ASME 2010 International Mechanical Engineering Congress and Exposition
IMECE 2010
November 12-18, 2010, Vancouver, British Columbia, Canada

IMECE2010-39319 (DRAFT)

DERIVATION OF THE SMOOTH VARIABLE STRUCTURE INFORMATION FILTER

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ABSTRACT

An information filter is one that propagates the inverse of the state error covariance, which is used in the state and parameter estimation process. The term 'information' is based on the Cramer-Rao lower bound (CRLB), which states that the mean square error of an estimator cannot be smaller than an amount based on its corresponding likelihood function. The most common information filter (IF) is derived based on the inverse of the Kalman filter (KF) covariance. This paper introduces preliminary work completed on developing the information form of the smooth variable structure filter (SVSF). The SVSF is a relatively new type of predictor-corrector estimator based on sliding mode concepts. A covariance derivation was recently proposed for the SVSF, such that now the information form can be finalized (i.e., SVSIF). The paper summarizes the recursive equations used in the smooth variable structure information filter.

INTRODUCTION

In the estimation field, filters may be classified as either covariance or information filters. A covariance filter is the most common type of filter used for estimating the states or parameters of a system. The most popular covariance filter is the Kalman filter **[1,2]**. It provides an elegant and statistically optimal solution for linear dynamic systems in the presence of Gaussian white noise **[3,1]**. However, the optimality of the KF comes at a price of stability and robustness. The KF assumes that the system model is known and is linear, the system and measurement noises are white, and the states have initial conditions that are modeled as random variables with known means and variances **[4,5]**. However, the previous assumptions do not always hold in real applications, particularly an exact knowledge of the system equations. If one of these assumptions is violated, the KF performance may yield suboptimal estimations and can even become unstable. Furthermore, the KF is sensitive to computer precision and the complexity of certain calculations (i.e., matrix inversions) **[1]**.

An information filter is one that propagates the inverse of the state error covariance, instead of using the normal covariance in the gain calculation like with the KF **[6]**. The term 'information' is based on the Cramer-Rao lower bound (CRLB), where the Fisher information matrix (FIM) is calculated as the inverse of the covariance matrix **[4]**. The CRLB states that the mean square error of an estimator cannot be smaller than an amount based on its corresponding likelihood function **[4]**. A filter is considered efficient if its variance is equal to the CRLB. There are certain advantages to using the information formulation of a filter. For example, if no prior information is available, one may initialize the information matrix using zeros such that no bias exists in the a priori estimate **[1]**. Furthermore, the observational update of the information matrix is more robust than the covariance filter form, which makes it a more attractive method when round-off errors may be an issue **[1]**. It is well known that the largest complaint to information filtering is the lack of physical understanding that 'information states' may bring. However, one may invert the information matrix such that the states may be interpreted physically **[1]**.

REVIEW OF ESTIMATION METHODS

a) Kalman Filter

As previously mentioned and presented in **[7]**, the KF provides an elegant and statistically optimal solution for linear dynamic systems in the presence of Gaussian white noise. It is an estimation method that utilizes measurements linearly related to the states or parameters of the systems, and error covariance matrices, to generate a gain referred to as the Kalman gain. This gain is applied to the a priori state estimate, thus creating an a posteriori (i.e., updated) estimate of the states. The estimation process is iterative and continues in a predictor-corrector fashion while maintaining a statistically minimal state error covariance matrix (for linear systems).

A typical linear dynamic system and measurement model are defined by using the following two equations, respectively:

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$
$$z_{k+1} = Hx_{k+1} + v_{k+1} \tag{2}$$

Please refer to the end of the paper for a list of pertinent nomenclature and variable definitions. The following five equations form the core of the KF algorithm, and are used in an iterative fashion. Equation (3) defines the a priori estimate based on the system definition, and (4) is the corresponding state error covariance matrix. The Kalman gain is defined by (5), and is used to update the state estimate shown in (6). The a posteriori state error covariance matrix is calculated by (7).

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{3}$$
$$P_{k+1|k} = HP_{k|k}H^T + Q_k \tag{4}$$
$$K_{k+1} = P_{k+1|k}H^T\left[HP_{k+1|k}H^T + R_{k+1}\right]^{-1} \tag{5}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left[z_{k+1} - H\hat{x}_{k+1|k}\right] \tag{6}$$
$$P_{k+1|k+1} = [I - K_{k+1}H]P_{k+1|k} \tag{7}$$

The effects due to mismodeling can be negative, as both the Kalman gain and covariance matrix calculations are dependent on the system and measurement matrices. Furthermore, the performance and stability of the KF may also be dependent on the definition of the process and measurement noise, made through covariance matrices **[1,8]**. Overlooked nonlinearities in the system may also cause the KF to become unstable. The EKF may be used for nonlinear systems (as described by the first two equations). It is conceptually similar to the iterative KF process, described above. The nonlinear system and measurement matrices are linearized according to its corresponding Jacobian, which is a first-order partial derivative. This linearization can sometimes cause instabilities when implementing the EKF **[1]**. The unscented Kalman filter (UKF) is able to handle a higher order of the nonlinearities by using a set of deterministically chosen sample points (i.e., sigma points) which are applied on a transformation such that it captures the true mean and covariance up to the second order of nonlinearity **[9]**. A great deal of research has been published on these filters, including on their robustness and numerical stability **[10,11,12,13,14,15]**.

b) Information Filter

The main information filter (IF) used in literature is based on the KF, where one utilizes the inverse of the covariance matrices. The information states are defined as functions of the covariance inverses and the true state vectors, as follows **[3]**:

$$\hat{a}_{k+1|k} = P_{k+1|k}^{-1}\hat{x}_{k+1|k} \tag{8}$$
$$\hat{a}_{k+1|k+1} = P_{k+1|k+1}^{-1}\hat{x}_{k+1|k+1} \tag{9}$$

Applying the matrix inversion lemma to (4) and (7) yields the corresponding information matrices, as presented in **[3]**:

$$P_{k+1|k}^{-1} = \left[I - A_p\right](A^{-1})^T P_{k|k}^{-1}A^{-1} \tag{10}$$
$$P_{k+1|k+1}^{-1} = P_{k+1|k}^{-1} + HR_{k+1}^{-1}H^T \tag{11}$$

Where:

$$A_p = (A^{-1})^T P_{k|k}^{-1}A^{-1}\left[(A^{-1})^T P_{k|k}^{-1}A^{-1} + Q_{k+1}^{-1}\right]^{-1} \tag{12}$$

It has been shown already in literature that the gain associated with the IF (simplified) is as follows **[3,6]**:

$$K_{IF_{k+1}} = AP_{k+1|k+1}^{-1}HR_{k+1}^{-1} \tag{13}$$

Using the above gain and information matrices, one may be able to determine (refer to **[3,6]**) the predicted and updated information vectors used by the information filter, respectively:

$$\hat{a}_{k+1|k} = \left[I - A_p\right]A^{-T}\hat{a}_{k|k} \tag{14}$$
$$\hat{a}_{k+1|k+1} = \hat{a}_{k+1|k} + HR_{k+1}^{-1}z_k \tag{15}$$

Equations (10) through (15) constitute the main formulas used in the information filter. Furthermore, note that the actual inverses do not have to be calculated, as the states are solved in a recursive manner.

c) Smooth Variable Structure Filter

As presented in **[7]**, sliding mode control and estimation techniques have been around for quite a few decades, and are mainly popular due to their relative ease of implementation and robustness to modeling uncertainties **[16,17]**. In a typical sliding mode control scenario, one utilizes a discontinuous switching plane along some desired trajectory **[18]**. This plane is quite often referred to as a sliding surface, in which the purpose is to keep the state values along this surface by minimizing the state errors (between the desired trajectory and the estimated or actual values). Ideally, if the state value is off or away from the surface, a switching gain would be used to push the state towards the sliding surface. Once upon the surface, the motion of the system as the states slide along the surface is called a sliding mode **[18]**. The discontinuous switching brings an inherent amount of stability to the control or estimation strategy, while also introducing excessive chattering (i.e., high-frequency switching) which may be undesirable in control since it may excite un-modeled dynamics. A boundary layer may be introduced along the sliding surface in order to saturate and smooth out the chattering within the boundary region. These sliding mode concepts are based on variable structure control, in which one alters the nonlinear dynamics of a system by the introduction of high-frequency switching **[16]**.

The variable structure filter (VSF) was first proposed in 2003, and was introduced as a new type of predictor-corrector estimator based on sliding mode concepts **[19]**. It is a type of sliding mode estimator, where gain switching is used to ensure that the estimates converge to within a boundary of the true state values (i.e., existence subspace). An internal model of the system, either linear or nonlinear, is used to predict an a priori state estimate. A corrective term (i.e., gain) is then applied to calculate the a posteriori state estimate, and the estimation process is repeated iteratively. The smooth variable structure filter (SVSF) was later derived from the VSF, and uses a much simpler and less complex gain calculation **[20]**. In its present form, the SVSF is stable and robust to modeling uncertainties and noise, given an upper bound on the level of un-modeled dynamics or knowledge of the magnitude of noise.

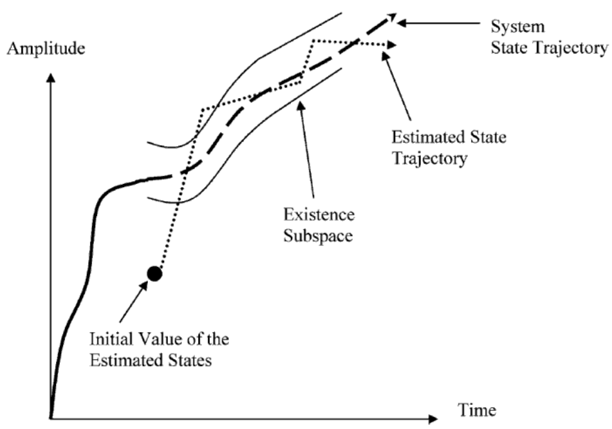

Fig. 1. The smooth variable structure filter estimation concept is shown in the above figure **[20]**.

The basic estimation concept of the SVSF is shown in the above figure. Some initial values of the estimated states are made based on probability distributions or designer knowledge. An area around the true system state trajectory is defined as the existence subspace. Through the use of the SVSF gain, the estimated state will be forced to within this region. Once the value enters the existence subspace, the estimated state is forced into switching along the system state trajectory. A saturation term may be used in this region to reduce the magnitude of chattering or smooth-out the result. As previously mentioned, the SVSF gain introduces a certain amount of chattering which brings an inherent amount of stability. This makes the estimation strategy an attractive method for control problems when not all of the dynamics are well known or defined correctly.

The SVSF method is model based and applies to smooth nonlinear dynamic equations. The estimation process is iterative and may be summarized by the following set of equations (for a linear control or estimation problem). Like the KF, the system model is used to calculate a priori state and measurement estimates. A corrective term, referred to as the SVSF gain, is calculated as a function of the error in the predicted output and a smoothing boundary layer. This gain is then used to update the state estimate. The estimation process is stable due to the gain calculation of (19). Furthermore, the switching found within the existence subspace is smoothed out by using the saturation term of (19), which is defined by the a priori output error and some predetermined boundary layer width.

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{16}$$
$$\hat{z}_{k+1|k} = \hat{H}\hat{x}_{k+1|k} \tag{17}$$
$$e_{z_{k+1|k}} = z_{k+1} - \hat{z}_{k+1|k} \tag{18}$$
$$K_{k+1} = \left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right) \tag{19}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \tag{20}$$
$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \tag{21}$$

Two critical variables in the SVSF estimation process are the a priori and a posteriori output error estimates, defined by (18) and (21), respectively. The estimation process is stable and convergent if the following lemma is satisfied:

$$|e_k| > |e_{k+1}| \tag{22}$$

The proof, as defined in **[20]**, is such that if one defines η_{k+1} as a random but bounded amplitude such that $\eta_{k+1} \leq \beta$, then $|e_k| > |e_{k+1}|$ applies to the phase where $|e_k| > \beta$ that is defined by the reachability phase. Let a Lyapunov function be defined such that $v_{k+1} = e_{k+1}^2$. Hence, the estimation process is stable if $(\Delta v_{k+1} = e_{k+1}^2 - e_k^2) < 0$. Furthermore, note that this stability condition is satisfied by (22) **[20,21]**.

As presented in **[20]**, the SVSF is not a classical filter in the sense that it does not have or make use of a covariance matrix. More recently, as shown in **[7]**, a revised form of the SVSF estimation process was proposed. This derivation included covariance calculations, which enables one to create an information form of the SVSF. The modified SVSF process (which is solved recursively) was defined as follows:

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{23}$$
$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{24}$$
$$\hat{z}_{k+1|k} = \hat{H}\hat{x}_{k+1|k} \tag{25}$$
$$e_{z_{k+1|k}} = z_{k+1} - \hat{z}_{k+1|k} \tag{26}$$
$$K_{k+1} = diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right)\right]\left[diag\left(e_{z_{k+1|k}}\right)\right]^{-1} \tag{27}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z_{k+1|k}} \tag{28}$$
$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{29}$$
$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \tag{30}$$

The adaptations of (27) and (28) do not affect the general nature of the SVSF (i.e., the gain may be divided by the a priori measurement error, but it gets multiplied out at the next step when updating the estimate). Introducing this notation enables one to derive a much simpler covariance derivation. Furthermore, the proof of stability for the SVSF is not affected **[20]**. It is interesting to note that the a priori and a posteriori state error covariance matrices for the SVSF are similar to the KF for linear systems.

DERIVATION OF THE SVSIF

In this section, a new information filter based on the SVSF is derived, and is referred to as the smooth variable structure information filter (SVSIF). To begin, one again defines the information vectors as follows:

$$\hat{a}_{k+1|k} = P_{k+1|k}^{-1}\hat{x}_{k+1|k} \tag{31}$$

$$\hat{a}_{k+1|k+1} = P_{k+1|k+1}^{-1}\hat{x}_{k+1|k+1} \tag{32}$$

Next, one needs to determine the inverse of the covariances defined by (24) and (29). Consider the following definition:

$$A_q = A^{-T}P_{k|k}^{-1}A^{-1} \tag{33}$$

Such that, from (24):

$$P_{k+1|k}^{-1} = \left[A_q^{-1} + IQ_{k+1}I^T\right]^{-1} \tag{34}$$

Writing (34) in such a manner allows one to use the matrix inversion lemma, which is defined by **[3]**:

$$[a_1^{-1} + a_2a_3a_2^T]^{-1} = [I - a_1a_2(a_2^Ta_1a_2 + a_3^{-1})^{-1}a_2^T]a_1 \tag{35}$$

This yields a more complete form of (34), or the a priori information matrix, as follows:

$$P_{k+1|k}^{-1} = \left[I - A_q\left(A_q + Q_{k+1}^{-1}\right)^{-1}\right]A_q \tag{36}$$

Alternatively, in other words:

$$P_{k+1|k}^{-1} = [I - A_r]A_q \tag{37}$$

$$A_r = A_q\left(A_q + Q_{k+1}^{-1}\right)^{-1} \tag{38}$$

Next, the a posteriori information matrix needs to be solved:

$$P_{k+1|k+1}^{-1} = \left[(I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T\right]^{-1} \tag{39}$$

Similar to before, one can define the following:

$$A_s = \left[(I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T\right]^{-1} \tag{40}$$

Such that:

$$A_s = (I - K_{k+1}H)^{-T}P_{k+1|k}^{-1}(I - K_{k+1}H)^{-1} \tag{41}$$

The a posteriori information matrix can then be rewritten as follows:

$$P_{k+1|k+1}^{-1} = [A_s^{-1} + K_{k+1}R_{k+1}K_{k+1}^T]^{-1} \tag{42}$$

Doing so allows one to use the matrix inversion lemma, to solve for the complete form of (39):

$$P_{k+1|k+1}^{-1} = [I - A_sK_{k+1}(K_{k+1}^TA_sK_{k+1} + R_{k+1}^{-1})^{-1}K_{k+1}^T]A_s \tag{43}$$

Alternatively, in other words:

$$P_{k+1|k+1}^{-1} = [I - A_tK_{k+1}^T]A_s \tag{44}$$

$$A_t = A_sK_{k+1}(K_{k+1}^TA_sK_{k+1} + R_{k+1}^{-1})^{-1} \tag{45}$$

Now that both the information matrices have been determined, the next step in deriving the SVSIF is to formulate the prediction and update equations for the information vectors. Substitution of (37) into (31) and noting that $\hat{x}_{\mathbf{k+1}|k} = A\hat{x}_{k|k} + Bu_k$ (assuming a system input) yields the following:

$$\hat{a}_{k+1|k} = [I - A_r]A_q\left(A\hat{x}_{k|k} + Bu_k\right) \tag{46}$$

Substitution of (33) into (46) yields:

$$\hat{a}_{k+1|k} = [I - A_r]A^{-T}P_{k|k}^{-1}A^{-1}A\hat{x}_{k|k} + [I - A_r]A^{-T}P_{k|k}^{-1}A^{-1}Bu_k \tag{47}$$

Simplifying (47) and substituting (32) yields the a priori information vector equation:

$$\hat{a}_{k+1|k} = [I - A_r]A^{-T}\left(\hat{a}_{k|k} + P_{k|k}^{-1}A^{-1}Bu_k\right) \tag{48}$$

The same approach may be used to solve for the update equation, starting with manipulating (32):

$$\hat{a}_{k+1|k+1} = \left[P_{k+1|k}^{-1} + K_{k+1}\right]\hat{x}_{k+1|k+1} \tag{49}$$

Substitution of (28) into (49) yields:

$$\hat{a}_{k+1|k+1} = \left[P_{k+1|k}^{-1} + K_{k+1}^{-1}\right]\left(\hat{x}_{k+1|k} + K_{k+1}e_{z_{k+1|k}}\right) \tag{50}$$

Expanding (50) gives:

$$\hat{a}_{k+1|k+1} = P_{k+1|k}^{-1}\hat{x}_{k+1|k} + P_{k+1|k}^{-1}K_{k+1}e_{z_{k+1|k}} + K_{k+1}^{-1}\hat{x}_{k+1|k} + e_{z_{k+1|k}} \tag{51}$$

Simplifying (51) further yields the following solution for the a posteriori information vector equation:

$$\hat{a}_{k+1|k+1} = \hat{a}_{k+1|k} + K_{k+1}^{-1}z_{k+1} \tag{52}$$

Equations (31) through (52) represent the derivation of the smooth variable structure information filter.

The following sets of equations summarize the iterative process for the smooth variable structure information filter:

$$\hat{a}_{k+1|k} = [I - A_r]A^{-T}\left(\hat{a}_{k|k} + P_{k|k}^{-1}A^{-1}Bu_k\right) \quad (53)$$

$$P_{k+1|k}^{-1} = \left[I - A_q\left(A_q + Q_{k+1}^{-1}\right)^{-1}\right]A_q \quad (54)$$

$$\hat{a}_{k+1|k+1} = \hat{a}_{k+1|k} + K_{k+1}^{-1}z_{k+1} \quad (55)$$

$$P_{k+1|k+1}^{-1} = [I - A_s K_{k+1}(K_{k+1}^T A_s K_{k+1} + R_{k+1}^{-1})^{-1}K_{k+1}^T]A_s \quad (56)$$

Where the support equations include the following:

$$A_q = A^{-T}P_{k|k}^{-1}A^{-1} \quad (57)$$

$$A_r = A_q\left(A_q + Q_{k+1}^{-1}\right)^{-1} \quad (58)$$

$$A_s = (I - K_{k+1}H)^{-T}P_{k+1|k}^{-1}(I - K_{k+1}H)^{-1} \quad (59)$$

$$A_t = A_s K_{k+1}(K_{k+1}^T A_s K_{k+1} + R_{k+1}^{-1})^{-1} \quad (60)$$

CONCLUSIONS

This paper introduced a new information filter based on the smooth variable structure filter (SVSF). After a review of estimation methods, a detailed description of the derivation was provided, which outlined the main steps taken to formulate the SVSIF. The difference between the IF and the SVSIF processes stem from the gain formulation of the SVSF. Future work will provide a thorough comparison of the IF and the SVSIF. In the estimation literature, the SVSF has been shown to be a more robust method, particularly when compared with the Kalman filter (KF). The enhanced robustness is due to the boundary layer concepts brought forth through the SVSF gain, which enables the estimate to converge within a region around the true state trajectory. Due to this, it is expected that the SVSIF will be more stable in terms of modeling errors and uncertainties, which will make it a more reliable information filter.

NOMENCLATURE

The following is a list of the nomenclature used throughout this paper.

Table 1. List of Nomenclature

Parameter	Definition
x	State vector or values
z	Measurement (system output) vector or values
w	System noise vector
v	Measurement noise vector
A	Linear system transition matrix
B	Input gain matrix
H	Linear measurement (output) matrix
K	Filter gain matrix (i.e., KF or SVSF)
P	State error covariance matrix
Q	System noise covariance matrix
R	Measurement noise covariance matrix
e	Measurement (output) error vector
$diag(a)$	Defines a diagonal matrix of some vector *a*
$sat(a)$	Defines a saturation of the term *a*
γ	SVSF 'convergence' or memory parameter
ψ	SVSF boundary layer width
$\|a\|$	Absolute value of some parameter *a*
T	Transpose of some vector or matrix
$\sim$	Denotes error or difference
^	Estimated vector or values
$k+1\|k$	A priori time step (i.e., before applied gain)
$k+1\|k+1$	A posteriori time step (i.e., after update)

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