System and Mathematical Modeling of Quadrotor Dynamics

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ABSTRACT

Unmanned aerial systems (UAS) are becoming increasingly visible in our daily lives; and range in operation from search and rescue, monitoring hazardous environments, and to the delivery of goods. One of the most popular UAS are based on a quad-rotor design. These are typically small devices that rely on four propellers for lift and movement. Quad-rotors are inherently unstable, and rely on advanced control methodologies to keep them operating safely and behaving in a predictable and desirable manner. The control of these devices can be enhanced and improved by making use of an accurate dynamic model. In this paper, we examine a simple quadrotor model, and note some of the additional dynamic considerations that were left out. We then compare simulation results of the simple model with that of another comprehensive model.

Keywords: Multirotor, quadrotor, quadcopter, modeling, control, UAV, blade flapping

1. INTRODUCTION

Unmanned aerial systems (UAS) can be traced back at least as early as 1896 with Samuel Langley's unmanned 'aerodome' which flew for almost a mile over the Potomac River [1]. Since then, UAS' have made significant progress and are now a rapidly expanding area of technological research and development. Unlike traditional aircraft, UAS are not limited by the need for an onboard human operator. This fact allows designers to consider a wide variety of unconventional airframes, new mission possibilities, and an entirely new flight envelope. Coupled with great advances in computer, sensor, and other relevant technology, UAS development is considered to be 'exploding'. Applications for UAS' are many and varied. Current military use of UAS is significant, with platforms developed for reconnaissance, surveillance, target acquisition, strike and other missions. Civilian use has seen significant expansion in recent years with applications in such areas as land surveying and mapping, agriculture, data collection, power grid maintenance, search and rescue operations, police surveillance, security systems, and even prototype parcel delivery systems.

One particular UAS platform of great interest is the multirotor. The multirotor is a type of rotorcraft distinguished from the traditional helicopter in that it possesses at least three rotors. Multirotors have the advantage that they can achieve flight control simply by varying the speed of their individual rotors, eliminating the need for mechanical pitch control. Recent developments in battery and motor technology as well as sensor and control algorithms, have accelerated development and proliferation of these vehicles. Current usages range from platforms for advanced robotics research to aerial photography to toys for children. Among multirotors, the four rotor quadrotor (a.k.a. quadrotor, or also known as quadcopter) is probably the most popular; its straightforward design makes it relatively easy to produce, build, and fly.

A good model of the quadrotor, or any relevant platform for that matter, is of tremendous value. Models are of critical importance in the design of the flight controller itself; a robust model will allow a designer to tune the PID values, or whatever relevant control scheme being used, without having to resort to expensive and potentially dangerous physical testing. Some control methodologies such as model-based predictive control incorporate the actual model in the flight control scheme [2, 3]. Accurate models are also very useful in the development of autonomous control systems, general robotics research, flight simulation, and any application where one needs a working prediction of a particular quadrotor's flight dynamics and performance.

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Typically, modeling can work one of two ways. The first is based on mathematics, where one derives the relevant equations based on an understanding of the physical system, making any necessary simplifying assumptions. The second takes an existing platform and measures its inputs and outputs, and creates a model without any real knowledge of the physics or dynamics. The latter 'black box' approach, or system identification, is often the best option when dealing with highly complex systems where a useful mathematical model would be difficult or impossible to obtain. The objective of this paper is to provide an overview of the most popular quadrotor models and dynamics. A simulation and comparison of models is also provided. Section 2 provides an overview of quadrotor flight (2.1), and a basic mathematical model (2.2) accounting only for the basic forces, thrusts, and torques involved. A PD flight controller is designed for the model and is described (2.3). Some of the more realistic dynamic considerations that appear in other models are also described (2.4). In section 3, MATLAB code is developed to simulate a quadrotor flight, and a simple model is compared with another popular model. The main findings are summarized and future work is discussed in the last section.

2. QUADROTOR MODELING

2.1 Quadrotor Flight

A quadrotor is controlled by carefully managing the speed of its rotors. The rotors generate both thrusts and reaction moments which act on the airframe. By controlling the four different rotor speeds in specific configurations, six degree of freedom flight can be achieved. Since there are only four adjustable inputs, the system is considered to be under-actuated.

The four rotors are configured into two clockwise and counter clockwise rotating pairs. The counter rotation of each pair serves to cancel out the net reaction moment when all rotors spin at the same angular velocity. In some quadrotors, the arrangement is that of a cross (+), where the primary directions of lateral flight (forward/reverser, left/right) are parallel to the rotor arms. In most other quadrotors, the configuration is that of an X, where the rotor arms are offset 45° from these principle directions. Figure 1 depicts some of the main maneuvers of an X configured quadrotor, and how they are achieved by varying rotor thrust.

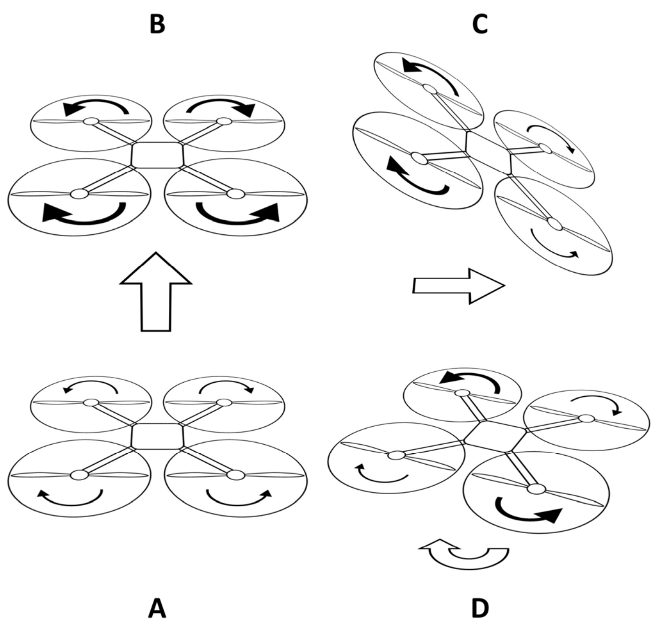

Figure 1. Quadrotor flight: A. Hover, B. Vertical climb, C. Lateral flight, D. Yaw maneuver

When all rotors spin at the same speed and generate enough thrust to overcome the weight of the airframe, a stationary hover is achieved (Fig. 1A). When thrust on all rotors are increased equally, the quadrotor climbs vertically (Fig. 1B). Lateral flight is achieved by increasing two rotors' speed relative to their opposites, generating a net moment on the airframe which pitches the craft. The change in pitch vectors some of thrust into the horizontal plane, pushing the quadrotor forward (Fig. 1C). Orientation control is achieved by changing a given rotor pair's speed with respect to the second, resulting in net moment and causing the quadrotor to yaw about its center of gravity (Fig. 1D).

2.2 Simple Model

Two basic approaches can been taken to model the quadrotor [4]: one using the Newton-Euler equations of motion [5, 6, 7, 8, 9] and the other the Lagrange-Euler equation [10, 11, 12]. The referenced sources are only a few examples of models employing either method. Many quadrotor models have been developed in recent years, the majority take the Newton-Euler approach, likely on account of its more intuitive appeal.

The following model is based on the Newton-Euler approach, and follows the basic equations developed in [13]. Only the most significant forces acting on the system are considered, which include: gravity, the thrusts generated by the rotors, and the reaction moments of the rotors due to aerodynamic drag. Consider a body fixed frame (x_b, y_b, z_b) and a world inertial frame (x, y, z). The location of the body frame within the world frame is given by a position vector ξ and its orientation with respect to the world frame defined by a set of three Euler angles (θ, ψ, ϕ), representing the pitch, roll, and yaw, respectively (as per Figure 2).

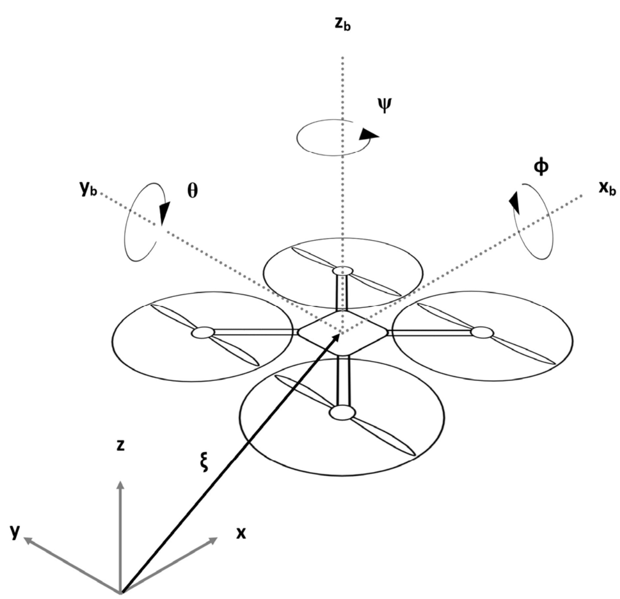

Figure 2. Basic coordinate system for a quadrotor.

Consider a ZYX rotation matrix R to express the attitude of the body frame with respect to the inertial frame:

$$R = \begin{pmatrix} c\psi c\theta & c\psi s\theta s\phi - c\phi s\psi & s\psi s\phi + c\psi c\phi s\theta \\ c\theta s\psi & c\psi c\phi + s\psi s\theta s\phi & c\phi s\psi s\theta - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{pmatrix} \quad (1)$$

where *c* and *s* refer to cosine and sine, respectively.

Consider the basic equations of motion by balancing forces and moments. Assume that the center of gravity of the body is positioned in the same plane as the quadrotor arms. This assumption simplifies the equations, although it is not without its drawbacks, as discussed later. Figure 3 depicts principle forces and moments acting on the quadrotor, l is the length of a single rotor arm measured from the center of gravity to the rotor hub. All rotor arms are assumed to have the same length.

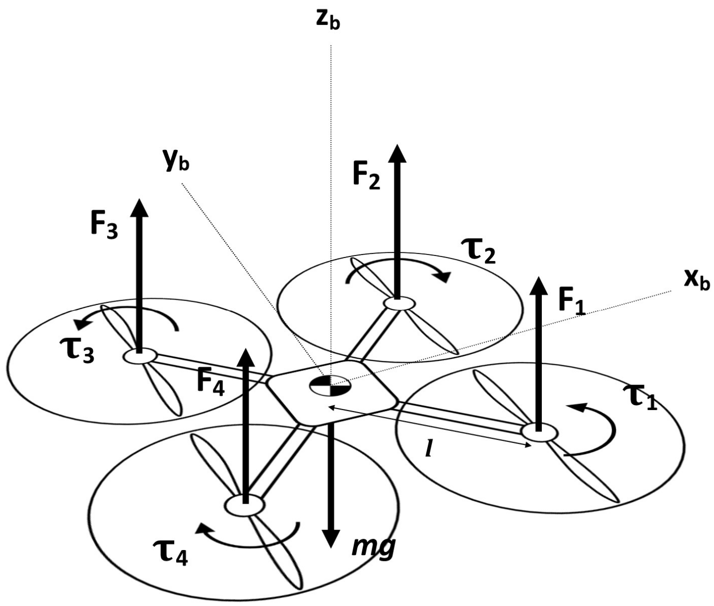

Figure 3. Force and moment balance applied on a quadrotor.

Based on Fig. 3, the basic equations of motion are defined as follows:

$$\ddot{x} = \frac{(\sum_{n=1}^{4} F_n)(\sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta) - K_1\dot{x}}{m}$$
$$\ddot{y} = \frac{(\sum_{n=1}^{4} F_n)(\cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi) - K_2\dot{y}}{m}$$
$$\ddot{z} = \frac{(\sum_{n=1}^{4} F_n)(\cos\theta\cos\phi) - mg - K_3\dot{z}}{m} \tag{2}$$
$$\ddot{\theta} = l(-F_1 - F_2 + F_3 + F_4 - K_4\dot{\theta})/J_1$$
$$\ddot{\phi} = l(-F_1 + F_2 + F_3 - F_4 - K_5\dot{\psi})/J_2$$
$$\ddot{\psi} = (\tau_1 - \tau_2 + \tau_3 - \tau_4 - K_6\dot{\phi})/J_3$$

where J_i are the moments of inertia with respect to the axes and l represents the length of an individual rotor arm from the axis origin. K_i refer to the drag coefficients. For the purposes of this model, assume that the drag is zero, which may be reasonable at low speeds. Define the controller inputs, as direct functions of the required forces. Furthermore, assume a linear relationship between the moments generated by the motors and the thrusts, defining a force-torque scaling factor C, such that $F = C\tau$. The inputs u_i can be expressed as follows:

$$
\begin{aligned}
u_1 &= (F_1 + F_2 + F_3 + F_4)/m \\
u_2 &= (-F_1 - F_2 + F_3 + F_4)/J_1 \\
u_3 &= (-F_1 + F_2 + F_3 - F_4)/J_2 \\
u_4 &= C(F_1 - F_2 + F_3 - F_4)/J_3
\end{aligned} \tag{3}
$$

Examining these inputs, u_1 controls the total thrust along the vertical axis, u_2 controls the pitch, u_3 the roll, and u_4 the yaw. Applying these definitions to (2), the equations of motion can be rewritten as:

$$
\begin{aligned}
\ddot{x} &= u_1(\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi) \\
\ddot{y} &= u_1(\sin\theta \sin\psi \cos\phi - \cos\psi \sin\phi) \\
\ddot{z} &= u_1(\cos\theta \cos\phi) - g \\
\ddot{\theta} &= u_2 l \\
\ddot{\phi} &= u_3 l \\
\ddot{\psi} &= u_4
\end{aligned} \tag{4}
$$

2.3 Simple Model Control

For the control scheme, a basic proportional-derivative (PD) controller is applied. This system can be divided into a fully-actuated subsystem and an under-actuated subsystem as follows:

$$
\begin{bmatrix} \ddot{z} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} u_1(cos\theta cos\phi) - g \\ u_4 \end{bmatrix} \tag{5}
$$

$$
\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} u_1 cos\phi & u_1 sin\phi \\ u_1 sin\phi & -u_1 cos\phi \end{bmatrix} \begin{bmatrix} sin\theta cos\psi \\ sin\psi \end{bmatrix} \tag{6}
$$

$$
\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} u_2 l \\ u_3 l \end{bmatrix} \tag{7}
$$

Equation (5) is a fully-actuated subsystem whereas (6) and (7) is an under-actuated subsystem. In order to control the under-actuated subsystem, x and y can be controlled by using ϕ and θ indirectly. The PD controller can be designed with desired input $[x_d, y_d, z_d, \psi_d]$, and (6) and (7). The block diagram of the controller is depicted in Figure 4.

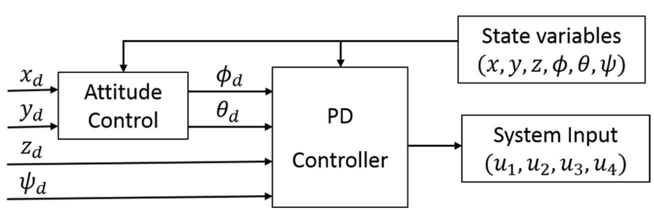

Figure 4. Block diagram of the PD controller.

The inputs can be expressed as follows:

$$
\begin{aligned}
u_1 &= k_{p,z}(z_d - z) + k_{d,z}(\dot{z_d} - \dot{z}) + g \\
u_2 &= k_{p,\theta}(\theta_d - \theta) + k_{d,\theta}(\dot{\theta_d} - \dot{\theta}) \\
u_3 &= k_{p,\phi}(\phi_d - \phi) + k_{d,\phi}(\dot{\phi_d} - \dot{\phi}) \\
u_4 &= k_{p,\psi}(\psi_d - \psi) + k_{d,\psi}(\dot{\psi_d} - \dot{\psi})
\end{aligned} \tag{8}
$$

where k_p and k_d are proportional and derivative gains, respectively.

2.4 Towards a More Accurate and Robust Model

The aforementioned model is highly simplified, accounting for only the most basic dynamic considerations. Additionally, control inputs are expressed with regards to the required thrusts, which is not the full picture of the system. In reality, the thrusts and moments are a result of rotor aerodynamics, which are a function of the rotational velocity of the rotors. Rotational velocity is driven by the DC motor and gearing, which are in turn driven by electronic speed controllers (ESC) and their associated input signals. Some models have been developed which incorporate all of these steps [3, 14].

Other physical effects act on the system as well, including gyroscopic effects [15, 6, 14], blade flapping and induced drag [14, 16, 8, 3, 17], reaction moments due to inertia [15, 6], Coriolis forces [6], the hub force [15, 10], rolling moment of rotor [10], ground effect [10], overall aerodynamic drag on the frame [3, 9], and position of the center of gravity. For a full treatment of some of the main aerodynamic forces involved with quadrotors see [18]. A brief explanation of some of these factors are considered here.

2.4.1 Principle Thrusts

A spinning rotor generates thrust as a function of rotational velocity ω. Momentum theory is often used to express this relationship. The thrust generated by a given rotor in a stationary of hover can be expressed as [16]:

$$T_n = C_T \rho A_{rn} r_n^2 \omega_n^2 \tag{9}$$

where C_T is the rotor's thrust coefficient, ρ is the density of air, A_{rn} is the rotor disk area, r_n is the radius, and ω_n the angular velocity. For simplicity, form a simple lumped parameter model [16]:

$$T_n = c_T \omega_n^2 \tag{10}$$

As the rotor spins, in addition to the thrust it generates, it also experiences an aerodynamic drag. The DC motor must exert a consistent torque to overcome this drag and maintain the rotor at a given rotational velocity. As a result, the motor shaft experiences a reaction moment, which in turn is transmitted to the entire frame of the quadrotor. This drag can be also expressed as a lumped parameter model:

$$Q_n = c_Q \omega_n^2 \tag{11}$$

Both c_T and c_Q can be determined experimentally for a given rotor. In more advanced modeling, momentum theory combined with blade element theory is used to form a more complete description of rotor thrust. A thorough treatment of some of these more advanced thrust concepts is presented in [19, 18].

2.4.2 Gyroscopic Effects

When the quadrotor is level, the rotor's axis of rotation is parallel to the vertical axis. When the quadrotor rolls or pitches, the orientation of the rotor plane changes. As a spinning rotor is essentially a gyroscope, the rotor will have a tendency to resist this torque and generate a counter-torque to align the rotor spin axis with the precession axis [6] (i.e. the axis of the applied moment).

This same effect can occur on the entire quadrotor frame itself. For example, when the quadrotor is in a yaw maneuver it has a particular angular momentum. If the quadrotor also begins a rolling maneuver, the spin (yaw) axis, will attempt to align with the precession axis, and the quadrotor will experience a counter moment, approximately 90° out of phase of the applied rolling moment.

2.4.3 Blade Flapping and Induced Drag

As a rotor travels through the air at a given speed, the rotor experiences an apparent wind. In this situation the rotor blade which sweeps toward the apparent wind is known as the advancing blade while the blade which sweeps away is the retreating blade. The advancing blade experiences a higher velocity due to its own velocity plus the velocity of the apparent wind. Conversely the velocity of retreating blade is its own velocity minus the velocity of the apparent wind. As the advancing blade has more velocity it generates more lift than the retreating blade, a condition known as dissymmetry of lift. The effect is counteracted by the fact the increased lift on the advancing blade causes it to bend upward, effectively lowering its angle of attack and the amount of lift it can generate. This is termed blade flapping or rotor flapping.

Due to the same gyroscopic conditions discussed above, the rotor plane ends up with a positive tilt facing the apparent wind. More detailed treatment of this effect as well as equations to determine the actual flapping angle are available in [16, 19]. The flapping angle is important, as the thrust vector, which is perpendicular to the rotor plane, will have components in the horizontal plane of the quadrotor, affecting flight characteristics.

Another factor to be considered is the induced drag. Since typical quadrotors have fairly rigid blades, the rotor does not bend enough to completely counteract the dissymmetry of lift. The resulting lift imbalance translates into a net induced drag on the rotor. As a result of these effects, quadrotors tend to have a noticeable horizontal drag, even at slow speeds [16].

2.4.4 Position of Center of Gravity

Many models make the assumption that the center of gravity (COG) lies in the same plane as the rotors. While this simplifies the equations, it is physically unrealistic; an average quadrotor by default has the rotor plane elevated from the rest of the craft for the simple reason that the spinning rotors must clear the rotor arms. Additionally, many quadrotor designs intentionally try to lower the center of gravity, attaching the relatively heavy battery packs to the bottom of the airframe. This position offers stabilization benefits as the quadrotor has a natural tendency to resist pitching and rolling moments. While the simplifying assumption may be justified, some well-developed models do account for offset of the COG from the rotor plane [3, 8, 19].

3. SIMULATION RESULTS

This section compares the initial simple model with a more developed model. The latter is a result of the work of [20] and is freely available as part of the Robotics Toolbox for MATLAB [21]. The second model includes some of the previously discussed gyroscopic effects and blade-flapping considerations, as well as an adjustable center of gravity. The two models are simulated in MATLAB. The specified mission is a flight to the target point (0, 0, 0) from the initial position (1.5, 1.5, 1.5) via the point (5, -3, 3) where the heading angle is changed from 0° to 90°. Parameters used in simulations are as follows.

Table 1. Model parameters used in the simulation.

Parameter	**Value (Units)**
J_1, J_2	0.05 $kg\ m^2$
J_3	0.10 $kg\ m^2$
m	4 kg
l_1	0.315 m
l_2	1 m
g	9.18 m/s^2
t_f	20 sec

The first case shows the flight of Model 1 (section 2.2) and Model 2 (section 2.4) with the same initial values and only l_1. Figures 5A and 5B show the results of trajectory of Model 1 and Model 2, respectively. Also, the four state variables of two models are depicted in Fig. 6. These two models follow the reference input very well, with some deviations. Based on this scenario, it appears the second model yields, for the most part, a very smooth flight trajectory. However, the first model yields a trajectory that responds more quickly to the reference or desired flight trajectory.

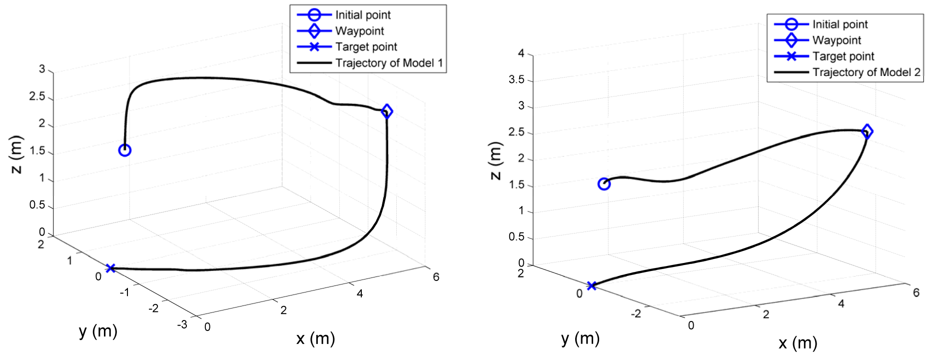

Figure 5. Flight trajectories for models 1 and 2, respectively.

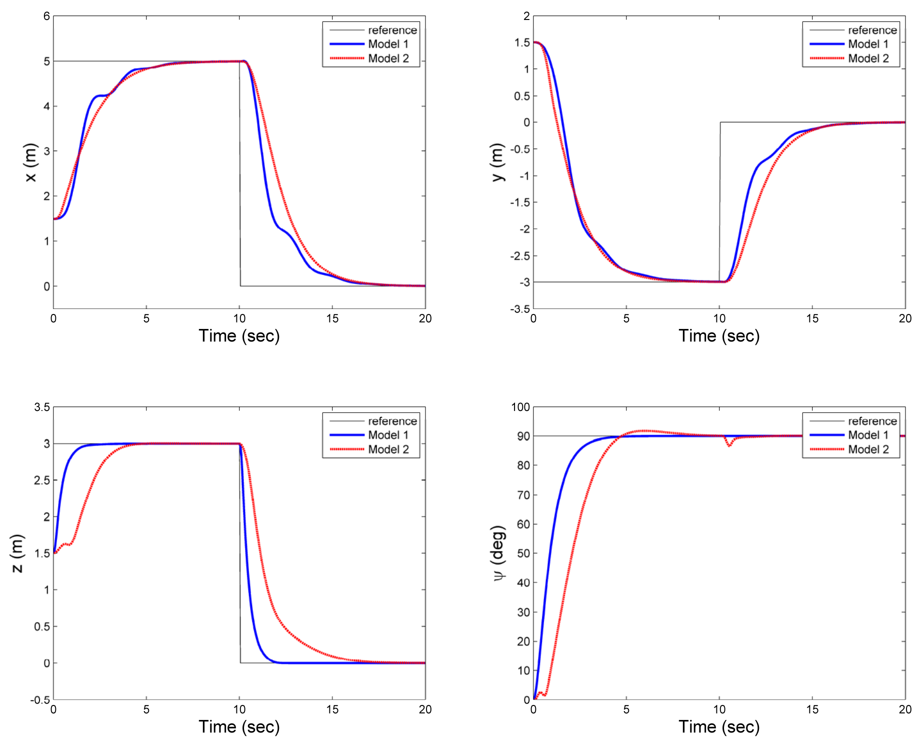

Figure 6. Positions and heading angle using Models 1 and 2.

To obtain a numerical result, the root-mean-square error (RMSE) was calculated as follows:

$$RMSE = \sum_{n=0}^{t_f} \left(x_{a,n} - x_{d,n}\right)^2. \tag{12}$$

The total RSME for Model 1 was 1890.3, whereas the total RMSE of Model 2 was 2338.3. The Model 1 has an RMSE value about 19.16% less than that of Model 2. In general, each state RMSE value differenced by 15% to 20%.

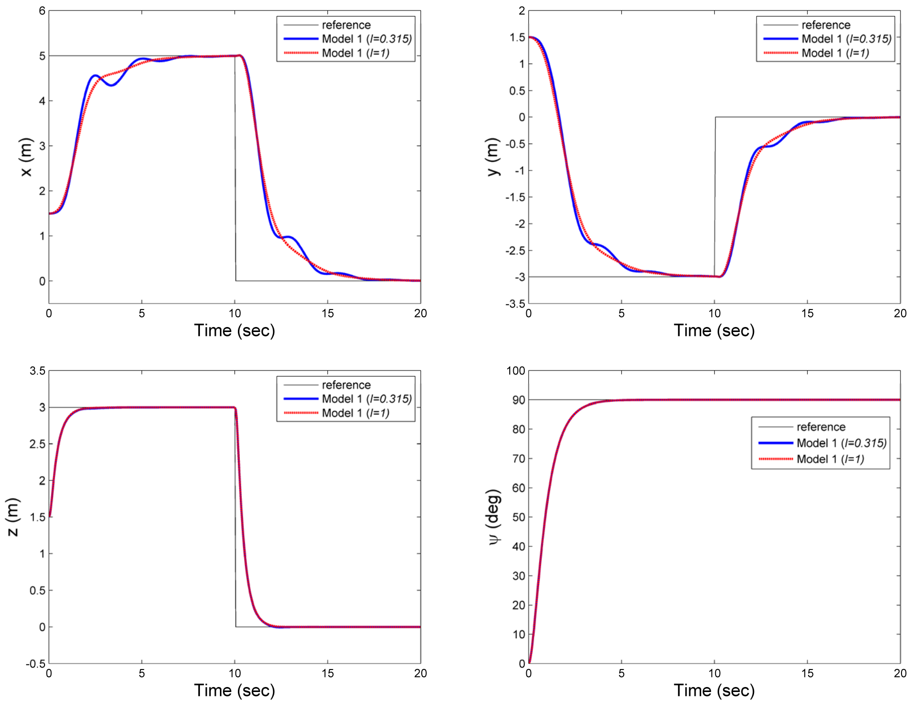

Figure 7. Positions and heading angle of quadrotor with Model 1 and varying lengths.

In an effort to study the models in more depth, consider a second case where the length from the center of the quadrotor to the rotor l_2 was changed. The state variables are presented in Figs. 7 and 8. Model 2 failed to achieve the mission with a new l_2 whereas the mission is completed wih l_1. Model 2 lose its control then crashed to the ground after 8 seconds. However, Model 1 followed the reference inputs well for both cases.

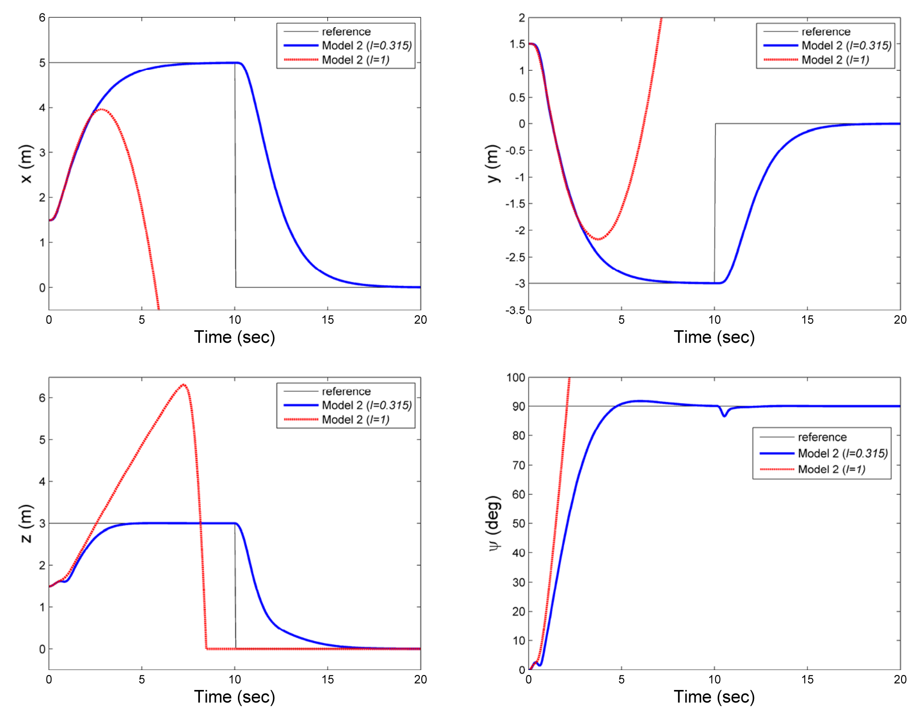

Figure 8. Positions and heading angle of quadrotor with Model 2 and varying lengths.

4. CONCLUSIONS AND FUTURE WORK

In this paper, unmanned aerial system (UAS) dynamics and popular models were introduced and studied. One of the most popular UAS are based on a quadrotor design. These are typically small devices that rely on four propellers for lift and movement. Quadrotors are inherently unstable, and rely on advanced control methodologies to keep them operating safely and behaving in a predictable and desirable manner. The control of these devices can be enhanced and improved by making use of an accurate dynamic model. This paper implemented two different quadrotor models with a PD controller. Both models were able to follow the desired flight trajectory. Furthermore, the results demonstrated that the simpler model yielded a more stable result at the presence of modeling uncertainty. Future work will look at developing additional dynamic models, and implement the models on experimental quadrotor setups.

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