft turns. The difficult experienced is primarily due to the difference between the model used by the EKF and the actual model experienced by the target. The estimation error of each filter is shown in Figure 4. It is easily noticeable from this figure that the SR-SIF yielded relatively similar results, regardless of the type of model implemented. This is primarily attributed to the robust estimation process inherent to the switching gain involved in the SIF. A second case was studied, in which the measurement at 50 seconds was increased a thousand-fold, and it was also determined that this case further demonstrates the robustness of the SR-SIF. The SR-EKF was unable to overcome the measurement error, however the SR-SIF was able to maintain the true target state trajectory. This behaviour is further illustrated by Figures 5 and 6.

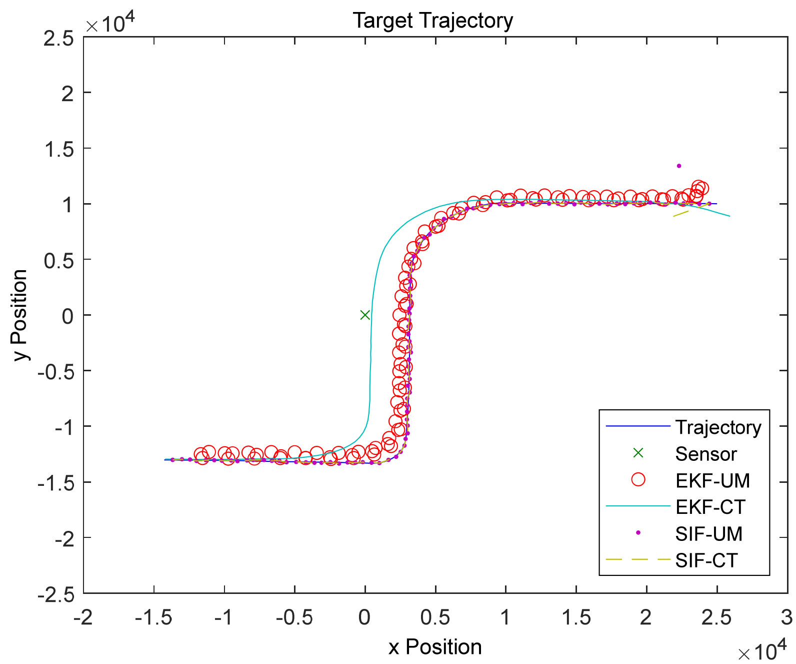

Figure 3. True and estimated target trajectories for the nonlinear estimation problem.

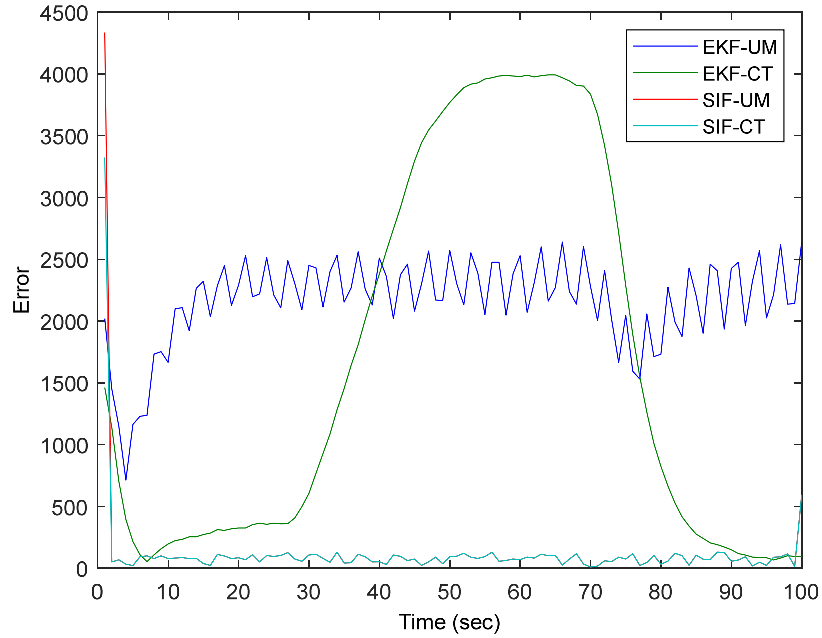

Figure 4. Estimation errors for the nonlinear estimation problem.

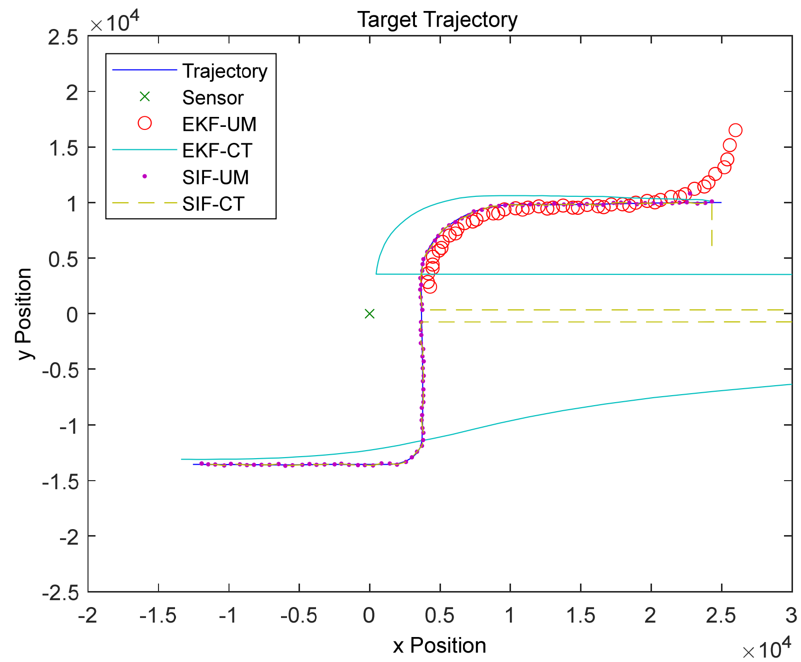

Figure 5. True and estimated target trajectories with the presence of measurement errors.

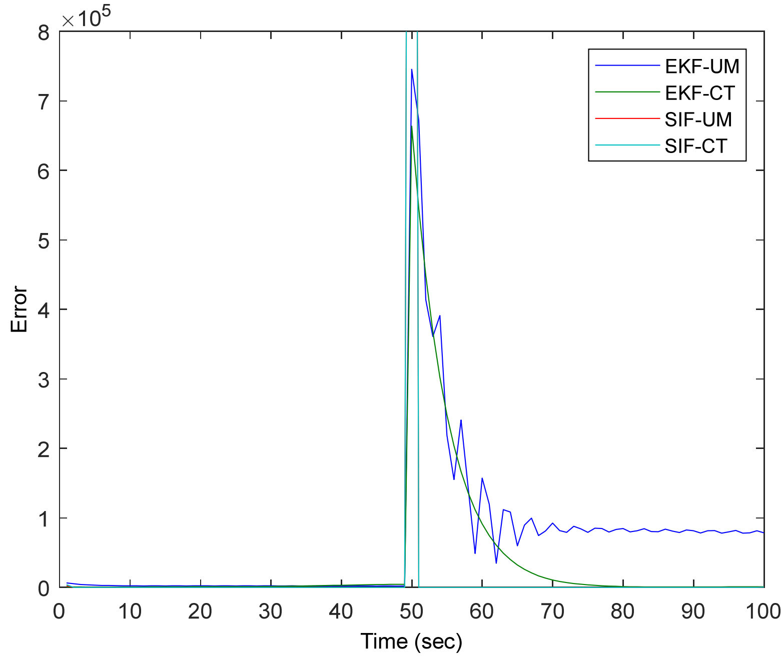

Figure 6. Estimation errors for the square-root filters with presence of measurement errors.

5. CONCLUSIONS

This paper introduced a new version of the sliding innovation filter (SIF) based on Potter's square-root algorithm. The new methodology, referred to simply as the SR-SIF, was applied on a nonlinear target tracking problem. The results were compared with the popular Kalman filter strategy. It was determined that the robustness of the SIF switching gain yielded a stable and accurate estimation of the target. The estimates were found to be founded to the true state trajectory. At the presence of measurement errors, the KF-based strategy failed to yield a good result, however the SIF-based strategy remained stable. Future work includes implementing the SR-SIF on real-life data and benchmark problems.

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