Lattice Kalman Filters

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***Abstract*—In this paper, a new filter in the nonlinear Kalman filtering framework is proposed. The new filter is referred to as the lattice Kalman filter (LKF) and is based on a class of quasi-Monte Carlo (QMC) methods known as lattice rules. The proposed LKF method uses the Korobov type lattice rule to deterministically generate sample points that are randomly shifted based on the Cranley-Patterson shift method in order to approximate multi-dimensional integrals in the Gaussian filtering context. The mathematical formulation of the proposed LKF method as well as its error bound propagation are discussed. To evaluate the efficiency of the LKF, it is applied on a nonlinear aerospace system and compared with four other well-known methods presented in the literature. Simulation results demonstrate LKF uses significantly fewer sampling points yielding a significantly lower computational burden than another variant of QMC filter while maintaining the estimation accuracy. Furthermore, it provides asymptotically similar results to the unscented Kalman filter (UKF) but with less computational complexity, which is an important consideration in real applications.**

***Index Terms*—Estimation theory, Kalman filter, lattice rules, quasi-Monte Carlo methods, randomly shifted lattice points.**

I. Introduction

CONSIDER a noisy nonlinear dynamic system that can be modelled using the following system and measurement equations, respectively:

$$x_k = f\left(x_{k-1}, u_{k-1}\right) + w_k \tag{1}$$
$$z_k = h\ \left(x_k\right) + v_k \tag{2}$$

where x_k, u_k, and z_k are the state, input, and measurement vector, respectively. The nonlinear system f_{k-1} and nonlinear measurement h_k contain system and measurement noise referred to as w_{k-1} and v_k, respectively. f_{k-1}, h_k, and u_k are assumed to be known, and w_{k-1} and v_k are considered to be mutually independent noises [1]. The available information is a set of measurements z_k that relate to x_k, obtained by one or more sensors from time step 1 to k, denoted by [2]:

$$Z_{1:k} = \{z_1,\ z_2, \ldots,\ z_k\} \tag{3}$$

The main objective of estimation theory is to extract the true state values from noisy measurements. Estimates of the states can then be used for predicting system health or future events. One of the most well-known filtering paradigms is Bayesian filtering, where the objective is to recursively calculate the posterior density which is used to describe the probability of an event or state [2]. In order to calculate the posterior density $p(x_k|Z_{1:k}\ )$, first, the predicted density is found [2], [3]:

$$p\ \left(x_k|\mathrm{Z}_{1:\mathrm{k}-1}\right) = \int p\left(x_k|x_{k-1}\right)\ p\left(x_{k-1}|\mathrm{Z}_{1:\mathrm{k}-1}\right)\ dx_{k-1} \tag{4}$$

where $p(x_{k-1}|Z_{1:k-1})$ is the previous posterior density at time step $k-1$, and $p(x_k|x_{k-1})$ is the state transition density which demonstrates how the states evolve throughout time. Next, the posterior density can be rewritten as $p\ (x_k|Z_{1:k}\ ) = p(x_k|z_k, \mathrm{Z1:k}-1)$ [2], [4]. Consequently, based on Bayes' law we have the following equation [2], [5]:

$$p\ \left(x_k|Z_{1:k}\ \right) = \frac{p\left(z_k|x_k\right)\ p\left(x_k|\mathrm{Z}_{1:\mathrm{k}-1}\right)}{p\left(z_k|\mathrm{Z}_{1:\mathrm{k}-1}\right)} \tag{5}$$

In (5), the factors $p(z_k|x_k)$ and $p(x_k|\mathrm{Z}_{1:\mathrm{k}-1})$ are named the likelihood and the predicted density, respectively. The factor $p(z_k|\mathrm{Z}_{1:\mathrm{k}-1})$ in the denominator is a normalizing term [2], [5]:

$$p\left(z_k|\mathrm{Z}_{1:\mathrm{k}-1}\right) \triangleq \int p\left(z_k|x_k\right) p\left(x_k|\mathbf{Z}_{1:k-1}\right) dx_k;\ k\ =\ 1,\ 2,\ \ldots \tag{6}$$

Based on a Gaussian assumption for all of the densities, the well-known Kalman filter (KF) formulation is constructed. This method recursively calculates the first and second statistical moments, also known as the mean and covariance [1], [5], [6]. This information is used to approximate the posterior density which is used to solve the estimation problem.

Another strategy to approximate the posterior density is based on the use of weighted sigma-points. According to [7], three types of moment approximation methods based on sigma-points are categorized: 1) to represent a probability distribution [8], [9]; 2) to calculate numerical integration [10], [11]; 3) to approximate nonlinear functions via interpolation [12], [13]. Based on the density approximation method presented in [8], [9], reduced forms of the unscented Kalman filter (UKF), including minimal skew simplex UKF (MSS-UKF) [14] and spherical simplex UKF (SS-UKF) [15], are developed to reduce the computational cost by decreasing the number of sample points. Based on the literature, a couple of other new methods have also been proposed to generate sigma-points [16], [17].

However, these approximation methods introduce errors in integral evaluations compared to Monte Carlo (MC) integration techniques with asymptotically accurate integral computations [18]. To address the high computational burden required by MC methods, due to a large number of sampling points, the authors in [19] utilized a class of quasi-Monte Carlo (QMC) methods known as digital sequence (DS). This approach asymptotically computes the multivariate integrals in the nonlinear KF context; although the estimation results of DSKF have acceptable accuracy, the number of sampling points remains high.

Manuscript received March 21, 2021; revised May 23, 2021; accepted June 5, 2021. Date of publication June 16, 2021; date of current version July 13, 2021. This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant RGPIN-2017-04087. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. Gloria Menegaz. *(Corresponding author: S. Andrew Gadsden.)*

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Digital Object Identifier 10.1109/LSP.2021.3089935

In this paper, we propose the use of lattice rules to generate sampling points to reduce the computational complexity of computing Gaussian-weighted integrals in the nonlinear KF framework. Lattice rules are discussed in Section II, and the proposed lattice Kalman filter (KF) is presented in Section III. In Section IV, the LKF is applied on an aerospace system and compared with other well-known filtering methods. Concluding remarks are made in Section V.

II. Multi-Dimensional Numerical Integration

Consider that we want to approximate a multivariate integral over a unit hypercube, as per:

$$I(f) := \int_{[0,1)^d} f(x)\, dx \tag{7}$$

In order to obtain a certain level of accuracy, it would not be feasible to apply the iterated classical quadrature rule for the moderately high number of dimensions d because of the so-called curse of dimensionality (or intractability) issues [20]. In this case, an equal-weight cubature rule was presented [20]:

$$Q(f; P_N) := \frac{1}{N} \sum_{j=1}^{N} f(x_j) \tag{8}$$

where P_N is a point set of x_j where $j = 1, \ldots, N$, and all points have the equal weight of $1/\mathrm{N}$. If the points are randomly and independently chosen, the method is known as MC integration. However, if these points are deterministically drawn and uniformly distributed (the so-called low-discrepancy points), the resultant integration method is referred to as QMC. In experimental problems, it can be shown that the error bound of low-discrepancy points generated by QMC methods may be observed to be in the form of $O(N^{-1})$, even for a very large number of dimensions. Hence, the error of integral approximation using the QMC methods can be smaller than using MC methods $O(N^{-1/2})$ if we choose a well-defined rule to generate low-discrepancy points [20].

There are generally two main categories of QMC methods: digital nets/sequences [21] and lattice points [20]. In this paper, we consider the rank-1 lattice rule [22] which is based on the modular arithmetic to produce low-discrepancy points.

A. Rank-1 Lattice Rule

The rank-1 lattice rule presented in [22] is a simple and efficient type of lattice rule used to generate low-discrepancy points defined as follows:

$$x_j^L = \left\{ \frac{g\, j}{N} \right\} := \frac{g\, j}{N} mod\, 1 = \frac{g\, j\ mod\ N}{N}; j = 0, \ldots, N-1 \tag{9}$$

where g is the generating vector consisting of d integers $g = [g_1\ g_2\ \ldots\ g_d]^T$ and modulo N is applied component-wise. However, given the number of points N, the selection of generating vector $g \in Z^d$ indicates if the generated lattice points are well-generated. Generally, the generating vector is chosen based on the well-known Korobov type lattice rule to generate the lattice points [22], which is defined as follows:

$$g = \left[a^0\ a^2\ a^3 \ldots\ a^{d-1}\right]^T \tag{10}$$

where a is an integer which is coprime with N. The selection of a generating vector to create lattice points is based on a worst-case error concept, and is comprehensively discussed in [20].

B. Randomly Shifted Lattice Points

To obtain an error estimator which can be used to increase the accuracy of the lattice methods, we can apply independent and identically distributed (IID) random shifts to the lattice points. In this paper, we use the Cranley-Patterson shift method (Modula 1 shift) [23] to generate randomly shifted lattice points defined as follows:

$$x_j^{SL} = \left(x_j^L + \Delta\right)\ mod\, 1;\ j = 0, \ldots, N-1 \tag{11}$$

where Δ is a shift that is uniformly and independently distributed over hypercube $[0,1)^d$. Applying this shift to the lattice points is necessary to ensure that the resulting integral computations are unbiased [22], [23].

C. Gaussian Points Generated Based on Lattice Points

In the nonlinear KF context, we frequently encounter multi-dimensional integrals of the following form:

$$I(f) = \int f(x)\, p(x)\, dx \tag{12}$$

where $p(x)$ is a Gaussian density, whereby $x \sim \mathcal{N}(x;\ \mu, \Sigma)$. A mapping function ϕ that projects the integration region to the unit hypercube is required. For a one-dimensional variable x with standard normal distribution, the function ϕ can be defined as a normal cumulative distribution defined as per [19], [24]:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt \tag{13}$$

Therefore, the multivariate integral $I(f)$ can be approximated utilizing the generated (randomly shifted) lattice points x_j^{SL} and the inverse mapping function ϕ^{-1} as follows:

$$I(f) = \int_{[0,1)^d} f\left(\phi^{-1}\left(x^{SL}\right)\right) dx \approx \frac{1}{N} \sum_{j=0}^{N-1} f\left(\phi^{-1}\left(x_j^{SL}\right)\right) \tag{14}$$

Note that ϕ^{-1} is the inverse function of the normal cumulative distribution function. We can transform the lattice point set $x^{SL} = \{x_j^{SL}, j = 0,\ 1,\ 2,\ \ldots,\ N-1\}$ defined in the unit hypercube $[0,1)^d$ into a quasi-Gaussian point set $X = \{X_j, j = 0,\ 1,\ 2,\ \ldots,\ N-1\}$ with the mean μ and covariance matrix Σ by:

$$X_j = \mu + SY_j; j = 0,\ 1,\ 2,\ \ldots, N-1 \tag{15}$$

where S is obtained by applying Cholesky decomposition on Σ ($\Sigma = \mathrm{S}^T\ \mathrm{S}$) and Y_j is computed by applying the inverse mapping function on the lattice points:

$$Y_j = \phi^{-1}\left(x_j^{SL}\right);\ j = 0,\ 1,\ 2,\ \ldots, N-1 \tag{16}$$

Eventually, the integral $I(f)$ may be asymptotically computed using the lattice-based generated Gaussian points as follows:

$$I(f) = \int f(x)\, p(x)\, dx \approx \frac{1}{N} \sum_{j=0}^{N-1} f(X_j) \tag{17}$$

III. Lattice Kalman Filter (LKF)

A. LKF Estimation Process

In (1) and (2), it is assumed that the system and measurement noise are independent and zero mean with Gaussian distributions defined by covariances Q and R, respectively. The LKF is formulated as predictor-corrector estimator in the nonlinear filtering framework with a Gaussian assumption.

In the prediction stage, the predicted state $\hat{x}_{k|k-1}$ and state error covariance matrix $P_{k|k-1}$ are computed as per [1], [3], [6]:

$$\hat{x}_{k|k-1} = \int_{\mathbb{R}^{n_x}} f\left(x_{k-1}, u_{k-1}\right) p\left(x_{k-1}|\mathrm{Z}_{1:\mathrm{k}-1}\right) dx_{k-1} \quad (18)$$

$$P_{k|k-1} = \int_{\mathbb{R}^{n_x}} \left[x_k - \hat{x}_{k|k-1}\right]\left[x_k - \hat{x}_{k|k-1}\right]^T \ldots p\left(x_{k-1}|\mathrm{Z}_{1:\mathrm{k}-1}\right)\ dx_{k-1} + Q_k \quad (19)$$

Based on (17), (18) and (19) can be computed as follows:

$$\hat{x}_{k|k-1} \approx \frac{1}{N}\sum_{j\,=\,0}^{N-1} f\left(X_{j,\ k-1|k-1}, u_{k-1}\right) \quad (20)$$

$$P_{k|k-1} \approx Q_k + \frac{1}{N}\sum_{j\,=\,0}^{N-1}\left[f\left(X_{j,k-1|k-1}, u_{k-1}\right) - \hat{x}_{k|k-1}\right] \ldots \left[f\left(X_{j,k-1|k-1}, u_{k-1}\right) - \hat{x}_{k|k-1}\right]^T \quad (21)$$

where $X_{j,\ k-1|k-1}; j\ =\ 0, 1, \ldots,\ N-1$ are lattice-based Gaussian points generated using the previous updated mean $\hat{x}_{k-1|k-1}$ and covariance matrix $P_{k-1|k-1}$.

The predicted measurement $\hat{z}_{k|k-1}$, innovation covariance $P_{zz,\ k|k-1}$, and cross-covariance $P_{xz,\ k|k-1}$ are computed next, respectively as follows [1], [3], [6]:

$$\hat{z}_{k|k-1} = \int_{\mathbb{R}^{n_x}} h\left(x_k, u_k\right) \times p\left(x_k|\mathrm{Z}_{1:\mathrm{k}-1}\right)\ dx_k \quad (23)$$

$$P_{zz,\ k|k-1} = \int_{\mathbb{R}^{n_x}} \left[h\left(x_k, u_k\right) - \hat{z}_{k|k-1}\right] \ldots \left[h\left(x_k, u_k\right) - \hat{z}_{k|k-1}\right]^T p\left(x_k|\mathrm{Z}_{1:\mathrm{k}-1}\right)\ dx_k + R_k \quad (24)$$

$$P_{xz,\ k|k-1} = \int_{\mathbb{R}^{n_x}} \left[x_k - \hat{x}_{k|k-1}\right]\left[h\left(x_k, u_k\right) - \hat{z}_{k|k-1}\right]^T \ldots p\left(x_k|\mathrm{Z}_{1:\mathrm{k}-1}\right)\ dx_{k-1} \quad (25)$$

Applying (17) to the multi-dimensional integrals in (23) through (25) yields the following:

$$\hat{z}_{k|k-1} \approx \frac{1}{N}\sum_{j\,=\,0}^{N-1} h\left(X_{j,\ k|k-1}, u_k\right) \quad (26)$$

$$P_{zz,\ k|k-1} \approx \frac{1}{N}\sum_{j\,=\,0}^{N-1}\left[h\left(X_{j,\ k|k-1}, u_k\right) - \hat{z}_{k|k-1}\right] \ldots \left[h\left(X_{j,\ k|k-1}, u_k\right) - \hat{z}_{k|k-1}\right]^T + R_k \quad (27)$$

$$P_{xz,\ k|k-1} \approx \frac{1}{N}\sum_{j\,=\,0}^{N-1}\left[X_{j,\ k|k-1} - \hat{x}_{k|k-1}\right] \ldots \left[h\left(X_{j,\ k|k-1}, u_k\right) - \hat{z}_{k|k-1}\right]^T \quad (28)$$

where $X_{j,\ k|k-1}; j\ =\ 0, 1,\ \ldots,\ N-1$ are lattice-based Gaussian points generated using the predicted mean $\hat{x}_{k|k-1}$ and state error covariance $P_{k|k-1}$. In the update stage, the LKF gain K_k, estimates $\hat{x}_{k|k}$, and state error covariance $P_{k|k}$ are computed respectively as follows [1], [3], [6]:

$$K_k = P_{xz,\ k|k-1}\ P^{-1}_{zz,\ k|k-1} \quad (29)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1}\ + K_k\left(z_k - \hat{z}_{k|k-1}\right) \quad (30)$$

$$P_{k|k} = P_{k|k-1}\ - K_k P_{zz,k|k-1} K_k^T \quad (31)$$

The LKF strategy is summarized by equations (20) and (21), and (26) through (31). The process is repeated recursively at each time step.

B. Error Bound Propagation

In this subsection, the bound of error between the true posterior density $p(x_k|\mathrm{Z}_{1:\mathrm{k}})$ and the approximated posterior $\tilde{p}(x_k|\mathrm{Z}_{1:\mathrm{k}})$ density computed using the LKF is investigated and estimated. To achieve this goal, based on the simplification provided in [19], the true and approximated posterior densities are defined, respectively, as follows:

$$p\ \left(x_k|\mathrm{Z}_{1:\mathrm{k}}\right) = UP_k\ \left(PR_k\left(p\left(x_{k-1}|\mathrm{Z}_{1:\mathrm{k}-1}\right)\right)\right) \quad (32)$$

$$\widetilde{p}\ \left(x_k|\mathrm{Z}_{1:\mathrm{k}}\right) = \widetilde{UP}_k\ \left(\widetilde{PR}_k\left(\tilde{p}\left(x_{k-1}|\mathrm{Z}_{1:\mathrm{k}-1}\right)\right)\right) \quad (33)$$

where PR_k and UP_k represent the prediction and update stage of Bayesian recursive formulas in (4) and (5), respectively. Note that $\widetilde{PR}_k$ and $\widetilde{UP}_k$ denote the Gaussian approximation of PR_k using (20)-(21), and of UP_k using (29)-(31), respectively.

If the prediction PR_k and update UP_k operators satisfy the Lipschitz continuous condition for any given density $p(x)$ and $q(x)$:

$$\left|PR_k\left(p\right) - PR_k\left(q\right)\right| \leq \rho_1\left|p - q\right| \quad (34)$$

$$\left|UP_k\left(p\right) - UP_k\left(q\right)\right| \leq \rho_2\left|p - q\right| \quad (35)$$

Then the error propagation at time step k will have the subsequent upper bound:

$$e_k \leq \left(\rho_1\rho_2\right)^k e_0 + \sum_{j\,=\,1}^{k}\left(\rho_1\rho_2\right)^{k-j}\left(e_{j,UP} + \rho_2 e_{j,PR}\right) \quad (36)$$

where $e_0 = \left|\tilde{p}(x_0) - p(x_0)\right|$ is the initial error. Note that $e_{j,PR}$ and $e_{j,UP}$ are the errors produced based on the approximation in the prediction step $\widetilde{PR}_k$ and update step $\widetilde{UP}_k$, respectively:

$$e_{j,PR} = \left|\widetilde{PR}_j(\tilde{p}(x_{j-1}|\mathrm{Z}_{1:\mathrm{j}-1})) - PR_j(\tilde{p}(x_{j-1}|\mathrm{Z}_{1:\mathrm{j}-1}))\right| \quad (37)$$

$$e_{j,UP} = \left|\widetilde{UP}_j(\tilde{p}(x_j|\mathrm{Z}_{1:\mathrm{j}-1})) - Up_j(\tilde{p}(x_j|\mathrm{Z}_{1:\mathrm{j}-1}))\right| \quad (38)$$

As well, note that $|.|$ can be defined as any kind of norm. Due to space limitations, we do not prove the error bound presented in (36); however, it may be proven using the definition of the error and applying the Cauchy-Schwarz inequality and Lipschitz continuous condition. Overall, according to (36), the accumulated error is dependent on the initial state values and process and measurement functions.

IV. Computer Experiment and Results

A. Nonlinear System

In order to evaluate the efficiency of the proposed LKF estimation method, we studied a nonlinear aerospace system known as an electrohydrostatic actuator (EHA) [25]. The following state space model defines the nonlinear system [25]:

$$x_{1,k} = x_{1,k-1}\ + Tx_{2,k-1} + w_{1,k-1} \quad (39)$$

$$x_{2,k} = x_{2,k-1}\ + Tx_{3,k-1} + w_{2,k-1} \quad (40)$$

$$x_{3,k} = \left[1 - T\frac{a_2V_0 + M\beta_e L}{MV_0}\right]\ x_{3,k-1} - T\frac{\left(A_E^2 + a_2 L\right)\beta_e}{MV_0}\ x_{2,k-1} \ldots - T\frac{2a_1V_0x_{2,k-1}x_{3,k-1} + L\beta_e\left(a_1x_{2,k-1}^2 + a_3\right)}{MV_0}\mathrm{sgn}\left(x_{2,k-1}\right) \ldots + T\frac{A_E\beta_e}{MV_0}u_{k-1} + \ w_{3,k-1} \quad (41)$$

$$x_{4,k} = A_E^2\ \left(a_2x_{2,k-1} + \left(a_1x_{2,k-1}^2 + a_3\right) sgn\left(x_{2,k-1}\right)\right) \ldots + w_{4,k-1} \quad (42)$$

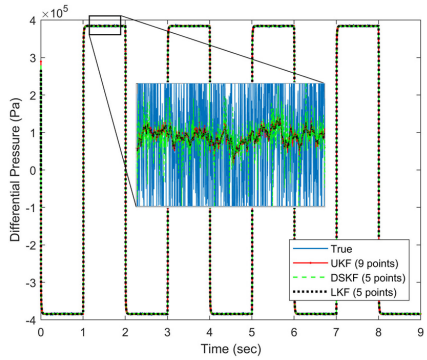
The input signal (the flow rate per volume) is defined by:

$$u_{k-1} = D_p\,\omega_{p,k-1} - \mathrm{sgn}\left(x_{4,k-1}\right) Q_{L0} \quad (43)$$

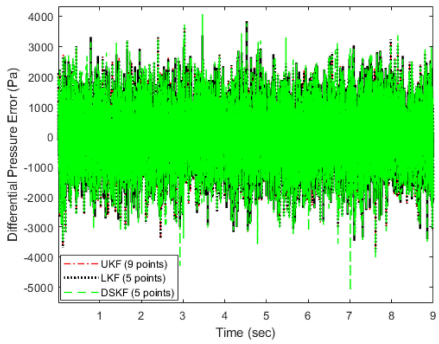
where A_E is the piston cross-sectional area, β_e is the fluid stiffness of the hydraulic circuit, and D_p and ω_p are pump

TABLE I
RMSE RESULTS FOR THE NONLINEAR EHA BENCHMARK ESTIMATION PROBLEM

States	MSS-UKF	SS-UKF	UKF	DSKF	LKF	LKF
Position (m)	5.67×10^{-3}	1.45×10^{-3}	5.32×10^{-5}	2.93×10^{-4}	5.47×10^{-5}	4.58×10^{-5}
Velocity (m/s)	5.83×10^{-4}	3.53×10^{-4}	1.43×10^{-5}	5.16×10^{-5}	2.28×10^{-5}	1.52×10^{-5}
Acceleration (m/s^2)	4.82×10^{-2}	2.99×10^{-3}	6.72×10^{-4}	1.53×10^{-3}	8.01×10^{-4}	7.70×10^{-4}
Diff. Pressure (Pa)	6.227×10^{3}	4.115×10^{3}	1.032×10^{3}	1.962×10^{3}	1.048×10^{3}	1.027×10^{3}
Number of Samples	$d+1=5$	$d+2=6$	$2d+1=9$	5	5	9
Average Time (sec)	1.4727	1.5173	1.5825	1.4586	1.3749	1.4607



(a) Estimated and true differential pressure.



(b) Differential pressure estimation error for each filter.

Fig. 1. Results of applying the LKF and various forms of the UKF on the nonlinear EHA estimation problem. Figure (a) shows the true and estimated differential pressure values (all four filtering methods). Figure (b) shows the differential pressure error over time. Note the UKF and LKF are nearly overlapping.(a) Estimated and true differential pressure. (b) Differential pressure estimation error for each filter.

displacement and pump angular speed, respectively. M refers to the weight of the cylinders, V_0 is the initial volume of the cylinder, L and Q_{L0} are leakage coefficients, and a_1, a_2, and a_3 are friction coefficients [25]. The values of these parameters are presented in Table III of [25] and were determined based on experimentation as per [26]. The linear measurement equation is defined as follows:

$$z_k = Cx_k + v_k \tag{44}$$

where $C = I_{4\times4}$ (4×4 identity matrix). We set the system noise covariance as $Q = diag([10^{-12}\ 10^{-10}\ 10^{-9}\ 10^{6}])$ where $diag$ refers to a 4×4 diagonal matrix [26]. We also defined the measurement covariance as $R = 10^3\ Q$. Note w_k and v_k were defined as white noise with zero means and covariance matrices of Q and R, respectively. For all filters, the initial state estimates were set to zero and the error covariance matrix was set to $10Q$. In order to generate the input signal in (43), the angular speed of the pump ω_p was defined as a $\pm 100\ rad/sec$ square wave.

B. Results and Discussion

In this study, the proposed LKF algorithm was compared with the standard UKF [8], [9], MSS-UKF [14], SS-UKF [15], and DSKF [19]. These methods were implemented to estimate the states of the nonlinear EHA system. The number of sample points generated based on the lattice rule for the LKF was set to $N = 5$ with $a = 7$ (tuned manually). However, the minimum number of points which yielded acceptable estimation error results was $N = 4$. Furthermore, increasing the number of lattice points led to a reduction of the estimation error before being saturated at a specific number of points (see Table I).

Fig. 1 depicts the estimated differential pressure of the proposed LKF (with 5 sample points) along with the corresponding estimation error compared with the standard UKF and DSKF (with 5 sample points). As observed, LKF yielded similar estimation results as the standard UKF. However, with the same number of sample points as LKF, the DSKF method yielded a larger estimation error for the differential pressure estimates (as per Fig. 1.b).

Table I summarizes the root mean squared error (RMSE) between estimated and true values computed for all of the filters based on averaging over 100 Monte Carlo simulations. The RMSE results of the position and differential pressure estimate of the LKF were nearly identical to the UKF, and were found to be significantly better than those of the other filters. Overall, the proposed LKF provided more accurate estimates than the DSKF (with the same number of sample points). It also gave similar results as the UKF but with a lower number of sample points. This decreases the computational complexity of the LKF (see Table I), which would be more noticeable for high-dimensional systems in other real-world problems. Note that LKF robustness, similar to UKF, declines against uncertainties.

V. CONCLUSION

In this paper, we proposed a new filtering algorithm called the lattice Kalman filter (LKF). The proposed filter is called LKF because it utilizes a Korobov type (rank-1) lattice rule to deterministically generate low-discrepancy points to compute Gaussian-weighted multi-dimensional integrals in the nonlinear Kalman filtering context. In order to evaluate the accuracy of the proposed LKF, it was applied to a nonlinear estimation problem. The results were compared with four other estimation methods: DSKF, UKF, MSS-UKF, and SS-UKF. In this case, the results demonstrated that the LKF outperformed the DSKF and the reduced UKFs. The LKF had similar performance to the standard UKF. However, the LKF required fewer sample points (5 compared to 9) which reduces computational complexity and burden. This advantage provides us with the opportunity to apply the LKF to high-dimensional systems while minimizing computation complexity issues. Future work will study higher-order systems as well as methods for improved robustness and stability to uncertain systems and external disturbances.

REFERENCES

[1] H. H. Afshari, S. A. Gadsden, and S. Habibi, "Gaussian filters for parameter and state estimation: A general review of theory and recent trends," *Signal Process.*, vol. 135, pp. 218–238, Jun. 2017.

[2] M. Lundgren, "Bayesian filtering for automotive applications," Ph.D. dissertation, Dept. Signals Syst.,Chalmers Univ. Technol., 2015.

[3] H. Fang, N. Tian, Y. Wang, M. Zhou, and M. A. Haile, "Nonlinear Bayesian estimation: From Kalman filtering to a broader horizon," *IEEE/CAA J. Automatica Sinica*, vol. 5, no. 2, pp. 401–417, Mar. 2018.

[4] P. Stano, Z. Lendek, J. Braaksma, R. Babuška, C. De Keizer, and A. J. Den Dekker, "Parametric Bayesian filters for nonlinear stochastic dynamical systems: A survey," *IEEE Trans. Cybern.*, vol. 43, no. 6, pp. 1607–1624, Dec. 2013.

[5] Y. Bar-Shalom, X.-R. Li, and T. Kirubarajan, *Estimation With Applications to Tracking and Navigation: Theory, Algorithms and Software*. New York, NY, USA: Wiley, 2001.

[6] M. Roth, "Advanced Kalman filtering approaches to Bayesian state estimation," Linköping Univ. Electron. Press, 2017.

[7] M. Roth, G. Hendeby, and F. Gustafsson, "Nonlinear Kalman filters explained: A tutorial on moment computations and sigma point methods," *J. Adv. Inf. Fusion*, 2016.

[8] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation," *Proc. IEEE*, vol. 92, no. 3, pp. 401–422, Mar. 2004.

[9] S. Julier, J. Uhlmann, and H. F. Durrant-Whyte, "A new method for the nonlinear transformation of means and covariances in filters and estimators," *IEEE Trans. Autom. Control*, vol. 45, no. 3, pp. 477–482, Mar. 2000.

[10] I. Arasaratnam and S. Haykin, "Cubature Kalman smoothers," *Automatica*, vol. 47, no. 10, pp. 2245–2250, Oct. 2011.

[11] I. Arasaratnam, S. Haykin, and T. R. Hurd, "Cubature Kalman filtering for continuous-discrete systems: Theory and simulations," *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 4977–4993, Oct. 2010.

[12] T. S. Schei, "A finite-difference method for linearization in nonlinear estimation algorithms," *Automatica*, vol. 33, no. 11, pp. 2053–2058, Nov. 1997.

[13] M. Nørgaard, N. K. Poulsen, and O. Ravn, "New developments in state estimation for nonlinear systems," *Automatica*, vol. 36, no. 11, pp. 1627–1638, Nov. 2000.

[14] S. J. Julier and J. K. Uhlmann, "Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations," in *Proc. Amer. Control Conf.*, vol. 2, pp. 887–892, 2002.

[15] S. J. Julier, "The spherical simplex unscented transformation," in *Proc. Amer. Control Conf.*, 2003, vol. 3, pp. 2430–2434.

[16] R. Radhakrishnan, A. Yadav, P. Date, and S. Bhaumik, "A new method for generating sigma points and weights for nonlinear filtering," *IEEE Control Syst. Lett.*, vol. 2, no. 3, pp. 519–524, Jul. 2018.

[17] Z. Wang, X. Wang, Y. Liang, and F. Yang, "A new Sigma-point filter–uniform random sampling Kalman filter," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 3853–3858, Jul. 2017.

[18] J. Dunik, O. Straka, and M. Šimandl, "Stochastic integration filter," *IEEE Trans. Autom. Control.*, vol. 58, no. 6, pp. 1561–1566, Jun. 2013.

[19] D. Guo and X. Wang, "Quasi-Monte Carlo filtering in nonlinear dynamic systems," *IEEE Trans. Signal Process.*, vol. 54, no. 6 , pp. 2087–2098, Jun. 2006.

[20] D. Nuyens, *Fast Construction of Good Lattice Rules*, Apr. 2007.

[21] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*. Philadelphia, PA, USA: SIAM, 1992.

[22] I. H. Sloan and S. Joe, *Lattice Methods for Multple Integration*, New York, NY, USA: Oxford Univ. Press

[23] R. Cranley and T. N. L. Patterson, "Randomization of number theoretic methods for multiple integration," *SIAM J. Numer. Anal.*, vol. 13, no. 6, pp. 904–914, Dec. 1976.

[24] S. G. Henderson, B. A. Chiera, and R. M. Cooke, "Generating 'dependent' quasi-random numbers," in *Winter Simul. Conf. Proc.*, vol. 1, 2000, pp. 527–536.

[25] S. A. Gadsden and M. Al-Shabi, "The sliding innovation filter," *IEEE Access*, vol. 8, pp. 96129–96138, 2020.

[26] S. A. Gadsden, Y. Song, and S. R. Habibi, "Novel model-based estimators for the purposes of fault detection and diagnosis," *IEEE/ASME Trans. Mechatron.*, vol. 18, no. 4, pp. 1237–1249, Aug. 2013.