MAGNETIC MONOPOLES AND DYONS: THE LOW-ENERGY PERSPECTIVE

MAGNETIC MONOPOLES AND DYONS: THE LOW-ENERGY PERSPECTIVE

By SARA BOGOJEVIĆ, B.Sc., M.Sc.

A Thesis Submitted to the School of Graduate Studies in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2024)

Hamilton, Ontario, Canada (Physics and Astronomy)

TITLE:	Magnetic Monopoles and Dyons: The Low-Energy Perspective
AUTHOR:	Sara Bogojević B.Sc. (University of Belgrade) M.Sc. (University of Waterloo)
SUPERVISOR:	Dr. Cliff Burgess
NUMBER OF PAGES:	ix, 143

Lay Abstract

This thesis discusses magnetic monopoles and dyons – hypothetical particles carrying non-zero magnetic charge – which have long been thought to exist but have thus far proved elusive in experiments. These particles are generally predicted by a class of theories that could very well describe physics at energies beyond the reach of modern-day accelerators. Monopoles and dyons are rare examples of high-energy predictions that could be tested experimentally, since they catalyze certain reactions that were naively expected to proceed with rates suppressed by the large monopole or dyon mass. In this thesis, we explain how the phenomenon of monopole and dyon catalysis can be understood from the perspective of effective field theory. We further lay the groundwork for classifying the dominant low-energy interactions of these hypothetical particles with the known elementary particles, which could be used to inform future experiments.

Abstract

This thesis uses the framework of point-particle effective field theory (PPEFT) to describe the interactions of magnetic monopoles and dyons with low-energy relativistic fermions. Our main goal in doing so is to reconcile the apparent inconsistency between the decoupling principle – which states that short-distance physics decouples from long-distance observables – and the famous observation that monopole-fermion (or dyon-fermion) scattering need not be suppressed by the heavy monopole or dyon mass. We further use this effective field theory description to explore the long-distance complications associated with polarizing the fermionic vacuum exterior to a dyon and show in some circumstances how our methods can simplify calculations of low-energy fermion-dyon scattering in their presence. We propose an effective Hamiltonian governing how dyon excitations respond to fermion scattering in terms of a time-dependent vacuum angle and outline open questions remaining in its microscopic derivation. Although we predominantly focus on the simplest examples of monopole and dyon solutions, our methods lay the foundation for describing how more realistic monopoles and dyons – those arising in Grand Unified Theories – couple to Standard Model fields.

Acknowledgments

First and foremost, I would like to thank my supervisor, Dr. Cliff Burgess, for his guidance throughout my PhD and for his willingness to spend so much time in various discussions with me. Thank you for always pushing me to understand all aspects of a problem and for the many things you've taught me over the years.

To the members of my supervisory committee – Dr. Sung-Sik Lee and Dr. Duncan O'Dell – thank you for all the insightful comments and suggestions you've given on my research.

I'm also very grateful to all of the people at Perimeter Institute who have helped to make my time there so enjoyable – from the staff, who have always been eager to help with any problems, to the many many wonderful graduate students and postdocs I've met at PI. I'd especially like to single out Aiden, Juani and Natalie, without whom COVID lockdowns would have been significantly less fun than they turned out to be.

To my family – Mom, Dad and Teodora – thank you for always being there for me and for all your support and encouragement.

Finally, I would like to thank Rami for too many things to mention here, but most of all for being by my side all this time.

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Notation and Conventions

Notation

When it is necessary to differentiate between vectors acting in physical and gauge space, we denote the former in boldface *e.g.* $\hat{\boldsymbol{r}}$ and the latter with arrows *e.g.* $\vec{T}$.

Conventions

Signature	We use the 'mostly plus' signature, for which $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$.
Units	We work in fundamental units, for which $\hbar = c = 1$.
Gamma matrices	When necessary, we use the chiral basis of gamma matrices. See Appendix A for further details.

Declaration of Academic Achievement

I, Sara Bogojević, declare that this thesis titled "Magnetic Monopoles and Dyons: The Low-Energy Perspective" and the work presented therein are my own. The material presented in chapters §2 to §4 contains research from [1] which was authored by myself and Dr. Cliff Burgess. The idea for this work was conceived by Dr. Burgess, while almost all of the calculations have been performed by myself.

Chapter 1

Introduction

Magnetic monopoles boast a rich and extensive history in physics. Remarkably, the idea that these hypothetical particles exist has proved so appealing that they have been studied for close to a century, despite a lack of observational evidence.

Interest in this topic was first sparked by Dirac [2] who showed that some monopoles – those whose magnetic charges q_{M} satisfy the relation $2q_e\, q_{M} \in \mathbb{Z}$, where q_e is any allowed electric charge – can be consistently described in quantum theory. The condition imposed on q_{M}, known as the *Dirac quantization condition*, suggests a surprising solution to the problem of why observed electric charges are quantized: If a single magnetic monopole exists in nature, all electric charges are forced to be integer multiples of an elementary unit[1]. This realization, along with its implications for the duality between electricity and magnetism [3–6], made for a compelling case in favour of monopoles.

The modern perspective is that electric charge quantization likely has a different origin. One popular potential explanation of this phenomenon is that it is a consequence of the unification of the electromagnetic, weak and strong forces at high energies. To understand where this idea comes from and how it relates to monopoles, we first consider the low-energy (or large-distance) limit, in which fundamental interactions are well understood. At energies below[2] $\sim 246\,\mathrm{GeV}$, all known interactions between subatomic particles with the exception of gravity are mediated by the electromagnetic, weak

[1] A monopole of magnetic charge $q_{M} = (2e)^{-1}$ could explain why *most* observed electric charges are integer multiples of the electron charge, $-e$. The only known exceptions to this are quarks, whose charges are nonetheless consistent with a generalized version of the Dirac quantization condition which we discuss later.

[2] Or equivalently for distances larger than $\sim 10^{-16}\,\mathrm{cm}$.

and strong forces and can be described within the Standard Model of particle physics (see *e.g.* [7] for a review). The Standard Model (SM) builds on the already established theories of quantum electrodynamics and Fermi's theory of weak interactions and has correctly predicted the existence and properties of several new particles (such as the W, Z and Higgs bosons and the top and charm quarks), as well as the form of weak neutral current interactions. Practically all particle physics experimental data agrees with this theory[3], which makes it tremendously successful [9].

Nevertheless, the Standard Model is likely not a comprehensive theory of non-gravitational fundamental interactions as several of its features appear to be fairly arbitrary. For instance, it describes the electroweak and strong interactions as a gauge theory governed by the group $SU_c(3) \times SU_L(2) \times U_Y(1)$ but offers no explanation for the complicated form of this group. Some of the Model's other drawbacks are that it depends on nineteen free parameters, which is considered excessively large for a fundamental theory, and that it does not explain the observed baryon asymmetry in nature [10] or the quantization of electric charge[4]. These (and other) reasons have led to the belief that the Standard Model is an effective field theory (EFT) [11–13] (see [14] for a review of EFTs) that is the low-energy limit of some other, more fundamental theory.

This 'new' theory is generally expected to have a larger symmetry group than the Standard Model at high energies since the presence of additional symmetries could constrain some of the Model's arbitrary features [15]. Although there is no consensus on what the appropriate high-energy theory should be, a promising set of candidates can be constructed by taking an idea already used in the Standard Model to its natural limit. The idea in question is the mechanism of spontaneous symmetry breaking [16–18], which explains how electromagnetism arises in the Standard Model, but could equally be used to explain how the gauge group $SU_c(3) \times SU_L(2) \times U_Y(1)$ is obtained from a larger group G, which unifies all three of the SM interactions. Theories in which this type of unification occurs are called Grand Unified Theories (GUTs) [19, 20], and their supersymmetric variants are currently believed to be the most likely candidates for describing beyond-SM physics [9]. In addition to explaining the origin of the SM gauge group, GUTs often propose solutions to some other unexplained aspects of the Standard Model. In particular, if the GUT gauge group G

[3]The observed phenomenon of neutrino oscillations [8] is not consistent with the usual formulation of the Standard Model. This does not pose a significant problem, however, since neutrino masses and so also neutrino oscillations can be accounted for by minimal extensions to the theory.

[4]Under certain assumptions, charge quantization can be explained by the cancellation of anomalies in the SM.

is simple, electric charge is guaranteed to be quantized since the generators of G, and so also the electric charge operator, will have discrete eigenvalues [21].

The arguments of the above paragraph pose a problem since they appear to undermine our initial motivation for studying magnetic monopoles. Fortunately, it turns out that the very same conditions that lead to the quantization of electric charge also imply the existence of magnetic monopoles [22, 23]. Monopoles as well as dyons – particles which carry both electric and magnetic charge – arise in GUTs as stable, static, finite-energy solutions to the GUT field equations [24, 25] (see [26–28] for reviews). These particles are generally expected to be very small and superheavy, with a core size of order $R \sim m_g^{-1}$ where $m_g \sim 10^{14}\,\text{GeV}$ is the unification scale[5] [29], and masses of order $M \sim 4\pi\, m_g/g^2$, where g is the GUT gauge coupling. Since modern-day accelerators can only reach energies of up to $10^4\,\text{GeV}$, producing GUT monopoles and dyons in collider experiments appears to be next to impossible.

The huge difference in magnitude between the minimum energy of a monopole and accessible energies in modern-day experiments presents a significant problem for monopole searches. In a scenario with such a large hierarchy of scales, the *decoupling principle* – the statement that very little needs to be known about short-distance physics in order to calculate long-distance observables – implies that the interactions between the heavy and light degrees of freedom are highly suppressed by the heavy scale [30]. This suggests that the usual difficulties encountered when testing new predictions of high-energy theories apply in the case of monopoles as well, leaving us with little hope of being able to detect them. Remarkably, there is good reason to believe that this expectation fails for monopoles and dyons. As shown by Rubakov and Callan [31, 32], monopole-fermion scattering is not necessarily suppressed by the GUT scale and can instead proceed with strong-interaction cross sections. This observation suggests that monopoles and dyons could catalyze proton or nucleon decay at strong-interaction rates, since their interactions with fermions generally violate the conservation of baryon number[6].

For this reason, monopole-fermion (and dyon-fermion) scattering is often said to violate the

[5] The GUT scale estimate in [29] relies on the desert hypothesis *i.e.* the assumption that no new physics appears between the electroweak and GUT scales, with the potential exception of GUT multiplets containing only superheavy $m \sim \Lambda_{GUT}$ particles or containing only particles with masses of order $m \sim 10\,\text{GeV}$. An even higher value of $\sim 10^{16}\,\text{GeV}$ is expected for supersymmetric GUTs [9].

[6] Baryon number is conserved in the Standard Model and this guarantees that the proton, being the lightest baryon, is stable.

decoupling principle. This would be extremely surprising if true since decoupling is thought to hold generally in nature, as evidenced by examples across fields ranging from atomic to gravitational physics. For instance, the energy levels of an electron in an atom depend on the nuclear mass and charge but not on the details of how the nucleus is held together by the strong force; the structure of atoms is largely irrelevant for describing the gravitational motion of planets and stars *etc.*

One of the objectives of this thesis is to show that the decoupling principle still holds in the case of monopoles and dyons. We do this by constructing an effective field theory (the derivation of which hinges on the validity of the decoupling principle) describing low-energy dyon-fermion interactions, which successfully reproduces the relevant results from the literature.

The specific ‘flavour’ of effective field theory we use is called point-particle effective field theory (PPEFT), which is adapted to situations in which a massive, small object interacts with particles whose wavelengths are much larger than the point-like particle’s radius. PPEFT techniques [33–35] have been tested extensively in applications for which answers are known by other methods, by using it to describe the influence of finite nuclear size (and various nuclear moments) for atomic energy levels [36–38], absorption by hot wires in atom traps [39] and to describe the gravitational back-reaction of codimension-two branes in various dimensions [40, 41].

The goals of this thesis are as follows: (*i*) to resolve the apparent contradiction between the large cross sections for fermion-monopole (and fermion-dyon) scattering and the decoupling principle, (*ii*) to understand what sets monopoles and dyons apart from *e.g.* nuclei, which are described by similar EFTs but which predict analogous cross sections that are suppressed by the relevant heavy scale, (*iii*) to develop a formalism that can be used to classify the dominant interactions of a GUT magnetic monopole with low-energy Standard Model fermions, so that contact can be made with present and future experiments.

We wrap up the introduction with a short overview of the current status of monopole searches. The supermassive GUT monopoles and dyons we discuss in this thesis could only have been produced in the early universe, as no present-day process is energetic enough to create them [26]. A potential monopole detected on Earth would then necessarily have originated from cosmic rays and can be measured either directly in the ray or indirectly if it binds to any matter. Several techniques for detecting monopoles have been developed and these often make use of the monopoles’ electromagnetic properties *e.g.* by attempting to measure the current a moving monopole induces in a superconducting ring [42]. Alternatively - scintillators, gas chamber detectors and others can be used to exploit the fact that monopoles lose electromagnetic energy at a much faster rate than known particles. Other approaches also exist, such as those that look for evidence of the catalysis of nucleon decay, as predicted by the Callan-Rubakov effect. While no detection of a monopole or dyon has been confirmed to date[7], searches are still ongoing at several experiments at the LHC, the IceCube experiment and others (see for example [44, 45] and [9] for a recent review).

We now briefly introduce some relevant background related to magnetic monopoles, dyons as well as effective field theory techniques before moving on to the main results.

1.1 Magnetic monopoles and dyons

Magnetic monopoles are particles that source a radial, Coulomb-like magnetic field of the form

$$\mathbf{B}_{M} = \frac{q_{M}}{r^2}\,\hat{\boldsymbol{r}}, \tag{1.1.1}$$

where q_M is the particle's magnetic charge. At first glance, this magnetic field appears to be inconsistent with the potential formulation of electrodynamics – and so also with quantum electrodynamics – since the definition of the magnetic field in terms of the vector potential, $\mathbf{A}$, implies $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$.

As pointed out by Dirac [2], the above argument does not rule out the existence of monopoles. To see why, consider a vector potential of the form

$$\mathbf{A}_{D} = \frac{q_{M}(1-\cos\theta)}{r\sin\theta}\,\hat{\boldsymbol{\phi}}, \tag{1.1.2}$$

also known as the *Dirac potential*, which corresponds to precisely the desired magnetic field $\mathbf{B}_M$ everywhere except the $\theta = \pi$ half-line, along which the vector potential diverges. A careful treatment of the half-line singularity shows that it produces a magnetic field of its own, corresponding to the field of an infinitesimally thin, never-ending solenoid for all $z < 0$. The total magnetic field is then

[7] A single candidate event of a monopole detection has been reported [43], but has not been reproduced since.

given by

$$\mathbf{B}_{D} = \mathbf{B}_{M} + \mathbf{B}_s = \frac{q_{M}}{r^2}\,\hat{\mathbf{r}} - 4\pi q_{M}\,\Theta(-z)\delta(x)\delta(y)\,(-\hat{\boldsymbol{z}}), \tag{1.1.3}$$

where $\mathbf{B}_s$ is the field of the string (or half-line). The Dirac potential avoids any inconsistency with Maxwell's equations since the flux of the half-line precisely cancels the flux of the monopole field

$$\frac{1}{4\pi}\int \mathbf{B}\cdot \mathrm{d}^2\mathbf{S} = q_{M} - q_{M} = 0, \tag{1.1.4}$$

but this was achieved by introducing a seemingly unphysical term in the magnetic field.

Thus far, the potential (1.1.2) does not appear to be very useful since it is both singular and describes a semi-infinitely long and infinitesimally thin solenoid as opposed to a monopole. Both of these problems would be resolved if the singular string were somehow unobservable and this would be the case if its contribution to the Aharonov-Bohm phase [46] acquired by a particle of electric charge q_e traversing a closed path enclosing the string is a multiple of 2π. The Aharonov-Bohm phase due to the string, ϑ^s_{A-B}, must then be given by

$$\vartheta^s_{A-B} \coloneqq q_e \int_{\mathcal{R}} \mathbf{B}_s \cdot \mathrm{d}^2\mathbf{S} = -4\pi\, q_e\, q_{M} = 2\pi n, \tag{1.1.5}$$

where $\mathcal{R}$ is the region enclosed by the path in question and $n \in \mathbb{Z}$. This leads us to the famous Dirac quantization condition:

$$2q_e\, q_{M} \in \mathbb{Z}. \tag{1.1.6}$$

An alternative prescription, due to Wu and Yang, avoids the singularity altogether at the expense of introducing non-trivial topology into the problem [47]. This is done by defining the vector potential separately in two overlapping coordinate patches:

$$\mathbf{A}^{\pm}_{D} = \frac{q_{M}(\pm 1 - \cos\theta)}{r\sin\theta}\,\hat{\boldsymbol{\phi}}, \tag{1.1.7}$$

where the labels $\pm$ refer to the two regions of space $R_{\pm}$, defined as

$$R_+ : \left\{0 \le \theta < \frac{\pi}{2} + \delta,\, r > 0,\, 0 \le \phi < 2\pi\right\}, \tag{1.1.8}$$

as well as

$$R_- : \left\{ \frac{\pi}{2} - \delta < \theta \le \pi,\, r > 0,\, 0 \le \phi < 2\pi \right\}, \tag{1.1.9}$$

and the parameter δ satisfies $0 < \delta \le \frac{\pi}{2}$ and is otherwise arbitrary. In the overlapping region, the vector potentials $\mathbf{A}_D^{\pm}$ must be equal, up to a gauge transformation. The gauge function corresponding to the desired transformation is given by $e^{i\omega(\phi)} = e^{2iq_e q_M \phi}$, where q_e is the minimal allowed electric charge in the theory, as can be seen from

$$(A_D^+)_\phi - (A_D^-)_\phi = -\frac{i}{q_e r \sin\theta} e^{-i\omega(\phi)} \partial_\phi e^{i\omega(\phi)} = \frac{2q_M}{r \sin\theta}. \tag{1.1.10}$$

The Dirac quantization condition again emerges in this alternative formulation by requiring the gauge function $e^{i\omega(\phi)}$ to be single-valued.

When working in the Wu-Yang formalism, the wave function of a particle in the monopole background[8] becomes a *section* as opposed to an ordinary function. This implies that the wave function must be defined such that its explicit forms in the regions R_+ and R_-, which we denote ψ_+ and ψ_- respectively, are related in the overlap region by a gauge transformation, given in this instance by

$$\psi_+(x) = e^{2iq_\psi q_M \phi} \psi_-(x), \quad \text{for} \quad x \in R_+ \cap R_- \tag{1.1.11}$$

where q_ψ is the particle electric charge.

Nonabelian gauge theory monopoles: An $SU(2)$ toy model

The simplest example of a nonabelian gauge theory that admits magnetic monopole and dyon solutions is the $SU(2)$ Georgi-Glashow model [48]. Although the low-energy limit of this theory disagrees with the Standard Model, it is nonetheless a useful starting point since GUT monopole solutions can often be constructed by embedding the monopole solutions of this model into the GUT gauge group.

The $SU(2)$ Georgi-Glashow model is an $SU(2)$ gauge theory coupled to a Higgs field in the

[8] Or more generally, any solution to the equation of motion of a particle in the background of a monopole or dyon.

adjoint representation. This theory is described by the action

$$S = - \int \mathrm{d}^4 x \left[\frac{1}{4} F^a_{\mu\nu} \, F^{\mu\nu}_a + \frac{1}{2} D_\mu \Phi^a \, D^\mu \Phi_a - \frac{\mu^2}{2} \, \Phi^a \, \Phi_a + \frac{\lambda}{4} \, (\Phi^a \, \Phi_a)^2 \right] , \tag{1.1.12}$$

where the nonabelian field strength is $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e \, \epsilon^{abc} A_{b\mu} \, A_{c\nu}$ and the covariant derivative is given by $(D_\mu \Phi)_a = \partial_\mu \Phi_a + \epsilon_{abc} \, e A^b_\mu \, \Phi^c$. The gauge generators are defined in terms of the Pauli matrices by $T_a = \frac{1}{2} \tau_a$ and so satisfy the $SU(2)$ commutation relation $[T_a, T_b] = i \epsilon_{abc} T_c$. The model's parameters λ, e and μ^2 are all real and positive and the choice $\mu^2 > 0$ ensures the vacuum expectation value $w^a := \langle \Phi^a \rangle$ satisfies $w^2 := w^a w_a = \mu^2 / \lambda \neq 0$ so that the gauge symmetry is spontaneously broken down to $U(1)$.

With respect to the gauge field gauging rotations in *e.g.* the T_3 direction, we see that the two gauge bosons spanned by A^1_μ and A^2_μ have charge $\pm e$, while the components of the adjoint Higgs multiplet Φ^a similarly carry charges zero and $\pm e$ for this symmetry. The two charged gauge bosons 'eat' the two charged fields in Φ^a and by doing so acquire nonzero mass $m_g \simeq ew = \beta \mu$, where $\beta^2 := e^2 / \lambda$. The remaining spin-one particle is massless and so can be regarded as the 'photon' that gauges the unbroken $U(1)$ symmetry. The uneaten physical scalar is also neutral under the unbroken gauge symmetry and has mass $m_s \simeq \sqrt{2\lambda} \, w = \sqrt{2} \, \mu$.

The Julia-Zee dyon

The $SU(2)$ Georgi-Glashow model predicts the existence of magnetic monopoles and dyons at the classical level, as solutions to the model's field equations. One such solution is the so-called Julia-Zee dyon [24, 49, 47] which has the form

$$e \mathcal{A}_i = (\hat{\boldsymbol{r}} \times \vec{T})_i \left[\frac{1 - \mathcal{K}(r)}{r} \right] , \quad e \mathcal{A}_0 = \hat{\boldsymbol{r}} \cdot \vec{T} \, \frac{\mathcal{J}(r)}{r} \quad \text{and} \quad e \varphi = \hat{\boldsymbol{r}} \cdot \vec{T} \, \frac{\mathcal{H}(r)}{r} , \tag{1.1.13}$$

where the dimensionless functions $\mathcal{K}(r)$, $\mathcal{H}(r)$ and $\mathcal{J}(r)$ depend on r only through the combination μr. These functions satisfy second-order coupled ordinary differential equations given explicitly in [49] that depend on the other two parameters of the model, e and λ, only through the ratio $e^2 / \lambda = \beta^2$.

The integration constants are chosen to ensure boundedness at the origin — and so $\mathcal{K} \to 1$ and

$\mathcal{J}, \mathcal{H} \to 0$ as $r \to 0$. They also have a finite-energy falloff to the vacuum solution as $r \to \infty$, implying the asymptotic behaviour

$$\mathcal{K}(r) \to 0\,, \quad \mathcal{J}(r) \to e(Q - vr) \quad \text{and} \quad \mathcal{H}(r) \to hr \quad \text{as } r \to \infty\,, \tag{1.1.14}$$

where $h = ew = \beta\mu$ and only one of Q or v is an independent parameter because one combination of Q and v is a calculable function of β and μ. The approach to these asymptotic forms is exponential in μr. For instance, the function $\mathcal{K}(r)$ behaves for large r as $\mathcal{K}(r) \sim e^{-ar}$ with $a = \sqrt{h^2 - (ev)^2}$, which shows the solutions damp exponentially provided $h > ev$. The Julia-Zee solution then describes a localized field configuration, in the sense that it differs appreciably from the vacuum solution only inside a region of size $R \sim \mu^{-1}$.

The stability of this field configuration is guaranteed by topological arguments [24, 50]. Specifically, the large r form of the Higgs field defines a map from the sphere at spatial infinity, S^2_∞, to the sphere $S^2 := \{\varphi : \varphi^a \varphi_a = w^2\}$ in gauge space. Any such map is characterized by its *winding number*, defined as

$$n_{\rm w} = \frac{1}{8\pi w^3} \int_{S^2_\infty} {\rm d}^2 S_i \, \epsilon^{ijk} \epsilon_{abc} \, \varphi^a \, \partial_j \varphi^b \, \partial_k \varphi^c, \tag{1.1.15}$$

which counts the number of times the sphere S^2 is covered after a single turn around S^2_∞ and is necessarily an integer. The winding number of a field configuration is a topological invariant – *i.e.* it does not change under smooth deformations of the field – and this explains why the Julia-Zee solution (for which $n_{\rm w} = 1$) does not decay to the vacuum of the theory (for which $n_{\rm w} = 0$). As shown in [50], $n_{\rm w}$ is related to the dyon's magnetic charge through $q_{\scriptscriptstyle M} = n_{\rm w}/e$. This is yet another manifestation of the Dirac quantization condition, which we explicitly confirm is satisfied by calculating $q_{\scriptscriptstyle M}$ later in this section.

The functions $\mathcal{K}(r), \mathcal{J}(r)$ and $\mathcal{H}(r)$ must generally be determined numerically, but in the special case where $\lambda, e \to 0$ with $\beta^2 = e^2/\lambda$ and μ fixed, these functions can be solved in closed form [51]. In this *Prasad-Sommerfield* limit the solution that is regular at the origin and for which $\varphi^a \varphi_a$

approaches a constant[9] at infinity is given explicitly by

$$\mathcal{K}(r) = \frac{cr}{\sinh(cr)} \qquad \text{and} \qquad \mathcal{J}(r) = \sinh\varpi \Big[1 - cr\coth(cr)\Big], \tag{1.1.16}$$

as well as

$$\mathcal{H}(r) = \cosh\varpi \Big[cr\coth(cr) - 1\Big], \tag{1.1.17}$$

with ϖ an arbitrary real constant and $c = \beta\mu$. Comparing with the large-r limit of $\mathcal{J}(r)$ shows that v and Q are given in terms of these constants by $ev = c\sinh\varpi$ and $eQ = \sinh\varpi$ and so $Q = v/c$.

At any particular spacetime point the nonzero fields break the $SU(2)$ gauge invariance down to a $U(1)$ subgroup, so just like for the vacuum the dyon preserves a $U(1)$ symmetry and as a result one of the gauge modes – the photon – remains precisely masses. For the dyon, the particular embedding of the unbroken $U(1)$ within $SU(2)$ varies from place to place, as can be seen by the mixing of gauge and spatial indices in (1.1.13). This makes it convenient to change gauge to a form for which the massless gauge mode corresponds to the same gauge direction everywhere in spacetime.

If the asymptotic gauge field is chosen to point along the third direction in $SU(2)$ space, the required gauge transformation is

$$U(\boldsymbol{r}) = \frac{1}{\sqrt{2}}\left[\sqrt{1 - \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}}} + \frac{i\vec{\tau}\cdot(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}})}{\sqrt{1-\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}}}}\right]. \tag{1.1.18}$$

This is a singular gauge transformation inasmuch as it introduces a previously non-existent singularity into the asymptotic vector potential if it is performed everywhere. Because the singularity is a gauge artefact we can remove it by following the Wu-Yang prescription [47], described below equation (1.1.6).

That is, if the two overlapping regions R_+ and R_- are given by (1.1.8) and (1.1.9), we define the gauge field in region R_- using the gauge function $U(\boldsymbol{r})$ while the gauge field in R_+ is additionally transformed using $V(\boldsymbol{r}) = e^{i\phi\,\tau_3}$. With this choice of gauge, the Julia-Zee dyon field configuration is given by

$$e\mathcal{A}^{a\pm}_i(\boldsymbol{r}) = \frac{1}{r}\left[\frac{\pm 1 - \cos\theta}{\sin\theta}\,\hat{\phi}_i\,\delta^a_3 - \mathcal{K}(r)\,\zeta^a_{i\pm}\right] \quad \text{and} \quad e\mathcal{A}^a_0(\boldsymbol{r}) = -\frac{\mathcal{J}(r)}{r}\,\delta^a_3, \tag{1.1.19}$$

[9]When $\lambda \to 0$ it is no longer necessary to require $\mathcal{H}/r$ to approach the specific constant h as $r \to \infty$.

as well as

$$e\Phi^a(\boldsymbol{r}) = -\frac{\mathcal{H}(r)}{r}\,\delta^a_3. \tag{1.1.20}$$

In the above, $\zeta^a_{i\pm}$ is defined through

$$\boldsymbol{\zeta}_{i\pm} := \frac{1}{2}\,\zeta^a_{i\pm}\tau_a = \frac{1}{2}\left(i\hat{\theta}_i - \hat{\phi}_i\right)e^{\pm i\phi}\tau_+ - \frac{1}{2}\left(i\hat{\theta}_i + \hat{\phi}_i\right)e^{\mp i\phi}\tau_-, \tag{1.1.21}$$

and arises as the gauge transform of $\frac{1}{2}\hat{\boldsymbol{r}} \times \vec{\tau}$. Here the subscripts $\pm$ refer to the regions $R_\pm$ and $\tau_\pm$ are defined as usual by $\tau_\pm := \frac{1}{2}(\tau_1 \pm i\tau_2)$.

The asymptotic large-r form of the solution in this gauge makes its electromagnetic properties explicit,

$$e\mathcal{A}^{a\pm}_i \to \frac{\pm 1 - \cos\theta}{r\sin\theta}\hat{\phi}_i\,\delta^a_3 \qquad \text{and} \quad e\mathcal{A}^a_0 \to \left(ev - \frac{eQ}{r}\right)\delta^a_3 \qquad \text{as} \quad r \to \infty\,, \tag{1.1.22}$$

since using this in Gauss' law

$$q_{\scriptscriptstyle E} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{E}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = Q \quad \text{and} \quad q_{\scriptscriptstyle M} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{B}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = \frac{1}{e}, \tag{1.1.23}$$

reveals it to be a dyon that carries magnetic charge $q_{\scriptscriptstyle M} = 1/e$ and electric charge $q_{\scriptscriptstyle E} = Q$. Once fermions in the fundamental representation are added to the theory (as in later chapters), the Julia-Zee dyon manifestly satisfies the Dirac quantization condition $2q_{\scriptscriptstyle M}q_e \in \mathbb{Z}$, since the doublet components have electric charge $q_e = \pm e/2$.

The parameter v is the electrostatic potential difference between the origin and infinity, which in the general case is not independent of Q, with

$$ev = \mu\,\mathcal{Z}\left(\beta, eQ\right)\,, \tag{1.1.24}$$

for a calculable order-unity function $\mathcal{Z}$. (For instance in the Prasad-Sommerfield limit (1.1.16), (1.1.17) we have $\mathcal{Z} = eQ\beta$.) The monopole limit $Q \to 0$ corresponds to taking $v \to 0$ so $\mathcal{Z}(\beta, 0) = 0$. Notice that the asymptotic form of the vector potential in this gauge, given by (1.1.22), exactly matches the Wu-Yang potential (1.1.7), which shows that the monopole limit of the Julia-Zee solution is equivalent to the Dirac monopole at large distances from the origin.

The classical dyon mass is given by evaluating the energy of the classical solution:

$$M \simeq \frac{4\pi\beta\mu}{e^2}\, F\left(\beta, v/\mu\right) \tag{1.1.25}$$

where $m_g \simeq \beta\mu = ew$ is the gauge boson mass, $\alpha = e^2/4\pi$ is the fine-structure constant and F is an explicitly calculable function that is order unity when $e^2 \sim \lambda$ and $v \sim \mu$.

Solitons in quantum field theory

The Julia-Zee dyon is an example of a *soliton*: a solution to the classical field equations of a given theory which is stable, static, localized and has finite energy (see *e.g.* [52] for a review). Soliton field configurations share many of the properties we typically ascribe to particles, yet do not arise in the same way as the elementary particles of a field theory. The implications of soliton solutions for a quantum theory are well-understood in the *semiclassical* limit, which for the Georgi-Glashow model corresponds to the case where the dimensionless couplings e, λ are both taken to be small[10], with $\beta^2 = e^2/\lambda \sim \mathcal{O}(1)$. In this limit, the mass of the dyon satisfies

$$M \sim \mathcal{O}\left(\frac{m_g}{\alpha}\right), \tag{1.1.26}$$

and is related to the dyon size $R \sim m_g^{-1}$ by $MR \sim 4\pi/e^2 \gg 1$. The dyon becomes a nearly classical object, since its Compton wavelength is much smaller than its radius.

The semiclassical limit of a field theory corresponds to the case where the fields are reasonably well described by the soliton configuration, similarly to how a particle can have an approximately classical position when it is in a state in which its position and momentum uncertainties can be treated as small [52]. Because of this, it is convenient to expand the fields around the soliton solution

$$A^a_\mu(x) = \mathcal{A}^a_\mu(x) + \delta\mathcal{A}^a_\mu(x) \quad \text{and} \quad \Phi^a(x) = \varphi^a(x) + \delta\varphi^a(x), \tag{1.1.27}$$

[10]The semiclassical expansion is formally an expansion in $\hbar$. Rescaling the fields in (1.1.12) by e^{-1} shows that every appearance of $\hbar$ in *e.g.* a perturbatively calculated amplitude will be accompanied by a factor of e^2. This means that – in units where $\hbar = 1$ – quantum corrections are suppressed in the small e^2 limit. The rescaling also trades the coupling constant λ for λ/e^2 which implies the semiclassical approximation is only reliable for small λ, with β taken to be fixed in the classical limit.

where $\delta\mathcal{A}^a_\mu(x)$ and $\delta\varphi^a(x)$ are fluctuations around the classical configuration. Before performing this semiclassical expansion, one typically first separates out a subset of the field fluctuations corresponding to symmetry transformations that act non-trivially on the background solution. This is done (for reasons we address later) by parametrizing the soliton solution with a set of *collective coordinates*, so that this 'special' subset of field fluctuations is captured by variations in the collective coordinates instead of $\delta\mathcal{A}^a_\mu(x)$, $\delta\varphi^a(x)$. For the Julia-Zee dyon, the relevant symmetry transformations are translations and global gauge rotations in the unbroken $U(1)$ direction and so the dyon collective coordinates are just its center-of-mass position, $\boldsymbol{y}(t)$, and a charge degree of freedom, $\mathfrak{a}(t)$. These can be regarded as the special cases where A^a_μ and Φ^a are obtained by transforming the dyon by a time-dependent spatial translation or a time-dependent gauge rotation in the unbroken $U(1)$ gauge symmetry direction (generated by τ_3 in the gauge (1.1.19)),

$$\delta\mathcal{A}^a_\mu(x) = y^i\partial_i\mathcal{A}^a_\mu + \partial_\mu y^i\mathcal{A}^a_i + \frac{1}{e}\,\delta^a_3\,\partial_\mu\mathfrak{a} - \mathfrak{a}\,\epsilon^{a3b}\mathcal{A}^b_\mu \quad \text{and} \quad \delta\varphi^a(x) = y^i\partial_i\varphi^a - \mathfrak{a}\,\epsilon^{a3b}\varphi^b, \qquad (1.1.28)$$

The parameters $\boldsymbol{y}(t)$ and $\mathfrak{a}(t)$ become operators describing the dyon degrees of freedom upon quantization.

The above arguments illustrate the fact that soliton solutions have particle counterparts in the quantum theory when the semiclassical limit is applicable. In the remainder of this thesis we work in this limit and so take e^2, λ to be small, with β^2 fixed.

Angular momentum in a monopole background

The spherical gauge used in equation (1.1.13) shows that the Julia-Zee dyon is not invariant under general global gauge or spatial rotations, unless these are performed in unison. As a result, the background dyon breaks the freedom to independently perform gauge and spatial rotations and the angular momentum operator, which characterizes the transformation properties of a spin-$\vec{S}$ particle with respect to rotations, is given by

$$\vec{J} = \vec{L} + \vec{S} + \vec{T}, \qquad (1.1.29)$$

where[11] $\vec{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the usual orbital angular momentum and $\vec{T}$ is the gauge isospin. Defined this way, the angular momentum operator is conserved and satisfies the expected algebra, $[J_i, J_j] = i\epsilon_{ijk}\, J_k$. In the abelian gauge of (1.1.19), the total angular momentum is instead given by

$$\vec{J} = \boldsymbol{r} \times (\boldsymbol{p} - e\boldsymbol{\mathcal{A}}) + \vec{S} - \hat{\boldsymbol{r}}\, T_3, \tag{1.1.30}$$

where $\boldsymbol{\mathcal{A}}$ is the spatial part of the gauge potential in (1.1.19), and the 'extra' term $-\hat{\boldsymbol{r}}\, T_3$ can be interpreted as the angular momentum of the dyon-particle (or monopole-particle) electromagnetic field.

Our primary interest lies in describing how dyons or monopoles interact with isodoublet fermions. As we now show, the additional term in the definition of $\vec{J}$ in (1.1.29) or (1.1.30) implies that isodoublet fermions have integer instead of half-integer angular momentum. Remarkably, a fermion in a monopole/dyon background can then have zero total angular momentum and it is precisely this partial wave that leads to many of the unexpected monopole properties we mention earlier. In the case of an isodoublet fermion, $\vec{T}^2$ and $\vec{S}^2$ both have eigenvalue $\frac{3}{4}$ as appropriate for a spin-half contribution to the total angular momentum. The usual rules for combining angular momenta show that the combination $\vec{S} + \vec{T}$ can either carry spin zero or spin one, and writing the eigenvalues of $\vec{L}^2$ and $\vec{J}^2$ as $\ell(\ell+1)$ and $j(j+1)$ respectively, we see that $j = 0$ can be obtained by combining the spin and 'magnetic' angular momentum with either of the two orbital angular momenta $\ell = 0, 1$.

S-wave states *i.e.* states with vanishing total angular momentum are 'special' because they experience no angular momentum barrier and so can reach the dyon core even at low energies. This can be seen by rewriting the Dirac equation in the $r \to \infty$ limit as follows

$$\begin{aligned} &\gamma^\mu(\partial_\mu - ie\mathcal{A}_\mu)\psi \\ &= \left\{\gamma^0 \left[\partial_t - i\left(\frac{eQ}{r} - ev\right) T_r\right] + \gamma^r \partial_r - \frac{i}{r}\boldsymbol{\gamma} \cdot \left(\hat{\boldsymbol{r}} \times (\vec{L} + \vec{T})\right)\right\} \psi \\ &= \frac{1}{r}\left\{\gamma^0 \left[\partial_t - i\left(\frac{eQ}{r} - ev\right) T_r\right] + \gamma^r \partial_r - \frac{1}{r}\left[\gamma^r + i\boldsymbol{\gamma} \cdot \left(\hat{\boldsymbol{r}} \times (\vec{L} + \vec{T})\right)\right]\right\} (r\psi) = 0, \end{aligned} \tag{1.1.31}$$

where $\boldsymbol{\gamma} \coloneqq \gamma^i \mathbf{e}_i$ and we work in the spherical gauge of (1.1.13) for convenience. The last term in the

[11] We denote the vectors $\vec{L}$, $\vec{S}$ with arrows even though they act in physical space so that the notation is consistent with that used for the gauge isospin, $\vec{T}$.

third line of this equation gives rise to the centrifugal barrier once the Dirac equation is written in a Klein-Gordon like form. This term can be rewritten as[12] $[\gamma^r + i\boldsymbol{\gamma}\cdot(\hat{\boldsymbol{r}}\times(\vec{L}+\vec{T}))]/r = i(\boldsymbol{\gamma}\times\hat{\boldsymbol{r}})\cdot\vec{J}/r$, which implies that the angular momentum barrier vanishes for S-wave states.

The absence of a centrifugal barrier allows for the possibility of dyon (or monopole)-mediated catalysis wherein S-wave fermions scatter off the dyon at rates that are unsuppressed by the dyon's small size. This realization was made by Rubakov and Callan [31, 32] who independently showed that the interactions of a monopole with massless S-wave fermions induce various fermionic condensates – *i.e.* non-zero expectation values of fermion bilinear operators – which fall off as a power law as one moves away from the dyon. Surprisingly, these condensates do not exhibit an exponential suppression by the heavy scale, $e^{-r/R}$, that is typical of correlation functions and this led to the conclusion that the GUT scale need not always suppress low-energy observables. Although the lack of a centrifugal barrier plays a key role, it is clearly not a sufficient condition for catalysis since *e.g.* the scattering of a spinless particle off a nucleus is suppressed by the small nuclear size. We discuss why these two cases are so different in chapter §2.

Grand Unified Theory monopoles and dyons

While realistic Grand Unified Theories generally involve additional complications compared to the $SU(2)$ Georgi-Glashow model, the monopole and dyon sectors of GUTs and the toy model share many similarities. This is due to the fact that Grand Unified Theory dyons can be constructed by embedding and slightly generalizing the Julia-Zee dyon into the GUT gauge group G.

One of the simplest examples of such GUT solutions is the Dokos-Tomaras dyon [53] which arises

[12] This can be shown by using the cyclic property of the triple product (when γ matrices aren't interchanged) and writing the spin operator as $S^i = -\frac{i}{4}\epsilon^{ijk}\gamma_j\gamma_k = \frac{1}{2}\gamma^5\gamma^0\gamma^i$, since γ^r can then be written as $i\boldsymbol{\gamma}\cdot(\hat{\boldsymbol{r}}\times\vec{S}) = -i\hat{\boldsymbol{r}}\cdot(\vec{S}\times\boldsymbol{\gamma}) = -\frac{i}{2}\gamma^5\gamma^0\hat{r}^i\epsilon_{ijk}\gamma^j\gamma^k = \gamma^5\gamma^0\gamma^5\gamma^0\hat{r}^i\gamma_i = \gamma^r$.

in the $SU(5)$ Georgi-Glashow model and is given by[13]

$$
\begin{aligned}
g\mathcal{A}_i(\boldsymbol{r}) \;&=\; (\hat{\boldsymbol{r}} \times \vec{t}\,)_i \left[\frac{1-\mathcal{K}(r)}{r}\right] , \qquad (1.1.33) \\
g\mathcal{A}_0(\boldsymbol{r}) \;&=\; \hat{\boldsymbol{r}} \cdot \vec{t}\, \frac{\mathcal{J}(r)}{r} + \frac{1}{r} \begin{pmatrix} \mathcal{J}_1(r)\, I_{2\times 2} & & \\ & \mathcal{J}_2(r)\, I_{2\times 2} & \\ & & -2\left[\mathcal{J}_1(r) + \mathcal{J}_2(r)\right] \end{pmatrix} ,
\end{aligned}
$$

where g is the gauge coupling, $I_{n\times n}$ is an n by n unit matrix acting in a subset of $SU(5)$ gauge space, as well as

$$
\begin{aligned}
g\varrho^a(\boldsymbol{r}) \;&=\; \mathcal{G}(r)\, \delta^a_5 , \qquad (1.1.34) \\
g\varphi(\boldsymbol{r}) \;&=\; \hat{\boldsymbol{r}} \cdot \vec{t}\, \frac{\mathcal{H}(r)}{r} + \frac{1}{r} \begin{pmatrix} \mathcal{H}_1(r)\, I_{2\times 2} & & \\ & \mathcal{H}_2(r)\, I_{2\times 2} & \\ & & -2\left[\mathcal{H}_1(r) + \mathcal{H}_2(r)\right] \end{pmatrix} ,
\end{aligned}
$$

where ϱ is the background field configuration of an additional Higgs field in the fundamental representation, which is introduced to further break the Standard Model gauge group down to $SU_c(3) \times U_{EM}(1)$ at low energies. In the above, $\vec{t}$ corresponds to the following embedding of $SU(2)$ generators into $SU(5)$

$$
t^k = \frac{1}{2} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \tau_k & \\ & & & 0 \end{pmatrix} , \qquad (1.1.35)
$$

where τ_k are again the Pauli matrices.

The Dokos-Tomaras solution must be bounded at the origin and this implies that $\mathcal{K}(r) \to 1$ and $\mathcal{J}, \mathcal{J}_\alpha \to 0$, $\mathcal{H}, \mathcal{H}_\alpha \to 0$ as $r \to 0$, where $\alpha = 1, 2$. As we are interested in dyons with finite energy, these functions must also approach the vacuum at large distances and so

$$
\mathcal{K}(r) \to 0 \,, \quad \mathcal{J}(r) \to e(Q - vr) \quad \text{and} \quad \mathcal{J}_\alpha(r) \to 0 \quad \text{as } r \to \infty \,, \qquad (1.1.36)
$$

[13] In our conventions, the generators of $SU_c(3)$, $SU_L(2)$ and $U_Y(1)$ are embedded into $SU(5)$ as follows:

$$
\Lambda_i = \frac{1}{2} \begin{pmatrix} \lambda_i & \\ & 0_{2\times 2} \end{pmatrix} , \quad \Sigma_j = \frac{1}{2} \begin{pmatrix} 0_{3\times 3} & \\ & \sigma_j \end{pmatrix} \quad \text{and} \quad Y = \frac{1}{\sqrt{60}} \begin{pmatrix} 2\, I_{3\times 3} & \\ & -3\, I_{2\times 2}, \end{pmatrix} \qquad (1.1.32)
$$

where $I_{n\times n}, 0_{n\times n}$ are n by n unit and zero matrices, respectively; $i = 1, \cdots 8$, $j = 1, \cdots 3$ and λ_i are the Gell-Mann matrices, while σ_j are the Pauli matrices.

while the Higgs field configurations asymptotically satisfy

$$\mathcal{H}_1(r) \to hr\,, \quad \mathcal{H}_2(r) \to \left(\varepsilon - \frac{1}{2}\right)\frac{hr}{2} \quad \text{and} \quad \mathcal{H}(r) \to \left(\varepsilon - \frac{5}{2}\right) hr \quad \text{as } r \to \infty\,, \tag{1.1.37}$$

as well as $\mathcal{G}(r) \to \nu$ as $r \to \infty$, where h/g and ν/g are related to the adjoint representation and fundamental representation Higgs vevs[14] and $\varepsilon = \mathcal{O}(\nu/h) \sim 10^{-14}$.

Notice that, apart from differences in their respective Higgs sectors, the Dokos-Tomaras dyon is asymptotically equivalent to the Julia-Zee solution embedded into $SU(5)$ using (1.1.35). This particular embedding is chosen since the corresponding solution has the smallest possible magnetic charge and so also the smallest mass. As in (1.1.19), the $SU(5)$ dyon can be gauge rotated so that it is asymptotically abelian and points along *e.g.* the t_3 direction in gauge space. Since the embedding we use acts in both the colour and electroweak sectors, the dyon gives rise to both ordinary and colour electromagnetic charges. To see this explicitly, note that the generators of $SU_c(3) \times U_{EM}(1)$ are given by

$$\Lambda_i = \frac{1}{2}\begin{pmatrix} \lambda_i & \\ & 0_{2\times 2}\end{pmatrix} \quad \text{and} \quad T_{EM} = -\sqrt{\frac{1}{24}}\begin{pmatrix} I_{3\times 3} & & \\ & -3 & \\ & & 0\end{pmatrix}, \tag{1.1.38}$$

where λ_i are the Gell-Mann matrices and $0_{n\times n}$ is an n by n zero matrix. The matrix t_3 can be expanded in terms of these generators as

$$t_3 = -\frac{1}{2}\left(\sqrt{\frac{8}{3}}\,T_{EM} + \frac{2}{\sqrt{3}}\Lambda_8\right) = -\frac{1}{2}\left(\frac{g}{e}\,T_{EM} + \frac{2}{\sqrt{3}}\Lambda_8\right), \tag{1.1.39}$$

where the second equality uses the fact that the $SU(5)$ gauge coupling g is related to the electromagnetic coupling through $(g/e)^2 = 8/3$. Gauge transforming the solution with a generalization[15] of (1.1.18) and using Gauss' law as well as (1.1.39) then implies the dyon's 'ordinary' electric and

[14] We determine the asymptotic behaviour of $\mathcal{H}, \mathcal{H}_\alpha$ by imposing that the Dokos-Tomaras solution asymptotically tends to the vacuum, once gauge transformed to the 'abelian' gauge. The Higgs field vacuum expectation values are $g\,\langle\rho\rangle = \nu\,(0\ \ 0\ \ 0\ \ 0\ \ 1)^T$ and $g\,\langle\Phi\rangle = h\,\mathrm{diag}\left(1,1,1,-\frac{3}{2}+\varepsilon,-\frac{3}{2}-\varepsilon\right)$.

[15] The relevant gauge function in the R_- region is given by $U(\boldsymbol{r}) = \begin{pmatrix} I_{2\times 2} & & \\ & \sqrt{\frac{1-\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}}}{2}}\,I_{2\times 2} & \\ & & 1\end{pmatrix} + \sqrt{\frac{2}{1-\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}}}}\,i\vec{t}\cdot(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}})$.

magnetic charges are given by

$$q_E^{EM} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{E}^{EM}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = -\frac{Q}{2} \quad \text{and} \quad q_M^{EM} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{B}^{EM}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = -\frac{1}{2e}, \tag{1.1.40}$$

while its colour electric and magnetic charges are

$$q_E^{c} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{E}^{c}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = -\frac{Q}{2\sqrt{2}} \quad \text{and} \quad q_M^{c} = \frac{1}{4\pi}\int_0^{4\pi} \mathbf{B}^{c}\cdot\hat{\mathbf{r}}\,\mathrm{d}^2S = -\frac{1}{\sqrt{3}g}. \tag{1.1.41}$$

A Grand Unified Theory dyon might be expected to have many more collective coordinates than its $SU(2)$ counterpart as the GUT symmetry group is larger. This is not always the case, however, since the non-trivial topology of magnetic monopoles and dyons leads to ambiguities in defining global nonabelian gauge transformations. As shown in [71], gauge transformations can be defined globally only if they act trivially on the long-range monopole field because precisely this subset of transformations preserves the GUT generalization of the condition (1.1.11). For the Dokos-Tomaras dyon coupled to fermions in the fundamental representation, the condition (1.1.11) becomes

$$\psi_+(x) = e^{2i\phi\, t_3}\psi_-(x) \quad \text{for} \quad x \in R_+ \cap R_-, \tag{1.1.42}$$

and the compatibility of gauge transformations with the above decreases the number of potential $SU(5)$ dyon degrees of freedom down to precisely those of the Julia-Zee dyon, where the charge collective coordinate $\mathfrak{a}$ is now related to the unbroken $U(1)$ gauge group, generated by t_3.

Grand Unified Theory dyons and monopoles satisfy a generalized version of Dirac's quantization condition. This can most easily be seen by switching to a gauge in which the singular string is reintroduced, such as by choosing a gauge in which the dyon is asymptotically abelian. As in the case of the abelian monopole, the singular string introduces an extra magnetic field, which now has both a colour and ordinary magnetic field contribution

$$\mathbf{B}_s^{c} = \frac{4\pi}{\sqrt{3}g}\,\Theta(-z)\delta(x)\delta(y)\,(-\hat{\boldsymbol{z}}) \quad \text{and} \quad \mathbf{B}_s^{EM} = \frac{2\pi}{e}\,\Theta(-z)\delta(x)\delta(y)\,(-\hat{\boldsymbol{z}}). \tag{1.1.43}$$

Imposing that the string magnetic fields are unobservable gives

$$\vartheta^{s}_{A-B} = q^{c}_{\psi} \int \mathbf{B}^{c}_{s} \cdot \mathrm{d}^2\boldsymbol{S} + q^{EM}_{\psi} \int \mathbf{B}^{EM}_{s} \cdot \mathrm{d}^2\boldsymbol{S} = -4\pi(q^{c}_{M}\, q^{c}_{\psi} + q^{EM}_{M}\, q^{EM}_{\psi}) = 2\pi n, \tag{1.1.44}$$

for a particle with colour electric charge q^{c}_{ψ} and ordinary electric charge q^{EM}_{ψ}. This generalized quantization condition is satisfied by all the fundamental particles of the Standard Model, if the GUT dyon has magnetic charges given by (1.1.40) and (1.1.41).

1.2 Effective field theories

Having summarized the relevant properties of magnetic monopoles and dyons, we now discuss in what ways heavy degrees of freedom such as these can influence physics at significantly lower energies. In principle, this can be seen by expanding the observables of our chosen theory in powers of a small ratio of energies E/M, where E is the energy of a light degree of freedom and $M \gg E$ is the heavy particle mass. This section describes an alternative and much more efficient framework, known as effective field theory [11–13] (see [14] for a review), that was designed to tackle problems such as these which exhibit a large hierarchy of scales.

The starting point when working in the EFT formalism is usually the construction of an effective or *Wilsonian* action, S_W. The Wilson action can be used to compute the low-energy expansion of *any* observable using the standard techniques of quantum field theory. When the high-energy theory is known, S_W can be explicitly derived by integrating out the degrees of freedom with energies above a certain threshold. The result of this procedure is an effective action with the following properties: (*i*) It can be written as an expansion in powers of M^{-1} *i.e.* of the inverse of the heavy scale. To any fixed order in this expansion, it is a local functional – meaning S_W is given by $S_W = \int \mathrm{d}^4x\, \mathcal{L}_W(x)$ – of the light degrees of freedom and their derivatives; (*ii*) S_W generally respects the symmetries of the original high-energy theory, although these symmetries need not be realized in the same way in the Wilson and original actions. Note, however, that if a gauge symmetry of the high-energy theory is spontaneously broken at low energies, a convenient choice of gauge will make the symmetry explicitly broken in the low-energy regime. This is equivalent to working in a more general gauge in which the symmetry is present at low energies, but acts nonlinearly on the fields. (*iii*) The Wilson action contains an infinite number of operators, including non-renormalizable ones, but can nonetheless be used to extract physical predictions once the desired accuracy of a calculation is specified. This can be done because choosing the accuracy required of a result is equivalent to determining the order in the small ratio of scales, E/M, to which one works and because only a finite number of operators in S_W can contribute to observables at any fixed order in E/M.

While S_W can often be derived as we schematically describe above, it is usually simpler and sometimes even necessary to obtain it using an alternative approach. Specifically, since any Wilson action satisfies the properties $(i)-(iii)$, S_W can be constructed as the most general action consistent with these properties. Although the coupling constants of such an action are formally arbitrary, they can be determined by comparing the observables of the effective theory with their measured (or computed) values in what is called a matching procedure.

We now turn to a simple toy model to illustrate why the above EFT prescription works. Consider the following action

$$S[\psi,\overline{\psi},h] := -\int \mathrm{d}^4x \left[\overline{\psi}\,\gamma^\mu\partial_\mu\,\psi + \frac{1}{2}h(-\partial^\mu\partial_\mu + M^2)h + \lambda h\overline{\psi}\,\psi\right], \tag{1.2.1}$$

describing a massive scalar field, h, coupled to a massless fermion ψ via a Yukawa potential with dimensionless coupling λ. The partition function of this theory is given by

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\psi\,\mathcal{D}\overline{\psi}\,\mathcal{D}h\, e^{iS[\psi,\overline{\psi},h]} \\ &= \int \mathcal{D}\psi\,\mathcal{D}\overline{\psi}\, e^{iS_0[\psi,\overline{\psi}]}\int \mathcal{D}h\,\exp\left\{-i\int \mathrm{d}^4x\,\left(\frac{1}{2}h\Delta h + \lambda h\overline{\psi}\psi\right)\right\},\end{aligned} \tag{1.2.2}$$

where we separate the free fermion action, $S_0[\psi,\overline{\psi}] := -\int \mathrm{d}^4x\,\overline{\psi}\,\gamma^\mu\partial_\mu\,\psi$, from the remainder of $S[\psi,\overline{\psi},h]$ and introduce $\Delta := -\partial_\mu\partial^\mu + M^2 = -\Box + M^2$. A Wilson action for this theory can be obtained from $\mathcal{Z}$ after integrating out the massive scalar h

$$\begin{aligned}e^{iS_W[\psi,\overline{\psi}]} &:= e^{iS_0[\psi,\overline{\psi}]}\int \mathcal{D}h\, e^{-i\int \mathrm{d}^4x\,\left(\frac{1}{2}h\Delta h + \lambda h\overline{\psi}\psi\right)} \\ &\propto\; (\det\Delta)^{-1/2}\, e^{iS_0[\psi,\overline{\psi}]}\exp\left\{-\frac{\lambda^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\overline{\psi}(x)\psi(x)G(x-y)\overline{\psi}(y)\psi(y)\right\},\end{aligned} \tag{1.2.3}$$

where we drop ψ-independent numerical factors. In the above, $G(x-y)$ is the scalar field propagator, given by

$$G(x-y) := \langle 0|Th(x)h(y)|0\rangle = -i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+M^2}, \tag{1.2.4}$$

which can be expanded as follows

$$G(x-y) = -\frac{i}{M^2}\sum_{j=0}^{\infty}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{-p^2}{M^2}\right)^j e^{ip(x-y)} = -\frac{i}{M^2}\sum_{j=0}^{\infty}\left(\frac{\Box_y}{M^2}\right)^j \delta^4(x-y). \tag{1.2.5}$$

Using the Taylor-expanded form of the propagator in (1.2.3) and integrating by parts to 'move' derivatives onto the fermion fields shows that the Wilson action can be written as

$$S_W[\psi,\overline{\psi}] = -\int \mathrm{d}^4x\left(\overline{\psi}\,\gamma^\mu\partial_\mu\,\psi - \frac{\lambda^2}{2M^2}\sum_{j=0}^{\infty}\overline{\psi}\psi\left(\frac{\Box}{M^2}\right)^j\overline{\psi}\psi\right), \tag{1.2.6}$$

where we drop the ψ-independent factor of $(\det\Delta)^{-1/2}$.

The above Wilson action is manifestly seen to be a local functional of the light field and its derivatives. Locality is achieved by expanding the scalar field propagator in powers of the heavy mass scale, but comes at the expense of introducing an infinite number of operators in S_W, some of which are non-renormalizable. As can be seen from (1.2.6), the higher the mass dimension of an operator in the Wilson action, the more it will be suppressed by inverse powers of the heavy mass scale, M^2, so that only a finite number of terms need to be taken into account in realistic applications. While the suppression of operators in the action suffices to justify the small size of tree-level amplitudes at low energies, more care needs to be taken when evaluating the contributions of loops to observables. The appearance of the heavy scale in any amplitude (including loop contributions) can be tracked by the use of power-counting arguments, which should be used to single out the operators that contribute to observables at any fixed order in the inverse heavy scale.

Since the toy model (1.2.1) does not exhibit spontaneous symmetry breaking, the EFT prescription dictates that its low-energy theory be invariant under the symmetries of the original action. In our case, the relevant symmetries are Poincaré invariance and invariance under global $U(1)$ transformations, both of which clearly 'survive' in the low-energy limit.

The effective action of equation (1.2.6) does not quite describe the low-energy limit of the original theory, since it was obtained by integrating out only a subset of the heavy fields. To capture the effects of *all* virtual heavy degrees of freedom on physics at energies $E \ll M$, we must also integrate out the high-energy fermion modes. This is typically done by imposing a UV cutoff Λ, which satisfies $E \ll \Lambda \ll M$ but is otherwise arbitrary[16], and integrating out fermionic modes with energies $\omega \gg \Lambda$. The dependence of any observable on the unphysical parameter Λ is ultimately removed after renormalization.

Alternatively, the low-energy Wilson action can be constructed as the most general Poincaré and $U(1)$-invariant action that is a local functional of ψ and its derivatives, to the desired order in powers of M^{-2}. Up to and including (mass) dimension-5 operators, the resulting Wilson action is given by

$$S_W[\psi,\overline{\psi}] = -\int \mathrm{d}^4x \left(\overline{\psi}\,\gamma^\mu\partial_\mu\,\psi + c_0^s\overline{\psi}\psi + c_0^{ps}\overline{\psi}\gamma^5\psi + \frac{c_2^s}{M^2}\overline{\psi}\Box\psi + i\,\frac{c_2^{ps}}{M^2}\overline{\psi}\gamma^5\Box\psi\right), \tag{1.2.7}$$

where c_0^s, c_0^{ps} can be determined by matching. The value of the remaining coupling constants, c_2^s and c_2^{ps}, can be set to zero since the corresponding operators turn out to be redundant[17]. In this approach, we need not integrate out the high-energy modes of the fermion field ψ, since (1.2.7) captures the dominant contribution of *any* high-energy physics to low-energy amplitudes. Different UV completions of the low-energy theory then imply different values for the coefficients c_0^s, c_0^{ps}.

The low-energy action given in (1.2.7) illustrates why the decoupling principle works: If the maximum energy reached in experiments is negligible compared to M, *i.e.* if $E/M \sim 0$, the heavy scalar field can influence low-energy physics only through the two operators c_0^s, c_0^{ps}. Any observable of the low-energy theory can be expressed in terms of scattering amplitudes, which can be computed from the Wilson action in the usual way *i.e.* by differentiating the generating functional

$$\mathcal{Z}[j,\overline{j}] = \int \mathcal{D}\psi\,\mathcal{D}\overline{\psi}\; e^{iS_W[\psi,\overline{\psi}] + i\int \mathrm{d}^4x\,\left(\overline{j}\psi + \overline{\psi}j\right)}, \tag{1.2.8}$$

where $j, \overline{j}$ are Grassman variable sources.

[16]When imposing an explicit momentum cutoff, amplitudes are suppressed by positive powers of E/Λ as well as Λ/M which is why the cutoff is chosen such that $E \ll \Lambda \ll M$.

[17]Redundant operators can be removed at a fixed order in M^{-2} by an appropriately chosen field redefinition. In this case a field redefinition of the form $\delta\psi = -\frac{1}{2}\left(c_2^s/M^2 - i\gamma^5 c_2^{ps}/M^2\right)\gamma^\mu\partial_\mu\psi$ shifts the $\sim M^{-2}$ terms in (1.2.7) to order $\sim M^{-4}$. We also drop total derivative terms from the Wilson action, since there are no boundaries.

Point-particle EFTs

A subset of EFTs that is particularly well-suited to our purposes goes by the name of point-particle effective field theories (PPEFTs). The hierarchy of scales exploited in these theories is that between the small diameter, R, of a point-like object and the large wavelengths, λ, of the low-energy particles it interacts with. We now discuss how the PPEFT framework can be adapted to describe the interactions of point-like monopoles and dyons with low-energy fermions.

To start, we go back to the $SU(2)$ Georgi-Glashow model, now additionally coupled to a complex doublet of massless fermions

$$S = -\int \mathrm{d}^4x \left[\frac{1}{4} F^a_{\mu\nu}\, F^{\mu\nu}_a + \frac{1}{2} D_\mu \Phi^a\, D^\mu \Phi_a + \overline{\psi}\,\gamma^\mu D_\mu \psi - \frac{\mu^2}{2}\,\Phi^a\,\Phi_a + \frac{\lambda}{4}\,\left(\Phi^a\,\Phi_a\right)^2 \right], \tag{1.2.9}$$

where $D_\mu \psi = \left(\partial_\mu - \frac{i}{2}\, e A^a_\mu \tau_a\right)\psi$. As before, this theory admits soliton solutions like the Julia-Zee dyon (with vanishing ψ) which correspond to particles when the semiclassical limit is taken. To lowest order in the semiclassical expansion, the fermionic sector of the theory describes free fermions propagating in the background of a Julia-Zee dyon, since expanding the action (1.2.9) gives

$$S_\psi = -\int \mathrm{d}^4x\, \overline{\psi}(x)\gamma^\mu \left[\partial_\mu - ie\mathcal{A}^a_\mu(\boldsymbol{x}-\boldsymbol{y};\mathfrak{a})\,T_a\right]\psi(x), \tag{1.2.10}$$

where $\mathcal{A}^a_\mu(\boldsymbol{x}-\boldsymbol{y};\mathfrak{a})$ is the Julia-Zee solution parametrized by its collective coordinates $\boldsymbol{y}, \mathfrak{a}$.

Our goal is to describe how the dyon affects the fermion fields at distances much larger than its radius $r \gg R$. Although fermionic observables can be calculated in the full theory $-e.g.$ from the fermion field, once it is expanded in terms of solutions to the Dirac equation in the dyon background – EFT techniques once again offer an alternative and simpler approach in which the $r \gg R$ hierarchy is exploited early on. Specifically, a Wilson action for the system can be obtained by integrating out the short-distance physics and this will replace the fermion-dyon interactions in the core with a set of effective interactions defined on the dyon worldline.

The next step in the PPEFT formalism is to match the Wilson action on the worldline to a boundary action, S_b, which can be used to derive the boundary conditions satisfied by the fermion fields. This introduces a new unphysical boundary into the problem, which we define at a radius $r = \epsilon$ that satisfies $R \ll \epsilon \ll \lambda$ but is otherwise arbitrary. The position of the boundary plays a similar role as the UV cutoff of the EFT toy model and, just as the UV cutoff, does not enter into physical observables once the couplings of the Wilsonian action are appropriately renormalized.

The PPEFT formalism described above is particularly useful in cases where the exact form of the interactions in the core is not known, such as when the general form of the Julia-Zee solution (and not its Prasad-Sommerfield limit) is of interest. When this is satisfied, the Wilson action must be built from the 'bottom-up' *i.e.* as the most general action consistent with the EFT principles discussed above. Before this can be done we must first identify all the low-energy degrees of freedom, which in our case consist of the fermion and photon fields as well as the dyon collective coordinates. The collective coordinates $\boldsymbol{y}, \mathfrak{a}$ belong in the low-energy theory because they are Goldstone bosons for the broken translation and global $U(1)$ symmetries. Integrating them out along with the high-energy degrees of freedom would lead to a divergent result[18] and this, along with the simplicity they bring in implementing symmetry transformations, is why they are treated differently to other fluctuations when performing the semiclassical expansion. Notice that, even though the Julia-Zee solution breaks the rotational and boost invariance of the Georgi-Glashow model (in addition to breaking the translation and $U(1)$ symmetries), no collective coordinates are introduced to restore these symmetries. This is because rotating or boosting the dyon solution does not lead to new degrees of freedom, since a rotation of the dyon can be 'undone' by a gauge rotation and a Lorentz boost merely changes the value of $\dot{\boldsymbol{y}}$.

A PPEFT example

We now illustrate how the PPEFT procedure works on a simpler example of a spinless source with vanishing electric and magnetic charge, interacting with bulk fermion fields. This scenario is discussed in [35] and will prove useful in later chapters when comparing dyons to less exotic point-particles, such as nuclei.

We wish to couple a Dirac fermion to a point-like source located at the origin

$$S = -\int \mathrm{d}^4x \left[\overline{\psi}(\not{D} + m)\psi + \overline{\psi}N\psi\, \delta^3(x)\right], \tag{1.2.11}$$

[18] The divergence comes about because the fluctuations $\delta\mathcal{A}_\mu, \delta\varphi$ which relate the dyon solution to its translated and gauge rotated equivalent are zero modes of the high-energy theory, see for example [14].

where N is a Dirac matrix. For definiteness, we consider the rotational and parity invariant case $N = \hat{\mathfrak{c}}_s + i\hat{\mathfrak{c}}_v\gamma^0$ where $\hat{\mathfrak{c}}_s$ and $\hat{\mathfrak{c}}_v$ are coupling constants. The ψ equation of motion including the coupling to the source is

$$(\not{D} + m)\psi + N\psi\,\delta^3(x) = 0\,. \tag{1.2.12}$$

Formally we'd like to trade the delta function for a near-source boundary condition, and following usual practice this would be obtained by integrating (1.2.12) over a small Gaussian pillbox, P, of radius ϵ centred on the source. This gives, in the limit $\epsilon \to 0$ of vanishingly small pillbox, the result

$$\int_{\partial P} \mathrm{d}^2x\; n_\mu\gamma^\mu\psi = \int \mathrm{d}^2\Omega\,\epsilon^2\;\gamma^r\,\psi \simeq -N\psi(0)\,, \tag{1.2.13}$$

where the solid-angle measure is $\mathrm{d}^2\Omega = \sin\theta\,\mathrm{d}\theta\mathrm{d}\phi$ and n_μ is an outward-pointing unit normal to the pillbox so in polar coordinates $n_\mu\mathrm{d}x^\mu = \mathrm{d}r$ and $\psi(0)$ denotes the value of the field at the position of the source. The final approximate equality drops the $m\psi$ term, as is appropriate for a sufficiently small pillbox since this vanishes as $\epsilon \to 0$ provided ψ is sufficiently smooth near the origin.

The problem with the formal argument is that bulk fields like ψ are typically *not* smooth as $r \to 0$, which both complicates the neglect of the $m\psi$ term when integrating (1.2.12) over the pillbox and makes $\psi(0)$ undefined. The PPEFT way for dealing with both of these issues is essentially to regulate the source action by replacing it by a boundary action on the boundary of the pillbox ∂P at $r = \epsilon$. For configurations that are spherically symmetric[19] very near the source this is particularly simple to do by replacing the world-line action by its value integrated over ∂P:

$$\int_{r=0} \mathrm{d}t\;\overline{\psi}N\psi \to \frac{1}{4\pi\epsilon^2}\int_{\partial P} \mathrm{d}^2\Omega\,\mathrm{d}t\;\epsilon^2\,\overline{\psi}N\psi\,. \tag{1.2.14}$$

This procedure makes the replacement $N\psi(0) \to (4\pi\epsilon^2)^{-1}\int \mathrm{d}^2\Omega\,\epsilon^2 N(\epsilon)\psi(\epsilon)$ on the right-hand side of (1.2.13), leading (in the limit where ϵ is much smaller than all other scales of interest) to the regulated boundary condition

$$\int_{r=\epsilon} \mathrm{d}^2\Omega\;\left[\gamma^r + \frac{N}{4\pi\epsilon^2}\right]\psi = \int_{r=\epsilon} \mathrm{d}^2\Omega\;\left[\gamma^r + \frac{1}{4\pi\epsilon^2}\Big(\hat{\mathfrak{c}}_s + i\hat{\mathfrak{c}}_v\gamma^0\Big)\right]\psi = 0\,. \tag{1.2.15}$$

[19]Non-spherically symmetric configurations can also be handled by decomposing into spherical harmonics and treating each harmonic separately on ∂P.

Notice this boundary condition is trivially satisfied if we'd tried to make the same derivation using a pillbox that does *not* contain the source position. This is because in this case the $N\psi$ term is no longer present and so the boundary condition states $\int \mathrm{d}^2\Omega\, \gamma^r \psi = 0$. Since ψ varies very slowly in a small enough region not containing the source, it can be taken to be approximately constant across the pillbox and so the integral over all directions of γ^r gives zero trivially without restricting ψ.

Returning to the case where the pillbox does enclose the source, the boundary condition (1.2.15) can be written as $\int \mathrm{d}^2\Omega\ B_\epsilon\, \psi(\epsilon) = 0$ where

$$B_\epsilon := \gamma^r + \frac{N}{4\pi\epsilon^2} = \gamma^r + \hat{\mathcal{C}}_s + i\hat{\mathcal{C}}_v\gamma^0 = \begin{pmatrix} \hat{\mathcal{C}}_s & \hat{\mathcal{C}}_v - i\sigma^r \\ \hat{\mathcal{C}}_v + i\sigma^r & \hat{\mathcal{C}}_s \end{pmatrix}, \tag{1.2.16}$$

where $\hat{\mathcal{C}}_s = \hat{\mathfrak{c}}_s/(4\pi\epsilon^2)$ and $\hat{\mathcal{C}}_v = \hat{\mathfrak{c}}_v/(4\pi\epsilon^2)$ are now dimensionless effective couplings. The subscript ϵ on B_ϵ is meant to emphasize that the constants $\hat{\mathcal{C}}_a$ (and in general also the original couplings $\hat{\mathfrak{c}}_i$ themselves) must carry an implicit ϵ-dependence if physical quantities are to remain unchanged as ϵ is varied (more about which below). In terms of the left- and right-handed parts of ψ the boundary condition becomes

$$-\hat{\mathcal{C}}_s \int_\epsilon \mathrm{d}^2\Omega\, \psi_L^\pm = \int_\epsilon \mathrm{d}^2\Omega \left(\hat{\mathcal{C}}_v - i\sigma^r\right)\psi_R^\pm \quad \text{and} \quad -\int_\epsilon \mathrm{d}^2\Omega \left(\hat{\mathcal{C}}_v + i\sigma^r\right)\psi_L^\pm = \hat{\mathcal{C}}_s \int_\epsilon \mathrm{d}^2\Omega\, \psi_R^\pm\,. \tag{1.2.17}$$

To see what these boundary conditions imply, imagine solving the bulk equation $(\not{D} + m)\psi = 0$ for $r > \epsilon$ and decomposing the result into rotation and parity eigenstates. The parity-even solutions are

$$\psi^+ = \begin{pmatrix} \psi_L^+ \\ \psi_R^+ \end{pmatrix} = \begin{pmatrix} f_+(r)\, U^+(\theta,\phi) + ig_+(r)\, U^-(\theta,\phi) \\ f_+(r)\, U^+(\theta,\phi) - ig_+(r)\, U^-(\theta,\phi) \end{pmatrix}, \tag{1.2.18}$$

while the parity-odd ones are

$$\psi^- = \begin{pmatrix} \psi_L^- \\ \psi_R^- \end{pmatrix} = \begin{pmatrix} f_-(r)\, U^-(\theta,\phi) + ig_-(r)\, U^+(\theta,\phi) \\ f_-(r)\, U^-(\theta,\phi) - ig_-(r)\, U^+(\theta,\phi) \end{pmatrix}, \tag{1.2.19}$$

where $U^\pm$ are the spinor harmonics that combine the particle's spin-half with orbital angular momenta $\ell = j \mp \frac{1}{2}$ to give total angular momentum $j = \frac{1}{2}, \frac{3}{2}, \cdots$. The radial functions $f_\pm(r)$ and

$g_\pm(r)$ with mode frequency ω solve the radial equations

$$f_+' = (m+\omega)\, g_+ \quad \text{and} \quad g_+' + \frac{2g_+}{r} = (m-\omega)\, f_+\,, \tag{1.2.20}$$

together with

$$g_-' = (m-\omega)\, f_- \quad \text{and} \quad f_-' + \frac{2f_-}{r} = (m+\omega)\, g_-\,, \tag{1.2.21}$$

where primes denote differentiation with respect to r. The boundary conditions (1.2.17) fix the ratio of the functions f and g at $r = \epsilon$ once the angular integrations are performed, giving

$$\hat{\mathcal{C}}_s + \hat{\mathcal{C}}_v = \left(\frac{g_+}{f_+}\right)_{r=\epsilon} \qquad \text{and} \qquad \hat{\mathcal{C}}_s - \hat{\mathcal{C}}_v = \left(\frac{f_-}{g_-}\right)_{r=\epsilon}. \tag{1.2.22}$$

Eq. (1.2.22) provides the solution to how the properties of the source influence the bulk solutions for ψ in the source's vicinity. Given the general solution,

$$f_\pm(r) = C_1^\pm f_{1\pm}(r) + C_2^\pm f_{2\pm}(r) \qquad \text{and} \qquad g_\pm(r) = C_1^\pm g_{1\pm}(r) + C_2^\pm g_{2\pm}(r)\,, \tag{1.2.23}$$

to the radial part of the Dirac field equation we see that (1.2.22) show that the couplings $\hat{\mathcal{C}}_s$ and $\hat{\mathcal{C}}_v$ determine the ratios of integration constants C_2^+/C_1^+ and C_2^-/C_1^- that specify $(g_\pm/f_\pm)_{r=\epsilon}$. Energy levels for states of either parity and scattering amplitudes are then determined by the values of $C_2^\pm/C_1^\pm$.

But it is still a potential puzzle why physical predictions can depend on the radius, $r = \epsilon$, of the Gaussian pillbox which is not a physical scale (arising just as a way to regularize the boundary conditions). The precise value of ϵ must therefore drop out of predictions for observables (unlike the physical size, R, of the underlying source, say). In detail, this happens because any explicit ϵ-dependence arising in a calculation of an observable cancels an implicit ϵ-dependence buried within the 'bare' quantities $\hat{\mathfrak{c}}_s$ and $\hat{\mathfrak{c}}_v$. Physical predictions remain ϵ-independent if $\hat{\mathfrak{c}}_s(\epsilon)$ and $\hat{\mathfrak{c}}_v(\epsilon)$ are chosen to ensure the ratios $C_2^\pm/C_1^\pm$ are held fixed as ϵ is varied.

This gives us another way to interpret eq. (1.2.22). Rather than reading (1.2.22) as fixing $f_\pm/g_\pm$

at a specific radius given known values of ϵ, $\hat{\mathfrak{c}}_s$ and $\hat{\mathfrak{c}}_v$ we can instead read the equations

$$\hat{\mathfrak{c}}_s(\epsilon) = \left[\frac{g_+(\epsilon)}{f_+(\epsilon)} + \frac{f_-(\epsilon)}{g_-(\epsilon)}\right] 2\pi\epsilon^2 \quad \text{and} \quad \hat{\mathfrak{c}}_v(\epsilon) = \left[\frac{g_+(\epsilon)}{f_+(\epsilon)} - \frac{f_-(\epsilon)}{g_-(\epsilon)}\right] 2\pi\epsilon^2 \,, \tag{1.2.24}$$

as telling us how $\hat{\mathfrak{c}}_s(\epsilon)$ and $\hat{\mathfrak{c}}_v(\epsilon)$ must depend on ϵ in order to ensure that $C_2^\pm/C_1^\pm$ remains ϵ-independent. Since we choose ϵ much smaller than the typical scale of the external problem (such as the Bohr radius, for applications to atoms), it suffices to use the leading small-r form of the solutions $f_\pm$ and $g_\pm$ when using (1.2.24). In this regime solutions are usually well described by power laws, with (1.2.23) reducing to

$$f_\pm(r) = C_1^\pm \left(\frac{r}{a}\right)^{\mathfrak{z}-1} + C_2^\pm \left(\frac{r}{a}\right)^{-\mathfrak{z}-1} \qquad \text{and} \qquad g_\pm(r) = \widetilde{C}_1^\pm \left(\frac{r}{a}\right)^{\mathfrak{z}-1} + \widetilde{C}_2^\pm \left(\frac{r}{a}\right)^{-\mathfrak{z}-1} \,, \tag{1.2.25}$$

for some power[20] $\mathfrak{z}$ with $\widetilde{C}_i^\pm \propto C_i^\pm$ in a way that depends on the relative small-r asymptotic behaviour of $f_i(r)$ and $g_i(r)$. For such solutions the choice of $C_2^\pm/C_1^\pm$ controls the precise radius at which one of these solutions dominates the other one, and as a result the RG evolution of the couplings implied by (1.2.24) in this regime describes the cross-over between these two types of evolution.

1.3 Outline

The rest of this thesis is organized as follows. In chapter §2, we derive the fermion-dyon PPEFT and the induced near-dyon boundary conditions on S-wave fermion fields; §2.4 shows why many of the PPEFT operators that dominate at low energies are redundant, which explains why observables depend on fewer parameters than the initial number of lowest-dimension effective operators. §2.5 explores how the boundary couplings change as the position of the boundary is varied and shows in more detail why the kinematics of S-wave dyon-fermion scattering leads to scale-invariant cross sections even though the same does not happen for small objects without magnetic charge (such as nuclei). This section also discusses how the fermion condensation effects described in [68] can be incorporated within the PPEFT framework. Finally, §4 uses the PPEFT effective couplings to

[20]Within atoms, for instance, the relevant power is $\mathfrak{z} = \left[\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2\right]^{\frac{1}{2}}$ where the nuclear charge is Ze and α is the fine-structure constant.

compute fermion polarization and fermion-dyon scattering within the Born-Oppenheimer approximation, after first warming up by computing similar results perturbatively in the fermion-dyon interactions in §3. Our conclusions are briefly summarized in §5.

Chapter 2

Dyon Effective Theory

In this chapter, we construct the point-particle effective theory that describes the dominant low-energy interactions of relativistic S-wave fermions with a Julia-Zee dyon. Section §2.1 starts by reviewing some details of the semiclassical expansion and of fermionic S-wave states not covered in the introduction. In §2.2, we derive the effective fermion-dyon interactions on the dyon worldline and match them to a boundary action in the $j = 0$ fermionic sector, defined at radius $r = \epsilon$. This section also derives the boundary condition satisfied by S-wave fermions at the $r = \epsilon$ boundary. Although we disregard dyon recoil effects so that the dyon is approximately static, the boundary condition on fermion fields still explicitly depends on the charge collective coordinate, $\mathfrak{a}$, which complicates its use. Section §2.3 discusses two possible solutions to this problem – the first of these treats a subset of the dyon-fermion interactions (including all interactions with $\mathfrak{a}$) perturbatively, while the second makes use of the Born-Oppenheimer approximation and replaces $\mathfrak{a}(t)$ with a classical variable when calculating fermionic observables. Finally, §2.5 compares the dyon EFT of this chapter to other cases where the PPEFT formalism can be applied (such as when the point-particle is a nucleus) and explains why observables of the fermion-dyon theory need not be suppressed by the small point-particle radius R, unlike observables in the nuclear case.

2.1 Julia-Zee dyon in the semiclassical approximation

Once again, we consider the $SU(2)$ Georgi-Glashow model coupled to a massless Dirac isodoublet fermion[1]

$$S = -\int \mathrm{d}^4x \left[\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi_a + \frac{1}{2} \overline{\psi} \gamma^\mu \overleftrightarrow{D}_\mu \psi - \frac{\mu^2}{2} \Phi^a \Phi_a + \frac{\lambda}{4} (\Phi^a \Phi_a)^2 \right] , \tag{2.1.1}$$

where we use the symmetric covariant derivative $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$ to ensure that the action is real[2].

As explained in §1.1, the interactions of a dyon with other degrees of freedom are found by performing a semiclassical expansion of the quantum fields around the classical dyonic background, with

$$A^a_\mu(x) = \mathcal{A}^a_\mu(x) + \widehat{A}^a_\mu(x) \quad \text{and} \quad \Phi^a(x) = \varphi^a(x) + \widehat{\Phi}^a(x), \tag{2.1.2}$$

where the fields $\widehat{A}^a_\mu(x)$ and $\widehat{\Phi}^a(x)$ join the fermion field $\psi(x)$ as quantum operators. The semiclassical expansion proceeds by using (2.1.2) in the action (2.1.1) and Taylor expanding in the fluctuation fields. The classical contribution to the action (with $\widehat{A}^a_\mu = \widehat{\Phi}^a = \psi = 0$) is order $4\pi/e^2$ in the same way that eq. (1.1.25) implies that M/m_g is order $4\pi/e^2$. The leading dependence on fluctuations is quadratic and describes free quantum fields evolving within the dyonic background. Because the quadratic term is independent of e the masses of the quanta destroyed by the fluctuation fields are suppressed compared to the dyon mass by $e^2/4\pi$. It is because successive terms in this expansion are suppressed by still more powers of $e^2/4\pi$ that semiclassical methods are under control in the weakly coupled regime.

In particular, the kinetic term for the collective coordinate $\mathfrak{a}$ is found by evaluating the Maxwell kinetic term using the ansatz

$$e\mathcal{A}^\pm_i = \frac{\pm 1 - \cos\theta}{2r\sin\theta} \hat{\phi}_i \tau_3 - \frac{\mathcal{K}(r)}{2r} \Big[(i\hat{\theta}_i - \hat{\phi}_i) e^{i\mathfrak{a}(t)} e^{\pm i\phi} \tau_+ - (i\hat{\theta}_i + \hat{\phi}_i) e^{-i\mathfrak{a}(t)} e^{\mp i\phi} \tau_- \Big] \tag{2.1.3}$$

[1] Because this corresponds to two pseudoreal doublets, it is the minimal anomaly-free fermion content [70].
[2] The distinction between the symmetric and the usual one-sided derivatives in the Dirac action is a total derivative, but much of the later discussion hinges on being careful with total derivatives and boundary terms.

for which the field strength contains

$$\mathcal{F}_{0i} \ni \partial_0 \mathcal{A}_i^{\pm} = -\frac{i\dot{\mathfrak{a}}\,\mathcal{K}(r)}{2er}\Big[(i\hat{\theta}_i - \hat{\phi}_i)e^{i\mathfrak{a}}e^{\pm i\phi}\tau_+ + (i\hat{\theta}_i + \hat{\phi}_i)e^{-i\mathfrak{a}}e^{\mp i\phi}\tau_-\Big]\,, \tag{2.1.4}$$

and so

$$-\mathrm{Tr}(\mathcal{F}_{0i}\mathcal{F}^{0i}) = \dot{\mathfrak{a}}^2(-i)^2\left[\frac{\mathcal{K}(r)}{er}\right]^2(i\hat{\theta}_i - \hat{\phi}_i)\cdot(i\hat{\theta}_i + \hat{\phi}_i)\,\frac{\mathrm{Tr}(\tau_+\tau_-)}{2} = \left[\frac{\mathcal{K}(r)}{er}\right]^2\dot{\mathfrak{a}}^2\,. \tag{2.1.5}$$

This is localized near the position of the dyon because $\mathcal{K}(r)$ falls to zero for large r. This can be written as $\frac{1}{2}\,\mathcal{I}\dot{\mathfrak{a}}^2\,\delta^3(x)$ if the constant $\mathcal{I}$ is given by

$$\mathcal{I} = \frac{8\pi}{e^2}\int_0^\infty \mathrm{d}r\;\mathcal{K}^2(r) \sim \frac{1}{\alpha m_g}\,, \tag{2.1.6}$$

and so is parametrically large compared to the dyon size $R \sim m_g^{-1}$.

In the absence of fermion interactions the kinetic lagrangian for $\mathfrak{a}$ is $\frac{1}{2}\,\mathcal{I}\,\dot{\mathfrak{a}}^2$ and because $\mathfrak{a}$ is a periodic variable it behaves like a quantum rigid rotor, whose canonical momentum $\mathfrak{p}$ is both conserved and quantized. Because $\mathfrak{a}$ shifts under electromagnetic gauge transforms, with $\delta\mathfrak{a} = \omega$ when $\delta A^3_\mu = \frac{1}{e}\partial_\mu\omega$, the conserved rotor momentum is proportional to the rotor's contribution to the dyon-localized electric charge[3]

$$\mathcal{Q}_{D} = -e\,\mathfrak{p}\,, \tag{2.1.7}$$

and for $\mathfrak{a}$ identified with $\mathfrak{a}+2\pi$ the canonical rotor momentum takes integer values $\mathfrak{p}|n\rangle = n|n\rangle$ (and so $\mathcal{Q}_{D} = -en$). Unit steps in the rotor levels differ in charge by $\delta\mathcal{Q}_{D} = \pm e$ and differ in energy by

$$\delta E_n = E_{n+1} - E_n = \frac{2n+1}{\mathcal{I}} \sim \frac{e^2 m_g}{4\pi} \ll m_g\,. \tag{2.1.8}$$

These estimates show that dyonic excitations belong in the effective theory for energies below m_g precisely because the characteristic loop-counting factor $e^2/4\pi$ suppresses δE relative to m_g. But because the steps in rotor energy are proportional to $e^2/4\pi$ changes to rotor energies due to transitions can be negligible at a given order in the semiclassical expansion, so care must be taken

[3]We return to a more precise statement of rotor quantization including the effects of a vacuum angle in §4.2 below.

about the ordering of the limits $E/m_g \to 0$ and $e^2/4\pi \to 0$ when working at low energies within the semiclassical limit.

Fermion S-wave modes

Our main focus in this thesis is on how the bulk fermion interacts with the dyon, so we now consider the leading dependence of the action on the fermions, which is given by

$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4x\, \overline{\boldsymbol{\psi}}\, \gamma^\mu \left[\overleftrightarrow{\partial}_\mu - i\, e\mathcal{A}^a_\mu \tau_a\right] \boldsymbol{\psi}. \tag{2.1.9}$$

At low energies, the partial waves that dominate in fermion-dyon scattering are those that minimize the centrifugal barrier that must be penetrated in order to reach the dyon core. As argued in §1.1, this is achieved by S-wave fermions which do not experience the barrier at all since their total angular momentum vanishes. The explicit form for this type of S-wave fermion mode for the Julia-Zee dyon was found for the solution (1.1.13) by Jackiw and Rebbi [72], and writing the spin index i and isospin index a as a matrix M_{ia} for each chirality of field the S-wave configuration has the structure $M(x) = s(r,t) + ip(r,t)\,\hat{\boldsymbol{r}}\cdot\vec{\tau}$ and so ψ_L and ψ_R each involve two independent functions. Once transformed to the gauge where the solution has the form (1.1.19) the result becomes

$$\boldsymbol{\psi}(x) = \frac{1}{r}\begin{bmatrix}\psi_+(x)\\ \psi_-(x)\end{bmatrix} \quad \text{where} \quad \psi_\pm(x) = \begin{pmatrix} f_\pm(r,t)\, \eta_\pm(\theta,\phi) \\ g_\pm(r,t)\, \eta_\pm(\theta,\phi)\end{pmatrix}, \tag{2.1.10}$$

where square brackets denote gauge isodoublets and round brackets denote 4-component Dirac spinors in a basis for which $\gamma_5 = \mathrm{diag}(I, -I)$. The 2-component Weyl spinors $\eta_\pm$ are defined to satisfy $\sigma^r\eta_\pm = \pm\eta_\pm$ (see Appendix A for our Dirac matrix conventions) and so are given explicitly by

$$\eta_+(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\cos\frac{\theta}{2}\, e^{-i\phi}\\ \sin\frac{\theta}{2}\end{pmatrix} \quad \text{and} \quad \eta_-(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\, e^{i\phi}\end{pmatrix} \tag{2.1.11}$$

in the region R_- and by $\eta'_\pm(\theta,\phi) = \eta_\pm(\theta,\phi)\, e^{\pm i\phi}$ in the region R_+.

At large distances from the dyon core only the far-field electromagnetic parts of the monopole fields matter and so the S-wave sector Dirac equation simplifies to

$$\left(-\partial_t + \gamma^0\gamma^r\partial_r + \frac{i}{2}\,e\mathcal{A}_0^3\,\tau_3\right)(r\psi) = \begin{bmatrix} \left(-\partial_t + \partial_r + \frac{i}{2}\,e\mathcal{A}_0^3\right) f_+\eta_+ \\ \left(-\partial_t - \partial_r + \frac{i}{2}\,e\mathcal{A}_0^3\right) g_+\eta_+ \\ \left(-\partial_t - \partial_r - \frac{i}{2}\,e\mathcal{A}_0^3\right) f_-\eta_- \\ \left(-\partial_t + \partial_r - \frac{i}{2}\,e\mathcal{A}_0^3\right) g_-\eta_- \end{bmatrix} = 0 \tag{2.1.12}$$

which uses $\gamma^0\gamma^r = \mathrm{diag}(\sigma^r, -\sigma^r)$ (see Appendix A) and $\sigma^r\eta_\pm = \pm\eta_\pm$. In principle we can simultaneously diagonalize γ_5, $\gamma^0\gamma^r$ and τ_3, each of which has eigenvalues ±1, leading to eight possible unique sets of quantum numbers, $\{\mathfrak{c}, \mathfrak{h}, \mathfrak{s}\}$, where we denote the eigenvalues of γ_5, $\gamma^0\gamma^r$ and τ_3 respectively by $\mathfrak{c}$ (chirality), $\mathfrak{h}$ (helicity) and $\mathfrak{s}$ (for τ_3). Of these, the S-wave condition (2.1.10) says that the eigenvalues of τ_3 and $\gamma_5\gamma^0\gamma^r = \mathrm{diag}(\sigma^r, \sigma^r)$ are the same and so $\mathfrak{s} = \mathfrak{c}\mathfrak{h}$. Also, eq. (2.1.12) shows that the direction of motion (radially ingoing or radially outgoing) is correlated with the eigenvalue of $\gamma^0\gamma^r$ with $\mathfrak{h} = +1$ corresponding to infalling modes and $\mathfrak{h} = -1$ pairing with outgoing modes.

For an S-wave fermion passage through the origin inevitably brings a change of radial fermion direction and this must change the sign of $\mathfrak{h}$. Because $\mathfrak{h} = \mathfrak{c}\mathfrak{s}$ we see why S-wave scattering famously must involve a change in chirality ($\mathfrak{c}$) or in electric charge ($\mathfrak{s}$). If it should be true that the microscopic properties of the dyon preserve chirality – as is indeed the case *e.g.* when these interactions are modeled by solving the Dirac equation in the presence of a fixed dyon background [61, 73, 74, 62] – then $\mathfrak{c}$ cannot change and so only charge changing processes $\mathfrak{s}' \neq \mathfrak{s}$ are possible. Famously, the rotor-fermion interactions can significantly modify the fermionic vacuum which can in turn allow more complicated behaviour [31, 32, 63–66, 68, 56–59] (more about which later). For this reason we do not assume below that it is $\mathfrak{s}$ that must change during S-wave fermion-dyon scattering.

We label the four independent modes by their quantum numbers $\mathfrak{s}$ and $\mathfrak{c}$ (from which the S-wave condition implies $\mathfrak{h} = \mathfrak{c}\mathfrak{s}$), and then integrate to find the mode functions with frequency ω once $\mathcal{A}_0^3$ is evaluated using (1.1.19) specialized to the asymptotic form (1.1.22). This gives the following explicit basis of far-field positive-frequency solutions $u_{\mathfrak{s}\mathfrak{c}\omega}(x)$,

$$\begin{bmatrix} u_{+\mathfrak{c}\omega}(x) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u_{-\mathfrak{c}\omega}(x) \end{bmatrix} \quad \text{where} \quad u_{\mathfrak{s}\mathfrak{c}\omega}(x) := \frac{1}{r}\,\xi_{\mathfrak{c}} \otimes \eta_{\mathfrak{s}}(\theta,\phi)\,e^{-i\omega t} e^{-i(\mathfrak{s}\omega + \frac{1}{2}ev)\mathfrak{c}r} \left(\frac{r}{r_0}\right)^{i\mathfrak{c}eQ/2}, \tag{2.1.13}$$

with $\sigma_3\xi_{\mathfrak{c}} = \mathfrak{c}\,\xi_{\mathfrak{c}}$ and we write $\gamma_5 = \sigma_3 \otimes I$. The length scale r_0 is arbitrary and is introduced on dimensional grounds due to the singular Coulomb phase. Here ω is bounded from below, but the asymptotic voltage v implies the floor is $\omega \geq -\frac{1}{2}\mathfrak{s}ev$. The oscillatory factors in this expression can be equivalently written $e^{-i(k-\frac{1}{2}\mathfrak{s}ev)t}e^{-i\mathfrak{s}\mathfrak{c}kr}$ for $k \geq 0$. The negative frequency counterparts similarly are

$$\begin{bmatrix} v_{+\mathfrak{c}\omega}(x) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_{-\mathfrak{c}\omega}(x) \end{bmatrix} \quad \text{where} \quad v_{\mathfrak{s}\mathfrak{c}\omega}(x) := \frac{1}{r}\,\xi_{\mathfrak{c}} \otimes \eta_{\mathfrak{s}}(\theta,\phi)\, e^{i\omega t} e^{i(\mathfrak{s}\omega - \frac{1}{2}ev)\,\mathfrak{c}\,r} \left(\frac{r}{r_0}\right)^{i\mathfrak{c}\,eQ/2}. \tag{2.1.14}$$

Frequency ω is again bounded from below, but in this case the floor is $\omega \geq +\frac{1}{2}\mathfrak{s}ev$ and the oscillatory factors can be written $e^{i(k+\frac{1}{2}\mathfrak{s}ev)t}e^{i\mathfrak{s}\mathfrak{c}kr}$ for $k \geq 0$.

Keeping in mind that particle probability flux points in the opposite direction to the momentum for negative-frequency states (see for example [75]), the sign of the radial direction of motion for these modes is given by $-\mathfrak{s}\mathfrak{c}$ for particles and by $+\mathfrak{s}\mathfrak{c}$ for antiparticles. These modes are normalized so that $i\overline{u}_{\mathfrak{s}\mathfrak{c}\omega}\gamma^0 u_{\mathfrak{s}\mathfrak{c}\omega} = i\overline{v}_{\mathfrak{s}\mathfrak{c}\omega}\gamma^0 v_{\mathfrak{s}\mathfrak{c}\omega} = (4\pi r^2)^{-1}$ and as a result $i\overline{u}_{\mathfrak{s}\mathfrak{c}\omega}\gamma^r u_{\mathfrak{s}\mathfrak{c}\omega} = i\overline{v}_{\mathfrak{s}\mathfrak{c}\omega}\gamma^r v_{\mathfrak{s}\mathfrak{c}\omega} = -\mathfrak{s}\mathfrak{c}/(4\pi r^2) = -\mathfrak{h}/(4\pi r^2)$.

2D formulation

As has been often noted elsewhere purely S-wave dynamics can be usefully rewritten as a Dirac equation in 1+1 dimensions [31, 32], and we pause here to establish our conventions. In this 1+1 dimensional formulation the only spatial direction corresponds to the radial direction in the 3+1 dimensional formulation, and so runs only along the half-line corresponding to positive r, with the dyon interior providing a more complicated background solution only within a distance of order μ^{-1} about $r = 0$.

Within the full theory fields are required to be nonsingular at the origin (deep within the dyon) and this can be converted in the 2D formulation into a boundary condition for fields at $r = 0$ designed so that reflection at $r = 0$ describes the effect in the 4D theory of passing through the origin. Alternatively we can instead describe S-wave scattering within the 2D picture by extending the spatial direction to cover the entire real line and think of, say, $x < 0$ as describing incoming waves and $x > 0$ describing outgoing waves.

In either formulation 4D fermions are described by independent 2D spinors, χ, for each fermion flavour and chirality in the following way[4]

$$\chi_+(r,t) = \begin{pmatrix} f_+(r,t) \\ g_+(r,t) \end{pmatrix} \quad \text{and} \quad \chi_-(r,t) = \begin{pmatrix} g_-(r,t) \\ f_-(r,t) \end{pmatrix} = i\Gamma^0 \begin{pmatrix} f_-(r,t) \\ g_-(r,t) \end{pmatrix}, \tag{2.1.15}$$

with the subscript $\pm$ indicating both the isospin and the $U(1)$ charge of each spinor (see Appendix A for our 2D Dirac-matrix conventions). Comparing to (2.1.12) shows that their equations of motion can then be compactly written as a 2D Dirac equation in the background potential $\mathcal{A}_0^3(r)$ of the form:

$$\left(-\partial_t + \Gamma^0\Gamma^1\partial_1 + \frac{i\mathfrak{s}}{2}\, e\mathcal{A}_0^3\right)\chi_{\mathfrak{s}} = 0\,, \tag{2.1.16}$$

such as would follow from the bulk action

$$S_2 = -\frac{1}{2}\sum_{\mathfrak{s}=\pm}\int \mathrm{d}^2x\; \overline{\chi}_{\mathfrak{s}}\,\Gamma^\alpha \overleftrightarrow{D}_\alpha \chi_{\mathfrak{s}}\,, \tag{2.1.17}$$

where $\overline{\chi} := i\chi^\dagger\Gamma^0$ and $D_\alpha\chi_{\mathfrak{s}} = (\partial_\alpha - \frac{1}{2}ie\mathfrak{s}\mathcal{A}_\alpha^3)\chi_{\mathfrak{s}}$.

Notice that (2.1.10) shows that the eigenvalue of the 2D chirality $\Gamma_c := \Gamma^0\Gamma^1 = \sigma_3$ in this representation is $\mathfrak{c}\mathfrak{s}$ where $\mathfrak{c}$ is the eigenvalue of 4D chirality and so is the same as $\mathfrak{h}$ and therefore should correlate with the direction of motion. This correlation in the 2D language is a direct consequence of the Dirac equation because $\Gamma^0(\Gamma^\alpha\partial_\alpha) = -\partial_t + \Gamma_c\partial_1$.

Dimensional reduction of the S-wave modes strips away their angular content, but otherwise leaves them much as described in (2.1.13) and (2.1.14) above. For instance, writing the doublet built from χ_+ and χ_- by $\boldsymbol{\chi}$ (and continuing to denote the isodoublets by square brackets and spinor doublets with round brackets), the positive-frequency basis of solutions $\mathfrak{u}_{\mathfrak{s}\mathfrak{c}\omega}(x)$ corresponding to (2.1.13) are

$$\begin{bmatrix} \mathfrak{u}_{+\mathfrak{c}\omega}(x) \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ \mathfrak{u}_{-\mathfrak{c}\omega}(x) \end{bmatrix} \quad \text{where} \quad \mathfrak{u}_{\mathfrak{s}\mathfrak{c}\omega}(x) := \xi_{\mathfrak{s}\mathfrak{c}}\, e^{-i\omega t} e^{-i(\mathfrak{s}\omega + \frac{1}{2}ev)\,\mathfrak{c}\, r}\left(\frac{r}{r_0}\right)^{i\mathfrak{c}\, eQ/2}, \tag{2.1.18}$$

[4]This is to be compared with the 4D expression (2.1.10). These definitions ensure that $\psi_{\mathfrak{s}}$ and $\chi_{\mathfrak{s}}$ transform in the same way under the remaining Lorentz transformation in 2D *i.e.* radial boosts. These are generated by $\gamma^0\gamma^r$ in 4D and $\Gamma^0\Gamma^1 = \Gamma_c$ in 2D, and since $\mathfrak{h} = -\mathfrak{c}$ for negatively charged fermions, the order of $f_-(r,t)$ and $g_-(r,t)$ is switched when going to 2D.

with $\Gamma_c\, \xi_{\mathfrak{s}\mathfrak{c}} = \sigma_3\, \xi_{\mathfrak{s}\mathfrak{c}} = \mathfrak{s}\mathfrak{c}\, \xi_{\mathfrak{s}\mathfrak{c}}$. The negative frequency counterparts similarly are

$$\begin{bmatrix} \mathfrak{v}_{+\mathfrak{c}\omega}(x) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathfrak{v}_{-\mathfrak{c}\omega}(x) \end{bmatrix} \quad \text{where} \quad \mathfrak{v}_{\mathfrak{s}\mathfrak{c}\omega}(x) := \xi_{\mathfrak{s}\mathfrak{c}}\, e^{i\omega t} e^{i(\mathfrak{s}\omega - \frac{1}{2}ev)\,\mathfrak{c}\, r} \left(\frac{r}{r_0}\right)^{i\mathfrak{c}\, eQ/2}. \tag{2.1.19}$$

The frequency ω is (as above) bounded from below, with floors $\omega \geq -\frac{1}{2}\mathfrak{s}ev$ and $\omega \geq +\frac{1}{2}\mathfrak{s}ev$ for particle and antiparticle respectively. These modes are normalized so that $i\overline{\mathfrak{u}}_{\mathfrak{s}\mathfrak{c}\omega}\Gamma^0\mathfrak{u}_{\mathfrak{s}\mathfrak{c}\omega} = i\overline{\mathfrak{v}}_{\mathfrak{s}\mathfrak{c}\omega}\Gamma^0\mathfrak{v}_{\mathfrak{s}\mathfrak{c}\omega} = 1$ and so the 2D and 4D fluxes are related by $i\overline{\mathfrak{u}}_{\mathfrak{s}\mathfrak{c}\omega}\Gamma^r\mathfrak{u}_{\mathfrak{s}\mathfrak{c}\omega} = 4\pi r^2(i\overline{u}_{\mathfrak{s}\mathfrak{c}\omega}\gamma^r u_{\mathfrak{s}\mathfrak{c}\omega})$ and similarly for $\mathfrak{v}$ and v.

2.2 S-wave PPEFT

The far-field modes given in (2.1.13) and (2.1.14) suffice to fully describe the state of both an incident and departing fermion (or antifermion) when it is far from the dyon, but scattering calculations relate the size of the departing wave to the initially incident one and in principle this requires solving the full Dirac equation obtained from (2.1.9) including the dyon's interior structure. It is only once this structure is included that the fermionic modes can be required to be nonsingular at the origin, providing the information that ultimately links the incoming and outgoing modes.

But solving the full Dirac equation including the dyon interior is difficult (see however [61, 73, 74, 62]) and also likely overkill, at least for incident fermion energy E small compared with the inverse dyon size μ. In this regime the dyon is so small that its physical implications should be capturable by a choice of boundary condition near the origin. But this is where EFT techniques can usefully be applied and to this end we apply here the PPEFT formalism [33–35] — a framework explicitly designed for the purpose of constructively deriving such boundary conditions starting from the low-energy effective action for the compact object (for a review see *e.g.* [14]). This section applies this formalism to determine the form of the boundary conditions required for the Julia-Zee dyon (and its more complicated counterparts), at leading order in the small ratio E/μ.

The strategy, as always for EFTs, starts by replacing the full dyon with an effective description that captures its low-energy interactions. The required EFT is defined along the dyonic world-line and describes the interactions between low-energy bulk fields and the low-energy dyonic collective coordinates. This begins with an enumeration of all possible lowest-dimension interactions allowed by the assumed symmetries and particle content.

The dyon-localized fields to be included in the dyonic effective action include the dyonic collective coordinates like the centre-of-mass position, $x^\mu = y^\mu(s)$, of the dyon's world-line W and the charge-excitation field $\mathfrak{a}(s)$, where s is an arbitrary parameter along the world-line. The 'bulk' fields[5] to be included are those with masses much smaller than μ: the fermion doublet $\boldsymbol{\psi}(x)$ and the massless bulk electromagnetic field $\widehat{A}_\mu(x) = \widehat{A}^3_\mu(x)$, and (in principle) the spacetime metric $g_{\mu\nu}(x)$.

The symmetries to be imposed are (i) world-line reparameterizations of s; (ii) Poincaré invariance in spacetime (or general coordinate invariance if a general metric $g_{\mu\nu}$ is included); (iii) invariance under electromagnetic gauge transformations; and (iv) any low-energy flavour and/or discrete symmetries. Spacetime symmetries are built in by constructing the action using the pull-backs of spacetime tensors to the world-line,[6]

$$\psi(s) = \psi[y(s)]\,, \quad \widehat{A}(s) = \dot{y}^\mu(s)\widehat{A}_\mu[y(s)] \quad \text{and} \quad \gamma(s) = \dot{y}^\mu(s)\dot{y}^\nu(s)g_{\mu\nu}[y(s)]\,, \tag{2.2.1}$$

where over-dots denote differentiation with respect to s. One builds from these a reparameterization invariant action in the standard way.

For these fields the electromagnetic gauge transformations are

$$\delta\boldsymbol{\psi}(s) = \frac{i}{2}\,\omega(s)\,\tau_3\boldsymbol{\psi}(s)\,, \quad \delta\widehat{A}(s) = \frac{1}{e}\,\dot{\omega}(s) \quad \text{and} \quad \delta\mathfrak{a}(s) = \omega(s)\,, \tag{2.2.2}$$

where $\omega(s)$ is related to the spacetime dependent $SU(2)$ gauge transformation parameters $\omega^a(x)$ by $\omega(s) = \omega^3[y(s)]$. This symmetry requires derivatives of $\mathfrak{a}$ to appear within a covariant derivative

$$D\mathfrak{a} := \dot{\mathfrak{a}} - e\widehat{A}\,. \tag{2.2.3}$$

[5] Here 'bulk' fields mean fields that are defined everywhere in spacetime and not only along the dyon's world-line.

[6] Interactions involving the normals to the world-line can also be constructed but do not play any role in what follows.

2.2.1 Dyon world-line effective action

For these variables the lowest-dimension interactions involving the rotor field $\mathfrak{a}$ and the dyon displacement y^μ located at the dyonic position become

$$S_{\rm dyon} = \int_W {\rm d}s \left\{ \sqrt{-\gamma} \left[-M - \frac{\mathcal{I}}{2\gamma}(D\mathfrak{a})^2 - \frac{1}{2}\, \overline{\psi}\, \mathfrak{C}(\mathfrak{a})\, \psi + \cdots \right] + \frac{\vartheta}{2\pi} D\mathfrak{a} \right\} \tag{2.2.4}$$

where M is the classical dyon mass while $\mathcal{I}$ is the 'rotor' coefficient (2.1.6) and the ellipses include a variety of other higher-dimension terms whose contributions to physics should be suppressed at low energies and so whose detailed form is not required in what follows.[7] The most general fermion bilinear consistent with the field content, gauge and spacetime symmetries is

$$\begin{aligned} \mathfrak{C}(\mathfrak{a}) \;:=\; & \hat{\mathfrak{c}}_1^s + i\,\hat{\mathfrak{c}}_1^{ps}\gamma_5 + i\,\hat{\mathfrak{c}}_1^{v}\gamma_\mu \dot{y}^\mu + i\,\hat{\mathfrak{c}}_1^{pv}\gamma_5\gamma_\mu \dot{y}^\mu + \Big(\hat{\mathfrak{c}}_3^s + i\,\hat{\mathfrak{c}}_3^{ps}\gamma_5 + i\,\hat{\mathfrak{c}}_3^{v}\gamma_\mu \dot{y}^\mu + i\,\hat{\mathfrak{c}}_3^{pv}\gamma_5\gamma_\mu \dot{y}^\mu \Big)\tau_3 \\ & + \Big(\hat{\mathfrak{c}}_+^s + i\,\hat{\mathfrak{c}}_+^{ps}\gamma_5 + i\,\hat{\mathfrak{c}}_+^{v}\gamma_\mu \dot{y}^\mu + i\,\hat{\mathfrak{c}}_+^{pv}\gamma_5\gamma_\mu \dot{y}^\mu \Big) e^{i\mathfrak{a}}\tau_+ \\ & + \Big(\hat{\mathfrak{c}}_-^s + i\,\hat{\mathfrak{c}}_-^{ps}\gamma_5 + i\,\hat{\mathfrak{c}}_-^{v}\gamma_\mu \dot{y}^\mu + i\,\hat{\mathfrak{c}}_-^{pv}\gamma_5\gamma_\mu \dot{y}^\mu \Big) e^{-i\mathfrak{a}}\tau_- \,, \end{aligned} \tag{2.2.5}$$

where (as before) $\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$.

This action is meant to capture the low-energy interactions of the underlying dyon and the quantities M, $\mathfrak{C}$ and so on are found by matching to the properties of the dyon's microscopic description. In particular, this relates the parameter ϑ to the vacuum angle appearing in the bulk electromagnetic theta-term

$$\mathcal{L}_\vartheta = \frac{\vartheta e^2}{4\pi^2}\, \mathbf{E}\cdot\mathbf{B}\,, \tag{2.2.6}$$

since both ultimately descended from the underlying theta-term for the microscopic nonabelian $SU(2)$ gauge interactions. (Equivalently, this connection can also be established using invariance of the system under large gauge transformations.)

To interpret (2.2.4) it is convenient to specialize to a dyon that is perturbatively close to being at rest so $\dot{y}^\mu(s) = \delta^\mu_0 + \delta\dot{y}^\mu(s)$, choose the Minkowski metric and choose the parameter $s = t$ to be time in the background dyon's rest frame. In this case $-\gamma = -\eta_{\mu\nu}\dot{y}^\mu\dot{y}^\nu = 1 - \dot{\boldsymbol{y}}\cdot\dot{\boldsymbol{y}} + \cdots$ (where $\boldsymbol{y}$

[7] Notice we only include here the couplings of the fluctuation fields and not also the dyon interactions giving rise to the background fields themselves. This is why no term like $Q\widehat{A}$ appears describing the dyon's classical charge, and why no term is required expressing the dyon's magnetic charge [76]. The same could also have been done for the metric if we'd expanded about the dyon's gravitational back-reaction, but we do not do so.

is the spatial part of δy^μ) and (2.2.4) becomes

$$S_{\rm dyon} \simeq \int_W {\rm d}t \left[\frac{M}{2} \dot{\boldsymbol{y}} \cdot \dot{\boldsymbol{y}} + \frac{\mathcal{I}}{2} \Big(\dot{\mathfrak{a}} - e\widehat{A}_0 \Big)^2 - \frac{1}{2} \, \overline{\boldsymbol{\psi}} \, \mathfrak{C}(\mathfrak{a}) \, \boldsymbol{\psi} + \frac{\vartheta}{2\pi} \Big(\dot{\mathfrak{a}} - e\widehat{A}_0 \Big) + \cdots \right] \tag{2.2.7}$$

where to leading order – *i.e.* neglecting dyon recoil effects – the integral is evaluated along the dyon's world-line, which we choose to be located at $\boldsymbol{r} = \boldsymbol{0}$. In the same approximation the matrix $\mathfrak{C}(\mathfrak{a})$ controlling the fermion-dyon couplings becomes

$$\begin{aligned} \mathfrak{C}(\mathfrak{a}) \; := \; & \hat{\mathfrak{c}}_1^s + i \, \hat{\mathfrak{c}}_1^{ps} \gamma_5 - i \, \hat{\mathfrak{c}}_1^{v} \gamma^0 - i \, \hat{\mathfrak{c}}_1^{pv} \gamma_5 \gamma^0 + \Big(\hat{\mathfrak{c}}_3^s + i \, \hat{\mathfrak{c}}_3^{ps} \gamma_5 - i \, \hat{\mathfrak{c}}_3^{v} \gamma^0 - i \, \hat{\mathfrak{c}}_3^{pv} \gamma_5 \gamma^0 \Big) \tau_3 \\ & + \Big(\hat{\mathfrak{c}}_+^s + i \, \hat{\mathfrak{c}}_+^{ps} \gamma_5 - i \, \hat{\mathfrak{c}}_+^{v} \gamma^0 - i \, \hat{\mathfrak{c}}_+^{pv} \gamma_5 \gamma^0 \Big) e^{i\mathfrak{a}} \tau_+ + \Big(\hat{\mathfrak{c}}_-^s + i \, \hat{\mathfrak{c}}_-^{ps} \gamma_5 - i \, \hat{\mathfrak{c}}_-^{v} \gamma^0 - i \, \hat{\mathfrak{c}}_-^{pv} \gamma_5 \gamma^0 \Big) e^{-i\mathfrak{a}} \tau_- \, , \end{aligned} \tag{2.2.8}$$

which drops $\boldsymbol{\gamma} \cdot \dot{\boldsymbol{y}}$ dyon-recoil terms.

On dimensional grounds all sixteen of the effective couplings[8] $\hat{\mathfrak{c}}$ have dimensions (length)^2. All of these effective fermion interactions conserve electric charge, and this dictates the $\mathfrak{a}$-dependence of $\mathfrak{C}(\mathfrak{a})$. If treated perturbatively these interactions describe fermion scattering from the dyon, possibly associated with $\mathfrak{a}$ excitation. The perturbative dyon response to fermion scattering can be seen because the canonical momentum for $\mathfrak{a}$ is also its conserved charge,

$$\mathfrak{p} := \frac{\delta S_{\rm dyon}}{\delta \dot{\mathfrak{a}}} = \mathcal{I} D \mathfrak{a} + \frac{\vartheta}{2\pi} = - \frac{\mathcal{Q}_{{}_D}}{e} \, , \tag{2.2.9}$$

where – see *e.g.* eq. (2.2.11) – $\mathcal{Q}_{{}_D}$ is the contribution of dyonic excitations to the fluctuations' electric charge: $\mathcal{Q} = \mathcal{Q}_{{}_D} + \mathcal{Q}_{{}_F}$ (where $\mathcal{Q}_{{}_F}$ is the electric charge carried by the fermions). The canonical commutation relation $[\mathfrak{p}(t) \, , \mathfrak{a}(t)] = -i$ therefore implies the quantities $e^{\pm i\mathfrak{a}}$ act as raising/lowering operators for $\mathcal{Q}_{{}_D}$, since

$$\Big[\mathcal{Q}_{{}_D}/e \, , e^{\pm i \mathfrak{a}} \Big] = \mp e^{\pm i\mathfrak{a}} \, .$$

Interactions proportional to $e^{\pm i\mathfrak{a}}$ therefore describe transitions that raise or lower the dyon charge by e, as required by charge conservation for the reactions involving $\tau_\pm$ in (2.2.7) (which change the

[8]Since the size of $\mathfrak{c}$ might naturally be expected to be set by the dyon size $R \sim \mu^{-1}$ this is usually where the suppression by the monopole scale would naively enter into observable scattering. Part of the purpose of this exercise is to understand why this suppression does *not* actually arise in fermion-dyon scattering, and how general a phenomenon this is.

fermion charge from $-\frac{1}{2}e$ to $+\frac{1}{2}e$ or vice versa).

Because these fermion-$\mathfrak{a}$ couplings are not suppressed by powers of small couplings like e they are not intrinsically negligible even at leading loop order, which does not justify expanding the exponentials in powers of $\mathfrak{a}$. Furthermore the energy exchanged by exciting or de-exciting $\mathfrak{a}$ is order $\mathcal{I}^{-1}$ which (2.1.8) reveals is suppressed by $\alpha = e^2/(4\pi)$ compared to the characteristic scale μ and so has little intrinsic cost in the semiclassical limit. Such transitions can be important to fermion scattering, though there is an order-of-limits issue when working both at low fermion energies, $E \ll \mu$, and at leading nontrivial order in the loop expansion, $\alpha = e^2/(4\pi) \ll 1$, because it matters in practice whether or not the fermion energy E is larger or smaller than the dyon excitation scale $\alpha\mu$. As argued below, a natural way to handle this fermion-dyon dynamics in the effective theory – at least in the regime $\alpha\mu \ll E \ll \mu$ – is through the Born-Oppenheimer approximation [78] (in which the fermions play the role of the 'fast' degrees of freedom while the large size of $\mathcal{I}$ makes the dyonic excitations 'slow').

The interactions in (2.2.5) can also be classified by how they transform under global 'flavour' transformations acting on the fermion field. In the present instance the limited field content restricts this to two types of such symmetries: an axial symmetry for which $\delta_A \boldsymbol{\psi} = i\omega_A\, \gamma_5\, \boldsymbol{\psi}$ and a fermion-number 'baryon' symmetry for which $\delta_B \boldsymbol{\psi} = i\omega_B\, \boldsymbol{\psi}$ (notice the difference between this and the gauge transformation (2.2.2)). We denote the corresponding conserved charges by $\mathcal{Q}_A$ and $\mathcal{Q}_B$ to distinguish them from the fermionic contribution to the gauge charge $\mathcal{Q}_F$. The Noether currents for these two symmetries are $J^\mu_B = i\overline{\boldsymbol{\psi}}\gamma^\mu\boldsymbol{\psi}$ and $J^\mu_A = i\overline{\boldsymbol{\psi}}\gamma^\mu\gamma_5\boldsymbol{\psi}$ and satisfy

$$\partial_\mu J^\mu_B = 0 \quad \text{and} \quad \partial_\mu J^\mu_A = \frac{e^2}{32\pi^2}\, \epsilon^{\mu\nu\lambda\rho} F^a_{\mu\nu} F^a_{\lambda\rho} = -\frac{e^2}{4\pi^2}\, \mathbf{E}^a \cdot \mathbf{B}^a \,, \tag{2.2.10}$$

which shows that the axial symmetry is anomalous. Table 2.1 identifies which of the various effective interactions of (2.2.5) preserves each of these two flavour symmetries.

Induced boundary conditions

In practice we wish to excise the dyon from the external world and replace it with a gaussian 'pillbox', $\mathcal{P}_\epsilon$, whose radius is chosen much larger than the dyon's, $\epsilon \gg R \sim m_g^{-1}$, but also much smaller than the distances of interest for the bulk fields used for low-energy probes of the dyon. The idea is to

	$\Delta\mathcal{Q}_F = \Delta\mathcal{Q}_A = 0$	$\Delta\mathcal{Q}_F = 0$, $\Delta\mathcal{Q}_A \neq 0$	$\Delta\mathcal{Q}_A = 0$ and $\Delta\mathcal{Q}_F \neq 0$	$\Delta\mathcal{Q}_A \neq 0$ and $\Delta\mathcal{Q}_F \neq 0$
$\hat{\mathfrak{c}}$	$\hat{\mathfrak{c}}_1^v, \hat{\mathfrak{c}}_3^v, \hat{\mathfrak{c}}_1^{pv}, \hat{\mathfrak{c}}_3^{pv}$	$\hat{\mathfrak{c}}_1^s, \hat{\mathfrak{c}}_3^s, \hat{\mathfrak{c}}_1^{ps}, \hat{\mathfrak{c}}_3^{ps}$	$\hat{\mathfrak{c}}_\pm^v, \hat{\mathfrak{c}}_\pm^{pv}$	$\hat{\mathfrak{c}}_\pm^s, \hat{\mathfrak{c}}_\pm^{ps}$

Table 2.1: Classification of interaction terms in the worldline action according to whether they conserve the fermionic charges $\mathcal{Q}_F$ (electric charge) and $\mathcal{Q}_A$ (axial charge) as defined in the main text. All interactions conserve $\mathcal{Q}_B$ (fermion number). Total electric charge (including the contribution from $\mathfrak{a}$) is also conserved by all interactions.

replace the dyon with a set of near-dyon boundary conditions on the surface $r = \epsilon$ of this pillbox, whose detailed form is chosen to reproduce the interactions implied by (2.2.4). It is these boundary conditions that communicate dyonic physics to the external world from which the dyon is excised.

The boundary conditions can be obtained by a variety of means [33–35], such as by integrating the field equations with the dyon-localized interaction written proportional to $\delta^3(x)$ and then regularizing the appearance of divergent bulk fields, like $\widehat{A}(0)$, with its value on the surface of the pillbox, $\widehat{A}(\epsilon)$. Applied to (2.2.7) this leads to a familiar expression: Gauss' law itself, in the form

$$\oint \mathrm{d}^2\Omega \left(r^2\partial_r \widehat{A}_0\right)_{r=\epsilon} = 4\pi\left(r^2\partial_r\widehat{A}_0\right)_{r=\epsilon} = \left(\frac{\delta S_{\rm dyon}}{\delta \widehat{A}_0}\right)_{r=\epsilon} = -e\mathfrak{p} = -e\left(\mathcal{I}D\mathfrak{a} + \frac{\vartheta}{2\pi}\right)\,. \tag{2.2.11}$$

Comparing this to the bulk far-field Coulomb solution $\widehat{A}_0 \simeq -\mathcal{Q}_D/(4\pi r)$ with integration constant $\mathcal{Q}_D$ given by the dyonic fluctuation's charge, this boundary condition fixes $\mathcal{Q}_D = -e\mathfrak{p}$ as used above (and in passing shows how the dyon acquires a ϑ-dependent charge through the Witten effect [79]).

We wish to make the same argument for the bulk fermion field. The same steps [35] lead to the boundary condition

$$\frac{1}{2}\oint \mathrm{d}^2\Omega \left(r^2\gamma^r\boldsymbol{\psi}\right)_{r=\epsilon} = \left(\frac{\delta S_{\rm dyon}}{\delta\overline{\boldsymbol{\psi}}}\right)_{r=\epsilon} = -\frac{1}{2}\Big[\mathfrak{C}(\mathfrak{a})\,\boldsymbol{\psi}\Big]_{r=\epsilon}\,, \tag{2.2.12}$$

once projected onto the S wave state. This can formally be recast as the non-differential condition

$$\begin{aligned} 0 = \Big\{\Big[\gamma^r &+ \left(\hat{\mathcal{C}}_1^s + i\,\hat{\mathcal{C}}_1^{ps}\gamma_5 - i\,\hat{\mathcal{C}}_1^v\gamma^0 - i\,\hat{\mathcal{C}}_1^{pv}\gamma_5\gamma^0\right) + \left(\hat{\mathcal{C}}_3^s + i\,\hat{\mathcal{C}}_3^{ps}\gamma_5 - i\,\hat{\mathcal{C}}_3^v\gamma^0 - i\,\hat{\mathcal{C}}_3^{pv}\gamma_5\gamma^0\right)\tau_3 \\ &+\left(\hat{\mathcal{C}}_+^s + i\,\hat{\mathcal{C}}_+^{ps}\gamma_5 - i\,\hat{\mathcal{C}}_+^v\gamma^0 - i\,\hat{\mathcal{C}}_+^{pv}\gamma_5\gamma^0\right)e^{i\mathfrak{a}}\tau_+ + \left(\hat{\mathcal{C}}_-^s + i\,\hat{\mathcal{C}}_-^{ps}\gamma_5 - i\,\hat{\mathcal{C}}_-^v\gamma^0 - i\,\hat{\mathcal{C}}_-^{pv}\gamma_5\gamma^0\right)e^{-i\mathfrak{a}}\tau_-\Big]\boldsymbol{\psi}\Big\}_{r=\epsilon} \end{aligned} \tag{2.2.13}$$

where the dimensionless coefficients $\hat{\mathcal{C}}^A_I$ are related to the coefficients $\hat{\mathfrak{c}}^A_I$ by

$$\hat{\mathcal{C}}^A_I = \frac{\hat{\mathfrak{c}}^A_I}{4\pi\epsilon^2}\,, \tag{2.2.14}$$

for all $A = s, ps, v, pv$ and $I = 1, 3, +, -$.

This expression is only 'formal' because the projection onto the S-wave while regulating by displacing the field to $r = \epsilon$ is more subtle because of the different angular dependence imposed on ψ_+ relative to ψ_- by the presence of the magnetic monopole background configuration. These issues are summarized in Appendix B and the result is most conveniently expressed in terms of the two-dimensional fields χ whose bulk dynamics is described by (2.1.17). In this 2D formulation the fermionic terms in the dyon's point-particle world line action given by (2.2.7) and (2.2.5) can be written

$$S_{\rm dyon2} = -\frac{1}{2}\sum_{\mathfrak{s},\mathfrak{s}'=\pm}\int {\rm d}t\,\overline{\chi}_{\mathfrak{s}}\Big(\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} + i\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'}\Gamma_c + i\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}\Gamma_\alpha\dot{y}^\alpha + i\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'}\Gamma^\alpha\epsilon_{\alpha\beta}\dot{y}^\beta\Big)\,\chi_{\mathfrak{s}'}\;e^{\frac{i}{2}(\mathfrak{s}-\mathfrak{s}')\,\mathfrak{a}}\,, \tag{2.2.15}$$

where the fields are evaluated at $r = \epsilon$. Here (as above) $\Gamma_c := \Gamma^0\Gamma^1$ has eigenvalues $\mathfrak{h} = \pm 1$ (and is diagonal in the basis used here) while the 2D Dirac matrices satisfy $\Gamma_c\Gamma^\alpha = \epsilon^{\alpha\beta}\Gamma_\beta$ where our Levi-Civita convention chooses $\epsilon^{01} = +1$. The coefficients $\mathcal{C}^A_{ij}$ are dimensionless in the same way that the $\hat{\mathcal{C}}^A_I$ are, and must satisfy $\mathcal{C}^{A*}_{\mathfrak{s}\mathfrak{s}'} = \mathcal{C}^A_{\mathfrak{s}'\mathfrak{s}}$ if the action $S_{\rm dyon2}$ is real (in which case they contain 16 independent real parameters).

Combining this 'boundary' action with the 'bulk' action (2.1.17) leads to the following near-dyon boundary condition for the 2D fermions near a static dyon (for which $\dot{y}^\alpha \simeq \delta^\alpha_0$)

$$\sum_{\mathfrak{s}'}\Big(\delta_{\mathfrak{s}\mathfrak{s}'}\Gamma^1 + \mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} + i\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'}\Gamma_c - i\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}\Gamma^0 + i\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'}\Gamma^1\Big)\,e^{\frac{i}{2}(\mathfrak{s}-\mathfrak{s}')\mathfrak{a}(t)}\chi_{\mathfrak{s}'}(\epsilon,t) = 0, \tag{2.2.16}$$

for all t. Equivalently

$$\Big[\Gamma^1 + O_{\mathcal{B}}(\mathfrak{a})\Big]\begin{bmatrix}\chi_+\\ \chi_-\end{bmatrix}_{r=\epsilon} = 0, \tag{2.2.17}$$

where

$$O_{\mathcal{B}}(\mathfrak{a}) = \begin{pmatrix} \mathcal{C}^s_{++} & \mathcal{C}^s_{+-}e^{i\mathfrak{a}} \\ \mathcal{C}^s_{-+}e^{-i\mathfrak{a}} & \mathcal{C}^s_{--} \end{pmatrix} + i \begin{pmatrix} \mathcal{C}^{ps}_{++} & \mathcal{C}^{ps}_{+-}e^{i\mathfrak{a}} \\ \mathcal{C}^{ps}_{-+}e^{-i\mathfrak{a}} & \mathcal{C}^{ps}_{--} \end{pmatrix} \Gamma_c$$
$$-i \begin{pmatrix} \mathcal{C}^v_{++} & \mathcal{C}^v_{+-}e^{i\mathfrak{a}} \\ \mathcal{C}^v_{-+}e^{-i\mathfrak{a}} & \mathcal{C}^v_{--} \end{pmatrix} \Gamma^0 + i \begin{pmatrix} \mathcal{C}^{pv}_{++} & \mathcal{C}^{pv}_{+-}e^{i\mathfrak{a}} \\ \mathcal{C}^{pv}_{-+}e^{-i\mathfrak{a}} & \mathcal{C}^{pv}_{--} \end{pmatrix} \Gamma^1 . \tag{2.2.18}$$

Some intuition about the physical meaning of this boundary condition can be obtained by considering what it implies for the flux of fermionic currents through the surface at $r = \epsilon$. Consider, for example, a fermion current of the form $j^\alpha = i\overline{\boldsymbol{\chi}}\Gamma^\alpha M\boldsymbol{\chi}$ where M is some matrix in spin and isospin space. When the action (2.1.17) is real the boundary condition allows $\Gamma^1\boldsymbol{\chi}(\epsilon,t)$ to be replaced by terms involving the effective couplings $\mathcal{C}^A_{ij}$ and this can be used to learn something about the flux $j^1(\epsilon,t)$. As applied to the currents for fermion number, $j^\alpha_{B} = i\overline{\boldsymbol{\chi}}\Gamma^\alpha\boldsymbol{\chi}$, electric charge, $j^\alpha_{F} = \frac{1}{2}ie\overline{\boldsymbol{\chi}}\Gamma^\alpha\tau_3\boldsymbol{\chi}$ and axial symmetry[9] $j^\alpha_{A} = i\overline{\boldsymbol{\chi}}\Gamma^\alpha\Gamma_c\tau_3\boldsymbol{\chi}$ (for real boundary action) the boundary conditions imply[10] $j^r_{B}(\epsilon,t) = 0$,

$$j^r_{F}(\epsilon,t) = -\tfrac{1}{4}ie\,\overline{\boldsymbol{\chi}}(\epsilon,t)\Big[\tau_3, O_{\mathcal{B}}(\mathfrak{a})\Big]\boldsymbol{\chi}(\epsilon,t) \quad \text{and} \quad j^r_{A}(\epsilon,t) = \tfrac{1}{2}i\,\overline{\boldsymbol{\chi}}(\epsilon,t)\Big\{\Gamma_c\tau_3, O_{\mathcal{B}}(\mathfrak{a})\Big\}\boldsymbol{\chi}(\epsilon,t)\,, \tag{2.2.19}$$

where $O_{\mathcal{B}}$ is defined by (2.2.18). These show that none of the dyon-fermion interactions transfer fermion number to or from the dyon while the terms in the boundary action involving $\tau_\pm$ contribute nonzero flux of electrical current at $r = \epsilon$, as required by charge conservation when exciting or de-exciting the dyonic field $\mathfrak{a}$.

Several things about the boundary condition (2.2.17) are noteworthy (for more details and a discussion of how things look for a non-dyonic source see §1.2 and [35]).

- *Linearity:* This boundary condition is linear in $\boldsymbol{\psi}$ (or $\boldsymbol{\chi}$) because the action (2.2.7) is quadratic in $\boldsymbol{\psi}$ (or (2.2.15) is quadratic in $\boldsymbol{\chi}$). Furthermore, the action is quadratic because this is the lowest-dimension interaction consistent with the field content and symmetries and it is the lowest-dimension interactions that dominate at low energies within an effective theory. It is through arguments like this that PPEFTs explain the ubiquity of linear boundary conditions

[9] j^α_A is chosen in the 2D theory to match the 4D axial current $J^\mu_A = i\overline{\boldsymbol{\psi}}\gamma^\mu\gamma_5\boldsymbol{\psi}$ for S-wave states, up to factors of $4\pi r^2$.

[10] Divergences associated with evaluating these fermion bilinears can be regulated in a way that preserves this relation, as we show in appendix F.

for many systems as being a generic consequence of the low-energy limit.

- *Algebraic:* This boundary condition does not involve derivatives – unlike (2.2.11), for instance – because the fermion bulk action is linear in derivatives of the fields. The first term in (2.2.13) comes from integrating by parts in the bulk action, and when the same exercise is performed for bosonic fields one instead finds first-derivative Robin-style boundary conditions because the bulk field equations are second order in derivatives. We discuss some novel implications of non-derivative boundary conditions at some length below.

Implications for mode functions

We next work out how these boundary conditions influence the bulk fermion mode functions. This is complicated by the fact that the boundary condition (2.2.13) explicitly involves the collective coordinate $\mathfrak{a}$, and this is important because it shows how the dynamics of $\mathfrak{a}$ feeds back onto the fermionic field as it interacts with the dyon. As emphasized in [68] it is the nonperturbative nature of this fermion-dyon interaction that drives much of the unusual dynamics of the dyonic effective theory, generating effects that are correlated between the $\boldsymbol{\psi}$ and $\mathfrak{a}$ sectors. In practice this complicates things by making the fermion boundary condition (2.2.13) field-dependent. For the remainder of this section we regard the field $\mathfrak{a}$ to be a specified classical function while we explore the implications of (2.2.17) for the fermion field. §2.3 returns to justify how to interpret the resulting $\mathfrak{a}$-dependence of our results using the Born-Oppenheimer approximation.

Eqs. (2.1.13) and (2.1.14) provide the general solution for the $\boldsymbol{\psi}$ mode functions and so form a complete basis of solutions to the bulk field equations. Energy eigenstates in the presence of the dyon are found by asking these also to satisfy the near-dyon boundary condition (2.2.17), since this captures the implications of the dyon for its surroundings. Subtleties to do with the angle-dependence of the S-wave mode functions can be avoided if the boundary conditions are phrased in terms of the two-dimensional fields $\boldsymbol{\chi}$. For instance, if a general positive-frequency solution is given as a linear combination of the solutions (2.1.18), with coefficients $\boldsymbol{\kappa}_{\mathfrak{sc}}$, then a near-source boundary condition of the form (2.2.17) can be written

$$\mathcal{B}(\mathfrak{a})\begin{bmatrix}\boldsymbol{\kappa}_{++}\,\mathfrak{u}_{++\omega}(\epsilon,t)+\boldsymbol{\kappa}_{+-}\,\mathfrak{u}_{+-\omega}(\epsilon,t)\\ \boldsymbol{\kappa}_{-+}\,\mathfrak{u}_{-+\omega}(\epsilon,t)+\boldsymbol{\kappa}_{--}\,\mathfrak{u}_{--\omega}(\epsilon,t)\end{bmatrix}=0 \tag{2.2.20}$$

where $\mathcal{B}(\mathfrak{a}) := \Gamma^0[\Gamma^1 + O_{\mathcal{B}}(\mathfrak{a})]$ and so

$$\mathcal{B}(\mathfrak{a})=\begin{pmatrix} 1+i(\mathcal{C}^{pv}_{++}+\mathcal{C}^{v}_{++}) & -(\mathcal{C}^{ps}_{++}+i\mathcal{C}^{s}_{++}) & i(\mathcal{C}^{pv}_{+-}+\mathcal{C}^{v}_{+-})e^{i\mathfrak{a}} & -(\mathcal{C}^{ps}_{+-}+i\mathcal{C}^{s}_{+-})e^{i\mathfrak{a}} \\ \mathcal{C}^{ps}_{++}-i\mathcal{C}^{s}_{++} & -1-i(\mathcal{C}^{pv}_{++}-\mathcal{C}^{v}_{++}) & (\mathcal{C}^{ps}_{+-}-i\mathcal{C}^{s}_{+-})e^{i\mathfrak{a}} & -i(\mathcal{C}^{pv}_{+-}-\mathcal{C}^{v}_{+-})e^{i\mathfrak{a}} \\ i(\mathcal{C}^{pv}_{-+}+\mathcal{C}^{v}_{-+})e^{-i\mathfrak{a}} & -(\mathcal{C}^{ps}_{-+}+i\mathcal{C}^{s}_{-+})e^{-i\mathfrak{a}} & 1+i(\mathcal{C}^{pv}_{--}+\mathcal{C}^{v}_{--}) & -(\mathcal{C}^{ps}_{--}+i\mathcal{C}^{s}_{--}) \\ (\mathcal{C}^{ps}_{-+}-i\mathcal{C}^{s}_{-+})e^{-i\mathfrak{a}} & -i(\mathcal{C}^{pv}_{-+}-\mathcal{C}^{v}_{-+})e^{-i\mathfrak{a}} & \mathcal{C}^{ps}_{--}-i\mathcal{C}^{s}_{--} & -1-i(\mathcal{C}^{pv}_{--}-\mathcal{C}^{v}_{--}) \end{pmatrix}. \tag{2.2.21}$$

Notice that when the $\mathcal{C}^A_{ij}$ couplings are hermitian – *i.e.* when the dyon-fermion action (2.2.15) is real – the matrix $\mathcal{B}(\mathfrak{a})$ satisfies the useful identity

$$\mathcal{B}^\dagger = 2\begin{pmatrix}\Gamma_c & 0\\ 0 & \Gamma_c\end{pmatrix} - \mathcal{B} \tag{2.2.22}$$

for all $\mathfrak{a}$, where $\Gamma_c = \sigma_3$ is the diagonal Pauli matrix. The most general matrix satisfying this condition has 16 real parameters, corresponding to the 16 real effective couplings contained in the $\mathcal{C}^A_{ij}$.

The physical implications of this boundary condition depend sensitively on the rank of the matrix $\mathcal{B}$. For instance, if all couplings $\mathcal{C}^A_{ij}$ were independent of one another $\mathcal{B}$ would generically have rank 4, in which case the only solution to (2.2.20) is $\boldsymbol{\chi}(\epsilon,t)=0$. If $\mathcal{B}$ instead has rank $4-n$ for $n=0,1,2,3$ then there would be n linearly independent nonzero values for $\boldsymbol{\chi}(\epsilon,t)$ allowed by (2.2.20).

Now comes a key observation: when the boundary action S_{dyon2} is real – and so the constants $\mathcal{C}^A_{ij}$ are hermitian – then the rank of $\mathcal{B}$ cannot be smaller than two. This is because both of the following quantities are $\mathfrak{a}$-independent and always nonzero:

$$\begin{aligned}\begin{vmatrix}\mathcal{B}_{22} & \mathcal{B}_{24}\\ \mathcal{B}_{42} & \mathcal{B}_{44}\end{vmatrix} &= 1+i(\mathcal{C}^{pv}_{--}-\mathcal{C}^{v}_{--}+\mathcal{C}^{pv}_{++}-\mathcal{C}^{v}_{++})-(\mathcal{C}^{pv}_{++}-\mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--}-\mathcal{C}^{v}_{--})+\left|\mathcal{C}^{pv}_{+-}-\mathcal{C}^{v}_{+-}\right|^2,\\ \begin{vmatrix}\mathcal{B}_{11} & \mathcal{B}_{13}\\ \mathcal{B}_{31} & \mathcal{B}_{33}\end{vmatrix} &= 1+i(\mathcal{C}^{pv}_{--}+\mathcal{C}^{v}_{--}+\mathcal{C}^{pv}_{++}+\mathcal{C}^{v}_{++})-(\mathcal{C}^{pv}_{++}+\mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--}+\mathcal{C}^{v}_{--})+\left|\mathcal{C}^{pv}_{+-}+\mathcal{C}^{v}_{+-}\right|^2.\end{aligned} \tag{2.2.23}$$

To see why these cannot vanish, consider the first case. Notice that its imaginary part can only vanish if $\mathcal{C}^{pv}_{--} - \mathcal{C}^{v}_{--} = -(\mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})$, but if this is true then the real part becomes $1 + (\mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})^2 + \left|\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-}\right|^2 \geq 1$. A similar argument goes through also for the second case.

As a consequence it is always possible to solve for two of the $\boldsymbol{\kappa}$'s in terms of the other two. For instance, solving for $\boldsymbol{\kappa}_{++}$ and $\boldsymbol{\kappa}_{--}$ gives

$$\boldsymbol{\kappa}_{++} = -\boldsymbol{\kappa}_{+-} e^{2i(\omega + \frac{1}{2}ev)\epsilon} \left(\frac{\epsilon}{r_0}\right)^{-ieQ} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{12} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{32} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}} - \boldsymbol{\kappa}_{-+} e^{2i\omega\epsilon} e^{i\mathfrak{a}} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{14} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{34} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \tag{2.2.24}$$

and

$$\boldsymbol{\kappa}_{--} = -\boldsymbol{\kappa}_{+-} e^{2i\omega\epsilon} e^{-i\mathfrak{a}} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{12} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{32} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}} - \boldsymbol{\kappa}_{-+} e^{2i(\omega - \frac{1}{2}ev)\epsilon} \left(\frac{\epsilon}{r_0}\right)^{ieQ} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{14} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{34} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \tag{2.2.25}$$

which defines $\hat{\mathcal{B}}_{ij} := \mathcal{B}_{ij}(\mathfrak{a} = 0)$ so that all $\mathfrak{a}$-dependence is explicit. In later sections it is sometimes useful instead to solve for $\boldsymbol{\kappa}_{+-}$ and $\boldsymbol{\kappa}_{-+}$, which instead gives

$$\boldsymbol{\kappa}_{+-} = -\boldsymbol{\kappa}_{++} e^{-2i(\omega + \frac{1}{2}ev)\epsilon} \left(\frac{\epsilon}{r_0}\right)^{ieQ} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{21} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{41} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}} - \boldsymbol{\kappa}_{--} e^{-2i\omega\epsilon} e^{i\mathfrak{a}} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{23} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{43} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}, \tag{2.2.26}$$

and

$$\boldsymbol{\kappa}_{-+} = -\boldsymbol{\kappa}_{++} e^{-2i\omega\epsilon} e^{-i\mathfrak{a}} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{21} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{41} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}} - \boldsymbol{\kappa}_{--} e^{-2i(\omega - \frac{1}{2}ev)\epsilon} \left(\frac{\epsilon}{r_0}\right)^{-ieQ} \frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{23} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{43} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}. \tag{2.2.27}$$

If $\mathcal{B}$ is rank two then this is all that can be learned because the other two equations in (2.2.20) are not independent. When $\mathcal{B}$ has rank two the boundary condition has precisely enough information to determine 'out' states from the 'in' states (with no extra constraints) as is required for scattering problems, so we henceforth assume the effective couplings $\mathcal{C}^A_{ij}$ satisfy the conditions required to ensure rank$(\mathcal{B}) = 2$. This should automatically be the case when the microscopic physics of the source allows out states to be inferred from arbitrary in states – such as for fermion scattering in the classical dyon background of the full nonabelian theory – since the effective couplings obtained by matching must give a consistent description.[11] As is shown in Appendix C the requirement that $\mathcal{B}$ be rank two imposes a total of 8 conditions on its coefficients (four of which amount to unitarity conditions) and so removes half of the 16 real parameters that could have been encoded in $\mathcal{B}$ for general choices of hermitian $\mathcal{C}^A_{ij}$'s.

2.2.2 Scattering states

We next construct the explicit single-particle scattering states appropriate for fermion dyon scattering, assuming the rank of the matrix $\mathcal{B}$ is two. This allows us to identify which combinations of the effective couplings actually appear in scattering processes.

To this end we construct a basis of energy eigenmodes that either correspond to a single type of particle moving towards the dyon (*in* state) or a single type of particle moving away from the dyon (*out* state). These correspond to choosing specific couplings $\boldsymbol{\kappa}_{\mathfrak{s}\mathfrak{c}}$ to vanish, as described in detail below. Because the direction of motion of an S-wave state correlates with the value of the quantum number $\mathfrak{h} = \mathfrak{s}\mathfrak{c}$, it suffices to label in and out states using just $\mathfrak{s}$ and momentum k (or frequency ω), leading to positive (negative) frequency modes $\mathfrak{u}^{\rm in}_{\mathfrak{s},k}$ and $\mathfrak{v}^{\rm in}_{\mathfrak{s},k}$ or $\mathfrak{u}^{\rm out}_{\mathfrak{s},k}$ and $\mathfrak{v}^{\rm out}_{\mathfrak{s},k}$.

In modes

We define the positive-frequency in-states to be those modes for which there is only one component with incoming momentum (heading towards the dyon). Labelling these by the sign of the incoming

[11] We have examples of effective couplings for which rank$(\mathcal{B}) = 3$, but defer an exploration of their microscopic physical significance to future work.

particle's electric charge $\mathfrak{s}$, they are given explicitly by

$$\mathfrak{u}^{\text{in}}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{-ikr} \\ e^{ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \mathcal{T}^{++}_{\text{in}} \end{pmatrix} \\ \begin{pmatrix} 0 \\ e^{i(k-ev)r} e^{-i(2k-ev)\epsilon}\, e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\text{in}} \end{pmatrix} \end{bmatrix} e^{-i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2} \tag{2.2.28}$$

and

$$\mathfrak{u}^{\text{in}}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ e^{i(k+ev)r} e^{-i(2k+ev)\epsilon}\, e^{i\mathfrak{a}} \mathcal{T}^{+-}_{\text{in}} \end{pmatrix} \\ \begin{pmatrix} e^{-ikr} \\ e^{ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \mathcal{T}^{--}_{\text{in}} \end{pmatrix} \end{bmatrix} e^{-i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2} . \tag{2.2.29}$$

where $k = \omega + \frac{1}{2}\mathfrak{s}ev \geq 0$. (These can be obtained from (2.2.26) and (2.2.27) by choosing $\boldsymbol{\kappa}_{++} = 1, \boldsymbol{\kappa}_{--} = 0$ for $\mathfrak{u}^{\text{in}}_{+,k}$ and $\boldsymbol{\kappa}_{--} = 1, \boldsymbol{\kappa}_{++} = 0$ for $\mathfrak{u}^{\text{in}}_{-,k}$.) The negative-frequency in-modes are similarly defined by

$$\mathfrak{v}^{\text{in}}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{ikr} \\ e^{-ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \mathcal{T}^{++}_{\text{in}} \end{pmatrix} \\ \begin{pmatrix} 0 \\ e^{-i(k+ev)r} e^{i(2k+ev)\epsilon}\, e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\text{in}} \end{pmatrix} \end{bmatrix} e^{i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2} \tag{2.2.30}$$

and

$$\mathfrak{v}^{\text{in}}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ e^{-i(k-ev)r} e^{i(2k-ev)\epsilon}\, e^{i\mathfrak{a}} \mathcal{T}^{+-}_{\text{in}} \end{pmatrix} \\ \begin{pmatrix} e^{ikr} \\ e^{-ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \mathcal{T}^{--}_{\text{in}} \end{pmatrix} \end{bmatrix} e^{i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2} , \tag{2.2.31}$$

where $k = \omega - \frac{1}{2}\mathfrak{s}ev \geq 0$. In both of these expressions r_0 is an arbitrary length that contributes only to the overall Coulomb phase.

The coefficients $\mathcal{T}^{++}_{\text{in}}$, $\mathcal{T}^{+-}_{\text{in}}$, $\mathcal{T}^{-+}_{\text{in}}$ and $\mathcal{T}^{--}_{\text{in}}$ appearing in these expressions are $\mathfrak{a}$-independent quantities given as explicit functions of the fermion-dyon couplings by

$$\mathcal{T}^{++}_{\text{in}} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{21} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{41} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}},\ \mathcal{T}^{+-}_{\text{in}} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{23} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{43} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}},\ \mathcal{T}^{-+}_{\text{in}} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{21} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{41} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}},\ \mathcal{T}^{--}_{\text{in}} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{23} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{43} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}} . \tag{2.2.32}$$

Written directly in terms of the boundary couplings, these become

$$
\begin{aligned}
\mathcal{T}_{\rm in}^{++} &= \frac{(\mathcal{C}^{s}_{++} + i\mathcal{C}^{ps}_{++})(i - \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--}) + (\mathcal{C}^{s}_{-+} + i\mathcal{C}^{ps}_{-+})(\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^{v}_{--} - i)}, \\
\mathcal{T}_{\rm in}^{--} &= \frac{(\mathcal{C}^{s}_{--} + i\mathcal{C}^{ps}_{--})(i - \mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++}) + (\mathcal{C}^{s}_{+-} + i\mathcal{C}^{ps}_{+-})(\mathcal{C}^{pv}_{-+} - \mathcal{C}^{v}_{-+})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^{v}_{--} - i)}, \\
\mathcal{T}_{\rm in}^{-+} &= \frac{(\mathcal{C}^{s}_{-+} + i\mathcal{C}^{ps}_{-+})(i - \mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++}) + (\mathcal{C}^{s}_{++} + i\mathcal{C}^{ps}_{++})(\mathcal{C}^{pv}_{-+} - \mathcal{C}^{v}_{-+})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^{v}_{--} - i)}, \\
\mathcal{T}_{\rm in}^{+-} &= \frac{(\mathcal{C}^{s}_{+-} + i\mathcal{C}^{ps}_{+-})(i - \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--}) + (\mathcal{C}^{s}_{--} + i\mathcal{C}^{ps}_{--})(\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^{v}_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^{v}_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^{v}_{--} - i)}.
\end{aligned}
\tag{2.2.33}
$$

When the boundary action $S_{\rm dyon2}$ is real (and so the $\mathcal{C}^{A}_{ij}$ are hermitian) these satisfy the following unitarity conditions

$$
|\mathcal{T}_{\rm in}^{+-}|^2 = |\mathcal{T}_{\rm in}^{-+}|^2 = 1 - |\mathcal{T}_{\rm in}^{++}|^2 = 1 - |\mathcal{T}_{\rm in}^{--}|^2 \,, \tag{2.2.34}
$$

as well as

$$
\mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{-+\,*} + \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm in}^{++\,*} = 0, \tag{2.2.35}
$$

as identities (see Appendix C for a derivation). These relations imply that the *in* amplitudes only carry 4 real parameters' worth of information; they can always be written

$$
\mathcal{T}_{\rm in}^{++} = \rho\, e^{i\theta_{++}} \quad \text{and} \quad \mathcal{T}_{\rm in}^{--} = \rho\, e^{i\theta_{--}}, \tag{2.2.36}
$$

and

$$
\mathcal{T}_{\rm in}^{+-} = \sqrt{1-\rho^2}\, e^{i\theta_{+-}} \quad \text{and} \quad \mathcal{T}_{\rm in}^{-+} = -\sqrt{1-\rho^2}\, e^{i(\theta_{++}+\theta_{--}-\theta_{+-})}, \tag{2.2.37}
$$

where ρ, θ_{++}, θ_{+-} and θ_{--} are the four independent real parameters.

Out modes

A set of outgoing modes can be constructed in precisely the same way, with the positive frequency modes given by

$$\mathfrak{u}^{\text{out}}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{-ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \mathcal{T}^{++}_{\text{out}} \\ e^{ikr} \end{pmatrix} \\ \begin{pmatrix} e^{-i(k-ev)r} e^{i(2k-ev)\epsilon}\, e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\text{out}} \\ 0 \end{pmatrix} \end{bmatrix} e^{-i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2}, \tag{2.2.38}$$

and

$$\mathfrak{u}^{\text{out}}_{-,k} = \begin{bmatrix} \begin{pmatrix} e^{-i(k+ev)r} e^{i(2k+ev)\epsilon}\, e^{i\mathfrak{a}} \mathcal{T}^{+-}_{\text{out}} \\ 0 \end{pmatrix} \\ \begin{pmatrix} e^{-ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \mathcal{T}^{--}_{\text{out}} \\ e^{ikr} \end{pmatrix} \end{bmatrix} e^{-i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2}, \tag{2.2.39}$$

with $k = \omega + \frac{1}{2}\mathfrak{s}ev \geq 0$ (these can be obtained from (2.2.24) and (2.2.25) by choosing $\boldsymbol{\kappa}_{+-} = 1$, $\boldsymbol{\kappa}_{-+} = 0$ for $\mathfrak{u}^{\text{out}}_{+,k}$ and $\boldsymbol{\kappa}_{-+} = 1, \boldsymbol{\kappa}_{+-} = 0$ for $\mathfrak{u}^{\text{out}}_{-,k}$). The negative-frequency out-modes are

$$\mathfrak{v}^{\text{out}}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \mathcal{T}^{++}_{\text{out}} \\ e^{-ikr} \end{pmatrix} \\ \begin{pmatrix} e^{i(k+ev)r} e^{-i(2k+ev)\epsilon}\, e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\text{out}} \\ 0 \end{pmatrix} \end{bmatrix} e^{i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2}, \tag{2.2.40}$$

and

$$\mathfrak{v}^{\text{out}}_{-,k} = \begin{bmatrix} \begin{pmatrix} e^{i(k-ev)r} e^{-i(2k-ev)\epsilon}\, e^{i\mathfrak{a}} \mathcal{T}^{+-}_{\text{out}} \\ 0 \end{pmatrix} \\ \begin{pmatrix} e^{ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \mathcal{T}^{--}_{\text{out}} \\ e^{-ikr} \end{pmatrix} \end{bmatrix} e^{i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2}, \tag{2.2.41}$$

for which $k = \omega - \frac{1}{2}\mathfrak{s}ev \geq 0$.

The $\mathfrak{a}$-independent coefficients $\mathcal{T}^{++}_{\text{out}}, \mathcal{T}^{+-}_{\text{out}}, \mathcal{T}^{-+}_{\text{out}}, \mathcal{T}^{--}_{\text{out}}$ appearing in these expressions are defined in terms of the dyon-fermion couplings by

$$\mathcal{T}_{\rm out}^{++} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{12} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{32} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}},\ \mathcal{T}_{\rm out}^{-+} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{12} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{32} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}},\ \mathcal{T}_{\rm out}^{+-} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{14} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{34} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}},\ \mathcal{T}_{\rm out}^{--} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{14} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{34} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \quad (2.2.42)$$

and so

$$\begin{aligned}
\mathcal{T}_{\rm out}^{++} &= \frac{i(\mathcal{C}^{ps}_{++} + i\mathcal{C}^{s}_{++})(-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--}) + (-i\mathcal{C}^{ps}_{-+} + \mathcal{C}^{s}_{-+})(\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++} - i)}, \\
\mathcal{T}_{\rm out}^{--} &= \frac{i(\mathcal{C}^{ps}_{--} + i\mathcal{C}^{s}_{--})(-i + \mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++}) + (-i\mathcal{C}^{ps}_{+-} + \mathcal{C}^{s}_{+-})(\mathcal{C}^{pv}_{-+} + \mathcal{C}^{v}_{-+})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++} - i)}, \\
\mathcal{T}_{\rm out}^{-+} &= \frac{i(\mathcal{C}^{ps}_{-+} + i\mathcal{C}^{s}_{-+})(-i + \mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++}) + (-i\mathcal{C}^{ps}_{++} + \mathcal{C}^{s}_{++})(\mathcal{C}^{pv}_{-+} + \mathcal{C}^{v}_{-+})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++} - i)}, \\
\mathcal{T}_{\rm out}^{+-} &= \frac{i(\mathcal{C}^{ps}_{+-} + i\mathcal{C}^{s}_{+-})(-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--}) + (-i\mathcal{C}^{ps}_{--} + \mathcal{C}^{s}_{--})(\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^{v}_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^{v}_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^{v}_{++} - i)}.
\end{aligned} \quad (2.2.43)$$

When the boundary action $S_{\rm dyon2}$ is real (so the $\mathcal{C}^{A}_{ij}$'s are hermitian) these satisfy

$$|\mathcal{T}_{\rm out}^{+-}|^2 = |\mathcal{T}_{\rm out}^{-+}|^2 = 1 - |\mathcal{T}_{\rm out}^{++}|^2 = 1 - |\mathcal{T}_{\rm out}^{--}|^2 \,. \quad (2.2.44)$$

and

$$\mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{-+\,*} + \mathcal{T}_{\rm out}^{+-}\mathcal{T}_{\rm out}^{++\,*} = 0, \quad (2.2.45)$$

in the same way as found earlier for $\mathcal{T}_{\rm in}$. There is of course a good reason why $\mathcal{T}_{\rm out}$ and $\mathcal{T}_{\rm in}$ satisfy similar conditions: they are not independent of one another. Since both the *in* and *out* bases are complete, the $\mathcal{T}_{\rm out}$ amplitudes can be expressed in terms of the $\mathcal{T}_{\rm in}$ amplitudes. This leads to the relations (see Appendix C)

$$\mathcal{T}_{\rm out}^{++} = (\mathcal{T}_{\rm in}^{++})^*, \quad \mathcal{T}_{\rm out}^{--} = (\mathcal{T}_{\rm in}^{--})^*, \quad \mathcal{T}_{\rm out}^{+-} = (\mathcal{T}_{\rm in}^{-+})^* \quad \text{and} \quad \mathcal{T}_{\rm out}^{-+} = (\mathcal{T}_{\rm in}^{+-})^*\,, \quad (2.2.46)$$

and so the *out* amplitudes also depend only on the four parameters given in (2.2.36) and (2.2.37).

The implications of the $\mathcal{T}_{\rm in}$ and $\mathcal{T}_{\rm out}$ amplitudes for fermion-dyon scattering problems are explored in some detail in §3 and §4 below and suffice it to say that they carry all of the information about the dyon that is relevant to describing transitions amongst the S-wave fermion modes described above, at least at leading order at low energies. See §2.5.1 for a discussion of the values predicted for these quantities by specific microscopic choices for the underlying dyon solution.

But the above discussion introduces a minor puzzle: why are there only four free real parameters within, say, the $\mathcal{T}_{\rm in}$'s when there are 16 possible real parameters in the hermitian effective couplings $\mathcal{C}^A_{ij}$ appearing in $S_{\rm dyon2}$ in *e.g.* eqs. (2.2.7) and (2.2.8)? We argue in §2.4 that all but four of the effective couplings $\mathcal{C}^A_{ij}$ are redundant (in the precise EFT sense [14]), but before doing so the next section first develops a required tool. Along the way it also resolves a dangling technical issue: how to handle systematically the $\mathfrak{a}$-dependence that is embedded in boundary conditions like (2.2.17) or (2.2.20).

2.3 Dyonic response

Up to this point we have treated the matrix $\mathcal{B}$ as if it were specified completely once the effective couplings $\mathcal{C}^A_{ij}$ are, but the appearance of the field $\mathfrak{a}$ within $\mathcal{B}$ makes this not quite true. Having $\mathcal{B}$ be a function of $\mathfrak{a}$ complicates its use – as in (2.2.20) – to find fermionic mode functions. This section provides two complementary ways to handle this field-dependence: perturbation theory and the Born-Oppenheimer approximation.

Fermion-dyon perturbation theory

The first approach starts with the observation that not *every* term in the matrix $\mathcal{B}$ depends on $\mathfrak{a}$. If the coefficients $\mathcal{C}^A_{\mathfrak{s}\mathfrak{s}'}$ with $\mathfrak{s} \neq \mathfrak{s}'$ were for some reason much smaller than the others then they could be ignored when formulating the near-dyon fermion boundary conditions with the effects of $\mathfrak{a}$-dependent terms then included perturbatively.

For instance, suppose $S_{\rm dyon2}$ of (2.2.15) could be written $S^{(0)}_{\rm dyon2} + S^{\rm int}_{\rm dyon2}$ with[12]

$$S^{(0)}_{\rm dyon2} := -\frac{1}{2}\sum_{\mathfrak{s}=\pm}\int {\rm d}t\, i\overline{\chi}_{\mathfrak{s}}\Gamma_c\,\chi_{\mathfrak{s}}\,, \tag{2.3.1}$$

[12]We choose the unperturbed boundary terms here fairly arbitrarily and all that is important for our arguments is that the dominant piece be $\mathfrak{a}$-independent.

and

$$S^{\rm int}_{\rm dyon2} = -\frac{1}{2}\sum_{\mathfrak{s},\mathfrak{s}'=\pm}\int {\rm d}t\, \overline{\chi}_{\mathfrak{s}}\Big(\delta\mathcal{C}^{s}_{\mathfrak{s}\mathfrak{s}'} + i\delta\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'}\Gamma_c - i\delta\mathcal{C}^{v}_{\mathfrak{s}\mathfrak{s}'}\Gamma^0 + i\delta\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'}\Gamma^1\Big)\,\chi_{\mathfrak{s}'}\; e^{\frac{i}{2}(\mathfrak{s}-\mathfrak{s}')\,\mathfrak{a}}\,, \tag{2.3.2}$$

where the coefficients $\delta\mathcal{C}^A_{ij}$ are regarded as being perturbatively small[13] and we again specialize to the case of an approximately static dyon (for which $\dot{y}^\alpha \simeq \delta^\alpha_0$). In this case only $S^{(0)}_{\rm dyon2}$ need play a role in determining the boundary conditions of the free modes appearing in the field expansions within the interaction picture.

With this choice the boundary condition matrix of (2.2.18) becomes $O^{(0)}_{\mathcal{B}} = iI\otimes\Gamma_c$ and so the matrix appearing in (2.2.21) becomes

$$\mathcal{B}^{(0)} = \Gamma^0\left[\Gamma^1 + O^{(0)}_{\mathcal{B}}\right] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{2.3.3}$$

which clearly has rank two. The boundary condition satisfied by the free mode functions at $r=\epsilon$ therefore is

$$\Big(\Gamma^1 + i\Gamma_c\Big)\boldsymbol{\chi}(\epsilon,t) = 0\,. \tag{2.3.4}$$

The continuum normalized free *in* and *out* mode functions that satisfy (2.3.4) then are

$$\mathfrak{u}^{\rm in0}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{-ikr} \\ e^{ik(r-2\epsilon)}\left(\frac{\epsilon}{r}\right)^{ieQ} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} e^{-i(k-\frac{ev}{2})t}\left(\frac{r}{r_0}\right)^{ieQ/2}, \quad \mathfrak{u}^{\rm in0}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} e^{-ikr} \\ e^{ik(r-2\epsilon)}\left(\frac{r}{\epsilon}\right)^{ieQ} \end{pmatrix} \end{bmatrix} e^{-i(k+\frac{ev}{2})t}\left(\frac{r}{r_0}\right)^{-ieQ/2}, \tag{2.3.5}$$

$$\mathfrak{v}^{\rm in0}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{ikr} \\ e^{-ik(r-2\epsilon)}\left(\frac{\epsilon}{r}\right)^{ieQ} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} e^{i(k+\frac{ev}{2})t}\left(\frac{r}{r_0}\right)^{ieQ/2}, \quad \mathfrak{v}^{\rm in0}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} e^{ikr} \\ e^{-ik(r-2\epsilon)}\left(\frac{r}{\epsilon}\right)^{ieQ} \end{pmatrix} \end{bmatrix} e^{i(k-\frac{ev}{2})t}\left(\frac{r}{r_0}\right)^{-ieQ/2}, \tag{2.3.6}$$

[13] Although, as noted above, the $\mathcal{C}^A_{ij}$'s are not generically suppressed by the loop-counting parameter $\alpha = e^2/(4\pi)$ it is possible that in some circumstances some of them are suppressed by another small quantity.

(where $k \geq 0$) and

$$\mathfrak{u}^{\rm out0}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{-ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \\ e^{ikr} \end{pmatrix} \\ \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} e^{-i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2}, \quad \mathfrak{u}^{\rm out0}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} e^{-ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \\ e^{ikr} \end{pmatrix} \end{bmatrix} e^{-i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2}, \tag{2.3.7}$$

$$\mathfrak{v}^{\rm out0}_{+,k} = \begin{bmatrix} \begin{pmatrix} e^{ik(r-2\epsilon)} \left(\frac{r}{\epsilon}\right)^{ieQ} \\ e^{-ikr} \end{pmatrix} \\ \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix} e^{i(k+\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{-ieQ/2}, \quad \mathfrak{v}^{\rm out0}_{-,k} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} e^{ik(r-2\epsilon)} \left(\frac{\epsilon}{r}\right)^{ieQ} \\ e^{-ikr} \end{pmatrix} \end{bmatrix} e^{i(k-\frac{ev}{2})t} \left(\frac{r}{r_0}\right)^{ieQ/2}, \tag{2.3.8}$$

and so the unperturbed modes satisfy $\mathcal{T}^{++}_{\rm in} = \mathcal{T}^{--}_{\rm in} = 1$ and $\mathcal{T}^{+-}_{\rm in} = \mathcal{T}^{-+}_{\rm in} = 0$. These modes are normalized so that

$$\int_\epsilon^\infty {\rm d}r\, (\mathfrak{u}^{\rm in0}_{\mathfrak{s},k})^\dagger\, \mathfrak{u}^{\rm in0}_{\mathfrak{s}',k'} = \int_\epsilon^\infty {\rm d}r\, (\mathfrak{v}^{\rm in0}_{\mathfrak{s},k})^\dagger\, \mathfrak{v}^{\rm in0}_{\mathfrak{s}',k'} = 2\pi\delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}, \tag{2.3.9}$$

and

$$\int_\epsilon^\infty {\rm d}r\, (\mathfrak{u}^{\rm in0}_{\mathfrak{s},k})^\dagger\, \mathfrak{v}^{\rm in0}_{\mathfrak{s}',k'} = 2\pi\delta(k+k')\,\delta_{\mathfrak{s}\mathfrak{s}'}, \tag{2.3.10}$$

and similarly for the *out* modes.

From here on perturbation theory proceeds in the standard way by moving to the interaction picture and expanding the fermion field operator using these mode functions,

$$\begin{aligned} \boldsymbol{\chi}(x) &= \sum_{\mathfrak{s}=\pm} \int_0^\infty \frac{{\rm d}k}{\sqrt{2\pi}} \Big[\mathfrak{u}^{\rm in0}_{\mathfrak{s},k}(x)\, a^{\rm in}_{\mathfrak{s},k} + \mathfrak{v}^{\rm in0}_{\mathfrak{s},k}(x)\, (\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\Big] \\ &= \sum_{\mathfrak{s}=\pm} \int_0^\infty \frac{{\rm d}k}{\sqrt{2\pi}} \Big[\mathfrak{u}^{\rm out0}_{\mathfrak{s},k}(x)\, a^{\rm out}_{\mathfrak{s},k} + \mathfrak{v}^{\rm out0}_{\mathfrak{s},k}(x)\, (\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\Big], \end{aligned} \tag{2.3.11}$$

where the creation/annihilation operators satisfy the usual algebra $\{a^{\rm in}_{\mathfrak{s},k},\, (a^{\rm in}_{\mathfrak{s}',k'})^\star\} = \delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}$ and $\{\overline{a}^{\rm in}_{\mathfrak{s},k},\, (\overline{a}^{\rm in}_{\mathfrak{s}',k'})^\star\} = \delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}$ appropriate for fermions (and similarly for the *out* operators). Because both the *in* and *out* modes are complete, each can be expanded in terms of the other, allowing the *in* and *out* operators to be related by

$$a^{\rm out}_{\mathfrak{s},k} = e^{-2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{i\mathfrak{s}eQ} a^{\rm in}_{\mathfrak{s},k} \quad \text{and} \quad (\overline{a}^{\rm out}_{\mathfrak{s},k})^\star = e^{2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{i\mathfrak{s}eQ} (\overline{a}^{\rm in}_{\mathfrak{s},k})^\star, \tag{2.3.12}$$

and their adjoints. The above Bogoliubov relations depend on the arbitrary scale r_0. Physical predictions should not depend on r_0 and in §3 – which uses the perturbative framework set up here – we rephase the *out* operators to absorb the r_0 dependence.

The perturbative formalism is useful for some kinds of questions – such as the discussion of redundancy in §2.4 – and is used in §3 to compute some scattering processes (which can be compared with calculations performed in §4 using the alternative approach we describe next). But it is not guaranteed that specific microscopic realizations of the dyon must yield effective couplings for which such a perturbative approach is a good approximation. This is true in particular for massless fermion scattering in the classical nonabelian dyon background because in this case conservation of chirality implies the vanishing of all of the $\mathfrak{a}$-independent boundary couplings $\mathcal{C}^A_{\mathfrak{s}\mathfrak{s}'}$ with $\mathfrak{s}' = \mathfrak{s}$.

Born-Oppenheimer evolution of $\mathfrak{a}$

A more ambitious approach to computing the fermion-$\mathfrak{a}$ interactions takes advantage of the fact that the mass $\mathcal{I} \sim (\alpha m_g)^{-1}$ appearing in the $\mathfrak{a}$ kinetic term is much larger than the generic dyon scale m_g^{-1} in the semiclassical limit (for which $\alpha = e^2/4\pi \ll 1$). This means that the energy associated with $\mathfrak{a}$ excitation is order $\mathcal{I}^{-1} \sim \alpha m_g$ and so is negligible – even within the low energy theory – for fermion momenta in the range $\alpha m_g \ll k \ll m_g$.

Furthermore, the equations of motion say that $\dot{\mathfrak{a}} \propto \mathfrak{p}/\mathcal{I}$ and so the timescale for $\mathfrak{a}$ to respond to order-unity changes in $\mathfrak{p}$ are order $\tau \sim \mathcal{I}$. By contrast the timescale for a relativistic fermionic wave packet of size L to interact with a much smaller dyon is order L (which cannot be smaller than $1/k$ for fermions with momentum k). So the rotor response is slow compared to the fermion interaction time for fermion momenta in the range $\alpha m_g \ll L^{-1} \lesssim k$.

In the regime $\alpha m_g \ll k$ the 'rotor' field response to fermions is slow and costs little energy in much the same way that the position of heavy atomic nuclei respond to much lighter and faster electrons in everyday solids. This electron/nucleus analogy suggests using the Born-Oppenheimer approximation [78] to describe the interactions of relativistic fermions with the slower dyonic excitation $\mathfrak{a}$. In this approximation one first solves for the motion of the fast degrees of freedom (the light fermions) with the slow degrees of freedom (the nuclear positions $\mathbf{R}$ or the field $\mathfrak{a}$) imagined to be fixed classical variables. The idea is that the slow variables behave classically because they do not have time to respond at all when hit by the fast ones. Once the fast evolution is computed one calculates an effective Hamiltonian describing the dynamics of the slow degrees of freedom obtained by averaging over the fast degrees of freedom. This Hamiltonian is then used to solve for how the slower variables evolve in response to its interactions with the much faster system.

As applied to dyon-fermion interactions the first part of the Born-Oppenheimer procedure asks for a solution to fermion evolution with the field $\mathfrak{a}$ regarded as a fixed classical background. This is precisely what is done in §2.2.2 above when finding the fermion scattering states in the presence of an $\mathfrak{a}$-dependent boundary condition like (2.2.20). These are used in §4 to calculate scattering rates also for fixed dyon configurations. We return to the issue of how to determine the dyon reponse in §4.2 below, but first close this section by gathering together several loose ends and conceptual issues to do with using the PPEFT framework in the monopole/dyon setting.

2.4 Redundant interactions

In this section we return to the puzzle alluded to at the end of §2.2.2 above: why are the physical effects of the 16 real parameters in the couplings $\mathcal{C}^{A}_{ij}$ in the action $S_{\rm dyon2}$ all encoded in amplitudes $\mathcal{T}_{\rm in}$ (or $\mathcal{T}_{\rm out}$) that involve only 4 parameters. Why are there not more scattering options available given the number of possible effective interactions? Part of the reason for this reduction is the rank-2 condition we assume for $\mathcal{B}(\mathfrak{a})$, since – as shown in Appendix C.2 – this imposes 8 real conditions on the components of $\mathcal{B}(\mathfrak{a})$ (and so also on the $\mathcal{C}^{A}_{ij}$'s). But the fact that the remaining 8 independent effective couplings only produce a 4-parameter number of scattering outcomes strongly suggests that some of the remaining effective couplings are actually redundant (in the precise EFT sense reviewed for example in [14]).

One way in which an effective operator can be seen to be redundant is if it can be removed using a field redefinition. When working in perturbation theory there is a very simple diagnostic for when such a field redefinition exists: one asks whether the operator vanishes when it is evaluated at the solution to the leading order field equations. In the present instance the fermion field equation at the position $r = \epsilon$ is actually just the boundary condition itself, and this suggests a redundancy test that can be performed, at least if the dyon-fermion interactions are treated perturbatively (as they are in §2.3 when perturbing the dyon effective couplings around the action $S^{(0)}_{\rm dyon2}$).

When perturbing in this way we first ask what conditions the $\delta\mathcal{C}^{\mathcal{A}}_{ij}$'s must satisfy to ensure that the perturbed boundary condition remains rank two. Perturbing the rank-2 conditions like (C.2.4) of the appendix to linear order in the $\delta\mathcal{C}$s shows that $\mathcal{B}$ remains rank two only if the coefficients $\delta\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'}$ and $\delta\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'}$ all vanish. For the remaining couplings the redundancy test then asks how many of the effective couplings in (2.3.2) survive when simplified using the lowest-order fermion boundary condition $(\Gamma^1 + i\Gamma_c)\boldsymbol{\chi}(\epsilon) = 0$ that follows from the unperturbed dyon-fermion action (2.3.1). Notice that because $\Gamma_c = \Gamma^0\Gamma^1$ this is equivalent to the condition $i\Gamma^0\boldsymbol{\chi}(\epsilon) = \boldsymbol{\chi}(\epsilon)$ and because Γ^0 is antihermitian this implies $\boldsymbol{\chi}^\dagger(i\Gamma^0) = \boldsymbol{\chi}^\dagger$ when evaluated at $r = \epsilon$, and so also $\overline{\boldsymbol{\chi}} = \boldsymbol{\chi}^\dagger$ there. These conditions allow us to rewrite

$$\overline{\chi}_{\mathfrak{s}}(\epsilon)\chi_{\mathfrak{s}'}(\epsilon) = \chi^\dagger_{\mathfrak{s}}(\epsilon)i\Gamma^0\chi_{\mathfrak{s}'}(\epsilon) = \chi^\dagger_{\mathfrak{s}}(\epsilon)\chi_{\mathfrak{s}'}(\epsilon) = \overline{\chi}_{\mathfrak{s}}(\epsilon)i\Gamma^0\chi_{\mathfrak{s}'}(\epsilon), \tag{2.4.1}$$

where the first equality uses the definition $\overline{\boldsymbol{\chi}} := i\boldsymbol{\chi}^\dagger\Gamma^0$. This shows that within perturbative calculations the couplings $\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'}\overline{\chi}_{\mathfrak{s}}\chi_{\mathfrak{s}'} - i\delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}\overline{\chi}_{\mathfrak{s}}\Gamma^0\chi_{\mathfrak{s}'}$ appearing in expression (2.3.2) for $S^{\rm int}_{\rm dyon2}$ should only contribute to physical predictions through the combination $\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} - \delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}$.

This is consistent with the more general expressions like (2.2.33) for the amplitudes $\mathcal{T}_{\rm in}$ and $\mathcal{T}_{\rm out}$ within the Born-Oppenheimer framework, since for rank-2 perturbations about the unperturbed couplings $\mathcal{C}^{ps}_{++} = \mathcal{C}^{ps}_{--} = 1$ these give

$$\begin{aligned}
&\mathcal{T}^{++}_{\rm in} \simeq 1 + i(\delta\mathcal{C}^v_{++} - \delta\mathcal{C}^s_{++}), \quad \mathcal{T}^{--}_{\rm in} \simeq 1 + i(\delta\mathcal{C}^v_{--} - \delta\mathcal{C}^s_{--}),\\
&\mathcal{T}^{+-}_{\rm in} \simeq i(\delta\mathcal{C}^v_{+-} - \delta\mathcal{C}^s_{+-}), \quad \text{and} \quad \mathcal{T}^{-+}_{\rm in} \simeq i(\delta\mathcal{C}^v_{-+} - \delta\mathcal{C}^s_{-+}),
\end{aligned} \tag{2.4.2}$$

showing at linear order that $\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'}$ and $\delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}$ really do only appear in the combination $\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} - \delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}$.

These arguments suggest that within the perturbative framework one combination of $\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'}$ and $\delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}$ is redundant in the sense that it can be removed by performing a field redefinition at the position of the dyon worldline. Indeed, writing

$$\overline{\boldsymbol{\chi}}\mathcal{C}^s\boldsymbol{\chi} - i\overline{\boldsymbol{\chi}}\mathcal{C}^v\Gamma^0\boldsymbol{\chi} = \tfrac{1}{2}\overline{\boldsymbol{\chi}}(\mathcal{C}^s + \mathcal{C}^v)(1 - i\Gamma^0)\boldsymbol{\chi} + \tfrac{1}{2}\overline{\boldsymbol{\chi}}(\mathcal{C}^s - \mathcal{C}^v)(1 + i\Gamma^0)\boldsymbol{\chi}\,, \tag{2.4.3}$$

we seek a redefinition that removes the first term. But under a small variation $\boldsymbol{\chi} \to \boldsymbol{\chi} + \delta\boldsymbol{\chi}$ at $r = \epsilon$ the variation of the bulk action and the unperturbed boundary action (2.3.1) becomes

$$\delta S(r = \epsilon) = -\tfrac{1}{2}\overline{\boldsymbol{\chi}}\Big(-\Gamma^1 + i\Gamma_c\Big)\delta\boldsymbol{\chi} - \tfrac{1}{2}\delta\overline{\boldsymbol{\chi}}\Big(\Gamma^1 + i\Gamma_c\Big)\boldsymbol{\chi}\,, \tag{2.4.4}$$

which for $\delta\boldsymbol{\chi} = iA\,\Gamma_c\boldsymbol{\chi}$ (with hermitian A in flavour space) gives

$$\delta S(r = \epsilon) = -\tfrac{1}{2}i\,\overline{\boldsymbol{\chi}}A\Big[(-\Gamma^1 + i\Gamma_c)\Gamma_c + \Gamma_c(\Gamma^1 + i\Gamma_c)\Big]\boldsymbol{\chi} = \overline{\boldsymbol{\chi}}A\Big(1 - i\Gamma^0\Big)\boldsymbol{\chi}\,, \tag{2.4.5}$$

showing that A can be chosen to remove the $\mathcal{C}^s + \mathcal{C}^v$ term in (2.4.3), as claimed.

A very similar reduction in the number of independent couplings also happens for more prosaic applications of the PPEFT framework to fermions. When used to describe the influence of a non-pointlike nucleus on electronic energy levels within an atom there turn out to be more effective interactions describing various types of nuclear effective couplings on the nuclear worldline than there are independent nuclear contributions to electronic energy levels. In that case the redundancy of many of the apparent nuclear moments at low energies ensures that nuclear uncertainties enter into atomic calculations in fewer ways than one would naively expect [36–38].

2.5 RG methods and catalysis

We now turn to the issue of ϵ-dependence. The mode functions and scattering amplitudes found above depend in detail on the boundary conditions imposed at the surface ∂P of the gaussian pillbox that surrounds each source. But how can it be that physical quantities depend on an arbitrary radius $r = \epsilon$ that defines the boundary of this pillbox?

This question is precisely what PPEFT methods are good for: they show how effective couplings like the $\mathcal{C}^A_{ij}$'s appearing in the dyon's effective action must depend implicitly on (*i.e.* run with) ϵ in order to ensure that physical predictions are ϵ-independent. A side benefit of this discussion is that it shows how divergences that arise (even at the classical level) in the values of the fields as $\epsilon \to 0$ get renormalized into the effective couplings of the dyon action, along the lines described in [80]. In this language the physical scales of the UV physics – such as μ in the action (2.1.1) – are described within the EFT as RG-invariant scales associated with this running.

In this section we set up how this works for S-wave scattering from the dyon and in particular ask why this running makes scattering for monopoles so different from scattering from other small massive objects like nuclei in atoms (for which the corresponding issues are described in [35] and briefly summarized in Appendix 1.2). We discuss in turn two ways the running of effective couplings of fermions to dyons differ from their couplings to more run-of-the-mill compact objects: (i) dimensional scaling changes associated with the kinematics of the fermion S-wave (which lies at the root of why dyon-fermion scattering is insensitive to dyon size), and (ii) the large-scale effects to do with the nontrivial fermionic interacting vacuum (what is sometimes called the fermion 'condensate').

2.5.1 Scaling for compact objects

The most important difference between fermion-dyon and fermion-nucleus scattering is the size of the interaction rates to which one is led. For fermion-nucleus scattering the RG invariant scale that sets the size of scattering cross sections depends on the nucleus' small radius.[14] But for fermion-dyon scattering the S-wave cross sections are *not* suppressed by the small dyonic size $R \sim m_g^{-1}$ even at energies $E \ll m_g$. Although the mechanism for this (the special kinematics of S-wave scattering) has been well-understood for quite some time [31, 32] our discussion here embeds this understanding into the wider PPEFT framework and shows how it can be understood using the same standard EFT reasoning that also applies to nuclei.

As argued in §1.2, physical predictions of the fermion-nucleus PPEFT (*e.g.* for atomic energy levels or scattering cross sections) can be expressed in terms of $C_2^{\pm}/C_1^{\pm}$ *i.e.* in terms of ratios of the integration constants in (1.2.23). These ratios are in turn determined in terms of nuclear properties through the boundary conditions (1.2.22), which relate g_+/f_+ and f_-/g_- at $r = \epsilon$ to the effective couplings $\hat{\mathcal{C}}_s$ and $\hat{\mathcal{C}}_v$. Naively, this prescription appears to make observables depend on the

[14]More precisely, the RG scales expressing the implications of the nuclear strong interactions of pionic atoms, say, are set by the nuclear radius while the nucleus' electromagnetic effects relevant for electron-nuclear interactions are suppressed relative to this by powers of the fine-structure constant [36–38].

arbitrary scale ϵ, which is introduced to regularize the boundary conditions and need not be directly related to the underlying physical scales such as the size of the compact object within the pillbox, R. However, since the radius of the Gaussian pillbox is not a physical scale, it must drop out of physical predictions and this happens because any explicit ϵ-dependence arising in calculations of an observable cancels an implicit ϵ-dependence buried within the 'bare' quantities $\hat{\mathcal{C}}_s, \hat{\mathcal{C}}_v$.

Physical predictions remain ϵ-independent if quantities like $\hat{\mathcal{C}}_s, \hat{\mathcal{C}}_v$ are chosen to be ϵ-dependent in a way that ensures that ratios like $C_2^{\pm}/C_1^{\pm}$ remain fixed as ϵ is varied. The boundary condition (1.2.22) can then be reinterpreted as a prescription for how $\hat{\mathcal{C}}_s(\epsilon), \hat{\mathcal{C}}_v(\epsilon)$ must depend on ϵ, as opposed to a condition that determines the values of g_+/f_+ and f_-/g_- at $r = \epsilon$ as a function of the known effective couplings. Within this rereading physical observables depend only on the *trajectory* $(\epsilon, \hat{\mathcal{C}}_i(\epsilon))$ rather than depending on ϵ and $\hat{\mathcal{C}}_i(\epsilon)$ separately. Changes of ϵ with physical observables fixed can be regarded as defining a renormalization-group (RG) flow of the $\hat{\mathcal{C}}_i(\epsilon)$'s, and physical observables must be invariant with respect to this flow.

It turns out that this kind of RG flow defines a natural RG-invariant length scale, and it is this length scale that both appears in observables (such as for scattering cross sections) and is determined in terms of physical scales like R when matching effective couplings to the full UV theory of the source. To see how this scale arises in terms of radial mode functions like $f_{\pm}(r)$ and $g_{\pm}(r)$ it is convenient to choose ϵ such that $R \ll \epsilon \ll a$ where a is a characteristic length scale of the bulk theory far from the source (for instance, for nuclei in atoms we might have R of order the nuclear size and a of order the Bohr radius). Radial mode functions are often well-approximated by power laws in this regime, such as in (1.2.25) (repeated here for convenience)

$$f_{\pm}(r) = C_1^{\pm} \left(\frac{r}{a}\right)^{\mathfrak{z}-1} + C_2^{\pm} \left(\frac{r}{a}\right)^{-\mathfrak{z}-1} \qquad \text{and} \qquad g_{\pm}(r) = \widetilde{C}_1^{\pm} \left(\frac{r}{a}\right)^{\mathfrak{z}-1} + \widetilde{C}_2^{\pm} \left(\frac{r}{a}\right)^{-\mathfrak{z}-1}, \tag{2.5.1}$$

for some power $\mathfrak{z}$ with $\widetilde{C}_i \propto C_i$ in a way that depends on the relative small-r asymptotic behaviour of $f_i(r)$ and $g_i(r)$. One of these solutions dominates for sufficiently small r while the other wins when r is sufficiently big. The precise crossover radius $R_\star$ between these two regimes depends only on the value of the constants $C_2^{\pm}/C_1^{\pm}$, and so provides a convenient RG-invariant characterization of the coupling evolution, and one typically finds *e.g.* scattering cross sections with bulk fields of size $\sigma \sim \pi R_\star^2$ for scattering of long-wavelength modes ($kR_\star \ll 1$) despite couplings like $\hat{\mathcal{C}}_i$ being dimensionless – see [35].

With the above story in mind we can now use the same EFT language to see why S-wave scattering from dyons is so different from scattering from other compact objects. The key issue is not whether couplings like $\hat{\mathcal{C}}_i$ are dimensionless or not (they are dimensionless for both nuclei and dyons). The key issue is the size to be expected for RG-invariant scales like $R_\star$.

The main issue is the difference between eqs. (1.2.20) for nuclei and (2.1.12) for dyons. For nuclei (1.2.20) have two linearly independent solutions and so admit two different power-law type asymptotic forms like (1.2.25) the transition between which defines the RG-invariant scale $R_\star$. But for dyons the S-wave condition removes one of these solutions leaving just a single first-order equation (1.2.25) for each choice of external quantum numbers. This means there is never a transition between two asymptotic power-law regimes; there is only one power-law for each type of mode. As a result there is no RG-invariant scale $R_\star$ on which measurable things like scattering cross sections can depend.

For S-wave scattering from dyons the situation is similar to what would have happened for nucleons if for some reason we were required to choose $C_2^{\pm} = 0$. In this case using the asymptotic form (1.2.25) in (1.2.22) would imply that $\hat{\mathcal{C}}$ is ϵ-independent. The ϵ-independence of physical quantities for S-wave dyons similarly requires quantities like $\mathcal{T}_{\rm in}$ or $\mathcal{T}_{\rm out}$ to be ϵ-independent (as we see below explicitly), and this asks all of the dimensionless rank-two couplings $\mathcal{C}_i$ to themselves directly be ϵ-independent. This makes S-wave dyon scattering from massless fermions scale invariant and so its size is mainly set by the projection of any incoming wave onto the S-wave state, leading to cross sections that vary as $\sigma \propto \pi/k^2$ when $kR \ll 1$ rather than $\sigma \propto \pi R^2$ (as we verify explicitly below). This is true for any S-wave process regardless of whether or not the reaction in question violates a flavour symmetry (like baryon number).

Arguments like these relying on the uniqueness of the S-wave kinematics are standard ones [31, 32] for explaining the large size of catalysis cross sections. What the above arguments do is provide them with an EFT veneer that shows why they do not undermine the usual notions of decoupling (once these are carefully formulated).

Matching

To this point the effective couplings $\mathcal{C}_i$ and amplitudes $\mathcal{T}_{\rm in}$ have been treated as arbitrary parameters. But they really should be regarded as being functions of microscopic parameters in any specific theory and so take definite values once a given microscopic dyon construction is chosen. Physical predictions for things like fermion-dyon scattering cross sections within specific microscopic theories are then obtained by substituting the appropriate values for the $\mathcal{C}_i$'s (or $\mathcal{T}_{\rm in}$) into the general expressions for *e.g.* scattering cross sections given in §3 and §4 below.

For simple calculations of semiclassical scattering of massless fermions moving in a classical dyon field chirality is conserved by the Dirac equation. As discussed below eq. (2.1.12) the change of direction of radial motion required by S-wave scattering implies $\mathfrak{h} = \mathfrak{c}\mathfrak{s}$ must change sign during scattering and so conservation of $\mathfrak{c}$ implies $\mathfrak{s}$ must change. This means that fermion charge is always exchanged with the dyon and so $\mathcal{T}_{\rm in}^{++} = \mathcal{T}_{\rm in}^{--} = 0$ for this type of semiclassical scattering. This then implies – *c.f.* the unitarity conditions (2.2.34) – $\mathcal{T}_{\rm in}^{-+} = e^{-i\delta}, \mathcal{T}_{\rm in}^{+-} = e^{i\delta'}$ for some phases δ, δ'. This is the situation that applies to the majority of microscopic fermion scattering calculations (where typically $\delta' = \delta$) performed in the presence of a classical dyon [61, 73, 74, 62].

Alternatively Kazama *et.al.* [77] compute scattering processes where the fermion moving within the dyonic background has an anomalous magnetic moment. In this case chirality is not conserved and chirality-changing processes dominate, corresponding to the case $\mathcal{T}_{\rm in}^{++} = \mathcal{T}_{\rm in}^{--} = i\,\mathrm{sign}(\kappa)$ (where κ is a parameter of their model) and $\mathcal{T}_{\rm in}^{+-} = \mathcal{T}_{\rm in}^{-+} = 0$. In both chirality-preserving and chirality-breaking cases our expressions for cross sections and currents found in later sections agree with theirs once restricted to these choices.

2.5.2 Interaction effects

In practice the motion of a free fermion within a fixed dyonic background does not provide a good approximation to fermion-dyon scattering. The free-fermion-moving-in-a-fixed-background approximation breaks down because charge-changing fermion interactions with the rotor field $\mathfrak{a}$ described by the amplitudes $\mathcal{T}_{\rm in}^{+-}$ and $\mathcal{T}_{\rm in}^{-+}$ significantly distort the ground state within the fermionic sector (more about this in §4.1 below) and this distortion cannot be neglected [31, 32, 63–66, 68]. The radial extent of the fermionic vacuum polarization can extend outside the dyon to distances of order the fermion Compton wavelength and so can be much larger than the underlying classical dyon solution itself.

The back-reaction of this kind of dynamics can appreciably alter the RG flow of couplings like the $\mathcal{C}_i$'s once ϵ is large enough to include a significant component of fermionic polarization within the gaussian pillbox. In this case it is the new values for $\mathcal{C}_i(\epsilon)$ that are relevant when computing quantities like $\mathcal{T}_{\rm in}$ and the above conclusion that $\mathcal{T}_{\rm in}$'s are ϵ-independent changes. The results of this type of evolution are studied in [31, 32, 63, 68], and for general models the full interpretation of the resulting physics remains incomplete (see for example [58, 59] for recent discussions). In the specific model considered here, however, the upshot is fairly simple: the Coulomb energies associated with the fermionic vacuum distortions turn out to convert the UV semiclassical prediction $\mathcal{T}_{\rm in}^{++} = \mathcal{T}_{\rm in}^{--} = 0$ for ϵ of order the monopole scale into the new prediction $\mathcal{T}_{\rm in}^{+-} = \mathcal{T}_{\rm in}^{-+} = 0$ for ϵ large enough to include the fermionic vacuum distortions [68]. This conversion is intuitive inasmuch as the underlying distortion of the fermionic vacuum is driven by nonzero $\mathcal{T}_{\rm in}^{+-}$ and $\mathcal{T}_{\rm in}^{-+}$ (as we see in §4 below)

More generally the coefficients appropriate to any other particular microscopic dyon construction can in principle be obtained in a similar way by matching the EFT to the microscopic theory of interest, once this is known. When doing this matching we typically choose ϵ such that $R \ll \epsilon \ll a$. For the simplest applications $R \sim m_g^{-1}$ is of order the dyon size, but for applications including fermion condensation in the dyon field R is instead of order the fermion's Compton wavelength $R \sim m_\psi^{-1}$. In either case taking $R \ll \epsilon \ll a$ remains justified provided the low-energy focus is on sufficiently large a.

Chapter 3

Perturbative calculations

This chapter uses the PPEFT constructed in §2 to calculate some simple dyon-fermion reaction rates and cross sections, within the perturbative framework described in §2.3 above. While the specific approximation used in §2.3 may not be useful for particular microscopic descriptions of the underlying dyon, it nonetheless allows us to explore some perturbative consequences of the fermion-dyon interactions. We return to the more broadly applicable general case, which relies on the Born-Oppenheimer approximation, in chapter §4.

3.1 The perturbative framework

The dyon-localized part of the hamiltonian governing dyon-fermion interactions obtained from the lagrangian (2.2.7) (and including the electrostatic background and fluctuation field) is

$$H_{\text{dyon}} = e\mathfrak{p}\,\widehat{A}_0(r=\epsilon) + \frac{1}{2\mathcal{I}}\left(\mathfrak{p} - \frac{\vartheta}{2\pi}\right)^2 + \frac{1}{2}\left[\overline{\psi}\,\mathfrak{C}(\mathfrak{a})\,\psi\right]_{r=\epsilon} \tag{3.1.1}$$

where $\mathfrak{p}$ is the canonical momentum for $\mathfrak{a}$ given in (2.2.9). In this section we drop the Coulomb fluctuation $\widehat{A}_0$ and perturb the fermion boundary action about a simple $\mathfrak{a}$-independent boundary action along the lines described in §2.3. Specializing to the S-wave and switching to 2D fields, we split the bulk and boundary-localized hamiltonian into unperturbed and perturbed parts, $H_0 + H_{\text{int}}$.

The unperturbed hamiltonian is[1]

$$H_0 = \frac{1}{2\mathcal{I}}\left(\mathfrak{p} - \frac{\vartheta}{2\pi}\right)^2 + \frac{i}{2}\left(\overline{\boldsymbol{\chi}}\,\Gamma_c \boldsymbol{\chi}\right)_{r=\epsilon} + \frac{1}{2}\sum_{\mathfrak{s}=\pm}\int_{\epsilon}^{\infty} \mathrm{d}r\, \overline{\chi}_{\mathfrak{s}} \left[\Gamma^1 \overleftrightarrow{\partial}_1 - i\mathfrak{s}\,\Gamma^0\left(ev - \frac{eQ}{r}\right)\right]\chi_{\mathfrak{s}}, \qquad (3.1.2)$$

in which the first term describes the free rotor dynamics and the second term gives the boundary conditions at $r = \epsilon$ satisfied by the fermions, whose bulk dynamics in the presence of the background dyon charge is given by the last term.

The unperturbed boundary term $\frac{i}{2}\left(\overline{\boldsymbol{\chi}}\Gamma_c\boldsymbol{\chi}\right)_{r=\epsilon}$ is chosen such that (i) it does not involve the rotor field $\mathfrak{a}$; (ii) the corresponding boundary matrix $\mathcal{B}^{(0)}$ has rank 2 and (iii) the resulting modes describe fermion reflection from the dyon with no phase change at $r = \epsilon$: that is, $\mathcal{T}^{++} = \mathcal{T}^{--} = 1$ to zeroeth order in perturbation theory. This choice implies the interaction picture field $\boldsymbol{\chi}$ satisfies the boundary condition (2.3.4) and can be expanded in terms of either the modes $\mathfrak{u}^{\mathrm{in}0}_{\mathfrak{s},k}, \mathfrak{v}^{\mathrm{in}0}_{\mathfrak{s},k}$ or $\mathfrak{u}^{\mathrm{out}0}_{\mathfrak{s},k}, \mathfrak{v}^{\mathrm{out}0}_{\mathfrak{s},k}$, given in equations (2.3.5)-(2.3.8).

The interaction hamiltonian in the interaction picture is then given by

$$H_{\mathrm{int}} = \frac{1}{2}\sum_{\mathfrak{s},\mathfrak{s}'=\pm} \overline{\chi}_{\mathfrak{s}}\left(\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} + i\delta\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'}\Gamma_c - i\delta\mathcal{C}^{v}_{\mathfrak{s}\mathfrak{s}'}\Gamma^0 + i\delta\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'}\Gamma^1\right)\chi_{\mathfrak{s}'}\; e^{\frac{i}{2}(\mathfrak{s}-\mathfrak{s}')\,\mathfrak{a}_I}\,, \qquad (3.1.3)$$

where the fermion fields are evaluated at $r = \epsilon$ and $\mathfrak{a}_I$ is the interaction picture rotor field

$$\mathfrak{a}_I(t) := \; e^{\frac{i}{2\mathcal{I}}\Pi^2 t}\mathfrak{a}\, e^{-\frac{i}{2\mathcal{I}}\Pi^2 t}\,, \qquad (3.1.4)$$

with $\Pi := \mathfrak{p} - (\vartheta/2\pi)$. As discussed in §2.4 this perturbed problem also has rank two only when $\delta\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'} = \delta\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'} = 0$, which we henceforth assume (though this is not crucial for this perturbative discussion). In terms of creation and annihilation operators eq. (3.1.3) can be written in the normal-ordered form

$$\begin{aligned} H_{\mathrm{int}} \;=\; E_0 + \sum_{\mathfrak{s},\mathfrak{s}'=\pm}\int_0^{\infty}\frac{\mathrm{d}k\,\mathrm{d}k'}{2\pi}\left(\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}'} - \delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}'}\right)\Big[(a^{\mathrm{in}}_{\mathfrak{s},k})^\star a^{\mathrm{in}}_{\mathfrak{s}',k'}e^{i(k-k')t} + (a^{\mathrm{in}}_{\mathfrak{s},k})^\star(\overline{a}^{\mathrm{in}}_{\mathfrak{s}',k'})^\star e^{i(k+k')t} \\ +\overline{a}^{\mathrm{in}}_{\mathfrak{s},k}a^{\mathrm{in}}_{\mathfrak{s}',k'}e^{-i(k+k')t} - (\overline{a}^{\mathrm{in}}_{\mathfrak{s}',k'})^\star\overline{a}^{\mathrm{in}}_{\mathfrak{s},k}e^{-i(k-k')t}\Big]\; e^{\frac{i}{2\mathcal{I}}\Pi^2 t}e^{\frac{i}{2}(\mathfrak{s}-\mathfrak{s}')(\mathfrak{a}-evt)}e^{-\frac{i}{2\mathcal{I}}\Pi^2 t} \qquad (3.1.5) \end{aligned}$$

[1]We follow standard practice here and keep the rotor kinetic term despite it being order $(e^2/4\pi)\mu$ in magnitude and so nominally being a higher-loop size.

where we absorb an r_0-dependent phase using an appropriate constant shift of $\mathfrak{a}$. We also drop factors of $e^{i(k\pm k')\epsilon}$ because our EFT framework requires we choose ϵ to be smaller than the characteristic bulk length scales of interest, as described in §2.5. In particular this requires us to restrict our attention to the regime where $k\epsilon$, $k'\epsilon$ and $ev\epsilon$ are all much smaller than unity.

In (3.1.5) E_0 denotes the vacuum expectation value of the interacting hamiltonian $H_{\rm int}$, given by

$$E_0 := \langle 0| H_{\rm int} |0\rangle = \frac{eQ}{4\pi\epsilon}(\delta\mathcal{C}^s_{--} - \delta\mathcal{C}^v_{--} - \delta\mathcal{C}^s_{++} + \delta\mathcal{C}^v_{++}), \tag{3.1.6}$$

as shown in Appendix F. Although this diverges as $\epsilon \to 0$ it can be absorbed into the counterterm describing the mass of the dyon. Notice these expressions depend on the boundary couplings only through the combination $\delta\mathcal{C}^s - \delta\mathcal{C}^v$, as argued must be the case in §2.4.

The remainder of this section uses the above setup to calculate some scattering observables within this perturbative framework.

3.2 Processes involving dyon charge eigenstates

We start by recording the amplitudes for single-fermion scattering processes assuming the dyon is chosen to be in a charge – *i.e.* momentum – eigenstate at both the initial and final times and working to first order in perturbation theory. By focusing here on single-particle scattering we avoid the complications of multiparticle state-definitions in a monopole background described in [56, 57] and the twist operators described in [58, 59].

Pair production

Among the charge-changing processes mediated at leading order by (3.1.5) is the production or absorption of particle-antiparticle pairs carrying net charge. The amplitude obtained at leading order from (3.1.5) for the production of a pair of positive charge – *i.e.* for $|0\rangle\,|\Pi\rangle \to (a^{\rm out}_{+,k})^\star(\overline{a}^{\rm out}_{-,k'})^\star\,|0\rangle\,|\Pi'\rangle$ – is

$$\mathcal{A}\Big[\Pi \to \Pi' + f_+(k) + \bar{f}_-(k')\Big] = -\delta_{\Pi',\Pi+1}\delta\left(\omega_{+,k} + \overline{\omega}_{-,k'} + \frac{\Pi'^2 - \Pi^2}{2\mathcal{I}}\right) i\left(\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}\right), \tag{3.2.1}$$

where $\omega_{\mathfrak{s},k} = k - \frac{1}{2}\mathfrak{s}ev$ is the energy of a particle with charge $\mathfrak{s}e/2$ and momentum k and $\overline{\omega}_{\mathfrak{s},k} = k + \frac{1}{2}\mathfrak{s}ev$ is the energy of an antiparticle with charge $-\mathfrak{s}e/2$ and momentum k.

The amplitude for producing a negatively charged pair – *i.e.* for $|0\rangle\,|\Pi\rangle \to (a^{\rm out}_{-,k})^\star(\overline{a}^{\rm out}_{+,k'})^\star\,|0\rangle\,|\Pi'\rangle$ – similarly is

$$\mathcal{A}\Big[\Pi \to \Pi' + f_-(k) + \bar{f}_+(k')\Big] = -\delta_{\Pi',\Pi-1}\delta\left(\omega_{-,k} + \overline{\omega}_{+,k'} + \frac{\Pi'^2 - \Pi^2}{2\mathcal{I}}\right) i\left(\delta\mathcal{C}^s_{-+} - \delta\mathcal{C}^v_{-+}\right) \tag{3.2.2}$$

These show that the rotor level can only change by one unit, as required for it to absorb or emit the charge lost or gained by the fermions. The fermion similarly gains or loses the energy required by this transition. These processes cause rotor states with nonzero Π to decay towards $\Pi = 0$ by emitting fermion pairs. It is energetically possible to emit positively charged pairs when $k + k' = ev - \frac{1+2\Pi}{2\mathcal{I}} \geq 0$ and so is only possible when $\Pi \leq \mathcal{I}ev - \frac{1}{2}$. It is similarly possible to emit negatively charged pairs if $k + k' = -ev - \frac{1-2\Pi}{2\mathcal{I}} \geq 0$ and this is only possible when $\Pi \geq \mathcal{I}ev + \frac{1}{2}$. No pair production occurs for electrically neutral particle-antiparticle pairs since energy conservation implies this can only happen if $k = k' = 0$.

Writing $\mathcal{A} = \mathcal{M}\,\delta(E_f - E_i)$ and using these amplitudes in Fermi's golden rule gives the following differential decay rate

$$\mathrm{d}\Gamma = 2\pi|\mathcal{M}|^2\,\delta(E_f - E_i)\,\frac{\mathrm{d}k}{2\pi}\,\frac{\mathrm{d}k'}{2\pi}\,. \tag{3.2.3}$$

Using this and performing the final-state momentum integrals, the integrated rate for emitting positively charged fermions is nonzero for initial dyon momenta satisfying $\Pi < \mathcal{I}ev - \frac{1}{2} \sim (4\pi/e)(v/\mu)$, and is given by

$$\Gamma\Big[\Pi \to (\Pi+1) + f_+(k) + \bar{f}_-(k')\Big] = \frac{1}{2\pi}\left(ev - \frac{2\Pi+1}{2\mathcal{I}}\right)|\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}|^2\,. \tag{3.2.4}$$

The integrated rate for emitting negatively charged pairs is similarly nonzero when $\Pi > \mathcal{I}ev + \frac{1}{2}$ with

$$\Gamma\Big[\Pi \to (\Pi-1) + f_-(k) + \bar{f}_+(k')\Big] = \frac{1}{2\pi}\left(-ev + \frac{2\Pi-1}{2\mathcal{I}}\right)|\delta\mathcal{C}^s_{-+} - \delta\mathcal{C}^v_{-+}|^2\,. \tag{3.2.5}$$

Scattering cross sections

The Hamiltonian (3.1.5) also describes scattering processes. The amplitude for the charge-changing process $(a^{\rm in}_{+,k})^\star \left|0\right\rangle \left|\Pi\right\rangle \to (a^{\rm out}_{-,k'})^\star \left|0\right\rangle \left|\Pi'\right\rangle$ is given by

$$\mathcal{A}\Big[f_+(k) + \Pi \to \Pi' + f_-(k')\Big] = -\delta_{\Pi',\Pi-1}\delta\left(\omega_{-,k'} - \omega_{+,k} + \frac{{\Pi'}^2 - \Pi^2}{2\mathcal{I}}\right) i\left(\delta\mathcal{C}^s_{-+} - \delta\mathcal{C}^v_{-+}\right), \quad (3.2.6)$$

where (as before) $\omega_{\mathfrak{s},k} = k - \mathfrak{s}\frac{ev}{2}$. Similarly the amplitude for $(a^{\rm in}_{-,k})^\star \left|0\right\rangle \left|\Pi\right\rangle \to (a^{\rm out}_{+,k'})^\star \left|0\right\rangle \left|\Pi'\right\rangle$ is

$$\mathcal{A}\Big[f_-(k) + \Pi \to \Pi' + f_+(k')\Big] = -\delta_{\Pi',\Pi+1}\delta\left(\omega_{+,k'} - \omega_{-,k} + \frac{{\Pi'}^2 - \Pi^2}{2\mathcal{I}}\right) i\left(\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}\right). \quad (3.2.7)$$

The analogous amplitudes for charge-changing antiparticle transitions can be inferred from those above by crossing symmetry. These reactions can proceed so long as the initial fermion energy satisfies $\omega_i \geq \frac{1}{2\mathcal{I}}(\mathfrak{s}'2\Pi + 1) - \mathfrak{s}'\frac{ev}{2}$, where $\mathfrak{s}'$ is the charge of the final particle, since $\omega_f \geq -\mathfrak{s}'\frac{ev}{2}$ for single particle scattering.

The effective interactions in (3.1.5) also describe scattering processes that do not exchange charge or energy with the dyon (and so necessarily flip the 4D chirality). The amplitude for an incoming positively charged particle to scatter to an outgoing particle of the same charge is

$$\mathcal{A}\Big[f_+(k) + \Pi \to \Pi' + f_+(k')\Big] = -\delta_{\Pi',\Pi}\,\delta\left(\omega_{+,k'} - \omega_{+,k}\right) i\Big[\int_{-\infty}^{\infty} \mathrm{d}t\, \left\langle 0\right| H_{\rm int} \left|0\right\rangle + \delta\mathcal{C}^s_{++} - \delta\mathcal{C}^v_{++}\Big], \quad (3.2.8)$$

where we omit the zeroth order term in the perturbative expansion of the S-matrix. The above expression includes a contribution from the vacuum survival amplitude, given in terms of the vacuum expectation value of $H_{\rm int}$. Although we formally include this in amplitudes such as (3.2.8) and (3.2.9) below, we are primarily interested in *inclusive* scattering processes in which the number of pairs produced by the dyon is unmeasured, as explained in more detail in §4. In that case, the vacuum survival amplitude, which describes a process in which no pairs are produced, drops out of physical predictions such as cross section results. The corresponding amplitude for a negatively charged incoming particle is

$$\mathcal{A}\Big[f_-(k)+\Pi \to \Pi'+f_-(k')\Big] = -\delta_{\Pi',\Pi}\delta\left(\omega_{-,k'}-\omega_{-,k}\right) i\Big[\int_{-\infty}^{\infty} \mathrm{d}t\ \langle 0|\, H_{\rm int}\,|0\rangle + \delta\mathcal{C}^s_{--} - \delta\mathcal{C}^v_{--}\Big]. \quad (3.2.9)$$

The amplitudes for transitions between antiparticles of the same charge can be calculated similarly.

Fermion-dyon scattering reactions are most usefully described in terms of 4D cross sections rather than 2D scattering rates, so we pause to make the connection to these explicit. Because incoming initial states in 4D are plane waves far from the dyon, they are not prepared in the S-wave. Their scattering rates are therefore the product of the 2D S-wave scattering rate times the probability, p_s, of finding the incoming plane-wave in the S-wave. The 4D cross section then is obtained by dividing by the incoming 4D particle flux $\mathfrak{F}_i$.

Combining these factors leads to the following factorized expression for the 4D single-particle S-wave differential cross section $\mathrm{d}\sigma_s$ computed using the above amplitudes with $\mathcal{A} = \mathcal{M}\,\delta(E_f - E_i)$:

$$\mathrm{d}\sigma_s\Big[f_{\mathfrak{s}\mathfrak{c}}(k)+\Pi \to \Pi' + f_{\mathfrak{s}'\mathfrak{c}'}(k')\Big] = \frac{p_s}{\mathfrak{F}_i}\,\mathrm{d}\Gamma_2 = \frac{\pi}{k^2}\,\delta_{\mathfrak{s},\mathfrak{c}}\delta_{\mathfrak{s}',-\mathfrak{c}'}\,|\mathcal{M}|^2\,\delta(E_f-E_i)\,\mathrm{d}k'\,, \quad (3.2.10)$$

where the factors p_s and $\mathfrak{F}_i$ are computed explicitly in Appendix D, where we also show how the 2D rates are calculated. The factor $\delta_{\mathfrak{s},\mathfrak{c}}$ ensures the result is nonzero only when $\mathfrak{s}=\mathfrak{c}$ corresponding to the observation made below eq. (2.1.14) that $\mathfrak{s}\mathfrak{c}=+1$ for any incoming S-wave fermion. Similarly $\mathfrak{s}'\mathfrak{c}'=-1$ for any outgoing fermion.[2] Notice the proportionality to $1/k^2$ ensures the cross section scales with energy as does the unitarity bound (so long as the fermion mass is negligible).

For instance, using the amplitudes for fermion scattering given above we find the charge-changing cross section to be

$$\mathrm{d}\sigma_s\Big[f_{++}(k)+\Pi \to \Pi' + f_{-+}(k')\Big] = \frac{\pi}{k^2}\,\delta_{\Pi',\Pi-1}\delta\left(k'-k+ev+\frac{1-2\Pi}{2\mathcal{I}}\right)|\delta\mathcal{C}^s_{-+}-\delta\mathcal{C}^v_{-+}|^2\,\mathrm{d}k'\,. \quad (3.2.11)$$

Integrating over the final fermion momentum and marginalizing over the unmeasured final dyon momentum then gives the total charge-changing cross section

$$\sigma_s\Big[f_{++}(k)+\Pi \to (\Pi-1) + f_{-+}(k')\Big] = \Theta\left(k-ev+\frac{2\Pi-1}{2\mathcal{I}}\right)\frac{\pi}{k^2}\,|\,\delta\mathcal{C}^s_{-+}-\delta\mathcal{C}^v_{-+}|^2\,. \quad (3.2.12)$$

[2] We restore the chirality label $\mathfrak{c}'$ on the final S-wave state to make all the quantum numbers explicit in $\mathrm{d}\sigma_s$

In the same way for the charge-changing $f_{--}(k) + \Pi \to (\Pi+1) + f_{+-}(k')$ transition, we get

$$\sigma_s\Big[f_{--}(k) + \Pi \to (\Pi+1) + f_{+-}(k')\Big] = \Theta\left(k + ev - \frac{2\Pi+1}{2\mathcal{I}}\right)\frac{\pi}{k^2}|\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}|^2. \quad (3.2.13)$$

Similar expressions can be found for the cross section for processes where the 4D chirality of the fermions changes but not their charge (such as the amplitudes (3.2.8) and (3.2.9)) but we do not provide them explicitly here because they depend more sensitively on our initial choice of unperturbed boundary conditions. Expressions for these processes are instead derived in §4 below using the Born-Oppenheimer approximation.

3.3 Transitions between dyon field eigenstates

For the purposes of comparing with Born-Oppenheimer results in §4 it is worth computing the same processes as above but this time choosing the initial and final rotor states to be eigenstates of $\mathfrak{a}$ rather than $\mathfrak{p}$ or Π. Strictly speaking, the interaction picture operator

$$\mathfrak{a}_I(t) = \mathfrak{a} + \frac{\Pi\, t}{\mathcal{I}}\,, \quad (3.3.1)$$

is not conserved and so its eigenstates need not agree at different times. However because $\mathcal{I}^{-1} \sim \alpha\mu \ll \mu$ in the semiclassical limit, it is a good approximation to neglect the time-dependence of $\mathfrak{a}$ and so amplitudes for transitions between $\mathfrak{a}$ eigenstates simplify considerably, because $\mathfrak{a}$ is approximately conserved.

Pair production

For instance, for static $\mathfrak{a}$ the amplitude to produce a particle-antiparticle pair of positive charge is found by taking the matrix element of (3.1.3), leading to

$$\mathcal{A}\Big[\mathfrak{a} \to \mathfrak{a}' + f_+(k) + \bar{f}_-(k')\Big] \simeq -\delta_{\mathfrak{a}'\mathfrak{a}}\delta\left(\omega_{+,k} + \overline{\omega}_{-,k'}\right) e^{i\mathfrak{a}}\, i\left(\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}\right). \quad (3.3.2)$$

The corresponding amplitude for producing a negatively charged pair vanishes in this approximation because the energy conservation condition now implies $k + k' + ev = 0$, which is never satisfied for $k, k' \geq 0$, $ev > 0$. As we shall see, the above amplitudes exactly match the ones listed in §4 below, once these are evaluated in the perturbative regime.

The resulting total rate for producing fermion pairs (marginalized over their unmeasured quantum numbers and the dyon final state) then is

$$\Gamma\Big[\mathfrak{a} \to \mathfrak{a}' + f_+(k) + \bar{f}_-(k')\Big] \simeq \frac{ev}{2\pi}\Big|\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}\Big|^2. \tag{3.3.3}$$

Scattering

The S-wave amplitudes for charge-changing fermion scattering in the same static-$\mathfrak{a}$ limit are

$$\mathcal{A}\Big[f_+(k) + \mathfrak{a} \to \mathfrak{a}' + f_-(k')\Big] = -\delta_{\mathfrak{a}'\mathfrak{a}}\delta\left(\omega_{-,k'} - \omega_{+,k}\right) e^{-i\mathfrak{a}}\, i\left(\delta\mathcal{C}^s_{-+} - \delta\mathcal{C}^v_{-+}\right), \tag{3.3.4}$$

and

$$\mathcal{A}\Big[f_-(k) + \mathfrak{a} \to \mathfrak{a}' + f_+(k')\Big] = -\delta_{\mathfrak{a}'\mathfrak{a}}\delta\left(\omega_{+,k'} - \omega_{-,k}\right) e^{i\mathfrak{a}}\, i\left(\delta\mathcal{C}^s_{+-} - \delta\mathcal{C}^v_{+-}\right). \tag{3.3.5}$$

The corresponding total cross section for charge-exchange processes in which the dyon remains in an $\mathfrak{a}$ eigenstate then is

$$\sigma_s\Big[f_{\mathfrak{s}\mathfrak{s}}(k) + \mathfrak{a} \to \mathfrak{a} + f_{-\mathfrak{s}\mathfrak{s}}(k')\Big] = \Theta\left(k - \mathfrak{s}ev\right)\, \frac{\pi}{k^2}\left|\delta\mathcal{C}^s_{-\mathfrak{s}\mathfrak{s}} - \delta\mathcal{C}^v_{-\mathfrak{s}\mathfrak{s}}\right|^2. \tag{3.3.6}$$

Notice that both pair production and scattering involve only a fairly simple condition on k (if any) as opposed to the fairly complicated restrictions on Π that arose when the dyon was prepared in a charge eigenstate. This relative simplicity arises because the $\mathfrak{a}$ eigenstate always has an overlap with Π eigenstates for which the processes are energetically allowed. The $\mathfrak{a}$-eigenstate expressions are easier to compare with the Born-Oppenheimer, and agree within their common domain of validity with the cross sections found in §4.

Chapter 4

Born-Oppenheimer approximation

In this chapter, we compute the pair-production and scattering implied by the full mode functions described within the Born-Oppenheimer approximation of §2.2.2, for which the fermions are quantized with the bosonic field $\mathfrak{a}$ initially treated as a classical field. We start in §4.1 by examining how the fast degrees of freedom (the fermions) evolve in the presence of a static classical rotor field $\mathfrak{a}$, then continue in §4.2 with some observations about the rotor's response.

For these purposes recall that dyon interactions with S-wave fermions are described by the hamiltonian $H = H_{\rm dyon} + H_{\rm bulk}$ that is the sum of the dyon-localized terms of (3.1.1), reproduced for convenience here,[1]

$$H_{\rm dyon} = \frac{1}{2\mathcal{I}}\left[\mathfrak{p}(t) - \frac{\vartheta}{2\pi}\right]^2 + e\mathfrak{p}\,\widehat{A}_0(r=\epsilon,t) + \frac{1}{2}\Big[\overline{\boldsymbol{\chi}}\,O_{\mathcal{B}}(\mathfrak{a})\,\boldsymbol{\chi}\Big]_{r=\epsilon,t} \tag{4.0.1}$$

where $O_{\mathcal{B}}(\mathfrak{a})$ is given in terms of the couplings $\mathcal{C}^{A}_{\mathfrak{s}\mathfrak{s}'}$ by (2.2.18) and the bulk 2D hamiltonian is

$$H_{\rm bulk} = H_C + \frac{1}{2}\sum_{\mathfrak{s}=\pm}\int_\epsilon^\infty {\rm d}r\,\overline{\chi}_{\mathfrak{s}}\left[\Gamma^1\overset{\leftrightarrow}{\partial}_1 - i\mathfrak{s}\,\Gamma^0\left(ev - \frac{eQ}{r} + e\widehat{A}_0\right)\right]\chi_{\mathfrak{s}}\,, \tag{4.0.2}$$

with H_C denoting the part of the Maxwell action depending on the Coulomb field $\widehat{A}_0$. We follow previous work and perturb in the Coulomb interactions involving $\widehat{A}_0$, whose contributions to the

[1] We follow standard practice here and keep the rotor kinetic term despite it being order $(e^2/4\pi)m_g$ in magnitude and so nominally being higher-loop in size. It can make sense to do so to the extent that all of the rotor's responses arise at this same order (and are included).

energy are suppressed by powers of e^2. We differ from the warm-up calculations of §3 by *not* splitting $O_{\mathcal{B}}(\mathfrak{a})$ into an unperturbed and perturbed piece; instead treating the interaction with $\mathfrak{a}$ using Born-Oppenheimer methods (as motivated in §2.3). In practice this means that we treat $\mathfrak{a}$ as a fixed classical field when determining the fermionic response and then return to ask how this slower rotor field evolves in response to interactions with the faster fermions.

4.1 Fermion evolution

We start by calculating the Bogoliubov coefficients that relate *in* and *out* modes, neglecting the Coulomb back-reaction of the distortions of the fermion ground state. Subsequent subsections then consider some of the implications of this fermionic distortion such as to pair-production rates and scattering cross sections.

Bogoliubov relations

The starting point expands the S-wave fermion field in terms of the *in* and *out* bases for the fermion modes in the presence of a dyon background described in §2.2.2:

$$\begin{aligned}\boldsymbol{\chi}(x) &= \sum_{\mathfrak{s}=\pm}\int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\pi}}\left[\mathfrak{u}^{\text{in}}_{\mathfrak{s},k}(x)\, a^{\text{in}}_{\mathfrak{s},k} + \mathfrak{v}^{\text{in}}_{\mathfrak{s},k}(x)\,(\overline{a}^{\text{in}}_{\mathfrak{s},k})^\star\right]\\ &= \sum_{\mathfrak{s}=\pm}\int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\pi}}\left[\mathfrak{u}^{\text{out}}_{\mathfrak{s},k}(x)\, a^{\text{out}}_{\mathfrak{s},k} + \mathfrak{v}^{\text{out}}_{\mathfrak{s},k}(x)\,(\overline{a}^{\text{out}}_{\mathfrak{s},k})^\star\right] \end{aligned} \tag{4.1.1}$$

with mode functions defined in eqs. (2.2.28) through (2.2.31) and (2.2.38) through (2.2.41) and particle and antiparticle creation and annihilation operators satisfying

$$\begin{aligned} \left\{a^{\text{in}}_{\mathfrak{s},k},(a^{\text{in}}_{\mathfrak{s}',k'})^\star\right\} &= \left\{\overline{a}^{\text{in}}_{\mathfrak{s},k},(\overline{a}^{\text{in}}_{\mathfrak{s}',k'})^\star\right\} = \delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}\,,\\ \text{and}\quad \left\{a^{\text{out}}_{\mathfrak{s},k},(a^{\text{out}}_{\mathfrak{s}',k'})^\star\right\} &= \left\{\overline{a}^{\text{out}}_{\mathfrak{s},k},(\overline{a}^{\text{out}}_{\mathfrak{s}',k'})^\star\right\} = \delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}\,, \end{aligned} \tag{4.1.2}$$

with all other anticommutators vanishing.

Since both the *in* and *out* basis are complete, each can be expanded in terms of the other. For each $k \geq 0$ we have

$$\begin{aligned}
\mathfrak{u}^{\text{in}}_{-,k}(x) &= \mathcal{T}^{--}_{\text{in}} e^{-2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{-ieQ} \mathfrak{u}^{\text{out}}_{-,k}(x) + \mathcal{T}^{+-}_{\text{in}} e^{-i(2k+ev)\epsilon}\, e^{i\mathfrak{a}}\, \mathfrak{u}^{\text{out}}_{+,k+ev}(x) \\
\mathfrak{v}^{\text{in}}_{+,k}(x) &= \mathcal{T}^{++}_{\text{in}} e^{2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{ieQ} \mathfrak{v}^{\text{out}}_{+,k}(x) + \mathcal{T}^{-+}_{\text{in}} e^{i(2k+ev)\epsilon}\, e^{-i\mathfrak{a}}\, \mathfrak{v}^{\text{out}}_{-,k+ev}(x) \\
\mathfrak{u}^{\text{in}}_{+,k}(x) &= \mathcal{T}^{++}_{\text{in}} e^{-2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{ieQ} \mathfrak{u}^{\text{out}}_{+,k}(x) \\
&\quad + \mathcal{T}^{-+}_{\text{in}} e^{-i(2k-ev)\epsilon}\, e^{-i\mathfrak{a}}\Big[\Theta\left(k-ev\right) \mathfrak{u}^{\text{out}}_{-,k-ev}(x) + \Theta\left(-k+ev\right) \mathfrak{v}^{\text{out}}_{-,-k+ev}(x)\Big] \\
\mathfrak{v}^{\text{in}}_{-,k}(x) &= \mathcal{T}^{--}_{\text{in}} e^{2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{-ieQ} \mathfrak{v}^{\text{out}}_{-,k}(x) \\
&\quad + \mathcal{T}^{+-}_{\text{in}} e^{i(2k-ev)\epsilon}\, e^{i\mathfrak{a}}\Big[\Theta\left(k-ev\right) \mathfrak{v}^{\text{out}}_{+,k-ev}(x) + \Theta\left(-k+ev\right) \mathfrak{u}^{\text{out}}_{+,-k+ev}(x)\Big],
\end{aligned} \tag{4.1.3}$$

and similarly

$$\begin{aligned}
\mathfrak{u}^{\text{out}}_{-,k}(x) &= \mathcal{T}^{--}_{\text{out}} e^{2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{ieQ} \mathfrak{u}^{\text{in}}_{-,k}(x) + \mathcal{T}^{+-}_{\text{out}} e^{i(2k+ev)\epsilon}\, e^{i\mathfrak{a}}\, \mathfrak{u}^{\text{in}}_{+,k+ev}(x) \\
\mathfrak{v}^{\text{out}}_{+,k}(x) &= \mathcal{T}^{++}_{\text{out}} e^{-2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{-ieQ} \mathfrak{v}^{\text{in}}_{+,k}(x) + \mathcal{T}^{-+}_{\text{out}} e^{-i(2k+ev)\epsilon}\, e^{-i\mathfrak{a}}\, \mathfrak{v}^{\text{in}}_{-,k+ev}(x) \\
\mathfrak{u}^{\text{out}}_{+,k}(x) &= \mathcal{T}^{++}_{\text{out}} e^{2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{-ieQ} \mathfrak{u}^{\text{in}}_{+,k}(x) \\
&\quad + \mathcal{T}^{-+}_{\text{out}} e^{i(2k-ev)\epsilon}\, e^{-i\mathfrak{a}}\Big[\Theta\left(k-ev\right) \mathfrak{u}^{\text{in}}_{-,k-ev}(x) + \Theta\left(-k+ev\right) \mathfrak{v}^{\text{in}}_{-,-k+ev}(x)\Big] \\
\mathfrak{v}^{\text{out}}_{-,k}(x) &= \mathcal{T}^{--}_{\text{out}} e^{-2ik\epsilon}\left(\frac{\epsilon}{r_0}\right)^{ieQ} \mathfrak{v}^{\text{in}}_{-,k}(x) \\
&\quad + \mathcal{T}^{+-}_{\text{out}} e^{-i(2k-ev)\epsilon}\, e^{i\mathfrak{a}}\Big[\Theta\left(k-ev\right) \mathfrak{v}^{\text{in}}_{+,k-ev}(x) + \Theta\left(-k+ev\right) \mathfrak{u}^{\text{in}}_{+,-k+ev}(x)\Big].
\end{aligned} \tag{4.1.4}$$

These lead to the following Bogoliubov relations between the *in* and *out* operators for each $k \geq 0$ (see appendix D for derivation)

$$\begin{gathered}
a^{\text{out}}_{-,k} = \mathcal{T}^{--}_{\text{in}}\; a^{\text{in}}_{-,k} + \mathcal{T}^{-+}_{\text{in}}\, e^{-i\mathfrak{a}}\; a^{\text{in}}_{+,k+ev}\,, \qquad (\overline{a}^{\text{out}}_{+,k})^{\star} = \mathcal{T}^{++}_{\text{in}}\; (\overline{a}^{\text{in}}_{+,k})^{\star} + \mathcal{T}^{+-}_{\text{in}}\, e^{i\mathfrak{a}}\; (\overline{a}^{\text{in}}_{-,k+ev})^{\star} \\
a^{\text{out}}_{+,k} = \mathcal{T}^{++}_{\text{in}} a^{\text{in}}_{+,k} + \mathcal{T}^{+-}_{\text{in}}\, e^{i\mathfrak{a}}\Big[\Theta\left(k-ev\right) a^{\text{in}}_{-,k-ev} + \Theta\left(-k+ev\right) (\overline{a}^{\text{in}}_{-,-k+ev})^{\star}\Big] \\
(\overline{a}^{\text{out}}_{-,k})^{\star} = \mathcal{T}^{--}_{\text{in}} (\overline{a}^{\text{in}}_{-,k})^{\star} + \mathcal{T}^{-+}_{\text{in}}\, e^{-i\mathfrak{a}}\Big[\Theta\left(k-ev\right) (\overline{a}^{\text{in}}_{+,k-ev})^{\star} + \Theta\left(-k+ev\right) a^{\text{in}}_{+,-k+ev}\Big],
\end{gathered} \tag{4.1.5}$$

as well as their inverses

$$
\begin{aligned}
a^{\rm in}_{-,k} = \mathcal{T}^{--}_{\rm out}\; a^{\rm out}_{-,k} + \mathcal{T}^{-+}_{\rm out}\, e^{-i\mathfrak{a}}\; a^{\rm out}_{+,k+ev}\,, \qquad (\overline{a}^{\rm in}_{+,k})^\star = \mathcal{T}^{++}_{\rm out}\; (\overline{a}^{\rm out}_{+,k})^\star + \mathcal{T}^{+-}_{\rm out}\; e^{i\mathfrak{a}}\; (\overline{a}^{\rm out}_{-,k+ev})^\star \\
a^{\rm in}_{+,k} = \mathcal{T}^{++}_{\rm out} a^{\rm out}_{+,k} + \mathcal{T}^{+-}_{\rm out}\, e^{i\mathfrak{a}}\Big[\Theta\,(k-ev)\, a^{\rm out}_{-,k-ev} + \Theta\,(-k+ev)\,(\overline{a}^{\rm out}_{-,-k+ev})^\star\Big] \qquad (4.1.6) \\
(\overline{a}^{\rm in}_{-,k})^\star = \mathcal{T}^{--}_{\rm out}\,(\overline{a}^{\rm out}_{-,k})^\star + \mathcal{T}^{-+}_{\rm out}\; e^{-i\mathfrak{a}}\Big[\Theta\,(k-ev)\,(\overline{a}^{\rm out}_{+,k-ev})^\star + \Theta\,(-k+ev)\, a^{\rm out}_{+,-k+ev}\Big].
\end{aligned}
$$

These are consistent with the anticommutation relations (4.1.2) by virtue of the unitarity identities (2.2.34) and (2.2.35) – and their counterparts (2.2.44) and (2.2.45) – satsified by the $\mathcal{T}$'s. As in §3, we drop powers of $k\epsilon$, $k'\epsilon$ and $ev\epsilon$ and shift $\mathfrak{a}$ and rephase the out-state creation and-annihilation operators to remove an r_0-dependent phase.

Pair production

The mixing of creation and annihilation operators in the Bogoliubov transformations (4.1.5) and (4.1.6) shows that the system is unstable to pair production if $\mathcal{T}^{+-}_{\rm in/out}$ is nonzero.

To compute the pair-production rate define the *in* and *out* vacua $|0_{\rm in}\rangle$ and $|0_{\rm out}\rangle$ as usual:

$$
\boldsymbol{a}^{\rm in}_{\mathfrak{s},k}\,|0_{\rm in}\rangle = \overline{\boldsymbol{a}}^{\rm in}_{\mathfrak{s},k}\,|0_{\rm in}\rangle = 0 \quad \text{and} \quad \boldsymbol{a}^{\rm out}_{\mathfrak{s},k}\,|0_{\rm out}\rangle = \overline{\boldsymbol{a}}^{\rm out}_{\mathfrak{s},k}\,|0_{\rm out}\rangle = 0\,. \qquad (4.1.7)
$$

We switch here for convenience to discretely normalized momentum states, for which we denote the creation and annihilation operators using bold-faced fonts, as in $(\boldsymbol{a}, \overline{\boldsymbol{a}})$. (See appendix D for relation between discrete and continuum normalized states.)

Writing the total vacuum as a tensor product over momenta, $|0\rangle = \prod\limits_{k>0} |0^{\,k}\rangle$, and using the above Bogoliubov transformations implies for each $k \geq 0$ we have

$$
|0^k_{\rm in}\rangle = \Big\{\Theta\,(k-ev) + \Theta\,(-k+ev)\,\big[\mathcal{T}^{--}_{\rm in} + \mathcal{T}^{+-}_{\rm in} e^{i\mathfrak{a}}(\boldsymbol{a}^{\rm out}_{+,k})^\star(\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\big]\Big\}\,|0^k_{\rm out}\rangle \qquad (4.1.8)
$$

and

$$
|0^k_{\rm out}\rangle = \Big\{\Theta\,(k-ev) + \Theta\,(-k+ev)\,\big[\mathcal{T}^{--\,*}_{\rm in} + \mathcal{T}^{-+\,*}_{\rm in} e^{i\mathfrak{a}}(\boldsymbol{a}^{\rm in}_{+,k})^\star(\overline{\boldsymbol{a}}^{\rm in}_{-,-k+ev})^\star\big]\Big\}\,|0^k_{\rm in}\rangle\;, \qquad (4.1.9)
$$

where as usual $\Theta(x)$ denotes the Heaviside step function. See appendix E for derivation of the above expressions.

If $\mathcal{T}^{+-} = 0$ then (2.2.34) shows that $\mathcal{T}^{-+}$ also vanishes and $|\mathcal{T}^{--}| = |\mathcal{T}^{++}| = 1$. In this case (4.1.8) and (4.1.9) imply the *in* and *out* vacua are the same state. But when $\mathcal{T}^{+-} \neq 0$ for some momenta the *in* vacuum contains occupied *out* particles. This shows that when charge-changing effective dyon-fermion interactions are present the dyon spontaneously emits a fermion with charge $+\frac{1}{2}e$ and the antiparticle of the charge $-\frac{1}{2}e$ state, ensuring the net charge emission of $\frac{1}{2}e + \frac{1}{2}e = e$.

Total electric charge is conserved because this emission is accompanied by a transition between rotor levels for $\mathfrak{a}$ that removes one unit $+e$ of charge from the dyon, as can be seen[2] from the $\mathfrak{a}$-dependence of (4.1.10). The sign of the charge removed is dictated by the overall sign of the dyon charge Q which we've chosen to be positive: $Q > 0$. These pairs are produced through the Schwinger effect [81, 73] by the external voltage v of the dyon between $r \to \infty$ and $r = 0$, and act to discharge the dyon's net charge.

The energetics of the process is somewhat obscured within the Born-Oppenheimer approximation because the rotor is treated as a classical field, which if regarded as an eigenstate of $\mathfrak{a}$ is not an energy eigenstate. When calculated perturbatively in §3 we do find that fermion emission extracts an energy given by the spacing between rotor steps, which is of order $\mathcal{I}^{-1} \sim \alpha m_g$. But in a semiclassical approximation this energy transfer is the same size (relative to m_g) as loop corrections, which to this order we neglect, and this is why (4.1.10) allows pair-production for the entire momentum range $k \in (0, ev)$. Implicit in this treatment is the assumption that the fermion energies of interest are much higher than the rotor gap; one manifestation of the noncommuting order of limits discussed below eq. (2.1.8).

The amplitude for this emission (for fixed $\mathfrak{a}$) in a specific momentum mode is

$$\langle 0^k_{\rm out}|\, \overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev}\, \boldsymbol{a}^{\rm out}_{+,k}\, |0^k_{\rm in}\rangle = \Theta(-k+ev)\, \mathcal{T}^{+-}_{\rm in}\, e^{i\mathfrak{a}}\,, \tag{4.1.10}$$

which has squared modulus $\mathcal{P}_k = |\mathcal{T}^{+-}_{\rm in}|^2 = |\mathcal{T}^{+-}_{\rm out}|^2$ when $0 < k < ev$. The vacuum-survival

[2] This is made very explicit in §3 for those who need convincing.

amplitude for this specific mode is similarly

$$\langle 0^k_{\rm out}|0^k_{\rm in}\rangle = \Theta\left(k-ev\right)+\Theta\left(-k+ev\right)\mathcal{T}^{--}_{\rm in}\,, \tag{4.1.11}$$

which squares to unity when $k > ev$ but has squared modulus $|\mathcal{T}^{--}_{\rm in}|^2 = 1-|\mathcal{T}^{+-}_{\rm in}|^2 = 1-\mathcal{P}_k$ when $0 < k < ev$. The likelihood of producing zero or one pairs sums to unity because for fermions these are the only possible options.

The *exclusive* probability for the full vacuum to produce exactly one pair in a specific mode $k \in (0, ev)$ is given by combining the above result for all k, giving

$$P_{\rm pair}(k) = |\,\langle 0_{\rm out}|\,\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev}\,\boldsymbol{a}^{\rm out}_{+,k}\,|0_{\rm in}\rangle\,|^2 = \mathcal{P}_k\prod_{q\neq k}^{q<ev}(1-\mathcal{P}_q) = |\mathcal{T}^{+-}_{\rm in}(k)|^2\prod_{q\neq k}^{q<ev}|\mathcal{T}^{--}_{\rm in}(q)|^2. \tag{4.1.12}$$

The total probability factorizes because the likelihood of pair-production in each mode is independent of what happens for the other modes. More useful is the *inclusive* probability for pair production of a specific mode with the other modes unmeasured (and so marginalized over). The above expressions show this is given by

$$p_{\rm pair}(k) = \mathcal{P}_k = |\mathcal{T}^{+-}_{\rm in}|^2 \quad\text{and so}\quad p_{\rm no\,pair}(k) = 1-\mathcal{P}_k = |\mathcal{T}^{--}_{\rm in}|^2\,. \tag{4.1.13}$$

We can calculate the average number of particles produced by making use of the Bogoliubov relations in the form

$$\begin{aligned}\langle 0_{\rm in}|\,N_{\rm out}\,|0_{\rm in}\rangle &= \sum_{k=0}^{\infty}\langle 0_{\rm in}|\,(\boldsymbol{a}^{\rm out}_{+k})^\star\,\boldsymbol{a}^{\rm out}_{+k}\,|0_{\rm in}\rangle \\ &= \sum_{k=0}^{ev}|\mathcal{T}^{+-}_{\rm in}|^2\,\langle 0_{\rm in}|\,\overline{\boldsymbol{a}}^{\rm in}_{-,-k+ev}(\overline{\boldsymbol{a}}^{\rm in}_{-,-k+ev})^\star\,|0_{\rm in}\rangle = \frac{ev}{\Delta k}|\mathcal{T}^{+-}_{\rm in}|^2, \end{aligned}\tag{4.1.14}$$

where $\Delta k = \pi/L$ is the spacing between momentum states (when these are discretely normalized).[3] The number of particles per unit length is obtained by dividing by $2L$, and this has a sensible continuum limit as $L\to\infty$, with $\mathrm{d}\langle N\rangle/\mathrm{d}x = ev|\mathcal{T}^{+-}_{\rm in}|^2/(2\pi)$. Since each produced particle moves

[3] We discretize momenta by putting the system in a box $-L < r < L$ with near-dyon boundary conditions imposed at $r = \epsilon \approx 0$. In these conventions the relevant length of the system is $2L$, and the density of states is $2\pi/(2L) = \pi/L$

to larger r at the speed of light this means that these particles emerge at infinity with a rate $\Gamma_\infty = ev|\mathcal{T}_{\rm in}^{+-}|^2/(2\pi)$. A similar counting also applies to the number of produced antiparticles. Because each pair contains one positively charged particle and one positively charged antiparticle the number of produced *pairs* is also

$$\Gamma_{\rm pair} = \frac{ev}{2\pi}|\mathcal{T}_{\rm in}^{+-}|^2\,. \tag{4.1.15}$$

This result is independently computed below using the expectation values for the fermionic currents.

Eq. (4.1.15) can also be compared with (3.3.3) (when restricted to the perturbative domain). To this end we must expand the coefficients $\mathcal{T}_{\rm in}$ perturbatively to the same order in the $\delta\mathcal{C}$s used in §3. To linear order the $\mathcal{T}_{\rm in}$ amplitudes are given by

$$\mathcal{T}_{\rm in}^{\mathfrak{s}\mathfrak{s}} = 1 + i(\delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}} - \delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}}) \quad \text{and} \quad \mathcal{T}_{\rm in}^{+-} = -\mathcal{T}_{\rm in}^{-+\,*} = i(\delta\mathcal{C}^v_{+-} - \delta\mathcal{C}^s_{+-}), \tag{4.1.16}$$

once the rank-2 conditions $\delta\mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}'} = \delta\mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}'} = 0$ are used. Using these in (4.1.15) then agrees with (3.3.3).

Vacuum currents

An alternative characterization of dyonic pair production that lends itself to the continuum limit is the integrated contribution of the produced pairs to current flow in the fermion sector. To display these currents we evaluate the expectation value of the various conserved currents in the *in* vacuum. Since these expectation values in general diverge we regulate them by point-splitting the two fermion fields involved by a distance ε, writing

$$\overline{\boldsymbol{\chi}}(r,t)M\boldsymbol{\chi}(r,t) \to \overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)M\boldsymbol{\chi}(r-\varepsilon/2,t)\,, \tag{4.1.17}$$

with $\varepsilon \to 0$ taken at the end, after renormalizing. We quote here expressions for the regularized currents – see appendix F for details of the matrix-element calculations.

The regularized components of the fermion number current[4] $j_B^\alpha = i\overline{\boldsymbol{\chi}}\Gamma^\alpha\boldsymbol{\chi}$ are given by

$$\langle 0_{\rm in}|\, j_B^0(x)\,|0_{\rm in}\rangle = \langle 0_{\rm out}|\, j_B^0(x)\,|0_{\rm out}\rangle = 0 \quad \text{and} \quad \langle 0_{\rm in}|\, j_B^1(x)\,|0_{\rm in}\rangle = \langle 0_{\rm out}|\, j_B^1(x)\,|0_{\rm out}\rangle = \frac{2i}{\pi\varepsilon}, \tag{4.1.18}$$

while those of the fermionic electromagnetic current $j_F^\alpha = \frac{1}{2}ie\overline{\boldsymbol{\chi}}\Gamma^\alpha\tau_3\boldsymbol{\chi}$ are

$$\begin{aligned} \langle 0_{\rm in}|\, j_F^0(r,t)\,|0_{\rm in}\rangle &= \langle 0_{\rm out}|\, j_F^0(r,t)\,|0_{\rm out}\rangle = \frac{e^2 v}{2\pi}|\mathcal{T}_{\rm in}^{+-}|^2 - \frac{e^2 Q}{2\pi r} \\ \text{and} \quad \langle 0_{\rm in}|\, j_F^1(r,t)\,|0_{\rm in}\rangle &= -\langle 0_{\rm out}|\, j_F^1(r,t)\,|0_{\rm out}\rangle = \frac{e^2 v}{2\pi}|\mathcal{T}_{\rm in}^{+-}|^2 \,. \end{aligned} \tag{4.1.19}$$

We define the axial current by $j_A^\alpha = i\overline{\boldsymbol{\chi}}\Gamma^\alpha\Gamma_A\boldsymbol{\chi}$ with $\Gamma_A := \Gamma_c\,\tau_3$ rather than Γ_c because this agrees with the 4D axial current $i\overline{\boldsymbol{\psi}}\gamma^\mu\gamma_5\boldsymbol{\psi}$ for S-wave states (up to the usual 2D normalization factor of $4\pi r^2$). Its vacuum matrix elements are

$$\begin{aligned} \langle 0_{\rm in}|\, j_A^0(r,t)\,|0_{\rm in}\rangle &= -\langle 0_{\rm out}|\, j_A^0(r,t)\,|0_{\rm out}\rangle = -\frac{ev}{\pi}|\mathcal{T}_{\rm in}^{+-}|^2 \\ \text{and} \quad \langle 0_{\rm in}|\, j_A^1(r,t)\,|0_{\rm in}\rangle &= \langle 0_{\rm out}|\, j_A^1(r,t)\,|0_{\rm out}\rangle = -\frac{ev}{\pi}|\mathcal{T}_{\rm in}^{+-}|^2 + \frac{eQ}{\pi r} \,. \end{aligned} \tag{4.1.20}$$

Recalling that the 2D modes are normalized so that the 2D flux j^1 gives the integrated radial flux $4\pi r^2 J^r$ for the corresponding 4D S-wave current, we see that at spatial infinity there is a nonzero flux of both electric and axial charge:

$$\langle 0_{\rm in}|\, j_F^1(\infty,t)\,|0_{\rm in}\rangle = e\mathfrak{F} \quad \text{and} \quad \langle 0_{\rm in}|\, j_A^1(\infty,t)\,|0_{\rm in}\rangle = -2\mathfrak{F} \quad \text{where} \quad \mathfrak{F} = \frac{ev}{2\pi}|\mathcal{T}_{\rm in}^{+-}|^2 \,. \tag{4.1.21}$$

This has a simple interpretation as a flux of pair-produced particles, since each such pair carries electric charge e and axial charge[5] -2 (and no net fermion number), with integrated particle flux (or rate with which particle pairs appear at infinity) given by $\Gamma_{\rm pair} = \mathfrak{F}$, in agreement with (4.1.15). Our result for the flux of the axial current is consistent with [73], when the $\mathcal{T}$ amplitudes are chosen to match theirs[6] *i.e.* when $|\mathcal{T}^{+-}| = 1$.

[4] We do not go through the exercise of renormalizing the fermion number current here, since we only use $\langle 0_{\rm in}|\, j_B^1(x)\,|0_{\rm in}\rangle$ to evaluate the conservation equations for j_F^α, j_A^α.

[5] See appendix D for discussion of asymptotic charges of *in*, *out* states.

[6] Note that the axial current in [73] has a relative minus sign compared to our definition.

These expressions are also consistent with (anomalous) current conservation. As shown in appendix F, the above currents satisfy

$$\begin{aligned} \partial_\alpha j^\alpha_{\scriptscriptstyle B}(x) &= \frac{eQ}{2r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon\,\overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)\Gamma^0\,\tau_3\boldsymbol{\chi}(r-\varepsilon/2,t)\Big] = -\frac{iQ}{r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon j^0_{\scriptscriptstyle F}(x)\Big] = 0, \\ \partial_\alpha j^\alpha_{\scriptscriptstyle F}(x) &= \frac{e^2Q}{4r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon\,\overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)\Gamma^0\boldsymbol{\chi}(r-\varepsilon/2,t)\Big] = -\frac{ie^2Q}{4r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon j^0_{\scriptscriptstyle B}(x)\Big] = 0, \\ \partial_\alpha j^\alpha_{\scriptscriptstyle A}(x) &= -\frac{eQ}{2r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon\,\overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)\Gamma^1\boldsymbol{\chi}(r-\varepsilon/2,t)\Big] = \frac{ieQ}{2r^2}\lim_{\varepsilon\to 0}\Big[\varepsilon j^1_{\scriptscriptstyle B}(x)\Big] = -\frac{eQ}{\pi r^2}\,. \end{aligned} \tag{4.1.22}$$

These agree with the standard 2D anomaly expressions, which in the present instance tell us that $j^\alpha_{\scriptscriptstyle B}$ and $j^\alpha_{\scriptscriptstyle F}$ are anomaly free and give

$$\partial_\alpha j^\alpha_{\scriptscriptstyle A} = \left[\frac{e}{2} - \left(-\frac{e}{2}\right)\right]\frac{1}{2\pi}\epsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta} = \frac{e}{\pi}\Big[\partial_0\mathcal{A}_r - \partial_r\mathcal{A}_0\Big] = \frac{1}{\pi}\partial_r\left(\frac{eQ}{r} - ev\right) = -\frac{eQ}{\pi r^2}, \tag{4.1.23}$$

when evaluated with the background dyonic Coulomb field.

It is also noteworthy that (4.1.19) implies that particle production significantly polarizes the fermionic ground state, inducing a charge density with the opposite sign to the dyon charge that (for massless fermions) falls off only as a power law as one moves away from the dyon. As has been remarked elsewhere [26, 62–66], such charging of the fermion ground state puffs up the dyon into a much bigger object than was the underlying classical dyon configuration.

Dyon-fermion scattering

The Bogoliubov relations in the 2D EFT also allow us to calculate the cross section for any S-wave fermion-dyon scattering process, by evaluating amplitudes of the form $\langle A_{\rm out}|B_{\rm in}\rangle$. The Bogoliubov transformation provides a succinct listing of the options for $\langle A_{\rm out}|$ that give nonzero amplitudes given a single-particle $|B_{\rm in}\rangle$ that can be seen by expanding the incoming state $(a^{\rm in}_{\mathfrak{s},k})^\star$ in terms of *out* operators – see (4.1.6). The presence of pair-production means that any such process can be accompanied by some number of spontaneously produced pairs.

For instance the 4D chirality-changing amplitude for $f_-(k) \to f_-(k')$ accompanied by the emission of n pairs is given by

$$\begin{aligned} &\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1}\; a^{\rm out}_{-,k'} (a^{\rm in}_{-,k})^\star \,|0_{\rm in}\rangle \\ &\quad = \delta(k-k') \mathcal{T}^{--}_{\rm in} \,\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1} \,|0_{\rm in}\rangle \,, \end{aligned} \tag{4.1.24}$$

where $n = 0, 1, 2, \cdots$. The amplitude for charge-changing processes like $f_\pm(k) \to f_\mp(k')$ (accompanied by n spontaneously produced pairs) is similarly

$$\begin{aligned} &\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1}\; a^{\rm out}_{-,k'} (a^{\rm in}_{+,k})^\star \,|0_{\rm in}\rangle \\ &\quad = \delta(k'+ev-k) e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\rm in} \,\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1} \,|0_{\rm in}\rangle \,, \end{aligned} \tag{4.1.25}$$

and

$$\begin{aligned} &\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1}\; a^{\rm out}_{+,k'} (a^{\rm in}_{-,k})^\star \,|0_{\rm in}\rangle \\ &\quad = \delta(k'-ev-k) e^{i\mathfrak{a}} \mathcal{T}^{+-}_{\rm in} \,\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1} \,|0_{\rm in}\rangle \,. \end{aligned} \tag{4.1.26}$$

In particular the different voltages seen by the two charge states imply the reaction $f_+(k) \to f_-(k')$ (plus pair production) vanishes unless $k > ev$ for want of fermion final states with the required energy.

The remaining reaction obtained with an initial incoming positively charged fermion is slightly more complicated. On one hand the 4D chirality-flipping process $f_+(k) \to f_+(k')$ (plus the production of n pairs) proceeds much as above. However when $k < ev$ the part of the Bogoliubov transformation relating $(a^{\rm in}_{+,k})^\star$ to $\overline{a}^{\rm out}_{-,-k+ev}$ also contributes to give an amplitude for spontaneously emitting $n+1$ pairs from the vacuum (with one of the pairs corresponding to a particle and antiparticle of momentum k' and $-k+ev$, respectively), leading to:

$$\begin{aligned} &\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1}\; a^{\rm out}_{+,k'} (a^{\rm in}_{+,k})^\star \,|0_{\rm in}\rangle \\ &\quad = \delta(k-k') \mathcal{T}^{++}_{\rm in} \,\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1} \,|0_{\rm in}\rangle \\ &\qquad + \Theta\left(-k+ev\right) e^{-i\mathfrak{a}} \mathcal{T}^{-+}_{\rm in} \,\langle 0_{\rm out}|\, \overline{a}^{\rm out}_{-,-q_n+ev} a^{\rm out}_{+,q_n} ... \overline{a}^{\rm out}_{-,-q_1+ev} a^{\rm out}_{+,q_1} a^{\rm out}_{+,k'} \overline{a}^{\rm out}_{-,-k+ev} \,|0_{\rm in}\rangle \,. \end{aligned} \tag{4.1.27}$$

In equations (4.1.24)-(4.1.27), the momenta $k', q_1, \cdots q_n$ are all distinct and satisfy $ev > q_1, \cdots q_n > 0$ as well as $k, k' > 0$. Similar formulae can be derived for the amplitudes with a single antiparticle in the initial state.

The next step is to evaluate the pair production amplitudes appearing in (4.1.24)-(4.1.27) by using the *out* particle content of the *in* vacuum, as in appendix E. When counting pairs it is more convenient to switch to a discrete normalization for momentum eigenstates, as we now do, in which case the maximum number of *out* pairs in the *in* vacuum, N, is a finite but large number[7]. The discrete normalization analogue of the amplitude for emitting n pairs appearing in (4.1.24) to (4.1.26) and the first line of the right-hand side of (4.1.27) evaluates to

$$\boldsymbol{\mathcal{A}}^n_{\rm pair} = \langle 0_{\rm out}|\, \overline{\boldsymbol{a}}^{\rm out}_{-,-q_n+ev} \boldsymbol{a}^{\rm out}_{+,q_n} ... \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev} \boldsymbol{a}^{\rm out}_{+,q_1} \,|0_{\rm in}\rangle = \left(\mathcal{T}^{--}_{\rm in}\right)^{N-n} \left(\mathcal{T}^{+-}_{\rm in}\right)^{n} e^{in\mathfrak{a}}, \tag{4.1.28}$$

for $n \le N$, and vanishes otherwise.[8] For $k < ev$ the amplitude for emitting $n+1$ pairs encountered in the second line of (4.1.27) similarly corresponds to

$$\begin{aligned} \boldsymbol{\mathcal{A}}^{n+1}_{\rm pair}(k,k') &= \langle 0_{\rm out}|\, \overline{\boldsymbol{a}}^{\rm out}_{-,-q_n+ev} \boldsymbol{a}^{\rm out}_{+,q_n} ... \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev} \boldsymbol{a}^{\rm out}_{+,q_1} \boldsymbol{a}^{\rm out}_{+,k'} \overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev} \,|0_{\rm in}\rangle \\ &= -\delta_{kk'} \left(\mathcal{T}^{--}_{\rm in}\right)^{N-(n+1)} \left(\mathcal{T}^{+-}_{\rm in}\right)^{n+1} e^{i(n+1)\mathfrak{a}}, \end{aligned} \tag{4.1.29}$$

when $n < N$ and vanishes otherwise. The two lines on the right-hand side of (4.1.27) then combine to become

$$\begin{aligned} &\langle 0_{\rm out}|\, \overline{\boldsymbol{a}}^{\rm out}_{-,-q_n+ev} \boldsymbol{a}^{\rm out}_{+,q_n} ... \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev} \boldsymbol{a}^{\rm out}_{+,q_1}\, \boldsymbol{a}^{\rm out}_{+,k'} (\boldsymbol{a}^{\rm in}_{+,k})^\star \,|0_{\rm in}\rangle \\ &\qquad = \delta_{kk'} \Big[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\Big] \left(\mathcal{T}^{--}_{\rm in}\right)^{N-(n+1)} \left(\mathcal{T}^{+-}_{\rm in}\right)^{n} e^{in\mathfrak{a}}, \end{aligned} \tag{4.1.30}$$

for $k < ev$ and $n < N$, when evaluated using discretely normalized states. The corresponding continuum normalization amplitudes have a very similar form.

Notice that the amplitudes (4.1.24)-(4.1.26) factorize into a product of a single-particle transition amplitude, $\mathcal{A}_{sc}$, from a particle with quantum numbers $\mathfrak{s}, k$ to one with quantum numbers $\mathfrak{s}', k'$,

[7] N is defined as the maximum value of $\langle 0_{\rm in}|\, N_{\rm out}\,|0_{\rm in}\rangle = ev/\Delta k\, |\mathcal{T}^{+-}_{\rm in}|^2$ and so is given by $N = ev/\Delta k$. In the continuum limit, N goes to infinity as the spacing Δk between states vanishes (see Appendix D for details).

[8] We denote transition amplitudes between discretely normalized states by $\boldsymbol{\mathcal{A}}$ and their continuum normalization counterparts by $\mathcal{A}$.

times a product of pair-production (or vacuum survival) amplitudes for all the modes. The amplitude (4.1.27) factorizes in the same way for initial momenta $k > ev$, while for $k < ev$ the product over pair production (or vacuum survival) amplitudes runs over all but the k-th mode[9]. The amplitudes factorize in this way because scattering and pair production for different modes are statistically independent. Inspection of the above formulae shows that the single-particle transition amplitudes $\mathcal{A}_{sc}[f_{\mathfrak{s}}(k) \to f_{\mathfrak{s}'}(k')]$ are given by

$$\mathcal{A}_{sc}[f_-(k) \to f_-(k')] = \mathcal{T}_{\rm in}^{--}\ \delta(k-k'), \tag{4.1.31}$$

$$\mathcal{A}_{sc}[f_+(k) \to f_-(k')] = e^{-i\mathfrak{a}}\mathcal{T}_{\rm in}^{-+}\ \delta(k'+ev-k), \tag{4.1.32}$$

$$\mathcal{A}_{sc}[f_-(k) \to f_+(k')] = e^{i\mathfrak{a}}\mathcal{T}_{\rm in}^{+-}\ \delta(k'-ev-k)\,, \tag{4.1.33}$$

and

$$\mathcal{A}_{sc}[f_+(k) \to f_+(k')] = \Big[\Theta(k-ev)\mathcal{T}_{\rm in}^{++} + \Theta\left(-k+ev\right)\left[\mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm in}^{--} - \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm in}^{-+}\right]\Big]\,\delta(k-k')\,. \tag{4.1.34}$$

Notice that the unitarity constraints on the $\mathcal{T}_{\rm in}$ amplitudes given in (2.2.34) and (2.2.35) imply that the expression $\mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm in}^{--} - \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm in}^{-+}$ simplifies to a phase.

Low-energy scattering rates can now be computed much as in §3 by projecting any incoming plane wave onto the S-wave. 4D cross sections can then be computed by dividing by the appropriate incident particle flux. For instance, factoring out the energy-conserving delta function from amplitudes (4.1.31)-(4.1.33) as $\mathcal{A}_{sc} = \mathcal{M}\,\delta(E_f - E_i)$, the 4D cross section for scattering with no pair production is

$$\mathrm{d}\sigma^s_{\rm exclusive}\Big[f_{\mathfrak{s}\mathfrak{c}}(k) \to f_{\mathfrak{s}'\mathfrak{c}'}(k')\Big] = \frac{p_s}{\mathfrak{F}_i}\,\mathrm{d}\Gamma_{\rm exclusive} = \frac{\pi}{k^2}\,\delta_{\mathfrak{s},\mathfrak{c}}\delta_{\mathfrak{s}',-\mathfrak{c}'}\,|\mathcal{M}|^2\,|\,\langle 0_{\rm out}|0_{\rm in}\rangle\,|^2\,\delta(E_f-E_i)\,\mathrm{d}k'\,, \tag{4.1.35}$$

[9]For $k < ev$, the amplitude (4.1.27) factorizes differently than (4.1.24)-(4.1.26) because it describes two processes in which the number of produced pairs is not the same. We define $\mathcal{A}_{sc}$ for this process as in (4.1.34) so that the single-particle amplitude captures the relevant contribution of (4.1.27) to inclusive observables.

where the probability for the plane wave to be found in an S-wave, p_s and the initial particle flux $\mathfrak{F}_i$ are defined as in §3 and are calculated in Appendix D along with the 2D exclusive differential interaction rate $\mathrm{d}\Gamma_{\text{exclusive}}$. Similarly, the cross section for scattering with no additional pair production for the final amplitude (4.1.34) is

$$\mathrm{d}\sigma^s_{\text{exclusive}}\Big[f_{++}(k) \to f_{+-}(k')\Big] = \frac{\pi}{k^2}\,|\mathcal{M}|^2\,|\langle 0_{\text{out}}|0_{\text{in}}\rangle|^2\,\delta(E_f - E_i)\,\mathrm{d}k' \times\Big[\Theta(k-ev) + \Theta(-k+ev)\left|\mathcal{T}^{--}_{\text{in}}\right|^{-2}\Big], \tag{4.1.36}$$

where $\mathcal{M}$ is defined through $\mathcal{A}_{sc} = \mathcal{M}\,\delta(E_f - E_i)$ and the factor of $\left|\mathcal{T}^{--}_{\text{in}}\right|^{-2}$ is cancelled by similar factors in the overlap $|\langle 0_{\text{out}}|0_{\text{in}}\rangle|^2$, making the cross section finite even in the $\mathcal{T}^{--}_{\text{in}} \to 0$ limit.

Of more practical interest are *inclusive* cross sections for which the number of associated pair productions is unmeasured and so marginalized over. Appendix E – see the discussion below eq. (E.2.1) – explicitly performs the marginalization over the number of produced pairs and shows that the resulting cross section can be expressed purely in terms of the single-particle scattering amplitudes $\mathcal{A}_{sc}$, given in (4.1.31)-(4.1.34). The ability to do so is a consequence of unitarity, and is also the reason the parameters n and N drop out of our final results. The inclusive 4D cross section becomes

$$\mathrm{d}\sigma_s\Big[f_{\mathfrak{s}\mathfrak{c}}(k) \to f_{\mathfrak{s}'\mathfrak{c}'}(k')\Big] = \frac{p_s}{\mathfrak{F}_i}\,\mathrm{d}\Gamma_{\text{inclusive}} = \frac{\pi}{k^2}\,\delta_{\mathfrak{s},\mathfrak{c}}\delta_{\mathfrak{s}',-\mathfrak{c}'}\,|\mathcal{M}|^2\,\delta(E_f - E_i)\,\mathrm{d}k'\ , \tag{4.1.37}$$

where the 2D inclusive differential interaction rate $\mathrm{d}\Gamma_{\text{inclusive}}$ is defined in Appendix D.

Combining results and integrating over the final momentum k' leads to the total inclusive S-wave cross sections. For charge-exchange processes with $\mathfrak{s} = \mathfrak{c}$ we have

$$\sigma_s[f_\pm \to f_\mp] = \frac{\pi}{k^2}\left|\mathcal{T}^{+-}_{\text{in}}\right|^2\Theta(k - \mathfrak{s}ev). \tag{4.1.38}$$

Similarly for processes in which the 4D chirality changes, but the charge doesn't the cross section when $\mathfrak{s} = \mathfrak{c} = -$ is

$$\sigma_s[f_- \to f_-] = \frac{\pi}{k^2}\left|\mathcal{T}^{--}_{\text{in}}\right|^2, \tag{4.1.39}$$

and

$$\sigma_s[f_+ \to f_+] = \frac{\pi}{k^2}\Big[\Theta(k-ev)\left|\mathcal{T}^{++}_{\text{in}}\right|^2 + \Theta\left(-k+ev\right)\Big], \tag{4.1.40}$$

when $\mathfrak{s} = \mathfrak{c} = +$. These agree with the corresponding perturbative expressions when their domains of validity overlap, and are consistent with the cross section results given in [77], when we restrict to their choices: $|\mathcal{T}_{\rm in}^{++}| = |\mathcal{T}_{\rm in}^{--}| = 1$, $\mathcal{T}_{\rm in}^{+-} = \mathcal{T}_{\rm in}^{-+} = 0$.

These cross sections display catalysis inasmuch as they are independent of the scale $R \sim m_g^{-1}$ of the underlying classical dyon [26–28]. They also do not depend directly on the dyon magnetic or electric charge (though there is a large Coulomb phase that drops out of the cross section that does see the dyon's electric charge). Their size scales like the unitarity bound $\sigma \propto 1/k^2$ whose origin comes purely from the projection of the incoming plane wave onto the S-wave that dominates at low energies. Finally the rates are directly controlled by the size of the effective couplings hidden within the $\mathcal{T}_{\rm in}$ amplitudes, rather than through the more microscopic scales associated with the RG-invariants that would normally be needed once couplings are renormalized to remove the spurious ϵ-dependence. In the present instance – and just for the special kinematics of the monopole S-wave – the existence of only a single solution to the radial equation implies that the dimensionless magnitudes $|\mathcal{T}_{\rm in}|$ themselves are already RG-invariant.

4.2 Effective dyon dynamics

Having described fermion behaviour in the approximation where the 'slow' degree of freedom $\mathfrak{a}$ is a fixed background we here return to the question of how $\mathfrak{a}$ responds to fermion scattering on longer time-scales. The leading dynamics of $\mathfrak{a}$ is governed by the hamiltonian (4.0.1)

$$H_{\rm dyon} = \frac{1}{2\mathcal{I}} \left[\mathfrak{p}(t) - \frac{\vartheta}{2\pi} \right]^2 + \frac{1}{2} \Big[\overline{\boldsymbol{\chi}}\, O_{\mathcal{B}}(\mathfrak{a})\, \boldsymbol{\chi} \Big]_{r=\epsilon,t} \tag{4.2.1}$$

where $\mathfrak{p}$ is the conjugate momentum for $\mathfrak{a}$ given in (2.2.9), repeated here for convenience:

$$\mathfrak{p} := \frac{\vartheta}{2\pi} + \mathcal{I}\dot{\mathfrak{a}}\,. \tag{4.2.2}$$

Charge conservation and rotor evolution

In the Heisenberg picture the momentum $\mathfrak{p}$ satisfies the equation of motion

$$e\,\dot{\mathfrak{p}}(t) = \frac{i}{2}\left(\overline{\chi}\,\left[O_{\mathcal{B}}(\mathfrak{a}(t)), \frac{e}{2}\tau_3\right]\,\chi\right)_{r=\epsilon} = j^1_F(\epsilon, t)\,, \tag{4.2.3}$$

where $O_{\mathcal{B}}(\mathfrak{a})$ is given in (2.2.18) and the last equality equating the result to the fermionic electromagnetic current flux is obtained by using the boundary condition (2.2.17) *c.f.* eq. (2.2.19):

$$\left[\Gamma^1 + O_{\mathcal{B}}(\mathfrak{a}(t))\right]\chi(\epsilon, t) = 0. \tag{4.2.4}$$

Eq. (4.2.3) expresses conservation of total electric charge in the sense that it equates the change in the dyon charge $-e\mathfrak{p}(t)$ to the radial flux of fermion electric charge $j^1_F(\epsilon, t)$ evaluated near the dyon.

Equation (4.2.3) integrates to give

$$\mathfrak{p}(t) = \mathfrak{p}(t_0) + \frac{1}{e}\int_{t_0}^{t} \mathrm{d}t' j^1_F(\epsilon, t'), \tag{4.2.5}$$

and this result can be used in (4.2.2) to evolve $\mathfrak{a}$, leading to

$$\begin{aligned} \mathfrak{a}(t) &= \mathfrak{a}(t_0) + \frac{1}{\mathcal{I}}\int_{t_0}^{t} \mathrm{d}t' \left[\mathfrak{p}(t_0) - \frac{\vartheta}{2\pi} + \frac{1}{e}\int_{t_0}^{t'} \mathrm{d}t'' j^1_F(\epsilon, t'')\right] \\ &= \mathfrak{a}(t_0) + \frac{t - t_0}{\mathcal{I}}\left[\mathfrak{p}(t_0) - \frac{\vartheta}{2\pi}\right] + \frac{1}{e\mathcal{I}}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' j^1_F(\epsilon, t''). \end{aligned} \tag{4.2.6}$$

These expressions show how the rotor operator acquires a component that acts within the fermionic part of the Hilbert space for times $t > t_0$.

Now comes the main point. We wish to use the above expressions to determine how the slow variable $\mathfrak{a}$ evolves in response to its interactions with the relativistic fermion field. If we use how nuclei are handled within the Born-Oppenheimer approximation applied to atoms, the answer seems simple. For atoms one first computes the electronic state $|\Psi(\mathbf{R})\rangle$ as a function of fixed classical nuclear positions, $\mathbf{R}$, and then determines the nuclear positions by minimizing the nuclear energy $V(\mathbf{R}) = \langle\Psi(\mathbf{R})|H|\Psi(\mathbf{R})\rangle$ given these atomic states. In the present instance the first step corresponds to computing the fermion state $|\psi_i; \mathfrak{a}\rangle$ as a function of a classical initial value for $\mathfrak{a}$. Then we compute an interaction Hamiltonian that captures the correct dynamics within the fermionic state $|\psi_i;\mathfrak{a}\rangle$ and use it to find how the field $\mathfrak{a}$ evolves.

Suppose we assume that the rotor and fermion sectors are initially unrelated to one another at $t = t_0$, at which point $\mathfrak{a}(t_0) = \mathfrak{a}_0$. The above reasoning suggests these slow-moving rotor degrees of freedom see only an average over the fast variables and so (4.2.3) is approximately given by

$$\dot{\mathfrak{p}}_{\rm eff}(t) \simeq \frac{1}{e}\left\langle\mathfrak{a}_0;\psi_i\right| j^1_{\scriptscriptstyle F}(\epsilon,t)\left|\psi_i;\mathfrak{a}_0\right\rangle\,, \tag{4.2.7}$$

which integrates to

$$\mathfrak{p}_{\rm eff}(t) = \mathfrak{p}(t_0) + \frac{1}{e}\int_{t_0}^{t}{\rm d}t'\left\langle\mathfrak{a}_0;\psi_i\right| j^1_{\scriptscriptstyle F}(\epsilon,t')\left|\psi_i;\mathfrak{a}_0\right\rangle, \tag{4.2.8}$$

as well as

$$\mathfrak{a}_{\rm eff}(t) = \mathfrak{a}_0 + \frac{t-t_0}{\mathcal{I}}\left[\mathfrak{p}(t_0) - \frac{\vartheta}{2\pi}\right] + \frac{1}{e\mathcal{I}}\int_{t_0}^{t}{\rm d}t'\int_{t_0}^{t'}{\rm d}t''\left\langle\mathfrak{a}_0;\psi_i\right| j^1_{\scriptscriptstyle F}(\epsilon,t'')\left|\psi_i;\mathfrak{a}_0\right\rangle. \tag{4.2.9}$$

For instance, for fermions initially prepared in the vacuum state $|\psi_i;\mathfrak{a}_0\rangle = |0_{\rm in}\rangle$ eq. (4.1.19) gives a time-independent current expectation value

$$\dot{\mathfrak{p}}_{\rm eff}(t) \simeq \frac{1}{e}\left\langle 0_{\rm in}\right| j^1_{\scriptscriptstyle F}(\epsilon,t)\left|0_{\rm in}\right\rangle = \frac{ev}{2\pi}|\mathcal{T}^{+-}_{\rm in}|^2\,, \tag{4.2.10}$$

and so the time-evolution of $\mathfrak{a}$ and its conjugate momentum $\mathfrak{p}$ are approximately given by

$$\mathfrak{p}_{\rm eff}(t) = \mathfrak{p}(t_0) + \frac{ev(t-t_0)}{2\pi}|\mathcal{T}^{+-}_{\rm in}|^2, \tag{4.2.11}$$

and

$$\mathfrak{a}_{\rm eff}(t) = \mathfrak{a}(t_0) + \frac{t-t_0}{\mathcal{I}}\left[\mathfrak{p}(t_0) - \frac{\vartheta}{2\pi}\right] + \frac{ev(t-t_0)^2}{4\pi\mathcal{I}}|\mathcal{T}^{+-}_{\rm in}|^2 \simeq \mathfrak{a}_0\,, \tag{4.2.12}$$

where the final approximate equality drops α-supressed terms involving $\mathcal{I}^{-1} \sim \alpha m_g$ (which also assumes $t - t_0$ is not too large).

Effective rotor hamiltonian

One can ask: is there an effective rotor hamiltonian whose equations of motion have the same form as eqs. (4.2.2) and (4.2.3)? Strictly speaking, such a hamiltonian need not exist because the rotor is an open quantum system once the fermion degrees of freedom are ignored. For such systems an effective hamiltonian only exists to the extent that there is a mean-field description for which fluctuations in the ignored degrees of freedom – the 'environment' – can be neglected relative to the mean evolution (for a review of these issues, including a more precise statement of the mean-field criterion – see [14]).

But even if an effective rotor hamiltonian exists, its description is likely not as simple as generating a potential $V(\mathfrak{a})$ for $\mathfrak{a}$, which would be the analogy expected based on the Born-Oppenheimer description of the energetics of nuclear positions within an atom. In particular any such hamiltonian must produce nonzero $\dot{\mathfrak{p}}$ that is independent of $\mathfrak{a}$. But while it is true that Hamilton's equation $\dot{\mathfrak{p}} = -\partial H/\partial\mathfrak{a}$ seems to imply that nonzero $\dot{\mathfrak{p}}$ requires H to depend on $\mathfrak{a}$ – such as by adding a potential $V(\mathfrak{a})$ to H – there is also no choice for a potential satisfying $V(\mathfrak{a}+2\pi) = V(\mathfrak{a})$ that can produce a nonzero $\dot{\mathfrak{p}}$ that is independent of $\mathfrak{a}$.

To explore what a successful choice for a rotor Hamiltonian would look like consider as a starting point the rotor dynamics implied by the Lagrangian of (2.2.7), repeated here (including the coupling to $\widehat{A}_0$)

$$L_{\rm dyon} \ni \frac{\vartheta_{\rm eff}(t)}{2\pi}\Big(\dot{\mathfrak{a}} - e\widehat{A}_0\Big) + \frac{\mathcal{I}}{2}\Big(\dot{\mathfrak{a}} - e\widehat{A}_0\Big)^2 + \cdots\,, \tag{4.2.13}$$

which writes the terms in order of dominance at low energies. For later purposes we temporarily entertain the possibility that $\vartheta_{\rm eff} = \vartheta_{\rm eff}(t)$ is a specified function of time.

Consider first keeping just the leading term,

$$S_{\rm rotor} = \int_W {\rm d}t\; \frac{\vartheta_{\rm eff}(t)}{2\pi}\Big(\dot{\mathfrak{a}} - e\widehat{A}_0\Big)\,, \tag{4.2.14}$$

in which case the canonical momentum and Hamiltonian are

$$\mathfrak{p} := \frac{\delta S_{\rm rotor}}{\delta\dot{\mathfrak{a}}} = \frac{\vartheta_{\rm eff}(t)}{2\pi} \quad \text{and} \quad H = \mathfrak{p}\,\dot{\mathfrak{a}} - L_{\rm rotor} = \frac{e\vartheta_{\rm eff}(t)}{2\pi}\,\widehat{A}_0\,, \tag{4.2.15}$$

where the momentum equation can be regarded as a constraint. The Hamiltonian reveals the sole energy associated with this interaction to be the Coulomb energy due to the dyon acquiring an additional induced charge $-e\vartheta_{\rm eff}/2\pi$ (as expected from the Witten effect [79]). The evolution equation for $\mathfrak{p}$ is then

$$\dot{\mathfrak{p}} = \frac{\dot{\vartheta}_{\rm eff}}{2\pi}. \tag{4.2.16}$$

Interestingly, this equation does agree with (4.2.7) provided we identify

$$\vartheta_{\rm eff}(t) \simeq \vartheta_0 + \frac{2\pi}{e}\int_{t_0}^{t} {\rm d}t' \;\langle \mathfrak{a}_0;\psi_i|\, j^1_{\scriptscriptstyle F}(\epsilon,t')\,|\psi_i;\mathfrak{a}_0\rangle \;. \tag{4.2.17}$$

In the special case where $|\psi_i;\mathfrak{a}_0\rangle = |0_{\rm in}\rangle$ this becomes

$$\vartheta_{\rm eff}(t) \simeq \vartheta_0 + ev|\mathcal{T}^{+-}_{\rm in}|^2(t-t_0)\,. \tag{4.2.18}$$

The idea that fermion scattering might cause vacuum angle evolution was earlier discussed in [69].

Extending the above to include the subdominant kinetic term for $\mathfrak{a}$ appearing in (4.2.13) – and again entertaining the possibility that $\vartheta_{\rm eff} = \vartheta_{\rm eff}(t)$ is a function of time – instead leads to the canonical momentum and Hamiltonian

$$\mathfrak{p} := \frac{\delta S}{\delta\dot{\mathfrak{a}}} = \frac{\vartheta_{\rm eff}(t)}{2\pi} + \mathcal{I}(\dot{\mathfrak{a}} - e\widehat{A}_0) \quad \text{and} \quad H = \mathfrak{p}\,\dot{\mathfrak{a}} - L = e\mathfrak{p}\,\widehat{A}_0 + \frac{1}{2\mathcal{I}}\left(\mathfrak{p} - \frac{\vartheta_{\rm eff}}{2\pi}\right)^2 , \tag{4.2.19}$$

which shows how the previous momentum constraint $\mathfrak{p} = \vartheta_{\rm eff}/(2\pi)$ emerges as the momentum choice that minimizes the energy, for fixed $e\widehat{A}_0$. To the extent that the rotor evolution minimizes its energy one expects

$$\dot{\mathfrak{p}} \simeq \frac{\dot{\vartheta}_{\rm eff}}{2\pi} \quad \text{and} \quad \dot{\mathfrak{a}} = \frac{1}{\mathcal{I}}\left(\mathfrak{p} - \frac{\vartheta_{\rm eff}}{2\pi}\right) \simeq 0\,, \tag{4.2.20}$$

which agree with eqs. (4.2.2) and (4.2.10) in the limit that $\mathcal{I}^{-1}$ can be neglected (so $\dot{\mathfrak{a}} \simeq 0$), when $\vartheta_{\rm eff}(t)$ is given by (4.2.18). Although the rotor charge $\mathfrak{p}$ evolves as it follows $\vartheta_{\rm eff}$ the minimized value of the energy remains unchanged.

The above picture apparently relies on $\mathfrak{a}$ and $\mathfrak{p}$ approximately behaving as slow classical variables so that $\mathfrak{p}$ can evolve continuously with time as $\vartheta_{\rm eff}$ does. This is indeed a good approximation for the fermion energies $\alpha m_g \ll E \ll m_g$ for which the Born-Oppenheimer approximation applies. In a fuller quantum treatment the initial value $\mathfrak{a}_0$ appearing in (4.2.9) should be regarded as an operator satisfying $[\mathfrak{a}_0\,,\mathfrak{p}_0]=i$, and commutes with the fermion degrees of freedom evaluated at t_0. The requirement that $\mathfrak{a}_0$ also be a periodic variable with $\mathfrak{a}_0 \sim \mathfrak{a}_0 + 2\pi$ implies that its canonical momentum $\mathfrak{p}_0$ is quantized with eigenvalues $\lambda_n = n$ where n is an arbitrary integer. Evaluating the hamiltonian of (4.2.19) as a function of $\mathfrak{p}_{\rm eff}(t)$ given in (4.2.11) with $\vartheta_{\rm eff}(t)$ given by (4.2.18) then shows – in the absence of $\widehat{A}_0$ – that

$$H = \frac{1}{2\mathcal{I}}\left[\mathfrak{p}_{\rm eff}(t) - \frac{ev}{2\pi}|\mathcal{T}^{+-}_{\rm in}|^2(t-t_0) - \frac{\vartheta_0}{2\pi}\right]^2 = \frac{1}{2\mathcal{I}}\left(\mathfrak{p}_0 - \frac{\vartheta_0}{2\pi}\right)^2 \tag{4.2.21}$$

and so has energy eigenvalues

$$E_n = \frac{1}{2\mathcal{I}}\left(n - \frac{\vartheta_0}{2\pi}\right)^2\,, \tag{4.2.22}$$

that are independent of time and quantized with step size of order $\mathcal{I}^{-1} \sim \alpha m_g$ even as $\mathfrak{p}_{\rm eff}(t)$ varies continuously.

Vacuum-angle evolution

The upshot of the previous section is that – somewhat surprisingly – an effective Hamiltonian can exist that captures the rotor's evolution equations (4.2.2) and (4.2.3) if tracing out the fermion were to produce an effective contribution to the effective lagrangian of the form $\Delta L = \frac{1}{2\pi}\vartheta_{\rm eff}(t) D\mathfrak{a}$ with $\vartheta_{\rm eff}(t)$ satisfying the matching condition (4.2.17). How might such an effective interaction actually be generated when explicitly integrating out the bulk fermion?[10]

An effective lagrangian of the form (4.2.13) with a time-dependent $\vartheta_{\rm eff}$ might arise if the fermion-dyon interaction involves operators that contain one extra derivative relative to those in (2.2.7), such as

$$\Delta L = \mathcal{O}(\overline{\chi},\chi)\, D\mathfrak{a}\,, \tag{4.2.23}$$

where $\mathcal{O}$ is an operator built from the bulk fermion field and $D\mathfrak{a} = \dot{\mathfrak{a}} - e\widehat{A}_0$ as before. Taking the expectation value of this in the *in* vacuum shows that $\langle\mathcal{O}\rangle$ plays the role of $\vartheta_{\rm eff}/(2\pi)$ and this can

[10] This issue is also discussed in [69], though in a way that invokes a bulk coupling of $\mathfrak{a}$ to fermions.

be time-dependent (and calculable) if $\langle \mathcal{O} \rangle$ is. In principle the operator we seek should satisfy

$$\partial_t \mathcal{O} \simeq \frac{1}{e} j^1_{\scriptscriptstyle F}(\epsilon, t') = -\frac{1}{2} j^0_{\scriptscriptstyle A}(\epsilon, t') \,. \tag{4.2.24}$$

in order to ensure that (4.2.17) is true.

This last condition suggests a guess for what the operator $\mathcal{O}$ should be. In 1+1 dimensions our two Dirac fermions $\chi_\mathfrak{s}$ can be bosonized into two real scalars $\phi_\mathfrak{s}$, where the map between bosons and fermions implies – see also (2.2.19)

$$j^\alpha_{\scriptscriptstyle A}(\epsilon, t) \propto \partial^\alpha \phi_+ - \partial^\alpha \phi_- \,. \tag{4.2.25}$$

Comparing this to (4.2.24) suggests that the operator $\mathcal{O}$ we seek has a simple expression in terms of the bosonized field: $\mathcal{O} \propto \phi_+ - \phi_-$. If so then (4.2.23) represents a dyon-localized kinetic mixing between $\phi_\mathfrak{s}$ and $\mathfrak{a}$.

We have not yet found a convincing derivation of why ΔL of the form (4.2.23) is generated once the fermions are integrated out, or why it arises with the right coefficients. But the above discussion reinforces earlier work [32, 63, 26, 69] that suggests that the bosonized formulation of the scattered fermions might be more useful for understanding how the dyonic excitation $\mathfrak{a}$ evolves.

Chapter 5

Conclusion

In this thesis, we construct an effective field theory that describes the dominant interactions of a dyon (or magnetic monopole) with relativistic fermions in the low-energy regime. We focus on the dyon's interactions with fermions because they sometimes lead to catalysis *i.e.* the phenomenon that dyon-fermion scattering cross sections can be much larger than the dyon's geometrical size. This sets fermions apart from other light fields, since the dyon's influence on observables that don't exhibit catalysis is generally expected to be negligible due to the huge hierarchy of scales between present-day energies and the superheavy dyon mass.

Our approach relies on the framework of point-particle effective field theory (PPEFT), which describes how massive compact sources (such as nuclei, dyons *etc.*) influence low-energy 'bulk' fields whose wavelengths are too large to resolve the source's interior structure. In this regime, the relevant degrees of freedom of the point particle are its collective coordinates such as the position of its centre of mass and (in the case of a dyon) a charge degree of freedom localized on the dyon's world-line.

The effective action that dictates how fermions interact with the dyon's collective coordinates (and other light fields) is given by (2.2.4) and is dominated at low energies by operators that are similar to those found in PPEFTs describing interactions between electrons and nuclei in atoms [35]. The PPEFT formalism relates the effective action on the dyon world-line to a set of near-dyon boundary conditions, imposed on the bulk fields at radius $r = \epsilon$ – such as (2.2.12) or (2.2.13) – and this is often a more useful way to encode interactions.

The main difference between dyons or monopoles and sources without magnetic charge such as nuclei lies in the peculiar kinematics of the lowest fermion partial wave, not the type of interactions allowed on the source's world-line. Specifically, the dyon's magnetic field contributes to a charged particle's angular momentum and this makes it possible for fermions to have zero total angular momentum. Such S-wave states are special because they exhibit only one type of power-law behaviour which forbids the emergence of an RG-invariant length scale, R_*, in the way it typically arises in less exotic PPEFT applications *i.e.* as the crossover radius between two types of power-law behaviour in angular momentum mode functions. The observables of a PPEFT must be expressed in terms of RG-invariant quantities such as R_* and (since these depend on the underlying physical scales) this is how the suppression by the small source size enters into observables. Since no analog of R_* exists for S-wave fermions, the size of fermion-dyon cross sections in the S-wave sector is instead mainly set by the incoming fermion's momentum.

We show how the resulting dynamics of the bulk fermion coupled to the dyon's 'rotor' mode captures the well-known fermion-dyon physics, but in a way that is very generally characterized by only a few parameters – *e.g.* the $\mathcal{T}_{\rm in}^{\mathfrak{s}\mathfrak{s}'}$ of (2.2.33) – whose values can be obtained by matching with microscopic physics for specific dyon configurations. Our general expressions for *e.g.* scattering as a function of the $\mathcal{T}_{\rm in}^{\mathfrak{s}\mathfrak{s}'}$ reduce to those of the literature once these matched values are used. These expressions also capture how scattering includes the nonperturbative effects of fermion-rotor interactions as studied in [68], which can be regarded as providing a more complicated matching prescription to the microscopic physics that changes the values found for the $\mathcal{T}_{\rm in}^{\mathfrak{s}\mathfrak{s}'}$ but not the expressions for how observables depend on these parameters.

We also explore how the fermion scattering causes the dyonic excitions to evolve and identify an effective hamiltonian that captures the dynamics required by the model's conservation laws at low energy. Although we do not yet have a microscopic derivation of this Hamiltonian we explore several preliminary options.

A natural next step is to apply our methods to monopoles and dyons arising in realistic Grand Unified Theories. Within this context, the EFT tools we develop can be used to classify the dominant low-energy interactions of GUT monopoles and dyons with Standard Model fermions, with the goal of guiding future experiments. Another avenue for future research is to explore the significance of

EFTs for which the boundary matrix has rank three. These EFTs are not prohibited by unitarity but naively appear to be unphysical, although we are not aware of any fundamental reason why they should be ruled out. Yet another future direction is to explicitly show how the renormalization of effective couplings described in [68] can take place within our PPEFT framework.

Appendix A

Gamma-matrix conventions

This appendix summarizes our Dirac matrix conventions in both four and two dimensions.

A.1 4D Dirac matrices

We follow Weinberg's spinor and metric conventions, so our metric has signature $(-,+,+,+)$ and a Weyl representation for the gamma matrices in four dimensions is given by

$$\gamma^0 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \tag{A.1.1}$$

where I is the 2 by 2 unit matrix and σ_j are Pauli matrices (acting in spin space, as opposed to the Pauli matrices τ_a acting on gauge doublet indices). These satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ where $\eta^{\mu\nu}$ is the inverse Minkowski metric, and the representation is chosen to diagonalize

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{A.1.2}$$

Dirac conjugation in these conventions is given by $\overline{\psi} = i\psi^\dagger\gamma^0$. In these conventions the left- and right-handed chirality projectors are $\gamma_{\scriptscriptstyle L} := \frac{1}{2}(1+\gamma_5)$ and $\gamma_{\scriptscriptstyle R} := \frac{1}{2}(1-\gamma_5)$.

Spherical coordinates

The gamma matrices adapted to spherical coordinates are given in terms of the unit coordinate vectors

$$\begin{aligned}
\hat{\boldsymbol{r}} := \hat{r}_i\,\mathbf{e}_i &= \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z, \\
\hat{\boldsymbol{\theta}} := \hat{\theta}_i\,\mathbf{e}_i &= \cos\theta\cos\phi\,\mathbf{e}_x + \cos\theta\sin\phi\,\mathbf{e}_y - \sin\theta\,\mathbf{e}_z, \\
\hat{\boldsymbol{\phi}} := \hat{\phi}_i\,\mathbf{e}_i &= -\sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_y.
\end{aligned} \tag{A.1.3}$$

by $\gamma^r := \gamma^i\,\hat{r}_i$, $\gamma^\theta := \gamma^i\,\hat{\theta}_i$ and $\gamma^\phi := \gamma^i\,\hat{\phi}_i/\sin\theta$, and so

$$\gamma^r = \begin{pmatrix} 0 & -i\sigma^r \\ i\sigma^r & 0 \end{pmatrix}, \quad \gamma^\theta = \begin{pmatrix} 0 & -i\sigma^\theta \\ i\sigma^\theta & 0 \end{pmatrix} \quad \text{and} \quad \gamma^\phi = \frac{1}{\sin\theta}\begin{pmatrix} 0 & -i\sigma^\phi \\ i\sigma^\phi & 0 \end{pmatrix}. \tag{A.1.4}$$

where the Pauli matrices in spherical coordinates are defined by

$$\sigma^r = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}, \ \sigma^\theta = \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix} \text{ and } \sigma^\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}. \tag{A.1.5}$$

A.2 2D Dirac matrices

In $D = 1+1$ spacetime dimensions we label coordinates with $x^\alpha = \{t, r\}$ for $\alpha = 0, 1$ and use the following representation of gamma matrices

$$\Gamma^0 = -i\sigma_1, \quad \Gamma^1 = \sigma_2\,. \tag{A.2.1}$$

These satisfy the algebra $\{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^{\alpha\beta}$, where $\eta^{\alpha\beta} = \mathrm{diag}(-1, 1)$ is the inverse Minkowski metric in $1+1$ dimensions. The chiral matrix in 1+1 dimensions we then define to be

$$\Gamma_c := \Gamma^0\Gamma^1 = \sigma_3\,, \tag{A.2.2}$$

which has eigenvalues ± 1 (and is diagonal in the basis used here). Notice that these definitions imply $\Gamma_c\Gamma^\alpha = \epsilon^{\alpha\beta}\Gamma_\beta$ where our Levi-Civita convention chooses $\epsilon^{01} = +1$. Dirac conjugation is again given by $\overline{\chi} = i\chi^\dagger\Gamma^0$ and in these conventions the left- and right-handed helicity projectors are $\Gamma_+ := \frac{1}{2}(1+\Gamma_c)$ and $\Gamma_- := \frac{1}{2}(1-\Gamma_c)$.

Appendix B

Regularized codimension-1 boundary action

This Appendix addresses the question of how to regularize the fermion-dyon interactions defined on the dyon world-line, which appear in equations such as (2.2.7).

As argued in the main text, the lowest-dimension interactions between the fermion doublet $\boldsymbol{\psi}(x)$ and the dyon collective coordinates $y^\mu(t), \mathfrak{a}(t)$ are given by

$$\begin{aligned} S^{\text{int}}_{\text{dyon}} &= -\frac{1}{2}\int \mathrm{d}t\, \overline{\boldsymbol{\psi}}\, \mathfrak{C}(\mathfrak{a})\, \boldsymbol{\psi} \\ &= -\frac{1}{2}\int \mathrm{d}t\, \overline{\boldsymbol{\psi}}\Big[\hat{\mathfrak{c}}^s_1 + i\,\hat{\mathfrak{c}}^{ps}_1\gamma_5 - i\,\hat{\mathfrak{c}}^v_1\gamma^0 - i\,\hat{\mathfrak{c}}^{pv}_1\gamma_5\gamma^0 + \Big(\hat{\mathfrak{c}}^s_3 + i\,\hat{\mathfrak{c}}^{ps}_3\gamma_5 - i\,\hat{\mathfrak{c}}^v_3\gamma^0 - i\,\hat{\mathfrak{c}}^{pv}_3\gamma_5\gamma^0\Big)\tau_3 \\ &\quad + \Big(\hat{\mathfrak{c}}^s_+ + i\,\hat{\mathfrak{c}}^{ps}_+\gamma_5 - i\,\hat{\mathfrak{c}}^v_+\gamma^0 - i\,\hat{\mathfrak{c}}^{pv}_+\gamma_5\gamma^0\Big)e^{i\mathfrak{a}}\tau_+ + \Big(\hat{\mathfrak{c}}^s_- + i\,\hat{\mathfrak{c}}^{ps}_-\gamma_5 - i\,\hat{\mathfrak{c}}^v_-\gamma^0 - i\,\hat{\mathfrak{c}}^{pv}_-\gamma_5\gamma^0\Big)e^{-i\mathfrak{a}}\tau_-\Big]\,\boldsymbol{\psi} \end{aligned} \tag{B.0.1}$$

where $\boldsymbol{\psi}$ is evaluated at $\boldsymbol{r} = 0$ and we specialize to the dyon rest frame and neglect dyon recoil effects so that $\dot{y}^\mu \approx \delta^\mu_0$. We can regulate the operators appearing in $S^{\text{int}}_{\text{dyon}}$ by replacing them with appropriate interaction terms, defined on the boundary of a Gaussian pillbox at $r = \epsilon$. Appendix 1.2 shows that for fermion field configurations which are spherically symmetric near the source, the regularization procedure amounts to replacing fermion bilinears such as $\overline{\boldsymbol{\psi}}M\boldsymbol{\psi}$, where M is a matrix in spin and isospin space, with their average over the pillbox boundary $(4\pi\epsilon^2)^{-1}\int \mathrm{d}^2\Omega\, \epsilon^2\, \overline{\boldsymbol{\psi}}(\epsilon)M\boldsymbol{\psi}(\epsilon)$. This is a valid prescription for operators in (B.0.1) describing ψ_+, ψ_- self-interactions (mediated by

$\hat{\mathfrak{c}}_1, \hat{\mathfrak{c}}_3$). For operators that couple ψ_+ and ψ_- we must use another prescription however, since the angular dependence of the two components of the doublet is different, even when restricting to the same partial wave. As a result, regularized fermion bilinears such as $\overline{\boldsymbol{\psi}}(\epsilon)\,\hat{\mathfrak{c}}^s_+\tau_+\boldsymbol{\psi}(\epsilon)$ are generally not rotation-invariant and so can vanish after integration over the boundary of the pillbox. This can be seen explicitly by specializing to S-wave states for which we get *e.g.*:

$$\frac{1}{4\pi\epsilon^2}\int \mathrm{d}^2\Omega\,\epsilon^2\,\overline{\boldsymbol{\psi}}(\epsilon)\,\hat{\mathfrak{c}}^s_+\tau_+\boldsymbol{\psi}(\epsilon) = \frac{f_+^*(\epsilon,t)\,g_-(\epsilon,t) + g_+^*(\epsilon,t)\,f_-(\epsilon,t)}{4\pi\epsilon^2}\hat{\mathfrak{c}}^s_+\int \mathrm{d}^2\Omega\,\eta_+^\dagger\,\eta_- = 0, \tag{B.0.2}$$

in the gauge where the Julia-Zee solution has the form (1.1.19) and S-wave fermions are given by (2.1.10). That the form of angular momentum eigenstates depends on their electric charge can be traced back to the expression for the angular momentum operator $\vec{J}$, which includes a contribution from the gauge isospin $\vec{T}$, as in (1.1.29).

To couple the two components of the doublet at the $r = \epsilon$ boundary, it suffices to introduce additional matrices $P_\pm$, which turn the angular dependence of ψ_+ into that of ψ_- and vice versa. For S-wave states and in the 'abelian' gauge of (1.1.19), these matrices act in spin space and are defined through:

$$P_+\,\frac{1}{r}\begin{pmatrix} f_+(r,t)\,\eta_+ \\ g_+(r,t)\,\eta_+ \end{pmatrix} = \frac{1}{r}\begin{pmatrix} f_+(r,t)\,\eta_- \\ g_+(r,t)\,\eta_- \end{pmatrix}, \quad \text{and} \quad P_-\,\frac{1}{r}\begin{pmatrix} f_-(r,t)\,\eta_- \\ g_-(r,t)\,\eta_- \end{pmatrix} = \frac{1}{r}\begin{pmatrix} f_-(r,t)\,\eta_+ \\ g_-(r,t)\,\eta_+ \end{pmatrix}. \tag{B.0.3}$$

Since these equations do not uniqely determine $P_\pm$, we further choose:

$$P_-\,\frac{1}{r}\begin{pmatrix} f_+(r,t)\,\eta_+ \\ g_+(r,t)\,\eta_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad P_+\,\frac{1}{r}\begin{pmatrix} f_-(r,t)\,\eta_- \\ g_-(r,t)\,\eta_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{B.0.4}$$

which leads us to the following expressions for $P_\pm$ in the R_- patch

$$P_+ = -\frac{ie^{i\phi}}{2}\begin{pmatrix} i\sigma_\theta + \sigma_\phi & 0 \\ 0 & i\sigma_\theta + \sigma_\phi \end{pmatrix}, \quad \text{and} \quad P_- = -\frac{ie^{-i\phi}}{2}\begin{pmatrix} i\sigma_\theta - \sigma_\phi & 0 \\ 0 & i\sigma_\theta - \sigma_\phi \end{pmatrix}, \tag{B.0.5}$$

as well as $P'_\pm \coloneqq e^{\mp 2i\phi} P_\pm$ in the R_+ patch. It follows that for S-wave states, the world-line interactions in (B.0.1) can be regulated using the following boundary action:

$$
\begin{aligned}
I^{\text{int}}_{\text{dyon}} \;=\; -\frac{1}{2}\int\limits_{r=\epsilon} \mathrm{d}t\,\mathrm{d}^2\Omega\,\epsilon^2\,\overline{\boldsymbol{\psi}}\Big[&\hat{\mathcal{C}}^s_1 + i\,\hat{\mathcal{C}}^{ps}_1\gamma_5 - i\,\hat{\mathcal{C}}^v_1\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_1\gamma_5\gamma^0 + \Big(\hat{\mathcal{C}}^s_3 + i\,\hat{\mathcal{C}}^{ps}_3\gamma_5 - i\,\hat{\mathcal{C}}^v_3\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_3\gamma_5\gamma^0\Big)\tau_3 \\
&+\Big(\hat{\mathcal{C}}^s_+ + i\,\hat{\mathcal{C}}^{ps}_+\gamma_5 - i\,\hat{\mathcal{C}}^v_+\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_+\gamma_5\gamma^0\Big)e^{i\mathfrak{a}}P_-\,\tau_+ + \Big(\hat{\mathcal{C}}^s_- + i\,\hat{\mathcal{C}}^{ps}_-\gamma_5 - i\,\hat{\mathcal{C}}^v_-\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_-\gamma_5\gamma^0\Big)e^{-i\mathfrak{a}}P_+\,\tau_-\Big]\,\boldsymbol{\psi}
\end{aligned}
\tag{B.0.6}
$$

where we introduce the boundary couplings $\hat{\mathcal{C}}^A_I = \frac{1}{4\pi\epsilon^2}\,\hat{\mathfrak{c}}^A_I$. The boundary condition satisfied by S-wave fermions is then given by

$$
\begin{aligned}
0 = \Big\{\Big[&\gamma^r + \Big(\hat{\mathcal{C}}^s_1 + i\,\hat{\mathcal{C}}^{ps}_1\gamma_5 - i\,\hat{\mathcal{C}}^v_1\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_1\gamma_5\gamma^0\Big) + \Big(\hat{\mathcal{C}}^s_3 + i\,\hat{\mathcal{C}}^{ps}_3\gamma_5 - i\,\hat{\mathcal{C}}^v_3\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_3\gamma_5\gamma^0\Big)\tau_3 \\
&+\Big(\hat{\mathcal{C}}^s_+ + i\,\hat{\mathcal{C}}^{ps}_+\gamma_5 - i\,\hat{\mathcal{C}}^v_+\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_+\gamma_5\gamma^0\Big)e^{i\mathfrak{a}}P_-\,\tau_+ + \Big(\hat{\mathcal{C}}^s_- + i\,\hat{\mathcal{C}}^{ps}_-\gamma_5 - i\,\hat{\mathcal{C}}^v_-\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_-\gamma_5\gamma^0\Big)e^{-i\mathfrak{a}}P_+\,\tau_-\Big]\boldsymbol{\psi}\Big\}_{r=\epsilon},
\end{aligned}
\tag{B.0.7}
$$

instead of (2.2.13) and is equivalent to the 2D boundary condition (2.2.16) when the 4D and 2D boundary couplings are related as follows:

$$
\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}} = \hat{\mathcal{C}}^s_1 + \mathfrak{s}\hat{\mathcal{C}}^s_3, \quad \mathcal{C}^{ps}_{\mathfrak{s}\mathfrak{s}} = \mathfrak{s}\hat{\mathcal{C}}^{ps}_1 + \hat{\mathcal{C}}^{ps}_3, \quad \mathcal{C}^v_{\mathfrak{s}\mathfrak{s}} = \hat{\mathcal{C}}^v_1 + \mathfrak{s}\hat{\mathcal{C}}^v_3, \quad \mathcal{C}^{pv}_{\mathfrak{s}\mathfrak{s}} = -(\mathfrak{s}\hat{\mathcal{C}}^{pv}_1 + \hat{\mathcal{C}}^{pv}_3) \tag{B.0.8}
$$

as well as

$$
\mathcal{C}^s_{+-} = -\hat{\mathcal{C}}^v_+, \quad \mathcal{C}^{ps}_{+-} = i\hat{\mathcal{C}}^{pv}_+, \quad \mathcal{C}^v_{+-} = -\hat{\mathcal{C}}^s_+, \quad \mathcal{C}^{pv}_{+-} = i\hat{\mathcal{C}}^{ps}_+ \tag{B.0.9}
$$

and

$$
\mathcal{C}^s_{-+} = -\hat{\mathcal{C}}^v_-, \quad \mathcal{C}^{ps}_{-+} = -i\hat{\mathcal{C}}^{pv}_-, \quad \mathcal{C}^v_{-+} = -\hat{\mathcal{C}}^s_-, \quad \mathcal{C}^{pv}_{-+} = -i\hat{\mathcal{C}}^{ps}_-. \tag{B.0.10}
$$

Notice that there is a mismatch between 2D and 4D chirality, which explains why some (pseudo)scalar and (pseudo)vector 4D couplings correspond to (pseudo)vector and (pseudo)scalar 2D couplings respectively, as well as the presence of additional minus signs in the matching of pseudoscalar and pseudovector couplings.

The S-wave boundary action could have equivalently been formulated in the original spherical gauge of (1.1.13), in which we get:

$$I^{\rm int}_{\rm dyon} \;=\; -\frac{1}{2}\int\limits_{r=\epsilon} {\rm d}t\,{\rm d}^2\Omega\,\epsilon^2\,\overline{\boldsymbol{\psi}}\Big[\hat{\mathcal{C}}^s_1 + i\,\hat{\mathcal{C}}^{ps}_1\gamma_5 - i\,\hat{\mathcal{C}}^v_1\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_1\gamma_5\gamma^0 - \Big(\hat{\mathcal{C}}^s_3 + i\,\hat{\mathcal{C}}^{ps}_3\gamma_5 - i\,\hat{\mathcal{C}}^v_3\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_3\gamma_5\gamma^0\Big)\tau_r \quad \text{(B.0.11)}$$
$$-\Big(\hat{\mathcal{C}}^s_+ + i\,\hat{\mathcal{C}}^{ps}_+\gamma_5 - i\,\hat{\mathcal{C}}^v_+\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_+\gamma_5\gamma^0\Big)e^{i\mathfrak{a}}P_-\,\tau^{-r}_+ - \Big(\hat{\mathcal{C}}^s_- + i\,\hat{\mathcal{C}}^{ps}_-\gamma_5 - i\,\hat{\mathcal{C}}^v_-\gamma^0 - i\,\hat{\mathcal{C}}^{pv}_-\gamma_5\gamma^0\Big)e^{-i\mathfrak{a}}P_+\,\tau^{-r}_-\Big]\,\boldsymbol{\psi},$$

where $P_\pm$ are given by (B.0.5) and $\tau^{-r}_\pm := \frac{1}{2}e^{\pm i\phi}\,(\tau_\theta \mp i\tau_\phi)$ act as raising and lowering operators on eigenstates of $-\tau_r$, that is

$$\tau^{-r}_+\eta_+(\theta,\phi) = \eta_-(\theta,\phi), \quad \tau^{-r}_-\eta_-(\theta,\phi) = \eta_+(\theta,\phi) \tag{B.0.12}$$

and

$$\tau^{-r}_-\eta_+(\theta,\phi) = \tau^{-r}_+\eta_-(\theta,\phi) = 0, \tag{B.0.13}$$

where $\eta_\pm(\theta,\phi)$ are vectors in isospin space given by (2.1.11), which satisfy $(-\tau_r)\eta_\pm(\theta,\phi) = \mp\eta_\pm(\theta,\phi)$.

Appendix C

Properties of the amplitudes $\mathcal{T}_{\rm in}, \mathcal{T}_{\rm out}$

This appendix derives and summarizes several useful properties of the amplitudes $\mathcal{T}_{\rm in}$ and $\mathcal{T}_{\rm out}$ that appear in the construction of the *in* and *out* states. We assume when doing so that the dyon-fermion action is real (and so the couplings $\mathcal{C}^A_{ij}$ are hermitian).

C.1 Definition of the amplitudes

The main text shows that the $\mathcal{T}_{\rm in}, \mathcal{T}_{\rm out}$ amplitudes are given in terms of the boundary matrix $\hat{\mathcal{B}}$ by

$$\mathcal{T}_{\rm in}^{++} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{21} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{41} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}, \mathcal{T}_{\rm in}^{+-} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{23} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{43} & \hat{\mathcal{B}}_{44} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}, \mathcal{T}_{\rm in}^{-+} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{21} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{41} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}, \mathcal{T}_{\rm in}^{--} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{23} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{43} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{22} & \hat{\mathcal{B}}_{24} \\ \hat{\mathcal{B}}_{42} & \hat{\mathcal{B}}_{44} \end{vmatrix}}, \quad \text{(C.1.1)}$$

and

$$\mathcal{T}_{\rm out}^{++} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{12} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{32} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \mathcal{T}_{\rm out}^{-+} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{12} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{32} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \mathcal{T}_{\rm out}^{+-} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{14} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{34} & \hat{\mathcal{B}}_{33} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}, \mathcal{T}_{\rm out}^{--} = -\frac{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{14} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{34} \end{vmatrix}}{\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}}. \tag{C.1.2}$$

Written directly in terms of the boundary couplings, these become

$$\begin{aligned}
\mathcal{T}_{\rm in}^{++} &= \frac{(\mathcal{C}^s_{++} + i\mathcal{C}^{ps}_{++})(i - \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--}) + (\mathcal{C}^s_{-+} + i\mathcal{C}^{ps}_{-+})(\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^v_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^v_{--} - i)}, \\
\mathcal{T}_{\rm in}^{--} &= \frac{(\mathcal{C}^s_{--} + i\mathcal{C}^{ps}_{--})(i - \mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++}) + (\mathcal{C}^s_{+-} + i\mathcal{C}^{ps}_{+-})(\mathcal{C}^{pv}_{-+} - \mathcal{C}^v_{-+})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^v_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^v_{--} - i)}, \\
\mathcal{T}_{\rm in}^{-+} &= \frac{(\mathcal{C}^s_{-+} + i\mathcal{C}^{ps}_{-+})(i - \mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++}) + (\mathcal{C}^s_{++} + i\mathcal{C}^{ps}_{++})(\mathcal{C}^{pv}_{-+} - \mathcal{C}^v_{-+})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^v_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^v_{--} - i)}, \\
\mathcal{T}_{\rm in}^{+-} &= \frac{(\mathcal{C}^s_{+-} + i\mathcal{C}^{ps}_{+-})(i - \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--}) + (\mathcal{C}^s_{--} + i\mathcal{C}^{ps}_{--})(\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-})}{-|\mathcal{C}^{pv}_{+-} - \mathcal{C}^v_{+-}|^2 + (-i + \mathcal{C}^{pv}_{++} - \mathcal{C}^v_{++})(\mathcal{C}^{pv}_{--} - \mathcal{C}^v_{--} - i)},
\end{aligned} \tag{C.1.3}$$

and

$$\begin{aligned}
\mathcal{T}_{\rm out}^{++} &= \frac{i(\mathcal{C}^{ps}_{++} + i\mathcal{C}^s_{++})(-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--}) + (-i\mathcal{C}^{ps}_{-+} + \mathcal{C}^s_{-+})(\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++} - i)}, \\
\mathcal{T}_{\rm out}^{--} &= \frac{i(\mathcal{C}^{ps}_{--} + i\mathcal{C}^s_{--})(-i + \mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++}) + (-i\mathcal{C}^{ps}_{+-} + \mathcal{C}^s_{+-})(\mathcal{C}^{pv}_{-+} + \mathcal{C}^v_{-+})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++} - i)}, \\
\mathcal{T}_{\rm out}^{-+} &= \frac{i(\mathcal{C}^{ps}_{-+} + i\mathcal{C}^s_{-+})(-i + \mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++}) + (-i\mathcal{C}^{ps}_{++} + \mathcal{C}^s_{++})(\mathcal{C}^{pv}_{-+} + \mathcal{C}^v_{-+})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++} - i)}, \\
\mathcal{T}_{\rm out}^{+-} &= \frac{i(\mathcal{C}^{ps}_{+-} + i\mathcal{C}^s_{+-})(-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--}) + (-i\mathcal{C}^{ps}_{--} + \mathcal{C}^s_{--})(\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-})}{|\mathcal{C}^{pv}_{+-} + \mathcal{C}^v_{+-}|^2 - (-i + \mathcal{C}^{pv}_{--} + \mathcal{C}^v_{--})(\mathcal{C}^{pv}_{++} + \mathcal{C}^v_{++} - i)}.
\end{aligned} \tag{C.1.4}$$

C.2 Rank-2 conditions

The above definitions assume the denominators do not vanish. This is satisfied when the $\mathcal{C}^A_{ij}$ are all hermitian, as shown in equation (2.2.23) and the text directly after it.

We next ask what is required to ensure that $\text{rank}(\mathcal{B}) = 2$. When this is true one pair of linearly independent columns of $\mathcal{B}$ can be written as a linear combination of the other two linearly independent columns. When the $\mathcal{C}^A_{ij}$ are hermitian we've seen the linearly independent pairs consist of the first and third columns of $\mathcal{B}$ and the second and fourth columns of $\mathcal{B}$. Consequently there exist nonzero coefficients A_1, A_3, B_1 and B_3 such that

$$\begin{aligned} \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{21} & \mathcal{B}_{31} & \mathcal{B}_{41} \end{pmatrix} &= A_1 \begin{pmatrix} \mathcal{B}_{12} & \mathcal{B}_{22} & \mathcal{B}_{32} & \mathcal{B}_{42} \end{pmatrix} + B_1 \begin{pmatrix} \mathcal{B}_{14} & \mathcal{B}_{24} & \mathcal{B}_{34} & \mathcal{B}_{44} \end{pmatrix}, \\ \begin{pmatrix} \mathcal{B}_{13} & \mathcal{B}_{23} & \mathcal{B}_{33} & \mathcal{B}_{43} \end{pmatrix} &= A_3 \begin{pmatrix} \mathcal{B}_{12} & \mathcal{B}_{22} & \mathcal{B}_{32} & \mathcal{B}_{42} \end{pmatrix} + B_3 \begin{pmatrix} \mathcal{B}_{14} & \mathcal{B}_{24} & \mathcal{B}_{34} & \mathcal{B}_{44} \end{pmatrix}, \end{aligned} \tag{C.2.1}$$

The required coefficients are given by

$$A_1 = \frac{\begin{vmatrix} \mathcal{B}_{21} & \mathcal{B}_{24} \\ \mathcal{B}_{41} & \mathcal{B}_{44} \end{vmatrix}}{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{24} \\ \mathcal{B}_{42} & \mathcal{B}_{44} \end{vmatrix}}, \quad B_1 = \frac{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{21} \\ \mathcal{B}_{42} & \mathcal{B}_{41} \end{vmatrix}}{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{24} \\ \mathcal{B}_{42} & \mathcal{B}_{44} \end{vmatrix}}, \quad A_3 = \frac{\begin{vmatrix} \mathcal{B}_{23} & \mathcal{B}_{24} \\ \mathcal{B}_{43} & \mathcal{B}_{44} \end{vmatrix}}{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{24} \\ \mathcal{B}_{42} & \mathcal{B}_{44} \end{vmatrix}}, \quad B_3 = \frac{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{23} \\ \mathcal{B}_{42} & \mathcal{B}_{43} \end{vmatrix}}{\begin{vmatrix} \mathcal{B}_{22} & \mathcal{B}_{24} \\ \mathcal{B}_{42} & \mathcal{B}_{44} \end{vmatrix}}. \tag{C.2.2}$$

Comparing these solutions to (C.1.1) implies

$$A_1 = -\mathcal{T}_{\rm in}^{++}, \quad B_1 = -\mathcal{T}_{\rm in}^{-+} e^{-i\mathfrak{a}}, \quad A_3 = -\mathcal{T}_{\rm in}^{+-} e^{i\mathfrak{a}}, \quad B_3 = -\mathcal{T}_{\rm in}^{--}, \tag{C.2.3}$$

and using these to trade $A_{1,3}, B_{1,3}$ for the $\mathcal{T}_{\rm in}$'s in equation (C.2.1) gives four tautologies as well as the following four complex conditions that can be regarded as conditions required if the matrix $\mathcal{B}$ is to have rank 2:

$$\begin{aligned} \mathcal{E}_1 &:= \hat{\mathcal{B}}_{13} + \mathcal{T}_{\rm in}^{+-} \hat{\mathcal{B}}_{12} + \mathcal{T}_{\rm in}^{--} \hat{\mathcal{B}}_{14} = 0, \\ \mathcal{E}_2 &:= \hat{\mathcal{B}}_{11} + \mathcal{T}_{\rm in}^{++} \hat{\mathcal{B}}_{12} + \mathcal{T}_{\rm in}^{-+} \hat{\mathcal{B}}_{14} = 0, \\ \mathcal{E}_3 &:= \hat{\mathcal{B}}_{33} + \mathcal{T}_{\rm in}^{+-} \hat{\mathcal{B}}_{32} + \mathcal{T}_{\rm in}^{--} \hat{\mathcal{B}}_{34} = 0, \\ \mathcal{E}_4 &:= \hat{\mathcal{B}}_{31} + \mathcal{T}_{\rm in}^{++} \hat{\mathcal{B}}_{32} + \mathcal{T}_{\rm in}^{-+} \hat{\mathcal{B}}_{34} = 0. \end{aligned} \tag{C.2.4}$$

These four complex equations amount to eight real conditions. Four of these state

$$\Re(\mathcal{E}_2) = 0, \quad \Re(\mathcal{E}_3) = 0, \quad \text{and} \quad \mathcal{E}_1 + \mathcal{E}_4^* = 0, \tag{C.2.5}$$

and, once expressions (C.1.1) are used, can be written purely in terms of the $\mathcal{T}_{\rm in}$ which must therefore satisfy

$$\begin{aligned} \Re(\mathcal{E}_2) &= -(|\mathcal{T}_{\rm in}^{++}|^2 + |\mathcal{T}_{\rm in}^{-+}|^2 - 1) = 0, \\ \Re(\mathcal{E}_3) &= -(|\mathcal{T}_{\rm in}^{--}|^2 + |\mathcal{T}_{\rm in}^{+-}|^2 - 1) = 0, \\ \mathcal{E}_1 + \mathcal{E}_4^* &= -2(\mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{-+\,*} + \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm in}^{++\,*}) = 0, \end{aligned} \tag{C.2.6}$$

as used in the main text. In deriving the above expressions we use the identity (2.2.22), which follows when the boundary coupling constants are hermitian. The relations (C.2.6) simply express the unitarity of the S matrix since the first two of these equations impose that the fermion number density of the *in* modes is conserved during scattering processes, while the third imposes that the off-diagonal matrix elements of $S^\dagger S$ vanish.

The remaining four rank $(\mathcal{B}) = 2$ conditions within (C.2.4) are

$$\mathcal{E}_1 = 0, \quad \Im(\mathcal{E}_2) = 0, \quad \text{and} \quad \Im(\mathcal{E}_3) = 0. \tag{C.2.7}$$

These depend on more than just the $\mathcal{T}_{\rm in}$ amplitudes but can be expressed in terms of both $\mathcal{T}_{\rm in}$ and $\mathcal{T}_{\rm out}$. To see why the rank-two conditions (C.2.4) can be written entirely in terms of $\mathcal{T}_{\rm in}, \mathcal{T}_{\rm out}$ take the following linear combinations

$$\begin{aligned} -\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}^{-1} \Big(\hat{\mathcal{B}}_{33}\,\mathcal{E}_1 - \hat{\mathcal{B}}_{13}\,\mathcal{E}_3\Big) &= \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm out}^{+-} + \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm out}^{++} = 0, \\ \begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}^{-1} \Big(\hat{\mathcal{B}}_{31}\,\mathcal{E}_1 - \hat{\mathcal{B}}_{11}\,\mathcal{E}_3\Big) &= \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm out}^{--} + \mathcal{T}_{\rm in}^{+-}\mathcal{T}_{\rm out}^{-+} - 1 = 0, \\ \begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}^{-1} \Big(\hat{\mathcal{B}}_{31}\,\mathcal{E}_2 - \hat{\mathcal{B}}_{11}\,\mathcal{E}_4\Big) &= \mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm out}^{--} + \mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm out}^{-+} = 0, \\ -\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix}^{-1} \Big(\hat{\mathcal{B}}_{33}\,\mathcal{E}_2 - \hat{\mathcal{B}}_{13}\,\mathcal{E}_4\Big) &= \mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm out}^{+-} + \mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm out}^{++} - 1 = 0. \end{aligned} \tag{C.2.8}$$

where we again use $\begin{vmatrix} \hat{\mathcal{B}}_{11} & \hat{\mathcal{B}}_{13} \\ \hat{\mathcal{B}}_{31} & \hat{\mathcal{B}}_{33} \end{vmatrix} \neq 0$. These can be solved to give the $\mathcal{T}_{\rm out}$ amplitudes in terms of $\mathcal{T}_{\rm in}$

amplitudes and vice versa, giving

$$\begin{aligned}
\mathcal{T}_{\rm out}^{+-} &= \frac{\mathcal{T}_{\rm in}^{+-}}{\mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-} - \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{++}}, \qquad & \mathcal{T}_{\rm out}^{--} &= -\frac{\mathcal{T}_{\rm in}^{++}}{\mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-} - \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{++}} \\
\mathcal{T}_{\rm out}^{++} &= -\frac{\mathcal{T}_{\rm in}^{--}}{\mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-} - \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{++}}, \qquad & \mathcal{T}_{\rm out}^{-+} &= \frac{\mathcal{T}_{\rm in}^{-+}}{\mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-} - \mathcal{T}_{\rm in}^{--}\mathcal{T}_{\rm in}^{++}}
\end{aligned} \tag{C.2.9}$$

and

$$\begin{aligned}
\mathcal{T}_{\rm in}^{+-} &= \frac{\mathcal{T}_{\rm out}^{+-}}{\mathcal{T}_{\rm out}^{-+}\mathcal{T}_{\rm out}^{+-} - \mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{++}}, \qquad & \mathcal{T}_{\rm in}^{--} &= -\frac{\mathcal{T}_{\rm out}^{++}}{\mathcal{T}_{\rm out}^{-+}\mathcal{T}_{\rm out}^{+-} - \mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{++}} \\
\mathcal{T}_{\rm in}^{++} &= -\frac{\mathcal{T}_{\rm out}^{--}}{\mathcal{T}_{\rm out}^{-+}\mathcal{T}_{\rm out}^{+-} - \mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{++}}, \qquad & \mathcal{T}_{\rm in}^{-+} &= \frac{\mathcal{T}_{\rm out}^{-+}}{\mathcal{T}_{\rm out}^{-+}\mathcal{T}_{\rm out}^{+-} - \mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{++}}.
\end{aligned} \tag{C.2.10}$$

Modulus and phase of $\mathcal{T}_{\rm in}$ and $\mathcal{T}_{\rm out}$

The constraint conditions (C.2.6) imply that the four complex $\mathcal{T}_{\rm in}$ amplitudes satisfy four real constraints and so actually only contain four independent real parameters. To see why notice that the first two conditions in (C.2.6) can be used to write the general amplitudes as

$$\mathcal{T}_{\rm in}^{++} = \rho_+ \, e^{i\theta_{++}}, \quad \mathcal{T}_{\rm in}^{-+} = \sqrt{1-\rho_+^2}\, e^{i\theta_{-+}}, \quad \mathcal{T}_{\rm in}^{--} = \rho_- \, e^{i\theta_{--}}, \quad \text{and} \quad \mathcal{T}_{\rm in}^{+-} = \sqrt{1-\rho_-^2}\, e^{i\theta_{+-}}, \tag{C.2.11}$$

for the six real parameters $\rho_\pm \geq 0$ and $\theta_{++}, \theta_{+-}, \theta_{-+}, \theta_{--}$. Plugging these expressions into the third condition in (C.2.6) then gives

$$\rho_- \sqrt{1-\rho_+^2}\, e^{i(\theta_{--}-\theta_{-+})} = \rho_+ \sqrt{1-\rho_-^2}\, e^{-i(\theta_{++}-\theta_{+-}+\pi)}, \tag{C.2.12}$$

and so taking the modulus squared of each side of the equation shows $\rho_+ = \rho_-$. Additionally, the four phases are not independent because they satisfy

$$\theta_{++} + \theta_{--} - (\theta_{-+} + \theta_{+-}) + \pi = 2\pi n, \tag{C.2.13}$$

where n is an integer. Consequently one of the phases can be eliminated in favor of the other three. We see that the most general form of the $\mathcal{T}_{\rm in}$ amplitudes for a hermitian theory with a rank 2 boundary matrix is then

$$\mathcal{T}_{\rm in}^{++} = \rho\, e^{i\theta_{++}}, \quad \mathcal{T}_{\rm in}^{+-} = \sqrt{1-\rho^2}\, e^{i\theta_{+-}}, \quad \mathcal{T}_{\rm in}^{--} = \rho\, e^{i\theta_{--}} \quad \text{and} \quad \mathcal{T}_{\rm in}^{-+} = -\sqrt{1-\rho^2}\, e^{i(\theta_{++}+\theta_{--}-\theta_{+-})}. \tag{C.2.14}$$

Notice that these imply $|\mathcal{T}_{\rm in}^{++}|^2 = |\mathcal{T}_{\rm in}^{--}|^2$ and $|\mathcal{T}_{\rm in}^{+-}|^2 = |\mathcal{T}_{\rm in}^{-+}|^2$ and that the denominator appearing in (C.2.9) is a pure phase:

$$\mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-} - \mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm in}^{--} = -e^{i(\theta_{++}+\theta_{--})}\,. \tag{C.2.15}$$

Implications for $\mathcal{T}_{\rm out}$

Plugging the most general form (C.2.14) for $\mathcal{T}_{\rm in}$ into (C.2.9) and using (C.2.15) gives

$$\mathcal{T}_{\rm out}^{++} = \rho\, e^{-i\theta_{++}} = (\mathcal{T}_{\rm in}^{++})^*, \quad \mathcal{T}_{\rm out}^{--} = \rho\, e^{-i\theta_{--}} = (\mathcal{T}_{\rm in}^{--})^*, \tag{C.2.16}$$

and

$$\mathcal{T}_{\rm out}^{+-} = -\sqrt{1-\rho^2}\, e^{i(\theta_{+-}-\theta_{++}-\theta_{--})} = (\mathcal{T}_{\rm in}^{-+})^*, \quad \mathcal{T}_{\rm out}^{-+} = \sqrt{1-\rho^2}\, e^{-i\theta_{+-}} = (\mathcal{T}_{\rm in}^{+-})^*. \tag{C.2.17}$$

It is clear that the unitarity conditions satisfied by $\mathcal{T}_{\rm in}$ are also satisfied by the $\mathcal{T}_{\rm out}$ amplitudes:

$$\begin{aligned} |\mathcal{T}_{\rm out}^{++}|^2 = |\mathcal{T}_{\rm out}^{--}|^2 = 1 - |\mathcal{T}_{\rm out}^{+-}|^2 = 1 - |\mathcal{T}_{\rm out}^{-+}|^2, \\ \mathcal{T}_{\rm out}^{--}\mathcal{T}_{\rm out}^{-+\,*} + \mathcal{T}_{\rm out}^{+-}\mathcal{T}_{\rm out}^{++\,*} = 0. \end{aligned} \tag{C.2.18}$$

Appendix D

Scattering states

In this appendix, we list some useful properties of the S-wave *in* and *out* states defined in the main text and derive the Bogoliubov relations given in §4. Additionally, we calculate the probability for a plane-wave state to be found in an S-wave, p_s, and derive the cross-section formulas in §3 and §4.

D.1 Properties of *in* and *out* states

Orthogonality and normalization relations

The unitarity constraints on *in* amplitudes (2.2.34)-(2.2.35) as well as the analogous *out* amplitude constraints (2.2.44)-(2.2.45) can be used to show that the *in* and *out* modes satisfy the orthogonality and normalization relations

$$\int_{\epsilon}^{\infty} \mathrm{d}r\, (\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k})^{\dagger}\, \mathfrak{u}^{\mathrm{d}}_{\mathfrak{s}',k'} = \int_{\epsilon}^{\infty} \mathrm{d}r\, (\mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k})^{\dagger}\, \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s}',k'} = 2\pi\delta(k-k')\,\delta_{\mathfrak{s}\mathfrak{s}'}, \tag{D.1.1}$$

and

$$\int_{\epsilon}^{\infty} \mathrm{d}r\, (\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k})^{\dagger}\, \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s}',k'} = 2\pi\delta(k+k')\,\delta_{\mathfrak{s}\mathfrak{s}'}. \tag{D.1.2}$$

where the label d indicates the direction of motion *i.e.* $\mathrm{d} = \{\mathrm{in}, \mathrm{out}\}$. These relations can be used to show that, when the S-wave fermion field is expanded in terms of *in*, *out* modes as

$$\boldsymbol{\chi}(x) \;=\; \sum_{\mathfrak{s}=\pm} \int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\pi}} \left[\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}(x)\, a^{\mathrm{d}}_{\mathfrak{s},k} + \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k}(x)\, (\overline{a}^{\mathrm{d}}_{\mathfrak{s},k})^\star \right], \tag{D.1.3}$$

the particle and antiparticle creation and annihilation operator anticommutation relations are given by

$$\left\{ a^{\mathrm{d}}_{\mathfrak{s},k}, (a^{\mathrm{d}}_{\mathfrak{s}',k'})^\star \right\} = \left\{ \overline{a}^{\mathrm{d}}_{\mathfrak{s},k}, (\overline{a}^{\mathrm{d}}_{\mathfrak{s}',k'})^\star \right\} = \delta(k-k')\, \delta_{\mathfrak{s}\mathfrak{s}'}, \tag{D.1.4}$$

with all other anticommutators vanishing.

For some applications, it is preferable to consider discretely normalized states. We define the discretely normalized *in*, *out* modes as

$$\boldsymbol{u}^{\mathrm{d}}_{\mathfrak{s},k}(x) \coloneqq \frac{1}{\sqrt{2L}} \mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}(x), \quad \boldsymbol{v}^{\mathrm{d}}_{\mathfrak{s},k}(x) \coloneqq \frac{1}{\sqrt{2L}} \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k}(x), \tag{D.1.5}$$

since the orthogonality and normalization relations for these modes become

$$\int_\epsilon^{L+\epsilon} \mathrm{d}r\, (\boldsymbol{u}^{\mathrm{d}}_{\mathfrak{s},k})^\dagger\, \boldsymbol{u}^{\mathrm{d}}_{\mathfrak{s}',k'} = \int_\epsilon^{L+\epsilon} \mathrm{d}r\, (\boldsymbol{v}^{\mathrm{d}}_{\mathfrak{s},k})^\dagger\, \boldsymbol{v}^{\mathrm{d}}_{\mathfrak{s}',k'} = \delta_{kk'}\, \delta_{\mathfrak{s}\mathfrak{s}'}, \tag{D.1.6}$$

and

$$\int_\epsilon^{L+\epsilon} \mathrm{d}r\, (\boldsymbol{u}^{\mathrm{d}}_{\mathfrak{s},k})^\dagger\, \boldsymbol{v}^{\mathrm{d}}_{\mathfrak{s}',k'} = \delta_{k,-k'}\, \delta_{\mathfrak{s}\mathfrak{s}'}, \tag{D.1.7}$$

in the large L limit. The S-wave fermion field can be expanded in terms of the discretely normalized bases:

$$\boldsymbol{\chi}(x) \;=\; \sum_{\mathfrak{s}=\pm} \sum_{k\geq 0}^{\infty} \left[\boldsymbol{u}^{\mathrm{d}}_{\mathfrak{s},k}(x)\, \boldsymbol{a}^{\mathrm{d}}_{\mathfrak{s},k} + \boldsymbol{v}^{\mathrm{d}}_{\mathfrak{s},k}(x)\, (\overline{\boldsymbol{a}}^{\mathrm{d}}_{\mathfrak{s},k})^\star \right], \tag{D.1.8}$$

where the discrete normalization particle and antiparticle creation and annihilation operators satisfy

$$\left\{ \boldsymbol{a}^{\mathrm{d}}_{\mathfrak{s},k}, (\boldsymbol{a}^{\mathrm{d}}_{\mathfrak{s}',k'})^\star \right\} = \left\{ \overline{\boldsymbol{a}}^{\mathrm{d}}_{\mathfrak{s},k}, (\overline{\boldsymbol{a}}^{\mathrm{d}}_{\mathfrak{s}',k'})^\star \right\} = \delta_{kk'}\, \delta_{\mathfrak{s}\mathfrak{s}'}, \tag{D.1.9}$$

with all other anticommutators vanishing.

Bogoliubov relations

The *in* creation and annihilation operators can be expanded in terms of the corresponding *out* operators and vice versa, as in the Bogoliubov relations (4.1.5) and (4.1.6). To see this, note that the *in* operators satisfy

$$a^{\rm in}_{\mathfrak{s},k} = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \mathrm{d}t\, (\mathfrak{u}^{\rm in}_{\mathfrak{s},k}(r,t))^\dagger \boldsymbol{\chi}(r,t), \quad \text{and} \quad (\overline{a}^{\rm in}_{\mathfrak{s},k})^\star = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} \mathrm{d}t\, (\mathfrak{v}^{\rm in}_{\mathfrak{s},k}(r,t))^\dagger \boldsymbol{\chi}(r,t), \tag{D.1.10}$$

which can be shown by expanding $\boldsymbol{\chi}(x)$ in terms of the *in* basis. Since $\boldsymbol{\chi}(x)$ can equivalently be expanded in terms of *out* states, the above equations imply that *e.g.* $a^{\rm in}_{+,k}$ is given by

$$\begin{aligned} a^{\rm in}_{+,k} \;&=\; \frac{1}{\sqrt{8\pi}} \sum_{\mathfrak{s}'=\pm} \int_{-\infty}^{\infty} \mathrm{d}t \int_0^\infty \frac{\mathrm{d}k'}{\sqrt{2\pi}} (\mathfrak{u}^{\rm in}_{+,k}(r,t))^\dagger \; \left(\mathfrak{u}^{\rm out}_{\mathfrak{s}',k'}(x)\, a^{\rm out}_{\mathfrak{s}',k'} + \mathfrak{v}^{\rm out}_{\mathfrak{s}',k'}(x)\, (\overline{a}^{\rm out}_{\mathfrak{s}',k'})^\star \right) \\ &=\; \int_0^\infty \mathrm{d}k' \left(\mathcal{T}^{++}_{\rm out} e^{2ik\epsilon} \left(\frac{\epsilon}{r_0}\right)^{-ieQ} \delta(\omega_{+,k} - \omega_{+,k'})\, a^{\rm out}_{+,k'} \right. \\ &\qquad \left. + \mathcal{T}^{+-}_{\rm out} e^{i(2k-ev)\epsilon} e^{i\mathfrak{a}} \left[\delta(\omega_{+,k} - \omega_{-,k'})\, a^{\rm out}_{-,k'} + \delta(\omega_{+,k} + \overline{\omega}_{-,k'})\, (\overline{a}^{\rm out}_{-,k'})^\star \right] \right), \end{aligned} \tag{D.1.11}$$

where $\omega_{\mathfrak{s},k} = k - \mathfrak{s}ev/2$ ($\overline{\omega}_{\mathfrak{s},k} = k + \mathfrak{s}ev/2$) is the energy of a particle (antiparticle) with quantum numbers $\mathfrak{s}, k$. The above equation reduces to one of the Bogoliubov relations, once the integral over momentum is evaluated:

$$a^{\rm in}_{+,k} \;=\; \mathcal{T}^{++}_{\rm out}\, a^{\rm out}_{+,k} + \mathcal{T}^{+-}_{\rm out}\, e^{i\mathfrak{a}} \left[\Theta(k-ev)\, a^{\rm out}_{-,k-ev} + \Theta(-k+ev)\, (\overline{a}^{\rm out}_{-,-k+ev})^\star \right], \tag{D.1.12}$$

where we drop powers of $k\epsilon$ and $ev\epsilon$, shift $\mathfrak{a}$ and rephase the *out* operators to absorb the r_0-dependent phase, as in the main text. The remaining Bogoliubov relations can be derived in a similar way.

It is sometimes convenient to write the Bogoliubov relations as in the last line of (D.1.11) since this shows that they are consistent with energy conservation, which is enforced through the delta functions that appear after the time integral is performed. Specifically, this form is useful for evaluating scattering amplitudes such as $\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\, (a^{\rm in}_{+,k})^\star\, |0_{\rm in}\rangle$, which can be written as:

$$\begin{aligned} \langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\, (a^{\rm in}_{+,k})^\star\, |0_{\rm in}\rangle \;&=\; \mathcal{T}^{+-\,*}_{\rm out} e^{-i\mathfrak{a}} \int_0^\infty \mathrm{d}p\, \delta(\omega_{+,k} - \omega_{-,p})\, \langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\, (a^{\rm out}_{-,p})^\star\, |0_{\rm in}\rangle \\ &=\; \mathcal{T}^{+-\,*}_{\rm out}\, \langle 0_{\rm out}|0_{\rm in}\rangle\, e^{-i\mathfrak{a}} \Theta(k-ev) \delta(\omega_{+,k} - \omega_{-,k'}), \end{aligned} \tag{D.1.13}$$

where in the last line we use the 'momentum-conserving' delta function $\delta(p-k')$ implicit in the overlap $\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\,(a^{\rm out}_{-,p})^\star\,|0_{\rm in}\rangle$ to perform the integral over p. The surviving delta function is the original energy-conserving one from (D.1.11) which is regularized by a factor of the duration of the interaction, T, when probabilities are calculated.

The above argument shows that the Heisenberg picture transition probabilities (which can be obtained from the amplitudes listed in §4) depend on T in the same way as their interaction picture counterparts, and so rates and cross sections can be defined in much the same way in both pictures. This can also be seen in a simpler way, by re-evaluating the amplitude $\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\,(a^{\rm in}_{+,k})^\star\,|0_{\rm in}\rangle$ using the 'standard' form of the relevant Bogoliubov relation, (D.1.12), in the following way

$$\begin{aligned}\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\,(a^{\rm in}_{+,k})^\star\,|0_{\rm in}\rangle &= \mathcal{T}^{+-\,*}_{\rm out} e^{-i\mathfrak{a}}\,\Theta(k-ev)\,\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\,(a^{\rm out}_{-,k-ev})^\star\,|0_{\rm in}\rangle \\ &= \mathcal{T}^{+-\,*}_{\rm out}\,\langle 0_{\rm out}|0_{\rm in}\rangle\, e^{-i\mathfrak{a}}\,\Theta(k-ev)\delta(k-k'-ev) \\ &= \mathcal{T}^{+-\,*}_{\rm out}\,\langle 0_{\rm out}|0_{\rm in}\rangle\, e^{-i\mathfrak{a}}\,\Theta(k-ev)\delta(\omega_{+,k}-\omega_{-,k'}), \end{aligned} \tag{D.1.14}$$

where the delta function in the second line comes from the overlap $\langle 0_{\rm out}|\, a^{\rm out}_{-,k'}\,(a^{\rm out}_{-,k-ev})^\star\,|0_{\rm in}\rangle$ and is rewritten in the third line by performing a change of variables from momentum to energy. Since the final expression in (D.1.14) involves an energy-conserving delta function, we can regulate it with a factor of T per the usual procedure.

Expectation values of energy and electric, axial charge

The *in* and *out* states are eigenstates of the S-wave fermionic hamiltonian (with $\widehat{A}_0=0$):

$$H_F = \frac{1}{2}\Big[\overline{\boldsymbol{\chi}}\, O_{\mathcal{B}}(\mathfrak{a})\,\boldsymbol{\chi}\Big]_{r=\epsilon,t} + \frac{1}{2}\sum_{\mathfrak{s}=\pm}\int_\epsilon^\infty {\rm d}r\,\overline{\chi}_{\mathfrak{s}}\left[\Gamma^1\overleftrightarrow{\partial}_1 - i\mathfrak{s}\,\Gamma^0\left(ev-\frac{eQ}{r}\right)\right]\chi_{\mathfrak{s}}, \tag{D.1.15}$$

when $\mathfrak{a}$ is treated as a classical variable within the Born-Oppenheimer approximation. This can be seen from the expansion of H_F in terms of *in* or *out* states (see appendix F for details of how calculations involving fermion bilinears are done)

$$H_F = E^{\rm d}_{0,F} + \sum_{\mathfrak{s}=\pm}\int_0^\infty {\rm d}k\,\left[\left(k-\frac{1}{2}\mathfrak{s}ev\right)(a^{\rm d}_{\mathfrak{s},k})^\star\, a^{\rm d}_{\mathfrak{s},k} + \left(k+\frac{1}{2}\mathfrak{s}ev\right)(\overline{a}^{\rm d}_{\mathfrak{s},k})^\star\,\overline{a}^{\rm d}_{\mathfrak{s},k}\right], \tag{D.1.16}$$

which implies that, relative to the vacuum, the single particle and antiparticle states have energies

$$\omega_{\mathfrak{s},k} = k - \frac{1}{2}\mathfrak{s}ev \quad \text{and} \quad \overline{\omega}_{\mathfrak{s},k} = k + \frac{1}{2}\mathfrak{s}ev, \tag{D.1.17}$$

where $\omega_{\mathfrak{s},k}$ ($\overline{\omega}_{\mathfrak{s},k}$) is the energy of the particle state $(a^{\mathrm{d}}_{\mathfrak{s},k})^\star |0_{\mathrm{d}}\rangle$ (antiparticle state $(\overline{a}^{\mathrm{d}}_{\mathfrak{s},k})^\star |0_{\mathrm{d}}\rangle$) relative to the $|0_{\mathrm{d}}\rangle$ vacuum. The energies of the *in* and *out* vacuum are given by $E^{\mathrm{d}}_{0,F} \coloneqq \langle 0_{\mathrm{d}}| H_F |0_{\mathrm{d}}\rangle$.

Generally, the *in* and *out* states are not eigenstates of the fermion electric charge $\mathcal{Q}_F$ or of the axial charge operator $\mathcal{Q}_A$. To see this, we expand $\mathcal{Q}_F, \mathcal{Q}_A$ in terms of the *in* basis:

$$\begin{aligned}
\mathcal{Q}_F = \quad & Q^{\mathrm{in}}_{0,F} + |\mathcal{T}^{++}_{\mathrm{in}}|^2 \int_0^\infty \mathrm{d}k \sum_{\mathfrak{s}=\pm} \frac{1}{2}\mathfrak{s}e \left((a^{\mathrm{in}}_{\mathfrak{s},k})^\star a^{\mathrm{in}}_{\mathfrak{s},k} - (\overline{a}^{\mathrm{in}}_{\mathfrak{s},k})^\star \overline{a}^{\mathrm{in}}_{\mathfrak{s},k} \right) \\
& + |\mathcal{T}^{+-}_{\mathrm{in}}|^2 \sum_{\mathfrak{s}=\pm} \mathfrak{s}e \int_0^\infty \frac{\mathrm{d}k}{2\pi} PV \Bigg[\int_0^\infty \mathrm{d}k' \Bigg((a^{\mathrm{in}}_{\mathfrak{s},k})^\star a^{\mathrm{in}}_{\mathfrak{s},k'} \frac{ie^{i(k-k')(\epsilon+t)}}{k-k'} + (a^{\mathrm{in}}_{\mathfrak{s},k})^\star (\overline{a}^{\mathrm{in}}_{\mathfrak{s},k'})^\star \frac{ie^{i(k+k')(\epsilon+t)}}{k+k'} \\
& \qquad + \overline{a}^{\mathrm{in}}_{\mathfrak{s},k} a^{\mathrm{in}}_{\mathfrak{s},k'} \frac{ie^{-i(k+k')(\epsilon+t)}}{-k-k'} - (\overline{a}^{\mathrm{in}}_{\mathfrak{s},k'})^\star \overline{a}^{\mathrm{in}}_{\mathfrak{s},k} \frac{ie^{-i(k-k')(\epsilon+t)}}{-k+k'} \Bigg) \Bigg] \\
& + e\,\mathcal{T}^{+-}_{\mathrm{in}} \mathcal{T}^{++\,*}_{\mathrm{in}} e^{i\mathfrak{a}} \int_0^\infty \frac{\mathrm{d}k\,\mathrm{d}k'}{2\pi} \lim_{\beta\to 0+} \Bigg[(a^{\mathrm{in}}_{+,k})^\star a^{\mathrm{in}}_{-,k'} \frac{ie^{i(k-k'-ev)t} e^{i(k-k')\epsilon}}{k'-k+ev+i\beta} \\
& \qquad + (a^{\mathrm{in}}_{+,k})^\star (\overline{a}^{\mathrm{in}}_{-,k'})^\star \frac{ie^{i(k+k'-ev)t} e^{i(k+k')\epsilon}}{-k'-k+ev+i\beta} + \overline{a}^{\mathrm{in}}_{+,k} a^{\mathrm{in}}_{-,k'} \frac{ie^{i(-k-k'-ev)t} e^{-i(k+k')\epsilon}}{k'+k+ev+i\beta} \\
& \qquad - (\overline{a}^{\mathrm{in}}_{-,k'})^\star \overline{a}^{\mathrm{in}}_{+,k} \frac{ie^{i(-k+k'-ev)t} e^{-i(k-k')\epsilon}}{-k'+k+ev+i\beta} \Bigg] \\
& - e\,\mathcal{T}^{+-\,*}_{\mathrm{in}} \mathcal{T}^{++}_{\mathrm{in}} e^{-i\mathfrak{a}} \int_0^\infty \frac{\mathrm{d}k\,\mathrm{d}k'}{2\pi} \lim_{\beta\to 0+} \Bigg[(a^{\mathrm{in}}_{-,k})^\star a^{\mathrm{in}}_{+,k'} \frac{ie^{-i(k'-k-ev)t} e^{i(k-k')\epsilon}}{-k'+k+ev-i\beta} \\
& \qquad + (a^{\mathrm{in}}_{-,k})^\star (\overline{a}^{\mathrm{in}}_{+,k'})^\star \frac{ie^{-i(-k'-k-ev)t} e^{i(k+k')\epsilon}}{k'+k+ev-i\beta} + \overline{a}^{\mathrm{in}}_{-,k} a^{\mathrm{in}}_{+,k'} \frac{ie^{-i(k'+k-ev)t} e^{-i(k+k')\epsilon}}{-k'-k+ev-i\beta} \\
& \qquad - (\overline{a}^{\mathrm{in}}_{+,k'})^\star \overline{a}^{\mathrm{in}}_{-,k} \frac{ie^{-i(-k'+k-ev)t} e^{-i(k-k')\epsilon}}{k'-k+ev-i\beta} \Bigg],
\end{aligned} \tag{D.1.18}$$

and

$$
\begin{aligned}
\mathcal{Q}_A = \quad & Q^{\rm in}_{0,A} + |\mathcal{T}^{+-}_{\rm in}|^2 \int_0^\infty {\rm d}k \sum_{\mathfrak{s}=\pm} \mathfrak{s} \left((a^{\rm in}_{\mathfrak{s},k})^\star \, a^{\rm in}_{\mathfrak{s},k} - (\overline{a}^{\rm in}_{\mathfrak{s},k})^\star \, \overline{a}^{\rm in}_{\mathfrak{s},k} \right) \\
& + 2|\mathcal{T}^{++}_{\rm in}|^2 \sum_{\mathfrak{s}=\pm} \mathfrak{s} \int_0^\infty \frac{{\rm d}k}{2\pi} \, PV \Bigg[\int_0^\infty {\rm d}k' \Bigg((a^{\rm in}_{\mathfrak{s},k})^\star \, a^{\rm in}_{\mathfrak{s},k'} \, \frac{ie^{i(k-k')(\epsilon+t)}}{k-k'} + (a^{\rm in}_{\mathfrak{s},k})^\star \, (\overline{a}^{\rm in}_{\mathfrak{s},k'})^\star \, \frac{ie^{i(k+k')(\epsilon+t)}}{k+k'} \\
& \qquad\qquad + \overline{a}^{\rm in}_{\mathfrak{s},k} a^{\rm in}_{\mathfrak{s},k'} \, \frac{ie^{-i(k+k')(\epsilon+t)}}{-k-k'} - (\overline{a}^{\rm in}_{\mathfrak{s},k'})^\star \, \overline{a}^{\rm in}_{\mathfrak{s},k} \, \frac{ie^{-i(k-k')(\epsilon+t)}}{-k+k'} \Bigg) \Bigg] \\
& - 2\, \mathcal{T}^{+-}_{\rm in} \mathcal{T}^{++\,*}_{\rm in} e^{i\mathfrak{a}} \int_0^\infty \frac{{\rm d}k\, {\rm d}k'}{2\pi} \lim_{\beta \to 0+} \Bigg[(a^{\rm in}_{+,k})^\star \, a^{\rm in}_{-,k'} \, \frac{ie^{i(k-k'-ev)t} e^{i(k-k')\epsilon}}{k'-k+ev+i\beta} \\
& \qquad\qquad + (a^{\rm in}_{+,k})^\star \, (\overline{a}^{\rm in}_{-,k'})^\star \, \frac{ie^{i(k+k'-ev)t} e^{i(k+k')\epsilon}}{-k'-k+ev+i\beta} + \overline{a}^{\rm in}_{+,k} \, a^{\rm in}_{-,k'} \, \frac{ie^{i(-k-k'-ev)t} e^{-i(k+k')\epsilon}}{k'+k+ev+i\beta} \\
& \qquad\qquad - (\overline{a}^{\rm in}_{-,k'})^\star \, \overline{a}^{\rm in}_{+,k} \, \frac{ie^{i(-k+k'-ev)t} e^{-i(k-k')\epsilon}}{-k'+k+ev+i\beta} \Bigg) \\
& + 2\, \mathcal{T}^{+-\,*}_{\rm in} \mathcal{T}^{++}_{\rm in} e^{-i\mathfrak{a}} \int_0^\infty \frac{{\rm d}k\, {\rm d}k'}{2\pi} \lim_{\beta \to 0+} \Bigg[(a^{\rm in}_{-,k})^\star \, a^{\rm in}_{+,k'} \, \frac{ie^{-i(k'-k-ev)t} e^{i(k-k')\epsilon}}{-k'+k+ev-i\beta} \\
& \qquad\qquad + (a^{\rm in}_{-,k})^\star \, (\overline{a}^{\rm in}_{+,k'})^\star \, \frac{ie^{-i(-k'-k-ev)t} e^{i(k+k')\epsilon}}{k'+k+ev-i\beta} + \overline{a}^{\rm in}_{-,k} \, a^{\rm in}_{+,k'} \, \frac{ie^{-i(k'+k-ev)t} e^{-i(k+k')\epsilon}}{-k'-k+ev-i\beta} \\
& \qquad\qquad - (\overline{a}^{\rm in}_{+,k'})^\star \, \overline{a}^{\rm in}_{-,k} \, \frac{ie^{-i(-k'+k-ev)t} e^{-i(k-k')\epsilon}}{k'-k+ev-i\beta} \Bigg],
\end{aligned}
\tag{D.1.19}
$$

where $Q^{\rm in}_{0,F} \coloneqq \langle 0_{\rm in} | \, \mathcal{Q}_F \, | 0_{\rm in} \rangle$, $Q^{\rm in}_{0,A} \coloneqq \langle 0_{\rm in} | \, \mathcal{Q}_A \, | 0_{\rm in} \rangle$, PV refers to the Cauchy principal value of an integral and we shift $\mathfrak{a}$ to absorb an r_0-dependent phase, as in the main text. The above expansions show that the *in* states are eigenstates of $\mathcal{Q}_F$ only if $\mathcal{T}^{+-}_{\rm in} = \mathcal{T}^{-+}_{\rm in} = 0$, *i.e.* when only 4D chirality-changing processes are possible at the boundary, and are eigenstates of $\mathcal{Q}_A$ in the special case where $\mathcal{T}^{++}_{\rm in} = \mathcal{T}^{--}_{\rm in} = 0$ and the boundary action only allows for charge-exchange processes. The expectation values of $\mathcal{Q}_F, \mathcal{Q}_A$ in the single particle *in* states are

$$
\begin{aligned}
\langle 0_{\rm in} | \, a^{\rm in}_{\mathfrak{s},k} \, \mathcal{Q}_F \, (a^{\rm in}_{\mathfrak{s},k})^\star \, | 0_{\rm in} \rangle &= \left(Q^{\rm in}_{0,F} + \frac{1}{2} \mathfrak{s} e |\mathcal{T}^{++}_{\rm in}|^2 \right) \langle 0_{\rm in} | \, a^{\rm in}_{\mathfrak{s},k} \, (a^{\rm in}_{\mathfrak{s},k})^\star \, | 0_{\rm in} \rangle \,, \\
\langle 0_{\rm in} | \, a^{\rm in}_{\mathfrak{s},k} \, \mathcal{Q}_A \, (a^{\rm in}_{\mathfrak{s},k})^\star \, | 0_{\rm in} \rangle &= \left(Q^{\rm in}_{0,A} + \mathfrak{s} |\mathcal{T}^{+-}_{\rm in}|^2 \right) \langle 0_{\rm in} | \, a^{\rm in}_{\mathfrak{s},k} \, (a^{\rm in}_{\mathfrak{s},k})^\star \, | 0_{\rm in} \rangle \,,
\end{aligned}
\tag{D.1.20}
$$

and are given by

$$\langle 0_{\rm in}|\,\overline{a}^{\rm in}_{\mathfrak{s},k}\,\mathcal{Q}_F\,(\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle = \left(Q^{\rm in}_{0,F} - \frac{1}{2}\mathfrak{s}e|\mathcal{T}^{++}_{\rm in}|^2\right)\langle 0_{\rm in}|\,\overline{a}^{\rm in}_{\mathfrak{s},k}\,(\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle\,,$$
$$\langle 0_{\rm in}|\,\overline{a}^{\rm in}_{\mathfrak{s},k}\,\mathcal{Q}_A\,(\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle = \left(Q^{\rm in}_{0,A} - \mathfrak{s}|\mathcal{T}^{+-}_{\rm in}|^2\right)\langle 0_{\rm in}|\,\overline{a}^{\rm in}_{\mathfrak{s},k}\,(\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle\,, \tag{D.1.21}$$

for *in* states with a single antiparticle. *Out* states similarly have a definite electric charge only if $\mathcal{T}^{+-}_{\rm out} = \mathcal{T}^{-+}_{\rm out} = 0$ and a definite axial charge only when $\mathcal{T}^{++}_{\rm out} = \mathcal{T}^{--}_{\rm out} = 0$. For more general choices of the $\mathcal{T}_{\rm out}$ amplitudes, the expectation values of the electric and axial charge in single particle or antiparticle *out* states are

$$\langle 0_{\rm out}|\,a^{\rm out}_{\mathfrak{s},k}\,\mathcal{Q}_F\,(a^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle = \left(Q^{\rm out}_{0,F} + \frac{1}{2}\mathfrak{s}e|\mathcal{T}^{++}_{\rm out}|^2\right)\langle 0_{\rm out}|\,a^{\rm out}_{\mathfrak{s},k}\,(a^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle\,,$$
$$\langle 0_{\rm out}|\,a^{\rm out}_{\mathfrak{s},k}\,\mathcal{Q}_A\,(a^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle = \left(Q^{\rm out}_{0,A} - \mathfrak{s}|\mathcal{T}^{+-}_{\rm out}|^2\right)\langle 0_{\rm out}|\,a^{\rm out}_{\mathfrak{s},k}\,(a^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle\,, \tag{D.1.22}$$

and

$$\langle 0_{\rm out}|\,\overline{a}^{\rm out}_{\mathfrak{s},k}\,\mathcal{Q}_F\,(\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle = \left(Q^{\rm out}_{0,F} - \frac{1}{2}\mathfrak{s}e|\mathcal{T}^{++}_{\rm out}|^2\right)\langle 0_{\rm out}|\,\overline{a}^{\rm out}_{\mathfrak{s},k}\,(\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle\,,$$
$$\langle 0_{\rm out}|\,\overline{a}^{\rm out}_{\mathfrak{s},k}\,\mathcal{Q}_A\,(\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle = \left(Q^{\rm out}_{0,A} + \mathfrak{s}|\mathcal{T}^{+-}_{\rm out}|^2\right)\langle 0_{\rm out}|\,\overline{a}^{\rm out}_{\mathfrak{s},k}\,(\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle\,. \tag{D.1.23}$$

Equations (D.1.20)-(D.1.23) show that a measurement of the electric charge in a single-particle state $(a^{\rm d}_{\mathfrak{s},k})^\star\,|0_{\rm d}\rangle$ or $(\overline{a}^{\rm d}_{\mathfrak{s},k})^\star\,|0_{\rm d}\rangle$ (relative to the vacuum) does not generally yield the result one would naively expect, namely $\frac{1}{2}\mathfrak{s}e$ for particles and $-\frac{1}{2}\mathfrak{s}e$ for antiparticles. This is because the *in* and *out* states agree with the usual notion of particles only in the asymptotic past and future, respectively. More precisely, the state $(a^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle$ ($(\overline{a}^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle$) describes a particle (antiparticle) that approaches the dyon with momentum k, charge $\frac{1}{2}\mathfrak{s}e$ ($-\frac{1}{2}\mathfrak{s}e$) and 4D chirality $\mathfrak{s}$ (-$\mathfrak{s}$) in the asymptotic past and then scatters either to a particle (antiparticle) with different electric charge and momentum or to one with different 4D chirality. The *out* states $(a^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle$ ($(\overline{a}^{\rm out}_{\mathfrak{s},k})^\star\,|0_{\rm out}\rangle$) similarly describe particles (antiparticles) of momentum k, charge $\frac{1}{2}\mathfrak{s}e$ ($-\frac{1}{2}\mathfrak{s}e$) and 4D chirality $-\mathfrak{s}$ ($\mathfrak{s}$) in the asymptotic future and have a more complicated description at early times. This can be shown explicitly by writing single-particle states as the infinitesimally narrow limit of a wave-packet peaked around a given momentum, and evaluating the fermionic current expectation values in these states in the asymptotic past (for *in* states) or asymptotic future (*out* states).

D.2 S-wave projection of plane-wave states

For scattering problems, the initial states of interest correspond to 4D plane waves far from the dyon and so are not prepared in the S-wave. We now show how these plane wave states can be projected onto the S-wave sector and use this to compute the cross sections for single-particle S-wave scattering given in the main text.

To start, we note that the Dirac equation simplifies considerably at large distances from the dyon, since the background Julia-Zee potential becomes constant *i.e.* $e\mathcal{A}^3_\mu(x) \underset{r\to\infty}{\sim} ev\,\delta^0_\mu$. The full fermion field can then be expanded in terms of particle and antiparticle mode functions $U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x), V_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x)$ whose asymptotic form is equivalent to that of plane-wave spinors, up to a phase[1]. The particle mode functions then satisfy

$$U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x) \underset{r\to\infty}{\sim} \mathcal{N}\, e^{-i(k-\mathfrak{s}\,ev/2)t} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\Phi^{\mathfrak{s},\mathfrak{c}}_U(x)} \xi_{\mathfrak{s}} \otimes \nu^U_{\mathfrak{c}}\,, \tag{D.2.1}$$

where $\tau_3\,\xi_{\mathfrak{s}} = \mathfrak{s}\,\xi_{\mathfrak{s}}$, $\nu^U_+ = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T$, $\nu^U_- = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^T$ and $\mathcal{N}$ is a normalization factor. Similarly, the antiparticle mode functions are asymptotically given by

$$V_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x) \underset{r\to\infty}{\sim} \mathcal{N}\, e^{i(k+\mathfrak{s}\,ev/2)t} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\Phi^{\mathfrak{s},\mathfrak{c}}_V(x)} \xi_{\mathfrak{s}} \otimes \nu^V_{\mathfrak{c}}\,, \tag{D.2.2}$$

where $\nu^V_+ = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$, $\nu^V_- = \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^T$. The functions $\Phi^{\mathfrak{s},\mathfrak{c}}_{U,V}(x)$ are phases whose radial dependence is $(r/r_0)^{i\mathfrak{s}eQ/2}$ [2], and whose angular dependence we discuss shortly. We choose the normalization factor, $\mathcal{N}$, such that the mode functions satisfy the following orthogonality and normalization relations

$$\int_{-\infty}^{\infty} \mathrm{d}^3x\, U^\dagger_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}\, U_{\mathfrak{s}',\mathfrak{c}',\boldsymbol{k}'} = \int_{-\infty}^{\infty} \mathrm{d}^3x\, V^\dagger_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}\, V_{\mathfrak{s}',\mathfrak{c}',\boldsymbol{k}'} = (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\mathfrak{s}\mathfrak{s}'}\delta_{\mathfrak{c}\mathfrak{c}'}\,, \tag{D.2.3}$$

[1] Note that this phase has to be defined such that $U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x)$, $V_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x)$ are sections *i.e.* it should be defined differently in the R_+ and R_- region.

[2] This choice ensures the asymptotic radial dependence of the incoming spherical waves in U, V and in partial wave solutions match.

as well as

$$\int_{-\infty}^{\infty} \mathrm{d}^3x \, U^{\dagger}_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}} \, V_{\mathfrak{s}',\mathfrak{c}',\boldsymbol{k}'} = 0, \tag{D.2.4}$$

The full fermion field can be expanded in terms of the $U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x)$, $V_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x)$ basis as follows:

$$\boldsymbol{\psi}(x) = \sum_{\mathfrak{s},\mathfrak{c}=\pm} \int_{-\infty}^{\infty} \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}} \, U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x) + (\overline{a}_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}})^{\star} \, V_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x) \right), \tag{D.2.5}$$

where the plane-wave particle and antiparticle creation and annihilation operators satisfy

$$\{a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}, (a_{\mathfrak{s}',\mathfrak{c}',\boldsymbol{k}'})^{\star}\} = \{\overline{a}_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}, (\overline{a}_{\mathfrak{s}',\mathfrak{c}',\boldsymbol{k}'})^{\star}\} = \delta(\boldsymbol{k} - \boldsymbol{k}')\delta_{\mathfrak{s}\mathfrak{s}'}\delta_{\mathfrak{c},\mathfrak{c}'}, \tag{D.2.6}$$

with all other anticommutators being zero. We can also expand $\boldsymbol{\psi}(x)$ in terms of partial wave states,

$$\boldsymbol{\psi}(x) = \sum_{\mathfrak{s}=\pm} \int_{0}^{\infty} \frac{\mathrm{d}k}{\sqrt{2\pi}} \left(a^{\mathrm{in}}_{\mathfrak{s},k} \, u^{\mathrm{in}}_{\mathfrak{s},k}(x) + (\overline{a}^{\mathrm{in}}_{\mathfrak{s},k})^{\star} \, v^{\mathrm{in}}_{\mathfrak{s},k}(x) \right) + \sum_{j>0} \boldsymbol{\psi}_j(x) \tag{D.2.7}$$

where $u^{\mathrm{in}}_{\mathfrak{s}k}(x), v^{\mathrm{in}}_{\mathfrak{s}k}(x)$ are the 4D equivalents of the 2D *in* modes[3] and $\boldsymbol{\psi}_j(x)$ is the j-th fermion partial wave. Since both the plane-wave and partial wave bases are complete, we can expand the plane-wave particle creation operators $(a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}})^{\star}$ in terms of partial wave creation operators

$$(a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}})^{\star} = \sum_{\mathfrak{s}'=\pm} \int_{0}^{\infty} \mathrm{d}k' \, \mathcal{D}^{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}_{\mathfrak{s}',k'} \, (a^{\mathrm{in}}_{\mathfrak{s}',k'})^{\star} + \sum_{j>0,\kappa_j} \mathcal{D}^{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}_{j,\kappa_j} \, (a_{j,\kappa_j})^{\star}, \tag{D.2.8}$$

where $(a_{j,\kappa_j})^{\star}$ is a creation operator for a state in the j-th partial wave with quantum numbers κ_j. To project the plane-wave state $(a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}})^{\star} |0\rangle$ onto the S-wave sector we simply need to determine the coefficients $\mathcal{D}^{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}_{\mathfrak{s}',k'}$, which can be done by expanding the corresponding plane-wave mode function in terms of S-wave mode functions,

$$\begin{aligned} U_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}(x) &= (2\pi)^{3/2} \langle 0| \, \boldsymbol{\psi}(x) \, (a_{\mathfrak{s},\mathfrak{c},\boldsymbol{k}})^{\star} \, |0\rangle \\ &= 2\pi \sum_{\mathfrak{s}'=\pm} \int_{0}^{\infty} \mathrm{d}k' \, \mathcal{D}^{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}_{\mathfrak{s}',k'} \, u^{\mathrm{in}}_{\mathfrak{s}',k'}(x) + \sum_{j>0,\kappa_j} (2\pi)^{3/2} \langle 0| \, \boldsymbol{\psi}(x) \, \mathcal{D}^{\mathfrak{s},\mathfrak{c},\boldsymbol{k}}_{j,\kappa_j} (a_{j,\kappa_j})^{\star} \, |0\rangle \, , \end{aligned} \tag{D.2.9}$$

where we use the fact that the S-wave annihilation operators also annihilate the full vacuum, $|0\rangle$.

[3] The 4D *in*, *out* modes are given by (2.1.10) where $f_{\pm}(r,t), g_{\pm}(r,t)$ are components of the corresponding 2D basis, as in (2.1.15).

Antiparticle states can be projected onto the lowest partial wave in much the same way.

The coefficients $\mathcal{D}^{\mathfrak{s},\mathfrak{c},k\hat{z}}_{\mathfrak{s}',k'}$

For concreteness, we now focus on particle plane-wave states which approach the dyon along the $\hat{\boldsymbol{z}}$ axis. As is usually done in scattering problems, we will determine the coefficients $\mathcal{D}^{\mathfrak{s},\mathfrak{c},k\hat{z}}_{\mathfrak{s}',k'}$ by matching terms containing incoming spherical waves on the left side of (D.2.9) to terms with $j = 0$ incoming spherical waves on the right side of this equation, at large distances from the dyon. To do this, we first isolate the incoming terms in $U_{\mathfrak{s},\mathfrak{c},k\hat{z}}$ and find their asymptotic form. Using the plane wave expansion and the asymptotic form of spherical Bessel functions, we get:

$$\left(U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)\right)_{\text{inc.}} \underset{r\to\infty}{\sim} -\mathcal{N}\frac{e^{-ikr}}{2ikr}\, e^{-i(k-\mathfrak{s}\,ev/2)t} e^{i\Phi_U^{\mathfrak{s},\mathfrak{c}}(x)} \xi_{\mathfrak{s}} \otimes \nu_{\mathfrak{c}}^U \sum_{\ell=0}^{\infty}(2\ell+1)(-1)^\ell P_\ell(\cos\theta), \tag{D.2.10}$$

where the subscript 'inc.' refers to only those terms in $U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)$ which include spherical waves that are incident on the dyon. This expression can be simplified further by noticing that $(-1)^\ell = P_\ell(-1)$ and using the Legendre polynomial completeness relation $\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(-1)P_\ell(x) = 2\delta(1+x)$

$$\left(U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)\right)_{\text{inc.}} \underset{r\to\infty}{\sim} -\mathcal{N}\frac{e^{-ikr}}{ikr}\, e^{-i(k-\mathfrak{s}\,ev/2)t}\, (r/r_0)^{i\mathfrak{s}eQ/2}\, e^{i\varphi_U^{\mathfrak{s},\mathfrak{c}}(\theta)} \xi_{\mathfrak{s}} \otimes \nu_{\mathfrak{c}}^U \delta(1+\cos\theta), \tag{D.2.11}$$

where we now specialize to the R_- region and rewrite $e^{i\Phi_U^{\mathfrak{s},\mathfrak{c}}(x)}$ as $(r/r_0)^{i\mathfrak{s}eQ/2}\, e^{i\varphi_U^{\mathfrak{s},\mathfrak{c}}(\theta)}$ in this region[4]

We can expand the top and bottom two components of each Dirac spinor in $U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)$ in terms of eigensections of the total angular momentum and its third component (J^2, J_z)

$$\begin{aligned}\left(U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)\right)_{\text{inc.}} \underset{r\to\infty}{\sim} \; & -\mathcal{N}\frac{e^{-ikr}}{ikr}\, e^{-i(k-\mathfrak{s}\,ev/2)t}\, (r/r_0)^{i\mathfrak{s}eQ/2}\, \xi_{\mathfrak{s}} \\ & \otimes \left[\begin{pmatrix} A^{\mathfrak{c}}_{0,\mathfrak{s}}\eta_{\mathfrak{s}}(\theta,\phi) \\ B^{\mathfrak{c}}_{0,\mathfrak{s}}\eta_{\mathfrak{s}}(\theta,\phi)\end{pmatrix} + \sum_{j>0}^{\infty}\sum_{j_z=-j}^{j}\sum_{i=1}^{2} \begin{pmatrix} A^{\mathfrak{c},(i)}_{j,j_z,\mathfrak{s}}\eta^{(i)}_{j,j_z,\mathfrak{s}}(\theta,\phi) \\ B^{\mathfrak{c},(i)}_{j,j_z,\mathfrak{s}}\eta^{(i)}_{j,j_z,\mathfrak{s}}(\theta,\phi)\end{pmatrix} \right],\end{aligned} \tag{D.2.12}$$

where $\eta_{\mathfrak{s}}(\theta,\phi)$ are defined as in equation (2.1.11) and the sentence below it and the higher partial wave angular momentum eigensections $\eta^{(1)}_{j,j_z,\mathfrak{s}}(\theta,\phi)$, $\eta^{(2)}_{j,j_z,\mathfrak{s}}(\theta,\phi)$ are given by the eigensections $\phi^{(1)}_{j,j_z}(\theta,\phi)$, $\phi^{(2)}_{j,j_z}(\theta,\phi)$ from [77] respectively, with an implied 'monopole strength' of $q = \frac{\mathfrak{s}}{2}$. We

[4] We choose the ϕ-dependence this way since the plane-wave travels along the $+\hat{\boldsymbol{z}}$ axis and so is incident on the dyon in the R_- region.

find the coefficients of interest $A^{\mathfrak{c}}_{0,\mathfrak{s}}, B^{\mathfrak{c}}_{0,\mathfrak{s}}$ by multiplying the above equation with $\xi^\dagger_{\mathfrak{s}} \otimes \left(\eta^\dagger_{\mathfrak{s}}(\theta,\phi)\ 0\right)$, $\xi^\dagger_{\mathfrak{s}} \otimes \left(0\ \eta^\dagger_{\mathfrak{s}}(\theta,\phi)\right)$ and performing the angular integrals. This gives

$$A^{\mathfrak{c}}_{0,\mathfrak{s}} = \delta_{\mathfrak{c},+} \int \mathrm{d}^2\Omega\, \eta^\dagger_{\mathfrak{s}}(\theta,\phi) \begin{pmatrix} 0 \\ e^{i\varphi^{\mathfrak{s}\mathfrak{c}}_U(\theta)} \end{pmatrix} \delta(1+\cos\theta) = \delta_{\mathfrak{c},+}\delta_{\mathfrak{s},+} e^{i\varphi^{\mathfrak{s}\mathfrak{c}}_U(\pi)}\sqrt{\pi}, \tag{D.2.13}$$

and

$$B^{\mathfrak{c}}_{0,\mathfrak{s}} = \delta_{\mathfrak{c},-} \int \mathrm{d}^2\Omega\, \eta^\dagger_{\mathfrak{s}}(\theta,\phi) \begin{pmatrix} e^{i\varphi^{\mathfrak{s}\mathfrak{c}}_U(\theta)} \\ 0 \end{pmatrix} \delta(1+\cos\theta) = -\delta_{\mathfrak{c},-}\delta_{\mathfrak{s},-} e^{i\varphi^{\mathfrak{s}\mathfrak{c}}_U(\pi)}\sqrt{\pi}, \tag{D.2.14}$$

with $\eta_{\mathfrak{s}}(\theta,\phi)$ given by (2.1.11) and where we use the fact that $\eta_{\mathfrak{s}}(\theta,\phi), \eta^{(1)}_{j,j_z,\mathfrak{s}}(\theta,\phi), \eta^{(2)}_{j,j_z,\mathfrak{s}}(\theta,\phi)$ are orthonormal. Using (D.2.13), (D.2.14) in equation (D.2.12) shows that

$$(U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x))_{\text{inc.}} \underset{r\to\infty}{\sim} \mathcal{N}\, \frac{i\sqrt{\pi}}{k} e^{i\varphi^{\mathfrak{s},\mathfrak{c}}_U(\pi)} \left(\delta_{\mathfrak{c},+}\delta_{\mathfrak{s},+}\, u^{\text{in}}_{+,k}(x) - \delta_{\mathfrak{c},-}\delta_{\mathfrak{s},-}\, u^{\text{in}}_{-,k}(x)\right)_{\text{inc.}} + (F(x))_{\text{inc.}}, \tag{D.2.15}$$

and so also that

$$U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x) \underset{r\to\infty}{\sim} \mathcal{N}\, \frac{i\sqrt{\pi}}{k} e^{i\varphi^{\mathfrak{s},\mathfrak{c}}_U(\pi)} \left(\delta_{\mathfrak{c},+}\delta_{\mathfrak{s},+}\, u^{\text{in}}_{+,k}(x) - \delta_{\mathfrak{c},-}\delta_{\mathfrak{s},-}\, u^{\text{in}}_{-,k}(x)\right) + F(x), \tag{D.2.16}$$

where $F(x)$ has no projection onto the S-wave. Note that the S-wave projection vanishes unless $\mathfrak{c}\mathfrak{s} = +1$, as expected since S-wave states satisfy $\mathfrak{h} = \mathfrak{c}\mathfrak{s}$ and we match $\mathfrak{h} = +1$ spherical waves on each side of the above equation. The coefficients $\mathcal{D}^{\mathfrak{s},\mathfrak{c},k\hat{z}}_{\mathfrak{s}',k'}$ are then:

$$\mathcal{D}^{\mathfrak{s},\mathfrak{c},k\hat{z}}_{\mathfrak{s}',k'} = \frac{i\mathcal{N}}{\sqrt{4\pi}k} e^{i\varphi^{\mathfrak{s},\mathfrak{c}}_U(\pi)} \delta(k-k')\delta_{\mathfrak{s}\mathfrak{s}'} \left(\delta_{\mathfrak{c},+}\delta_{\mathfrak{s},+} - \delta_{\mathfrak{c},-}\delta_{\mathfrak{s},-}\right). \tag{D.2.17}$$

S-wave scattering cross sections

We can now calculate the probability for a particle in a plane-wave state to be found in the S-wave, which we denote p_s in the main text. Since the probability of being in a specific momentum eigenstate tends to zero in the continuum limit, we compute p_s for a state described by a phase-space distribution function $\rho(\boldsymbol{k})$ which is normalized so that $\mathrm{dn} = \rho(\boldsymbol{k})\,\mathcal{V}\,\mathrm{d}^3k/(2\pi)^3$ is the number of momentum states in a volume d^3k around momentum $\boldsymbol{k} = k\,\hat{\boldsymbol{z}}$. For such an initial state (with electric charge $\mathfrak{s}$ and 4D chirality $\mathfrak{c}$), p_s is given by

$$\begin{aligned} p_s &= \left[\sum_{\mathfrak{s}'=\pm}\int_0^\infty \left|\langle 0_{\rm in}|\, a^{\rm in}_{\mathfrak{s}',k'}(a_{\mathfrak{s},\mathfrak{c},k\hat{z}})^\star\,|0\rangle\right|^2\,{\rm d}k'\right]\rho(\boldsymbol{k})\,{\rm d}^3k \\ &= \left[\sum_{\mathfrak{s}'=\pm}\int_0^\infty \left|\langle 0_{\rm in}|\, a^{\rm in}_{\mathfrak{s}',k'}(a^{\rm in}_{\mathfrak{s},k})^\star\,|0_{\rm in}\rangle\right|^2\,{\rm d}k'\right]\delta_{\mathfrak{s},\mathfrak{c}}\,\mathcal{N}^2/(4\pi k^2)\,\rho(\boldsymbol{k})\,{\rm d}^3k, \end{aligned} \tag{D.2.18}$$

which evaluates to

$$p_s = \delta_{\mathfrak{s},\mathfrak{c}}\,\frac{\mathcal{N}^2}{4\pi k^2}\,\frac{L}{\pi}\rho(\boldsymbol{k})\,{\rm d}^3k. \tag{D.2.19}$$

In the above, the factor of L/π is introduced to regulate the momentum-conserving delta function and is cancelled in the final cross section result by a similar factor in the 2D scattering rate. The differential S-wave scattering rates ${\rm d}\Gamma_2$ and ${\rm d}\Gamma_{\rm inclusive}$ can be calculated from the perturbative amplitudes listed in §3 and single-particle scattering amplitudes given in §4 respectively, both of which can be written as $\mathcal{A} = \mathcal{M}\,\delta(E_f - E_i)$. Using these amplitudes in Fermi's golden rule gives

$${\rm d}\Gamma = 2\pi|i\mathcal{M}/(2L)|^2\delta(E_f - E_i)\frac{{\rm d}k'\,L}{\pi} = |\mathcal{M}|^2\delta(E_f - E_i)\,{\rm d}k'/(2L), \tag{D.2.20}$$

for both the interaction picture and inclusive Heisenberg picture differential rate ${\rm d}\Gamma_2$ and ${\rm d}\Gamma_{\rm inclusive}$, respectively. The factor of $1/(2L)$ in the above differential rate again comes about because we cannot choose the initial S-wave state to have a specific momentum when working in the continuum limit. In principle we could remove this factor in the same way as above, by redefining our initial state in terms of a distribution function in momentum space, however we do not do so here as the definition of the initial S-wave state is purely an intermediate step in the cross section calculation. Exclusive 2D rates, $\Gamma_{\rm exclusive}$, can be calculated similarly with *e.g.* differential rates for processes in which no pairs are produced given by (D.2.20) where $\mathcal{A}$ is an exclusive amplitude such as (4.1.24) - (4.1.27).

The 4D differential rates for the scattering processes described here are then independent of the system length $2L$, as advertised, and become

$${\rm d}\Gamma_{4D} = \frac{\mathcal{N}^2}{8\pi^2 k^2}\,\delta_{\mathfrak{s},\mathfrak{c}}\,|\mathcal{M}|^2\delta(E_f - E_i)\,{\rm d}k'\,\rho(\boldsymbol{k})\,{\rm d}^3k. \tag{D.2.21}$$

The differential cross section is obtained from ${\rm d}\Gamma_{4D}$ by factoring out the flux of initial particles, given by $\mathfrak{F}_i = \mathrm{d}n\, v_{\mathrm{rel}} = \mathrm{d}n$, where $v_{\mathrm{rel}} = 1$ is the relative velocity between the dyon and incident fermions and $\mathrm{d}n = \mathcal{N}^2[\rho(\boldsymbol{k})/(2\pi)^3]\mathrm{d}^3k$ is the number density of incident fermions. We define $\mathrm{d}n$ as the product of $\mathrm{dn}/\mathcal{V}$, the density of states, and $i\overline{U}_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x)\gamma^0 U_{\mathfrak{s},\mathfrak{c},k\hat{z}}(x) = \mathcal{N}^2$, the contribution of a plane wave state to the fermion number density. Finally, we get that the 4D single-particle S-wave differential cross sections $\mathrm{d}\sigma_s$ simplify to

$$\mathrm{d}\sigma_s[f_{\mathfrak{s},\mathfrak{c}}(k) \to f_{\mathfrak{s}'}(k')] = \frac{\mathrm{d}\Gamma_{4D}}{\mathfrak{F}_i} = \frac{p_s}{\mathfrak{F}_i}\mathrm{d}\Gamma = \frac{\pi}{k^2}\delta_{\mathfrak{s},\mathfrak{c}}|\mathcal{M}|^2\delta(E_f - E_i)\,\mathrm{d}k', \tag{D.2.22}$$

as we claim in the main text.

Appendix E

The *in* and *out* vacuum

The Bogoliubov relations imply that the *in* and *out* vacua, defined as

$$\boldsymbol{a}^{\text{in}}_{\mathfrak{s},k}\,|0_{\text{in}}\rangle = \overline{\boldsymbol{a}}^{\text{in}}_{\mathfrak{s},k}\,|0_{\text{in}}\rangle = 0, \quad \boldsymbol{a}^{\text{out}}_{\mathfrak{s},k}\,|0_{\text{out}}\rangle = \overline{\boldsymbol{a}}^{\text{out}}_{\mathfrak{s},k}\,|0_{\text{out}}\rangle = 0, \tag{E.0.1}$$

do not necessarily coincide due to the possibility of pair production. This appendix shows how to expand the *in* vacuum in terms *out* states and vice versa. We also show how these expansions can be used to evaluate *inclusive* scattering observables, such as the cross sections given in §4.

E.1 Expansion of *in* vacuum in terms of *out* states

We first note that all *out* particles and antiparticles with momentum $k > ev$ annihilate the *in* vacuum, that is

$$\boldsymbol{a}^{\text{out}}_{\mathfrak{s},k}\,|0_{\text{in}}\rangle = \overline{\boldsymbol{a}}^{\text{out}}_{\mathfrak{s},k}\,|0_{\text{in}}\rangle = 0, \quad \text{for} \quad k > ev. \tag{E.1.1}$$

For smaller momenta, the above equation remains satisfied only for negatively charged particles and antiparticles

$$\boldsymbol{a}^{\text{out}}_{-,k}\,|0_{\text{in}}\rangle = \overline{\boldsymbol{a}}^{\text{out}}_{+,k}\,|0_{\text{in}}\rangle = 0, \quad \text{for} \quad 0 < k < ev, \tag{E.1.2}$$

meaning the *in* vacuum only contains positively charged particles and antiparticles with momenta in the range $0 < k < ev$.

Since the total vacuum can be written as a tensor product over the vacua for each momentum, $|0_{\rm in}\rangle = \prod_{k>0} |0^k_{\rm in}\rangle$, the expansion of the total vacuum in terms of *out* states can be done on a mode-by-mode basis. As explained above, for momenta $k > ev$ the single-momentum vacuum must be equal to the corresponding *out* vacuum up to a phase

$$|0^k_{\rm in}\rangle = e^{i\delta_k}\,|0^k_{\rm out}\rangle\,, \quad \text{for} \quad k > ev \tag{E.1.3}$$

where δ_k is an arbitrary phase. For momenta $k < ev$, it is convenient to define the single-mode vacuum $|0^k_{\rm in}\rangle$ in the following way

$$\boldsymbol{a}^{\rm in}_{+,k}\,|0^k_{\rm in}\rangle = \overline{\boldsymbol{a}}^{\rm in}_{+,k}\,|0^k_{\rm in}\rangle = 0 \quad \text{and} \quad \boldsymbol{a}^{\rm in}_{-,-k+ev}\,|0^k_{\rm in}\rangle = \overline{\boldsymbol{a}}^{\rm in}_{-,-k+ev}\,|0^k_{\rm in}\rangle = 0, \tag{E.1.4}$$

as opposed to $\boldsymbol{a}^{\rm in}_{\mathfrak{s},k}\,|0^k_{\rm in}\rangle = \overline{\boldsymbol{a}}^{\rm in}_{\mathfrak{s},k}\,|0^k_{\rm in}\rangle = 0$, since the Bogoliubov relations (which impose energy conservation) relate $\mathfrak{s} = +$ particle operators of momentum k to $\mathfrak{s} = -$ antiparticle operators of momentum $-k+ev$. With this definition, the most general form of the single-mode vacuum is:

$$|0^k_{\rm in}\rangle = \Big[B_{0;0} + B_{1;0}(\boldsymbol{a}^{\rm out}_{+,k})^\star + B_{0;1}(\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star + B_{1;1}(\boldsymbol{a}^{\rm out}_{+,k})^\star(\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\Big]\,|0^k_{\rm out}\rangle\,. \tag{E.1.5}$$

The action of any *out* operator on the single-mode vacuum $|0^k_{\rm in}\rangle$ can be calculated in more than one way: Directly, by using the above expansion in terms of *out* states, or by rewriting the operator using the Bogoliubov relations and then acting on the state. This can be used to determine the $B_{i;j}$ coefficients in (E.1.5), as we now show. Consider the state $\boldsymbol{a}^{\rm out}_{+,k}\,|0^k_{\rm in}\rangle$, which is equal to:

$$\boldsymbol{a}^{\rm out}_{+,k}\,|0^k_{\rm in}\rangle = \Big[B_{1;0} + B_{1;1}(\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\Big]\,|0^k_{\rm out}\rangle\,, \tag{E.1.6}$$

but can also be rewritten as

$$
\begin{aligned}
&\boldsymbol{a}^{\text{out}}_{+,k}\,|0^k_{\text{in}}\rangle = \mathcal{T}^{+-}_{\text{in}}\,e^{i\mathfrak{a}}\,(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k+ev})^\star\,|0^k_{\text{in}}\rangle \\
=\;& \mathcal{T}^{+-}_{\text{in}}\,e^{i\mathfrak{a}}\,\Big(\mathcal{T}^{--}_{\text{out}}\,(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star + \mathcal{T}^{-+}_{\text{out}}\,e^{-i\mathfrak{a}}\boldsymbol{a}^{\text{out}}_{+,k}\Big) \\
&\qquad\qquad \times\Big[B_{0;0} + B_{1;0}(\boldsymbol{a}^{\text{out}}_{+,k})^\star + B_{0;1}(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star + B_{1;1}(\boldsymbol{a}^{\text{out}}_{+,k})^\star(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star\Big]\,|0^k_{\text{out}}\rangle \\
=\;& \mathcal{T}^{+-}_{\text{in}}\mathcal{T}^{--}_{\text{out}}\,e^{i\mathfrak{a}}\Big[B_{0;0}(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star - B_{1;0}(\boldsymbol{a}^{\text{out}}_{+,k})^\star(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star\Big]\,|0^k_{\text{out}}\rangle \\
&\qquad\qquad\qquad + \mathcal{T}^{+-}_{\text{in}}\mathcal{T}^{-+}_{\text{out}}\Big[B_{1;0} + B_{1;1}(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star\Big]\,|0^k_{\text{out}}\rangle\,.
\end{aligned}
\tag{E.1.7}
$$

In the above, we use the Bogoliubov relations first to rewrite $\boldsymbol{a}^{\text{out}}_{+,k}$ and then again to rewrite $(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k+ev})^\star$. Comparing (E.1.6) and (E.1.7) shows that, when $\mathcal{T}^{+-}_{\text{in}}\mathcal{T}^{--}_{\text{out}} \neq 0$[1]

$$
B_{1;0} = 0 \quad \text{and} \quad \frac{B_{0;0}}{B_{1;1}} = \frac{1 - \mathcal{T}^{+-}_{\text{in}}\mathcal{T}^{-+}_{\text{out}}}{\mathcal{T}^{+-}_{\text{in}}\mathcal{T}^{--}_{\text{out}}}\,e^{-i\mathfrak{a}} = \frac{\mathcal{T}^{--}_{\text{in}}}{\mathcal{T}^{+-}_{\text{in}}}\,e^{-i\mathfrak{a}},
\tag{E.1.8}
$$

where in the last line we rewrite the $\mathcal{T}_{\text{out}}$ amplitudes in terms of $\mathcal{T}_{\text{in}}$ amplitudes and use (2.2.34). Similarly, the state $\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev}\,|0^k_{\text{in}}\rangle$ is equal to:

$$
\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev}\,|0^k_{\text{in}}\rangle = \Big[B_{0;1} - B_{1;1}(\boldsymbol{a}^{\text{out}}_{+,k})^\star\Big]\,|0^k_{\text{out}}\rangle\,,
\tag{E.1.9}
$$

but can also be rewritten as

$$
\begin{aligned}
&\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev}\,|0^k_{\text{in}}\rangle = \mathcal{T}^{-+\,*}_{\text{in}}\,e^{i\mathfrak{a}}\,(\boldsymbol{a}^{\text{in}}_{+,k})^\star\,|0^k_{\text{in}}\rangle \\
=\;& \mathcal{T}^{-+\,*}_{\text{in}}\,e^{i\mathfrak{a}}\,\Big(\mathcal{T}^{++\,*}_{\text{out}}(\boldsymbol{a}^{\text{out}}_{+,k})^\star + \mathcal{T}^{+-\,*}_{\text{out}}\,e^{-i\mathfrak{a}}\,\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev}\Big) \\
&\qquad\qquad \times\Big[B_{0;0} + B_{0;1}(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star + B_{1;1}(\boldsymbol{a}^{\text{out}}_{+,k})^\star(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star\Big]\,|0^k_{\text{out}}\rangle \\
=\;& \mathcal{T}^{-+\,*}_{\text{in}}\mathcal{T}^{++\,*}_{\text{out}}\,e^{i\mathfrak{a}}\Big[B_{0;0}(\boldsymbol{a}^{\text{out}}_{+,k})^\star + B_{0;1}(\boldsymbol{a}^{\text{out}}_{+,k})^\star(\overline{\boldsymbol{a}}^{\text{out}}_{-,-k+ev})^\star\Big]\,|0^k_{\text{out}}\rangle \\
&\qquad\qquad\qquad + \mathcal{T}^{-+\,*}_{\text{in}}\mathcal{T}^{+-\,*}_{\text{out}}\Big[B_{0;1} - B_{1;1}(\boldsymbol{a}^{\text{out}}_{+,k})^\star\Big]\,|0^k_{\text{out}}\rangle\,,
\end{aligned}
\tag{E.1.10}
$$

which implies that

$$
B_{0;1} = 0,
\tag{E.1.11}
$$

when compared to (E.1.9). The remaining coefficients $B_{0;0}$ and $B_{1;1}$ can be determined up to a

[1] This condition also implies that $B_{1;1} \neq 0$, otherwise $B_{1;0} = 0$ would imply $\boldsymbol{a}^{\text{out}}_{+,k}\,|0^k_{\text{in}}\rangle = 0$ which is not consistent since $\boldsymbol{a}^{\text{out}}_{+,k}\,|0^k_{\text{in}}\rangle = \mathcal{T}^{+-}_{\text{in}}\,e^{i\mathfrak{a}}\,(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k+ev})^\star\,|0^k_{\text{in}}\rangle \neq 0$.

phase, by imposing the normalization condition $\langle 0^k_{\rm in}|0^k_{\rm in}\rangle = 1$ and using (E.1.8). For $\mathcal{T}^{+-}_{\rm in}, \mathcal{T}^{--}_{\rm in} \neq 0$, we then get:

$$|0^k_{\rm in}\rangle = e^{i\delta_k}\left[\mathcal{T}^{--}_{\rm in} + \mathcal{T}^{+-}_{\rm in} e^{i\mathfrak{a}} (\boldsymbol{a}^{\rm out}_{+,k})^\star (\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\right]|0^k_{\rm out}\rangle\,, \tag{E.1.12}$$

for momenta $k < ev$, where δ_k is an arbitrary phase. When $\mathcal{T}^{+-}_{\rm in} = 0$ or $\mathcal{T}^{--}_{\rm in} = 0$, we define the expansion of $|0^k_{\rm in}\rangle$ in terms of *out* states as the $\mathcal{T}^{+-}_{\rm in} \to 0$ and $\mathcal{T}^{--}_{\rm in} \to 0$ limit of the above equation respectively, where the surviving amplitude is necessarily a phase.

For general momentum k and choice of $\mathcal{T}_{\rm in}$ amplitudes, the single-mode vacuum is:

$$|0^k_{\rm in}\rangle \;=\; e^{i\delta_k}\left\{\Theta(k-ev) + \Theta(-k+ev)\left[\mathcal{T}^{--}_{\rm in} + \mathcal{T}^{+-}_{\rm in} e^{i\mathfrak{a}} (\boldsymbol{a}^{\rm out}_{+,k})^\star (\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\right]\right\}|0^k_{\rm out}\rangle\,. \tag{E.1.13}$$

Finally, we can write the full *in* vacuum in the *out* basis as

$$\begin{aligned} |0_{\rm in}\rangle \;&=\; \prod_{k>0}|0^k_{\rm in}\rangle = \left(\prod_{k>ev}|0^k_{\rm in}\rangle\right)\left(\prod_{k>0}^{k<ev}|0^k_{\rm in}\rangle\right) \\ &=\; e^{i\delta}\left(\prod_{k>0}^{k<ev}\left[\mathcal{T}^{--}_{\rm in} + \mathcal{T}^{+-}_{\rm in} e^{i\mathfrak{a}} (\boldsymbol{a}^{\rm out}_{+,k})^\star (\overline{\boldsymbol{a}}^{\rm out}_{-,-k+ev})^\star\right]\right)|0_{\rm out}\rangle\,, \end{aligned} \tag{E.1.14}$$

where $\delta \coloneqq \sum\limits_{k>0}\delta_k$ is an arbitrary phase which we set to 0 in the main text, as it is not observable[2]. The product over all $k < ev$ modes evaluates to

$$\begin{aligned} |0_{\rm in}\rangle \;=\; e^{i\delta}\sum_{n=0}^{\frac{ev}{\Delta k}}\frac{1}{n!}(\mathcal{T}^{--}_{\rm in})^{\frac{ev}{\Delta k}-n}(\mathcal{T}^{+-}_{\rm in})^n\, e^{in\mathfrak{a}} \\ \times\left[\sum_{k_1=0}^{ev}\ldots\sum_{k_n=0}^{ev}(\boldsymbol{a}^{\rm out}_{+,k_1})^\star(\overline{\boldsymbol{a}}^{\rm out}_{-,-k_1+ev})^\star\ldots(\boldsymbol{a}^{\rm out}_{+,k_n})^\star(\overline{\boldsymbol{a}}^{\rm out}_{-,-k_n+ev})^\star\right]|0_{\rm out}\rangle \end{aligned} \tag{E.1.15}$$

The full *in* vacuum is normalized, since the coefficients $B_{i;j}$ were chosen such that the expansion of each single-mode *in* vacuum corresponds to a normalized state.

[2] The expansion of *in* states in terms of the *out* basis is only relevant when evaluating transition amplitudes. Since all amplitudes include the $e^{i\delta}$ factor, no interference experiment can be constructed to measure δ.

Expansion of *out* vacuum in terms of *in* states

The above procedure can be repeated for the *out* vacuum. We similarly get:

$$\begin{aligned}|0_{\text{out}}\rangle \;&=\; e^{-i\delta}\sum_{n=0}^{\frac{ev}{\Delta k}}\frac{1}{n!}(\mathcal{T}^{--}_{\text{out}})^{\frac{ev}{\Delta k}-n}(\mathcal{T}^{+-}_{\text{out}})^{n}\,e^{in\mathfrak{a}}\\&\qquad\times\Big[\sum_{k_1=0}^{ev}\cdots\sum_{k_n=0}^{ev}(\boldsymbol{a}^{\text{in}}_{+,k_1})^{\star}(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k_1+ev})^{\star}\ldots(\boldsymbol{a}^{\text{in}}_{+,k_n})^{\star}(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k_n+ev})^{\star}\Big]\,|0_{\text{in}}\rangle\\&=\; e^{-i\delta}\sum_{n=0}^{\frac{ev}{\Delta k}}\frac{1}{n!}(\mathcal{T}^{--\,*}_{\text{in}})^{\frac{ev}{\Delta k}-n}(\mathcal{T}^{-+\,*}_{\text{in}})^{n}\,e^{in\mathfrak{a}}\\&\qquad\times\Big[\sum_{k_1=0}^{ev}\cdots\sum_{k_n=0}^{ev}(\boldsymbol{a}^{\text{in}}_{+,k_1})^{\star}(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k_1+ev})^{\star}\ldots(\boldsymbol{a}^{\text{in}}_{+,k_n})^{\star}(\overline{\boldsymbol{a}}^{\text{in}}_{-,-k_n+ev})^{\star}\Big]\,|0_{\text{in}}\rangle\end{aligned}\tag{E.1.16}$$

E.2 Calculating inclusive observables

In the main text, we remark that the scattering observables of physical interest are often *inclusive* observables, for which the number of pairs produced by the dyon is unmeasured. We now show how such observables can be calculated, focusing on the inclusive S-wave scattering cross sections of §4. For convenience, we work in discrete normalization as in the rest of this appendix.

To start, we note that in most cases of interest in the main text, *exclusive* amplitudes $\boldsymbol{\mathcal{A}}_n$ can be written as

$$\boldsymbol{\mathcal{A}}_n = \boldsymbol{\mathcal{A}}_{sc}\,\boldsymbol{\mathcal{A}}^{n}_{\text{pair}},\tag{E.2.1}$$

that is they factorize into a single-particle scattering amplitude $\boldsymbol{\mathcal{A}}_{sc}$ and an amplitude to produce n pairs, $\boldsymbol{\mathcal{A}}^{n}_{\text{pair}}$. As a result, the exclusive probability for the process to happen, P_n, will be the product of the probability for the single-particle scattering event, p_{sc}, and the probability to produce n pairs, P^{n}_{pair},

$$P_n = p_{sc}\,P^{n}_{\text{pair}}.\tag{E.2.2}$$

The inclusive probability for the single-particle scattering process is given by

$$p = p_{sc}\,p_{\text{vac}} = p_{sc}\sum_{n=0}^{N}P^{n}_{\text{pair}},\tag{E.2.3}$$

where $N := \frac{ev}{\Delta k}$ is the maximum number of *out* pairs in the *in* vacuum, while $p_{\rm vac}$ is defined through the above equation and is equal to

$$\begin{aligned} p_{\rm vac} \;=\; & |\, \langle 0_{\rm out} | 0_{\rm in} \rangle \,|^2 + \sum_{q_1=0}^{ev} |\, \langle 0_{\rm out} | \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\, \boldsymbol{a}^{\rm out}_{+,q_1} | 0_{\rm in} \rangle \,|^2 + \cdots \\ & + \frac{1}{n!} \sum_{q_1,\cdots q_n=0}^{ev} |\, \langle 0_{\rm out} | \overline{\boldsymbol{a}}^{\rm out}_{-,-q_n+ev}\, \boldsymbol{a}^{\rm out}_{+,q_n} \cdots \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\, \boldsymbol{a}^{\rm out}_{+,q_1} | 0_{\rm in} \rangle \,|^2 + \cdots \\ & + \frac{1}{N!} \sum_{q_1,\cdots q_N=0}^{ev} |\, \langle 0_{\rm out} | \overline{\boldsymbol{a}}^{\rm out}_{-,-q_N+ev}\, \boldsymbol{a}^{\rm out}_{+,q_N} \cdots \overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\, \boldsymbol{a}^{\rm out}_{+,q_1} | 0_{\rm in} \rangle \,|^2 . \end{aligned} \tag{E.2.4}$$

In the above, the combinatorial prefactors are added to ensure we only sum over distinct final states. Keeping in mind that *out* states with doubly-occupied pairs do not contribute to the momentum sums in (E.2.4), $p_{\rm vac}$ simplifies to

$$\begin{aligned} p_{\rm vac} \;=\; & \left(|\mathcal{T}^{--}_{\rm in}|^2 \right)^N + N \left(|\mathcal{T}^{--}_{\rm in}|^2 \right)^{N-1} |\mathcal{T}^{+-}_{\rm in}|^2 + \cdots \\ & + \frac{N!}{n!(N-n)!} \left(|\mathcal{T}^{--}_{\rm in}|^2 \right)^{N-n} \left(|\mathcal{T}^{+-}_{\rm in}|^2 \right)^{n} + \cdots + \left(|\mathcal{T}^{+-}_{\rm in}|^2 \right)^{N} \\ =\; & \left(|\mathcal{T}^{--}_{\rm in}|^2 + |\mathcal{T}^{+-}_{\rm in}|^2 \right)^N = 1. \end{aligned} \tag{E.2.5}$$

This shows that the inclusive probability for the scattering process described by $\boldsymbol{\mathcal{A}}_{sc}$ is

$$p = p_{sc}, \tag{E.2.6}$$

and so the corresponding inclusive rates and cross sections can be computed in the usual way, using only the single-particle amplitude $\boldsymbol{\mathcal{A}}_{sc}$ or its continuum-normalization counterpart $\mathcal{A}_{sc}$.

The previous argument can be used to justify the inclusive cross section formulas given in §4, for all but one of the processes considered in that section. The exception is the process described by the amplitude (4.1.27), when the momentum of the initial particle, k, belongs in the pair production range $k < ev$. In this case, the exclusive amplitude factorizes as follows

$$\begin{aligned}\boldsymbol{\mathcal{A}}_n &= \langle 0_{\rm out}|\, \overline{\boldsymbol{a}}^{\rm out}_{-,-q_n+ev}\boldsymbol{a}^{\rm out}_{+,q_n}...\overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\boldsymbol{a}^{\rm out}_{+,q_1}\, \boldsymbol{a}^{\rm out}_{+,k'}(\boldsymbol{a}^{\rm in}_{+,k})^\star\, |0_{\rm in}\rangle \\ &= \delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\left(\mathcal{T}^{--}_{\rm in}\right)^{N-(n+1)}\left(\mathcal{T}^{+-}_{\rm in}\right)^n e^{in\mathfrak{a}} \\ &= \delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\left[\boldsymbol{\mathcal{A}}^n_{\rm pair}/\mathcal{T}^{--}_{\rm in}\right], \end{aligned} \tag{E.2.7}$$

when $n < N$ and $k', q_1, \cdots, q_n$ are distinct momenta, and vanishes otherwise. The inclusive probability for the relevant single-particle process is given by

$$\begin{aligned} p \;=\; \sum_{k'}\Bigg[&|\,\langle 0_{\rm out}|0_{\rm in}\rangle\,|^2 + \sum_{q_1=0}^{ev}|\,\langle 0_{\rm out}|\overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\,\boldsymbol{a}^{\rm out}_{+,q_1}|0_{\rm in}\rangle\,|^2 + \cdots \\ &+\frac{1}{(N-1)!}\sum_{q_1,\cdots q_{N-1}=0}^{ev}|\,\langle 0_{\rm out}|\overline{\boldsymbol{a}}^{\rm out}_{-,-q_{N-1}+ev}\,\boldsymbol{a}^{\rm out}_{+,q_{N-1}}\cdots\overline{\boldsymbol{a}}^{\rm out}_{-,-q_1+ev}\,\boldsymbol{a}^{\rm out}_{+,q_1}|0_{\rm in}\rangle\,|^2\Bigg] \\ &\times|\mathcal{T}^{--}_{\rm in}|^{-2}\left|\delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\right|^2. \end{aligned} \tag{E.2.8}$$

To evaluate the above sums, note that the Bogoliubov relations impose that the momentum k' is equal to the momentum of the initial particle, k, and so lies in the pair production range $0 < k' < ev$. As one of the particle states in the pair production range is already occupied, there are now at most $N-1$ different momenta that each q_i can be equated to. The inclusive probability, p, is then equal to

$$\begin{aligned} p \;=\; \sum_{k'}\Bigg[&\left(|\mathcal{T}^{--}_{\rm in}|^2\right)^{N-1} + (N-1)\left(|\mathcal{T}^{--}_{\rm in}|^2\right)^{N-2}|\mathcal{T}^{+-}_{\rm in}|^2 + \cdots + \left(|\mathcal{T}^{+-}_{\rm in}|^2\right)^{N-1}\Bigg] \\ &\times\left|\delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\right|^2, \end{aligned} \tag{E.2.9}$$

which simplifies to

$$\begin{aligned} p \;&=\; \sum_{k'}\left(|\mathcal{T}^{--}_{\rm in}|^2 + |\mathcal{T}^{+-}_{\rm in}|^2\right)^{N-1}\left|\delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\right|^2 \\ &= \sum_{k'}\left|\delta_{kk'}\left[\mathcal{T}^{++}_{\rm in}\mathcal{T}^{--}_{\rm in} - \mathcal{T}^{-+}_{\rm in}\mathcal{T}^{+-}_{\rm in}\right]\right|^2. \end{aligned} \tag{E.2.10}$$

The last line of (E.2.10) shows that the inclusive cross section for this process can be calculated in the usual way, if the single-particle amplitude

$$\boldsymbol{\mathcal{A}}_{sc} = \delta_{kk'}\Big[\mathcal{T}_{\rm in}^{++}\mathcal{T}_{\rm in}^{--} - \mathcal{T}_{\rm in}^{-+}\mathcal{T}_{\rm in}^{+-}\Big], \tag{E.2.11}$$

or its continuum-normalization counterpart (4.1.34) (which is valid for all k), is used instead of exclusive amplitudes.

Appendix F

Bilinear currents

In this appendix, we show how one can regularize local fermion bilinear operators and compute the *in* and *out* vacuum expectation values of the fermion number, electric charge and axial currents, as well as the interaction picture interacting hamiltonian. We further derive the conservation (or non-conservation) equations satisfied by the fermionic currents and show how the boundary condition can be used to directly evaluate their radial components at $r = \epsilon$.

F.1 Vacuum expectation values of fermion bilinears

Writing

$$\boldsymbol{\chi}(x) = \sum_{\mathfrak{s}=\pm} \int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\pi}} \left[\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}(x)\; a^{\mathrm{d}}_{\mathfrak{s},k} + \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k}(x)\; (\overline{a}^{\mathrm{d}}_{\mathfrak{s},k})^{\star} \right] \tag{F.1.1}$$

where $\mathrm{d} = \{\mathrm{in}, \mathrm{out}\}$, we seek the expectation values of fermion bilinears like

$$\begin{aligned} \overline{\boldsymbol{\chi}} M \boldsymbol{\chi} \;=\; \sum_{\mathfrak{s},\mathfrak{s}'=\pm} \int_0^\infty \frac{\mathrm{d}k\,\mathrm{d}k'}{2\pi} & \Big[\overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k} M\, \mathfrak{u}^{\mathrm{d}}_{\mathfrak{s}',k'} \left((a^{\mathrm{d}}_{\mathfrak{s},k})^{\star}\, a^{\mathrm{d}}_{\mathfrak{s}',k'} \right) + \overline{\mathfrak{v}}^{\mathrm{d}}_{\mathfrak{s},k} M\, \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s}',k'} \left(\overline{a}^{\mathrm{d}}_{\mathfrak{s},k}\, (\overline{a}^{\mathrm{d}}_{\mathfrak{s}',k'})^{\star} \right) \\ & + \overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k} M\, \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s}',k'} \left((a^{\mathrm{d}}_{\mathfrak{s},k})^{\star}\, (\overline{a}^{\mathrm{d}}_{\mathfrak{s}',k'})^{\star} \right) + \overline{\mathfrak{v}}^{\mathrm{d}}_{\mathfrak{s},k} M\, \mathfrak{u}^{\mathrm{d}}_{\mathfrak{s}',k'} \left(\overline{a}^{\mathrm{d}}_{\mathfrak{s},k}\, a^{\mathrm{d}}_{\mathfrak{s}',k'} \right) \Big] \end{aligned} \tag{F.1.2}$$

where we use the operator-ordering notation for fermions: $(AB) := \frac{1}{2}[A, B]$ that ensures classically hermitian expressions remain hermitian.

The *in* and *out* vacuum expectation values of local bilinear operators are formally given by

$$\begin{aligned}\langle 0_{\mathrm{d}}|\,\overline{\boldsymbol{\chi}}M\boldsymbol{\chi}\,|0_{\mathrm{d}}\rangle &= \sum_{\mathfrak{s},\mathfrak{s}'=\pm}\int_0^\infty \frac{\mathrm{d}k\,\mathrm{d}k'}{4\pi}\Big[-\overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k}M\,\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s}',k'}\,\langle 0_{\mathrm{d}}|\,a^{\mathrm{d}}_{\mathfrak{s}',k'}(a^{\mathrm{d}}_{\mathfrak{s},k})^\star\,|0_{\mathrm{d}}\rangle + \overline{\mathfrak{v}}^{\mathrm{d}}_{\mathfrak{s},k}M\,\mathfrak{v}^{\mathrm{d}}_{\mathfrak{s}',k'}\,\langle 0_{\mathrm{d}}|\,\overline{a}^{\mathrm{d}}_{\mathfrak{s},k}(\overline{a}^{\mathrm{d}}_{\mathfrak{s}',k'})^\star\,|0_{\mathrm{d}}\rangle\Big]\\ &= \sum_{\mathfrak{s}=\pm}\int_0^\infty \frac{\mathrm{d}k}{4\pi}\Big[-\overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k}M\,\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k} + \overline{\mathfrak{v}}^{\mathrm{d}}_{\mathfrak{s},k}M\,\mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k}\Big]. \qquad \text{(F.1.3)}\end{aligned}$$

At face value, the above expectation values vanish for any matrix M acting in spin and isospin space, since $\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}(x) = \mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},-k}(x)$ and so $\overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k}(x)M\,\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}(x) = \overline{\mathfrak{v}}^{\mathrm{d}}_{\mathfrak{s},k}(x)M\,\mathfrak{v}^{\mathrm{d}}_{\mathfrak{s},k}(x)$ for both *in* and *out* modes[1]. This turns out not to be the case, since the presence of anomalies means more care must be taken when evaluating local operators. To this end, we regularize the fermion bilinear operator $\overline{\boldsymbol{\chi}}(x)M\,\boldsymbol{\chi}(x)$ by evaluating $\boldsymbol{\chi},\overline{\boldsymbol{\chi}}$ at slightly different points, namely $\overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)M\,\boldsymbol{\chi}(r-\varepsilon/2,t)$[2], and take the limit $\varepsilon\to 0$ after calculating any matrix elements of interest. Note that the point-split operator $\overline{\boldsymbol{\chi}}(r+\varepsilon/2,t)M\,\boldsymbol{\chi}(r-\varepsilon/2,t)$ is gauge invariant without the addition of a Wilson line, since we point split only in the radial direction along which the gauge field vanishes, $\mathcal{A}_r(x)=0$. More explicitly, we calculate vacuum expectation values through

$$\langle 0_{\mathrm{d}}|\,\overline{\boldsymbol{\chi}}\,M\boldsymbol{\chi}\,|0_{\mathrm{d}}\rangle = \lim_{\varepsilon\to 0}\Big[\langle 0_{\mathrm{d}}|\,\overline{\chi}_+(x_+)\,M\,\chi_+(x_-)\,|0_{\mathrm{d}}\rangle + \langle 0_{\mathrm{d}}|\,\overline{\chi}_-(x_+)\,M\,\chi_-(x_-)\,|0_{\mathrm{d}}\rangle\Big], \qquad \text{(F.1.4)}$$

where $x_\pm \coloneqq (r_\pm,t) = (r\pm\frac{1}{2}\varepsilon,t)$. Applying this to the radial component of the axial current (for which $M = i\Gamma^1\Gamma_A$) gives the following contributions to the *in* vacuum expectation value:

$$\begin{aligned}\langle 0_{\mathrm{in}}|\,\overline{\chi}_+(x_+)\,i\Gamma^1\Gamma_A\,\chi_+(x_-)\,|0_{\mathrm{in}}\rangle &= \int_0^\infty \frac{\mathrm{d}k}{4\pi}\Big[-\overline{\mathfrak{u}}^{\mathrm{in}}_{+,k}(x_+)i\Gamma^1\Gamma_A\,\mathfrak{u}^{\mathrm{in}}_{+,k}(x_-) + \overline{\mathfrak{v}}^{\mathrm{in}}_{+,k}(x_+)i\Gamma^1\Gamma_A\,\mathfrak{v}^{\mathrm{in}}_{+,k}(x_-)\Big]\\ = \left[\left(-1-e^{iev\varepsilon}|\mathcal{T}^{-+}_{\mathrm{in}}|^2\right)\left(\frac{r_+}{r_-}\right)^{-\frac{ieQ}{2}} + |\mathcal{T}^{++}_{\mathrm{in}}|^2\left(\frac{r_+}{r_-}\right)^{\frac{ieQ}{2}}\right] &\int_0^\infty \frac{\mathrm{d}k}{4\pi}\left(e^{-ik\varepsilon}-e^{ik\varepsilon}\right), \qquad \text{(F.1.5)}\end{aligned}$$

as well as

$$\begin{aligned}\langle 0_{\mathrm{in}}|\,\overline{\chi}_-(x_+)\,i\Gamma^1\Gamma_A\,\chi_-(x_-)\,|0_{\mathrm{in}}\rangle &= \int_0^\infty \frac{\mathrm{d}k}{4\pi}\Big[-\overline{\mathfrak{u}}^{\mathrm{in}}_{-,k}(x_+)i\Gamma^1\Gamma_A\,\mathfrak{u}^{\mathrm{in}}_{-,k}(x_-) + \overline{\mathfrak{v}}^{\mathrm{in}}_{-,k}(x_+)i\Gamma^1\Gamma_A\,\mathfrak{v}^{\mathrm{in}}_{-,k}(x_-)\Big]\\ = \left[\left(1+e^{-iev\varepsilon}|\mathcal{T}^{+-}_{\mathrm{in}}|^2\right)\left(\frac{r_+}{r_-}\right)^{\frac{ieQ}{2}} - |\mathcal{T}^{--}_{\mathrm{in}}|^2\left(\frac{r_+}{r_-}\right)^{-\frac{ieQ}{2}}\right] &\int_0^\infty \frac{\mathrm{d}k}{4\pi}\left(e^{-ik\varepsilon}-e^{ik\varepsilon}\right). \qquad \text{(F.1.6)}\end{aligned}$$

[1]This is true for M such that $\overline{\mathfrak{u}}^{\mathrm{d}}_{\mathfrak{s},k}M\,\mathfrak{u}^{\mathrm{d}}_{\mathfrak{s},k}$ is independent of k.
[2]At $r=\epsilon$, we instead consider the operators $\overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)M\,\boldsymbol{\chi}(\epsilon,t)$ since $r\geq\epsilon$.

The momentum integrals in the above equations can be evaluated using the integral representation of the Heaviside theta function

$$\int_0^\infty \frac{\mathrm{d}k}{4\pi} e^{\pm ik\varepsilon} = \lim_{\beta\to 0+} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k\,\mathrm{d}\tau}{4\pi(\tau+i\beta)} e^{-ik(\tau\mp\varepsilon)} = \lim_{\beta\to 0+} \frac{i}{4\pi(\pm\varepsilon+i\beta)}, \tag{F.1.7}$$

an using (F.1.7) in (F.1.5) and (F.1.6) shows that the *in* vacuum expectation value of the radial component of the axial current can be written as

$$\begin{aligned}\langle 0_{\rm in}|\,\overline{\boldsymbol{\chi}}\, i\Gamma^1\Gamma_{A}\boldsymbol{\chi}\,|0_{\rm in}\rangle \;=\; \lim_{\varepsilon\to 0}\left(-\frac{i}{2\pi\varepsilon}\right)\Bigg[&\left(1+e^{-iev\varepsilon}|\mathcal{T}_{\rm in}^{+-}|^2+|\mathcal{T}_{\rm in}^{++}|^2\right)\left(\frac{r_+}{r_-}\right)^{\frac{ieQ}{2}} \\ &-\left(1+e^{iev\varepsilon}|\mathcal{T}_{\rm in}^{-+}|^2+|\mathcal{T}_{\rm in}^{--}|^2\right)\left(\frac{r_+}{r_-}\right)^{-\frac{ieQ}{2}}\Bigg].\end{aligned} \tag{F.1.8}$$

Finally, we simplify the above equation by expanding in ε and using (2.2.34) to get:

$$\langle 0_{\rm in}|\, j_{A}^1(x)\,|0_{\rm in}\rangle = \langle 0_{\rm in}|\,\overline{\boldsymbol{\chi}}\, i\Gamma^1\Gamma_{A}\boldsymbol{\chi}\,|0_{\rm in}\rangle = -\frac{ev}{\pi}|\mathcal{T}_{\rm in}^{+-}|^2+\frac{eQ}{\pi r}. \tag{F.1.9}$$

The *in* vacuum expectation value of other current operators of interest can similarly be written as

$$\begin{aligned}\langle 0_{\rm in}|\,\overline{\boldsymbol{\chi}}M\,\boldsymbol{\chi}\,|0_{\rm in}\rangle \;=\; \lim_{\varepsilon\to 0}\left(-\frac{i}{2\pi\varepsilon}\right)\Bigg[&\left(q_-^{M,\,\rm in}-q_{+-}^{M,\,\rm in}e^{-iev\varepsilon}|\mathcal{T}_{\rm in}^{+-}|^2-q_{++}^{M,\,\rm in}|\mathcal{T}_{\rm in}^{++}|^2\right)\left(\frac{r_+}{r_-}\right)^{\frac{ieQ}{2}} \\ &+\left(q_+^{M,\,\rm in}-q_{-+}^{M,\,\rm in}e^{iev\varepsilon}|\mathcal{T}_{\rm in}^{-+}|^2-q_{--}^{M,\,\rm in}|\mathcal{T}_{\rm in}^{--}|^2\right)\left(\frac{r_+}{r_-}\right)^{-\frac{ieQ}{2}}\Bigg],\end{aligned} \tag{F.1.10}$$

where $q_{\mathfrak{s}}^{M,\,\rm in}, q_{\mathfrak{s}\mathfrak{s}'}^{M,\,\rm in}$ are listed in table F.1 for several choices of the matrix M. The calculation of current vacuum expectation values goes through in the same way for the *out* vacuum. We get:

$$\begin{aligned}\langle 0_{\rm out}|\,\overline{\boldsymbol{\chi}}M\,\boldsymbol{\chi}\,|0_{\rm out}\rangle \;=\; \lim_{\varepsilon\to 0}\frac{i}{2\pi\varepsilon}\Bigg[&\left(q_-^{M,\,\rm out}-q_{+-}^{M,\,\rm out}e^{iev\varepsilon}|\mathcal{T}_{\rm out}^{+-}|^2-q_{++}^{M,\,\rm out}|\mathcal{T}_{\rm out}^{++}|^2\right)\left(\frac{r_+}{r_-}\right)^{-\frac{ieQ}{2}} \\ &+\left(q_+^{M,\,\rm out}-q_{-+}^{M,\,\rm out}e^{-iev\varepsilon}|\mathcal{T}_{\rm out}^{-+}|^2-q_{--}^{M,\,\rm out}|\mathcal{T}_{\rm out}^{--}|^2\right)\left(\frac{r_+}{r_-}\right)^{\frac{ieQ}{2}}\Bigg],\end{aligned} \tag{F.1.11}$$

where $q_{\mathfrak{s}}^{M,\,\rm out}, q_{\mathfrak{s}\mathfrak{s}'}^{M,\,\rm out}$ are listed in table F.2.

We see that the regularized vacuum matrix elements of the components of the fermion number

M	$q_{+}^{M,\,\text{in}}$	$q_{-}^{M,\,\text{in}}$	$q_{++}^{M,\,\text{in}}$	$q_{--}^{M,\,\text{in}}$	$q_{-+}^{M,\,\text{in}}$	$q_{+-}^{M,\,\text{in}}$
$i\Gamma^0$	1	1	1	1	1	1
$i\Gamma^1$	-1	-1	1	1	1	1
$\frac{ie}{2}\Gamma^0\tau_3$	$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$	-$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$
$\frac{ie}{2}\Gamma^1\tau_3$	-$\frac{e}{2}$	$\frac{e}{2}$	$\frac{e}{2}$	-$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$
$i\Gamma^0\Gamma_A$	1	-1	-1	1	1	-1
$i\Gamma^1\Gamma_A$	- 1	1	-1	1	1	-1

Table F.1: Numerical values of $q_{\mathfrak{s}}^{M,\,\text{in}}, q_{\mathfrak{s}\mathfrak{s}'}^{M,\,\text{in}}$, defined by eq. (F.1.10), for various choices of the matrix M.

current are given by

$$\langle 0_{\text{in}}|\, j_B^0(x)\,|0_{\text{in}}\rangle = \langle 0_{\text{out}}|\, j_B^0(x)\,|0_{\text{out}}\rangle = 0, \tag{F.1.12}$$

as well as

$$\langle 0_{\text{in}}|\, j_B^1(x)\,|0_{\text{in}}\rangle = \langle 0_{\text{out}}|\, j_B^1(x)\,|0_{\text{out}}\rangle = \frac{2i}{\pi\varepsilon}, \tag{F.1.13}$$

while those of the electromagnetic current are

$$\begin{aligned} \langle 0_{\text{in}}|\, j_F^0(r,t)\,|0_{\text{in}}\rangle \;&=\; \langle 0_{\text{out}}|\, j_F^0(r,t)\,|0_{\text{out}}\rangle = \frac{e^2 v}{2\pi}|\mathcal{T}_{\text{in}}^{+-}|^2 - \frac{e^2 Q}{2\pi r} \\ \text{and}\quad \langle 0_{\text{in}}|\, j_F^1(r,t)\,|0_{\text{in}}\rangle \;&=\; -\langle 0_{\text{out}}|\, j_F^1(r,t)\,|0_{\text{out}}\rangle = \frac{e^2 v}{2\pi}|\mathcal{T}_{\text{in}}^{+-}|^2\,. \end{aligned} \tag{F.1.14}$$

Finally, the axial current vacuum matrix elements are

$$\begin{aligned} \langle 0_{\text{in}}|\, j_A^0(r,t)\,|0_{\text{in}}\rangle \;&=\; -\langle 0_{\text{out}}|\, j_A^0(r,t)\,|0_{\text{out}}\rangle = -\frac{ev}{\pi}|\mathcal{T}_{\text{in}}^{+-}|^2 \\ \text{and}\quad \langle 0_{\text{in}}|\, j_A^1(r,t)\,|0_{\text{in}}\rangle \;&=\; \langle 0_{\text{out}}|\, j_A^1(r,t)\,|0_{\text{out}}\rangle = -\frac{ev}{\pi}|\mathcal{T}_{\text{in}}^{+-}|^2 + \frac{eQ}{\pi r}\,. \end{aligned} \tag{F.1.15}$$

The vacuum expectation value of the interaction hamiltonian in the interaction picture (3.1.3) (with $\delta\mathcal{C}^{pv} = \delta\mathcal{C}^{ps} = 0$) can be calculated similarly. This expectation value is defined as

$$\begin{aligned} \langle 0|\, H_{\text{int}}\,|0\rangle \;=\; \lim_{\varepsilon\to 0}\sum_{\mathfrak{s}=\pm}\int_0^\infty \frac{\mathrm{d}k}{8\pi}\Big[&-\overline{\mathfrak{u}}_{\mathfrak{s},k}^{\text{in}0}(\epsilon+\varepsilon,t)\Big(\delta\mathcal{C}_{\mathfrak{s}\mathfrak{s}'}^{s} - i\delta\mathcal{C}_{\mathfrak{s}\mathfrak{s}}^{v}\Gamma^0\Big)\,\mathfrak{u}_{\mathfrak{s},k}^{\text{in}0}(\epsilon,t) \\ &+\overline{\mathfrak{v}}_{\mathfrak{s},k}^{\text{in}0}(\epsilon+\varepsilon,t)\Big(\delta\mathcal{C}_{\mathfrak{s}\mathfrak{s}'}^{s} - i\delta\mathcal{C}_{\mathfrak{s}\mathfrak{s}}^{v}\Gamma^0\Big)\,\mathfrak{v}_{\mathfrak{s},k}^{\text{in}0}(\epsilon,t)\Big], \end{aligned} \tag{F.1.16}$$

M	$q_{+}^{M,\,\mathrm{out}}$	$q_{-}^{M,\,\mathrm{out}}$	$q_{++}^{M,\,\mathrm{out}}$	$q_{--}^{M,\,\mathrm{out}}$	$q_{-+}^{M,\,\mathrm{out}}$	$q_{+-}^{M,\,\mathrm{out}}$
$i\Gamma^0$	1	1	1	1	1	1
$i\Gamma^1$	1	1	-1	-1	-1	-1
$\frac{ie}{2}\Gamma^0\tau_3$	$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$	-$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$
$\frac{ie}{2}\Gamma^1\tau_3$	$\frac{e}{2}$	-$\frac{e}{2}$	-$\frac{e}{2}$	$\frac{e}{2}$	$\frac{e}{2}$	-$\frac{e}{2}$
$i\Gamma^0\Gamma_{A}$	-1	1	1	-1	-1	1
$i\Gamma^1\Gamma_{A}$	- 1	1	-1	1	1	-1

Table F.2: Numerical values of $q_{\mathfrak{s}}^{M,\,\mathrm{out}}, q_{\mathfrak{s}\mathfrak{s}'}^{M,\,\mathrm{out}}$, defined by equation (F.1.11), for various choices of M.

where $|0\rangle = |0_{\mathrm{in}}\rangle = |0_{\mathrm{out}}\rangle$ in the interaction picture and is equal to

$$\begin{aligned}\langle 0|\, H_{\mathrm{int}}\, |0\rangle \;&=\; \lim_{\varepsilon\to 0}\sum_{\mathfrak{s}=\pm}(\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}} - \delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}})\left[\left(\frac{\epsilon+\varepsilon}{\epsilon}\right)^{i\mathfrak{s}eQ/2} - \left(\frac{\epsilon+\varepsilon}{\epsilon}\right)^{-i\mathfrak{s}eQ/2}\right]\int_0^\infty \frac{\mathrm{d}k}{8\pi}\,(e^{ik\varepsilon} - e^{-ik\varepsilon}) \\ &=\; -\frac{eQ}{4\pi\epsilon}\sum_{\mathfrak{s}=\pm}\mathfrak{s}\,(\delta\mathcal{C}^s_{\mathfrak{s}\mathfrak{s}} - \delta\mathcal{C}^v_{\mathfrak{s}\mathfrak{s}}),\end{aligned} \tag{F.1.17}$$

which matches (3.1.6).

F.2 Current conservation equations

The fermionic currents satisfy conservation (or non-conservation) equations which can be derived directly from the 2D Dirac equation. That is, the current $j^{\alpha}_{M}(x) := i\overline{\boldsymbol{\chi}}(x)\Gamma^{\alpha}M\boldsymbol{\chi}(x)$ satisfies the equation

$$\begin{aligned}\partial_t\, j^t_{M}(x) \;&=\; i\lim_{\varepsilon\to 0}\big[(\partial_t\overline{\boldsymbol{\chi}}(r_+,t))\,\Gamma^0\, M\boldsymbol{\chi}(r_-,t) + \overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\, M\,(\partial_t\boldsymbol{\chi}(r_-,t))\big] \\ &=\; i\lim_{\varepsilon\to 0}\Big[-(\partial_r\overline{\boldsymbol{\chi}}(r_+,t))\,\Gamma_c\,\Gamma^0\, M\boldsymbol{\chi}(r_-,t) + \overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\, M\,\Gamma_c\,(\partial_r\boldsymbol{\chi}(r_-,t)) \\ &\qquad -\frac{ie}{2}\Big(\mathcal{A}^3_0(r_+,t)\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\tau_3\, M\boldsymbol{\chi}(r_-,t) - \mathcal{A}^3_0(r_-,t)\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\, M\tau_3\boldsymbol{\chi}(r_-,t)\Big)\Big] \\ &=\; i\lim_{\varepsilon\to 0}\Big[\partial_r\left(\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\Gamma_c\, M\,\boldsymbol{\chi}(r_-,t)\right) - \frac{ie\varepsilon}{2}(\partial_r\mathcal{A}^3_0(r))\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\, M\tau_3\boldsymbol{\chi}(r_-,t)\Big] \\ &=\; -\partial_r j^r_{M}(x) + \lim_{\varepsilon\to 0}\frac{\varepsilon eQ}{2r^2}\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\, M\tau_3\boldsymbol{\chi}(r_-,t),\end{aligned} \tag{F.2.1}$$

where $M \in \{1, \frac{e}{2}\tau_3, \Gamma_{A}\}$ and we use $[M, \Gamma_c] = [M, \tau_3] = 0$. This can be equivalently rewritten as

$$\begin{aligned}
\partial_\alpha j^\alpha_{B}(x) &= \frac{eQ}{2r^2}\lim_{\varepsilon\to 0}\left(\varepsilon\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\,\tau_3\boldsymbol{\chi}(r_-,t)\right) = -i\frac{eQ}{er^2}\lim_{\varepsilon\to 0}\left(\varepsilon j^0_{F}(x)\right),\\
\partial_\alpha j^\alpha_{F}(x) &= \frac{e(eQ)}{4r^2}\lim_{\varepsilon\to 0}\left(\varepsilon\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^0\boldsymbol{\chi}(r_-,t)\right) = -i\frac{e(eQ)}{4r^2}\lim_{\varepsilon\to 0}\left(\varepsilon j^0_{B}(x)\right),\\
\partial_\alpha j^\alpha_{A}(x) &= -\frac{eQ}{2r^2}\lim_{\varepsilon\to 0}\left(\varepsilon\,\overline{\boldsymbol{\chi}}(r_+,t)\Gamma^1\boldsymbol{\chi}(r_-,t)\right) = i\frac{eQ}{2r^2}\lim_{\varepsilon\to 0}\left(\varepsilon j^1_{B}(x)\right),
\end{aligned} \tag{F.2.2}$$

for each choice of M, which shows that the conservation of the fermionic currents hinges on whether j^α_{B}, j^0_{F} are singular in the small ε limit, or not. The dominant contributions to the source terms in the above conservation equations come from the vacuum expectation values of j^α_{B}, j^0_{F}. Equations (F.1.12)-(F.1.14) then imply that the fermion number and electric charge current are conserved, while the axial current satisfies the anomaly equation

$$\partial_\alpha j^\alpha_{A}(x) = -\frac{eQ}{\pi r^2} = \left[\frac{e}{2} - \left(-\frac{e}{2}\right)\right]\frac{1}{2\pi}\epsilon^{\alpha\beta}\mathcal{F}^3_{\alpha\beta}(x), \tag{F.2.3}$$

which we rewrite in the last line to emphasize the fact that the top and bottom components of the doublet contribute the usual factor of $\frac{q_{\mathfrak{s}}}{2\pi}\epsilon^{\alpha\beta}\mathcal{F}^3_{\alpha\beta}(x)$ to the 4-divergence of the axial current, where $q_{\mathfrak{s}} = \frac{1}{2}\mathfrak{s}e$. We take the difference, as opposed to the sum, of these two contributions since $j^\alpha_{A}(x)$ is defined with an extra τ_3 matrix compared to the standard definition of an axial current in 2D.

F.3 Boundary currents

At $r = \epsilon$ the radial components of fermionic currents can be evaluated by using the boundary condition $\Gamma^1\boldsymbol{\chi}(\epsilon, t) = -O_{\mathcal{B}}(\mathfrak{a})\,\boldsymbol{\chi}(\epsilon, t)$, which can also be rewritten as $\overline{\boldsymbol{\chi}}(\epsilon, t)\Gamma^1 = \overline{\boldsymbol{\chi}}(\epsilon, t)O_{\mathcal{B}}(\mathfrak{a})$. We get:

$$\begin{aligned}
j^1_{B}(\epsilon,t) &= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\boldsymbol{\chi}(\epsilon,t) + \overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right) = \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon,t) - \overline{\boldsymbol{\chi}}(\epsilon,t)O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon,t)\right) = 0,\\
j^1_{F}(\epsilon,t) &= \frac{ie}{4}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\tau_3\boldsymbol{\chi}(\epsilon,t) + \overline{\boldsymbol{\chi}}(\epsilon,t)\tau_3\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right) = \frac{i}{2}\overline{\boldsymbol{\chi}}(\epsilon,t)\left[O_{\mathcal{B}}(\mathfrak{a}), \frac{e}{2}\tau_3\right]\boldsymbol{\chi}(\epsilon,t),\\
j^1_{A}(\epsilon,t) &= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\Gamma_{A}\boldsymbol{\chi}(\epsilon,t) - \overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma_{A}\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right) = \frac{i}{2}\overline{\boldsymbol{\chi}}(\epsilon,t)\left\{O_{\mathcal{B}}(\mathfrak{a}), \Gamma_{A}\right\}\boldsymbol{\chi}(\epsilon,t).
\end{aligned} \tag{F.3.1}$$

The above currents vanish when the boundary action (2.2.15) is invariant under field transformations of the form $\delta\boldsymbol{\chi}(\epsilon,t) = i\theta M\boldsymbol{\chi}(\epsilon,t)$, where θ is a constant parameter and $M = 1$ gives the transformation corresponding to the fermion number current, $M = \frac{e}{2}\tau_3$ to the electric charge current and $M = \Gamma_A$ to the axial current.

As argued earlier in this appendix, the fermion bilinears appearing in (F.3.1) should be regulated by *e.g.* a point-splitting procedure. Such a regularization procedure generally need not preserve the equality in (F.3.1). However, if we regularize the boundary currents in the following way

$$\begin{aligned}
(j^1_B(\epsilon,t))_{\rm reg} &:= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\boldsymbol{\chi}(\epsilon+\varepsilon,t) + \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right), \\
(j^1_F(\epsilon,t))_{\rm reg} &:= \frac{ie}{4}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\tau_3\boldsymbol{\chi}(\epsilon+\varepsilon,t) + \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)\tau_3\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right), \\
(j^1_A(\epsilon,t))_{\rm reg} &:= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)\Gamma^1\Gamma_A\boldsymbol{\chi}(\epsilon+\varepsilon,t) - \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)\Gamma_A\Gamma^1\boldsymbol{\chi}(\epsilon,t)\right),
\end{aligned} \tag{F.3.2}$$

and the remaining fermion bilinears in (F.3.1) as

$$\begin{aligned}
\left(\frac{i}{2}\overline{\boldsymbol{\chi}}(\epsilon,t)\left[O_{\mathcal{B}}(\mathfrak{a}),1\right]\boldsymbol{\chi}(\epsilon,t)\right)_{\rm reg} &:= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon+\varepsilon,t) - \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon,t)\right), \\
\left(\frac{i}{2}\overline{\boldsymbol{\chi}}(\epsilon,t)\left[O_{\mathcal{B}}(\mathfrak{a}),\frac{e}{2}\tau_3\right]\boldsymbol{\chi}(\epsilon,t)\right)_{\rm reg} &:= \frac{ie}{4}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)O_{\mathcal{B}}(\mathfrak{a})\tau_3\boldsymbol{\chi}(\epsilon+\varepsilon,t) - \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)\tau_3 O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon,t)\right), \\
\left(\frac{i}{2}\overline{\boldsymbol{\chi}}(\epsilon,t)\left\{O_{\mathcal{B}}(\mathfrak{a}),\Gamma_A\right\}\boldsymbol{\chi}(\epsilon,t)\right)_{\rm reg} &:= \frac{i}{2}\left(\overline{\boldsymbol{\chi}}(\epsilon,t)O_{\mathcal{B}}(\mathfrak{a})\Gamma_A\boldsymbol{\chi}(\epsilon+\varepsilon,t) + \overline{\boldsymbol{\chi}}(\epsilon+\varepsilon,t)\Gamma_A O_{\mathcal{B}}(\mathfrak{a})\boldsymbol{\chi}(\epsilon,t)\right),
\end{aligned} \tag{F.3.3}$$

then the regularization scheme does preserve the equality in (F.3.1), which can be seen by using the boundary conditions satisfied by $\boldsymbol{\chi}(\epsilon), \overline{\boldsymbol{\chi}}(\epsilon)$.

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