Square-Root Formulation of the SVSF with Applications to Nonlinear Target Tracking Problems

S. A. Gadsden*[a], M. Al-Shabi[b], and T. Kirubarajan[c]
[a] University of Maryland, Baltimore County (UMBC), Baltimore, MD, USA 21250;
[b] Philadelphia University, Amman, Jordan;
[c] McMaster University, Hamilton, ON, Canada L8S 4L7

ABSTRACT

The smooth variable structure filter (SVSF) is a state and parameter estimation strategy based on sliding mode concepts. It has seen significant development and research activity in recent years. In an effort to improve upon the numerical stability of the SVSF, a square-root formulation is derived. The square-root SVSF is based on Potter's algorithm. The proposed formulation is computationally more efficient and reduces the risks of failure due to numerical instability. The new strategy is applied on target tracking scenarios for the purposes of state estimation. The results are compared with the popular Kalman filter.

Keywords: Square-root, Kalman filter, smooth variable structure filter, target tracking

1. INTRODUCTION

Estimation theory is considered to be a part of statistics and signal processing [1]. The purpose of estimation is to extract knowledge of the true states typically from noise measurements or observations made of the system, and form state estimates. The name 'filter' is appropriate as it removes unwanted noise from the signal. In most estimation strategies, the estimate is updated or refined based on some gain [2]. One of the most popular estimation strategies is the Kalman filter (KF) [3]. It is formulated in a predictor-corrector fashion, and is considered to yield an optimal solution for linear estimation problems. The KF gain is optimized based on minimizing the state error covariance matrix [4]. As demonstrated in literature, the state error covariance matrix must be symmetric and positive-definite in order to properly represent the statistics for state vector components [5]. In linear algebra, a symmetric matrix is defined as a square matrix which is equal to its corresponding transpose. Consider a symmetric matrix P, then the following is satisfied: $P = P^T$. A symmetric matrix P is considered positive-definite if the following is satisfied: $b^T P b > 0$, where b is a non-zero vector with real entries. Essentially, the above two definitions ensure that the off-diagonal elements of the state error covariance matrix are equal to each other (i.e., $p_{ij} = p_{ji}$), and that the elements along the diagonal are real and positive values (i.e., the square of each estimation error is guaranteed to be positive).

As described in [6], square-root (or factored-form) filters help to ensure numerical stability [7, 8, 9]. The square-root formulation makes use of three powerful linear algebra techniques: QR decomposition, Cholesky factor updating, and efficient least squares [10, 11]. The covariance matrix is broken up into factored terms, which are propagated forward and updated at each measurement [5]. The factors are multiplied together reforming the covariance matrix, thus ensuring it to be positive definite. The two most popular square-root filters are Potter's square-root filter and Bierman-Thornton's UD filter [12]. The UD filter has similar accuracy to Potter's strategy, however is less computationally expensive [13]. Introduced in the late 1970s, UD filtering is based on transformation methods that involve an upper triangle covariance factorization (2.3.2.3) [14, 15]. Although the UD strategy is considered a type of square-root filter, no square roots are actually calculated; where the covariance P is defined by:

$$P = UDU^T \tag{1.1}$$

*gadsden@umbc.edu; phone 1 410 455-3307; umbc.edu

where U is an upper triangle matrix with diagonal elements that are unity (all 1), and $D = diag(d_1, \dots, d_n)$. The matrices U and D are referred to as the UD factors of the covariance matrix P. A number of different strategies exist to perform UD decomposition (i.e., to create U and D matrices) [16]. Further to the UD strategy, numerical stability for filtering strategies can be improved by factoring the covariance matrix into Cholesky factors [17]. This was discovered when attempting to improve the stability of the KF when dealing with finite-precision arithmetic [16]. Essentially the nature of the KF remains the same; however, an equivalent statistical parameter is used and is found to be less sensitive to round-off errors [18]. Increasing the arithmetic precision reduces the effects of round-off error, which improves the overall stability of the filter.

The paper is organized as follows. The Kalman filter (KF) and smooth variable structure filter (SVSF) and their equations are summarized in section 2. The square-root formulations of the KF is introduced in section 3. The new square-root SVSF is then introduced and summarized. In section 4, the target tracking scenario is described. The results of implementing the square-root KF and square-root SVSF are shown and compared. The paper is then concluded and future work is described.

2. ESTIMATION STRATEGIES

The Kalman Filter

The following equations form the core of the Kalman filter (KF) algorithm, and are used in an iterative fashion. Equations (2.1.1) and (2.1.2) define the a priori state estimate $\hat{x}_{k+1|k}$ based on knowledge of the system F and previous state estimate $\hat{x}_{k|k}$, and the corresponding state error covariance matrix $P_{k+1|k}$, respectively.

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Gu_k \tag{2.1.1}$$

$$P_{k+1|k} = FP_{k|k}F^T + Q_k \tag{2.1.2}$$

The Kalman gain K_{k+1} is defined by (2.1.3), and is used to update the state estimate $\hat{x}_{k+1|k+1}$ as shown in (2.1.4). The gain makes use of an innovation covariance S_{k+1}, which is defined as the inverse term found in (2.1.3).

$$K_{k+1} = P_{k+1|k}H^T\left(HP_{k+1|k}H^T + R_{k+1}\right)^{-1} \tag{2.1.3}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(z_{k+1} - H\hat{x}_{k+1|k}\right) \tag{2.1.4}$$

The a posteriori state error covariance matrix $P_{k+1|k+1}$ is then calculated by (2.1.5), and is used iteratively, as per (2.1.2).

$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{2.1.5}$$

The derivation of the KF is well documented, with details available in [19, 3, 20]. The KF gain is unique as it yields an optimal solution to the linear estimation problem, however it comes at a price of stability and robustness. Assumptions used in the derivation include: the system model is known and linear, the system and measurement noises are white, and the states have initial conditions with known means and variances [21, 13]. However, the previous assumptions often do not hold in a number of applications. If these assumptions are violated, the KF yields suboptimal results and can become unstable [22]. In addition, the KF is sensitive to computer precision and the complexity of computations involving matrix inversions [16]. However, modern computing power has reduced this drawback significantly. The extended Kalman filter (EKF) is a natural extension of the KF method. However, the EKF may be used for nonlinear systems and measurements, unlike the KF. Nonlinear system or measurement equations may be linearized according to its Jacobian. The partial derivatives are used to compute linearized system and measurement matrices F and H, respectively found as follows [23]:

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k|k},u_k} \tag{2.1.6}$$

$$H_{k+1} = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k+1|k}} \tag{2.1.7}$$

Equations (2.1.6) and (2.1.7) essentially linearize the nonlinear system or measurement functions around the current state estimate [3]. These values can then be used as per equations (2.1.1) through (2.1.5). This comes at a loss of optimality; as such, the EKF yields a suboptimal solution to the nonlinear estimation problem [20]. Other Kalman-based methods exist beyond the EKF, and include the unscented Kalman filter (UKF) and the cubature Kalman filter (CKF) [24]. Although these methods yield improvements on the EKF, a number of strict assumptions still apply. Modeling errors, uncertainties, and disturbances can still lead to unstable estimates.

The Smooth Variable Structure Filter

The SVSF was derived in 2007 and has been shown to be stable and robust to bounded disturbances, modeling uncertainties and noise [25, 26]. The basic estimation concept of the SVSF is shown in the following figure.

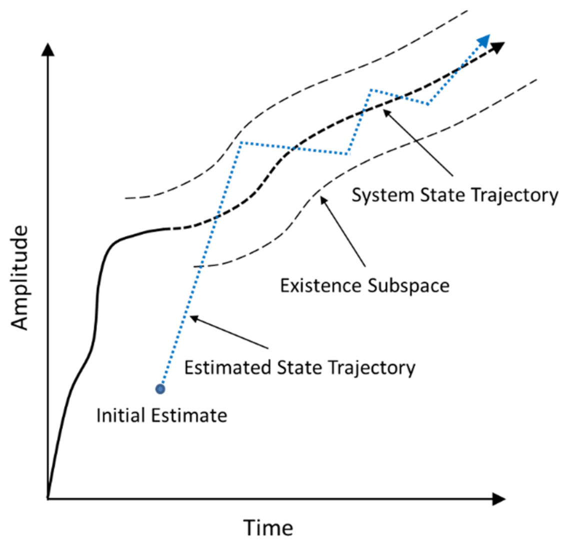

Figure 1. The above figure illustrates the SVSF estimation concept.

The SVSF method is model based and may be applied to differentiable linear or nonlinear dynamic system models [27, 28]. The original form of the SVSF as presented in [29] did not include covariance derivations. An augmented form of the SVSF was presented in [6, 4], which proposed a strategy for obtaining an error covariance matrix for the filter. The estimation process is iterative and may be summarized by the following set of equations. The predicted state estimates $\hat{x}_{k+1|k}$ and the error covariance matrix $P_{k+1|k}$ are first calculated as per the KF strategy.

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the predicted measurements $\hat{z}_{k+1|k}$, and the measurement errors $e_{z,k+1|k}$ may be calculated by (2.2.1) and (2.2.2) respectively.

$$\hat{z}_{k+1|k} = H\hat{x}_{k+1|k} \tag{2.2.1}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{2.2.2}$$

Notice how (2.2.1) and (2.2.2) are similar to the KF [30]. The SVSF process differs in how the gain is formulated. The SVSF gain is a function of: the a priori and the a posteriori measurement errors $e_{z,k+1|k}$ and $e_{z,k|k}$; the smoothing boundary layer widths ψ; the SVSF 'memory' or convergence rate γ; as well as the measurement matrix C. Refer to [29, 6] for a complete explanation on how the gain K_{k+1} is derived. The SVSF gain is defined as a diagonal matrix such that [4]:

$$K_{k+1} = C^{+} diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1} \tag{2.2.3}$$

The smoothing boundary layer term $\bar{\psi}$ in (2.2.3) is defined as:

$$\bar{\psi}^{-1} = \begin{bmatrix} \frac{1}{\psi_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\psi_m} \end{bmatrix} \tag{2.2.4}$$

where m is the number of measurements. This gain is used to calculate the updated state estimates $\hat{x}_{k+1|k+1}$ as well as the updated state error covariance matrix $P_{k+1|k+1}$, as per the KF strategy.

Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated, and are used in later iterations:

$$\hat{z}_{k+1|k+1} = C\hat{x}_{k+1|k+1} \tag{2.2.5}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{2.2.6}$$

The SVSF process results in the state estimates converging to within a region of the state trajectory [29, 4]. Thereafter, it switches back and forth across the state trajectory within a region referred to as the existence subspace, as shown earlier in Fig. 1. This switching effect brings about an inherent amount of stability and robustness in the estimation process, as will be demonstrated in the simulation.

3. SQUARE-ROOT FORMULATIONS

The Square-Root Kalman Filter

The square-root formulation of the KF was developed by James Potter and Angus Andrews [13]. The method described in this section is often referred to as Potter's algorithm [16]. As per Cholesky factorization, suppose that the square root of the state error covariance matrix P is available, such that $P = SS^T$. Modifying (2.1.2) yields:

$$P_{k+1|k} = S_{k+1|k}S_{k+1|k}^{\mathrm{T}} = FS_{k|k}S_{k|k}^{T}F^T + Q_k^{1/2}Q_k^{T/2} \tag{3.1.1}$$

Equation (3.1.1) is essentially (2.1.2). Modifying (2.1.3) yields:

$$K_{k+1} = S_{k+1|k}S_{k+1|k}^{T}H^T\left(HS_{k+1|k}S_{k+1|k}^{T}H^T + R_{k+1}\right)^{-1} \tag{3.1.2}$$

The updated state error covariance (2.1.5) then becomes:

$$P_{k+1|k+1} = (I - K_{k+1}H)S_{k+1|k}S_{k+1|k}^{T}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^{T} \tag{3.1.3}$$

Alternatively, this can be written as follows [13]:

$$P_{k+1|k+1} = S_{k+1|k}(I - a\phi\phi^T)S_{k+1|k}^{T} \tag{3.1.4}$$

where a and ϕ are defined as:

$$a = \left(\phi^T \phi + R_{i,k+1}\right)^{-1}$$
$$\phi = S_{k+1|k}^T H^T \tag{3.1.5}$$

Note that i refers to the i^{th} element of the corresponding matrix or vector. As per [13], the a posteriori square-root covariance matrix can be calculated as follows:

$$S_{k+1|k+1} = S_{k+1|k}(I - a\gamma\phi\phi^T) \tag{3.1.6}$$

where γ is given as [13]:

$$\gamma = \left(1 + \sqrt{aR_{i,k+1}}\right) \tag{3.1.7}$$

Equations (3.1.1) through (3.1.7) can be used in conjunction with the standard KF estimation process. The main difference is that the update equation is used to update S instead of P, and the process is repeatedly iteratively [13].

The Square-Root SVSF

This paper introduces the square-root formulation of the SVSF, hereafter referred as to SR-SVSF. It is based on the same approach as the square-root KF. For linear systems and measurements, the SR-SVSF estimation processed is summarized by the following set of equations. For nonlinear systems and measurements, the nonlinearities may be linearized as per the EKF methodology. The state estimates $\hat{x}_{k+1|k}$ and square-root covariance $S_{k+1|k}$ are first calculated, as follows:

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Gu_k \tag{3.2.1}$$

$$S_{k+1|k}S_{k+1|k}^T = FS_{k|k}S_{k|k}^T F^T + Q_k^{1/2} Q_k^{T/2} \tag{3.2.2}$$

The predicted measurement $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ are calculated next.

$$\hat{z}_{k+1|k} = H\hat{x}_{k+1|k} \tag{3.2.3}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{3.2.4}$$

Next, the gain K_{k+1} is calculated as per (3.2.5).

$$K_{k+1} = C^+ diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1} \tag{3.2.5}$$

The a posteriori square-root covariance matrix $S_{k+1|k+1}$ is calculated next as follows:

$$S_{k+1|k+1} = S_{k+1|k}(I - a\gamma\phi\phi^T) \tag{3.2.6}$$

where $a = \left(\phi^T\phi + R_{i,k+1}\right)^{-1}$, $\phi = S_{k+1|k}^T H^T$, and $\gamma = \left(1 + \sqrt{aR_{i,k+1}}\right)$. Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated, and are used in later iterations:

$$\hat{z}_{k+1|k+1} = C\hat{x}_{k+1|k+1} \tag{3.2.7}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{3.2.8}$$

The SR-SVSF estimation process is summarized by (3.2.1) through (3.2.8). It is important to note that in this case, the gain is not affected by the square-root covariance calculation. However, the SR-SVSF formulation sets the framework for future work and implementation in other types of SVSF that rely on the covariance [4].

4. COMPUTER EXPERIMENTS

Target Tracking Problem Setup

The target tracking problem is based on a generic air traffic control (ATC) scenario found in [21] and is as described in [4]. A radar stationed at the origin provides direct position only measurements, with a standard deviation of $50\ m$ in each coordinate. The following figure illustrates the average motion of the target.

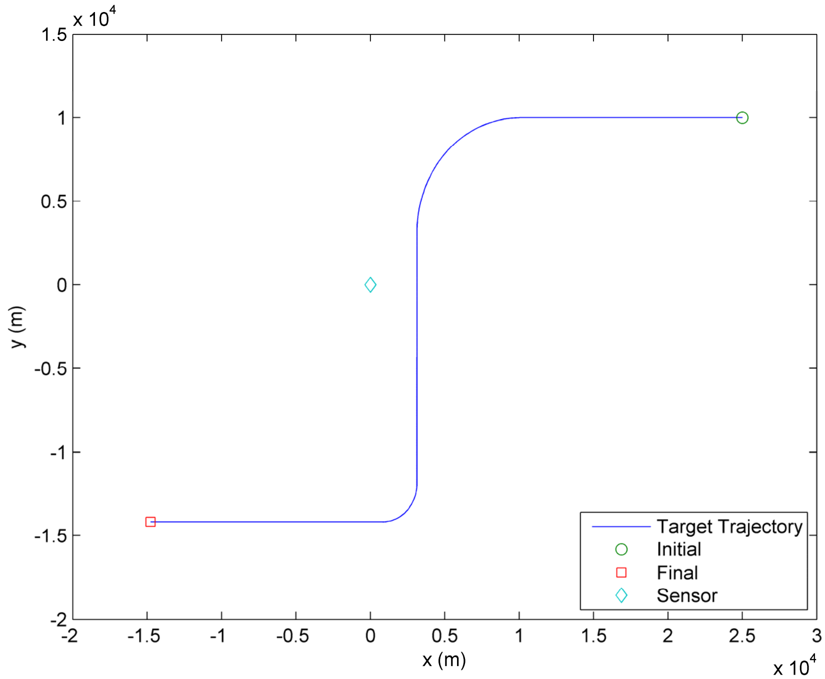

Figure 1. True target trajectory for the nonlinear estimation problem.

As shown in the previous figure, an aircraft starts from an initial position of $[25{,}000\ m, 10{,}000\ m]$ at time $t\ =\ 0\ s$, and flies westward at $120\ m/s$ for $125\ s$. The aircraft then begins a coordinated turn for a period of $90\ s$ at a rate of $1^{\circ}/s$. It then flies southward at $120\ m/s$ for $125\ s$, followed by another coordinated turn for $30\ s$ at $3^{\circ}/s$. The aircraft then continues to fly westward until it reaches its final destination.

In ATC scenarios, the behaviour of civilian aircraft may be modeled by two different modes: uniform motion (UM) which involves a straight flight path with a constant speed and course, and maneuvering which includes turning or climbing and descending [21]. In this case, maneuvering will refer to a coordinated turn (CT) model, where a turn is made at a constant turn rate and speed. The uniform motion model used for this target tracking problem is given by (4.1.1) [21, 31].

$$x_{k+1} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ 0 & \frac{1}{2}T^2 \\ T & 0 \\ 0 & T \end{bmatrix} w_k \tag{4.1.1}$$

The state vector of the aircraft may be defined as follows:

$$x_k = [\xi_k \quad \eta_k \quad \dot{\xi}_k \quad \dot{\eta}_k]^T \tag{4.1.2}$$

The first two states refer to the position along the x-axis and y-axis, respectively, and the last two states refer to the velocity along the x-axis and y-axis, respectively. The sampling time used in this simulation was 5 seconds. When using the CT model, the state vector needs to be augmented to include the turn rate, as shown in (4.1.3) [21]. The CT model may be considered nonlinear if the turn rate of the aircraft is not known. Note that a left turn corresponds to a positive turn rate, and a right turn has a negative turn rate. This sign convention follows the commonly used trigonometric convention (the opposite is true for navigation convention) [21]. As per [21, 31], the CT model is given by (4.1.4).

$$x_k = [\xi_k \quad \eta_k \quad \dot{\xi}_k \quad \dot{\eta}_k \quad \omega_k]^T \tag{4.1.3}$$

$$x_{k+1} = \begin{bmatrix} 1 & 0 & \frac{sin\omega_k T}{\omega_k} & -\frac{1-cos\omega_k T}{\omega_k} & 0 \\ 0 & 1 & \frac{1-cos\omega_k T}{\omega_k} & \frac{sin\omega_k T}{\omega_k} & 0 \\ 0 & 0 & cos\omega_k T & -sin\omega_k T & 0 \\ 0 & 0 & sin\omega_k T & cos\omega_k T & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2}T^2 & 0 & 0 \\ 0 & \frac{1}{2}T^2 & 0 \\ T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} w_k \tag{4.1.4}$$

Since the radar stationed at the origin provides direct position measurements only, the measurement equation may be formed linearly as follows:

$$z_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} x_k + v_k \tag{4.1.5}$$

Equations (4.1.1) through (4.1.5) were used to generate the true state values of the trajectory and the radar measurements for this target tracking scenario. As previously mentioned, the EKF uses a linearized form of the system and measurement matrices. In this case, the system defined by (4.1.4) is nonlinear, such that the Jacobian of it yields a linearized form as shown in (4.1.6). The terms in the last column of (4.1.6) are correspondingly defined in (4.1.7) [21].

$$\left[\nabla_x F_{k,x}^T\right]^T\Big|_{x_k=\hat{x}_k} = \begin{bmatrix} 1 & 0 & \frac{sin\widehat{\omega}_k T}{\widehat{\omega}_k} & -\frac{1-cos\widehat{\omega}_k T}{\widehat{\omega}_k} & F_{\widehat{\omega}1} \\ 0 & 1 & \frac{1-cos\widehat{\omega}_k T}{\widehat{\omega}_k} & \frac{sin\widehat{\omega}_k T}{\widehat{\omega}_k} & F_{\widehat{\omega}2} \\ 0 & 0 & cos\widehat{\omega}_k T & -sin\widehat{\omega}_k T & F_{\widehat{\omega}3} \\ 0 & 0 & sin\widehat{\omega}_k T & cos\widehat{\omega}_k T & F_{\widehat{\omega}4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1.6}$$

$$\begin{bmatrix} F_{\widehat{\omega}1} \\ F_{\widehat{\omega}2} \\ F_{\widehat{\omega}3} \\ F_{\widehat{\omega}4} \end{bmatrix} = \begin{bmatrix} \frac{(cos\widehat{\omega}_k T)T}{\widehat{\omega}_k}\hat{\dot{\xi}}_k - \frac{(sin\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\hat{\dot{\xi}}_k - \frac{(sin\widehat{\omega}_k T)T}{\widehat{\omega}_k}\widehat{\dot{\eta}_k} - \frac{(-1+cos\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\widehat{\dot{\eta}_k} \\ \frac{(sin\widehat{\omega}_k T)T}{\widehat{\omega}_k}\hat{\dot{\xi}}_k - \frac{(1-cos\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\hat{\dot{\xi}}_k - \frac{(cos\widehat{\omega}_k T)T}{\widehat{\omega}_k}\widehat{\dot{\eta}_k} - \frac{(sin\widehat{\omega}_k T)}{\widehat{\omega}_k^2}\widehat{\dot{\eta}_k} \\ -(sin\widehat{\omega}_k T)T\hat{\dot{\xi}}_k - (cos\widehat{\omega}_k T)T\widehat{\dot{\eta}_k} \\ (cos\widehat{\omega}_k T)T\hat{\dot{\xi}}_k - (sin\widehat{\omega}_k T)T\widehat{\dot{\eta}_k} \end{bmatrix} \tag{4.1.7}$$

To generate the results for this section, the following values were used for the initial state error covariance matrix $P_{0|0}$, the system noise matrix Q, and the measurement noise matrix R.

$$P_{0|0} = \begin{bmatrix} R_{11} & 0 & 0 & 0 & 0 \\ 0 & R_{22} & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1.8}$$

$$Q = L_1 \begin{bmatrix} \frac{T^3}{3} & 0 & \frac{T^2}{2} & 0 & 0 \\ 0 & \frac{T^3}{3} & 0 & \frac{T^2}{2} & 0 \\ \frac{T^2}{2} & 0 & T & 0 & 0 \\ 0 & \frac{T^2}{2} & 0 & T & 0 \\ 0 & 0 & 0 & 0 & \frac{L_2}{L_1}T \end{bmatrix} \tag{4.1.9}$$

$$R = 50^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.1.10}$$

Note that L_1 and L_2 are referred to as power spectral densities, and were defined as 0.16 and 0.01, respectively [31]. The system and measurement noise (w_k and v_k) were generated using their respective covariance values (Q and R). Also, when using the UM model, the fifth row and column of (4.1.8) and (4.1.9) were truncated. For the standalone SVSF estimation process, the limit on the smoothing boundary layer widths were defined as $\psi = [500 \quad 1{,}000 \quad 500 \quad 1{,}000 \quad 1]^T$, and the SVSF 'memory' or convergence rate was set to $\gamma = 0.1$. These parameters were tuned based on some knowledge of the uncertainties (i.e., magnitude of noise) and with the goal of decreasing the estimation error. It is required to transform the measurement matrix into a square matrix (i.e., identity), such that an 'artificial' measurement is created. It is possible to derive 'artificial' velocity measurements based on the available position measurements. For example, consider the following artificial measurement vector y_k for the SVSF:

$$y_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \left(z_{1,k+1} - z_{1,k}\right)/T \\ \left(z_{2,k+1} - z_{2,k}\right)/T \\ 0 \end{bmatrix} \tag{4.1.11}$$

The accuracy of (4.1.11) depends on the sampling rate T. Applying the above type of transformation to non-measured states allows a measurement matrix equivalent to the identity matrix. The estimation process would continue as in the previous section, where $H = I$. Note however that the artificial velocity measurements would be delayed one time step. Furthermore, it is assumed that the artificial turn rate measurement is set to 0, since no artificial measurement could be created based on the available measurements. A total of 500 Monte Carlo runs were performed, and the results were averaged.

Simulation Results

Both the square-root KF and the proposed SR-SVSF were applied on the target tracking problem. The algorithms were applied to the aforementioned setup. The target tracking results are shown in the following figure. The square-root based SVSF was able to follow the target trajectory, regardless of which flight model was implemented. However, the square-root based EKF experienced difficulty at the presence of the aircraft turns. This is primarily due to the difference between the model used by the filter and the model actually experienced by the target. The estimation error is shown in Fig. 3. Notice how the SR-SVSF yielded relatively similar results, regardless of which model was implemented. This is primarily due to the robust estimation process inherent to the switching gain. A second case was studied, where the measurement at 50 seconds was increased by 1,000 times. This case further demonstrated the robustness of the SR-SVSF. The SR-EKF was unable to overcome the measurement error, however the SR-SVSF was able to maintain the true state trajectory. This is further illustrated by Figs. 4 and 5.

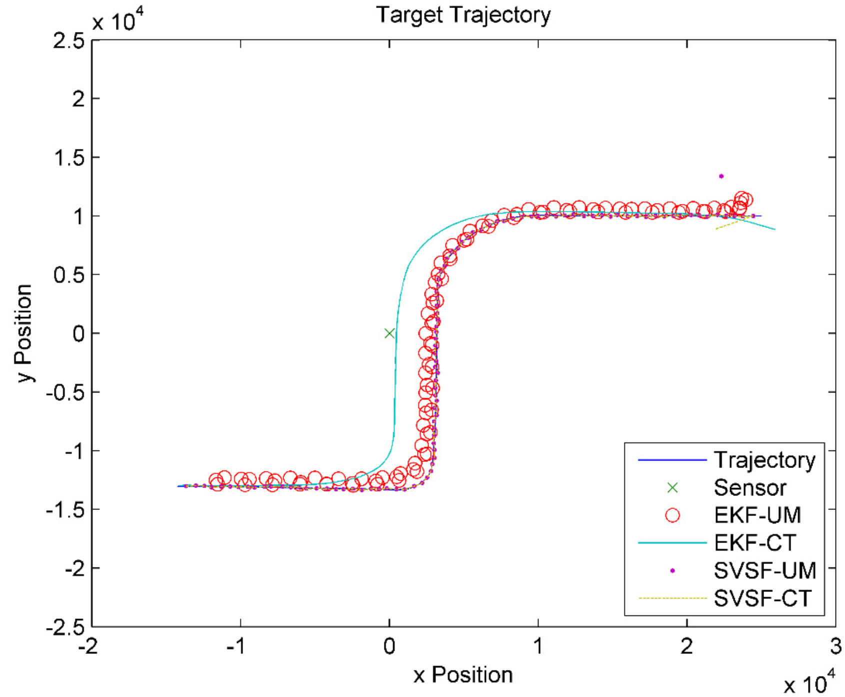

Figure 2. True and estimated target trajectories for the nonlinear estimation problem.

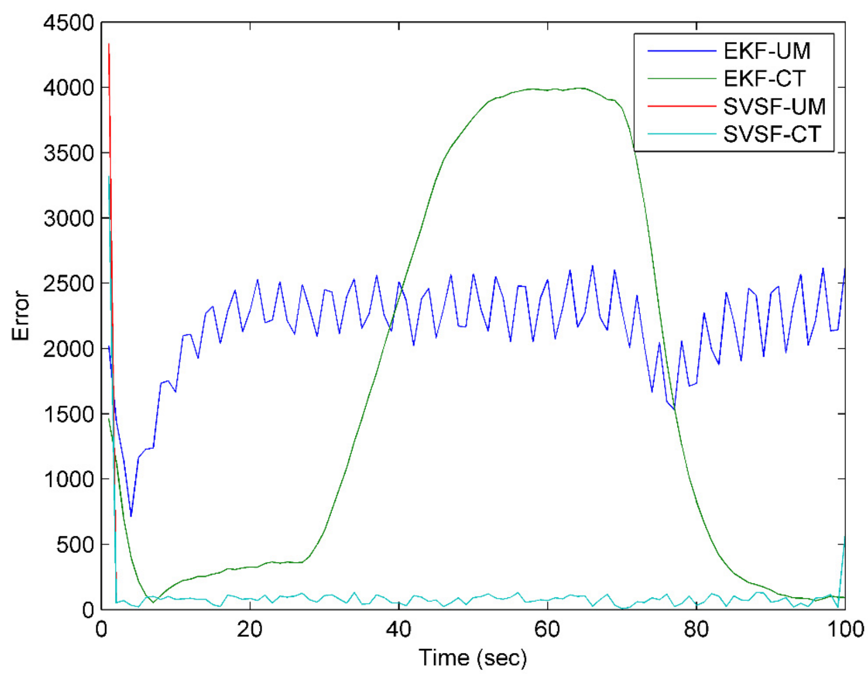

Figure 3. Estimation errors for the nonlinear estimation problem.

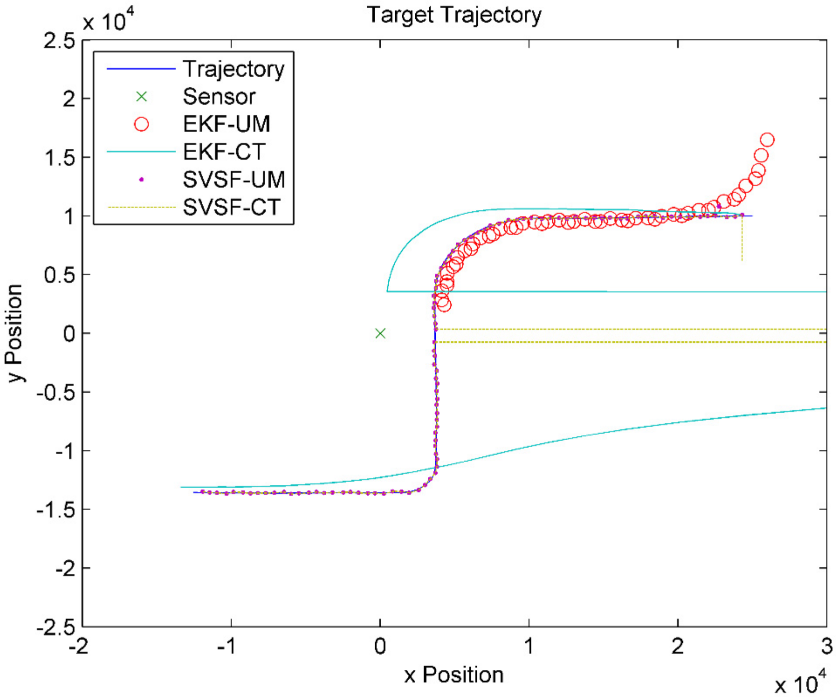

Figure 4. True and estimated target trajectories with the presence of measurement errors.

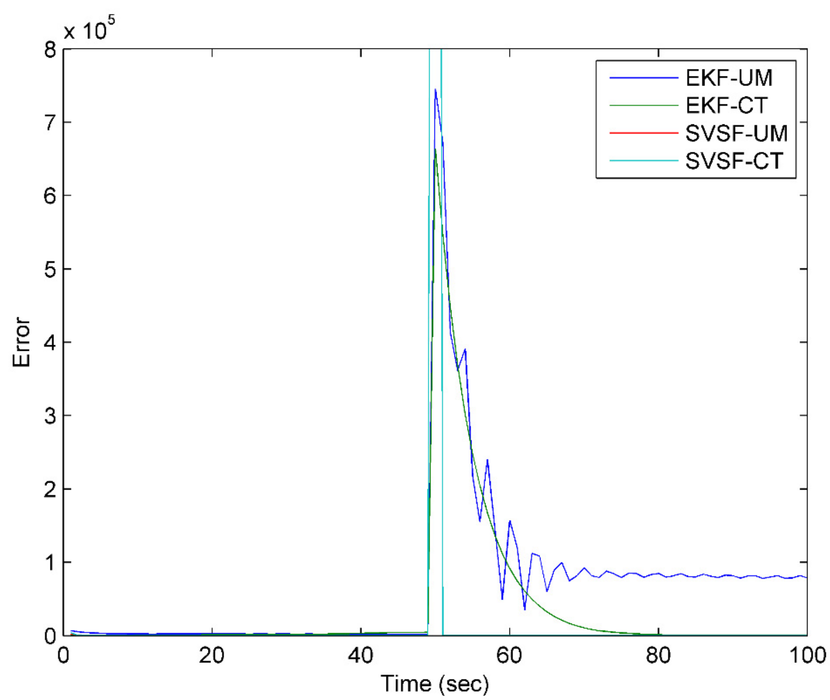

Figure 5. Estimation errors for the square-root filters with presence of measurement errors.

5. CONCLUSIONS

This paper introduced a new version of the smooth variable structure filter (SVSF) based on Potter's square-root algorithm. The new methodology, referred to simply as the SR-SVSF, was applied on a nonlinear target tracking problem. The results were compared with the popular Kalman filter strategy. It was determined that the robustness of the SVSF switching gain yielded a stable and accurate estimation of the target. The estimates were found to be founded to the true state trajectory. At the presence of measurement errors, the KF-based strategy failed to yield a good result, however the SVSF-based strategy remained stable. Future work includes implementing the SR-SVSF on real-life data and benchmark problems.

REFERENCES

[1] S. A. Gadsden and S. R. Habibi, "A New Robust Filtering Strategy for Linear Systems," *ASME Journal of Dynamic Systems, Measurement, and Control,* vol. 135, no. 1, 2011.

[2] S. A. Gadsden, Y. Song and S. R. Habibi, "Novel Model-Based Estimators for the Purposes of Fault Detection and Diagnosis," *IEEE/ASME Transactions on Mechatronics,* vol. 18, no. 4, 2013.

[3] B. D. O. Anderson and J. B. Moore, Optimal Filtering, Englewood Cliffs, NJ: Prentice-Hall, 1979.

[4] S. A. Gadsden, "Smooth Variable Structure Filtering: Theory and Applications," Ph.D. Thesis, McMaster University, Hamilton, Ontario, 2011.

[5] M. S. Grewal and K. James, "Kalman Filter Implementation with Improved Numerical Properties," *IEEE Transactions on Automatic Control,* vol. 55, no. 9, pp. 2058-2068, September 2010.

[6] S. A. Gadsden and S. R. Habibi, "A New Form of the Smooth Variable Structure Filter with a Covariance Derivation," in *IEEE Conference on Decision and Control*, Atlanta, Georgia, 2010.

[7] J. E. Potter and R. G. Stern, "Statistical Filtering of Space Navigation Measurements," in *Proceedings of 1963 AIAA Guidance and Control Conference*, New York, 1963.

[8] N. A. Carlson, "Fast Triangular Formulation of the Square Root Filter," *AIAA Journal,* vol. 11, no. 9, pp. 1259-1265, 1973.

[9] G. J. Bierman, Factorization Methods for Discrete Sequential Estimation, New York: Academic Press, Inc., 1977.

[10] R. Van der Merwe and E. A. Wan, "The Square-Root Unscented Kalman Filter for State and Parameter-Estimation," in *IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2001.

[11] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, Numerical Recipes: The Art of Scientific Computing, New York: Cambridge University Press, 2007.

[12] K. Lemon and B. W. Welch, "Comparison of Nonlinear Filtering Techniques for Lunar Surface Roving Navigation," Cleveland, Ohio, 2008.

[13] D. Simon, Optimal State Estimation: Kalman, H-Infinity, and Nonlinear Approaches, Wiley-Interscience, 2006.

[14] G. J. Bierman, "Measurement Updating Using the U-D Factorization," in *IEEE Conference on Decision and Control*, 1975.

[15] C. L. Thornton and G. J. Bierman, "Givens Transformation Techniques for Kalman Filtering," *Acta Astronautica,* vol. 4, no. 7-8, pp. 847-863, July-August 1977.

[16] M. S. Grewal and A. P. Andrews, Kalman Filtering: Theory and Practice Using MATLAB, 3 ed., New York: John Wiley and Sons, Inc., 2008.

[17] A. Andrews, "A Square Root Formulation of the Kalman Covariance Equations," *AIAA Journal,* vol. 6, pp. 1165-1166, 1968.

[18] J. M. Bennet, "Triangular Factors of Modified Matrices," *Numerische Mathematik,* vol. 7, pp. 217-221, 1963.

[19] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," *Journal of Basic Engineering, Transactions of ASME,* vol. 82, pp. 35-45, 1960.

[20] A. Gelb, Applied Optimal Estimation, Cambridge, MA: MIT Press, 1974.

[21] Y. Bar-Shalom, X. Rong Li and T. Kirubarajan, Estimation with Applications to Tracking and Navigation, New York: John Wiley and Sons, Inc., 2001.

[22] S. J. Julier, J. K. Ulhmann and H. F. Durrant-Whyte, "A New Method for Nonlinear Transformation of Means and Covariances in Filters and Estimators," *IEEE Transactions on Automatic Control,* vol. 45, pp. 472-482, March 2000.

[23] S. Haykin, Kalman Filtering and Neural Networks, New York: John Wiley and Sons, Inc., 2001.

[24] S. A. Gadsden, S. R. Habibi and T. Kirubarajan, "Kalman and Smooth Variable Structure Filters for Robust Estimation," *IEEE Transactions on Aerospace and Electronic Systems,* vol. 50, no. 2, 2014.

[25] S. R. Habibi and R. Burton, "The Variable Structure Filter," *Journal of Dynamic Systems, Measurement, and Control (ASME),* vol. 125, pp. 287-293, September 2003.

[26] S. R. Habibi and R. Burton, "Parameter Identification for a High Performance Hydrostatic Actuation System using the Variable Structure Filter Concept," *ASME Journal of Dynamic Systems, Measurement, and Control,* 2007.

[27] S. A. Gadsden, D. Dunne, S. R. Habibi and T. Kirubarajan, "Application of the Smooth Variable Structure Filter to a Multi-Target Tracking Problem," in *Signal Processing, Sensor Fusion, and Target Recognition, Society of Photo-Optical Instrumentation Engineers (SPIE)*, Orlando, Florida, 2011.

[28] S. A. Gadsden, S. R. Habibi and T. Kirubarajan, "A Novel Interacting Multiple Model Method for Target Tracking," in *13th International Conference on Information Fusion*, Edinburgh, UK, 2010.

[29] S. R. Habibi, "The Smooth Variable Structure Filter," *Proceedings of the IEEE,* vol. 95, no. 5, pp. 1026-1059, 2007.

[30] G. Welch and G. Bishop, "An Introduction to the Kalman Filter," 2006.

[31] H. Wang, T. Kirubarajan and Y. Bar-Shalom, "Precision Large Scale Air Traffic Surveillance Using an IMM Estimator with Assignment," *Transaction of the IEEE, Aerospace and Electronic Systems Journal,* vol. 35, no. 1, pp. 255-266, 1999.