THE CONTINUOUS-TIME SMOOTH VARIABLE STRUCTURE FILTER

S. A. Gadsden, M. A. El Sayed, and S. R. Habibi
Department of Mechanical Engineering
McMaster University
Hamilton, ON, Canada
gadsdesa@mcmaster.ca, abugabma@mcmaster.ca, and habibi@mcmaster.ca

ABSTRACT

State and parameter estimation techniques are important tools which provide accurate estimates of system states. This is important for the reliable and safe control of mechanical and electrical systems. Most estimation techniques are derived in discrete-time, due to the wide use of digital computers. However, continuous-time derivations do exist, and are particularly useful for studying estimation problems with small sampling intervals. The smooth variable structure filter (SVSF) is a relatively new estimation strategy based on sliding mode theory, and has been shown to be robust to modeling uncertainties. In this paper, a formulation of the SVSF is presented in continuous-time. The continuous-time SVSF is applied on an estimation problem, and the results are compared with the popular Kalman filter (KF).

THE KALMAN FILTER

A. Discrete-Time Formulation

The most popular and well-studied estimation method is the Kalman filter (KF), which was introduced in the 1960s [1,2]. The KF yields a statistically optimal solution for linear estimation problems, as defined by (1) and (2), in the presence of Gaussian noise where $P(w_k) \sim \mathcal{N}(0, Q_k)$ and $P(v_k) \sim \mathcal{N}(0, R_k)$. A typical model is represented by the following discrete-time equations:

$$x_{k+1} = F x_k + G u_k + w_k \tag{1}$$
$$z_{k+1} = H x_{k+1} + v_{k+1} \tag{2}$$

It is the goal of a filter to remove the effects that the system w_k and measurement v_{k+1} noise have on extracting the true state values x_{k+1} from the measurements z_{k+1}. The KF is formulated in a predictor-corrector manner. The states are first estimated using the system model, termed as a priori estimates, meaning 'prior to' knowledge of the observations. A correction term is then added based on the innovation (also called measurement errors), thus forming the updated or a posteriori (meaning 'subsequent to' the observations) state estimates.

The KF has been broadly applied to problems covering state and parameter estimation, signal processing, target tracking, fault detection and diagnosis, and even financial analysis [3,4]. The success of the KF comes from the optimality of the Kalman gain in minimizing the trace of the a posteriori state error covariance matrix. The trace is taken because it represents the state error vector in the estimation process [5]. The following five equations form the core of the KF algorithm, and are used in an iterative fashion. Equations (3) and (4) define the a priori state estimate $\hat{x}_{k+1|k}$ based on knowledge of the system F and previous state estimate $\hat{x}_{k|k}$, and the corresponding state error covariance matrix $P_{k+1|k}$, respectively.

$$\hat{x}_{k+1|k} = F \hat{x}_{k|k} + G u_k \tag{3}$$
$$P_{k+1|k} = H P_{k|k} H^T + Q_k \tag{4}$$

The Kalman gain K_{k+1} is defined by (5), and is used to update the state estimate $\hat{x}_{k+1|k+1}$ as shown in (6). The gain makes use of an innovation covariance S_{k+1}, which is defined as the inverse term found in (5).

$$K_{k+1} = P_{k+1|k} H^T \left[H P_{k+1|k} H^T + R_{k+1} \right]^{-1} \tag{5}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \left[z_{k+1} - H \hat{x}_{k+1|k} \right] \tag{6}$$

The a posteriori state error covariance matrix $P_{k+1|k+1}$ is then calculated by (7), and is used iteratively, as per (4).

$$P_{k+1|k+1} = [I - K_{k+1} H] P_{k+1|k} \tag{7}$$

A number of different methods have extended the classical KF to nonlinear systems, with the most popular and simplest method being the extended Kalman filter (EKF) [6,7]. The EKF is conceptually similar to the KF; however, the nonlinear system is linearized according to its Jacobian. This linearization process introduces uncertainties that can sometimes cause instability [7]. A list of the nomenclature used throughout this paper is provided in the Appendix.

B. Continuous-Time Formulation

A continuous-time Kalman filter (KF) was developed in the late 1950s in unpublished work by James Follin, A. G. Carlton, James Hanson, and Richard Bucy [6]. Later in 1961, Kalman and Bucy published their work [8]. The derivation of the continuous-time KF will not be provided here due to space constraints, but is readily available in literature [6,9]. The continuous-time system and measurement models may be defined respectively as follows:

$$\dot{x}(t) = F(t)x(t) + B(t)u(t) + w(t) \tag{8}$$
$$z(t) = H(t)x(t) + v(t) \tag{9}$$

For simplicity, the time component (t) will be left out of future equations. Like the standard discrete-time KF, a gain K is used to correct the state estimates:

$$K = PH^T R^{-1} \tag{10}$$

The state estimate $\hat{x}$ is updated as follows:

$$\dot{\hat{x}} = F\hat{x} + Bu + K[z - H\hat{x}] \tag{11}$$

Correspondingly, the state error covariance P is calculated forward in time by:

$$\dot{P} = FP + PF^T - PH^T R^{-1} HP + Q \tag{12}$$

The above three equations are used to derive estimates for the state values (8) given the available measurements (9). The principle is to minimize the effects of noise present in the system and measurements.

THE SMOOTH VARIABLE STRUCTURE FILTER

A. Discrete-Time Formulation

In 2007, the smooth variable structure filter (SVSF) was introduced based on variable structure theory and sliding mode concepts [10]. It implements a switching gain to converge the estimates to within a boundary of the true states (i.e., existence subspace). In its present form, the SVSF has been shown to be stable and robust to modeling uncertainties and noise [11,12]. The basic estimation concept of the SVSF is shown in Fig. 1. The SVSF method is model based and may be applied to differentiable linear or nonlinear dynamic equations. The original form of the SVSF as presented in [10] and did not include covariance derivations. An augmented form of the SVSF was presented in [13], which includes a full derivation for the filter. The purpose to provide a covariance derivation was to increase the number of applications for the SVSF (i.e., creating interacting multiple model forms). However, in this paper, the standard SVSF (with no covariance) will be expanded into continuous-time. The estimation process is iterative and may be summarized by the following sets of equations.

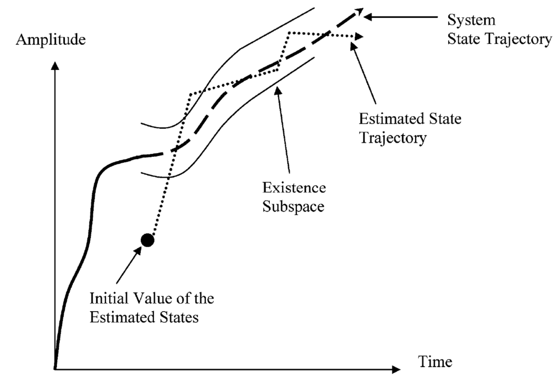

Fig. 1. The SVSF estimation concept [10].

The predicted state estimates $\hat{x}_{k+1|k}$ and state error covariances $P_{k+1|k}$ are first calculated respectively as follows:

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Gu_k \tag{13}$$
$$P_{k+1|k} = HP_{k|k}H^T + Q_k \tag{14}$$

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the corresponding predicted measurements $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ may be calculated:

$$\hat{z}_{k+1|k} = H\hat{x}_{k+1|k} \tag{15}$$
$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \tag{16}$$

The SVSF gain is a function of: the a priori and a posteriori measurement errors $e_{z,k+1|k}$ and $e_{z,k|k}$; the smoothing boundary layer widths ψ; the 'SVSF' memory or convergence rate γ; as well as the linear measurement matrix H. For the derivation of the SVSF gain K_{k+1}, refer to [10,13]. The SVSF gain is defined as a diagonal matrix such that:

$$K_{k+1} = H^{+} diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1} \tag{17}$$

This gain is used to calculate the updated state estimates $\hat{x}_{k+1|k+1}$ as well as the updated state error covariance matrix $P_{k+1|k+1}$:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z,k+1|k} \tag{18}$$
$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{19}$$

Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated, and are used in later iterations:

$$\hat{z}_{k+1|k+1} = H\hat{x}_{k+1|k+1} \tag{20}$$
$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \tag{21}$$

The estimation process is stable and convergent if the following lemma is satisfied:

$$|e_k| > |e_{k+1}| \tag{22}$$

The proof, as defined in [10], is such that if one defines η_{k+1} as a random but bounded amplitude such that $\eta_{k+1} \leq \beta$, then $|e_k| > |e_{k+1}|$ applies to the phase where $|e_k| > \beta$ that is defined by the reachability phase. Let a Lyapunov function be defined such that $v_{k+1} = e_{k+1}^2$. Hence, the estimation process is stable if $(\Delta v_{k+1} = e_{k+1}^2 - e_k^2) < 0$. Furthermore, note that this stability condition is satisfied by (22) [10,14].

B. Continuous-Time Formulation

The discrete-time SVSF gain was derived based on the assumption for stability, defined by (22). Likewise, one can derive a continuous-time form of the SVSF by starting with the following lemma for stability:

$$s\dot{s} < 0 \tag{23}$$

The sliding surface s and its derivative are defined respectively by:

$$s = \lim_{T \to 0} e_{z_{k+1|k+1}} \tag{24}$$

$$\dot{s} = \lim_{T \to 0} \frac{e_{z_{k+1|k+1}} - e_{z_{k|k}}}{T} \tag{25}$$

Hence, based on (24) and (24), the discrete-time a posteriori measurement error will be used in a continuous-time form to derive the new gain. From (20), the discrete-time a posteriori measurement error may be defined by:

$$e_{z_{k+1|k+1}} = z_{k+1} - H\left(\hat{x}_{k+1|k} + K_{k+1}\right) \tag{26}$$

Expanding (26) and simplifying yields:

$$e_{z_{k+1|k+1}} = e_{z_{k+1|k}} - HK_{k+1} \tag{27}$$

Likewise, the previous time-step's a posteriori measurement error (to be used later in (25)) may be defined as follows:

$$e_{z_{k|k}} = e_{z_{k|k-1}} - HK_k \tag{28}$$

From (24), and (27), one has:

$$s = \lim_{T \to 0} e_{z_{k+1|k+1}} = z_{k+1} - H\hat{x}_{k+1|k} - HK_{k+1} \tag{29}$$

Substituting (13) into (29) yields:

$$s = z_{k+1} - H\left(F\hat{x}_{k|k} + Gu_k\right) - HK_{k+1} \tag{30}$$

For small sample times, one can transfer the following matrices into continuous-time, as follows:

$$F \approx I + AT \tag{31}$$

$$G \approx BT \tag{32}$$

$$H = C \tag{33}$$

Hence, based on (31) – (33), the surface (30) becomes:

$$s = z_{k+1} - C\left([I + AT]\hat{x}_{k|k} + BTu_k\right) - CK_{k+1} \tag{34}$$

Solving, based on $T \to 0$, yields:

$$s = z - C(\hat{x} + K) \tag{35}$$

Next, the stability term $\dot{s}$ will be considered, based on a similar approach to solving (35):

$$\dot{s} = \frac{1}{T}\left(e_{z_{k+1|k}} - e_{z_{k|k-1}}\right) - \frac{H}{T}(K_{k+1} - K_k) \tag{36}$$

Expanding (36) yields:

$$\begin{aligned} \dot{s} = \frac{1}{T}[z_{k+1} - z_k] - \frac{1}{T}\left[C(I + AT)\left(\hat{x}_{k|k} - \hat{x}_{k-1|k-1}\right)\right] \\ - \frac{1}{T}[CBT(u_k - u_{k-1})] - \frac{C}{T}(K_{k+1} - K_k) \end{aligned} \tag{37}$$

Finally, simplifying (37) yields:

$$\dot{s} = \dot{z} - C\left(\dot{\hat{x}} + \dot{K}\right) \tag{38}$$

Now, consider (23) based on (35) and (38). If $s > 0$, then $\dot{s} < 0$ such that:

$$\dot{K} = C^{-1}\left(\dot{z} - C\dot{\hat{x}}\right) + \varepsilon \tag{39}$$

Where ε is some arbitrary positive constant. Similarly, if $s < 0$, then $\dot{s} > 0$ such that:

$$\dot{K} = C^{-1}\left(\dot{z} - C\dot{\hat{x}}\right) - \varepsilon \tag{40}$$

In general, the gain K may be formulated as follows:

$$\dot{K} = C^{-1}\left(\dot{z} - C\dot{\hat{x}}\right) + \varepsilon\, sign\left(z - C(\hat{x} + K)\right) \tag{41}$$

Next, the state estimate equation (update) needs to be calculated. Consider the following SVSF state update:

$$\hat{x}_{k+1|k+1} = F\hat{x}_{k|k} + Gu_k + K_{k+1} \tag{42}$$

Utilizing (31) – (33), (42) becomes:

$$\hat{x}_{k+1|k+1} - \hat{x}_{k|k} = AT\hat{x}_{k|k} + BTu_k + K_{k+1} \tag{43}$$

Finally, solving for the state update yields the following differential equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + K \tag{44}$$

Equations (41) and (44) form the backbone of the continuous-time SVSF estimation method.

SIMULATION PROBLEM

The continuous-time KF and SVSF strategies were applied on a simple second order spring-mass-damper system for demonstration purposes. The simulation was performed in Matlab's Simulink environment. In state-space, the system and measurement can be modeled as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + w \tag{45}$$

$$z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + v \tag{46}$$

The system and measurement noise that was added equals roughly 15% of the real system values. This is a considerable amount of noise. A constant force of 1 N was applied to the mass, resulting in a second-order system response (overshoot, etc.). The following figure shows the simulation results (estimation error over number of samples). The KF estimate was more sensitive to noise when compared with the estimate provided by the SVSF, as demonstrated by the 'spiked' estimates. The SVSF yielded a smoother result, which is beneficial for the purposes of control.

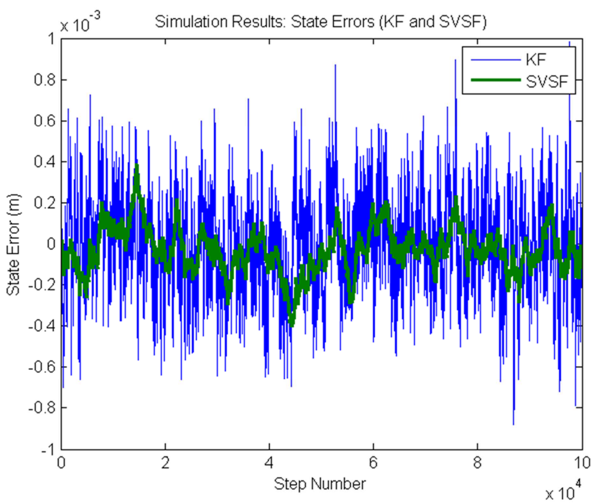

Fig. 2. Estimation results showing the state error over time.

CONCLUSIONS

In this paper, a continuous-time form of the smooth variable structure filter (SVSF) was derived. The KF and SVSF were applied on a simple simulation problem. In general, it was found that the continuous-time SVSF yielded comparable results to the KF. In the case studied, the SVSF was able to out-perform the KF in terms of estimation accuracy. Future work will involve studying the effects of the positive arbitrary constant that was derived in the continuous-time SVSF gain.

APPENDIX

The following is a table of important nomenclature used throughout this paper.

Parameter	Definition
x	State vector or values
z	Measurement (system output) vector or values
u	Input to the system
w	System noise vector
v	Measurement noise vector
A, F	Linear system transition matrix
B, G	Input gain matrix
C, H	Linear measurement (output) matrix
K	Filter gain matrix (i.e., KF or SVSF)
P	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix
e	Measurement (output) error vector
$diag(a)$ or $\bar{a}$	Defines a diagonal matrix of some vector *a*
$sat(a)$	Defines a saturation of the term *a*
γ	SVSF 'convergence' or memory parameter
ψ	SVSF boundary layer width
$\|a\|$	Absolute value of some parameter *a*
T	Transpose of some matrix or sample time
^	Estimated vector or values
$k+1\|k$	A priori time step (i.e., before applied gain)
$k+1\|k+1$	A posteriori time step (i.e., after update)

REFERENCES

[1] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," *Journal of Basic Engineering, Transactions of ASME*, vol. 82, pp. 35-45, 1960.

[2] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979.

[3] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*. Boston: Artech House, 2004.

[4] Simon Haykin, *Kalman Filtering and Neural Networks*. New York, U.S.A.: John Wiley and Sons, Inc., 2001.

[5] A. Gelb, *Applied Optimal Estimation*. Cambridge, MA: MIT Press, 1974.

[6] D. Simon, *Optimal State Estimation: Kalman, H-Infinity, and Nonlinear Approaches*.: Wiley-Interscience, 2006.

[7] G. Welch and G. Bishop, "An Introduction to the Kalman Filter," Department of Computer Science, University of North Carolina, 2006.

[8] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," *Journal of Basic Engineering, Transactions of ASME*, vol. 83, no. Series D, pp. 95-108, 1961.

[9] J. L. Crassidis and J. L. Junkins, *Optimal Estimation of Dynamic Systems*. New York, U.S.A.: Chapman and Hall (CRC), 2004.

[10] S. R. Habibi, "The Smooth Variable Structure Filter," *Proceedings of the IEEE*, vol. 95, no. 5, pp. 1026-1059, 2007.

[11] S. R. Habibi and R. Burton, "The Variable Structure Filter," *Journal of Dynamic Systems, Measurement, and Control (ASME)*, vol. 125, pp. 287-293, September 2003.

[12] S. R. Habibi and R. Burton, "Parameter Identification for a High Performance Hydrostatic Actuation System using the Variable Structure Filter Concept," *ASME Journal of Dynamic Systems, Measurement, and Control*, 2007.

[13] S. A. Gadsden and S. R. Habibi, "A New Form of the Smooth Variable Structure Filter with a Covariance Derivation," in *IEEE Conference on Decision and Control*, Atlanta, Georgia, 2010.

[14] P. A. Cook, *Nonlinear Dynamical Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1986.