Quadrature Kalman Filters with Applications to Robotic Manipulators

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***Abstract*— In this work, well known Quadrature Kalman Filters (QKFs), namely 2-point QKF (SeQKF), 3-point QKF (ThQKF), and 4-point QKF (FoQKF), were used to monitor a 4-degree of freedom prismatic-revolute-revolute-revolute (PRRR) manipulator. This manipulator represents a well-known industrial arm robot. These methods are applied on a PRRR robot, and are compared in terms of stability, robustness, computation time, complexity, and the quality of the optimality. For completeness, the results were compared to those obtained from the popular Unscented Kalman Filter (UKF) and a special form of the UKF known as the Cubature Kalman Filter (CKF).**

Keywords— Quadrature Point; Filtering; Estimation; Unscented Kalman Filter; Cubature Kalman Filter; Robotic Arm; PRRR.

I. INTRODUCTION

Since the last decade, robotic arms and manipulators are widely used in industrial applications due to their robustness, stability, and their ability to work in dull, dangerous, and dirty environments [1, 2, 3]. The enhancement of existing robotic designs commonly requires the use of high precision motors and sensors, which inevitably carries the burden of increased costs. In the effort of maintaining precision while balancing cost, estimation techniques, including filters, can be used as an alternative to high accuracy devices. By using such techniques, an industry may use less expensive components with minimal effects on overall performance. The prospect of achieving high performance with less capital investment has led many researchers to investigate and expand the use of estimation techniques within highly industrialized fields [4, 5, 6, 7].

One of the major branches in filtering is the optimal filter [8, 9]. These filters try to minimize the cost function to obtain an optimal solution; i.e. [8, 9, 10, 11]:

1- ***Minimum mean squared error*** (MMSE).

2- ***Maximum a posteriori*** (MAP).

The pioneer work of Rudolf Kalman in 1960 [12] is considered one of the major steps in estimation theory. It is a recursive, optimal and model based estimator formulated as predictor-corrector form. Some of the limitations associated with this filter is that it only applies to linear systems containing normally distributed uncertainties [9, 11, 12]. A Large number of researchers broadened that work and tried to modify the Kalman Filter (KF) to overcome these limitations. These efforts included the Joseph Stabilized KF [9], Sequential KF [9], Information KF [13, 14], Perturbation KF [15, 16], Extended KF (EKF) [17], Iterated EKF [9, 13], Higher Order EKF [9, 18], Unscented KF [9, 19, 20, 21, 22, 23], Central Difference KF [24, 18, 25], Quadrature KF [26, 27, 28, 29] and Cubature KF [28, 30, 31]. Additionally, vast amounts of research has been dedicated to combining the KF with other filtering methods [1, 3, 32, 33] and intelligent techniques [34, 35, 36, 37] to improve its robustness given uncertainty, and system and measurement noise.

This paper targets the Sigma-point Kalman filters (SPKF) which include the Unscented Kalman Filter (UKF) and Cubature Kalman Filter (CKF), in addition to the Quadrature Kalman Filter (QKFs). These popular estimation filters are widely known for their optimality/sub-optimality solutions. The filters have the ability to handle noise and the nonlinearity of a system without the use of linearization techniques. The two previously mentioned filters are commonly used for nonlinear applications, are widely used within common technology, and are highly recommended for industrial applications [1, 2, 32].

Within the subsequent sections, the use of SPKFs (specifically the UKF and CKF) and the QKFs (2, 3, and 4 point) to estimate the states of a 4-DoF PRRR robotic arm are discussed and compared on the basis of stability, robustness, the quality of optimality, complexity, and computation time.

The paper is divided into four sections, with section 2 dedicated to the QKFs, and the SPKFs while section 3 is assigned to the PRRR robotic system, results and discussion. Section 4 summarizes the effectiveness of the filters in estimating the states of the PRRR manipulator given the simulated results for three different uncertainty and noise scenarios.

Nomenclature

$\mathbf{a}^{-1}, \mathbf{a}^{T}$	Inverse, and transpose of the vector/matrix **a**, respectively.
$\hat{\mathbf{a}}, \bar{\mathbf{a}}$	The estimate and mean of **a**, respectively.
$\dot{\mathbf{a}}$	Derivative of **a** with respect to time. More dots indicates higher derivative.
$(\mathbf{a})_i$	The i row of **a.**
$\mathbf{e_m}$	The estimation error vectors in **m**.
$\mathbf{f}(.)$	System nonlinear matrix.
$\mathbf{g}(.)$	Measurement nonlinear matrix.
i, j	Subscripts used to identify elements.
$\mathbf{I}_{n\times n}$	The identity matrix with dimensions of $n \times n$.
k	Time step value.
$k\vert k-1$	The a priori value at time k.
$k\vert k$	The a posteriori value at time k.
$\mathbf{K}_X$	The correction gain of the filter X.
m	Number of measurements
n	Number of states
$\mathbb{N}(x\vert m, P)$	The Gaussian probability density function, with mean of m and standard deviation of P.
$\mathbf{P}_{\mathrm{xx}}$	The state's error covariance matrix.
$\mathbf{P}_{\mathrm{zz}}$	The output's error covariance matrix.
$\mathbf{P}$	The error covariance matrix.
q	The number of the quadrature points.
$\mathbf{Q}$	The process noise covariance matrix.
$\mathbf{R}$	The measurement noise covariance matrix.
T_s	Sampling time, and is equal to 0.001 sec.
$\mathbf{v}, \mathbf{w}$	The measurement and system noise, respectively.
W_i	The assigned weight for the i^{th} quadrature point.
$\mathbf{x}, \mathbf{z}$	The state and output vectors, respectively.
$\mathbf{X}_i$ and $\mathbf{Z}_i$	The estimate and its measurement for the i^{th} quadrature point, respectively.

II. The Quadrature Kalman Filters

The Quadrature Kalman Filters (QKFs) are predictor-corrector sub-optimal filters that linearize the nonlinear functions in the system using weighted statistical linear regression. They approximate the integration using a weighted summation of the function around specific points which are commonly referred to as the quadrature points [26, 27, 28, 29, 38, 39, 40, 41, 42, 43].

$$\int_{-\infty}^{\infty} f(x)\mathbb{N}(x|\bar{x}, P)dx = \sum_{i=1}^{n} W_i f(X_i) \tag{1}$$

The quadrature points and the weights used in this work are based on [38, 44], where the points are defined as:

$$\hat{\mathbf{X}}_i = \hat{\mathbf{x}} + \left(\Gamma_i\sqrt{\mathbf{P}}\right)_i^T \tag{2}$$

Γ_i is described by:

$$\Gamma_i = \sqrt{2}\psi_i \tag{3}$$

Where ψ_i is the ith eigenvalue of a symmetric tridiagonal matrix having zero diagonal elements while the entries located to the right of the diagonal are proportional to the row number (each having a value of $\frac{1}{\sqrt{2}}$ of the corresponding row value). The ith weighted value is defined as the squared value of the first element of the normalized eigenvector that corresponds to the ith eigenvalue.

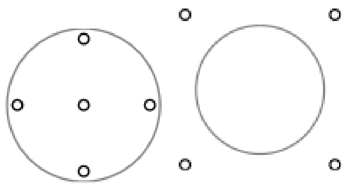
The resultant filter has the same structure of the Sigma-Point Kalman Filter (SPKF), although they have been derived from different respective sources [1, 11, 21, 45]. These filters differ in the number of points (q), their values (related to Γ_i) and their associated weights (W_i). A summary of the filters is shown in table 1. There is however some overlap between the presented methods as outlined by [46], which describes the CKF (a 2-point simplex QKF) as a special case of the UKF which falls within the SPKF category.

The QKFs and the SPKFs start by drawing out the points from the state's probability distribution function as shown in figure 1, and then projects them through the system's model to obtain the a priori estimates for the probability distributions. The a priori estimates are then fused together statistically as shown in figure 2 [8, 19, 22, 23, 47, 48, 49, 50, 51, 52], to create an improved a priori estimate vector. The same steps are used in the update stage, after replacing the system's model by the measurement's model in the propagation level, to obtain the a posteriori estimate vector and its covariance matrix as shown in figure 3.

Table 1: Differences between several SPKFs, and QKFs, [10, 11, 38]

	q	Γ_i	W_i
General UKF	$2n+1$	$\begin{cases} \sqrt{n+\lambda} & 1 \le i \le n \\ -\sqrt{n+\lambda} & n+1 \le i \le 2n \\ 0 & i = 0 \end{cases}$	$\begin{cases} \frac{\lambda}{n+\lambda} & i = 0 \\ \frac{1}{2(n+\lambda)} & i \neq 0 \end{cases}$

CKF	$2n$	$\begin{cases} \sqrt{n} & 1 \le i \le n \\ -\sqrt{n} & n+1 \le i \le 2n \end{cases}$	$\frac{1}{2n}$
Simplex UKF	$n+2$	which is obtained recursively as: $\Gamma_0{}^1 = 0$ and $\Gamma_1{}^1 = \Gamma_2{}^1 = \frac{-1}{\sqrt{2W_1}}$, (the superscript is the recursive index) for $l = 2, \cdots, n$ $\Gamma_i{}^l = \begin{cases} \begin{bmatrix} \Gamma_0{}^{l-1} \\ 0 \end{bmatrix} & i = 0 \\ \begin{bmatrix} \Gamma_i{}^{l-1} \\ \frac{-1}{\sqrt{2W_{l+1}}} \end{bmatrix} & 1 \le i \le l \\ \begin{bmatrix} \mathbf{0}_{l-1\times 1} \\ \frac{l}{\sqrt{2W_{l+1}}} \end{bmatrix} & i = l+1 \end{cases}$	$\begin{cases} 2^{-n}(1-W_0) & 1 \le i \le 2 \\ 2^{i-n}(W_1) & i > 2 \end{cases}$ W_0 is chosen as $W_0 \in [0,1)$
Spherical UKF	$n+2$	which is obtained recursively as: $\Gamma_0{}^1 = 0$ and $\Gamma_1{}^1 = \Gamma_2{}^1 = \frac{-1}{\sqrt{2W_1}}$, (the superscript is the recursive index) for $l = 2, \cdots, n$ $\begin{cases} \begin{bmatrix} \Gamma_0{}^{l-1} \\ 0 \end{bmatrix} & i = 0 \\ \begin{bmatrix} \Gamma_i{}^{l-1} \\ \frac{-1}{\sqrt{l(l+1)W_1}} \end{bmatrix} & 1 \le i \le l \\ \begin{bmatrix} \mathbf{0}_{l-1\times 1} \\ \frac{l}{\sqrt{l(l+1)W_1}} \end{bmatrix} & i = l+1 \end{cases}$	$\frac{1-W_0}{n+1}$ W_0 is chosen as $W_0 \in [0,1)$
QKF		$\Gamma_i = \sqrt{2}\psi_i$ And ψ_i is the ith eigenvalue of a symmetric tridiagonal matrix with zero diagonal elements and the elements that located to the right of the diagonal are proportional to the row number (each has a value of $\frac{1}{\sqrt{2}}$ of the corresponding row value).	$W_i = (\phi_{i,1})^2$, where ϕ is the first element of the normalized eigenvector of ψ_i.



(a) (b)

Fig 1: (a) Sigma Points, (b) 2-point Quadrature Points of a system with n=2 [44].

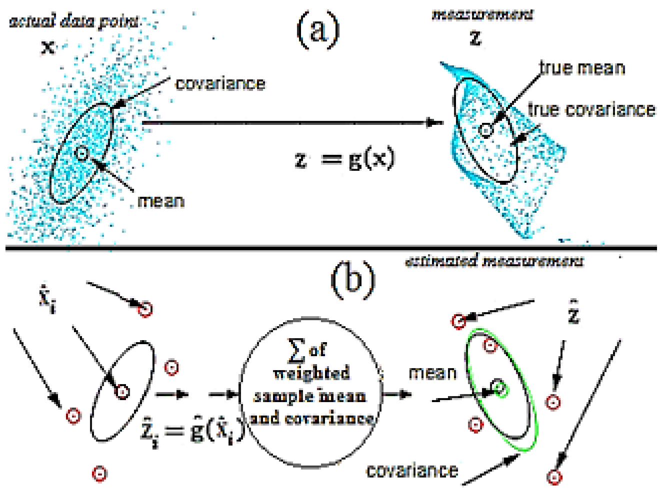


Fig 2 (a) The actual system states and their nonlinear measurement (b) The QKF/SPKF's estimates, [19].

Prediction Stage $k = k+1$

For $i = 1, \ldots, q$

$$\hat{\mathbf{X}}_{i_{k|k-1}} = \hat{\mathbf{f}}\left(\hat{\mathbf{x}}_{k-1|k-1} + \left(\Gamma_i\sqrt{\mathbf{P}_{k-1|k-1}}\right)_i^T, u_{k-1}\right),$$

End

$$\hat{\mathbf{x}}_{k|k-1} = \sum_{i=1}^{q} W_i \hat{\mathbf{X}}_{i_{k|k-1}}$$

$$P_{k|k-1} = \sum_{i=0}^{q} W_i \left(\hat{\mathbf{X}}_{i_{k|k-1}} - \hat{\mathbf{x}}_{k|k-1}\right)^2 + \mathbf{Q}_{k-1}$$

Update Stage

For $i = 1, \ldots, q$

$$\hat{\mathbf{X}}_{i_{k|k-1}} = \hat{\mathbf{x}}_{k|k-1} + \left(\Gamma_i\sqrt{\mathbf{P}_{k|k-1}}\right)_i^T$$

$$\hat{\mathbf{Z}}_{i_{k|k-1}} = \hat{\mathbf{g}}\left(\hat{\mathbf{X}}_{i_{k|k-1}}\right)$$

End

$$\hat{\mathbf{z}}_{k|k-1} = \sum_{i=0}^{q} W_i \hat{\mathbf{Z}}_{i_{k|k-1}}$$

$$\textit{Find } \mathbf{P}_{zz} = \sum_{i=0}^{q} W_i \left(\hat{\mathbf{Z}}_{i_{k|k-1}} - \hat{\mathbf{z}}_{k|k-1}\right)^2 + \mathbf{R}_k$$

$$\mathbf{P}_{xz} = \sum_{i=0}^{q} \left[W_i \left(\hat{\mathbf{X}}_{i_{k|k-1}} - \hat{\mathbf{x}}_{k|k-1}\right)\left(\hat{\mathbf{Z}}_{i_{k|k-1}}^T - \hat{\mathbf{z}}_{k|k-1}^T\right)^T\right]$$

$$\mathbf{K}_k = \mathbf{P}_{xz}\mathbf{P}_{zz}^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k\left(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}\right)$$

$$\boldsymbol{P}_{k|k} = \left(\mathbf{P}_{k|k-1} - \mathbf{K}_k\, \mathbf{P}_{zz}\mathbf{K}_k{}^T\right)$$

Fig 3: The Sigma/Quadrature-Point Kalman Filter, [11]

In this work, comparisons between different QKFs were established. The performance was measured by the Root Mean Square Error (RMSE), simulation time, stability, robustness, and the complexity of the estimation method. For completeness of work, the results of the QKFs were also compared to the simulated UKF and CKF results.

III. RESULTS & DISCUSSION

A. *Robotic Arm Dynamics Model*

The algorithms described in section II were tested on a system that was obtained in [1, 2, 3] using Simulink-MATLAB. The system is an industrial robotic arm that consists of a prismatic joint and three revolute joints as shown in figure 7 and 8. The dynamics model can be found in [1, 2, 3] where the states are $\left[d\ \dot{d}\ \theta_1\ \dot{\theta}_1\ \theta_2\ \dot{\theta}_2\ \theta_3\ \dot{\theta}_3\right]$, and the values of the variables are listed in table 2.

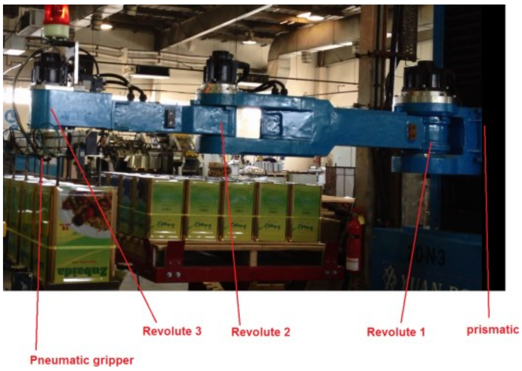


Fig.4: Industrial 4-DOF Robotic Arm [1, 2, 3]

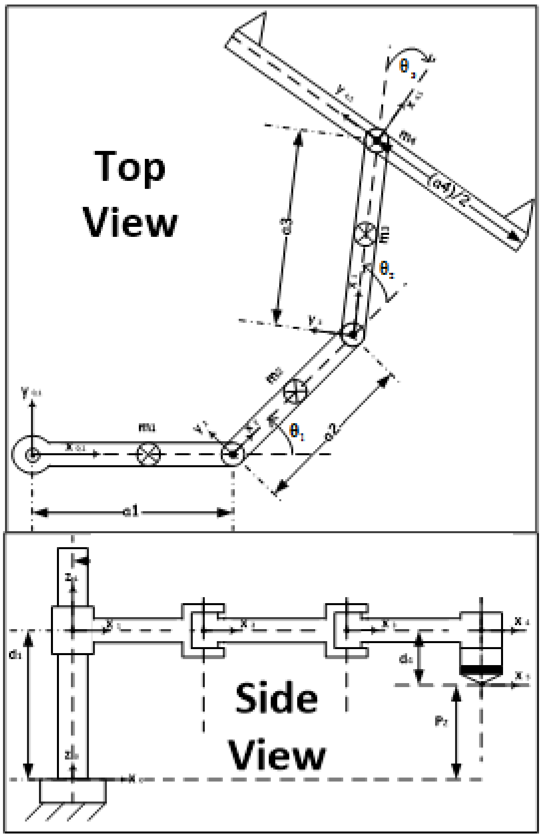


Fig.5: Top and side views of the Robotic Arm [1]

B. *Results*

The system was simulated given the three following scenarios: 1) no uncertainty has been injected to the filter model and noise levels are significantly small (Noise-to-Signal-Ratio $(NSR) \leq 5\%$), 2) no uncertainty has been injected to the filter model and noise levels are significantly large ($NSR \geq 10\%$), and 3) huge uncertainty (equal to 50% of system model) has been injected and noise levels are medium ($4\% \leq NSR \leq 9\%$). The root mean square errors (RMSE) for these experiments are presented in tables 3, 4 and 5, respectively.

Table 2: The values of robotic arm's parameters [32]

m_1	21.5 kg	a_1	0.25 m	I_1	1.04 kg.m^2
m_2	16 kg	a_2	1.2 m	I_2	13 kg.m^2
m_3	8.5 kg	a_3	0.8 m	I_3	3.12 kg.m^2
m_4	7.9 kg	a_4	1.2 m	I_4	1 kg.m^2
m_5	6.3 kg	a_5	0 m	I_5	0.84 kg.m^2

The results of the scenario 2 simulations are shown in figures 6-9 while the results of the scenario 3 simulations are provided in figures 10-13.

Table 3: The RMSE for scenario 1.

$\times 10^{-6}$	UKF	CKF	SeQKF	ThQKF	FoQKF
$d\ (m)$	6	6	6	6	6
$\dot{d}\ (m/s)$	4000	4000	4000	4000	4000
$\theta_1\ (rad)$	2	2	2	2	2
$\dot{\theta}_1\ (rad/s)$	500	500	500	700	1000
$\theta_2\ (rad)$	2	2	2	2	2
$\dot{\theta}_2\ (rad/s)$	700	700	700	300	300
$\theta_3\ (rad)$	2	2	2	2	2
$\dot{\theta}_3\ (rad/s)$	400	400	400	500	500

Table 4: The RMSE for scenario 2.

$\times 10^{-4}$	UKF	CKF	SeQKF	ThQKF	FoQKF
$d\ (m)$	7	7	7	7	7
$\dot{d}\ (m/s)$	4000	4000	4000	4000	4000
$\theta_1\ (rad)$	2	2	2	2	2
$\dot{\theta}_1\ (rad/s)$	70	70	70	70	70
$\theta_2\ (rad)$	2	2	2	2	2
$\dot{\theta}_2\ (rad/s)$	60	60	60	60	60
$\theta_3\ (rad)$	2	2	2	2	2
$\dot{\theta}_3\ (rad/s)$	50	50	50	50	50

Table 5: The RMSE for scenario 3.

$\times 10^{-4}$	UKF	CKF	SeQKF	ThQKF	FoQKF
$d\ (m)$	2	2	2	2	2
$\dot{d}\ (m/s)$	8000	8000	8000	8000	8000
$\theta_1\ (rad)$	0.2	0.2	0.2	0.3	0.3
$\dot{\theta}_1\ (rad/s)$	50	50	40	70	50
$\theta_2\ (rad)$	0.2	0.2	0.3	0.2	0.2
$\dot{\theta}_2\ (rad/s)$	100	100	80	90	90
$\theta_3\ (rad)$	0.2	0.2	0.2	0.2	0.2
$\dot{\theta}_3\ (rad/s)$	100	100	90	100	100

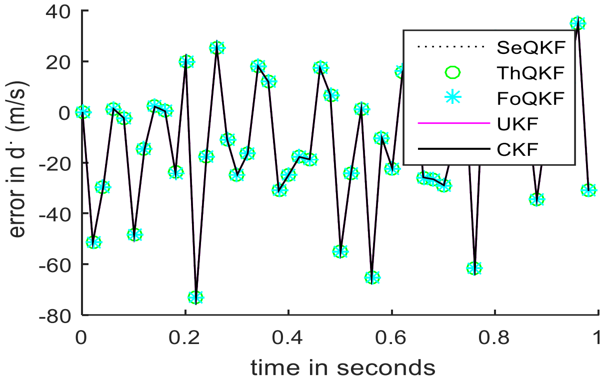


Fig 6: The error in estimating $\dot{d}$ for scenario 2.

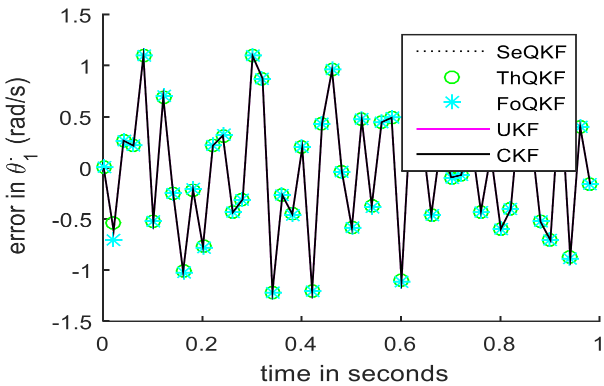


Fig 7: The error in estimating $\dot{\theta}_1$ for scenario 2.

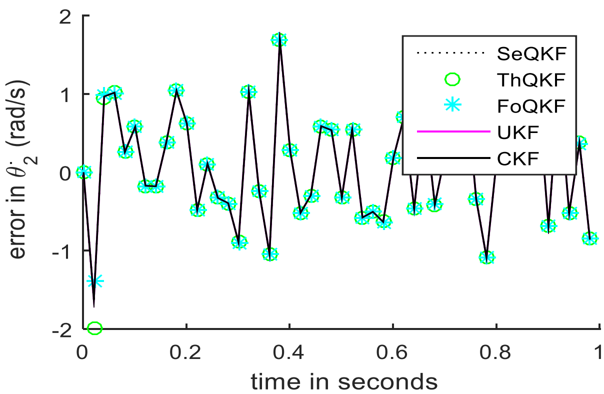


Fig 8: The error in estimating $\dot{\theta}_2$ for scenario 2.

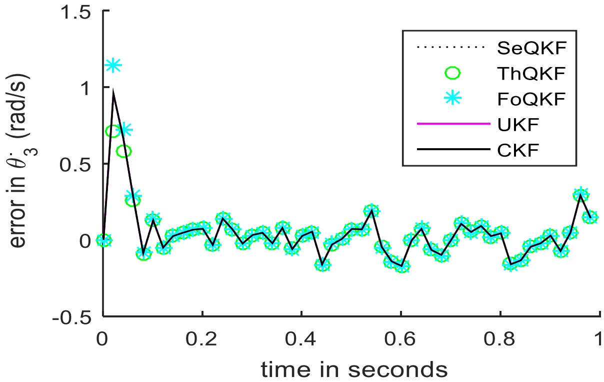


Fig 9: The error in estimating $\dot{\theta}_3$ for scenario 2.

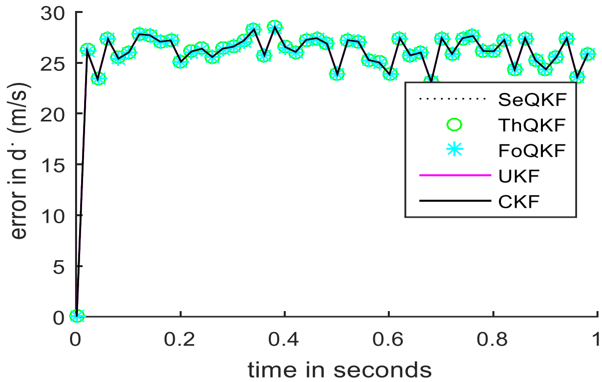


Fig 10: The error in estimating $\dot{d}$ for scenario 3.

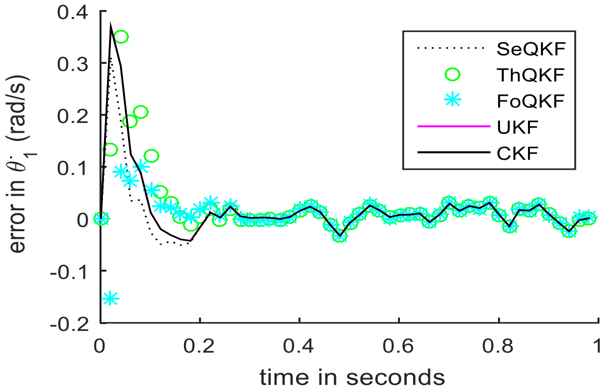


Fig 11: The error in estimating $\dot{\theta}_1$ for scenario 3.

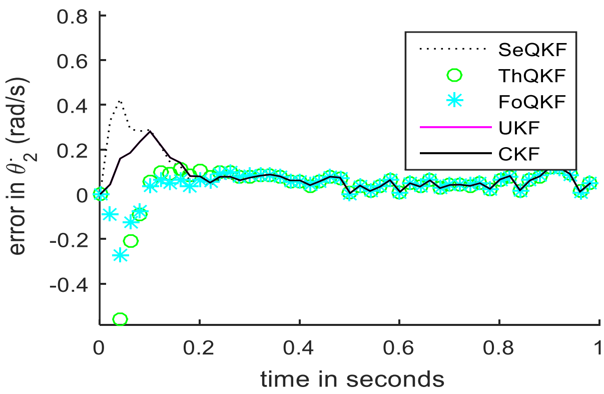


Fig 12: The error in estimating $\dot{\theta}_2$ for scenario 3.

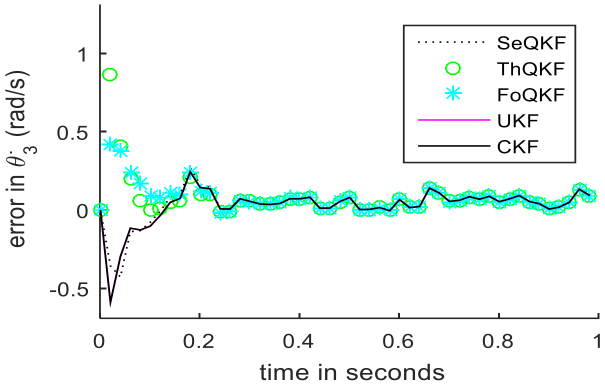


Fig 13: The error in estimating $\dot{\theta}_3$ for scenario 3.

C. Discussion

The results of tables 3, 4 and 5 and figures 6-12 show that the UKF and CKF are identical in performance. Both have the same stability, RMSE, computation time and complexity. The 2-point QKF (SeQKF) shows a performance that is similar to UKF and CKF. It has almost the same RMSE, however, it needs more computation time as UKF/CKF depends on $2n + 1$ points and SeQKF needs 2^n points. The system consists of eight states; $d, \dot{d}, \theta_1, \dot{\theta}_1, \theta_1, \dot{\theta}_2, \theta_2$ and $\dot{\theta}_3$, and therefore the UKF/CKF required 17 points while the SeQKF required 256 points. Using modern computers (with high performance I7 processer and 8 GB RAM) these numbers will not have a noticeable effect on the performance given a sampling time of 1 ms. The time required to evaluate the code is still acceptable, however, if a processor with low specification is used, then the use of SeQKF will be questionable. The SeQKF shows more robustness to uncertainty than the UKF and CKF. After increasing the modeling uncertainties, the UKF and CKF are the first to become unstable, while the SeQKF is the last.

The 3-point QKF (ThQKF) and the 4-point QKF (FoQKF) have a performance that is similar to UKF and CKF, and need more computation time compared to the previous filters. The ThQKF needs to compute the estimates using 6561 points while the FoQKF required 65536 points. This suggests that the ThQKF and FoQKF are more complicated and require more computation time. In actuality, the ThQKF and FoQKF needed 5-15 seconds to compute each (1 ms) time step. Due to the high computation time, the ThQKF and FoQKF are not suitable for online applications. Given increased uncertainties, the ThQKF and FoQKF outperform the other filters; however, they can suddenly become unstable and reach a point of instability sooner than the SeQKF. Among the QKFs used here, the FoQKF is first to become unstable, then the ThQKF, then the SeQKF. Due to the large number of points, the covariance matrix quickly approaches zero. This explains why the QKFs having the highest number of points are the first to become unstable. Conversely, the UKF and CKF tend towards instability as the covariance matrix approaches infinity.

By considering all factors, the SeQKF has the best performance among the tested filters. It is the most stable, and has a high degree of robustness with respect to uncertainties and noise. The UKF and CKF are the fastest algorithms, however, they are the first to become unstable. When applying estimation and filtering to robotic applications, it is undesirable to implement higher QKFs due to the long computation times and numerical instabilities associated with 3 and 4-point QKF. The 2-point QKF can capture the true mean and variance with acceptable computation times while maintaining stability and robustness.

By increasing the noise levels, all filters experienced very similar reductions in performance. Overall, the UKF and CKF are more sensitive to noise compared to QKF, which offers greater resilience when using a larger number of quadrature points.

IV. Conclusion

In this work, three estimation techniques— namely 2-point, 3-point and 4-point QKFs—were simulated to estimate the states of an industrial robotic arm of type PRRR. The results of the 2, 3, and 4 point QKF simulations were then compared to each other and the UKF and CKF simulations to determine the benefits of using QKF or SPKF techniques. The results demonstrated that the QKF methods were more stable and robust when compared to the UKF and CKF. When the system and measurement noise levels were increased, the performance became similar among all of the filters. This indicates that adding more quadrature points allows the filter to be less sensitive to noise. The 2-point QKF (SeQKF) showed superior performance when modeling error of the estimated states and would therefore be the optimal choice when compared to the other filters.

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