Combined Particle and Smooth Variable Structure Filtering for Nonlinear Estimation Problems

S. Andrew Gadsden
Department of Mechanical Engineering
McMaster University
Hamilton, Ontario, Canada
gadsdesa@mcmaster.ca

Saeid R. Habibi
Department of Mechanical Engineering
McMaster University
Hamilton, Ontario, Canada
habibi@mcmaster.ca

Darcy Dunne
Department of Electrical and Computer Eng.
McMaster University
Hamilton, Ontario, Canada
ddunne@grads.ece.mcmaster.ca

Thia Kirubarajan
Department of Electrical and Computer Eng.
McMaster University
Hamilton, Ontario, Canada
kiruba@mcmaster.ca

Abstract – In this paper, a new state and parameter estimation method is introduced based on the particle filter (PF) and the smooth variable structure filter (SVSF). The PF is a popular estimation method, which makes use of distributed point masses to form an approximation of the probability distribution function (PDF). The SVSF is a relatively new estimation strategy based on sliding mode concepts, formulated in a predictor-corrector format. It has been shown to be very robust to modeling errors and uncertainties. The combined method (PF-SVSF) utilizes the estimates and state error covariance of the SVSF to formulate the proposal distribution which generates the particles used by the PF. The PF-SVSF method is applied on a nonlinear target tracking problem, where the results are compared with other popular estimation methods.

Keywords: Particle filter, smooth variable structure filter, tracking, nonlinear estimation.

1 Introduction

In target tracking applications, one may be concerned with surveillance, guidance, obstacle avoidance or tracking a target given some measurements [1]. In a typical scenario, sensors provide a signal that is processed and output as a measurement. These measurements are related to the target state, and are typically noise-corrupted observations [1]. The target state usually consists of kinematic information such as position, velocity, and acceleration. The measurements are processed in order to form and maintain tracks, which are a sequence of target state estimates that vary with time. Multiple targets and measurements may yield multiple tracks. Gating and data association techniques help classify the source of measurements, and help associate measurements to the appropriate track [4]. Typically these gating techniques help to avoid extraneous measurements which would otherwise cause the estimation process to go unstable and fail. A tracking filter is used in a recursive manner to carry out the target state estimation.

The most popular and well-studied estimation method is the Kalman filter (KF), which was introduced in the 1960s [2,3]. The KF yields a statistically optimal solution for linear estimation problems, as defined by (1) and (2), in the presence of Gaussian noise where $P(w_k)\sim\mathcal{N}(0,Q_k)$ and $P(v_k)\sim\mathcal{N}(0,R_k)$. A typical linear model is represented by the following equations:

$$x_{k+1} = Fx_k + Gu_k + w_k \quad (1)$$

$$z_{k+1} = Hx_{k+1} + v_{k+1} \quad (2)$$

A list of the nomenclature used throughout this paper is provided in the Appendix. It is the goal of a filter to remove the effects that the system w_k and measurement v_k noise have on extracting the true state values x_k from the measurements z_k. The KF is formulated in a predictor-corrector manner. The states are first estimated using the system model, termed as a priori estimates, meaning 'prior to' knowledge of the observations. A correction term is then added based on the innovation (also called residuals or measurement errors), thus forming the updated or a posteriori (meaning 'subsequent to' the observations) state estimates. The KF has been broadly applied to problems covering state and parameter estimation, signal processing, target tracking, fault detection and diagnosis, and even financial analysis [1,4]. The success of the KF comes from the optimality of the Kalman gain in minimizing the trace of the a posteriori state error covariance matrix. The trace is taken because it represents the state error vector in the estimation process [5]. The KF estimation process and equations have been omitted from this paper due to page constraints and the fact that the KF is readily available in the literature [3].

In real-world situations, dynamic systems are often nonlinear. For nonlinear systems, the posterior density that encapsulates all the information about the current

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state cannot be described by a finite number of summary statistics and one has to be content with an approximate filtering solution. Popular suboptimal nonlinear filters include the extended Kalman filter (EKF) [5], the unscented Kalman filter (UKF) [6], the particle filter (PF) [1], and the recently introduced cubature Kalman filter (CKF) [7]. Of these filters, the CKF is reportedly the most numerically stable and accurate [7]. Equally importantly is that the CKF does not require Jacobians and is therefore applicable to a wide range of problems.

In an effort to further increase the estimation accuracy of the PF for nonlinear estimation problems, the PF has been combined with both the EKF and UKF [8,9,10]. The extended particle filter (EPF) and unscented particle filter (UPF) respectively utilize the EKF and UKF estimates and covariances to formulate the proposal distribution used to generate the particles [3,11]. This paper introduces a new PF combination, which makes use of the relatively new smooth variable structure filter (SVSF) [12]. This method is applied on a nonlinear target tracking problem, and is compared with the popular EKF, UKF and PF.

2 Particle Filter

The particle filter (PF) has many names: Monte Carlo filters, interacting particle approximations [13], bootstrap filters [14], condensation algorithm [15], and survival of the fittest [16], to name a few. Compared to the KF, it is a newer development, being introduced in 1993. Since then, the PF has become a very popular method for solving nonlinear estimation problems, ranging from predicting chemical processes to target tracking. The PF takes the Bayesian approach to dynamic state estimation, in which one attempts to accurately represent the probability distribution function (PDF) using values of interest [1].

The PF obtains its name from the use of weighted particles or 'point masses' (3) which are distributed throughout the state space to form an approximation of the PDF as in (4). These particles are used in a recursive manner to obtain new particles and importance weights, with the goal of creating a more accurate approximation of the PDF over time. In general, as the number of implemented particles becomes very large, the approximation of the PDF becomes more accurate [1].

$$\{x_k^{(i)}, \omega_k^{(i)}\}_{i=1}^N \tag{3}$$

$$p_k(x|Z^k) \approx \sum_{i=1}^{N} \omega_k^{(i)} \delta(x - x_k^{(i)}) \tag{4}$$

An important step in the PF is that of resampling, which eliminates particles with low weights and multiplies those with high weights [1]. This helps to avoid the degeneracy problem with the PF, which refers to only a few number of particles having significant importance weights after a large number of recursions. Resampling increases the accuracy of the PDF approximation by redistributing the particles and weights near those with higher weights, while eliminating those with lower weights.

The sequential importance resampling (SIR) algorithm is a very popular form of the PF, and may be summarized by the following sets of equations [14]. The first equation draws samples or particles from the proposal distribution, here chosen as the state transition function:

$$x_k^{(i)} = f(x_{k-1}^{(i)}, u_k^{(i)}) \tag{5}$$

Next, the importance weights are updated, up to a normalizing constant, as follows:

$$\widetilde{\omega}_k^{(i)} = f(x_k^{(i)} | x_{k-1}^{(i)}) \cdot \omega_{k-1}^{(i)} \tag{6}$$

The normalized weights are then calculated for each particle:

$$\omega_k^{(i)} = \frac{1}{\sum_{i=1}^{N} \omega_k^{(i)}} \cdot \widetilde{\omega}_k^{(i)} \tag{7}$$

Finally, a constant known as the effective number of particles is calculated as shown in (8). Resampling is performed if the effective number of particles is lower than some design threshold.

$$N_{eff} = \frac{1}{N \cdot \sum_{i=1}^{N} (\omega_k^{(i)})^2} \tag{8}$$

The final PF estimate of the states is typically calculated as a weighted sum of the particles, as follows:

$$\hat{x}_k = \sum_{i=1}^{N} \omega_k^{(i)} x_k^{(i)} \tag{9}$$

3 Smooth Variable Structure Filter

A new form of predictor-corrector estimator based on sliding mode concepts referred to as the variable structure filter (VSF) was introduced in 2003 [17]. Essentially this method makes use of the variable structure theory and sliding mode concepts. It uses a switching gain to converge the estimates to within a boundary of the true state values (i.e., existence subspace shown in Fig. 1). In 2007, the smooth variable structure filter (SVSF) was derived which makes use of a simpler and less complex gain calculation [12]. In its present form, the SVSF has been shown to be stable and robust to modeling uncertainties and noise, when given an upper bound on the level of un-modeled dynamics and noise [17,18].

The SVSF method is model based and may be applied to differentiable linear or nonlinear dynamic equations. The original form of the SVSF as presented in [12] did not include covariance derivations. An augmented form of the SVSF was presented in [19], which includes a full derivation for the filter.

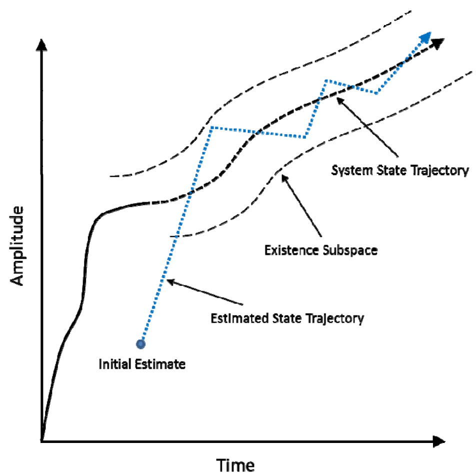

Figure 1. SVSF Estimation Concept

The estimation process is iterative and may be summarized by the following set of equations. The predicted state estimates $\hat{x}_{k+1|k}$ and state error covariances $P_{k+1|k}$ are first calculated respectively as follows:

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}, u_k) \tag{10}$$

$$P_{k+1|k} = F P_{k|k} F^T + Q_{k+1} \tag{11}$$

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the corresponding predicted measurements $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ may be calculated:

$$\hat{z}_{k+1|k} = h(\hat{x}_{k+1|k}) \tag{12}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{13}$$

The SVSF process differs from the KF in how the gain is formulated. The SVSF gain is a function of: the a priori and the a posteriori measurement errors $e_{z,k+1|k}$ and $e_{z,k|k}$; the smoothing boundary layer widths ψ; the 'SVSF' memory or convergence rate γ; as well as the linearized measurement matrix H. For the derivation of the gain K_{k+1}, refer to [12,19]. The SVSF gain is defined as a diagonal matrix such that:

$$K_{k+1} = H^{-1} diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma \left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1} \tag{14}$$

Where $\circ$ signifies Schur (or element-by-element) multiplication, and where $\bar{\psi}^{-1}$ is a diagonal matrix constructed from the smoothing boundary layer vector ψ, such that:

$$\bar{\psi}^{-1} = \begin{bmatrix} \frac{1}{\psi_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\psi_m} \end{bmatrix} \tag{15}$$

Note that m is the number of measurements, and the saturation function of (18) is defined by:

$$sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right) = \begin{cases} 1, & e_{z_i,k+1|k}/\psi_i \geq 1 \\ e_{z_i,k+1|k}/\psi_i, & -1 < e_{z_i,k+1|k}/\psi_i < 1 \\ -1, & e_{z_i,k+1|k}/\psi_i \leq -1 \end{cases} \tag{16}$$

This gain is used to calculate the updated state estimates $\hat{x}_{k+1|k+1}$ as well as the updated state error covariance matrix $P_{k+1|k+1}$:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} e_{z,k+1|k} \tag{17}$$

$$P_{k+1|k+1} = (I - K_{k+1}H) P_{k+1|k} (I - K_{k+1}H)^T + K_{k+1} R_{k+1} K_{k+1}^T \tag{18}$$

Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated, and are used in later iterations:

$$\hat{z}_{k+1|k+1} = h(\hat{x}_{k+1|k+1}) \tag{19}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{20}$$

The existence subspace shown in Figs. 1 and 2 represents the amount of uncertainties present in the estimation process, in terms of modeling errors or the presence of noise. The width of the existence space β is a function of the uncertain dynamics associated with the inaccuracy of the internal model of the filter as well as the measurement model, and varies with time [12]. Typically this value is not exactly known but an upper bound may be selected based on a priori knowledge. Once within the existence boundary subspace, the estimated states are forced (by the SVSF gain) to switch back and forth along the true state trajectory. High-frequency switching caused by the SVSF gain is referred to as chattering, and in most cases, is undesirable for obtaining accurate estimates [12]. However, the effects of chattering may be minimized by the introduction of a smoothing boundary layer ψ. The selection of the smoothing boundary layer width reflects the level of uncertainties in the filter and the disturbances.

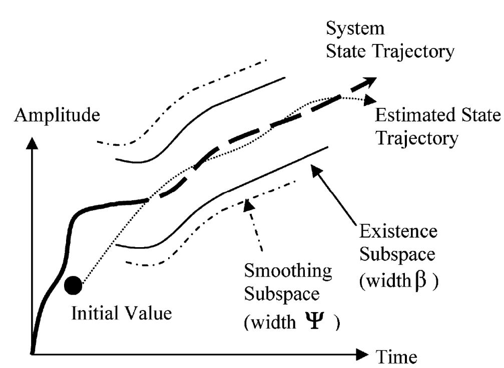

Figure 2a. Smoothed Estimated Trajectory ($\psi \geq \beta$)

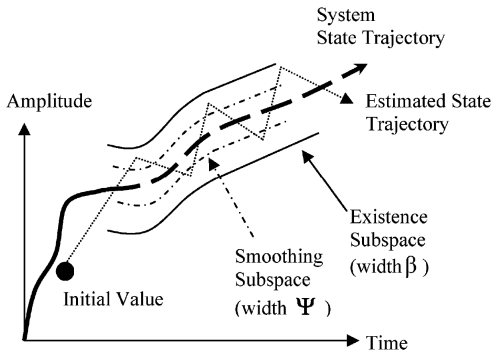

Figure 2b. Presence of Chattering Effect ($\psi < \beta$)

The effect of the smoothing boundary layer is shown in Fig. 2. When the smoothing boundary layer is defined larger than the existence subspace boundary, the estimated state trajectory is smoothed. However, when the smoothing term is too small, chattering remains due to the uncertainties being underestimated.

4 Combined Strategy

The SVSF provides an estimation process that is sub-optimal albeit robust and stable. It is hence beneficial to be able to combine the accurate performances of the KF with the stability of the SVSF; prior to combining it with the PF. A recent development, described in [20], provides a methodology for calculating a variable smoothing boundary layer ψ. The partial derivative of the a posteriori covariance (trace) with respect to the smoothing boundary layer term ψ is the basis for obtaining an optimal, time-varying strategy for the specification of ψ. In linear systems, this smoothing boundary layer yields an optimal gain, similar to the KF. Previous forms of the SVSF included a vector form of ψ, which had a single smoothing boundary layer term for each corresponding measurement error [12]. Essentially, the boundary layer terms were independent of each other such that the measurement errors would not mix when calculating the corresponding gain, leading to a loss of optimality. A 'near-optimal' formulation of the SVSF could be created using a vector form of ψ, however this would lead to a minimization of only the diagonal elements of the state error covariance matrix [21]. In an effort to obtain a smoothing boundary layer equation that yielded optimal state estimates, a full smoothing boundary layer matrix was proposed in [20]. Hence, consider the full matrix form of the smoothing boundary layer:

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1m} \\ \psi_{12} & \psi_{22} & \cdots & \psi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \psi_{m2} & \cdots & \psi_{mm} \end{bmatrix} \tag{21}$$

This definition includes terms that relate one smoothing boundary layer to another (i.e., off-diagonal terms). To solve for the optimal smoothing boundary layer based on (21), consider:

$$\frac{\partial\left(trace[P_{k+1|k+1}]\right)}{\partial\psi} = 0 \tag{22}$$

As described in [20], a solution for the smoothing boundary layer from (22) is defined as follows:

$$\psi_{k+1} = \left(\bar{A}^{-1} H P_{k+1|k} H^T S_{k+1}^{-1}\right)^{-1} \tag{23}$$

Where A is defined by:

$$A = \left(\left|e_{z_{k+1|k}}\right| + \gamma \left|e_{z_{k|k}}\right|\right) \tag{24}$$

Note that in (23), $\bar{A}$ refers to forming a diagonal matrix of elements consisting of A. The EKF and SVSF strategies will be combined using this smoothing boundary layer calculation. Consider the following sets of figures to help describe the overall implementation of the EK-SVSF strategy; prior to combining it with the PF.

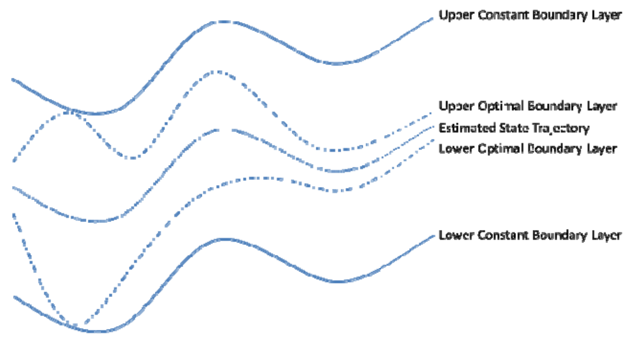

Figure 3. Boundary Layer Concept for the Well-Defined System Case [20]

Figure 3 illustrates the case when the constant smoothing boundary layer width used by the SVSF is defined larger than the optimal smoothing boundary layer (i.e., a conservative choice) calculated by (23). The difference between the constant and upper layers leads to a loss in optimality for the SVSF. Essentially, in this case, the EKF gain should be used to obtain the best result.

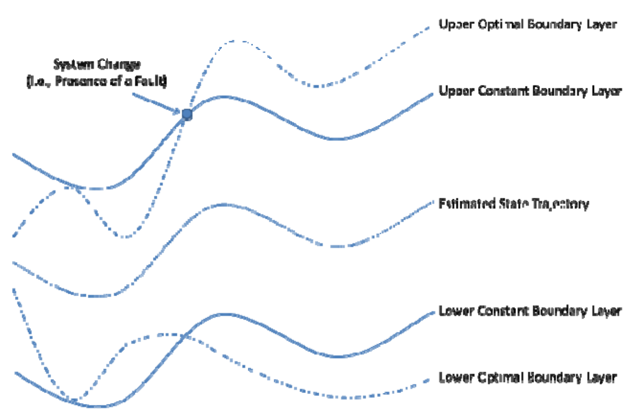

Figure 4. Presence of Fault or Poorly-Defined System Case [20]

Figure 4 illustrates the case when the optimal smoothing boundary layer is calculated to exist beyond the constant smoothing boundary layer. This typically occurs when there is modeling uncertainty (which leads to a loss in optimality) that exceed the limits of a constant smoothing boundary layer. The limits are set by the width of the existence subspace, which was discussed earlier. In a situation defined by Fig. 4, when $\psi_{opt} \geq \psi_{con}$, to ensure a stable estimate, the SVSF gain (14) should be used to update the state estimates. The smoothing boundary layer widths calculated by (23) are saturated at the constant values. This ensures a stable estimate, as defined by the proof of stability for the SVSF [12]. Furthermore, to improve the SVSF results (i.e., without the use of (23)), the averaged smoothing boundary layers (for the well-defined system) can be used to set the constant boundary layer widths. Doing so provides a well-tuned existence subspace that yields more accurate estimates.

Next, to combine the aforementioned SVSF strategy with the PF, a similar approach to formulating the EPF and UKF will be taken [11]. Essentially, the a posteriori state estimates (17) and state error covariance (18) are used to formulate the proposal distribution used by the PF to generate the particles, such that (5) becomes:

$$x_k^{(i)} = q(\hat{x}_{k+1|k+1}, P_{k+1|k+1}) \tag{25}$$

Following the distribution of the particles, the PF continues as normal [1].

5 Tracking Scenario

This section describes the tracking problem studied, and illustrates the estimation results. One of the most well studied aerospace applications involves ballistic objects on reentry [1]. In this paper, a ballistic target reentering the atmosphere is considered, as described in [1]. The following figure shows the experimental setup for ballistic target tracking.

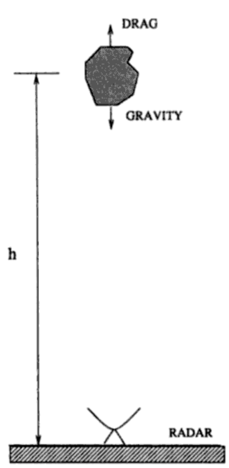

Figure 5. Ballistic Target Tracking Scenario [1]

Assuming that drag D and gravity g are the only forces acting on the object, the following differential equations govern its motion [1,22]:

$$\dot{h} = v \tag{26}$$

$$\dot{v} = -\frac{\rho(h) g v^2}{2\beta} + g \tag{27}$$

$$\dot{\beta} = 0 \tag{28}$$

The state vector is defined as $x = [h \quad v \quad \beta]^T$, which refers to the target altitude, velocity, and ballistic coefficient, respectively. The air density ρ is modeled as follows:

$$\rho = \gamma e^{-\eta h} \tag{29}$$

Where $\gamma = 1.754$ and $\eta = 1.49 \times 10^{-4}$. The discrete-time state equation is defined as follows [1]:

$$x_{k+1} = F x_k - G[D(x_k) - g] + w_k \tag{30}$$

With matrices F and G defined by:

$$F = \begin{bmatrix} 1 & -T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

$$G = [0 \quad T \quad 0]^T \tag{32}$$

Furthermore, the function for drag $D(x_k)$ (the only nonlinear term) is defined by:

$$D(x_k) = \frac{g\rho(x_{k,1}) x_{k,2}^2}{2x_{k,3}} \tag{33}$$

As in [1], the system noise w_k is assumed to be zero-mean Gaussian with a covariance matrix Q defined by:

$$Q \approx \begin{bmatrix} q_1 \frac{T^3}{3} & q_1 \frac{T^2}{2} & 0 \\ q_1 \frac{T^2}{2} & q_1 T & 0 \\ 0 & 0 & q_2 T \end{bmatrix} \tag{34}$$

Note that the parameters $q1$ and $q2$ respectively control the amount of system noise in the target dynamics and the ballistic coefficient [1]. As shown in Fig. 5, a radar is positioned on the ground below the target. The measurement equation in this scenario is defined by:

$$z_k = H x_k + v_k \tag{35}$$

Where it is assumed that two measurements are available, such that:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{36}$$

In this tracking scenario, the initial states are defined as follows: $x_{1,0} = 61{,}000\, m$, $x_{2,0} = 3{,}048\, m/s$, and $x_{3,0} = 19{,}161\, kg/ms^2$. Other notable parameters were defined as: $q_1 = 10^4$, $q_2 = 10$, $T = 0.1\, sec$, $R = diag([10^4 \quad 10^3])$, and $g = 9.81\, m/s^2$.

As per the earlier SVSF discussion, it is required to transform (36) into a square matrix (i.e., identity), such that an 'artificial' measurement is created. A number of methods exist, such as the reduced order or Luenberger's approach, which are presented in [12,23,24]. Consider a system model involving phase variables. It is possible to derive a third 'artificial' measurement $y_{3,k}$ based on the available measurements ($z_{1,k}$ and $z_{2,k}$).

In (36), the ballistic coefficient measurement is not available. If the system model (30) is known with complete confidence, then it is possible to derive an artificial measurement for the ballistic coefficient from the first two measurements. Hence, consider the following from (30):

$$y_{3,k} = \frac{Tg\gamma z_{2,k}^2}{2\left(z_{2,k+1} - z_{2,k} + Tg\right)e^{-\eta z_{1,k}}} \tag{37}$$

The accuracy of (37) depends on the sampling rate T. Applying (37) allows a measurement matrix equivalent to the identity matrix. The estimation process would continue as in the previous section, where a full measurement matrix was available. Note however that the artificial acceleration measurement would be delayed one time step. Furthermore, note that the artificial measurement would have to be initialized (i.e., 0 is a typical value). Equation (37) essentially propagates the known measurements through the system model to obtain the artificial ballistic coefficient measurement. It is conceptually similar to the method presented in [24] and creates a full measurement matrix.

The initial state estimates $\hat{x}_0$ were set 10% away from the true values x_0. The initial state error covariance matrix was set to $P_0 = 10Q$. The following figure shows the object altitude and estimates over time.

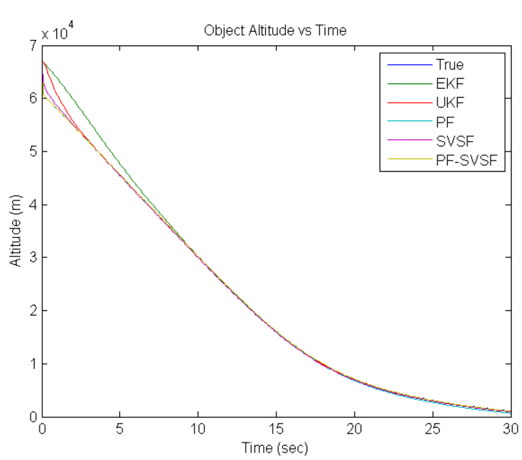

Figure 6. Object Altitude and Estimates

For this case, the filters performed relatively well, with the exception of the convergence rates. Looking at the first five seconds of Fig. 6, the PF-SVSF converged to the true state trajectory the fastest. Next was the PF, followed closely by the SVSF. The UKF was the second slowest, converging in about 3 seconds. The EKF was the last filter to converge, taking about 10 seconds. The root mean squared error (RMSE) was calculated for each filter, and is shown in the following table.

Table 1. RMSE of the Tracking Scenario

Filter	Altitude (m)	Velocity (m/s)	Ballistic (kg/ms^2)
EKF	1,902	286	1,916
UKF	957	64.5	2,329
PF	425	29.3	23,674
SVSF	468	92.8	1,917
PF-SVSF	352	19.4	1,698

The above table summarizes the RMSE for the tracking scenario defined earlier (note these results are repeatable). Overall, the proposed PF-SVSF algorithm provides the best result in terms of estimation accuracy. The PF performs very well, with the exception of the ballistic coefficient (it fails to provide a good estimate). The SVSF also performed well. The EKF provided the worst estimate, most likely due to the slower convergence rate. An interesting result occurs when one introduces modeling errors into the system model (30). As an example, in an effort to demonstrate the robustness of the PF-SVSF and SVSF to modeling uncertainties, consider the case when the gravity coefficient is doubled. The following figure shows the implications of modeling error being introduced at 15 seconds during the tracking scenario. The filters begin to diverge from the true state trajectory, with the PF being the furthest at 30 seconds.

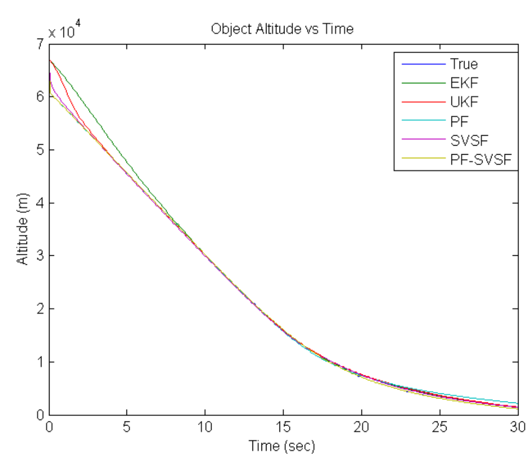

Figure 7. Object Altitude and Estimates with the Presence of Modeling Error at 15 seconds

The RMSE for this case was calculated for each filter, and is shown in the following table.

Table 2. RMSE of Tracking with Modeling Errors

Filter	Altitude (m)	Velocity (m/s)	Ballistic (kg/ms^2)
EKF	1,918	306	1,917
UKF	1,166	116	3,382
PF	619	88.9	39,814
SVSF	498	139	1,916
PF-SVSF	352	36.9	2,316

It is interesting to note that the PF-SVSF estimates remained relatively insensitive to the added modeling error. The combination of the SVSF and the PF improved the overall accuracy and stability of the PF estimation strategy.

6 Conclusions

In this paper, a new state and parameter estimation based on the combination of the PF and the SVSF was introduced. The combined method (PF-SVSF) utilizes the estimates and state error covariance of the SVSF to formulate the proposal distribution which generates the particles used by the PF. The PF-SVSF method was applied on a nonlinear target tracking problem. The results of this tracking scenario demonstrate the improved performance of the combined methodology. Future research work will involve studying other nonlinear estimation problems, and analyzing how the PF-SVSF performs.

Appendix

The following is a table of important nomenclature used throughout this paper.

Table 3. List of Nomenclature

Parameter	Definition
x	State vector or values
z	Measurement (system output) vector or values
y	Artificial measurement vector or values
u	Input to the system
w	System noise vector
v	Measurement noise vector
F	Linear system transition matrix
G	Input gain matrix
H	Linear measurement (output) matrix
K	Filter gain matrix (i.e., KF or SVSF)
P	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix
S	Innovation covariance matrix
e	Measurement (output) error vector
$diag(a)$ or $\bar{a}$	Defines a diagonal matrix of some vector *a*
$sat(a)$	Defines a saturation of the term *a*
γ	SVSF 'convergence' or memory parameter
ψ	SVSF boundary layer width
$\lvert a \rvert$	Absolute value of some parameter *a*
$E\{\cdot\}$	Expectation of some vector or value
T	Sample time, or transpose of some vector or matrix
^	Estimated vector or values
$x_k^{(i)}$	Particles used by the PF
$\omega_k^{(i)}$	Importance weights used by the PF
N_{eff}	Effective threshold for the PF
$k+1\|k$	A priori time step (i.e., before applied gain)
$k+1\|k+1$	A posteriori time step (i.e., after update)

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