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A REVIEW OF SMOOTH VARIABLE STRUCTURE FILTERS: RECENT ADVANCES IN THEORY AND APPLICATIONS

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Abstract—The smooth variable structure filter (SVSF) is a relatively new state and parameter estimation technique. Introduced in 2007, it is based on the sliding mode concept, and is formulated in a predictor-corrector fashion. The main advantages of the SVSF, over other estimation methods, are robustness to modeling errors and uncertainties, and its ability to detect system changes. Recent developments have looked at improving the SVSF from its original form. This review paper provides an overview of the SVSF, and summarizes the main advances in its theory.

I. INTRODUCTION

State and parameter estimation theory is an important branch of control theory and sciences. The ability to know the states of a system with confidence is critical for accurate control. A popular contributor was Andrei Kolmogorov, who helped formulate the mathematical basis of probability and random processes [1, 2]. His work, along with Norbert Wiener, founded the basics of estimation; including the theory of prediction, filtering, and smoothing [2]. The concept of prediction refers to estimation methods that use measurements or observations prior to the time that the state of the system is to be estimated, or $t_{obs} < t_{est}$ [3]. Filters use measurements up to and including the time of interest, or $t_{obs} \leq t_{est}$. Finally, smoothers make use of measurements beyond the desired time of interest, such that the estimate is refined further, or $t_{obs} > t_{est}$ [3].

As presented in [2], Wiener worked on developing an automatic controller for directing antiaircraft fire during the 1940s [1]. His work ultimately led to the derivation of an optimal estimator, based on the continuous-time framework [4]. Meanwhile, Kolmogorov independently derived an optimal linear predictor for discrete-time systems [3, 5]. The work that they performed would later be referred to as the Wiener-Kolmogorov filter, a predecessor to the popular Kalman filter (KF) [6]. The KF was introduced in the 1960's, and is–without a doubt–the most popular contribution to the estimation field [7, 8]. It quickly became the 'workhorse' of estimation, yielding solutions to many controls and engineering problems [3]. The KF yields a statistically optimal solution for linear estimation problems, and is formulated in a predictor-corrector fashion. The states (or parameters of interest) are first estimated using the system model

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and input, termed as a priori estimates. This means 'prior to' knowledge of the observations or measurements. A correction term is then used to adjust the a priori estimate based on the innovation term (also called residuals or measurement errors). This creates the updated or a posteriori state estimates, which means 'subsequent to' or based on the observations and measurements [2].

As presented in [2], the optimality of the KF comes at a price of stability and robustness. The KF assumes that the system model is known and linear, the system and measurement noises are white, and the states have initial conditions with known means and variances [3, 6]. However, the previous assumptions do not always hold in real applications, and often the KF will become suboptimal and unstable [2]. The smooth variable structure filter (SVSF) was developed in an effort to overcome the instability issues. The predecessors of the SVSF are presented in the next section, followed by the main SVSF summary and equations. Recent advances are then highlighted, and the review paper is concluded with future research directions.

II. PREDECESSORS TO THE SVSF

A. The Variable Structure Filter

The variable structure filter (VSF) was first presented in 2002, and is the predecessor to the smooth variable structure filter (SVSF) [9, 10]. Note that the discussion in this section is similar to that presented in [2]. It was a new model-based strategy that used concepts closely related to variable structure control. Variable structure control (VSC) theory can guarantee stability given some bounded parametric uncertainty [11, 12, 13]. The most popular form of VSC is that of sliding mode control (SMC), which utilizes a discontinuous switching plane along some desired trajectory [14, 15, 16, 17]. This plane is often referred to as the sliding surface, in which the objective is to keep the state values along this surface in order to minimize the trajectory errors. Ideally, if the state value is off or away from the surface, a switching gain would be used to push the state towards the sliding surface, creating a robust and stable control strategy [2]. Once on the surface, the states slide along the surface in what is called the sliding mode [17].

Sliding mode concepts have been used for state estimation, and are referred to as sliding mode observers (SMO) or estimators (SME) [18]. Although the VSF uses a discontinuous component to correct estimates like other sliding mode strategies, it differs in its formulation [9]. The VSF uses a predictor-corrector strategy similar to the KF. Given some knowledge of the system prior to time k, it calculates an a priori (or predicted) state estimate $\hat{x}_{k+1|k}$. This state estimate is then updated based on available measurements of the system, thus formulating an a posteriori state estimate $\hat{x}_{k+1|k+1}$. Consider the linear system and measurement equations of (2.1.1) and (2.1.2). The VSF estimation process is summarized as follows [9]. The a priori state estimate is first calculated using the previous time step's a posteriori state estimate and the estimated system model:

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{2.1.1}$$

A gain vector K_{k+1}^{VSF} is used to formulate an a posteriori state estimate, as follows:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}^{VSF}$$
(2.1.2)

Where the gain vector K_{k+1}^{VSF} is calculated as a function of the estimated system and measurement matrices \hat{A} and \hat{C} , a constant diagonal gain matrix Υ with elements $\Upsilon_{ii} \ge 1$, and an upper bound for both the system and measurement noises W_{Max} and V_{Max} [9]:

$$= \hat{A}^{-1}\hat{C}^{+}\left(\left|\left|\hat{C}\hat{A}\right|_{Abs}\left\{Y|\hat{C}\right|_{Abs}|e_{z,k+1|k}|_{Abs} + \left|\hat{A}^{-1}\hat{C}^{+}\tilde{\xi}_{Max}z_{k+1}\right|_{Abs} + \left|\hat{L}^{-1}\hat{C}^{+}\hat{\xi}_{Max}z_{k+1}\right|_{Abs} + \left|\hat{L}^{-1}\hat{C}^{+}\hat{C}\right|_{Abs}\left\{\tilde{\xi}_{Max} + l\right\} V_{Max} + \left|\hat{A}^{-1}\hat{C}^{+}\tilde{\delta}_{Max}u_{k}\right|_{Abs} + \left|\hat{A}^{-1}\hat{C}^{+}\hat{C}_{Max}\right|_{Abs} \right\} W_{Max} \right\} \right|_{Abs} \circ sign(e_{z,k+1|k}) \right)$$

$$(2.1.3)$$

Note that the modeling error is denoted by \sim , *I* refers to an identity matrix, and the subscript Max signifies an upper bound. Furthermore, consider that $\xi = CAC^+$, $\hat{\xi} = \hat{C}\hat{A}\hat{C}^+$, $\tilde{\xi} = \xi - \hat{\xi}$, $\delta = CB$, $\hat{\delta} = \hat{C}\hat{B}$, and $\tilde{\delta} = \delta - \hat{\delta}$. Finally, $\tilde{\xi}_{Max}$, $\tilde{\delta}_{Max}$, and \tilde{C}_{Max} are upper bounds on modeling uncertainties ξ , δ , and *C*, respectively. The a priori measurement error vector $e_{z,k+1|k}$ is defined as follows:

$$e_{z,k+1|k} = z_{k+1} - C\hat{x}_{k+1|k} \tag{2.1.4}$$

Furthermore, $sign(e_{z,k+1|k})$ represents a vector, with elements defined by:

$$sign(e_{z,k+1|k}) = \begin{bmatrix} sign(e_{z_1,k+1|k}) \\ \vdots \\ sign(e_{z_m,k+1|k}) \end{bmatrix}$$
(2.1.5)

Note that in general a *sign(e)* function is defined by:

$$sign(e) = \begin{cases} +1 & e > 0 \\ 0 & if & e = 0 \\ -1 & e < 0 \end{cases}$$
(2.1.6)

Furthermore, note that • refers to the Schur product, such that:

$$a \circ b = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{bmatrix}$$
(2.1.7)

Where a and b are column vectors with three elements each. Note that the VSF gain results in high frequency switching which limits the performance, as well as introduces chattering in the estimated states [9]. These results may be undesirable when smooth estimates are required. The chattering may be minimized and reduced by the introduction of a smoothing boundary layer ψ [9, 16]. It is important to note that outside this boundary layer, the sign function is maintained to ensure robustness and stability. Inside this boundary layer, the VSF gain is interpolated to obtain a smooth function [9]. Hence, consider the following change to the VSF gain:

$$sign(e_{z,k+1|k}) \rightarrow sat\left(\frac{e_{z_i,k+1|k}}{\psi_i}\right)$$
 (2.1.8)

Where the saturation function is defined by:

$$sat\left(\frac{e_{z_{i},k+1|k}}{\psi_{i}}\right) = \begin{cases} +1 & \frac{e_{z_{i},k+1|k}}{\psi_{i}} \ge 1\\ \frac{e_{z_{i},k+1|k}}{\psi_{i}} & if -1 < \frac{e_{z_{i},k+1|k}}{\psi_{i}} < 1 \quad (2.1.9)\\ -1 & \frac{e_{z_{i},k+1|k}}{\psi_{i}} \le -1 \end{cases}$$

As described in [9], for the purposes of stability, the VSF gain needs to be large enough to overcome the presence of uncertainties. There is a relationship between the magnitude of the VSF gain and the level of uncertainty. Furthermore, the smoothing boundary layer ψ width also needs to be sufficiently large, such that it encompasses the maximum VSF gain values present in the estimation process [9]. The width of this boundary layer also determines the average level of estimation accuracy. The larger the smoothing boundary layer width, the less accurate the estimate (i.e., more uncertainties present) [9]. This makes sense intuitively, since the presence of fewer uncertainties leads to a more accurate estimate. The boundary layer width is a function of the upper bounds associated with the uncertainties present in the estimation process (i.e., modeling errors, and the system and measurement noises) [9]:

$$= |\hat{C}^{+}\hat{A}^{-1}|_{Abs} |\hat{A}C|_{Abs} \left\{ |\hat{A}^{-1}\hat{C}^{+}\tilde{\xi}_{Max}Z_{Max}|_{Abs} + [|\hat{C}^{+}|_{Abs} + |\hat{A}^{-1}\hat{C}^{+}|_{Abs} (\hat{\xi} + \tilde{\xi}_{Max} + I)] V_{Max} + |\hat{A}^{-1}\hat{C}^{+}\hat{\delta}_{Max}u_{Max}|_{Abs} + [|\hat{A}^{-1}|_{Abs} + |\hat{A}^{-1}\hat{C}^{+}\tilde{C}_{Max}|_{Abs}] W_{Max} \right\}$$

$$(2.1.10)$$

· I.

The VSF offers a number of advantages. If the upper bounds of the system uncertainties and noise levels are well-defined, the VSF gain may be easily calculated as per (3.1.3) [19]. Furthermore, the VSF gain provides a robust estimation strategy, and has demonstrated stability to modeling uncertainties [9, 10]. However, the VSF strategy does have a few disadvantages. The strategy may only be applied to linear systems, and yields nonoptimal estimation results. Furthermore, the estimate may experience chattering, which may be undesirable, depending on the application [9]. Another important disadvantage is its rather large, and complicated, gain calculation.

B. The Extended VSF

In 2006, a modified form of the VSF was introduced, referred to as the extended variable structure filter (EVSF) [20]. As presented in [2], the EVSF method may be applied to nonlinear systems and measurements defined by (2.2.1) and (2.2.2) respectively.

$$x_{k+1} = f(x_k, u_k) + w_k \tag{2.2.1}$$

$$z_{k+1} = h(x_{k+1}) + v_{k+1} \tag{2.2.2}$$

The EVSF is formulated in a predictor-corrector fashion, and is conceptually similar to the VSF [20]. The state estimate is first predicted by using the estimated nonlinear system model, as follows:

$$\hat{x}_{k+1|k} = \hat{f}(\hat{x}_{k|k}, u_k)$$
(2.2.3)

The estimate $\hat{x}_{k+1|k}$ is obtained by using the previous state estimate $\hat{x}_{k|k}$, or the initial conditions x_0 at the start of the estimation process. The a priori state estimates are then used to calculate the a priori measurement estimates $\hat{z}_{k+1|k}$, as follows:

$$\hat{z}_{k+1|k} = \hat{h}(\hat{x}_{k+1|k})$$
 (2.2.4)

An EVSF corrective gain K_{k+1}^{EVSF} is then calculated, and used to refine the a posteriori state estimate as follows [20]:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}^{EVSF}$$
(2.2.5)

The EVSF strategy is similar to the extended Kalman filter (EKF), in the sense that it makes use of the linearized system and measurement functions, as follows:

$$\hat{F}_{k} = \frac{\partial \hat{f}}{\partial x} \bigg|_{\hat{x}_{k|k}, u_{k}}$$
(2.2.6)

$$\widehat{H}_{k+1} = \frac{\partial \widehat{h}}{\partial x} \bigg|_{\widehat{x}_{k+1|k}}$$
(2.2.7)

The linearization is performed in order to derive the EVSF corrective gain used in (2.2.8). The EVSF corrective gain is defined as follows [20]:

$$F_{k}^{EVSF} = \hat{F}_{k}^{+} \hat{H}_{k+1}^{+} \left(\left| \hat{H}_{k+1,Max} \right|_{Abs} \left| \hat{F}_{k,Max} \right|_{Abs} \left\{ Y \left| \hat{H}_{k+1,Max}^{+} \right|_{Abs} \right| e_{z,k+1|k} \right|_{Abs} + \left| \hat{F}_{k,Max}^{-1} \right|_{Abs} \left[\left| f(x_{k}, u_{k}) - \hat{f}(\hat{x}_{k|k}, u_{k}) \right|_{Abs} + W_{Max} + \left| \hat{H}_{k+1,Max}^{+} \right|_{Abs} V_{Max} \right] \circ sign(e_{z,k+1|k}) \right)$$

$$(2.2.8)$$

Note that it is assumed the system under consideration is observable, and has a system output matrix that is positive and constant in its linearized form such that $H_{k+1} = H_k$ [20]. The corrective gain (2.2.8) is conceptually similar to that presented for the VSF (2.1.3), however can be applied for nonlinear systems. The EVSF and VSF strategies have similar advantages and disadvantages [19]. However, one notable disadvantage for the EVSF method is that it requires linearization at each time step in order to calculate the EVSF gain. This results in an increase in numerical effort, as well as reduces the overall estimation accuracy due to the truncation of higher-order terms (i.e., Taylor series approximation) [20].

III. THE SMOOTH VARIABLE STRUCTURE FILTER

A revised form of the VSF, referred to as the smooth variable structure filter (SVSF), was presented in 2007 [21]. The SVSF strategy is also a predictor-corrector estimator based on sliding mode concepts, and can be applied on both linear or nonlinear systems and measurements. As shown in the following figure, and similar to the VSF, it utilizes a switching gain to converge the estimates to within a boundary of the true state values (i.e., existence subspace) [21]. The SVSF has been shown to be stable and robust to modeling uncertainties and noise, when given an upper bound on the level of un-modeled dynamics and noise [21, 9]. The origin of the SVSF name comes from the requirement that the system is differentiable (or 'smooth') [21, 19]. Furthermore, it is assumed that the system under consideration is observable [21, 22].

Consider the following process for the SVSF estimation strategy, as applied to a nonlinear system with a linear measurement equation. The predicted state estimates $\hat{x}_{k+1|k}$ are first calculated as follows:

$$\hat{x}_{k+1|k} = \hat{f}(\hat{x}_{k|k}, u_k)$$
 (3.1)

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the corresponding predicted measurements $\hat{z}_{k+1|k}$ and measurement error vector $e_{z,k+1|k}$ may be calculated:

$$\hat{z}_{k+1|k} = C\hat{x}_{k+1|k} \tag{3.2}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{3.3}$$



Fig. 1. SVSF estimation concept [2].

Next, the SVSF gain is calculated as follows [21]:

$$K_{k+1}^{SVSF} = C^+ \left(\left| e_{z,k+1|k} \right|_{Abs} + \gamma \left| e_{z,k|k} \right|_{Abs} \right) \circ sat\left(\frac{e_{z,k+1|k}}{\psi} \right)$$
(3.4)

The SVSF gain is a function of: the a priori and a posteriori measurement error vectors $e_{z,k+1|k}$ and $e_{z,k|k}$; the smoothing boundary layer widths ψ ; the SVSF 'memory' or convergence rate γ with elements $0 < \gamma_{ii} \le 1$; and the linear measurement matrix *C*. The SVSF gain is used to refine the state estimates as follows:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}^{SVSF}$$
(3.5)

Next, the updated measurement estimates $\hat{z}_{k+1|k+1}$ and corresponding errors $e_{z,k+1|k+1}$ are calculated:

$$\hat{z}_{k+1|k+1} = C\hat{x}_{k+1|k+1} \tag{3.6}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{3.7}$$

The SVSF process may be summarized by (3.1) through (3.7), and is repeated iteratively. According to [21], the estimation process is stable and converges to the existence subspace if the following condition is satisfied:

$$|e_{k|k}|_{Abs} > |e_{k+1|k+1}|_{Abs}$$
 (3.8)

Note that $|e|_{Abs}$ is the absolute of the vector e, and is equal to $|e|_{Abs} = e \cdot sign(e)$. The proof, as described in [21] and [19], yields the derivation of the SVSF gain from (3.8). The stability proof provided in [19] is very clear, and the interested reader is recommended to review it.



Fig. 2. Smoothed estimated trajectory ($\psi \ge \beta$) [21].

The SVSF results in the state estimates converging to within a region of the state trajectory, referred to as the existence subspace. Thereafter, it switches back and forth across the state trajectory, as shown earlier in Fig. 1. The existence subspace shown in Figs. 1-3 represents the amount of uncertainties present in the estimation process, in terms of modeling errors or the presence of noise. The width of the existence space β is a function of the uncertain dynamics associated with the inaccuracy of the internal model of the filter as well as the measurement model, and varies with time [21]. Typically this value is not exactly known but an upper bound may be selected based on a priori knowledge.



Fig. 3. Presence of chattering effect($\psi < \beta$) [21].

Once within the existence boundary subspace, the estimated states are forced (by the SVSF gain) to switch back and forth along the true state trajectory. As mentioned earlier, high-frequency switching caused by the SVSF gain is referred to as chattering, and in most cases, is undesirable for obtaining accurate estimates [21].

However, the effects of chattering may be minimized by the introduction of a smoothing boundary layer ψ . The selection of the smoothing boundary layer width reflects the level of uncertainties in the filter and the disturbances (i.e., system and measurement noise, and unmodeled dynamics). The effect of the smoothing boundary layer is shown in Figs. 2 and 3. When the smoothing boundary layer is defined larger than the existence subspace boundary, the estimated state trajectory is smoothed. However, when the smoothing term is too small, chattering remains due to the uncertainties being underestimated. Similar to the VSF strategy, the smoothing boundary layer ψ modifies the SVSF gain as follows [21]:

$$K_{k+1}^{SVSF} = C^{+} \left(\left| e_{z,k+1|k} \right|_{Abs} + \gamma \left| e_{z,k|k} \right|_{Abs} \right) \circ sat(e_{z,k+1|k}/\psi)$$
(3.9)

The SVSF gain is considerably less complex than its predecessor (VSF), which allows it to be implemented more easily (mathematically and conceptually). Furthermore, the SVSF estimation process is inherently robust and stable to modeling uncertainties due to the switching effect of the gain. This makes for a powerful estimation strategy, particularly when the system is not well known. Note that for systems that have fewer measurements than states, a 'reduced order' approach is taken to formulate a full measurement matrix [21, 23]. Essentially 'artificial measurements' are created and used throughout the estimation process.

IV. ADVANCES TO THE SVSF

The most relevant advances made to the SVSF since 2007 are summarized in this section.

A. Covariance Formulation

In its current form, the SVSF does not have or make use of a state error covariance matrix [2, 21]. A state error covariance is defined as the expectation of the error squared. It may be used for a variety of reasons: to determine an optimal value of the gain (i.e., such as in the case of the KF); for the implementation of multiple model (MM) methods; or to create other forms such as the information filter formulation (i.e., using the inverse of the covariance) [3]. The covariance form of the SVSF presented here was first described in [2] and [24]. A number of complex solutions were first proposed in [2]; however, in an effort to simplify the covariance derivation, a new SVSF update equation and gain was proposed [25]:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z,k+1|k}$$
(4.1.1)

$$K_{k+1} = C^{+} diag \left[\left(\left| e_{z,k+1|k} \right|_{Abs} + \gamma \left| e_{z,k|k} \right|_{Abs} \right) \circ sat \left(\frac{e_{z,k+1|k}}{\psi} \right) \right] \left[diag(e_{z,k+1|k}) \right]^{-1}$$
(4.1.2)

Rewriting the update equation (4.1.1) and the SVSF gain (4.1.2) greatly simplifies the SVSF covariance solution without changing its proof of stability. However, for numerical stability, it is important to ensure that one does not divide by zero in (4.1.2). This can be accomplished using a simple if statement with a very small threshold (i.e., 1×10^{-12}). The revised SVSF estimation strategy for linear systems and measurements is proposed as follows [2]. There are two stages: prediction and update. The first step is to predict the state estimates (4.1.3), calculate the a priori state error covariance (4.1.4), and find the corresponding estimation error (4.1.5).

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{4.1.3}$$

$$P_{k+1|k} = \hat{A}P_{k|k}\hat{A}^T + Q_k \tag{4.1.4}$$

$$e_{z,k+1|k} = z_{k+1} - C\hat{x}_{k+1|k} \tag{4.1.5}$$

As described in [2], the next step involves calculating the corresponding SVSF gain (4.1.6), updating the state estimate (4.1.7), finding the a posteriori state error covariance (4.1.8), and determining the a posteriori measurement error (4.1.9) which is to be used in the next iteration (recursively).

$$K_{k+1} = C^{+} diag \left[\left(\left| e_{z,k+1|k} \right|_{Abs} + \gamma \left| e_{z,k|k} \right|_{Abs} \right) \circ sat \left(\frac{e_{z,k+1|k}}{\psi} \right) \right] \left[diag(e_{z,k+1|k}) \right]^{-1} \qquad (4.1.6)$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z,k+1|k}$$
(4.1.7)

$$P_{k+1|k+1} = (I - K_{k+1}C)P_{k+1|k}(I - K_{k+1}C)^T + K_{k+1}R_{k+1}K_{k+1}^T$$
(4.1.8)

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$$e_{z,k+1|k+1} = z_{k+1} - C\hat{x}_{k+1|k+1} \tag{4.1.9}$$

The revised SVSF estimation strategy for linear systems may be summarized by (4.1.3) through (4.1.9). Nonlinear forms were also first presented in [2], and were derived similar to the extended (EKF), unscented (UKF), and cubature Kalman filters (CKF). It is interesting to point out that the calculation of $P_{k+1|k+1}$, as above, has no effect on the SVSF gain K_{k+1} . However, the SVSF gain does affect the final value of $P_{k+1|k+1}$. The introduction of the SVSF form with a covariance derivation further advances the development of the filter.

B. Time-Varying Smoothing Boundary Layer

The smoothing boundary layer ψ plays an important role in defining the level of uncertainties and modeling errors present in the estimation process. Therefore, obtaining an optimal value (which varies with time) proved to be an interesting task. A time-varying smoothing boundary layer was first presented in [2, 25], and the results are similarly presented here.

The partial derivative of the a posteriori covariance (trace) with respect to the smoothing boundary layer term ψ is the basis for obtaining a strategy for the specification of ψ . The approach taken is similar to determining an optimal gain for the KF. The following derivation is applicable to any measurement case provided that the measurement matrix is completely observable. For the case when there are fewer measurements than states, one needs to implement a reduced order form of the SVSF as shown in [2] and [21]. This allows the creation of a full measurement matrix, typically in the form of an identity. For the case when there are more measurements than states, the system output can be multiplied by the inverse of the measurement matrix, thus mapping the measurements to the states. One could then use a full measurement matrix (i.e., identity) in the estimation process [2].

Previous forms of the SVSF included a vector form of ψ , which had a single smoothing boundary layer term for each corresponding measurement error [21]. Essentially, the boundary layer terms were independent of each other such that the measurement errors would only directly be used for calculating its corresponding gain. The coupling effects are not explicitly considered thus preventing an optimal derivation. A 'nearoptimal' formulation of the SVSF could be created using a vector form of ψ , however this would lead to a minimization of only the diagonal elements of the state error covariance matrix [26]. In [2] and [21], in an effort to obtain a smoothing boundary layer equation that yields optimal state estimates for linear systems (like the KF), a full smoothing boundary layer matrix is proposed. Hence, the full matrix form of the smoothing boundary layer was proposed:

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1m} \\ \psi_{12} & \psi_{22} & \cdots & \psi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \psi_{m2} & \cdots & \psi_{mm} \end{bmatrix}$$
(4.2.1)

Note that the off-diagonal terms of (4.2.1) are zero for the standard SVSF (presented in [21]), whereas this is not the case for the algorithm presented in [2, 25]. This definition includes terms that relate one smoothing boundary layer to another (i.e., off-diagonal terms). To solve for a time-varying smoothing boundary layer (VBL) based on (4.2.1), consider:

$$\frac{\partial (trace[P_{k+1|k+1}])}{\partial \psi} = 0 \qquad (4.2.2)$$

To solve (4.2.2), first consider the following modification of the SVSF gain defined earlier by (4.1.6). Note that the gain structure remains the same as follows:

$$K_{k+1} = C^{-1} \{ diag(A) \cdot sat(\psi^{-1} diag[e_{z,k+1|k}]) \} [diag(e_{z,k+1|k})]^{-1} \qquad (4.2.3)$$

Where *A* is a 'vector of errors', defined as follows:

$$A = \left(\left| e_{z,k+1|k} \right|_{Abs} + \gamma \left| e_{z,k|k} \right|_{Abs} \right)$$
(4.2.4)

Solving (4.2.2) based on (4.1.8) and (4.2.3) yields the following equation for the time-varying smoothing boundary layer [2, 25]:

$$\psi_{k+1} = \left(\bar{A}^{-1}CP_{k+1|k}C^{T}S_{k+1}^{-1}\right)^{-1}$$
(4.2.5)

As described in [2, 25], the proposed smoothing boundary layer equation (4.2.5) is found to be a function of the a priori state error covariance $P_{k+1|k}$, measurement covariance S_{k+1} , measurement matrix C, a priori and previous a posteriori measurement error vectors $(e_{z,k+1|k} \text{ and } e_{z,k|k})$, and the convergence rate or SVSF 'memory' γ . It appears that the width of the smoothing boundary layer is therefore directly related to the level of modeling uncertainties (by virtue of the errors), as well as the estimated system and measurement noise (captured by $P_{k+1|k}$ and S_{k+1}).

C. Chattering Information

An important phenomenon caused by the SVSF gain (as shown in Fig. 3) is referred to as chattering. Chattering, in terms of sliding mode control (SMC), is typically defined as highfrequency switching about a sliding mode or trajectory of interest. It was discovered that the magnitude of this chattering contains relevant and useful information on the system [19]. For example, as described in [19], the chattering can be used to indicate the source and amplitude of modeling errors. This provided an opportunity to combine the SVSF using the chattering method with other filtering strategies [19].

D. Combinations of the SVSF with Filters

As per the results shown earlier and in [2, 25], it appears that the VBL for the SVSF yields the KF solution (gain) for linear systems. In this case, robustness to modeling uncertainties using the SVSF strategy with a VBL is lost. It is hence beneficial to propose a combined strategy where an accurate estimate is maintained (e.g., KF-based gain) while ensuring the estimate remains stable (i.e., SVSF gain) [2, 25].

The basic strategy for combining filters is as follows, and is presented as shown in [2, 25]. This strategy is implemented by imposing a saturation limit on the time-varying smoothing boundary layer as follows. Outside the limit the robustness and stability of the SVSF is maintained, while inside the boundary layer the optimal gain is applied. Consider the following sets of figures to help describe the overall implementation of the combined SVSF strategy.



Fig. 4. Well-defined system case for combining filters [2, 25].

As described in [2, 25], Fig. 4 illustrates the case when a limit is imposed on the smoothing boundary layer width (a conservative value) and the VBL (4.2.5) follows within this limit. In the standard SVSF, the smoothing boundary layer width is made equal to the limit; such that the difference between the limit and the VBL quantifies the loss in optimality [21]. Essentially, in this case, a KF-based gain (KF, EKF, UKF, or CKF) should be used to obtain the best estimation result [25, 27, 28, 29, 30]. Another way to simplify and understand this process is to consider the SVSF-VBL as using a time-varying boundary layer with saturated limits to ensure stability.



Fig. 5. Poorly-defined system case [2, 25].

As described in [2, 25], Fig. 5 illustrates the case when the VBL is larger than the limit imposed on the smoothing boundary layer. This typically occurs when there is modeling uncertainty (which leads to a loss in optimality) or when the limit on the smoothing boundary layer is underestimated. This strategy is useful for applications such as fault detection. The width of the smoothing boundary layer is directly related to the level of modeling uncertainties, as well as the estimated system and measurement noise (captured by $P_{k+1|k}$ and S_{k+1}) [2, 25]. Therefore, the VBL creates another indicator of performance for the SVSF: the widths may be used to determine the presence of modeling uncertainties, as well as detect any changes in the system [2, 25].

Essentially, in a well-defined case, the gain used to correct the estimate may be calculated by the KF (linear) or EKF, UKF, CKF (nonlinear). When the VBL goes beyond the limits, the smoothing boundary layer width requires saturation and the SVSF gain may be employed. This provides a relatively easy mechanism for combining the SVSF with other KF-based filters [2, 25].

E. Multiple-Model Formulation

Most systems actually behave according to a number of different models (modes, or operating regimes). As described in

[2], it is desirable to implement adaptive estimation algorithms, which 'adapt' themselves to certain types of uncertainties or models in an effort to minimize the state estimation error [3]. One type of adaptive estimation technique includes the 'multiple model' (MM) algorithm [31]; which include the following: static MM [32], dynamic MM [3], generalized pseudo-Bayesian (GPB) [33, 34, 35, 36], and the interacting multiple model (IMM) [3, 37, 38]. For the MM methods, a Bayesian framework is used (i.e., probability based). Essentially, based on some prior probabilities of each model being correct (i.e., the system is behaving according a finite number of modes), the corresponding updated probabilities are calculated [3].

The IMM method is one of the most popular MM strategies, as it is able to make use of more information and is relatively computationally efficient [2]. The IMM typically makes use of KF estimators that run in parallel. However, recent studies looked at combining the SVSF with the IMM [39, 40, 41]. This was made possible due to the covariance derivation of the SVSF. Since the SVSF is suboptimal albeit stable, it is intuitive to utilize the IMM strategy which increases the overall estimation accuracy [2]. An overview of the IMM-SVSF strategy may be found in the following figure.



Fig. 6. IMM-SVSF estimation concept [2].

As described in [2], the SVSF estimation strategy may be applied on a finite number of models. As an example, the above figure shows two models. The IMM-SVSF estimator consists of five main steps: calculation of the mixing probabilities, mixing stage, mode-matched filtering via the SVSF, mode probability update, and state estimate and covariance combination.

The first step involves calculating the mixing probabilities $\mu_{i|j,k|k}$ (i.e., the probability of the system currently in mode *i*, and switching to mode *j* at the next step). In addition to the mixing probabilities, the previous mode-matched states $\hat{x}_{i,k|k}$ and covariance's $P_{i,k|k}$ are also used to calculate the mixed initial conditions (states and covariance) for the filter matched to M_j (which consists of A_j and B_j). A number of SVSF filters are then run in parallel, and the innovation error and covariance are used to calculation a corresponding mode-matched likelihood function $\Lambda_{j,k+1}$. Utilizing the mode-matched likelihood functions, the mode probability $\mu_{i,k}$ may be updated [3].

The IMM-SVSF has been applied on target tracking examples, as well as fault detection and diagnosis problems. The results demonstrated an improved tracking performance when compared with the popular IMM-KF and IMM-EKF forms. Furthermore, in terms of fault detection, the IMM-SVSF strategy generally outperformed the IMM-KF in terms of estimation accuracy and mode probability determination. The 'false detection' probability was found to be lower for the IMM-SVSF than the IMM-KF strategy (i.e., detecting a fault when the EHA system was operating normally) [2].

F. Fault Detection and Diagnosis Techniques

Techniques in the area of fault detection and diagnosis are typically considered to be model-based or signal-based. As previously stated, the IMM-SVSF has been developed and successfully applied for fault detection problems. In terms of signal-based approaches, the SVSF has been combined with artificial neural networks (ANNs) [42, 43, 44]. ANN strategies are mathematical models inspired by biological systems. Essentially, they are used to find and model complex relationships between an input and some desired output. A hidden layer (or multiple) lay between the input and the output. Weights are used to give importance to different connections. These weights may be approximated using KF-based methods, or more recently, the SVSF method [42]. The results indicate that the NN-SVSF method provides a more accurate representation of the hidden layers, and yields a very powerful fault detection strategy [42, 43, 44].

V. CONCLUSIONS

The smooth variable structure filter (SVSF) is a relatively new predictor-corrector estimator, based on the sliding mode concept. Since its introduction in 2007, a significant amount of research has been performed. This review paper provided a very brief overview of the SVSF, and highlighted its main advances in theory and applications. Future research is concentrated in the area of predictive fault diagnosis and information extraction from the unique SVSF chattering signal.

NOTES FROM THE AUTHORS

The purpose of this paper was to provide an overview of the main advances in the SVSF estimator, and provide a source for readers to cross-reference. The readers are recommended to review the references for more detailed information on the SVSF and its latest developments.

APPENDIX

The following is a list of the main nomenclature used throughout the paper.

TABLE I. LIST OF IMPORTANT NOMENCLATURE AND PARAMETERS

Parameter	Definition
f	Nonlinear system function
h	Nonlinear measurement function
x	State vector or values
Z	Measurement (system output) vector or values
w	System noise vector
v	Measurement noise vector
Α	Linear system transition matrix
В	Linear input gain matrix
С	Linear measurement (output) matrix
Ε	SVSF error vector (or matrix)
K	Filter gain matrix
Р	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix

S	Innovation (measurement error) covariance matrix
ez	Measurement (output) error vector
γ	SVSF 'memory' or convergence rate
ψ	SVSF smoothing boundary layer
diag[a] or ā	Diagonal of some vector or matrix a
sat()	Saturation function
<i>a</i>	Absolute value of a
Т	Transpose of a vector (if superscript) or sample rate
+	Pseudoinverse of some non-square matrix
0	Denotes a Schur product (element-by-element
	multiplication)
~	Denotes error or difference
٨	Estimated vector or values

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