A Multiple Model Adaptive SVSF–KF Estimation Strategy

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ABSTRACT

State estimation strategies play a critical role in obtaining accurate information about the state of dynamic systems as they develop. Such information can be important on its own and critical for precise and predictable control of such systems. The Kalman filter (KF) is a classic algorithm and among the most powerful tools in state estimation. The Kalman filter however can be sensitive to modeling uncertainty and sudden changes in system dynamics. The Smooth Variable Structure Filter (SVSF) is a relatively new estimation strategy that operates on variable structure concepts. In general, the SVSF has the advantage that is can be quite robust to modeling uncertainty and sudden fault conditions. Recent advancements to the SVSF, such as the addition of a covariance formulation, and the derivation of a time varying smoothing boundary layer (VBL), have allowed for combined SVSF – KF strategies. In a typical SVSF-KF approach, the VBL is used to detect the presence of a system fault, and switch from the more optimal KF gain to the more robust SVSF gain. While this approach has been proven effective in several cases, there are circumstances where the VBL will fail to indicate the presence of an ongoing fault. A new form of the SVSF-KF is proposed, based on the framework of the Multiple Model Adaptive Estimator.

Keywords: Smooth Variable Structure Filter, SVSF-KF, Kalman Filter, MMAE, fault detection, robust estimation

1. INTRODUCTION

In many engineering problems we often need information about a dynamic system, either to accomplish a task or because the information is important in its own right. Typically, the means to obtain the information we need comes from sensor measurements. Often these measurements are corrupted by bias, noise, and other inaccuracies. In addition, in many cases we are unable to directly measure all the system states we need – we must somehow obtain them from the measurements we have.

Estimation theory is the discipline of extracting information about the states or parameters of a system from the information available. Estimation techniques of various kinds find ubiquitous implementation in all sorts of modern engineering applications. Arguably the most significant development in estimation theory of the 21st century was the Kalman filter. The Kalman filter was discovered by Rudolph Kalman as well as Richard Bucy and others around 1958[1]. Also termed the "Linear Least Squares Estimator" (LLSME) or the "Linear Quadratic Estimator" (LQE), the Kalman filter has been the workhorse of estimation for the last several decades, finding application in space flight, navigation, and many other engineering problems[2].

The Kalman filter though is not without its disadvantages. In its original formulation, it was only applicable to linear systems, a significant limitation when trying to solve real world problems. Additionally, the Kalman filter can be susceptible to computer round-off errors, causing it to lose stability and performance. Also, the Kalman filter depends on a good internal model of the system, as well as knowledge of the system's noise characteristics. In the presence of modeling uncertainty, or a sudden change to the system's dynamics, the performance of the Kalman filter can become degraded, or even fail altogether.

Significant research and development of the Kalman filter has resulted in the mitigation of many of these problems. Extensions to the Kalman filter such as the Extended Kalman filter, the Unscented Kalman filter, and the Cubature Kalman filter, have made the filter applicable to nonlinear systems as well. Square root filtering techniques have

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helped deal with stability and computer precision problems, making the Kalman filter an practical in small-scale embedded systems. Adaptive filtering as well as other strategies, have made the Kalman filter more robust to both uncertain and changing model dynamics. Beyond the Kalman filter, many additional estimation techniques have been put forward, each with its unique advantages and disadvantages.

More recently, a relatively new approach to the estimation problem has been presented. Called the Smooth Variable Structure Filter (SVSF), this estimator is based on variable structure theory concepts, similar to those found in sliding mode control and observation[3]. Like the Kalman filter, the SVSF takes the form of a predictor-corrector – making a model-based prediction and refining the prediction with measurement data using a corrective gain. The SVSF gain uses a nonlinear switching action to drive the state estimates to with a region of the true states known as the "existence subspace." The width of the existence subspace is unknown but assumed to be bounded. Once within the existence subspace, the state estimates are forced to remain there throughout the estimation process. Because of the nature of this switching action, an artificial high frequency noise called "chattering" is introduced. To mitigate the effects of this chattering, a smoothing boundary layer (SBL) term is added. The width of the smoothing boundary layer determines the extent to which the chattering is attenuated, but at the same time can result in degradation of the filter's performance[3,4].

The SVSF has some significant advantages. The SVSF can be applied to both linear and nonlinear estimation problems – so long that the system in question is observable and its differential equations are differentiable and smooth. In addition, the SVSF is highly robust to modeling uncertainty as well as sudden changes to the system dynamics. The SVSF also has multiple indicators of performance. The chattering signal in each of estimated states can be used for the purposes of fault detection, as well as in schemes for adaptive refinement of the system model.

Since its introduction in 2007, the SVSF has undergone several improvements, and continues to be an active area of academic research. In its initial formulation, the SVSF lacked a covariance term, used a fixed smoothing boundary layer width, and required a full measurement matrix. Some of the early improvements to the SVSF include a covariance derivation[5,6], an optimal time varying smoothing boundary width[3,4], and a strategy for better dealing with missing measurements[4]. In addition, the SVSF has been integrated with Interacting Multiple Model adaptive strategies[5], a second order SVSF formulation has been derived[7,8], as has seen many other improvements and applications[9,10,11,12,13,14,15,16].

With the derivation of an optimal time varying smoothing boundary layer (VBL), it was shown that the SVSF reduces to the Kalman filter for linear systems. Based on this result, a joint strategy was proposed. Called the SVSF-VBL or SVSF-KF, the idea was to combine the optimality of the linear KF, with the robustness of the SVSF. Under normal operating conditions, where the system behaves as expected, and the filter model is accurate the SVSF-KF uses the standard KF gain. In the presence of a system change or fault, the VBL width grows beyond its normal bounds, and this information is used to switch the filter to the more robust SVSF gain. This approach was demonstrated to be effective in multiple cases[5,9].

In this paper we note that the original formulation of the SVSF-KF may not be suited to all situations. As we demonstrate, the VBL at times cannot provide an ongoing indication of a fault, resulting in a failure to switch to the SVSF gain. Without the SVSF gain, the estimate ultimately drifts and fails to correctly track the system states. As a solution to this problem, we propose an alternate formulation of the SVSF-KF based on the Multiple Model Adaptive Estimator. In this approach, both the SVSF and KF are run in a parallel filter bank, with the respective filter innovations used to compute the probabilities that a given filter is correct. We demonstrate our results in a simple toy test scenario.

This paper is organized as follows: Part 1 is the introduction. In Part 2 we review the Kalman Filter, the SVSF in its original formulation, and the derivation of the covariance term for the SVSF along with the optimal time varying smoothing boundary layer. In Part 3 we discuss the original SVSF-KF strategy and note by a simulation example some of the issues discovered. In part 4 we present an alternative implementation of the SVSF-KF based on the MMAE. In part 6 we draw our conclusions.

2. BACKGROUND

2.1 The Kalman Filter

For a linear system, the system model can be expressed in state representation form as follows:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{w}_k \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{2}$$

In equation (1), $\boldsymbol{x}_k$ is the system state vector. $\boldsymbol{A}$ is the discretized linear system matrix, $\boldsymbol{B}$ is the input gain matrix, $\boldsymbol{u}_k$ is the input vector and $\boldsymbol{w}_k$ is the system noise. In equation (2), $\boldsymbol{z}_k$ is the measurement vector, $\boldsymbol{H}$ is the linear measurement matrix, and $\boldsymbol{v}_k$ represents the measurement noise.

The Kalman filter assumes that the system model is well known and linear, the initial states are known, and the measurement and system noise is white with zero mean and known respective covariance matrices[4]. The Kalman filter works as a predictor-corrector; the system model is used to obtain an *a priori* estimate of the states, whereupon measurements combined with the Kalman gain matrix are used to apply a correction term to create an updated *a posteriori* state estimate.

What makes the Kalman filter so effective is the ability of an appropriately computed Kalman Gain matrix to optimally minimize the estimation error. The steps and equations that form the basic Kalman filter algorithm are presented below[4]:

The *a priori* state estimate (3) is made based on the process model. The *a priori* state covariance matrix (4) is calculated based on the process model and the associated modeling noise covariance matrix $\boldsymbol{Q}_k$.

$$\widehat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{A}\widehat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{B}u_k \tag{3}$$

$$\widehat{\boldsymbol{P}}_{k+1|k} = \boldsymbol{A}\boldsymbol{P}_{k|k}\boldsymbol{A}^T + \boldsymbol{Q}_k \tag{4}$$

The Kalman gain (5) is computed based on (4) and is then used to update the state estimate (6):

$$\boldsymbol{K}_{k+1} = \frac{\boldsymbol{P}_{k+1|k}\boldsymbol{H}^T}{\boldsymbol{H}\boldsymbol{P}_{k+1|k}\boldsymbol{H}^T + \boldsymbol{R}_{k+1}} \tag{5}$$

$$\widehat{\boldsymbol{x}}_{k+1|k+1} = \widehat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1}\left(\boldsymbol{z}_k - \boldsymbol{H}\widehat{\boldsymbol{x}}_{k+1|k}\right) \tag{6}$$

The *a posteriori* state error covariance matrix is then calculated as per (7), and the process repeats iteratively.

$$\boldsymbol{P}_{k+1|k+1} = (\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H})\boldsymbol{P}_{k+1|k} \tag{7}$$

In a successful application of the Kalman filter, the state estimates will rapidly converge, providing the optimal statistical estimate based on the given information.

2.2 The Smooth Variable Structure Filter

The Smooth Variable Structure filter is a relatively recent development, appearing in 2007. Also formulated as a predictor corrector, it is based on variable structure theory as well as sliding mode concepts. The basic idea is to use a switching gain to drive estimates to within a defined boundary of the true states - termed the "existence subspace." Once within this subspace, estimates should remain confined throughout the estimation process. To reduce the effects of the chattering caused by the non-linear switching gain, as well as to reduce the sensitivity to measurement noise, a smoothing boundary layer (SBL) can be applied. The width of this boundary layer generally determines the overall performance of the SVSF. Too wide and the estimates become less accurate, too small and chattering and excessive measurement noise can corrupt the estimates.

The SVSF can be applied to both linear systems modeled as (1) or non-linear systems expressed as in (8):

$$\boldsymbol{x}_{k+1} = \boldsymbol{\mathcal{F}}(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k) \tag{8}$$

$$\boldsymbol{z}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{9}$$

Important prerequisites of the SVSF is that the system in question be differentiable as well as observable. In addition, a full rank linear measurement matrix $\boldsymbol{H}$ is required for the SVSF to operate. In situations where full measurements are not available, an augmented measurement matrix can be constructed using a reduced order observer strategy.

The steps and equations that form the essential SVSF algorithm are presented below[3]:

An a priori state estimate is determined using an approximate system model $\widehat{\boldsymbol{\mathcal{F}}}$ and the previous *a posteriori* (or initial) state estimate. In the case of linear systems, the *a priori* state can be obtained as with (3). The estimated *a priori* measurements are then calculated (11).

$$\widehat{\boldsymbol{x}}_{k+1|k} = \widehat{\boldsymbol{\mathcal{F}}}\left(\widehat{\boldsymbol{x}}_{k|k}, \boldsymbol{u}_k\right) \tag{10}$$

$$\widehat{\boldsymbol{z}}_{k+1|k} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{x}}_{k+1|k} \tag{11}$$

The *a priori* output error estimate is calculated

$$\boldsymbol{e}_{z_{k+1|k}} = \boldsymbol{z}_k - \widehat{\boldsymbol{z}}_{k+1|k} \tag{12}$$

The SVSF gain, $\boldsymbol{K}_{k+1}$ is calculated based on the *a priori* and *a posteriori* output error estimates. The *a priori* state estimate is then updated to the *a posteriori* state estimate using the SVSF gain

$$\widehat{\boldsymbol{x}}_{k+1|k+1} = \widehat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1} \tag{13}$$

The a priori measurements are updated to the *a posteriori* measurements:

$$\widehat{\boldsymbol{z}}_{k+1|k+1} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{x}}_{k+1|k+1} \tag{14}$$

The *a priori* output error estimate is updated to the *a posteriori* output error estimate. The process then repeats iteratively.

$$\boldsymbol{e}_{z_{k+1|k+1}} = \boldsymbol{z}_k - \widehat{\boldsymbol{z}}_{k+1|k+1} \tag{15}$$

The SVSF gain is derived based on Lyapunov stability condition. It can be shown that to achieve a stable estimation process, the estimation error must be reduced with each time step.

$$\left|\boldsymbol{e}_{k+1|k+1}\right| < \left|\boldsymbol{e}_{k|k}\right| \tag{16}$$

Using the above theorem, the following a set of conditions for the SVSF gain can be derived:

$$\left|\boldsymbol{e}_{z_{k+1|k}}\right| \leq \left|\widehat{\boldsymbol{H}}\boldsymbol{K}\right| < \left|\boldsymbol{e}_{k|k}\right| + \left|\boldsymbol{e}_{z_{k+1|k}}\right| \tag{17}$$

and

$$sign\left(\widehat{\boldsymbol{H}}\boldsymbol{K}\right) = sign\left(\boldsymbol{e}_{z_{k|k}}\right) \tag{18}$$

An SVSF gain that satisfies the above conditions can be expressed as:

$$\boldsymbol{K} = \widehat{\boldsymbol{H}}^{+}\left(\gamma\left|\boldsymbol{e}_{k|k}\right| + \left|\boldsymbol{e}_{k+1|k}\right|\right) \circ sign\left(\boldsymbol{e}_{z_{k+1|k}}\right) \tag{19}$$

Where $^+$ denotes the pseudo inverse, and $\circ$ denotes the Schur product. γ is a diagonal scalar matrix such that $0 < \gamma_{ii} < 1$.

As mentioned, to reduce the effects of chattering, as well as improve the overall quality of the state estimates, a Smoothing Boundary Layer (SBL) can be introduced. Inside the SBL the corrective action of the SVSF gain is interpolated based the ratio of the *a priori* estimation error and the smoothing boundary layer width $\boldsymbol{\Psi}$.

$$\boldsymbol{K} = \widehat{\boldsymbol{H}}^{+}\left(\gamma\left|\boldsymbol{e}_{k|k}\right| + \left|\boldsymbol{e}_{k+1|k}\right|\right) \circ sat\left(\boldsymbol{e}_{z_{k+1|k}}, \boldsymbol{\Psi}\right) \tag{20}$$

Above represents the original form of the SVSF. In this form, unlike many other estimation strategies, there is no covariance computed representing the overall uncertainty of the SVSF estimate. Also, the smoothing boundary layer width $\boldsymbol{\Psi}$ remains at a fixed conservative value throughout the estimation process.

2.3 Covariance and Time Varying Smoothing Boundary Layer for the SVSF

Recently[5,17], a covariance derivation for the SVSF for linear systems was presented. To derive the covariance, a revised SVSF update of the form (21) was proposed.

$$\hat{\boldsymbol{x}}_{k+1|k+1} = \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1}\boldsymbol{e}_{k|k-1} \tag{21}$$

The SVSF gain according to this formulation becomes

$$\boldsymbol{K} = \hat{\boldsymbol{H}}^{+} diag\left[\left(\gamma\left|\boldsymbol{e}_{k|k}\right| + \left|\boldsymbol{e}_{k+1|k}\right|\right) \circ sat\left(\boldsymbol{e}_{z_{k+1|k}}, \boldsymbol{\Psi}\right)\right]\left[diag\left(\boldsymbol{e}_{k+1|k}\right)\right]^{-1} \tag{22}$$

In this form, a covariance derivation similar in structure to that of the Kalman filter was obtained. The SVSF *a priori* and *a posteriori* covariance calculation was determined to be

$$\boldsymbol{P}_{k+1|k} = \boldsymbol{A}\boldsymbol{P}_{k|k}\boldsymbol{A}^{T} + \boldsymbol{Q}_{k} \tag{23}$$

$$\boldsymbol{P}_{k+1|k+1} = (\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H})\boldsymbol{P}_{k+1|k}(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H})^{T} + \boldsymbol{K}_{k+1}\boldsymbol{R}\boldsymbol{K}_{k+1}^{T} \tag{24}$$

Note that the covariance structure is identical to that of the Kalman filter – the latter being in the "Joseph form." While the basic structure is the same, the SVSF covariance will differ from that of the Kalman filter, on account of the SVSF gain.

With the availability of a covariance expressing the overall SVSF estimation uncertainty, effort was made to develop an optimal time varying smoothing boundary layer. Using the form of the SVSF gain expressed in (22), and considering only the region within the saturation limits, the SVSF gain can also be written as:

$$\boldsymbol{K} = \boldsymbol{H}^{-1}\bar{\boldsymbol{A}}\boldsymbol{\Psi}^{-1} \tag{25}$$

Where

$$\bar{\boldsymbol{A}} = diag\left[\left(\gamma\left|\boldsymbol{e}_{k|k}\right| + \left|\boldsymbol{e}_{k+1|k}\right|\right)\right] \tag{26}$$

Using the gain in the form of (25), one can determine an optimal time varying smooth boundary layer width by minimizing the trace of the *a posteriori* covariance with respect to the proposed VBL.

$$\frac{\partial(trace(\boldsymbol{P}_{k+1|k+1}))}{\partial\boldsymbol{\Psi}} = 0 \tag{27}$$

One arrives at the following calculation of the VBL

$$\boldsymbol{\Psi} = \left(\frac{\bar{\boldsymbol{A}}^{-1}\boldsymbol{H}\boldsymbol{P}_{k+1|k}\boldsymbol{H}^{T}}{\boldsymbol{H}\boldsymbol{P}_{k+1|k}\boldsymbol{H}^{T} + \boldsymbol{R}}\right)^{-1} \tag{28}$$

It was noted[5] that the optimal time varying smoothing boundary layer $\boldsymbol{\Psi}$ in the SVSF simply yields the Kalman gain. It was thus concluded that in a linear case, the optimal smoothing boundary layer reduced the SVSF to the Kalman filter. While the KF yields the optimal state estimate, the robust switching effect of the SVSF was lost.

3. THE SVSF-KF

3.1 Original formulation

Based on the above conclusions, a joint SVSF-KF approach was proposed[5,9,18]. The idea was to combine the benefits of the more optimal KF, with the robustness characteristics of the SVSF. In the original SVSF-KF, a gain switching algorithm was chosen. During normal operation of the filter, the KF gain would be used update the state – providing a better overall estimate. When encountering a sudden system change, the SVSF-KF would switch to the SVSF gain, ensuring that the filter remains stable. In this process VBL is used to detect the onset of modeling error. During normal operation, the VBL is expected to remain bounded within a fixed region. In the presence of a system change, the VBL will begin to grow beyond this bound. In setting up the filter, the designer chooses a fixed VBL limit – if the VBL grows beyond this limit, the filter switches to the SVSF gain. Figures 1 and 2 illustrate the overall concept.

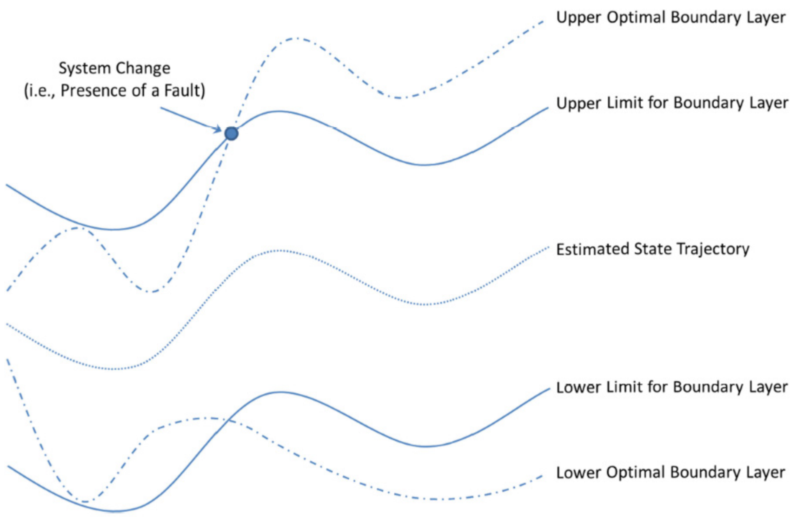

Figure 1. Behavior of Time Varying Smoothing Boundary Layer (VBL) during a fault condition[5].

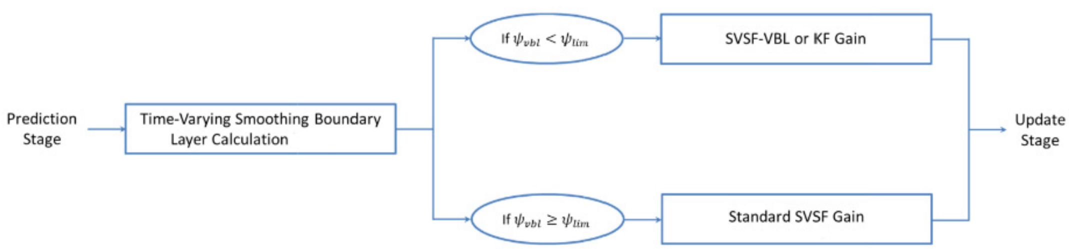

Figure 2. Illustration of overall SVSF-KF strategy. Gain switch determined by whether VBL is within prespecified limit[5].

The SVSF-KF strategy was applied and successfully demonstrated in a variety of cases, including target tracking problems, as well as an electro hydrostatic actuator system[5]. In addition, the overall strategy was shown not to be limited to just the linear KF gain. The SVSF-KF strategy was successfully demonstrated using several nonlinear approaches as well, such as the EKF, UKF, and CKF – termed the SVSF-EKF, SVSF-UKF, and SVSF-CKF respectively[5].

3.2 Issues with the standard SVSF-KF

In further exploring the SVSF-KF strategy, it has been discovered that under certain conditions the calculated VBL will fail to provide an ongoing indication of a sustained fault condition. The result of this is that the SVSF-KF will only temporarily switch to the robust SVSF gain, only switch back to the KF gain and ultimately fail to track the true states. Ideally, in the case of a sustained fault/modeling change, the overall filter strategy would switch to and continue to use the SVSF gain – so long as the fault conditioned remained.

To illustrate, we shall use a simple toy scenario. Consider the one-dimensional harmonic oscillator, formulated as a linear spring mass damper system. The state equations can be expressed as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{29}$$

Where k is the spring constant, m is the mass, and c is the damping coefficient. x and v are the position and velocity states respectively. We shall simulate this system for a 15 kg mass, with 5 N/m spring constant and 2 N s/m damping coefficient. Also, we shall introduce artificial measurement noise, Gaussian distributed with zero mean and a variance of 0.001. We shall assume no process noise in this example. The system shall be excited by assuming an initial displacement of 1 m.

We seek to estimate the natural decay response of the system from the noisy measurements. In this case, for the purposes of highlighting the SVSF-KF approach we shall introduce a sudden unmodeled fault condition. At $t = 20$ s, the actual mass of the system is suddenly increased from 15 kg to 30 kg. For our initial test we demonstrate both a standard SVSF with a fixed SBL as well as a Kalman filter. Both the SVSF and KF use the original spring damper model throughout the simulation, and thus will be working with an incorrect model after 20 s. Both filters were provided with reasonable assessments of initial covariance, in this case values of 1.2 and 0.2 for the position and velocity states respectively. Each filter was also provided with accurate process and measurement noise covariances.

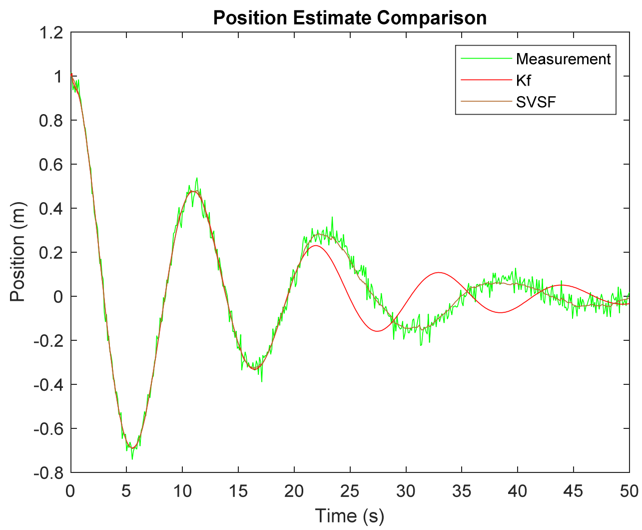

Figure 3. Position state estimates of a simple harmonic oscillator in free response. Sudden system change occurs after 20 seconds.

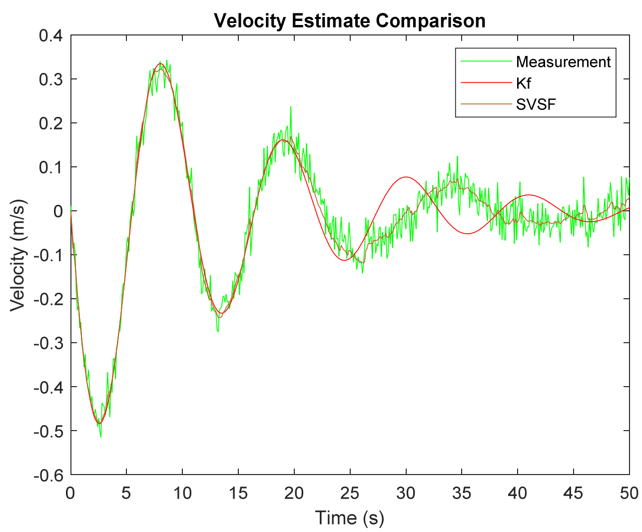

Figure 4. Velocity state estimates of a simple harmonic oscillator in free response. Sudden system change occurs after 20 seconds.

The KF provides the optimal estimate during the initial phase of the simulation prior to the onset of modeling error. The SVSF also provides a reasonable estimate, albeit not optimal. After 20 seconds however, the KF fails to track the states and begins to drift – putting excess confidence in its own filter model. The SVSF however continues to track the states well, remaining robust despite the fault.

We now consider the implementation of the SVSF-KF. The goal of this approach is to provide the optimal KF estimate, so long as the system behaves as expected, while switching to the more robust SVSF when encountering modeling uncertainty. As noted, the algorithm works by alternatively updating the state estimate with either the KF gain or the standard fixed SBL SVSF gain, depending on the size of the VBL. So long as the VBL remains within an expected designer defined range, the KF gain is used, when the VBL expands beyond this set limit, a modeling error is assumed and the fixed SBL SVSF gain is used. Figure 5 below shows the VBL for the position state. One can clearly see the onset of the fault condition around 20 s.

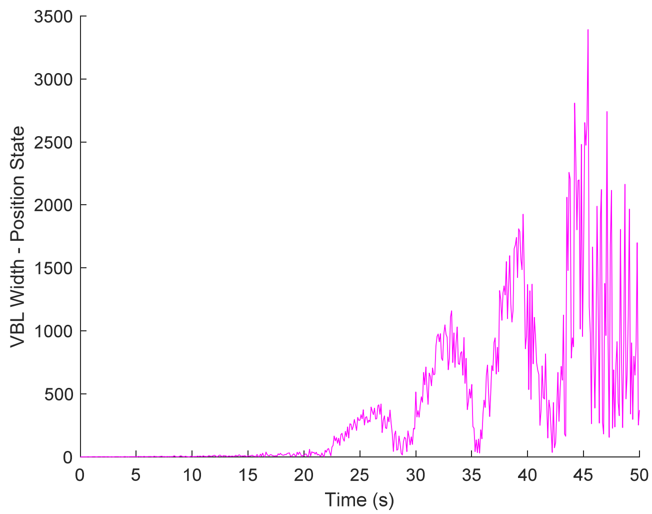

Figure 5. Behavior of the VBL throughout the simulation. Onset of modeling error after 20 s is readily apparent.

In the SVFS-KF strategy reported in the literature, the overall algorithm was successfully implemented using a fixed limit imposed on the VBL width as the determining factor in switching the gain. As we will note in this example, this may not always be effective. Considering the plot above, a VBL limit of around 100 seems be a reasonable choice to affect the gain switch soon after the occurrence of the fault. Figure 6 shows the state estimate produced by the SVSF-KF.

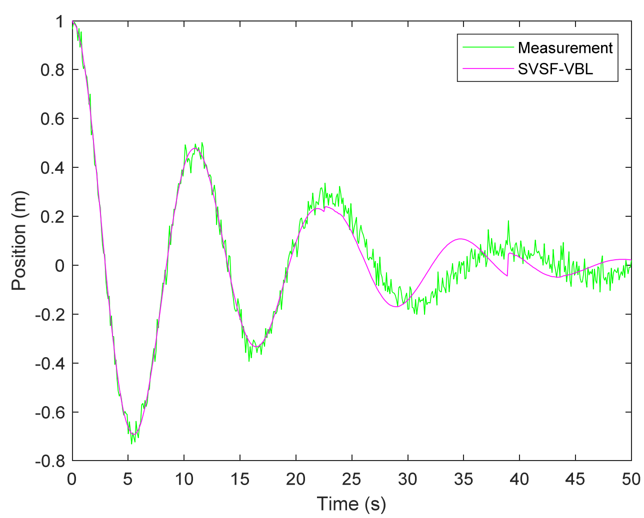

Figure 6. Position estimate of SVSF-KF strategy using a VBL limit of 100. Filter fails to maintain accurate state estimate during ongoing system change.

We note that the filter temporarily switches to the SVSF gain early after the fault as well as around 39 s, however the filter fails to remain in SVSF mode and mostly uses the inaccurate KF gain. Plotting the SVSF gain ON condition with respect to time (Figure 7) confirms this (0 KF gain in use, 1 for SVSF gain). We can see that the filter only switches to the SVSF gain twice during the simulation.

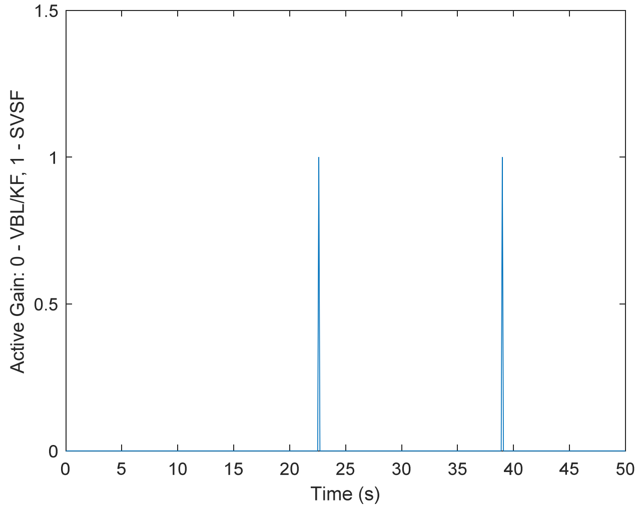

Figure 7. Plot showing which gain is in use during operation of SVSF-KF. 0 denotes the KF gain, 1 denotes the SVSF gain. SVSF gain is only triggered twice despite continued modelling error.

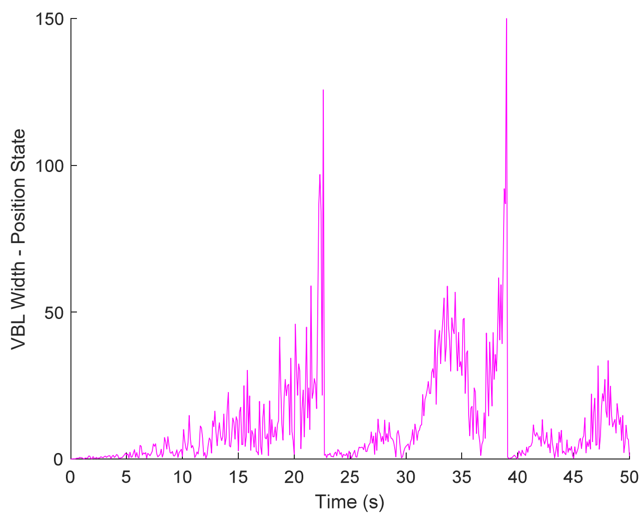

Figure 8. Behavior of VBL during active operation of SVSF-KF. VBL drops rapidly following each activation of SVSF gain.

In Figure 8 we replot the VBL with respect to time. As can be seen, each time the SVSF gain is activated, the VBL value is lowered significantly below the VBL set limit of 100. The SVSF gain thus only remains active briefly, whereupon the KF gain resumes and the estimate continues to drift. As is apparent, the VBL fails to grow fast enough to hit the VBL limit to retrigger the SVSF gain until about 15 seconds later. Attempts to increase the likelihood of switching to the SVSF gain by continually lowering the VBL limit generally do not help. The same phenomenon repeats itself, until the VBL fails to provide a meaning indication of the presence of the fault at all. The desired outcome of the optimal KF gain being used during normal operation, and a sustained switch to the SVSF gain after the onset of a fault appears difficult to achieve.

4. MMAE SVSF-KF

Considering the difficulties encountered using a single filter based SVSF-KF with a VBL driven gain switching action, we suggest that an alternative approach is needed. We propose a filter bank approach where a standard fixed SBL SVSF is run in parallel with the KF. In this new approach, we use the framework of the Multiple Model Adaptive Estimator (MMAE) originally proposed in 1965[19]. The MMAE has been shown to be effective in a variety of cases[20,21] and operates by running multiple filters in parallel with a probabilistic framework to switch between the filters. In a typical implementation, the MMAE uses several Kalman filters, each with a different system model. We propose using the MMAE to coordinate switching between a well-tuned Kalman filter and the SVSF - with the goal of optimizing both the overall state estimate, as well as its robustness.

The MMAE uses both the filter residuals, or innovations, as well as the innovation covariance to detect changes in the model, and asses the likelihood that one of the alternant filters has a better model. The innovation and innovation covariance can be expressed as follows.

$$\boldsymbol{\nu} = \boldsymbol{z}_k - \boldsymbol{H}\boldsymbol{A}\hat{\boldsymbol{x}}_{k+1|k} \tag{30}$$

$$\boldsymbol{P}_{\nu\nu,k} = \boldsymbol{H}\boldsymbol{P}_{k+1|k}\boldsymbol{H}^T + \boldsymbol{R}_{k+1} \tag{31}$$

These two values are used to compute a likelihood function for each filter running in the MMAE. We shall consider only the position estimate, so we can treat the innovations as a scalar value. The likelihood function for the i^{th} filter given a scalar innovation input can be expressed as[21]:

$$f_{(i),k} = \frac{1}{\sqrt{2\pi P_{(i)\,\nu\nu,k}}} \exp\left[\frac{0.5\nu_{(i)}^2}{P_{(i)\,\nu\nu,k}}\right] \tag{32}$$

For a bank of r filters in the algorithm, probabilities for each filter can computed using Bayes rule and the computed likelihood function of each filter.

$$p_{(i),k} = \frac{f_{(i),k}\,p_{(i),k-1}}{\sum_{i=2}^{r} f_{(i),k}\,p_{(i),k-1}} \tag{33}$$

The MMAE uses the computed probabilities to provide a final weighted output estimate.

$$\hat{\boldsymbol{x}}_{MMAE,k+1} = \sum_{i=2}^{r} \hat{\boldsymbol{x}}_{(i),k}\,p_{(i),k} \tag{34}$$

Note that the MMAE will require an initial probability for each filter.

We now simulate our same scenario using the MMAE approach. We assume an initial filter probability of 0.5 for both the KF and SVSF. The SVSF and KF are tuned the same way as shown in the original example. Results for the position state appear in Figure 9 below:

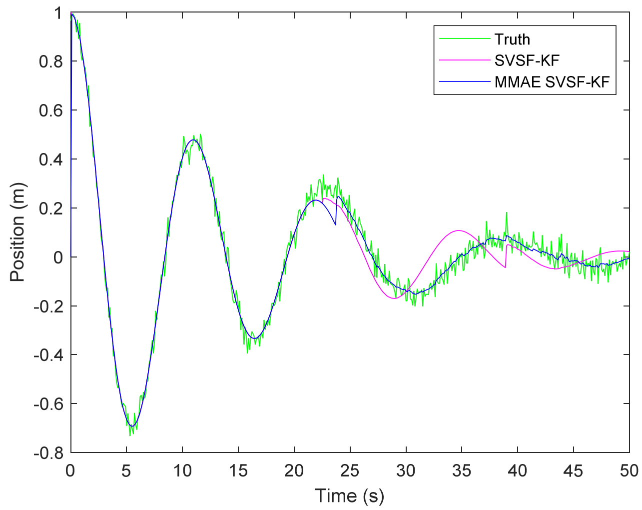

Figure 9. Comparison of regular SVSF-KF with proposed MMAE SVSF-KF. The latter successfully switches to and maintains the estimate using the SVSF.

As can be seen the MMAE approach successfully detects the modeling error less than 5 seconds after it occurs. The MMAE SVSF-KF also remains in the SVSF mode for the remainder of the simulation, ensuring a more stable (albeit noisier) estimate. Prior to the onset of the fault, the algorithm provided the more optimal Kalman filter estimate.

We provide a summary of the overall performance of each of the filtering methods listed above. We record the root mean square (RMSE) position error for each of the filter's discussed, based on a 500 Monte Carlo run simulation. To better compare overall performance, we list the RMSE prior to the modeling fault, after the modeling fault, and the total RMSE throughout the simulation. As to be expected the KF provides an excellent estimate prior to the system change but becomes much worse afterward. The standard SVSF provides the best overall performance, albeit its estimate in the prior stage is not as good. The MMAE SVSF-KF provides the best compromise approach – providing the optimal estimate in the prior stage, and a good estimate after the fault. Indeed, the MMAE SVSF-KF only faired worse than the standard SVSF in this simulation due to brief time taken for filter switch to occur. In longer running scenarios, one could easily see how the MMAE SVSF-KF could outperform the standard SVSF as well.

Table 1. RMSE filter performance comparison.

	KF	**SVSF**	**SVSF-KF**	**MMAE SVSF-KF**
RMSE Prior to System Change	0.0033	0.0068	0.0033	0.0032
RMSE After System Change	0.0956	0.0098	0.0647	0.0138
Total RMSE	0.0917	0.0109	0.0619	0.0190

5. CONCLUSIONS

As discussed in the literature, the aim of the SVSF-KF was to provide a robust estimation strategy which would use the optimal Kalman Filter gain during normal operation, and the more robust SVSF gain in the presence of a fault condition or system change. The strategy was based on a gain switching approach, determined by the size of the SVSF variable boundary layer size. As has been demonstrated, there are scenarios where when executing this approach, the VBL will fail to be a good indicator of a sustained fault condition. As an alternative to the current SVSF-KF, we have proposed a new approach based on the MMAE strategy. By running the filters independently and in parallel, and using a filter switching action based on the filter innovations, modeling error can be effectively detected and mitigated.

It is important to note that in our particular example, the Kalman filter drift could have easily be remediated by simply inflating the process noise. The example chosen was merely for demonstration purposes. In a real-world design situation however, inflation of the filter's process noise may be less than ideal or not an option at all. The new MMAE SVSF-KF strategy gives the designer an alternative approach to achieving optimal state estimates in normal operating conditions, while remaining stable and robust in the face of sudden system changes.

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