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# A New Form of the Smooth Variable Structure Filter with a Covariance Derivation

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*Abstract* — **State and parameter estimation is important for the control of systems, particularly when not all of the system information is available for the designer. Filters are used to extract state information from measurements, which are typically corrupted by noise. A common measure of the performance of an estimate by a filter is through the use of a covariance matrix. This essentially provides a measure of the error in the estimate. Furthermore, knowledge of this covariance can lead to a more accurate derivation and greater number of applications for the filter. Introduced in 2007, the smooth variable structure filter (SVSF) is a relatively new filter. It is a predictor-correct estimator based on sliding mode control and estimation. In its current form, the SVSF is not a classical filter in the sense that it does not have a covariance matrix. This paper introduces the SVSF in a new form without affecting its original proof of stability, and outlines the derivation of a covariance matrix that can be used for comparative purposes as well as other applications. A linear mechanical system referred to as an electrohydrostatic actuator (EHA) is used to numerically demonstrate the new SVSF. The results are compared with the classical Kalman filter (KF), which is the most common and efficient filtering strategy for linear systems.**

## I. Introduction

FOR many control applications, knowledge of the current states and parameters of the systems are essential for accurate and reliable control. Depending on the system, sensors are used to obtain measurements from the environment, typically taking readings of position, velocity, acceleration, force, and pressure. It is quite common for the measurements to be corrupted by noise, which are unwanted signals that reduce the quality of the information obtained [1]. A filter may be used to estimate the state of a dynamic system, whether linear or nonlinear. The word filter is used because when finding the best estimate, one has to filter out the noisy signals or uncertain observations [2]. Filters belong in the domain of estimation theory, which involves finding a value of some parameter of interest. For example, for space vehicles, inertial measuring units may be used to calculate the acceleration. However, since their alignment deteriorates over time, calculating the acceleration by other means (i.e., state estimation) may be desirable [3].

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In the case of many mechanical and electrical systems, the detection and diagnosis of faults prior to their inception is critical for a reliable and safe operation. An effective method of determining faults is by utilizing a model-based condition monitoring approach, where a set of physical mathematical models pertaining to each fault condition are used. The value of multiple parameters may be analyzed within the models, and could be used to diagnosis the type of fault [4]. An interesting example for the requirement of accurate fault detection and estimation is in electrohydrostatic actuators (EHAs). An EHA is an emerging type of actuator typically used in the aerospace industry, and are self-contained units comprised of their own pump, hydraulic circuit, and actuating cylinder [5]. A few important parameters of this system include internal and external leakage, friction, and bulk modulus. These system parameters may be related to specific faults that the EHA may experience during its operation, such as a leak in one of the fittings, a blown piston seal, contaminants stuck in a seal or a gear pump, or the presence of air pockets in the hydraulic oil.

These values cannot be measured directly and must be estimated using state and parameter estimation techniques such as the Kalman filter (KF) or the smooth variable structure filter (SVSF). The KF was introduced in 1960, and remains one of the most popular and studied filters to date [6]. It provides an elegant and statistically optimal solution for linear dynamic systems in the presence of Gaussian white noise [7,8]. However, the optimality of the KF comes at a price of stability and robustness. The KF assumes that the system model is known and is linear, the system and measurement noises are white, and the states have initial conditions that are modeled as random variables with known means and variances [2,9]. However, the previous assumptions do not always hold in real applications, particularly an exact knowledge of the system equations. If one of these assumptions is violated, the KF performance may yield suboptimal estimations and can even become unstable. Furthermore, the KF is sensitive to computer precision and the complexity of certain calculations (i.e., matrix inversions) [8]. The SVSF is a relatively new predictor-correct estimation method based on sliding mode theory [5]. It yields suboptimal estimates, however is extremely robust and stable to modeling uncertainties and errors. This makes it an attractive filter for applications where reliable estimates are required. Furthermore, it has been shown to be an effective method for helping to predict and diagnose faults in systems [4,5].

## II. ESTIMATION METHODS

### A. Kalman Filter

As previously mentioned, the KF provides an elegant and statistically optimal solution for linear dynamic systems in the presence of Gaussian white noise. It is an estimation method that utilizes measurements linearly related to the states or parameters of the systems, and error covariance matrices, to generate a gain referred to as the Kalman gain. This gain is applied to the a priori state estimate, thus creating an a posteriori (i.e., updated) estimate of the states. The estimation process is iterative and continues in a predictor-corrector fashion while maintaining a statistically minimal state error covariance matrix (for linear systems).

A typical linear dynamic system and measurement model are defined by using the following two equations, respectively:

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$
$$z_{k+1} = Hx_{k+1} + v_{k+1} \tag{2}$$

Please refer to the Appendix for a list of pertinent nomenclature and variable definitions. The following five equations form the core of the KF algorithm, and are used in an iterative fashion. Equation (3) defines the a priori estimate based on the system definition, and (4) is the corresponding state error covariance matrix. The Kalman gain is defined by (5), and is used to update the state estimate shown in (6). The a posteriori state error covariance matrix is then calculated by (7).

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{3}$$
$$P_{k+1|k} = HP_{k|k}H^T + Q_k \tag{4}$$
$$K_{k+1} = P_{k+1|k}H^T\left[HP_{k+1|k}H^T + R_{k+1}\right]^{-1} \tag{5}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left[z_{k+1} - H\hat{x}_{k+1|k}\right] \tag{6}$$
$$P_{k+1|k+1} = [I - K_{k+1}H]P_{k+1|k} \tag{7}$$

A number of different methods have extended the classical KF to nonlinear systems, with the most popular ones being the extended (EKF) and unscented (UKF) forms [9,10]. The EKF is conceptually similar to the KF; however, the nonlinear system and measurement matrices are linearized according to its Jacobian (i.e., first-order partial derivative). This linearization process sometimes causes instabilities when implementing the EKF [10]. The UKF is typically more accurate than the EKF since it is able to capture a higher order of the nonlinearities [8]. The UKF approximates the distribution of the states by a Gaussian density, using a set of deterministically chosen sample points which, after a transformation, captures the true mean and covariance up to the second order of nonlinearity [11]. The UKF is sometimes referred to as a type of linear regression Kalman filter since it is based on statistical linearization, rather than analytical linearization like the EKF.

### B. Smooth Variable Structure Filter

Sliding mode control and estimation techniques have been around for quite a few decades, and are mainly popular due to their relative ease of implementation and robustness to modeling uncertainties [12,13]. In a typical sliding mode control scenario, one utilizes a discontinuous switching plane along some desired trajectory [14]. This plane is quite often referred to as a sliding surface, in which the purpose is to keep the state values along this surface by minimizing the state errors (between the desired trajectory and the estimated or actual values). Ideally, if the state value is off or away from the surface, a switching gain would be used to push the state towards the sliding surface. Once upon the surface, the motion of the system as the states slide along the surface is called a sliding mode [14]. The discontinuous switching brings an inherent amount of stability to the control or estimation strategy, while also introducing excessive chattering (i.e., high-frequency switching) which may be undesirable in control since it may excite un-modeled dynamics. A boundary layer may be introduced along the sliding surface in order to saturate and smooth out the chattering within the boundary region. These sliding mode concepts are based on variable structure control, in which one alters the nonlinear dynamics of a system by the introduction of high-frequency switching [12].

The variable structure filter (VSF) was first proposed in 2003, and was introduced as a new type of predictor-corrector estimator based on sliding mode concepts [15]. It is a type of sliding mode estimator, where gain switching is used to ensure that the estimates converge to within a boundary of the true state values (i.e., existence subspace). An internal model of the system, either linear or nonlinear, is used to predict an a priori state estimate. A corrective term (i.e., gain) is then applied to calculate the a posteriori state estimate, and the estimation process is repeated iteratively. The smooth variable structure filter (SVSF) was later derived from the VSF, and uses a much simpler and less complex gain calculation [4]. In its present form, the SVSF is stable and robust to modeling uncertainties and noise, given an upper bound on the level of un-modeled dynamics or knowledge of the magnitude of noise. The basic estimation concept of the SVSF is shown in the following figure. Some initial values of the estimated states are made based on probability distributions or designer knowledge. An area around the true system state trajectory is defined as the existence subspace. Through the use of the SVSF gain, the estimated state will be forced to within this region. Once the value enters the existence subspace, the estimated state is forced into switching along the system state trajectory. A saturation term may be used in this region to reduce the magnitude of chattering or smooth-out the result. As previously mentioned, the SVSF gain introduces a certain amount of chattering which brings an inherent amount of stability. This makes the estimation strategy an attractive method for control problems when not all of the dynamics are well known or defined correctly.

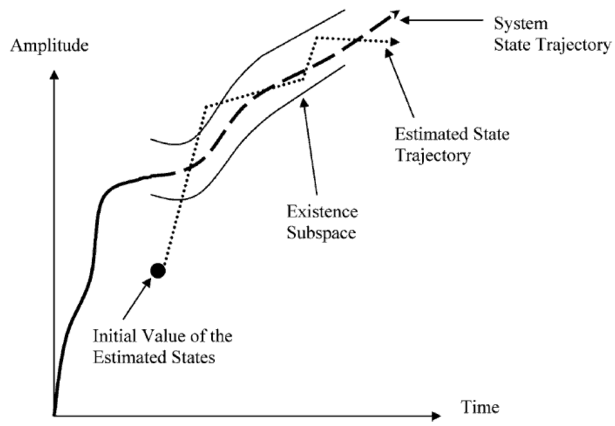


Fig. 1. The smooth variable structure filter estimation concept is shown in the above figure [4].

The SVSF method is model based and applies to smooth nonlinear dynamic equations. The estimation process is iterative and may be summarized by the following set of equations (for a linear control or estimation problem). Like the KF, the system model is used to calculate a priori state and measurement estimates. A corrective term, referred to as the SVSF gain, is calculated as a function of the error in the predicted output and a smoothing boundary layer. This gain is then used to update the state estimate. The estimation process is stable due to the gain calculation of (11). Furthermore, the switching found within the existence subspace is smoothed out by using the saturation term of (11), which is defined by the a priori output error and some predetermined boundary layer width.

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{8}$$

$$\hat{z}_{k+1|k} = \hat{H}\hat{x}_{k+1|k} \tag{9}$$

$$e_{z_{k+1|k}} = z_{k+1} - \hat{z}_{k+1|k} \tag{10}$$

$$K_{k+1} = \left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right) \tag{11}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \tag{12}$$

$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \tag{13}$$

Two critical variables in the SVSF estimation process are the a priori and a posteriori output error estimates, defined by (10) and (13), respectively. The estimation process is stable and convergent if the following lemma is satisfied:

$$|e_k| > |e_{k+1}| \tag{14}$$

The proof, as defined in [4], is such that if one defines $\eta_{k+1}$ as a random but bounded amplitude such that $\eta_{k+1} \leq \beta$, then $|e_k| > |e_{k+1}|$ applies to the phase where $|e_k| > \beta$ that is defined by the reachability phase. Let a Lyapunov function be defined such that $v_{k+1} = e_{k+1}^2$. Hence, the estimation process is stable if $(\Delta v_{k+1} = e_{k+1}^2 - e_k^2) < 0$. Furthermore, note that this stability condition is satisfied by (14) [4,16].

### C. A Revised Form of the SVSF

In its current form, the SVSF is not a classical filter in the sense that it does not have or make use of a covariance matrix. A covariance may be used for a variety of reasons: to determine an optimal value of the gain (i.e., such as in the case of the KF), for the implementation of interacting multiple model (IMM) methods that can be used for target tracking or fault detection and diagnosis, or to create other forms such as the information filter (i.e., inverse of the covariance) [2]. A covariance matrix is a function of state estimation errors (i.e., the difference between the actual and the estimated values), and may be defined as the expectation of the error squared, as follows:

$$P = \mathrm{E}\{\tilde{x}\tilde{x}^T\} = \mathrm{E}\{(x - \hat{x})(x - \hat{x})^T\} \tag{15}$$

In this paper, a revised form of the SVSF is introduced in order to obtain a simplified covariance matrix. It is proposed that the SVSF gain and update estimate be modified respectively as follows:

$$K_{k+1} = diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right)\right]\left[diag\left(e_{z_{k+1|k}}\right)\right]^{-1} \tag{16}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z_{k+1|k}} \tag{17}$$

Essentially the nature of the SVSF remains the same, as one divides the gain by the a priori output error, and then multiplies by it again in the a posteriori estimate equation. However, introducing this notation enables one to derive a much simpler covariance derivation. Furthermore, note that the proof of stability for the SVSF is not affected. It is interesting to note that the a priori and a posteriori state error covariance matrices for the SVSF are similar to the KF for linear systems.

## III. A Revised Smooth Variable Structure Filter

### A. SVSF Covariance Derivation

The following is the proof for the a priori state error covariance equation used in the SVSF for linear systems, for the case without modeling errors. The a priori state error covariance matrix may be defined as follows:

$$P_{k+1|k} = \mathrm{E}\left\{\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T\right\} \tag{18}$$

Where the a priori state error may be defined by:

$$\tilde{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} \tag{19}$$

The discrete model of the system may be described by the following equation:

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{20}$$

Furthermore, the uncertain estimation model for the system that is used in the prediction stage of the SVSF may be defined as follows:

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{21}$$

Substituting (20) and (21) into (19) yields:

$$\tilde{x}_{k+1|k} = Ax_k + Bu_k + w_k - \hat{A}\hat{x}_{k|k} - \hat{B}u_k \tag{22}$$

Adding and subtracting $\hat{A}x_k$ to both sides of (22) yields the following:

$$\tilde{x}_{k+1|k} = \left(A - \hat{A}\right)x_k + \hat{A}(x_k - \hat{x}_{k|k}) + (B - \hat{B})u_k + w_k \tag{23}$$

Simplifying (23) further yields the following a priori state error equation:

$$\tilde{x}_{k+1|k} = \tilde{A}x_k + \hat{A}\tilde{x}_{k|k} + \tilde{B}u_k + w_k \tag{24}$$

For this case, we have no model mismatch $(\hat{A} \approx A \text{ and } \hat{B} \approx B)$ such that the a priori state error equation simplifies to the following:

$$\tilde{x}_{k+1|k} = A\tilde{x}_{k|k} + w_k \tag{25}$$

Substituting (25) and its corresponding transpose into (18) yields the following definition for the a priori state error covariance:

$$P_{k+1|k} = \mathrm{E}\{[A\tilde{x}_{k|k} + w_k][\tilde{x}_{k|k}^T A^T + w_k^T]\} \tag{26}$$

Expanding the terms in (26) yields the following:

$$P_{k+1|k} = \mathrm{E}\{A\tilde{x}_{k|k}\tilde{x}_{k|k}^T A^T + A\tilde{x}_{k|k}w_k^T + w_k\tilde{x}_{k|k}^T A^T + w_k w_k^T]\} \tag{27}$$

Note that the system noise is typically modeled as Gaussian white noise, such that it is zero-mean with a covariance referred to as $Q$. Furthermore, it is assumed that the system noise and the state errors are independent of each other. Based on these assumptions we then have the following:

$$w_k \sim \mathcal{N}(0, Q_k) \tag{28}$$

$$\mathrm{E}\{w_k\} = \mathrm{E}\{w_k^T\} = 0 \tag{29}$$

$$\mathrm{E}\{w_k w_k^T\} = Q_k \tag{30}$$

$$\mathrm{E}\left\{w_k\tilde{x}_{k|k}^T\right\} = \mathrm{E}\{w_k\}\mathrm{E}\{\tilde{x}_{k|k}^T\} = 0 \tag{31}$$

$$\mathrm{E}\left\{\tilde{x}_{k|k}w_k^T\right\} = \mathrm{E}\left\{\tilde{x}_{k|k}\right\}\mathrm{E}\{w_k^T\} = 0 \tag{32}$$

Also, it is important to note the definition for the previous time step's a posteriori state error covariance:

$$P_{k|k} = \mathrm{E}\left\{\tilde{x}_{k|k}\tilde{x}_{k|k}^T\right\} \tag{33}$$

Applying the above six definitions to (27) yields the *a priori state error covariance* equation for the SVSF as follows:

$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{34}$$

In this case, it is shown that the a priori state error covariance is a function of the previous a posteriori state error covariance, the system model, and the system noise covariance. The proof for the a posteriori state error covariance equation may now be solved. The a posteriori state error covariance matrix may be defined as follows:

$$P_{k+1|k+1} = \mathrm{E}\left\{\tilde{x}_{k+1|k+1}\tilde{x}_{k+1|k+1}^T \middle| Z_{1:k+1}\right\} \tag{35}$$

Where the a posteriori state error may be defined by:

$$\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1} \tag{36}$$

From the SVSF, we have the state update and a priori measurement error equations, respectively as follows:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z_{k+1|k}} \tag{37}$$

$$e_{z_{k+1|k}} = z_{k+1} - H\hat{x}_{k+1|k} \tag{38}$$

Substitution of (37) and (38) into (36) yields:

$$\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k} - K_{k+1}\left(z_{k+1} - H\hat{x}_{k+1|k}\right) \tag{39}$$

The measurement update equation is defined as follows:

$$z_{k+1} = Hx_{k+1} + v_{k+1} \tag{40}$$

Substitution of (40) into (39) yields:

$$\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k} - K_{k+1}\left(Hx_{k+1} + v_{k+1} - H\hat{x}_{k+1|k}\right) \tag{41}$$

Based on the state error definitions we have:

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} - K_{k+1}\left(H\tilde{x}_{k+1|k} + v_{k+1}\right) \tag{42}$$

Simplifying (42) yields the following definition for the a posteriori state error equation:

$$\tilde{x}_{k+1|k+1} = (I - K_{k+1}H)\tilde{x}_{k+1|k} - K_{k+1}v_{k+1} \tag{43}$$

Substituting (43) and its corresponding transpose into (35) yields the following definition for the a posteriori state error covariance:

$$P_{k+1|k+1} = \mathrm{E}\Big\{\Big[(I - K_{k+1}H)\tilde{x}_{k+1|k} - K_{k+1}v_{k+1}\Big]\Big[\tilde{x}_{k+1|k}^T(I - K_{k+1}H)^T - v_{k+1}^T K_{k+1}^T\Big]\Big|Z_{1:k+1}\Big\} \tag{44}$$

Expanding the terms in (44) yields the following:

$$P_{k+1|k+1} = \mathrm{E}\{[(I - K_{k+1}H)\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T(I - K_{k+1}H)^T - (I - K_{k+1}H)\tilde{x}_{k+1|k}v_{k+1}^T K_{k+1}^T - K_{k+1}v_{k+1}\tilde{x}_{k+1|k}^T(I - K_{k+1}H)^T + K_{k+1}v_{k+1}v_{k+1}^T K_{k+1}^T]|Z_{1:k+1}\} \tag{45}$$

Note that the measurement noise is typically modeled as Gaussian white noise, such that it is zero-mean with a covariance referred to as $R$. Furthermore, it is assumed that the measurement noise and the state errors are independent of each other. Based on these assumptions we then have the following:

$$v_{k+1} \sim \mathcal{N}(0, R_{k+1}) \tag{46}$$
$$\mathrm{E}\{v_{k+1}\} = \mathrm{E}\{v_{k+1}^T\} = 0 \tag{47}$$
$$\mathrm{E}\{v_{k+1}v_{k+1}^T\} = R_{k+1} \tag{48}$$
$$\mathrm{E}\{v_{k+1}\tilde{x}_{k+1|k}^T\} = \mathrm{E}\{v_{k+1}\}\mathrm{E}\{\tilde{x}_{k+1|k}^T\} = 0 \tag{49}$$
$$\mathrm{E}\{\tilde{x}_{k+1|k}v_{k+1}^T\} = \mathrm{E}\{\tilde{x}_{k+1|k}\}\mathrm{E}\{v_{k+1}^T\} = 0 \tag{50}$$

Applying the above five definitions to (45) yields the *a posteriori state error covariance* equation for the SVSF as follows:

$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{51}$$

In this case, it is shown that the a posteriori state error covariance is a function of the a priori state error covariance, the measurement model, and the measurement noise covariance.

### *B. The Revised SVSF Estimation Process*

The following equations summarize the new estimation process for the SVSF, applied to linear systems:

$$\hat{x}_{k+1|k} = \hat{A}\hat{x}_{k|k} + \hat{B}u_k \tag{52}$$
$$P_{k+1|k} = AP_{k|k}A^T + Q_k \tag{53}$$
$$\hat{z}_{k+1|k} = \hat{H}\hat{x}_{k+1|k} \tag{54}$$
$$e_{z_{k+1|k}} = z_{k+1} - \hat{z}_{k+1|k} \tag{55}$$
$$K_{k+1} = diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\frac{e_{z_{k+1|k}}}{\psi}\right)\right]\left[diag\left(e_{z_{k+1|k}}\right)\right]^{-1} \tag{56}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}e_{z_{k+1|k}} \tag{57}$$
$$P_{k+1|k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{58}$$
$$e_{z_{k+1|k+1}} = z_{k+1} - \hat{H}\hat{x}_{k+1|k+1} \tag{59}$$

## IV. EHA Simulation

In this section, an electrohydrostatic actuator (EHA) is simulated. The EHA to be simulated is based on an actual prototype built for experimentation [4,5]. The purpose of this simple simulation is to demonstrate that the new estimation process for the SVSF is functional, and that the derived covariance matrix is numerically similar to that of the KF for linear systems. Note that for linear systems the KF will yield the optimal solution (i.e., best estimate).

The EHA is a third order system with state variables related to its position, velocity, and acceleration. It is assumed that all three states have measurements associated with them (i.e., full measurement matrix). The input to the system is a random normal distribution with magnitude 1. A step change is inserted into the input of the system half-way through the duration. The sample time of the system is 0.001 seconds. The entire EHA system description may be found in [5], however for the purpose of this paper, the discrete state-space model of the system is simply defined as follows:

$$x_{k+1} = \begin{bmatrix} 1 & 0.001 & 0 \\ 0 & 1 & 0.001 \\ -557.02 & -28.616 & 0.9418 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ 557.02 \end{bmatrix} u_k \tag{60}$$

The initial state values are set to zero. The system and measurement noises are considered to be Gaussian with maximum amplitude corresponding to 10% error ($W_{Max} = [0.01 \quad 1 \quad 10]^T$ and $V_{Max} = [0.1 \quad 10 \quad 100]^T$). The initial state error covariance, system noise covariance, and measurement noise covariance are defined respectively as follows:

$$P_{0|0} = 10Q \tag{61}$$
$$Q = diag([1 \quad 10 \quad 100]) \tag{62}$$
$$R = diag([0.1 \quad 100 \quad 1000]) \tag{63}$$

For the SVSF estimation process, the 'memory' or convergence rate was set to $\gamma = 0.1$, and the boundary layer widths were defined as $\psi = [0.1 \quad 10 \quad 50]^T$. These parameters were tuned by trial-and-error, with the goal of decreasing the estimation error. The main results of applying the KF and SVSF on the EHA problem are shown in the following figure. This figure shows the true position of the EHA with the KF and SVSF estimates. The estimation results of both filters are relatively the same. It is important to note, that even with a tuned SVSF, the KF provides the best estimate (i.e., optimal) for a linear system.

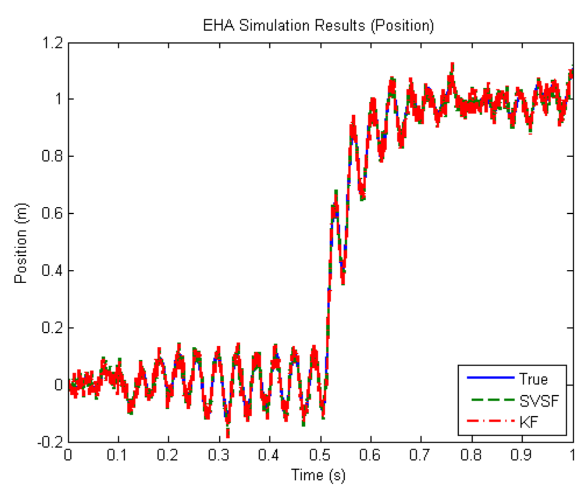


Fig. 2. The position estimates for the EHA simulation are shown in the above figure. Note that the lines are nearly concentric and thus are relatively difficult to distinguish.

The velocity and acceleration estimates were relatively the same as those shown in the previous figure (and were thus omitted for space constraints). The RMSE results of running the simulation are as follows:

TABLE I
RMSE SIMULATION RESULTS

| Filter | Position (m) | Velocity (m/s) | Acceleration ($m/s^2$) |
|---|---|---|---|
| KF | 0.0251 | 0.8218 | 28.43 |
| SVSF | 0.0254 | 1.7697 | 28.76 |

As shown in the above table, the KF provides the optimal result. However, the SVSF estimate remains excellent, with a notable difference in the velocity estimate. The final covariance estimates of the KF and the SVSF were found to be respectively as follows:

$$P_{KF\infty} = \begin{bmatrix} 0.09 & -0.04 & -0.11 \\ -0.04 & 18.1 & -9.75 \\ -0.11 & -9.75 & 975 \end{bmatrix} \tag{64}$$

$$P_{SVSF\infty} = \begin{bmatrix} 0.10 & 0.01 & -0.01 \\ 0.01 & 41.2 & -13.2 \\ -0.01 & -13.2 & 988 \end{bmatrix} \tag{65}$$

By analyzing and comparing the results of the RMSE and the covariance values, the SVSF in its revised form appears to be yielding an accurate calculation of its respective covariance. It is important to note that this covariance derivation is for linear systems. The revised SVSF would have to be further extended for nonlinear systems.

## V. CONCLUSIONS

This paper introduced a revised form of the smooth variable structure filter with a covariance derivation. The proposed estimation strategy was applied to an electrohydrostatic actuator for numerical comparison with the popular Kalman filter. For linear systems, the covariance of the revised SVSF was found to be similar to that of the Kalman filter. The addition of the covariance to the SVSF creates more of a well-rounded filter and introduces opportunity for further research. The revised method may now be combined with interacting multiple model methods which are used on target tracking, and fault detection and diagnosis problems.

## APPENDIX

The following is a table of important nomenclature used throughout this paper:

TABLE II
LIST OF NOMENCLATURE

| Parameter | Definition |
|---|---|
| $x$ | State vector or values |
| $z$ | Measurement (system output) vector or values |
| $w$ | System noise vector |
| $v$ | Measurement noise vector |
| $A$ | Linear system transition matrix |
| $B$ | Input gain matrix |
| $E\{a\}$ | Expectation of some value $a$ |
| $H$ | Linear measurement (output) matrix |
| $K$ | Filter gain matrix (i.e., KF or SVSF) |
| $P$ | State error covariance matrix |
| $Q$ | System noise covariance matrix |
| $R$ | Measurement noise covariance matrix |
| $e$ | Measurement (output) error vector |
| $diag(a)$ | Defines a diagonal matrix of some vector $a$ |
| $sat(a)$ | Defines a saturation of the term $a$ |
| $\mathcal{N}(\mu,\sigma)$ | Normal distribution with mean $\mu$ and variance $\sigma$ |
| $\gamma$ | SVSF 'convergence' or memory parameter |
| $\psi$ | SVSF boundary layer width |
| $\lvert a \rvert$ | Absolute value of some parameter $a$ |
| $T$ | Transpose of some vector or matrix |
| $\sim$ | Denotes error or difference |
| ^ | Estimated vector or values |
| $k+1\mid k$ | A priori time step (i.e., before applied gain) |
| $k+1\mid k+1$ | A posteriori time step (i.e., after update) |

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