ON THE 5D SOLITON SOLUTION TO THE VACUUM EINSTEIN EQUATIONS

AN ANALYSIS OF THE 5D STATIONARY BI-AXISYMMETRIC SOLITON SOLUTION TO THE VACUUM EINSTEIN EQUATIONS

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Lay Abstract

We study the geometry of 5D blackholes. These blackholes are idealized by certain spatial symmetries and time invariance. They are solutions to the vacuum Einstein equations. The unique characteristic of these blackholes is the range of behaviour they may exhibit at the boundary of the domain of outer communication. There could be a standard event horizon called a horizon rod or an axis rod where a certain part of the spatial symmetry becomes trivial. In this thesis we start by deriving the harmonic map equations which are satisfied in the interior of the domain of communication. Then we show how this boundary data affects the metric through the smoothness conditions. We then analyze the soliton example in a paper by Khuri, Weinstein and Yamada and show that it respects the smoothness conditions. We then provide a new example which is interesting in the fact it has non-constant twist potentials.

Abstract

We set out to analyze 5D stationary and bi-axisymmetric solutions to the vacuum Einstein equations. These are in the cohomogeneity 2 setting where the orbit space is a right half plane. They can have a wide range of behaviour at the boundary of the orbit space. The goal is to understand in detail the soliton example in Khuri, Weinstein and Yamada's paper "5-dimensional space-periodic solutions of the static vacuum Einstein equations". This example is periodic and has alternating axis rods as its boundary data. We start by deriving the harmonic equations which determines the behaviour of the metric in the interior of the orbit space. Then we analyze what conditions the boundary data imposes on the metric. These are called the smoothness conditions which we derive for solely the alternating axis rod case. We show that with an ellipticity assumption they predict that the twist potentials are constant and that the metric is of the form which appears in Khuri, Weinstein and Yamada's paper. We then analyze the Schwarzschild metric in its standard form which is cohomogeneity 1 and its Weyl form which is cohomogeneity 2. This Weyl form can be made periodic and this serves as an inspiration for the examples in Khuri, Weinstein and Yamada's paper. Finally we analyze the soliton example in detail and show that it satisfies the smoothness conditions. We then provide a new example which has a single axis rod on the boundary with non-constant twist potentials but that is missing a point on the boundary.

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Chapter 1

Introduction

In this thesis we aim to provide the necessary background to understand 5D stationary, bi-axisymmetric solutions to the vacuum Einstein equations. This allows us to further understand example 2 [18, p. 9-11], put forward by Khuri, Weinstein, and Yamada in "5-dimensional space periodic solutions of the vacuum Einstein equation", at a level of detail not included in the paper. This is done by providing mathematical proof of the exhibited properties of example 2 and confirming that it satisfies certain smoothness conditions although these smoothness conditions are not mentioned in the paper. Such smoothness conditions are obtained at the level of the manifold by imposing natural slice representations. From these smoothness conditions we show that the form of the metric they give in the solution in example 2 can't be made more general. At least if the metric is real analytic in the interior of the orbit space. Additionally we provide a new example with non-constant twist potentials but with the shortcoming that it is missing a point on the boundary. This singularity partly corresponds to a 1-point horizon rod.

With the background we are not only providing the necessary mathematical concepts but also illustrating previous models which led to the development of the examples brought forward by Khuri et al.. For this purpose, it is crucial to understand the distinction between cohomogeneity 1 and cohomogeneity 2. These refer to the codimension of the principal orbits [10, p. 111]. This is equal to the dimension of the orbit space. With a 1-dimensional orbit space of a connected manifold the boundary is made up of 1 or 2 points. But for a 2-dimensional orbit space the boundary, in the case of the paper in question, is a line [18, p. 2] and thus more can happen on the boundary. In this paper, the group acting is $T^2 \times \mathbb{R}$ [18, p. 2]. The group acts on the metric, g, by isometries; where T^2 corresponds to the bi-axisymmetric requirement and the $\mathbb{R}$ corresponds to the stationary requirement [18, p. 2]. The orbit space is the right half plane with coordinates (r, z). In the interior of the orbit space the metric is well understood being given by the vacuum Einstein equations [18, p. 3]. However when we move to the boundary that equation becomes undefined. It is thus necessary to impose smoothness conditions that apply to open sets around the boundary points. For this we need the boundary data. We divide the boundary into a sequence of axis rods and horizon rods [18, p. 2]. We take ∂_{φ_1} and ∂_{φ_2} to be vector fields tangent to the first circle and the second circle respectively. We take ∂_t to be vector field tangent to $\mathbb{R}$. For a (p, q) axis rod we have that $g(p\partial_{\varphi_1} + q\partial_{\varphi_2}, p\partial_{\varphi_1} + q\partial_{\varphi_2}) \to 0$ as you approach the axis rod. Here p and q are relatively prime integers. It is simpler to consider $(1, 0)$ and $(0, 1)$ axis rods. For a $(1, 0)$ rod, the first circle, S^1_1, shrinks to a point as you approach it. For a $(0, 1)$ rod, the second circle, S^1_2, shrinks to a point as you approach it. At the intersection of two rods there is a corner point. The admissibility condition requires that for a corner point between a (p, q) rod and a (k, l) rod that $\det \begin{pmatrix} p & q \\ k & l \end{pmatrix} = \pm 1$ [18, p. 2]. For a horizon rod we have that $g(\partial_t + \Omega_1\partial_{\varphi_1} + \Omega_2\partial_{\varphi_2}, \partial_t + \Omega_1\partial_{\varphi_1} + \Omega_2\partial_{\varphi_2})$ goes to 0 as you approach the horizon rod. Here Ω_1 and Ω_2 are constants called the angular velocities. We require that a horizon rod and an axis rod cannot occur simultaneously. We call an intersection between an axis rod and an horizon rod a pole. Moving back to cohomogeneity 1 case, the quintessential example is the Schwarzschild metric. The 4D Schwarzschild metric is static and spherically symmetric [30, p. 119]. The metric depends on a single parameter r and has a curvature singularity at $r = 0$ [30, p. 124]. This metric can be converted to its Weyl form and from that a new periodic Schwarzschild metric can be constructed. This is shown in the paper "Periodic Analog of the Schwarzschild Metric"[21] by Korotkin and Nicolai. This forms the inspiration for the periodic examples constructed by Khuri et al..

The layout of this document is as follows. In chapter 1 we summarize basic manifold theory, Lie groups and Lie algebras, concepts relating to group actions and representations. The notion of a principal fibre bundle is important in understanding the metric in the interior of the orbit space and the notions of slices and tubes are necessary for expressing the smoothness conditions. Schur's Lemma is used heavily in the section on the smoothness conditions. In chapter 2, using the framework of Riemannian submersions outlined by Besse [3], we derive the harmonic map equations from the Ricci flat condition, and the use of Killing fields which stem from alternate definitions of the stationary and bi-axisymmetric requirements. In chapter 3 we state the smoothness conditions and derive the conclusions about the behaviour of the metric near the axis rods and the corner points. This involves imposing separate slice representations for the slice at a point on a $(1,0)$ rod, a $(0,1)$ road and at a corner point. These slice representations are made up of two dimensional rotations and the identity representation. The smoothness conditions are only derived from the rod structure being a sequence of $(1,0)$ and $(0,1)$ rods alternating. These smoothness conditions are stated in terms of a different set of coordinates, (x_1, y_1, x_2, y_2), the radius $r_1 = \sqrt{x_1^2 + y_1^2}$ is thought of as the radius of S_1^1 and the other radius $r_2 = \sqrt{x_2^2 + y_2^2}$ is thought of as the radius of S_2^1. In chapter 4, we derive the Schwarzschild metric, convert to its Weyl form and derive the periodic Schwarzschild metric. We leave out some of the calculations for the derivation of the Schwarzschild metric out since they can be found in Wald's book [30]. In chapter 5 we analyze example 2. We show that it is periodic in z and using a Fourier series in terms of z we obtain its asymptotic behaviour. We also determine its behaviour near the axis rods and corner points as well as its topology. In chapter 6 we state our new example where the rod structure is a single $(1,0)$ rod. We check that it is a solution and satisfies the required properties. Additionally we provide a derivation and derive its topology. Again it has the shortfall of missing a point on the boundary.

Chapter 2

Background

2.1 A Summary of Basic Manifold Theory

2.1.1 Manifolds

We will assume a familiarity with the basic concepts of manifold theory but we provide a short summary. Roughly speaking a manifold is a space with a topology which is locally Euclidean. That is there are homeomorphisms, called *charts* which map open sets on the manifold to open sets in $\mathbb{R}^n$, where n is the dimension of the manifold [22, p. 52]. A manifold is thus a space with meaningful local coordinates. Additionally a manifold can be thought of as being C^∞ when we impose that all charts in its atlas are C^∞ related [22, p. 53]. We can construct C^∞ maps between manifolds [22, p. 56]. When the map's inverse is a C^∞ map it is called a *diffeomorphism* [22, p. 59]. Diffeomorphic manifolds are thought of as being geometrically equivalent and two manifolds being diffeomorphic is a stricter requirement than them being homeomorphic as topological spaces.

2.1.2 Tangent Spaces

Moving on to *tangent spaces*, in the setting of say a surface in $\mathbb{R}^3$, we consider the tangent space as being extrinsic to the surface and lying in $\mathbb{R}^3$. In the case of a general manifold it is not natural to think of the manifold as being embedded into $\mathbb{R}^n$. We therefore want a definition of a tangent space at a point that is intrinsic to the manifold but being a separate space and not lying in the manifold. After all a manifold is not equipped with a vector space structure.

Elements of a tangent space can be defined either as equivalence classes of

C^∞ curves [22, p. 67-68] or as linear derivations which act on functions which maps points on the manifold to $\mathbb{R}$ [22, p. 103-108]. Both view points are useful. If we have a map f between manifolds M and N then it induces a linear *tangent map*, $(f_*)_m$ between the tangent spaces $T_m M$ and $T_{f(m)}N$. This map can also be written as $df|_m$. Take X in $T_m M$ to be tangent to a curve γ at m, then we can write $X = \gamma'(0)$. Furthermore we have the useful identity: $(f_*)_m X = (f \circ \gamma)'(0)$ [22, p. 68]. The derivations view point gives us a way to describe X in terms of a coordinate basis. That is, $X = \Sigma_i X(x^i) \frac{\partial}{\partial x_i}|_m$[23, p. 45]. Of course when we perform a change of basis X is preserved. With the tangent map in mind we can further characterize maps between manifolds.

Definition 2.2. *[22, p. 69] A C^∞ map f between manifolds M and N is an immersion if for all m, the tangent map $(f_*)_m : T_m M \mapsto T_{f(m)}N$ is injective.*

Definition 2.3. *[22, p. 69] A C^∞ map f between manifolds M and N is a submersion if for all m, the tangent map $(f_*)_m : T_m M \mapsto T_{f(m)}N$ is surjective.*

2.3.1 Submanifolds

A subset N of an n-dimensional manifold M is a p-dimensional *submanifold* of M [22, p. 69] if for every $m \in N$ there exists an open neighbourhood of m, U in M, an open neighbourhood of $0 \in \mathbb{R}^n$, V, and a diffeomorphism f from U to V where we have the following.

$$f(U \cap N) = V \cap (\mathbb{R}^p \times \{0\})$$

Consider the following subset, N, of $\mathbb{R}^2$.

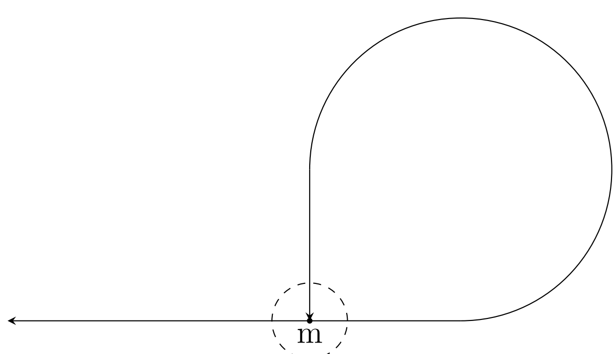

Figure 2.1:

We can show that N can't be a submanifold. Take an arbitrary subset U around m and suppose we have a diffeomorphism, f, where $f(U \cap N) = V \cap (\mathbb{R} \times \{0\})$ and $f(m) = 0$. We can take a ball centered at 0 in which is contained in V. Call this ball B. If we delete the point $(0,0)$ from $B \cap (\mathbb{R} \times \{0\})$ we obtain a set with 2 connected components. Take the preimage, $f^{-1}(B \cap (\mathbb{R} \times \{0\}))$. We require B to have a small radius to ensure that the preimage looks similar to the one in the diagram. If we delete $m = f^{-1}(0)$ from the preimage we obtain a set with at least 3 connected components. Since f is a diffeomorphism it is also a homeomorphism and even a homeomorphism with a point removed. So we have a contradiction since the number of connected components must be conserved under a homeomorphism.

Definition 2.4. *[22, p. 70] A C^∞ map f from manifolds M to N is an embedding if $f(M)$ is a submanifold of N and if f is a diffeomorphism from M to $f(M)$.*

2.4.1 Tangent Bundle

With the union of the tangent spaces at all points in the manifold we can form the *tangent bundle*, TM [22, p. 113]. Elements of the tangent bundle are vector fields. Strictly speaking vector fields assign each point in a subset of the manifold to a vector in the tangent space at the point. Of course we can let vector field X act on a function g by $(Xg)(m) = X(m)g$. Any diffeomorphism f induces a linear vector field map f_*. Where $f_*X(g) = X(g \circ f) \circ f^{-1}$ for functions g acting on the codomain of f [23, p. 62].

2.4.2 Lie Derivatives

We think of a particle being pushed by a vector field X and thus tracing out a curve. These curves are called the *integral curves of* X [22, p. 119]. The *flow* is just a map that takes you along the integral curve by a fixed amount. The flow is a local diffeomorphism and in addition we have that $(\varphi_s)_*X = X$ where φ_s is the flow of X [22, p. 123] (See proposition 3.37; this will be useful later).

Definition 2.5. *[29, p. 150] Let X and Y be vector fields on M. Let φ_t be the flow of Y. The Lie derivative of X is the vector field, L_XY, obtained by the following formula*

$$L_XY = \frac{d}{dt}\bigg|_{t=0}\Big((\varphi_t)_*X\Big) \tag{2.5.1}$$

Of course one can show that $L_XY = XY - YX = [X,Y]$ [22, p. 124].

We can define *1-forms* to be linear functionals which take a vector field as their input. These are elements of the dual of the tangent bundle. For example dx. We can further extend the notion of vector fields to *tensor fields* of type (r, s). These tensor fields are multilinear functional with r inputs which take in 1-forms and s inputs which take in vector fields [4, p. 118-120] We can define the Lie derivative of a tensor field in a similar way as vector fields. A Lie derivative of $(0, 2)$ tensor, T, has the following formula where X, Y and Z are vector fields [4, p. 130].

$$L_X T(Y, Z) = X(T(Y, Z)) - T(Y, L_X Z) - T(L_X Y, Z) \tag{2.5.2}$$

2.5.1 Metric Tensor

We now introduce the *metric* which plays a fundamental geometric role; allowing distances to be defined, acting as an inner product in the Riemannian case, and allowing curvature to be defined. A *Pseudo-Riemannian metric* is a symmetric $(0, 2)$ tensor field defined on all of a manifold, M, whose associated matrix has no 0 eigenvalues. The eigenvalues of the associated matrix are real since this matrix is symmetric. Furthermore since the eigenvalues are never 0, this matrix has the same signature everywhere. If the matrix is positive definite we say that the metric is *Riemannian*, if the signature is $(+, \ldots, +, -)$ or $(-, \ldots, -, +)$ we say that the metric is *Lorentz* [4, p, 110].

When we have a pseudo-Riemannian metric, g, it uniquely determines a *Levi Civita connection* D which satisfies the following two properties. Let X, Y and Z be vector fields. [6, p. 53-55]:

i D is compatible with the metric: $X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z)$

ii D is torsion-free: $D_X Y - D_Y X = [X, Y]$

2.5.2 Curvature Tensors

The Levi-Civita connection lets us define the curvature tensors. We begin with a mapping R which takes vector fields X and Y to the operator $R(X, Y)$ which maps vector fields to vector fields. Let Z be another vector field. We have that $R(X, Y)Z$ is given by the following [6, p. 89]:

$$R(X, Y)Z = D_Y(D_X Z)) - D_X(D_Y Z) + D_{[X,Y]} Z$$

We can use this operator to define *Riemann curvature tensor*. This tensor is type $(0, 4)$ with vector field inputs X, Y, Z, W and the output is denoted by $R(X, Y, Z, W)$. The formula is given by [6, p.91]:

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

In addition to the Riemann curvature tensor we have the *Ricci curvature tensor* defined using an orthonormal basis of vector fields on the manifold M, $\{e_i\}$. Let X and Y be vector fields on M [12, p. 135].

$$\mathrm{Ric}(X,Y) = \Sigma_{i=1}^{n} R(X, e_i, Y, e_i)$$

The *scalar curvature*, R, is defined to be $R = \Sigma_{i=1}^{n}\mathrm{Ric}(e_i, e_i)$ [12, p. 136].

2.6 Lie Groups and Lie Algebras

Lie groups are an important concept as they will be used later in the cohomogeneity 2 setting. We start with their definition.

Definition 2.7. *[31, p. 82] A Lie group G is a group which has the structure of a manifold and such that its group operation and inverse operation are both smooth. This can be summarized by checking that for g and h in G the function $f : G \times G \mapsto G$, where $f(g,h) = gh^{-1}$, is smooth in g and h.*

An example of a Lie group would be the real line, $\mathbb{R}$, with the group operation being addition. Another example would be S^1 thought of as lying in the complex plane. The group operation in that case would be complex multiplication. We now move on to the separate concept of a Lie algebra.

Definition 2.8. *[31, p. 84] A Lie algebra is a real vector space, V, equipped with a bilinear map, $[\cdot,\cdot] : V \times V \mapsto V$, which satisfies the following properties for all X, Y and Z in V.*

i $[X, X] = 0$

ii $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

However Lie groups and Lie algebras are not unrelated. For every real Lie group has a Lie algebra where the underlying vector space is the left invariant vector fields of the Lie group.

Definition 2.9. *[31, p. 85-86] Let G be a Lie group. Let g be in G and define the left translation map $L_g : G \mapsto G$ by $L_g(h) = gh$. We say a vector field over G, say X is left invariant if $(L_g)_*X = X$. The Lie algebra of G, Lie(G), is defined to be the set of all left invariant vector fields over G. The bracket will be the Lie bracket. One can check that this is well defined since if X and Y are left invariant vector fields then so is $[X, Y]$.*

We know that $\mathrm{GL}(\mathbb{R}, n)$ is a Lie group and its Lie algebra is isomorphic with $\mathfrak{gl}(\mathbb{R}, n)$. We can define maps between Lie groups.

Definition 2.10. *[31, p.89-90] Let G and H be Lie groups. A map $\varphi : G \to H$ is a (Lie group) homomorphism if φ is C^∞ and φ is a group homomorphism. We say φ is a (Lie group) isomorphism if it is also a diffeomorphism. A map ψ is a Lie algebra homomorphism between Lie algebras, $\mathfrak{g}$ and $\mathfrak{h}$, if it preserves the bracket and it is linear. That is to say for X and Y in $\mathfrak{g}$ that $[\psi(X), \psi(Y)] = [X, Y]$. If you have a Lie group homomorphism between Lie groups G and H, φ, then the derivative map $d\varphi$ is a Lie algebra homomorphism [31, p. 90]. If $H = Aut(V)$ for some vector space V then a homomorphism $\varphi : G \to H$ is called a representation of the Lie group G. An example for H would be $GL(\mathbb{R}, n)$ since matrices correspond to linear transformations.*

We now introduce the *exponential map*. Let G be a Lie group with a Lie algebra $\mathfrak{g}$. We take a homomorphism of the Lie algebra of $\mathbb{R}$ into $\mathfrak{g}$. Where X is in $\mathfrak{g}$ and λ is a real scalar.

$$\lambda\partial_r \to \lambda X$$

By Warner, there is a unique 1-parameter group which we will denote by $\exp_X : \mathbb{R} \to G$ such that the tangent map satisfies $d(\exp_X)(\lambda\partial_r) = \lambda X$ [31, p. 102]. We define the exponential map $\exp : \mathfrak{g} \to G$ by $\exp(X) = \exp_X(1)$. In the case where $G = \mathrm{GL}(\mathbb{R}, n)$, the exponential map is given by matrix exponentiation. Let A be in $\mathfrak{gl}(\mathbb{R}, n)$, then [31, p. 105]:

$$\exp(A) = e^A = I + A + \frac{A^2}{2!} + \dots$$

Clearly e^A is in $\mathrm{GL}(\mathbb{R}, n)$, since its eigenvalues are given by e^{γ_i} where γ_i are the eigenvalues for A. The reason being that $e^{\gamma_i} \neq 0$. We have the following theorem:

Theorem 2.11. *[31, p. 104] Let G and H be Lie groups. Let $\varphi : G \to H$ be a homomorphism. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \end{array}$$

We now introduce the *adjoint representation*. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then we define a map $a : G \times G \to G$ by $a(\sigma, \tau) = a_\sigma(\tau) = \sigma\tau\sigma^{-1}$. The map, $\sigma \to da_\sigma$, sends a group element to automorphisms of $\mathfrak{g}$ and is thus a representation of G. We call this map $\mathrm{Ad} : G \to Aut(\mathfrak{g})$. We define ad to be the derivative map $d(\mathrm{Ad})$. We denote $\mathrm{Ad}(\sigma) = \mathrm{Ad}_\sigma$ and $\mathrm{ad}(X) = \mathrm{ad}_X$[31, p. 113-114].

We have the following commutative diagrams from p.114 of Warren's book.

$$\begin{array}{ccc} G & \xrightarrow{Ad} & Aut(\mathfrak{g}) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{a_\sigma} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{Ad_\sigma} & \mathfrak{g} \end{array}$$

We have the following simplification when $G = GL(\mathbb{R}, n)$. Let B be in G and C in $\mathfrak{g} = \mathfrak{gl}(\mathbb{R}, n)$. We have that $\mathrm{Ad}_B(C) = da_B(C)$. It is clear that C is the tangent for the curve $\gamma = e^{Ct}$ at $t = 0$. We now use the formula $da_B(C) = \frac{d}{dt}\Big|_{t=0} (a_B \circ \gamma)$. From p. 114 of Warren's book we have the following:

$$\mathrm{Ad}_B(C) = da_B(C) = \frac{d}{dt}\Big|_{t=0} (B^{-1}e^{Ct}B) = \frac{d}{dt}\Big|_{t=0} (e^{B^{-1}CBt}) = B^{-1}CB$$

Lastly we have the following useful formula for ad [31, p. 115]. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let X and Y be in $\mathfrak{g}$.

$$\mathrm{ad}_X(Y) = [X, Y] \tag{2.11.1}$$

From this formula we can see that if the Lie group G is Abelian then Ad is constant. Thus ad is 0 and we can conclude that the bracket in the Lie algebra of the Lie group is 0.

2.12 Group Actions

In this section we introduce the concept of a *group action* which we will use frequently later on.

Definition 2.13 (Topological Group). *[5, p. 1] We say G is a topological group if it is a Hausdorff space with a continuous multiplication $G \times G \mapsto G$ which makes G a group and such that the map $g \mapsto g^{-1}$ from $G \mapsto G$ is continuous.*

Definition 2.14 (Topological Transformation Group). *[5, p. 32] A topological transformation group is a triple (G, M, Θ) where G is a topological group, M is a Hausdorff topological space and $\Theta : G \times M \mapsto M$ is a map that satisfies the* *following.*

$$\Theta(g, \Theta(h, x)) = \Theta(gh, x) \qquad \textit{For all } x \in M \textit{ and } g, h \in G$$
$$\Theta(e, x) = x \qquad \textit{For all } x \in M, \textit{ where } e \textit{ is the identity element of } G$$

We call Θ the *group action* and M a G-space. We distinguish between right and left G-spaces when the action is either written with the group element on the right or on the left. That is $\Theta(g, x) = gx$ or $\Theta(g, x) = xg$. When the type of G-space is not stated it will be assumed to be a left G-space. There is a notion of equivalence of topological transformation groups. To do so we need a map between the two G-spaces which commutes with the group action.

Definition 2.15 (Equivariant Map)**.** *[5, p. 35] An equivariant map $\varphi : M \mapsto N$ between G spaces M and N is a map which satisfies the following.*

$$\varphi(gx) = g\varphi(x) \qquad \textit{For all } x \in M \textit{ and all } g \in G.$$

When there is an equivariant map the actions on M and N are said to be equivalent.

Note that the inverse of an equivariant map is also an equivariant map when it's a homeomorphism.

We now move on to the *isotropy group*, G_x, where x is a point in M. This is the subgroup of G which fixes x defined by $G_x = \{g \in G \mid gx = x\}$ [5, p. 35]. We have that $G_{gx} = gG_xg^{-1}$ [5, p. 35]. If $G_x = \{e\}$ for all x in M then the group action is said to be *free* [5, p. 36].

A mapping Φ between topological spaces U and V is said to be *proper* if $\Phi^{-1}(K)$ is compact in U whenever K is compact in V. A group action is said to be *proper* if a mapping from $G \times M$ to $M \times M$ given by $(g, x) \to (gx, x)$ is a proper mapping [9, p. 53]. We can further analyze how G acts on M by looking at the subset of M that we get when we let the the whole group G act on a point x. This subset is called an *orbit* and is defined by: $G(x) = \{g(x) \in M \mid g \in G\}$ [5, p. 37]. It is clear that the orbits are disjoint otherwise non-disjoint orbits would combine together to form one orbit. Let M/G be the set of orbits of all points in M. Let $\pi : M \mapsto M/G$ map points to their orbit. We call M/G, endowed with the quotient topology (U is open in M/G iff $\pi^{-1}(U)$ is open in M), the *orbit space* [5, p. 37]. We can now imagine taking an orbit and mapping it to one of its points. We say that a *cross section for* $\pi : M \mapsto M/G$ is a continuous map $\sigma : M/G \mapsto M$ such that $\pi \circ \sigma$ is the identity on M/G. More often we have a *local cross section* which is defined for an open subset $U \subset M/G$ [5, p. 39].

Theorem 2.16. *[9, p. 94] Let x be in M. There is an equivariant bijection between G/G_x and $G(x)$ which we denote by B_x. It is given by $B_x(gG_x) = gx$.*

We can go further.

Theorem 2.17. *[9, p. 53] Let G be a Lie group, let M be a C^k manifold for $k \geq 1$ and the group action is proper and free. Then the orbit space M/G has a structure of a C^k manifold with dimension equal to the dimension of M minus the dimension of G. The topology of G/M is the quotient topology.*

We define the *type of an orbit* $G(x)$, Type($G(x)$), to be the equivalence class of $G(x)$ where equivalence between orbits occurs when there is equivariant bijection mapping between them [9, p. 107]. Since B_x is an equivariant map between $G(x)$ and G/G_x we can use G/G_x as the representative of the equivalence class Type($G(x)$). Now we use the following theorem in order to give an easier way of proving that two types are equivalent.

Theorem 2.18. *[9, p. 107] Let G be a Lie group and H and K be closed subgroups. Then there exists an equivariant map that maps G/H to G/K iff H is conjugate to a subgroup of K. This in turn implies that Type(G/G_x) = Type(G/G_y) iff G_x and G_y are conjugate.*

In Kolk he defines various equivalence relations of orbits but they turn out to all be the same as the definition of orbit types when we restrict ourselves to our cohomogeneity 2 situation. We simply state them as properties of orbit types to avoid confusion.

Theorem 2.19. *[9, p. 109] Let x and y be in our manifold M. Let $G(x)$ and $G(y)$ be the corresponding orbits. If Type($G(x)$) = Type($G(y)$) then there is a G-equivariant diffeomorphism from a neighbourhood U of x to a neighbourhood V of y.*

We define $M_x = \{y \in M \mid \text{Type}(G(x)) = \text{Type}(G(y))\}$ [9, p. 109].

Definition 2.20. *[9, p. 115-116] For a proper C^k action of a Lie group G on a manifold M, the orbit $G(x)$ at x in M is said to be a principal orbit if M_x is open in M.*

Theorem 2.21. *[9, p. 116] Let x and y be in M. If Type($G(x)$) = Type($G(y)$) then there exists a g in our Lie group G such that $Ad_{g^{-1}}(\mathfrak{g}_x) = \mathfrak{g}_y$. Where $\mathfrak{g}_x$ and $\mathfrak{g}_y$ are the Lie algebras at x and y respectively.*

Theorem 2.22. *[9, p. 117] If an orbit $G(x)$ at x in M is a principal orbit then the type of the orbits going through points in a neighbourhood of x are the same. This is equivalent to saying that the dimensions of the orbits near x are the same.*

Let $M_{reg} = \{x \in M \mid G(x)$ is a principal orbit$\}$ [9, p. 117]. Now we introduce the principal orbit theorem.

Theorem 2.23. *[9, p. 118] Suppose that a Lie group G is acting properly in a C^1 way on a connected manifold M. Then $M \backslash M_{reg}$ is the union of all points whose orbits are of codimension n, where $n \geq 2$. The subset M_{reg} is connected, open and dense. It follows that points in the orbit space which corresponds to principal orbits form a connected, open and dense subset of the orbit space. Furthermore, there is only 1 orbit type amongst the principal orbits*

Given a group action Θ with a Lie group G acting on a manifold M, we can define a tangent map at (g, x) which we will call $T_{(g,x)}\Theta$. Where g is in G and x is in M. Here $T_{(g,x)}\Theta : T_{(g,x)}(G \times M) \to T_{gx}M$. You can partly think of it has taking a tangent at a point and moving it onto a different point via the group action. However you also have to factor in the Lie algebra of G.

2.24 Principal Fibre Bundles

In this section we will define principal fibre bundles and connections over them.

Definition 2.25. *[20, p. 50] A principal fibre bundle consists of a total space, P, a base space M and a Lie group G which acts on P on the right. The principal fibre bundle is denoted by $P(M,G)$. The spaces M and P are smooth manifolds. The group action of G on P satisfies the following three properties*

i G acts freely on P.

ii M is the quotient space P/G. Therefore M consists of equivalence classes of elements in P. We say that two elements in P, x and x', are equivalent if there exists an a in G such that $x = x'a$. Furthermore, we have that the canonical projection, $\pi : P \to M$, which maps x in P to its equivalence class $\pi(x)$ is differentiable.

iii We have that P is locally trivial. That is for all m in M there is a neighbourhood U of m such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. It is isomorphic in the sense that there exists a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(x) = (\pi(x), \varphi(x))$. Here φ is equivariant in the sense that for all a in G and all x in P, we have that $\varphi(xa) = \varphi(x)a$.

We now present an example of a principal fibre bundle; the bundle of linear frames [20, p. 55-56]. Let M be the base manifold of dimension n. We take a set of all collections of n linearly independent tangent vectors at a point

m for all points in M. This manifold, $L(M)$, is the bundle of linear frames. For our group we take GL$(n, \mathbb{R})$ which acts on $L(M)$ in the following way. It keeps the point constant but changes the frame. Let $u = ((X_1, \ldots, X_n), m)$ be a linear frame at m. Then $ua = ((Y_1, \ldots, Y_n), m)$ where $Y_i = \Sigma_{j=1}^n a_i^j X_j$. We take our projection π to map a linear frame at a point m to the point itself. Clearly π is differentiable. This group action is free, consider a such that $ua = u$. Then we have that $X_i = \Sigma_{j=1}^n a_i^j X_j$. We can convert this into matrix form by setting the components of the matrix X to be $X_{ij} = X_j^i$, where X_j^i is the component of X_j with respect to some local coordinates. Then we have $X = aX$. Since the frame consists of linearly independent vectors we know that X is invertible so we have that $a = I$. We now check local triviality. Let $u = ((X_1, \ldots, X_n), m)$ be a linear frame at m and $\psi(u) = (\pi(u), \varphi(u))$ where ψ is defined on all of $L(M)$. We let $\varphi(u) = X \in$ GL$(n, (R))$. We take $ua = ((Y_1, \ldots, Y_n), m)$ where $Y_i = \Sigma_{j=1}^n a_i^j X_j$ and let the components of the matrix Y be $Y_{ik} = Y_k^i = \Sigma_{j=1}^n a_i^j X_{kj}$. It is clear by matrix multiplication that $\varphi(ua) = Y = Xa = \varphi(u)a$.

We now construct a fibre bundle associated with a principal bundle [20, p. 54-55]. Let $P(M, G)$ be a principal fibre bundle and let F be a manifold on which G acts on the left. We take the product manifold $P \times F$ and define a right group action on it. Let $(x, \xi) \in P \times F$ and let a be in G. Then $(x, \xi)a = (xa, a^{-1}\xi)$. We define $E = P \times_G F$ to be the quotient space under this group action. We can define a projection, π_E, of E onto M by $\pi_E([x, \xi]) = \pi(x)$. Clearly the output of π_E does not depend on the representative. Let m be a point in M and U be an open neighbourhood in U containing m. There is an isomorphism between $\pi_E^{-1}(U)$ and $U \times F$. To see this let $\psi([x, \xi]) = (\pi_E([x, \xi]), \varphi([x, \xi]))$. Let σ be a local cross section mapping U into P. Let $\varphi([x, \xi]) = a\xi$ where $x = va$ where v is a value of the cross section. We know that $\varphi([x, \xi])$ is well defined since if we take xb and $b^{-1}\xi$we have that $xb = v(ab)$ which implies $\varphi([xb, b^{-1}\xi]) = abb^{-1}\xi = a\xi = \varphi([x, \xi])$. Furthermore $\psi([x, \xi])$ is injective since if $\psi([x_1, \xi_1]) = \psi([x_2, \xi_2])$ then $\pi(x_1) = \pi(x_2)$. So we have that $x_2 = x_1 b = vab$. Therefore $a\xi_1 = ab\xi_2$ which implies $\xi_2 = b^{-1}\xi_1$. This in turn implies that $[x_1, \xi_1] = [x_2.\xi_2]$. Next we will check that $\psi[x, \xi]$ is surjective. Let $\psi[x, \xi] = (m, \zeta)$. There clearly exists x such that $\pi(x) = m$. Furthermore there exists ξ such that $a\xi = \zeta$. Namely $\xi = a^{-1}\zeta$. Therefore we can conclude that ψ is an isomorphism. The *fibre bundle associated with a principal bundle with standard fibre* F is denoted $E(M, F, G, P)$.

An example of such a fibre bundle is the tangent bundle of a manifold M, TM [20, p. 56]. Here the principal bundle is $L(M)$, the group is GL$(n, \mathbb{R})$) and the standard fibre is $\mathbb{R}^n$. Here $L(M)$ provides a basis, $\mathbb{R}^n$ provides the components and the group action takes care of equivalence under a change of basis. This example provides a succinct way of thinking of the tangent bundle.

We move on to connections on a principal fibre bundle. Let $P(M,G)$ be a principal fibre bundle. Let x be in P. We define G_x to be the tangent space through the fibre at x. Thus there is a natural way to define a vertical space. However, we must introduce the notion of a connection to assign a horizontal space at all x.

Definition 2.26. *[20, p. 63] We define a connection Γ on a principal bundle $P(M,G)$ to be an assignment to every x in P an horizontal space Q_x which has the following properties.*

i *We can uniquely decompose a tangent vector into horizontal part and a vertical part. That is $T_xP = G_x \oplus Q_x$*

ii *We have that Q_x respects right translation. Here $R_a(x) = xa$. We have that for all a and x that $Q_{xa} = (R_a)_*Q_x$.*

iii *Lastly, Q_x depend differentiably on x. That is if X is a differentiable vector field then so are its horizontal and vertical parts.*

For all A in $\mathfrak{g}$, the Lie algebra of G, we have that $a_t = e^{tA}$ is in G and we define the *fundamental vector field* of A to be given by $A^*_x = \frac{d}{dt}(xa_t)\Big|_{t=0}$ [20, p. 42]. Clearly A^* is vertical. For each X in T_xP, we define the *connection 1-form* ω by $\omega(X) = A$. Here A is the unique element in $\mathfrak{g}$ whose fundamental vector field is the vertical part of X. Clearly $\omega(X) = 0$ if and only if X is horizontal. For if $A = 0$ then a_t is constant so $A^*_x = 0$. We have that the connection 1-form satisfies the following properties [20, p. 64]

i $\omega(A^*) = A$, for all A in $\mathfrak{g}$.

ii $\omega((R_a)_*X) = ad(a^{-1})\omega(X)$ for all X in T_xP and all a in G.

As a consequence of i and ii, for every 1-form ω that satisfies i and ii there is a unique connection with its connection 1-form being ω. This is done by defining the vertical and horizontal parts of a vector X to be $\omega(X)$ and $X - \omega(X)$ respectively.

2.27 Slices and Tubes

We now return to group actions. Suppose we have a left group action acting on M via Lie group G. Denote $A_x : G \to M$ with $A_x(g) = gx$. Let $\mathfrak{g}$ be the Lie algebra of G. Let X be in $\mathfrak{g}$ and the map $\alpha_x : \mathfrak{g} \to T_xM$ be given by $\alpha_x(X) = dA_x(X)$. With this map α in mind we can define a slice at a point x_0.

Definition 2.28. *[9, p. 98] Let M be a manifold with a Lie group G acting on it in a C^k manner. A C^k slice S, at a point x_0 in M, is a C^k submanifold of M that goes through x_0 such that the following holds.*

i *The tangent at* x_0 *decomposes into a part tangent to* G *and to* S*;* $T_{x_0}M = \alpha_{x_0}(\mathfrak{g}) \oplus T_{x_0}S$*. And also for* x *in* S*, we have* $T_xM = \alpha_x(\mathfrak{g}) + T_xS$*, meaning the part tangent to* G *may have grown in dimension.*

ii *We have that* S *is* G_{x_0} *invariant. That is* $A_x(G_{x_0}) \subset S$ *for all* $x \in S$*.*

iii *If* $x \in S$ *,* $g \in G$ *and* $gx \in S$*. Then* $g \in G_{x_0}$*.*

Let S be a slice through x_0. Let $x \to x$ be the identity map from S to M. Then it induces a homeomorphism $G_{x_0} \cdot x \to G \cdot x$ mapping the orbit space S/G_{x_0} to the orbit space M/G. To see this we know that the map is well defined since if $y = gx$ where g is in G_{x_0}, then g is in G thus $G \cdot x = G \cdot y$. Also, we are using (ii) implicitly to guarantee that S is G_{x_0} invariant. For injectivity, suppose that $G \cdot y = G \cdot x$. Then there exists g in G such that $y = gx$. Now since y and x are in S we have that by (iii) of the above definition that g is in G_{x_0}. Therefore $G_{x_0} \cdot x = G_{x_0} \cdot y$. Surjectivity follows by restricting the codomain to the range. The homeomorphic part follows from the definition of the quotient topology which is the given topology on each of the orbit spaces and the fact that the inclusion map is a homeomorphism.

To showcase the local nature of a slice through a point we consider the real projective plane, $\mathbb{RP}^2$. The manifold $\mathbb{RP}^2$ is the quotient space of $\mathbb{R}^3\backslash\{0\}$ under the following equivalence relation. Let p and q be in $\mathbb{R}^3\backslash\{0\}$, then $p \sim q$ iff there exists a non-zero real number λ such that $p = \lambda q$. We introduce homogeneous coordinates for $\mathbb{RP}^2$ which are the equivalences classes, $[x, y, z]$. We call the quotient map, π. We can define 3 coordinate charts, φ_x, φ_y and φ_z with respective domains B_x, B_y and B_z.

$$\varphi_x([x,y,z]) = \left(\frac{y}{x}, \frac{z}{x}\right) \qquad B_x = \{[x,y,z] \,|\, x \neq 0\}$$
$$\varphi_y([x,y,z]) = \left(\frac{x}{y}, \frac{z}{y}\right) \qquad B_y = \{[x,y,z] \,|\, y \neq 0\}$$
$$\varphi_z([x,y,z]) = \left(\frac{x}{z}, \frac{y}{z}\right) \qquad B_z = \{[x,y,z] \,|\, z \neq 0\}$$

To see that φ_x is injective we note that if $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ and $\frac{z_1}{x_1} = \frac{z_2}{x_2}$ then $[x_1, y_1, z_1] = [x_2, y_2, z_2]$. For surjectivity we simply take outputs of points in $\mathbb{RP}^2$ of the form $[1, y, z]$. To check that is a homeomorphism, suppose that U is open in $\mathbb{RP}^2$. Then by definition of the quotient topology we have that the $\pi^{-1}(U)$ is open. We define that map $\Phi_x : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^2$ by $\Phi_x(x, y, z) = (\frac{y}{x}, \frac{z}{x})$. Clearly Φ_x is continuous on its domain. Therefore $\Phi_x(\pi^{-1}(U)) = \varphi_x(U)$ is open. Therefore φ_x is an open map. Now suppose V is open in $\mathbb{R}^2$. We have that $\varphi_x^{-1}(V)$ is open iff $\pi^{-1}(\varphi_x^{-1}(V)) = W$ is open in $\mathbb{R}^3\backslash\{0\}$. We have that $W = \{(x, y, z) \in \mathbb{R}^3\backslash\{0\} \,|\, \exists\lambda \neq 0 \,\&\, \exists(u, v) \in V \; s.t \, (x, y, z) = (\lambda, \lambda u, \lambda v)\}$. Clearly W is open in $\mathbb{R}^3\backslash\{0\}$. The same reasoning applies to φ_y and φ_z. To check compatibility let $(u, v) \in \varphi_x(B_x \cap B_y)$. Then $\varphi_x \circ \varphi_y^{-1}(u, v) = \varphi_x([u, 1, v]) = (\frac{1}{u}, \frac{v}{u})$. Clearly this transition function is C^∞. It's the same story for the other compatibility checks.

Now we are interested in the derivative of the quotient map, $d\pi : T_{(x,y,z)}\mathbb{R}^3\backslash\{0\} \to T_{[x,y,z]}\mathbb{RP}^2$. This is so that later on we can map vectors back and forth from the unit sphere to $\mathbb{RP}^2$. Suppose $x \neq 0$ and the coordinates for $\mathbb{RP}^2$ at $[x, y, z]$ are $\varphi_x = (y_x, z_x)$. We can write that $d\pi(\partial_x) = a_x\partial_{y_x} + b_x\partial_{z_x}$. We have that $a_x = \partial_x(y_x \circ \pi) = \partial_x\left(\frac{y}{x}\right) = -\frac{y}{x^2}$. And $b_x = \partial_x(z_x \circ \pi) = \partial_x(\frac{z}{x}) = -\frac{z}{x^2}$. So $d\pi(\partial_x) = -\frac{y}{x^2}\partial_{y_x} - \frac{z}{x^2}\partial_{z_x}$. Furthermore we can state that $d\pi(\partial_y) = a_y\partial_{y_x} + b_y\partial_{z_x}$. So $a_y = \partial_y\left(\frac{y}{x}\right) = \frac{1}{x}$ and $b_y = \partial_y\left(\frac{z}{x}\right) = 0$. Therefore $d\pi(\partial_y) = \frac{1}{x}\partial_{y_x}$. Finally we can state that $d\pi(\partial_z) = a_z\partial_{y_x} + b_z\partial_{z_x}$. Then $a_z = \partial_z\left(\frac{y}{x}\right) = 0$ and $b_z = \partial_z\left(\frac{z}{x}\right) = \frac{1}{x}$. Therefore $d\pi(\partial_z) = \frac{1}{x}\partial_{z_x}$.

We now introduce a group action on $\mathbb{RP}^2$. We spin about the x-axis. That is that S^1 is acting on $\mathbb{RP}^2$ by $\theta[x, y, z] = [x, y\cos(\theta) - z\sin(\theta), y\sin(\theta) + z\cos(\theta)]$. This group action is clearly well defined since rotation and scaling commute. Assume $x \neq 0$. Let's try to find the part of the tangent space that is tangent to the orbit. If $(y, z) = 0$ then the orbit is 0-dimensional so the tangent space is just the 0 vector. Now if $(y, z) \neq 0$, then the orbit is 1-dimensional. Let $p \in B_x$. So then X_θ tangent to the orbit is given by:

$$\begin{aligned}
X_\theta &= \partial_\theta(y_x \circ \theta p)\Big|_{\theta=0}\partial_{y_x} + \partial_\theta(z_x \circ \theta p)\Big|_{\theta=0}\partial_{z_x} \\
&= \partial_\theta\left(\frac{y\cos(\theta) - z\sin(\theta)}{x}\right)\Big|_{\theta=0}\partial_{y_x} + \partial_\theta\left(\frac{y\sin(\theta) + z\cos(\theta)}{x}\right)\Big|_{\theta=0}\partial_{z_x} \\
&= -\frac{z}{x}\partial_{y_x} + \frac{y}{x}\partial_{z_x}
\end{aligned}$$

We need to choose a metric on $\mathbb{RP}^2$. To do so we use the round metric on S^2. We take π_{S^2} to be the quotient map restricted to S^2. Let $q \in S^2$. We need the inverse of $(d\pi_{S^2})_q$ thus we will show that the derivative map is $(d\pi_{S^2})_q$ is injective. To see this let $X = a\partial_x + b\partial_y + c\partial_z$ be tangent to S^2. Then $X \cdot r = 0$, where $r = x\partial_x + y\partial_y + z\partial_z$. That is $a = -\frac{by+cz}{x}$. Let X and Y be tangent to S^2 suppose the following:

$$\begin{gathered}
(d\pi_{S^2})_q(X) = (d\pi_{S^2})_q(Y) \\
(d\pi_{S^2})_q(-\frac{by + cz}{x}\partial_x + b\partial_y + c\partial_z) = (d\pi_{S^2})_q(-\frac{b'y + c'z}{x}\partial_x + b'\partial_y + c'\partial_z) \\
-\frac{by + cz}{x}(-\frac{y}{x^2}\partial_{y_x} - \frac{z}{x^2}\partial_{z_x}) + \frac{b}{x}\partial_{y_x} + \frac{c}{x}\partial_{z_x} = -\frac{b'y + c'z}{x}(-\frac{y}{x^2}\partial_{y_x} - \frac{z}{x^2}\partial_{z_x}) + \frac{b'}{x}\partial_{y_x} + \frac{c'}{x}\partial_{z_x}
\end{gathered}$$

Comparing coefficients of ∂_{y_x} and ∂_{z_x} we obtain the following:

$$\frac{b(y^2+x^2)+czy}{x^2} = \frac{b'(y^2+x^2)+c'zy}{x^2}$$
$$\frac{c(z^2+x^2)+bzy}{x^2} = \frac{c'(y^2+x^2)+b'zy}{x^2}$$

Or equivalently:

$$(b-b')(y^2+x^2)+(c-c')zy = 0$$
$$(c-c')(z^2+x^2)+(b-b')zy = 0$$

We calculate the determinant for this linear system.

$$\begin{vmatrix} y^2+x^2 & zy \\ zy & x^2+z^2 \end{vmatrix} = (y^2+x^2)(z^2+x^2) - z^2y^2 = (y^2+z^2)x^2 + x^4 > 0$$

Therefore $(d\pi_{S^2})_q$ has an inverse. Let q be in S^2, g be the round metric on S^2 and X, Y be in $T_{[q]}\mathbb{RP}^2$. We will define a metric h on $\mathbb{RP}^2$.

$$h(X,Y)\Big|_{[q]} = g((d\pi|_q)^{-1}X, (d\pi|_q)^{-1}Y)\Big|_q$$

To check that this is well defined the right hand side must be the same if we choose $-q$. Let $\rho : S^2 \to S^2$ be defined by $\rho(q) = -q$. Clearly, $\pi = \pi \circ \rho$. So we have that:

$$\begin{aligned} g((d\pi|_q)^{-1}X, (d\pi|_q)^{-1}Y)\Big|_q &= g((d(\pi\circ\rho)|_q)^{-1}X, (d(\pi\circ\rho)|_q)^{-1}Y)\Big|_q \\ &= g((d\rho|_q)^{-1}\circ(d\pi|_{-q})^{-1}X, (d\rho|_q)^{-1}\circ(d\pi|_{-q})^{-1}Y)\Big|_q \\ &= g((d\pi|_{-q})^{-1}X, (d\pi|_{-q})^{-1}Y)\Big|_{-q} \end{aligned}$$

The last step follows since ρ is an isometry of the round metric. This enables us to find a tangent space normal to the orbit in $\mathbb{RP}^2$. Let Y_θ be in T_qS^2 be given by $Y_\theta = -z\partial_y + y\partial_z$. Clearly $d\pi_{S^2}(Y_\theta) = X_\theta$ and $Y_\theta \cdot r = 0$. Clearly $N_\theta = y\partial_y + z\partial_z + a\partial_x$ is orthogonal to Y_θ using the Euclidean metric on $\mathbb{R}^3$. For N_θ to be in T_qS^2 we must have that $a = -\frac{y^2+z^2}{x}$. Let's project it down to $\mathbb{RP}^2$.

$$\begin{aligned} (d\pi_{S^2})_q(N_\theta) &= (d\pi_{S^2})_q(y\partial_y + z\partial_z - \frac{y^2+z^2}{x}\partial_x) \\ &= \frac{y}{x}\partial_{y_x} + \frac{z}{x}\partial_{z_x} - \frac{y^2+z^2}{x}(-\frac{y}{x^2}\partial_{y_x} - \frac{z}{x^2}\partial_{z_x}) \\ &= \frac{x^2+y^2+z^2}{x^3}(y\partial_{y_x} + z\partial_{z_x}) \end{aligned}$$

To construct a slice at p in B_x we use the normal geodesics. These geodesic have the normal vector as their initial velocity. Because of our choice of metric on $\mathbb{RP}^2$ there is a 1:1 correspondence between the geodesics on S^2 and the geodesics on $\mathbb{RP}^2$. The geodesics on S^2 are the great circles. If we take a point whose orbit is 1 dimensional, the slice would correspond to a great circle which is orthogonal to the orbit. Note that the isotropy group at p is trivial. This geodesic can be extended arbitrarily close to $x = 0$ but can't be extended to $x = 0$ due to the coordinate restrictions. Therefore not every orbit intersects the slice. For instance the orbit whose points have 0 as their x coordinate. If we take a point whose orbit is 0-dimensional than the slice corresponds to a subset of $\mathbb{RP}^2$, call it S, where $S = \{[x, y, z] \mid x \neq 0\}$. We have that S is 2 dimensional but again not every orbit intersects S. Using points B_y and B_z one can analyze the other slices but I believe the local nature of a slice through a point is fully illustrated.

Definition 2.29. *[9, p. 98] A group action is said to be proper at x_0 in M, if for every sequence x_j in M and g_j in G such that $\lim_{j\to\infty} x_j = x_0$ and $\lim_{j\to\infty} g_j x_j = x_0$, there exists a subsequence $j = j(k)$ such that $g_{j(k)}$ converges in G as k goes to ∞.*

Here are some examples. Let $M = \mathbb{R}$ and $G = \mathbb{R}$ act by addition. Let the sequence $x_j \in M$ converge to x_0 and let the sequence $g_j \in G$ satisfy $\lim_{j\to\infty}(g_j \cdot x_j) = x_0$. Clearly $\lim_{j\to\infty}(g_j \cdot x_j) = \lim_{j\to\infty}(g_j + x_j) = x_0$ which implies $\lim_{j\to\infty}(g_j) = 0$. For a counter example take $G = \mathbb{R} - \{0\}$ be a group which acts by multiplication on $\mathbb{R}$. Let $x_j = \frac{1}{j^2}$ and $g_j = j$ where $j \geq 1$. Clearly $\lim_{j\to\infty} x_j = 0$ and $\lim_{j\to\infty}(g_j \cdot x_j) = 0$. But there is no convergent subsequence of g_j. Therefore the group action is not proper at $x = 0$. We have the following theorem.

Theorem 2.30. *[9, p. 99] Existence of a Slice: Consider a C^k group action of the Lie group G on a manifold M and suppose that the action is proper at x_0. Then there exists a C^k slice, S, at x_0.*

We now revisit associated fibre bundles. Let M be a manifold on which a Lie group H acts on the right and N be a manifold in which H acts on the left. Take $M \times_H N$ and we take another Lie group G acting on M from the left. This action is required to commute with the H action. We can then define a G-action on $M \times_H N$. This action is defined by $g[x, y] = [gx, y]$ where g is in G, x in M and y in N [9, p. 101]. This is action is well-defined due to the aforementioned commutativity. We now introduce the tube theorem.

Theorem 2.31. *[9, p. 102-103] Tube Theorem: Let Θ be a C^k action of a Lie group G on a manifold M which is proper at x_0. Then there exists a G invariant neighbourhood U of x_0 in M such that the G-action on U is equivalent to the G-action on $G \times_{G_{x_0}} B$. Where B is an open set containing 0 in $T_{x_0}M/\alpha_{x_0}(\mathfrak{g})$ on which G_{x_0} acts linearly via the tangent action modulo $\alpha_{x_0}(\mathfrak{g})$. Here we refer to $G \times_{G_{x_0}} B$ as the tube.*

Proof. We provide a sketch of the proof. Let S be a slice at x_0. Then by definition the tangent space at x, where x in S, is $\alpha_x(\mathfrak{g}) + T_x S$. In the special case which is our focus, the metric is G invariant. This means that the tangent space is of the above form on all points of the orbit $G \cdot x$. Therefore if we restrict our group action on M to domain $G \times S$ we will find that the tangent map $T_{(g,x)}\Theta$ is surjective. Therefore Θ is a submersion which means it must be an open map. Therefore $\Theta(G \times S)$ is an open G- invariant neighbourhood containing x_0. Next suppose x and y are in S and g and h are in G. Then if $gx = hy$ we have that $y = h^{-1}gx$. So by the third property of the slice we have that $h^{-1}g = k \in G_{x_0}$. Then we have that $(h, y) = (gk^{-1}, kx)$ which means that $[h, y] = [g, x] \in G \times_{G_{x_0}} S$. Next suppose there exist a k in G_{x_0} such that $(h, y) = (gk^{-1}, kx)$. This is well posed by the G_{x_0} invariant property of the slice. It is clear that $hy = hk^{-1}kx = gx$. Therefore we have a bijective equivariant map $\Phi : G \times_{G_{x_0}} S \to U$ defined by $\Phi([g, x]) = gx$. We have therefore established the equivalence of the G-actions while leaving out some minor details. Note that slice corresponds to an open set in the tangent space modulo α_{x_0} since the slice is constructed with the normal exponential map which gives a correspondence between normal vectors and points along the corresponding geodesic. You can see this in the $\mathbb{RP}^2$ example. □

2.32 Irreducible Representations

A real *representation* of a group G onto a finite dimensional real vector space V is a homomorphism $\rho : G \to GL(V)$ [11, p. .3] For instance if $G = S^1$ and $V = \mathbb{R}^2$ then ρ would map an element of S^1 to a rotation map which rotates a vector in $\mathbb{R}^2$. This rotation map would be given by a matrix and it is useful to think of elements of $GL(V)$ as 2x2 matrices. Let g be in G and v be in V. Then $\rho(g)v$ is denoted by gv. We often refer to representations by their vector space.

A *map between representations* V and W of G , φ, is a vector space map which commutes with the group [11, p. 103].

A *subrepresentation* of V is a subspace W of V which is invariant under G. This means that for all w in W and g in G, that gw is in W [11, p. 104]. A representation is *irreducible* if its only subrepresentations are 0 and the entire vector space [11, p. 104]. We can actually decompose any representation into irreducible subrepresentations.

Proposition 2.33. *[11, p. 6] Any representation of a compact Lie group is a direct sum of irreducible representations. The proof is in Fulton and Harris.*

This brings us to Schur's Lemma which we will heavily use later.

Lemma 2.34. *[11, p. 7] If V and W are irreducible representations of G and we have a G-equivariant map $\varphi : V \to W$, then we have that:*

i either φ is an isomorphism or $\varphi = 0$.

ii If $V = W$ as G-representations, then $\varphi = \lambda I$ where λ is a complex number and I is the identity map.

Let V be a finite dimensional vector space. We now present the correspondence between representations on the m-degree symmetric tensors, $S^m(V)$, and representations on m-degree symmetric homogeneous polynomials $P^m(V)$. To begin we consider $V = \mathbb{R}^2$ and $G = S^1$ acting on V by rotation. Let e_1 and e_2 be an orthonormal basis of V and a and b be the components. We have the following for g in S^1.

$$g\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

In particular $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $g(e_1) = \cos(\theta)e_1 + \sin(\theta)e_2$. Also, we have that $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $g(e_2) = -\sin(\theta)e_1 + \cos(\theta)e_2$. We now define a representation on the space of functions.

Definition 2.35. *Let V and W be G-spaces and $f : V \to W$. Let p be in V. Then the representation on functions is defined by the following [11, p. 4]*

$$(g(f))(p) = f(g^{-1}p) \tag{2.35.1}$$

We now return to the above example and consider linear functionals x and y being the duals of e_1 and e_2 respectively. We can express $g(x)$ and $g(y)$ in the following way:

$$g(x) = Ax + By \qquad g(y) = Cx + Dy$$

For instance we can determine A by using as input, e_1 into the first equation.

$$g(x)(e_1) = x(g^{-1}e_1) = x(\cos(\theta)e_1 + \sin(-\theta)e_2) = \cos(\theta)$$

Repeating this for the other coefficient yields the following equations:

$$g(x) = \cos(\theta)x + \sin(\theta)y \qquad g(y) = -\sin(\theta)x + \cos(\theta)y$$

This formula is the same as the one for the orthonormal basis. This will be exploited.

Definition 2.36. *Let V be a G-space and consider a tensor $v = v_1 \otimes \ldots \otimes v_k$ in $S^m(V)$. Then G acts on v by [11, p. 4]*

$$g(v) = g(v_1) \otimes \ldots \otimes g(v_k)$$

This can be extended to arbitrary tensors in $S^m(V)$ through linearity.

Returning to our $\mathbb{R}^2$ example we can map symmetric tensors in $S^m(V)$ comprising of tensor products of e_1 and e_2 to homogeneous symmetric polynomials involving x and y. This turns out to be an equivariant map. Here's an example using $S^2(V)$:

$$a(e_1 \otimes e_1) + b(e_1 \otimes e_2 + e_2 \otimes e_1) + c(e_2 \otimes e_2) \leftrightarrow ax^2 + 2bxy + cy^2$$

Now we wish to use this to find irreducible representations of $S^2(V)$. Let $T = a(e_1 \otimes e_1) + b(e_1 \otimes e_2 + e_2 \otimes e_1) + c(e_2 \otimes e_2)$ and $U = ax^2 + 2bxy + cy^2$. We can act on each part of U individually to obtain the following:

$$\begin{aligned}
g(x^2) &= (\cos(\theta)x + \sin(\theta)y)^2 = \cos^2(\theta)x^2 + \sin^2(\theta)y^2 + 2\cos(\theta)\sin(\theta)xy \\
&= \frac{\cos^2(\theta) - \sin^2(\theta)}{2}x^2 + \frac{\cos^2(\theta) + \sin^2(\theta)}{2}x^2 - \frac{\cos^2(\theta) - \sin^2(\theta)}{2}y^2 + \frac{\cos^2(\theta) + \sin^2(\theta)}{2}y^2 \\
&\ldots + 2\cos(\theta)\sin(\theta)xy \\
&= \cos(2\theta)\frac{x^2 - y^2}{2} + \frac{x^2 + y^2}{2} + \sin(2\theta)xy \\
g(y^2) &= (-\sin(\theta)x + \cos(\theta)y)^2 = \sin^2(\theta)x^2 + \cos^2(\theta)y^2 - 2\cos(\theta)\sin(\theta)xy \\
&= -\cos(2\theta)\frac{x^2 - y^2}{2} + \frac{x^2 + y^2}{2} - \sin(2\theta)xy \\
g(2xy) &= 2(\cos(\theta)x + \sin(\theta)y)(-\sin(\theta)x + \cos(\theta)y) = 2(-(x^2 - y^2)\cos(\theta)\sin(\theta) + (\cos^2(\theta) - \sin^2(\theta))xy) \\
&= -(x^2 - y^2)\sin(2\theta) + 2\cos(2\theta)xy
\end{aligned}$$

We can now express $g(U)$.

$$g(U) = \left(\frac{a-c}{2}\cos(2\theta) - 2b\sin(2\theta)\right)\frac{x^2 - y^2}{2} + 2\left(\frac{a-c}{2}\sin(2\theta) + 2b\cos(2\theta)\right)xy + (a+c)\frac{x^2 + y^2}{2} \tag{2.36.1}$$

Thus we have that g acts on Span$(x^2 - y^2, 2xy)$ by rotation through twice the angle θ and g acts on Span(x^2+y^2) by the identity map. Therefore the irreducible representations are ρ^2 and $\mathbb{1}$. Here $\rho^2(g) = \rho(gg)$ where g is in S^1. We note that an arbitrary element of Span$(x^2 - y^2, 2xy)$ is of the form $\frac{a-c}{2}\frac{x^2-y^2}{2} + 2bxy$.

Chapter 3

Harmonic Map Equations

3.1 Reduction of the form of the Metric

For ease of reference we state the following equations which are the conclusions of this chapter. First of all under certain assumptions we can state the metric g on our manifold M in the following way [18, p. 3].

$$g = e^{2\alpha}(dr^2 + dz^2) - f^{-1}r^2dt^2 + f_{ij}(d\phi^i + v^i dt)(d\phi^j + v^j dt) \tag{3.1.1}$$

The first equation below describes the vertical part of the Ricci flat condition. This is the harmonic map equation in matrix form. The second equation is for the partial derivatives of the exponent α of the conformal factor of the metric on the base. This equation follows from the horizontal part of the Ricci flat condition. The third equation relates the twist potentials to the components of the metric on the fibre. The next two equations are the harmonic map equations in component form. The last two equations are for the partial derivatives of α stated in component form. In the rest of the chapter will we make clear all the terms in these equations.

$$0 = \frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right) \tag{3.1.2}$$

$$\alpha_r = \frac{r}{8}\left(Tr(H^{-1}H_rH^{-1}H_r) - Tr(H^{-1}H_zH^{-1}H_z) - \frac{4}{r^2}\right) \qquad \alpha_z = \frac{r}{4}Tr(H^{-1}H_rH^{-1}H_z) \tag{3.1.3}$$

$$\omega_z = fr^{-1}v_r^TF \qquad \omega_r = -fr^{-1}v_z^TF \tag{3.1.4}$$

$$0 = \Delta_g f_{ij} - f^{kl}\nabla^n f_{ik}\nabla_n f_{lj} + f^{-1}\nabla^n\omega_i\nabla_n\omega_j \tag{3.1.5}$$

$$0 = \Delta_g \omega_i - f^{jk} \nabla^n f_{jk} \nabla_n \omega_i - f^{jk} \nabla^n f_{ki} \nabla_n \omega_j \tag{3.1.6}$$

$$\alpha_r = \frac{r}{8}\Big(\log f)_r^2 - \log(f)_z^2 - 4\frac{(\log f)_r}{r} + Tr(F^{-1}F_rF^{-1}F_r) - Tr(F^{-1}F_zF^{-1}F_z) \dots$$
$$\dots + \frac{2}{f}\omega_r F^{-1}\omega_r^T - \frac{2}{f}\omega_z F^{-1}\omega_z^T\Big) \tag{3.1.7}$$

$$\alpha_z = \frac{r}{4}\left((\log f)_r (\log f)_z - 2\frac{(\log f)_z}{r} + \frac{2}{r}\omega_z F^{-1}\omega_r^T + Tr(F^{-1}F_rF^{-1}F_z) \right) \tag{3.1.8}$$

We start off with a stationary bi-axisymmetric cohomogeneity 2 connected manifold M with metric g. The group in question, G, is $\mathbb{T}^2 \times \mathbb{R}$. The cohomogeneity 2 assumption means that the orbit space is 2-dimensional. Let $M_{reg} = \{p \in M \,|\, G(p)$ is a principal orbit $\}$, by the principal orbit theorem we have that it is dense, open and connected in M. We know by the principal orbit theorem that $\pi(M_{reg})$ is open connected and dense in the orbit space. We will assume the group action is proper it is free over M_{reg}. This implies that $\pi(M_{reg})$ is a manifold. We also have that $M_{reg}(\pi(M_{reg}), G)$ is a principal G-bundle.

Now we will assume the existence of a section, Σ which is a connected closed regularly embedded smooth submanifold which intersects every orbit orthogonally [25, p. 771]. More specifically we are stating that $G(\Sigma) = M$, that is the image of the section under the group covers M. And that for all p in Σ, we have that the tangent space to the group, T_pG, is orthogonal to the tangent space of Σ, $T_p\Sigma$ [25, p. 777]. Note that the existence of the section Σ implies that if we let a group element τ act on Σ then the result $\tau\Sigma$ is still a section [25, p. 777]. Thus we have many sections. An example of such a section entails S^1 acting on S^2 by rotation about the z-axis.

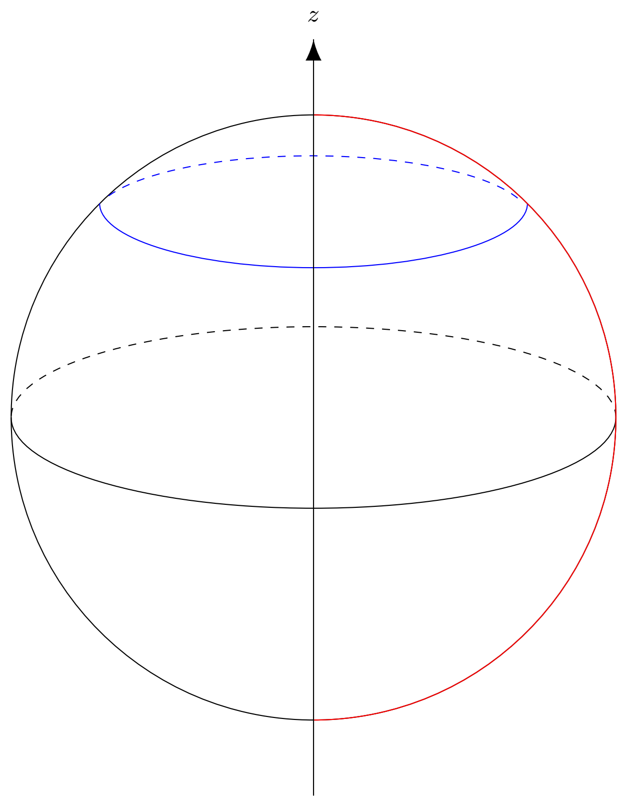

Figure 3.1:

A typical orbit is shown in blue. The cross section here would be the red meridian since it intersects each orbit orthogonally. Back to our manifold M, we have the stronger result that for all p in M_{reg} there is a unique section, Σ_p, which passes through p whose tangent space, $T_p\Sigma_p$, is orthogonal to T_pG [25, p. 778]. Let E be a vector field, we can decompose E into a part tangent to the fibre, $v(E)$, and a part tangent to the section, $h(E)$, which dynamically depends on the point. Let E_1 and E_2 be vector fields, we can write the metric as $g(E_1, E_2) = g(v(E_1), v(E_2)) + g(h(E_1), h(E_2))$.

Now enter the stationary and bi-axisymmetric assumptions, we have three 1-parameter groups which act by isometries. Associated to each of these 1-parameter groups is a Killing vector field. We will assume that these Killing fields commute in which case there is a global coordinate system where each Killing vector field is a coordinate vector field [23, p. 337]. The one associated to the stationary assumption is timelike and is denoted by ∂_t . The two Killing vector fields associated to the bi-axisymmetric assumptions are ∂_{φ^1} and ∂_{φ^2}. These Killing vector fields generate the group action by G. The fact that the Killing vectors commutes agrees with G being Abelian. Since M_{reg} is a principal G bundle, it has local coordinates $(t, \varphi_1, \varphi_2, \xi_1, \xi_2)$ where ξ_1 and ξ_2 are coordinates on the orbit space. Because of the the isometries we know that our metric g is solely a function of ξ_1 and ξ_2. To see this let E_1 and E_2 be coordinate vector fields.

$$\partial_t(g(E_1, E_2) = g(D_{\partial_t}E_1, E_2) + g(D_{\partial_t}E_2, E_1) = g(D_{E_1}\partial_t, E_2) + g(D_{E_2}\partial_t, E_1)$$
$$= -g(D_{E_2}\partial_t, E_1) + g(D_{E_2}\partial_t, E_1) = 0$$

The same holds true for ∂_{φ_1} and ∂_{φ_2}. Using our previously mentioned decomposition of g we can rewrite it as $g = g_f + g_b$, where g_f corresponds to the fibre and g_b corresponds to the orbit space. Let $\partial_{\varphi_0} = \partial_t$, then we define $r^2 = -\det H$ where $H_{ij} = g_f(\partial_{\varphi_{i-1}}, \partial_{\varphi_{j-1}})$. We want to show that $r > 0$ on M_{reg} and $r = 0$ on $M - M_{reg}$. To do this we will assume that points on $M - M_{reg}$ correspond to points whose orbit is on a $(1, 0)$ rod a $(0, 1)$ rod or a corner. We will assume that the lower right 2x2 block of H which we will call F is positive definite on M_{reg}. In order to use the theorem in Chrusciel's paper [7, p. 6] for our proof, we need the domain of outer communication. In order to have a domain of outer communication we need the existence of an Kaluza-Klein asymptotic end.

Definition 3.2. *[7, p. 3] We say that L_{ext} is an Kaluza Klein asymptotic end when it is diffeomorphic to $(\mathbb{R}^n - \overline{B(R)}) \times N$. Where $\overline{B(R)}$ is a closed ball of radius R [7, p. 3] and N is a compact subset. We take the Euclidean metric on $\mathbb{R}^n$, δ, and fix a Riemannian metric, ϵ, on N. We say that a Riemannian metric g_L on L_{ext} is Kaluza Klein asymptotically flat if there exists an $\beta > 0$ and an $j \geq 1$ such that the difference between g_L and $\delta + \epsilon$ on $\mathbb{R}^n$, δ, satisfies the following for $0 \leq k \leq j$*

$$\partial_{x_{i_1}} \dots \partial_{x_{i_k}}((g_L)_{pq} - (\delta_{pq} + \epsilon_{pq})) = O(s^{-\alpha-j})$$

Here x_i are coordinates on $\mathbb{R}^n$ and $s = \sqrt{x_1^2 + \ldots + x_n^2}$. We are assuming g_L and ϵ are solely functions of x_i.

Now we construct L_{ext} in our case which is 4-dimensional. We need the set, $P = \{p \in M_{reg} \,|\, t(p) = 0\}$, and the set $Q = \{p \in (M - M_{reg}) \,|\, t(p) = 0\}$. Here we are using the fact that the coordinate t is global. The metric on P, thinking of P as a hypersurface, is $g_P + g_b$. Where g_P is obtained by taking g_f and setting $t = 0$. The matrix corresponding to g_P is F which is positive definite on M_{reg}. Therefore g_P is Riemannian on P. On can show that g_b is Riemannian [17, p. 654]. Thus $g_b + g_P$ is Riemannian. For p in P we can construct a sequence of points, $\{p_i\}$ which starts at p and then converges to q in Q via a geodesic. We define the distance between p and q, $d(p, q)$, to be the supremum of the geodesic distance between p and p_i over all i. We define $L_{ext} = \{p \in P \,|\, d(p, q) > R$ for all $q \in Q\}$, here R is some non-zero constant. We assume that L_{ext} is diffeomorphic to $(\mathbb{R}^2 - \overline{B(R')}) \times T^2$. We fix the background metric on T^2 to be g_P. Therefore for the metric $g_P + g_b$ to be Kaluza Klein asymptotically flat we simply need to assume that g_b approaches the Euclidean metric in the sense of the definition above. In chapter 6 we check that g_b is asymptotically flat for the periodic soliton solution. We assume ∂_t has complete orbits and it approaches the time-like unit normal to L_{ext} as s goes to ∞. We transport the points in L_{ext} to everywhere where the flow of ∂_t takes it and we take the union of all these points to be M_{ext}. It is clear that $M_{ext} = \{(x_1, y_1, x_2, y_2, t) \in M \mid (x_1, y_1, x_2, y_2, 0) \in L_{ext}\}$. We define the *domain of outer communication*, $\langle\langle M_{ext}\rangle\rangle$, as follows:

$$\langle\langle M_{ext}\rangle\rangle = I^+(M_{ext}) \cap I^-(M_{ext})$$

Here $I^+(p)$ is the *chronological future* of a point p and $I^-(p)$ is the *chronological past* of a point p. When q is in $I^+(p)$, it means that there is a future directed timelike curve from p to q, also written as $p << q$. Conversely, q in $I^-(p)$ means there is a future directed timelike curve from q to p, $q << p$. When we take $I^{\pm}(S)$ for some set S we are taking $\cup_{p\in S} I^{\pm}(p)$. Now we show that $M_{reg} \subset \langle\langle M_{ext}\rangle\rangle$. We can make use of the fact that M_{ext} only contains complete orbits. We make the following assumption: for all p_1, p_2 in M_{reg}, there exists q in $G(p_2)$ such that $p_1 << q$. We also assume that for all p_1, p_2 in M_{reg}, there exists q in $G(p_2)$ such $q << p_1$. In other words, we are assuming that each orbit has a point further in the future (or the past) than a fixed point, and also that the points are chronologically connected. Thus it is easy to see with this assumption that $M_{reg} \subset \langle\langle M_{ext}\rangle\rangle$.

Earlier we assumed that we only have $(1,0)$ rods and $(0,1)$ rods on the set $M - M_{reg}$. In the definition of a $(1,0)$ rod we have that $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ approaches 0 as you approach the points on the corresponding singular orbit. In the definition of a $(0,1)$ rod we have that $g(\partial_{\varphi_2}, \partial_{\varphi_2})$ approaches 0 as you approach the points on the corresponding singular orbit. We will further assume that $g(\partial_{\varphi_1}, \partial_{\varphi_2})$ approaches 0 as you approach points in both types of singular orbits. This will later be shown to be true with the smoothness conditions. Therefore $f = f_{11}f_{22} - f_{12}^2$ approaches 0 as you approach $M - M_{reg}$.

We define the *null energy condition*. Let Y be a null vector, then the null energy condition states that $\mathrm{Ric}(Y, Y) \geq 0$. This is trivially satisfied in our Ricci flat case. Next we define the *orthogonal integrability condition* to be all for $i = 0, 1, 2$ that the following holds:

$$d(\partial^\flat_{\varphi_i}) \wedge \partial^\flat_{\varphi_0} \wedge \partial^\flat_{\varphi_1} \wedge \partial^\flat_{\varphi_2} = 0$$

Here $\partial^\flat_{\varphi_i} = \sum_{j=0}^{2} g_{ij} d\varphi_j$. We lack $d\xi_1$ and $d\xi_2$ terms because we have a product metric. We can show that the orthogonal integrability condition is satisfied.

$$d(\partial^\flat_{\varphi_i}) = \sum_{j=0}^{2}\sum_{k=1}^{2} \partial_{\xi_k}(g_{ij}) d_k \wedge d\varphi_j$$
$$\partial^\flat_{\varphi_0} \wedge \partial^\flat_{\varphi_1} \wedge \partial^\flat_{\varphi_2} = -r^2 dt \wedge d\varphi_1 \wedge d\varphi_2$$
$$d(\partial^\flat_{\varphi_i}) \wedge \partial^\flat_{\varphi_0} \wedge \partial^\flat_{\varphi_1} \wedge \partial^\flat_{\varphi_2} = 0$$

This looks similar to a definition of the twist 1-forms but the $\partial^\flat_{\varphi_0}$ term in the wedge product makes it easier to evaluate. We are now in position to use use a theorem by Chrusciel.

Theorem 3.3. *Suppose we have a spacetime (M, g) satisfying the null energy condition and containing a Kaluza-Klein asymptotically flat end L_{ext}. Suppose further that $\langle\langle M_{ext}\rangle\rangle$ is globally hyperbolic [26, p. 48]. Assume there is a group action by isometries G which looks like our group. Furthermore assume that (M, g) is I^+ regular [7, p. 5] and that the orthogonal integrability condition is satisfied. Let A be the subset of M such that $f = 0$. Then we have that on $\langle\langle M_{ext}\rangle\rangle - A$ that $r > 0$ and on $\partial\langle\langle M_{ext}\rangle\rangle \cup A$ that $r = 0$.*

Since f approaches 0 as you approach $M - M_{reg}$ we have that r approaches 0 as you approach $\partial\langle\langle M_{ext}\rangle\rangle \cup (M - M_{reg})$. Furthermore since $f > 0$ on M_{reg} we have that $r > 0$ on M_{reg}. This is of course under the assumptions of the above theorem.

3.4 Harmonicity of r

It clear that π is a pseudo-Riemannian submersion so we can use the equations developed by Besse [3, p. 236]. We will define the horizontal and vertical distributions. Our vertical distribution is made up of $\frac{\partial}{\partial\varphi^i}$, where $0 \le i \le 2$. The horizontal distribution is made up of $\frac{\partial}{\partial\xi_1}$ and $\frac{\partial}{\partial\xi_2}$. We will show that the vanishing of the vertical components of the Ricci curvature causes r to be harmonic with respect to g_b. For a vector field E we will denote its horizontal part by hE and its vertical part by vE. We will define the tensors A and T [3, p. 239]. Here E_1 and E_2 are arbitrary vector fields.

$$A_{E_1}E_2 = hD_{hE_1}(vE_2) + vD_{hE_1}(hE_2) \qquad T_{E_1}E_2 = hD_{vE_1}(vE_2) + vD_{vE_1}(hE_2) \tag{3.4.1}$$

Theorem 3.5. *Let Y and Z horizontal vector fields expressed as $Y^i\partial_{\xi_i}$ and $Z^j\partial_{\xi_j}$ respectively. Then because A is linear we can write $A_Y Z = Y^i Z^j A_{\partial_{\xi_i}}\partial_{\xi_j}$. Since $A_{\partial_{\xi_i}}\partial_{\xi_j} = v[\partial_{\xi_i}, \partial_{\xi_j}] = 0$ [3, p. 240], we have that $A_Y Z = 0$.*

3.5.1 Purely Vertical Components of the Ricci Curvature

We state the Ricci curvature where U and V are vertical vector fields [3, p. 244].

$$r(U,V) = r_f(U,V) - (N, T_U V) + (AU, AV) + (\tilde{\delta}T)(U,V) \tag{3.5.1}$$

Where r_f is the Ricci curvature using the connection of the vertical distribution. We have that N is the mean curvature vector, $N = \sum_{ij}\left(T_{\frac{\partial}{\partial \phi^i}} \frac{\partial}{\partial \phi^j} H^{ij}\right)$ [3, p. 243], where H^{ij} are the components of the inverse of H. Here, $(AU, AV) = \sum_{ij}(A_{\partial_{\xi_i}} U, A_{\partial_{\xi_j}} V) g_b^{ij}$ [3, p. 243]. Where g_b^{ij} are components of the inverse of the metric on the base. And finally $(\tilde{\delta}T)(U,V) = \sum_{kl} g_b^{kl}((D_{\partial_{\xi_k}} T)_U V, \partial_{\xi_l})$ [3, p. 243].

To check $(A\partial_{\varphi_k}, A\partial_{\varphi_l}) = 0$ we must check that $A_{\partial_{\xi_i}} \partial_{\varphi_k} = 0$. This amounts to checking that $(D_{\partial_{\xi_i}} \partial_{\varphi_k}, \partial_{\xi_j}) = 0$.

$$\begin{aligned}
&(D_{\partial_{\xi_i}} \partial_{\varphi_k}, \partial_{\xi_j}) = \partial_{\xi_i}(\partial_{\varphi_k}, \partial_{\xi_j}) - (\partial_{\varphi_k}, D_{\partial_{\xi_i}} \partial_{\xi_j}) = -(\partial_{\varphi_k}, D_{\partial_{\xi_j}} \partial_{\xi_i}) \\
&(D_{\partial_{\xi_i}} \partial_{\varphi_k}, \partial_{\xi_j}) = -(D_{\partial_{\xi_j}} \partial_{\varphi_k}, \partial_{\xi_i}) = (\partial_{\varphi_k}, D_{\partial_{\xi_j}} \partial_{\xi_i}) \\
&(\partial_{\varphi_k}, D_{\partial_{\xi_i}} \partial_{\xi_j}) = 0 \implies (D_{\partial_{\xi_i}} \partial_{\varphi_k}, \partial_{\xi_j}) = 0
\end{aligned}$$

Now for $r_f(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j})$ we are interested in the connection of the vertical distribution. This amounts to calculating the vertical components of covariant derivatives which involves only vertical coordinate vector fields. We have the following relation since $\frac{\partial}{\partial \varphi^j}$ are Killing vectors [3, p. 183]:

$$2\left(D_{\frac{\partial}{\partial \varphi^i}} \frac{\partial}{\partial \varphi^j}, \frac{\partial}{\partial \varphi^k}\right) = \left(\left[\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right], \frac{\partial}{\partial \varphi^k}\right) + \left(\left[\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^k}\right], \frac{\partial}{\partial \varphi^j}\right) + \left(\frac{\partial}{\partial \varphi^i}, \left[\frac{\partial}{\partial \varphi^j}, \frac{\partial}{\partial \varphi^k}\right]\right) = 0 \tag{3.5.2}$$

So we know that there are only two terms that make up (3.5.1).

$$r\left(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right) = -\left(N, T_{\frac{\partial}{\partial \varphi^i}} \frac{\partial}{\partial \varphi^j}\right) + (\tilde{\delta}T)\left(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right)$$

We will start with a useful relation involving T and the derivatives of H_{ij}

$$\begin{aligned}
\left(T_{\frac{\partial}{\partial\varphi^i}}\frac{\partial}{\partial\varphi^j},\frac{\partial}{\partial\xi_k}\right) &= \left(D_{\frac{\partial}{\partial\varphi^i}}\frac{\partial}{\partial\varphi^j},\frac{\partial}{\partial\xi_k}\right) = -\left(D_{\frac{\partial}{\partial\xi_k}}\frac{\partial}{\partial\varphi^j},\frac{\partial}{\partial\varphi^i}\right) \\
&= -\frac{\partial}{\partial\xi_k}\left(\frac{\partial}{\partial\varphi^i},\frac{\partial}{\partial\varphi^j}\right) + \left(\frac{\partial}{\partial\varphi^j},D_{\frac{\partial}{\partial\xi_k}}\frac{\partial}{\partial\varphi^i}\right) \\
&= -\frac{\partial}{\partial\xi_k}\left(\frac{\partial}{\partial\varphi^i},\frac{\partial}{\partial\varphi^j}\right) - \left(\frac{\partial}{\partial\xi_k},D_{\frac{\partial}{\partial\varphi^j}}\frac{\partial}{\partial\varphi^i}\right) \\
\left(T_{\frac{\partial}{\partial\varphi^i}}\frac{\partial}{\partial\varphi^j},\frac{\partial}{\partial\xi_k}\right) &= -\frac{\partial}{\partial\xi_k}\left(\frac{\partial}{\partial\varphi^i},\frac{\partial}{\partial\varphi^j}\right) - \left(T_{\frac{\partial}{\partial\varphi^i}}\frac{\partial}{\partial\varphi^j},\frac{\partial}{\partial\xi_k}\right) \\
&= -\frac{1}{2}\frac{\partial}{\partial\xi_k}\left(\frac{\partial}{\partial\varphi^i},\frac{\partial}{\partial\varphi^j}\right) = -\frac{1}{2}\partial_{\xi_k}(H_{ij})
\end{aligned}$$

We start by breaking down $(\tilde{\delta}T)(\partial_{\varphi_i},\partial_{\varphi_j})$.

$$(D_{\partial_{\xi_k}}T)_{\partial_{\varphi_i}}\partial_{\varphi_j} = D_{\partial_{\xi_k}}(T_{\partial_{\varphi_i}}\partial_{\varphi_j}) - T_{D_{\partial_{\xi_k}}\partial_{\varphi_i}}\partial_{\varphi_j} - T_{\partial_{\varphi_i}}(D_{\partial_{\xi_k}}\partial_{\varphi_j})$$

We will start by analyzing $T_{\partial_{\varphi_i}}\partial_{\varphi_j}$

$$\begin{aligned}
T_{\partial_{\varphi_i}}\partial_{\varphi_j} &= \sum_{nm} g_b^{mn}(T_{\partial_{\varphi_i}}\partial_{\varphi_j},\partial_{\xi_m})\partial_{\xi_n} \\
&= -\frac{1}{2}\sum_{nm} g_b^{mn}\partial_{\xi_m}(H_{ij})\partial_{\xi_n} \\
D_{\partial_{\xi_k}}(T_{\partial_{\varphi_i}}\partial_{\varphi_j}) &= -\frac{1}{2}\sum_{nm} D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n})\partial_{\xi_m}(H_{ij}) - \frac{1}{2}\sum_{nm} g_b^{mn}\partial^2_{\xi_m\xi_k}(H_{ij})\partial_{\xi_n}
\end{aligned}$$

We move on to $T_{D_{\partial_{\xi_k}}\partial_{\varphi_i}}\partial_{\varphi_j}$.

$$\begin{aligned}
D_{\partial_{\xi_k}}\partial_{\varphi_i} &= \sum_{pq} H^{pq}(D_{\partial_{\xi_k}}\partial_{\varphi_i},\partial_{\varphi_p})\partial_{\varphi_q} \\
&= \frac{1}{2}\sum_{pq} H^{pq}\partial_{\xi_k}(H_{ip})\partial_{\varphi_q} \\
T_{D_{\partial_{\xi_k}}\partial_{\varphi_i}}\partial_{\varphi_j} &= \frac{1}{2}\sum_{pq} H^{pq}\partial_{\xi_k}(H_{ip})T_{\partial_{\varphi_q}}\partial_{\varphi_j} \\
&= -\frac{1}{4}\sum_{nmpq} g_b^{mn}\partial_{\xi_m}(H_{jq})H^{pq}\partial_{\xi_k}(H_{ip})\partial_{\xi_n}
\end{aligned}$$

We can perform a similar calculation for $T_{\partial_{\varphi_i}}(D_{\partial_{\xi_k}}\partial_{\varphi_j})$.

$$\begin{aligned}
D_{\partial_{\xi_k}}\partial_{\varphi_j} &= \frac{1}{2}\sum_{pq} H^{pq}\partial_{\xi_k}(H_{jp})\partial_{\varphi_q} \\
T_{\partial_{\varphi_i}}(D_{\partial_{\xi_k}}\partial_{\varphi_j}) &= -\frac{1}{4}\sum_{nmpq} g_b^{mn}\partial_{\xi_m}(H_{iq})H^{pq}\partial_{\xi_k}(H_{jp})\partial_{\xi_n}
\end{aligned}$$

We now move on to $(N, T_{\partial_{\varphi_i}}\partial_{\varphi_j})$.

$$\begin{aligned}
N &= \sum_{mnpq} g_b^{mn} H^{pq} (T_{\partial_{\varphi_p}}\partial_{\varphi_q}, \partial_{\xi_n})\partial_{\xi_m} \\
&= -\frac{1}{2}\sum_{mnpq} g_b^{mn} H^{pq} \partial_{\xi_n}(H_{pq})\partial_{\xi_m} \\
&= -\frac{1}{2}\sum_{mn} g_b^{mn} Tr(H^{-1}\partial_{\xi_n} H)\partial_{\xi_m} \\
&= -\frac{1}{2}\sum_{mn} g_b^{mn} \partial_{\xi_n}(\log(r^2))\partial_{\xi_m} \\
&= -\sum_{mn} g_b^{mn} \frac{\partial_{\xi_n}(r)}{r}\partial_{\xi_m} \\
(N, T_{\partial_{\varphi_i}}\partial_{\varphi_j}) &= \frac{1}{2}\sum_{klmn} g_b^{mn} \frac{\partial_{\xi_n}(r)}{r} g_b^{kl}\partial_{\xi_k}(H_{ij})(\partial_{\xi_m}, \partial_{\xi_l}) \\
&= \frac{1}{2}\sum_{mn} g_b^{mn} \frac{\partial_{\xi_n}(r)}{r}\partial_{\xi_m}(H_{ij})
\end{aligned}$$

We now collect all terms and set the Ricci curvature to be 0.

$$\begin{aligned}
0 &= -\frac{1}{2}\sum_{mn} g_b^{mn}\frac{\partial_{\xi_n}(r)}{r}\partial_{\xi_m}(H_{ij}) + \frac{1}{2}\sum_{mpq} g_b^{mk}\partial_{\xi_m}(H_{iq})H^{pq}\partial_{\xi_k}(H_{jp})\ldots \\
&\ldots - \frac{1}{2}\sum_{nm}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_l})g_b^{kl}\partial_{\xi_m}(H_{ij}) - \frac{1}{2}\sum_{nm} g_b^{mk}\partial^2_{\xi_m\xi_k}(H_{ij}) \\
&= \sum_{imk}(g_b^{mk})(\partial_{\xi_m}(r)\partial_{\xi_k}(H_{ij})H^{il} - rH^{il}\partial_{\xi_m}(H_{iq})H^{pq}\partial_{\xi_k}(H_{jp}) + rH^{il}\partial^2_{\xi_m\xi_k}(H_{ij}))\ldots \\
&\ldots + \sum_{imn}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp} rH^{il}\partial_{\xi_m}(H_{ij}) \\
&= \sum_{lmn} g_b^{mk}\partial_{\xi_m}(rH^{-1}\partial_{\xi_k}H)_{lj} + \sum_{lmn}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp} r(H^{-1}\partial_{\xi_k}H)_{lj}
\end{aligned}$$

3.5.2 The Global Nature of r and z

We now take the trace of the above equation and notice that the H terms disapear.

$$\begin{aligned}
0 &= \sum_{mn} g_b^{mk}\partial_{\xi_m}(r\partial_{\xi_k}(\log(r^2)) + \sum_{mn}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp} r\partial_{\xi_k}(\log(r^2)) \\
0 &= \sum_{mn} g_b^{mk}\partial_{\xi_m}\partial_{\xi_k}(r) + \sum_{mn}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp}\partial_{\xi_k}(r)
\end{aligned}$$

We define the Laplace-Beltrami operator below in a general setting.

Theorem 3.6. *Let N be a manifold with metric h. Let c be a function on N.*

Let v_i be a local frame and let X be a vector field. Then div(X) is defined to be:

$$div(X) = \sum_i (D_{v_i} X)(v_i)$$

Let x_i be local coordinates. We define the gradient of c, ∇c to be:

$$\begin{aligned}\nabla c &= (dc)^{\#} \\ dc &= \sum_i \left(\frac{\partial c}{\partial x_i} dx_i \right) \\ (dc)^{\#} &= \sum_{ij} h^{ij} \frac{\partial c}{\partial x_i} \frac{\partial}{\partial x_j}\end{aligned}$$

Finally we define the Laplace-Beltrami operator by: $\Delta_h c =$ div(∇c).

Now we calculate $\Delta_{g_b} r$.

$$\begin{aligned}dr^{\#} &= \sum_{ij} g_b^{ij} \frac{\partial r}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \\ D_{\partial_{\xi_k}}(dr^{\#}) &= \sum_{ij} D_{\partial_{\xi_k}} \left(g_b^{ij} \frac{\partial r}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \right) \\ &= \sum_{ij} \partial_{\xi_k} \partial_{\xi_i}(r) g_b^{ij} \partial_{\xi_j} + \sum_{ij} \partial_{\xi_i}(r) D_{\partial_{\xi_k}}(g_b^{ij} \partial_{\xi_j}) \\ D_{\partial_{\xi_k}}(dr^{\#})(\xi_k) &= \sum_{ik} \partial_{\xi_k} \partial_{\xi_i}(r) g_b^{ik} + \sum_{ijk} \partial_{\xi_i}(r)(D_{\partial_{\xi_k}}(g_b^{ij} \partial_{\xi_j}), \partial_{\xi_l}) g_b^{lm} \partial_{\xi_m}(\xi_k) \\ \Delta_{g_b} r &= \sum_{ik} \partial_{\xi_k} \partial_{\xi_i}(r) g_b^{ik} + \sum_{ijk} \partial_{\xi_i}(r)(D_{\partial_{\xi_k}}(g_b^{ij} \partial_{\xi_j}), \partial_{\xi_l}) g_b^{lk}\end{aligned}$$

Thus, comparing to the trace of the Ricci curvature, we have shown that r is harmonic with respect to g_b. Let z be its harmonic conjugate. In order to define z we need to introduce the metric $\overline{g}_b = \frac{1}{\sqrt{\det(g_b)}} g_b$. We denote components of the inverse of $\overline{g}_b$ by $\overline{g}_b^{ij}$. It is clear that $\det(\overline{g}_b) = 1$ since g_b is 2-dimensional and also that they are conformal related since g_b is Riemannian. We define z as follows:

$$\sum_i \overline{g}_b^{1i} \partial_{\xi_i} z = \partial_{\xi_2} r \qquad \sum_i \overline{g}_b^{2i} \partial_{\xi_i} z = -\partial_{\xi_1} r$$

We let $\overline{D}$ be the connection for $\overline{g}_b$. Let's compute the Laplace Beltrami operator acting on z.

$$\begin{aligned}
\Delta_{\overline{g}_b} z &= \sum_{ki} (\overline{D}_{\partial_{\xi_k}}(\overline{g}_b^{i1}\partial_{\xi_i}(z)\partial_{\xi_1})(\xi_k) + \overline{D}_{\partial_{\xi_k}}(\overline{g}_b^{i2}\partial_{\xi_i}(z)\partial_{\xi_2})(\xi_k)) \\
&= \sum_{k} (\overline{D}_{\partial_{\xi_k}}(\partial_{\xi_2}(r)\partial_{\xi_1})(\xi_k) - \overline{D}_{\partial_{\xi_k}}(\partial_{\xi_1}(r)\partial_{\xi_2})(\xi_k)) \\
&= \partial_{\xi_1}\partial_{\xi_2}(r) - \partial_{\xi_2}\partial_{\xi_1}(r) + \sum_k \left(\partial_{\xi_2}(r)(\overline{D}_{\partial_{\xi_k}}\partial_{\xi_1})(\xi_k) - \partial_{\xi_1}(r)(\overline{D}_{\partial_{\xi_k}}\partial_{\xi_1})(\xi_k)\right) \\
&= \sum_k \left(\partial_{\xi_2}(r)(\overline{\Gamma}^k_{k1}) - \partial_{\xi_1}(r)(\overline{\Gamma}^k_{k2})\right) \\
&= \partial_{\xi_2}(r)\partial_{\xi_1}\left(\log\left(\sqrt{\det(\overline{g}_b)}\right)\right) - \partial_{\xi_1}(r)\partial_{\xi_2}\left(\log\left(\sqrt{\det(\overline{g}_b)}\right)\right) \\
&= 0
\end{aligned}$$

The formula used to justify the last step can be found in Sokolnikoff's book "Tensor Analysis Theory and Applications" [28, p. 81]. Therefore $\Delta_{g_b} z = 0$ since the metrics g_b and $\overline{g}_b$ are conformally related.

It can be shown that (r, z) form global coordinates due to harmonicity and the fact that the orbit space $\pi(M)$ is homeomorphic to the right half plane [17, p. 655-656]. We further assume that g_b expressed in these coordinates is conformal to the flat metric.

$$g_b = e^{2\alpha}(dr^2 + dz^2)$$

3.7 Metric on the Fibre

Armed with the coordinates r and z and the simplified form for g_b, we can derive the harmonic map equations; making some simplifications to the result found in the last section. We set $\xi_1 = r$ and $\xi_2 = z$.

$$\begin{aligned}
(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp} &= (\partial_{\xi_n}, D_{\partial_{\xi_k}}\partial_{\xi_p})g_b^{kp}g_b^{mn} \\
&= e^{-4\alpha}\Big((\partial_r, D_{\partial_r}\partial_r) + (\partial_r, D_{\partial_z}\partial_z) + (\partial_z, D_{\partial_r}\partial_r) + (\partial_z, D_{\partial_z}\partial_z)\Big) \\
&= e^{-4\alpha}\left(\frac{1}{2}\partial_r(\partial_r, \partial_r) - \frac{1}{2}\partial_r(\partial_z\partial_z) + \frac{1}{2}\partial_z(\partial_r, \partial_r) - \frac{1}{2}\partial_z(\partial_z\partial_z)\right) = 0 \\
0 &= \sum_{lmn} g_b^{mk}\partial_{\xi_m}(rH^{-1}\partial_{\xi_k}H)_{lj} + \sum_{lmn}(D_{\partial_{\xi_k}}(g_b^{mn}\partial_{\xi_n}), \partial_{\xi_p})g_b^{kp} r(H^{-1}\partial_{\xi_k}H)_{lj} \\
0 &= e^{-2\alpha}(\partial_r(rH^{-1}H_r)_{lj} + \partial_z(rH^{-1}H_z)_{lj}) \\
0 &= \partial_r(rH^{-1}H_r) + \partial_z(rH^{-1}H_z)
\end{aligned}$$

In the last step we went from component form to matrix form.

3.7.1 Derivation of the Block Matrix Form of the Harmonic Map Equations

We now derive equation (3.1.5) and (3.1.6), which correspond to equation (2.2) in the Khuri et al.'s paper [18]. To do this we utilize a block matrix form of H. Here v is a 2 dimensional column vector. We write H in the following form so we can easily takes its inverse and so we can explicitly see the twist potentials and the components of F.

$$H = \begin{pmatrix} -f^{-1}r^2 + v^T F v & (Fv)^T \\ Fv & F \end{pmatrix}$$

We now determine H^{-1}:

$$H^{-1} = \begin{pmatrix} -fr^{-2} & fr^{-2}v^T \\ fr^{-2}v & -fr^{-2}vv^T + F^{-1} \end{pmatrix}$$

We can now calculate the partial derivatives of H, H_r and H_z.

$$H_r = \begin{pmatrix} f^{-2}f_r r^2 - 2f^{-1}r + v^T(Fv)_r + (v^T)_r Fv & (v^T F)_r \\ (Fv)_r & F_r \end{pmatrix}$$

$$H_z = \begin{pmatrix} f^{-2}f_z r^2 + v^T(Fv)_z + (v^T)_z Fv & (v^T F)_z \\ (Fv)_z & F_z \end{pmatrix}$$

Take M to be a 3x3 matrix. The lower right 2x2 block will be denoted by $M_{\bullet\bullet}$ and the top left corner will be denoted by M_{11}. The remainder of the first row will be denoted by $M_{1\bullet}$, and the remainder of the first column will be denoted by $M_{\bullet 1}$. We now find the corresponding parts of the matrix $H^{-1}H_r$ and $H^{-1}H_z$.

$$
\begin{aligned}
(H^{-1}H_r)_{11} &= -f^{-1}f_r + 2r^{-1} - fr^{-2}(v^T(Fv)_r + v_r^TFv) + fr^{-2}v^T(Fv)_r \\
&= -f^{-1}f_r + 2r^{-1} - fr^{-2}v_r^TFv \\
(H^{-1}H_r)_{1\bullet} &= -fr^{-2}(v^TF)_r + fr^{-2}v^TF_r = -fr^{-2}v_r^TF \\
(H^{-1}H_r)_{\bullet 1} &= f^{-1}f_rv - 2r^{-1}v + fr^{-2}v((v^TF)_rv + v^TFv_r) - fr^{-2}vv^T(Fv)_r + F^{-1}(Fv)_r \\
&= (f^{-1}f_r - 2r^{-1})v + F^{-1}(Fv)_r + fr^{-2}(vv_r^TFv + vv^TF_rv + vv^TFv_r - vv^TF_rv - vv^TFv_r)) \\
&= (f^{-1}f_r - 2r^{-1})v + fr^{-2}(vv_r^TFv) + v_r + F^{-1}F_rv \\
(H^{-1}H_r)_{\bullet\bullet} &= fr^{-2}v(v^TF)_r - fr^{-2}vv^TF_r + F^{-1}F_r = fr^{-2}vv_r^TF + F^{-1}F_r \\
(H^{-1}H_z)_{11} &= -f^{-1}f_z - fr^{-2}(v^T(Fv)_z + v_z^TFv) + fr^{-2}v^T(Fv)_z \\
&= -f^{-1}f_z - fr^{-2}v_z^TFv \\
(H^{-1}H_z)_{1\bullet} &= -fr^{-2}(v^TF)_z + fr^{-2}v^TF_z = -fr^{-2}v_z^TF \\
(H^{-1}H_z)_{\bullet 1} &= f^{-1}f_zv + fr^{-2}v((v^TF)_zv + v^TFv_z) - fr^{-2}vv^T(Fv)_z + F^{-1}(Fv)_z \\
&= (f^{-1}f_z)v + F^{-1}(Fv)_z + fr^{-2}(vv_z^TFv + vv^TF_zv + vv^TFv_z - vv^TF_zv - vv^TFv_z)) \\
&= (f^{-1}f_z)v + fr^{-2}(vv_z^TFv) + v_z + F^{-1}F_zv \\
(H^{-1}H_z)_{\bullet\bullet} &= fr^{-2}v(v^TF)_z - fr^{-2}vv^TF_z + F^{-1}F_z = fr^{-2}vv_z^TF + F^{-1}F_z
\end{aligned}
\tag{3.7.1}
$$

We now compute the harmonic map equations for each part of the matrix, $\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right)$.

$$
\begin{aligned}
\left(\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right)\right)_{1\bullet} &= \frac{\partial}{\partial r}\left(-fr^{-1}v_r^TF\right) + \frac{\partial}{\partial z}\left(-fr^{-1}v_z^TF\right) = 0 \\
&= \frac{\partial}{\partial r}\left(fr^{-1}v_r^TF\right) - \frac{\partial}{\partial z}\left(-fr^{-1}v_z^TF\right) = 0
\end{aligned}
$$

Let $a = fr^{-1}v_r^TF$ and $b = -fr^{-1}v_z^TF$. Let $\tau = adz + bdr$, thus τ is a 2-dimensional row vector with 1 form values. We have the following:

$$d\tau = (\partial_r(a) - \partial_z(b))dr \wedge dz = 0$$

Thus since the orbit space is simply connected we have that τ is exact, i.e $\tau = d\omega$. We have that the partial derivatives of ω are given by the following:

$$\omega_z = fr^{-1}v_r^TF \qquad \omega_r = -fr^{-1}v_z^TF$$

We can solve for v in terms of ω.

$$v_r^T = f^{-1}r\omega_zF^{-1} \qquad v_z^T = -f^{-1}r\omega_rF^{-1} \tag{3.7.2}$$

We can use the integrability of v^T to find a PDE involving ω.

$$\begin{aligned}
v^T_{rz} &= r\left(-f^{-2}f_z\omega_z F^{-1} + f^{-1}\omega_{zz}F^{-1} - f^{-1}\omega_z F^{-1}F_z F^{-1}\right)\\
v^T_{zr} &= -f^{-1}\omega_r F^{-1} - r\left(-f^{-2}f_r\omega_r F^{-1} + f^{-1}\omega_{rr}F^{-1} - f^{-1}\omega_r F^{-1}F_r F^{-1}\right)\\
0 &= v^T_{rz} - v^T_{zr}\\
0 &= -f^{-1}f_z\omega_z + \omega_{zz} - \omega_z F^{-1}F_z + \frac{\omega_r}{r} - f^{-1}f_r\omega_r + \omega_{rr} - \omega_r F^{-1}f_r
\end{aligned} \tag{3.7.3}$$

$$0 = \Delta\omega - (\ln f)_r\omega_r - (\ln f)_z\omega_z - \omega_z F^{-1}F_z - \omega_r F^{-1}F_r \tag{3.7.4}$$

Where Δc is the laplacian of $c = c(r,z)$ in 3 dimensions with cylindrical coordinates, (r, z, θ). However it is also proportional to the laplacian with respect to the metric of the entire space g. We use theorem 3.6 to do the calculations. We start by computing $(D_{\partial_r}\boldsymbol{\nabla} c)(r)$.

$$\begin{aligned}
(D_{\frac{\partial}{\partial r}}\boldsymbol{\nabla} c)(r) &= D_{\frac{\partial}{\partial r}}\left(e^{-2\alpha}\frac{\partial c}{\partial r}\frac{\partial}{\partial r} + e^{-2\alpha}\frac{\partial c}{\partial z}\frac{\partial}{\partial z}\right)(r)\\
&= \frac{\partial}{\partial r}\left(e^{-2\alpha}\frac{\partial c}{\partial r}\right) + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial r}\left(D_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial z}\left(D_{\frac{\partial}{\partial r}}\frac{\partial}{\partial z},\frac{\partial}{\partial r}\right)\\
&= \left(-2\alpha_r e^{-2\alpha}\frac{\partial c}{\partial r} + e^{-2\alpha}\frac{\partial^2 c}{\partial r^2} + e^{-4\alpha}\frac{\partial c}{\partial r}e^{2\alpha}\alpha_r + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial z}e^{2\alpha}\alpha_z\right)\\
&= e^{-2\alpha}\left(\frac{\partial c}{\partial z}\alpha_z - \frac{\partial c}{\partial r}\alpha_r + \frac{\partial^2 c}{\partial r^2}\right)
\end{aligned}$$

Let's compute $(D_{\frac{\partial}{\partial z}}\boldsymbol{\nabla} c)(z)$.

$$\begin{aligned}
D_{\frac{\partial}{\partial z}}(\boldsymbol{\nabla} c)(z) &= D_{\frac{\partial}{\partial z}}\left(e^{-2\alpha}\frac{\partial c}{\partial r}\frac{\partial}{\partial r} + e^{-2\alpha}\frac{\partial c}{\partial z}\frac{\partial}{\partial z}\right)(z)\\
&= \frac{\partial}{\partial z}\left(e^{-2\alpha}\frac{\partial c}{\partial z}\right) + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial r}\left(D_{\frac{\partial}{\partial r}}\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right) + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial z}\left(D_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right)\\
&= \left(-2\alpha_z e^{-2\alpha}\frac{\partial c}{\partial z} + e^{-2\alpha}\frac{\partial^2 c}{\partial z^2} + e^{-4\alpha}\frac{\partial c}{\partial r}e^{2\alpha}\alpha_r + e^{-2\alpha}e^{-2\alpha}\frac{\partial c}{\partial z}e^{2\alpha}\alpha_z\right)\\
&= e^{-2\alpha}\left(\frac{\partial c}{\partial r}\alpha_r - \frac{\partial c}{\partial z}\alpha_z + \frac{\partial^2 c}{\partial z^2}\right)
\end{aligned}$$

We now compute $D_{\frac{\partial}{\partial \phi^i}}(\boldsymbol{\nabla} c)$.

$$\begin{aligned}
D_{\frac{\partial}{\partial \phi^i}}(\boldsymbol{\nabla} c) &= e^{-2\alpha}\frac{\partial c}{\partial r}D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial r} + e^{-2\alpha}\frac{\partial c}{\partial z}D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial z}\\
&= e^{-2\alpha}\left(\frac{\partial c}{\partial r}\left(D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial r},\frac{\partial}{\partial \phi^j}\right)H^{ij}\frac{\partial}{\partial \phi^i} + \frac{\partial c}{\partial z}\left(D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial z},\frac{\partial}{\partial \phi^j}\right)H^{ij}\frac{\partial}{\partial \phi^i}\right)\\
&= e^{-2\alpha}\left(\frac{1}{2}(H_{ij})_r H^{ij}\frac{\partial c}{\partial r} + \frac{1}{2}(H_{ij})_z H^{ij}\frac{\partial c}{\partial z}\right)\frac{\partial}{\partial \phi^i}
\end{aligned}$$

We now collect all the terms with the summation.

$$\begin{aligned}
\Delta_g c &= e^{-2\alpha}\left(\frac{\partial c}{\partial z}\alpha_z - \frac{\partial c}{\partial r}\alpha_r + \frac{\partial^2 c}{\partial r^2}\right) + e^{-2\alpha}\left(\frac{\partial c}{\partial r}\alpha_r - \frac{\partial c}{\partial z}\alpha_z + \frac{\partial^2 c}{\partial z^2}\right)\dots \\
&\dots + \Sigma_i e^{-2\alpha}\left(\frac{1}{2}(H_{ij})_r H^{ij}\frac{\partial c}{\partial r} + \frac{1}{2}(H_{ij})_z H^{ij}\frac{\partial c}{\partial z}\right) \\
&= e^{-2\alpha}\left(\frac{\partial^2 c}{\partial r^2} + \frac{\partial^2 c}{\partial z^2} + \frac{1}{2}Tr(H^{-1}H_r)\frac{\partial c}{\partial r} + \frac{1}{2}Tr(H^{-1}H_z)\frac{\partial c}{\partial z}\right) \\
&= e^{-2\alpha}\left(\frac{\partial^2 c}{\partial r^2} + \frac{\partial^2 c}{\partial z^2} + \frac{1}{2}\frac{\partial}{\partial r}(\ln(r^2))\frac{\partial c}{\partial r} + \frac{1}{2}\frac{\partial}{\partial z}(\ln(r^2))\frac{\partial c}{\partial z}\right) \\
&= e^{-2\alpha}\left(\frac{\partial^2 c}{\partial r^2} + \frac{\partial^2 c}{\partial z^2} + \frac{1}{r}\frac{\partial c}{\partial r}\right)
\end{aligned}$$

We now compute the harmonic map for the 2x2 block of $\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right)$.

$$\begin{aligned}
\left(\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right)\right)_{\bullet\bullet} &= \frac{\partial}{\partial r}\left(fr^{-1}vv_r^T F + rF^{-1}F_r\right) + \frac{\partial}{\partial z}\left(fr^{-1}vv_z^T F + rF^{-1}F_z\right) \\
&= \frac{\partial}{\partial r}\left(v\omega_z + rF^{-1}F_r\right) + \frac{\partial}{\partial z}\left(-v\omega_r + rF^{-1}F_z\right) \\
&= v_r\omega_z + v\omega_{zr} - v_z\omega_r - v\omega_{rz} + F^{-1}F_r - rF^{-1}F_rF^{-1}F_r\dots \\
&\dots + rF^{-1}F_{rr} - rF^{-1}F_zF^{-1}F_z + rF^{-1}F_{zz} \\
&= f^{-1}rF^{-1}\omega_z^T\omega_z + f^{-1}rF^{-1}\omega_r^T\omega_r + F^{-1}F_r - rF^{-1}F_rF^{-1}F_r\dots \\
&\dots + rF^{-1}F_{rr} - rF^{-1}F_zF^{-1}F_z + rF^{-1}F_{zz}
\end{aligned}$$

$$0 = f^{-1}\omega_z^T\omega_z + f^{-1}\omega_r^T\omega_r + \frac{1}{r}F_r + F_{rr} + F_{zz} - F_rF^{-1}F_r - F_zF^{-1}F_z \qquad (3.7.5)$$

The remaining parts of the matrix, $\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right)$, will turn out not to be independent of the previously computed parts. It's easy to see why the 11 part is not independent when we use the trace.

$$\begin{aligned}
(H^{-1}H_r)_{11} &= Tr(H^{-1}H_r) - Tr((H^{-1}H_r)_{\bullet\bullet}) = \frac{2}{r} - Tr((H^{-1}H_r)_{\bullet\bullet}) \\
(H^{-1}H_z)_{11} &= Tr(H^{-1}H_z) - Tr((H^{-1}H_z)_{\bullet\bullet}) = -Tr((H^{-1}H_z)_{\bullet\bullet}) \\
\frac{\partial}{\partial r}\left(r(H^{-1}H_r)_{11}\right) + \frac{\partial}{\partial z}\left(r(H^{-1}H_z)_{11}\right) &= 0 - Tr\left(\frac{\partial}{\partial r}\left(r(H^{-1}H_r)_{\bullet\bullet}\right) + \frac{\partial}{\partial z}\left(r(H^{-1}H_z)_{\bullet\bullet}\right)\right)
\end{aligned}$$

It is trickier to see that $\bullet 1$ part is not independent. To do so we must use all the previously shown parts and use equations (3.1.4) and (3.7.2)

$$
\begin{aligned}
0 &= \frac{\partial}{\partial r}\left(r(H^{-1}H_r)_{\bullet 1}\right) + \frac{\partial}{\partial z}\left(r(H^{-1}H_z)_{\bullet 1}\right)\\
0 &= \frac{\partial}{\partial r}\Big(r((f^{-1}f_r - 2r^{-1})v + fr^{-2}(vv_r^T Fv) + v_r + F^{-1}F_r v)\Big) + \frac{\partial}{\partial z}\Big(r((f^{-1}f_z)v + fr^{-2}(vv_z^T Fv) + v_z + F^{-1}F_z v)\Big)\\
0 &= \underbrace{-\frac{\partial}{\partial r}\Big(r(-(f^{-1}f_r - 2r^{-1}) - fr^{-2}v_r^T Fv)\Big) - \frac{\partial}{\partial z}\Big(r(-(f^{-1}f_z) - fr^{-2}v_z^T Fv)\Big)}_{\left(\frac{\partial}{\partial r}(rG^{-1}G_r)+\frac{\partial}{\partial z}(rG^{-1}G_z)\right)_{11}} v \dots\\
&\dots + \frac{\partial}{\partial r}\left(rF^{-1}F_r v\right) + \frac{\partial}{\partial z}\left(rF^{-1}F_z v\right)\dots\\
&\dots - r(-(f^{-1}f_r - 2r^{-1}) - fr^{-2}v_r^T Fv)v_r - r((-f^{-1}f_z) - fr^{-2}v_z^T Fv)v_z + \frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial r}(rv_z)\\
&= r((f^{-1}f_r - 2r^{-1}) + fr^{-2}v_r^T Fv)v_r + r(f^{-1}f_z + fr^{-2}v_z^T Fv)v_z + F^{-1}F_r v - rF^{-1}F_r F^{-1}F_r v\dots\\
&\dots + rF^{-1}F_{rr}v + rF^{-1}F_r v_r - rF^{-1}F_z F^{-1}F_z v + rF^{-1}F_{zz}v + rF^{-1}F_z v_z + rv_{rr} + v_r + rv_{zz}
\end{aligned}
$$

$$
\begin{aligned}
v_r^T &= f^{-1}r\omega_z F^{-1}\\
v_{rr}^T &= -f^{-2}f_r r\omega_z F^{-1} + f^{-1}\omega_z F^{-1} + f^{-1}r\omega_{zr}F^{-1} - f^{-1}r\omega_z F^{-1}F_r F^{-1}\\
&= -f_r f^{-1}v_r^T + v_r r^{-1} + f^{-1}\omega_{zr}F^{-1} - v_r^T F_r F^{-1}\\
v_z^T &= -f^{-1}r\omega_r F^{-1}\\
v_{zz}^T &= -f^{-2}f_z r\omega_r F^{-1} - f^{-1}r\omega_{zr}F^{-1} + f^{-1}r\omega_r F^{-1}F_z F^{-1}\\
&= f_z f^{-1}v_z^T - f^{-1}\omega_{zr}F^{-1} - v_z^T F_z F^{-1}
\end{aligned}
$$

We plug these into the previous formula.

$$
\begin{aligned}
0 &= r((f^{-1}f_r - 2r^{-1}) + fr^{-2}v_r^T Fv)v_r + r(f^{-1}f_z + fr^{-2}v_z^T Fv)v_z\dots\\
&\dots + r(-f_r f^{-1}v_r + v_r r^{-1} + (f^{-1}\omega_{zr}F^{-1})^T - F^{-1}F_r v_r) + r(f_z f^{-1}v_z - (f^{-1}\omega_{zr}F^{-1})^T - F^{-1}F_z v_z) + v_r\dots\\
&\dots + F^{-1}F_r v - rF^{-1}F_r F^{-1}F_r v + rF^{-1}F_{rr}v + rF^{-1}F_r v_r - rF^{-1}F_z F^{-1}F_z v + rF^{-1}F_{zz}v + rF^{-1}F_z v_z\\
&= r(fr^{-2}v_r^T Fv)v_r + rfr^{-2}v_z^T Fv)v_z + F^{-1}F_r v - rF^{-1}F_r F^{-1}F_r v + rF^{-1}F_{rr}v - rF^{-1}F_z F^{-1}F_z v + rF^{-1}F_{zz}v\\
&= (r(fr^{-2})v_r v_r^T F + rfr^{-2}v_z v_z^T F + F^{-1}F_r + F^{-1}F_{rr} + rF^{-1}F_{zz} - rF^{-1}F_r F^{-1}F_r - rF^{-1}F_z F^{-1}F_z)v\\
0 &= (\Delta F + fr^{-2}(f^{-1}rF^{-1}\omega_z^T)(f^{-1}r\omega_z F^{-1}F) + fr^{-2}(-f^{-1}rF^{-1}\omega_z^T)(-f^{-1}r\omega_z F^{-1})F\dots\\
&\dots - F^{-1}F_r F^{-1}F_r - F^{-1}F_z F^{-1}F_z)v\\
&= (\Delta F + \underbrace{f^{-1}\omega_z^T\omega_z + f^{-1}\omega_z^T\omega_z}_{\text{same as (3.7.5)}} - F_r F^{-1}F_r - F_z F^{-1}F_z)v
\end{aligned}
$$

3.7.2 Derivation of the Component Form of the Harmonic Map Equations

We now convert (3.7.4) and (3.7.5) from their matrix form to equation 2.2 found in the paper by Khuri et al.[18, p. 3]. We do so for (3.7.4) by taking the i'th component. And for (3.7.5) we take the ij'th component. We use the notation ∇_1 to be the r derivative and ∇_2 to be the z derivative. We raise the index of the derivative using the inverse of the metric on the base. Since the inverse is diagonal with components $e^{-2\alpha}$, we get that $\nabla^k = e^{-2\alpha}\nabla_k$.

$$
\begin{aligned}
0 &= \Delta f_{ij} - (F_r F^{-1} F_r)_{ij} - (F_z F^{-1} F_z)_{ij} + f^{-1}(\omega_r^T \omega_r)_{ij} + f^{-1}(\omega_z^T \omega_z)_{ij} \\
0 &= \Delta f_{ij} - (f_{ik})_r f^{kl}(f_{lj})_r - (f_{ik})_z f^{kl}(f_{lj})_z + f^{-1}(\omega_i)_r(\omega_j)_r + f^{-1}(\omega_i)_z(\omega_j)_z \\
0 &= \Delta_g f_{ij} - f^{kl}\nabla^n f_{ik}\nabla_n f_{lj} + f^{-1}\nabla^n \omega_i \nabla_n \omega_j
\end{aligned}
$$

We now convert equation (3.7.4) by taking its components. We use the fact that $(\ln f)_r = \mathrm{Tr}(F^{-1}F_r) = f^{jk}(f_{jk})_r$ and $(\ln f)_z = \mathrm{Tr}(F^{-1}F_z) = f^{jk}(f_{jk})_z$

$$
\begin{aligned}
0 &= \Delta\omega_i - f^{jk}(f_{jk})_z(\omega_i)_z - f^{jk}(f_{jk})_r(\omega_i)_r - (\omega_j)_z f^{jk}(f_{ki})_z - (\omega_j)_r f^{jk}(f_{ki})_r \\
0 &= \Delta_g \omega_i - f^{jk}\nabla^n f_{jk}\nabla_n \omega_i - f^{jk}\nabla^n f_{ki}\nabla_n \omega_j
\end{aligned}
$$

3.8 Metric on the Base

3.8.1 Solving for α_r and α_z

We now move on to derive the equations for the conformal factor α which appears in g_b. We start again with the Ricci flat conditions but this time using its purely horizontal components. We have from Besse that [3, p. 244]:

$$
r(X,Y) = r_b(X,Y) - 2(A_X, A_Y) - (TX, TY) + \frac{1}{2}((D_X N, Y) + (D_Y N, X)) \tag{3.8.1}
$$

Where r_b is the Ricci curvature of g_b. Let $g_0 = dr^2 + dz^2$ be the flat metric. Then $g_b = e^{2\alpha} g_0$. From Besse we have that $r_b = -(\Delta\alpha) g_0$. So $r_b = -(\alpha_{rr} + \alpha_{zz})(dr^2 + dz^2)$ [3, p. 59]. Looking at the next term we have that:

$$
(A_X, A_Y) = \sum_{i,j} H^{ij} \left(A_X \frac{\partial}{\partial\varphi_i}, A_Y \frac{\partial}{\partial\varphi_j} \right)
$$

We know that $A_X \frac{\partial}{\partial\varphi_i}$ is horizontal so we only need to determine its horizontal components. Let Z be a horizontal vector field, we have that $(A_X \frac{\partial}{\partial\varphi_i}, Z) = (\partial_{\varphi_i}, A_X Z)$, and by theorem 3.5 it is 0.

From the previous subsection we know that $N = -\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}$. We now calculate the term which contains N in (3.8.1) using all relevant combinations of $\frac{\partial}{\partial r}$ and

$\frac{\partial}{\partial z}$.

$$\begin{aligned}
\left(D_{\frac{\partial}{\partial r}}N, \frac{\partial}{\partial r}\right) &= \left(D_{\frac{\partial}{\partial r}}\left(-\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}\right), \frac{\partial}{\partial r}\right) \\
&= -\frac{e^{-2\alpha}}{2r}\frac{\partial}{\partial r}(e^{2\alpha}) + \frac{\partial}{\partial r}\left(-\frac{e^{-2\alpha}}{r}\right)e^{2\alpha} \\
&= -\frac{\alpha_r}{r} + \frac{1}{r^2} + \frac{2\alpha_r}{r} = \frac{\alpha_r}{r} + \frac{1}{r^2} \\
\left(D_{\frac{\partial}{\partial z}}N, \frac{\partial}{\partial z}\right) &= \left(D_{\frac{\partial}{\partial z}}\left(-\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}\right), \frac{\partial}{\partial z}\right) \\
&= -\frac{e^{-2\alpha}}{2r}\frac{\partial}{\partial r}(e^{2\alpha}) = -\frac{\alpha_r}{r} \\
\left(D_{\frac{\partial}{\partial r}}N, \frac{\partial}{\partial z}\right) &= \left(D_{\frac{\partial}{\partial r}}\left(-\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}\right), \frac{\partial}{\partial z}\right) = -\frac{e^{2\alpha}}{r}\left(-\left(\frac{\partial}{\partial r}, D_{\frac{\partial}{\partial r}}\frac{\partial}{\partial z}\right)\right) \\
&= \frac{e^{-2\alpha}}{2r}\frac{\partial}{\partial z}(e^{2\alpha}) = \frac{\alpha_z}{r} \\
\left(D_{\frac{\partial}{\partial z}}N, \frac{\partial}{\partial r}\right) &= \left(D_{\frac{\partial}{\partial z}}\left(-\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}\right), \frac{\partial}{\partial r}\right) \\
&= \frac{\partial}{\partial z}\left(-\frac{e^{-2\alpha}}{r}\right)e^{2\alpha} - \frac{e^{-2\alpha}}{2r}\frac{\partial}{\partial z}(e^{2\alpha}) \\
&= 2\frac{\alpha_z}{r} - \frac{\alpha_z}{r} = \frac{\alpha_z}{r}
\end{aligned}$$

We now examine the term (TX, TY). We have that:

$$(TX, TY) = \sum_{i,j} H^{ij}(T_{\frac{\partial}{\partial \phi^i}}X, T_{\frac{\partial}{\partial \phi^j}}Y)$$

We know that $T_{\frac{\partial}{\partial \phi^i}}X$ is vertical so we can work out what it is from calculations in the previous subsection. We'll let X be $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial z}$.

$$\begin{aligned}
\left(T_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi^k}\right) = \left(D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi^k}\right) = \left(D_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \phi^i}, \frac{\partial}{\partial \phi^k}\right) = -\left(D_{\frac{\partial}{\partial \phi^k}}\frac{\partial}{\partial \phi^i}, \frac{\partial}{\partial r}\right) = \frac{1}{2}(H_{ik})_r \\
\left(T_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial z}, \frac{\partial}{\partial \phi^k}\right) = \left(D_{\frac{\partial}{\partial \phi^i}}\frac{\partial}{\partial z}, \frac{\partial}{\partial \phi^k}\right) = \left(D_{\frac{\partial}{\partial z}}\frac{\partial}{\partial \phi^i}, \frac{\partial}{\partial \phi^k}\right) = -\left(D_{\frac{\partial}{\partial \phi^k}}\frac{\partial}{\partial \phi^i}, \frac{\partial}{\partial z}\right) = \frac{1}{2}(H_{ik})_z
\end{aligned}$$

We now work out $(T\frac{\partial}{\partial r}, T\frac{\partial}{\partial r})$. Let $K = H^{-1}H_r$ and $L = H^{-1}H_z$. We have the following two PDEs involving K and L. This is using (3.1.2).

$$\begin{aligned}
K_z &= -H^{-1}H_zH^{-1}H_r + H^{-1}H_{rz} \\
L_r &= -H^{-1}H_rH^{-1}H_z + H^{-1}H_{rz} \\
K_z - L_r &= [K, L]
\end{aligned}$$

$$0 = K_r + L_z + \frac{K}{r} \qquad 0 = K_z - L_r - [K, L] \tag{3.8.2}$$

$$\begin{aligned}
\left(T\frac{\partial}{\partial r}, T\frac{\partial}{\partial r}\right) &= \sum_{i,j} H^{ij}\left(\sum_{k,l}\left(T_{\frac{\partial}{\partial\phi^i}}\frac{\partial}{\partial r}, \frac{\partial}{\partial\phi^k}\right)H^{kl}\frac{\partial}{\partial\phi^l}, \sum_{m,n}\left(T_{\frac{\partial}{\partial\phi^j}}\frac{\partial}{\partial r}, \frac{\partial}{\partial\phi^m}\right)H^{mn}\frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}H^{kl}(H_{ik})_r H^{mn}(H_{jm})_r\left(\frac{\partial}{\partial\phi^l}, \frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_r H^{mn}H_{nl}H^{kl}(H_{ik})_r = \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_r H^{mk}(H_{ki})_r\\
&= \frac{1}{4}Tr(H^{-1}H_rH^{-1}H_r) = \frac{1}{4}Tr(K^2)
\end{aligned}$$

We now work out $(T\frac{\partial}{\partial z}, T\frac{\partial}{\partial z})$.

$$\begin{aligned}
\left(T\frac{\partial}{\partial z}, T\frac{\partial}{\partial z}\right) &= \sum_{i,j} H^{ij}\left(\Sigma_{k,l}\left(T_{\frac{\partial}{\partial\phi^i}}\frac{\partial}{\partial z}, \frac{\partial}{\partial\phi^k}\right)H^{kl}\frac{\partial}{\partial\phi^l}, \sum_{m,n}\left(T_{\frac{\partial}{\partial\phi^j}}\frac{\partial}{\partial z}, \frac{\partial}{\partial\phi^m}\right)H^{mn}\frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}H^{kl}(H_{ik})_z H^{mn}(H_{jm})_z\left(\frac{\partial}{\partial\phi^l}, \frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_z H^{mn}H_{nl}H^{kl}(H_{ik})_z = \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_z H^{mk}(H_{ki})_z\\
&= \frac{1}{4}Tr(H^{-1}H_zH^{-1}H_z) = \frac{1}{4}Tr(L^2)
\end{aligned}$$

We now work out $(T\frac{\partial}{\partial r}, T\frac{\partial}{\partial z})$.

$$\begin{aligned}
\left(T\frac{\partial}{\partial r}, T\frac{\partial}{\partial z}\right) &= \sum_{i,j} H^{ij}\left(\sum_{k,l}\left(T_{\frac{\partial}{\partial\phi^i}}\frac{\partial}{\partial r}, \frac{\partial}{\partial\phi^k}\right)H^{kl}\frac{\partial}{\partial\phi^l}, \sum_{m,n}\left(T_{\frac{\partial}{\partial\phi^j}}\frac{\partial}{\partial z}, \frac{\partial}{\partial\phi^m}\right)H^{mn}\frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}H^{kl}(H_{ik})_r H^{mn}(H_{jm})_z\left(\frac{\partial}{\partial\phi^l}, \frac{\partial}{\partial\phi^n}\right)\\
&= \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_r H^{mn}H_{nl}H^{kl}(H_{ik})_z = \frac{1}{4}\sum_{i,j,k,l,m,n} H^{ij}(H_{jm})_r H^{mk}(H_{ki})_z\\
&= \frac{1}{4}Tr(H^{-1}H_rH^{-1}H_z) = \frac{1}{4}Tr(KL)
\end{aligned}$$

Now we add together all the terms that make up $r(X, Y)$.

$$\begin{aligned}
0 &= r\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -(\alpha_{rr} + \alpha_{zz}) - \frac{1}{4}Tr(K^2) + \frac{\alpha_r}{r} + \frac{1}{r^2}\\
0 &= r\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = -(\alpha_{rr} + \alpha_{zz}) - \frac{1}{4}Tr(L^2) - \frac{\alpha_r}{r}\\
0 &= -\frac{1}{4}Tr(K^2) + \frac{1}{4}Tr(L^2) + 2\frac{\alpha_r}{r} + \frac{1}{r^2}\\
\alpha_r &= \frac{r}{8}\left(Tr(K^2) - Tr(L^2) - \frac{4}{r^2}\right)\\
0 &= r\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right) = \frac{\alpha_z}{r} - \frac{1}{4}Tr(KL)\\
\alpha_z &= \frac{r}{4}Tr(KL)
\end{aligned}$$

3.8.2 Verifying the Consistency of the Purely Horizontal Ricci Flat Equations

We now check that the equations for the Ricci curvature are consistent. This is because α_{rr} and α_{zz} appear earlier and cancelled when we solved for α_r. To check we take derivatives of the partials of α and plug them into the equations for the Ricci curvature. We use (3.8.2).

$$\begin{aligned}
\alpha_{rr} &= \frac{1}{8}(Tr(K^2) - Tr(L^2)) + \frac{1}{2r^2} + \frac{r}{8}(Tr((K^2)_r) - Tr((L^2)_r)) \\
&= \frac{1}{8}(Tr(K^2) - Tr(L^2)) + \frac{1}{2r^2} + \frac{r}{4}(Tr(K_rK) - Tr(L_rL)) \\
\alpha_{zz} &= \frac{r}{4}Tr(K_zL + KL_z) \\
r\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= -\frac{1}{8}(Tr(K^2) - Tr(L^2)) - \frac{1}{2r^2} - \frac{r}{4}(Tr(K_rK) - Tr(K_zL) + Tr([K,L]L)) \ldots \\
&\ldots - \frac{r}{4}Tr(K_zL + KL_z) - \frac{1}{4}Tr(K^2) + \frac{1}{8}\left(Tr(K^2) - Tr(L^2) - \frac{4}{r^2}\right) + \frac{1}{r^2} \\
&= -\frac{1}{4}Tr(K^2) - \frac{r}{4}(Tr(K_rK + KL_z) = -\frac{r}{4}\left(Tr(K(\frac{1}{r}K + K_r + L_z))\right) = 0 \\
r\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) &= -\frac{1}{8}(Tr(K^2) - Tr(L^2)) - \frac{1}{2r^2} - \frac{r}{4}(Tr(K_rK) - Tr(K_zL)) \ldots \\
&\ldots - \frac{r}{4}Tr(K_zL + KL_z) - \frac{1}{4}Tr(K^2) - \frac{1}{8}\left(Tr(L^2) - Tr(K^2) - \frac{4}{r^2}\right) \\
&= -\frac{r}{4}\left(Tr(K(\frac{1}{r}K + K_r + L_z))\right) = 0
\end{aligned}$$

3.8.3 Deriving the Block Matrix Form for α_r and α_z

We now work on converting the equations for the partials of α to the form presented in equation (2.5) in the Khuri et al.'s paper. We use the block matrix form of $H^{-1}H_r$ and $H^{-1}H_z$ that we calculated in the previous subsection. Using the commutative property of the trace we have the following:

$$\begin{aligned}
Tr(H^{-1}H_rH^{-1}H_r) &= Tr((H^{-1}H_r)_{11}^2 + (H^{-1}H_r)_{1\bullet}(H^{-1}H_r)_{\bullet 1} + (H^{-1}H_r)_{\bullet 1}(H^{-1}H_r)_{1\bullet} + (H^{-1}H_r)_{\bullet\bullet}^2) \\
&= (H^{-1}H_r)_{11}^2 + 2(H^{-1}H_r)_{1\bullet}(H^{-1}H_r)_{\bullet 1} + Tr((H^{-1}H_r)_{\bullet\bullet}^2) \\
(H^{-1}H_r)_{11}^2 &= (-f^{-1}f_r + 2r^{-1} - fr^{-2}(v_r^TFv))^2 \\
&= (f^{-1}f_r)^2 + \frac{4}{r^2} + \frac{f^2}{r^4}(v_r^TFv)^2 - \frac{4}{r}\frac{f}{f_r} + 2\frac{f_r}{r^2}v_r^TFv - 4\frac{f}{r^3}v_r^TFv \\
2(H^{-1}H_r)_{1\bullet}(H^{-1}H_r)_{\bullet 1} &= 2(-fr^{-2}v_r^TF)((f^{-1}f_r - 2r^{-1})v + fr^{-2}(vv_r^TFv) + v_r + F^{-1}F_rv) \\
&= -2\frac{f_r}{r^2}v_r^TFv + 4\frac{f}{r^3}v_r^TFv - 2\frac{f^2}{r^4}(v_r^TFv)^2 - 2\frac{f}{r^2}v_r^TF_rv - 2\frac{f}{r^2}v_r^TFv_r \\
Tr((H^{-1}H_r)_{\bullet\bullet}^2) &= Tr((fr^{-2}vv_r^TF + F^{-1}F_r)^2) \\
&= \frac{f^2}{r^4}Tr((vv_r^TF)^2) + 2\frac{f}{r^2}Tr(vv_r^TF_r) + Tr(F^{-1}F_rF^{-1}F_r) \\
&= \frac{f^2}{r^4}(v_r^TFv)^2 + 2\frac{f}{r^2}v_r^TF_rv + Tr(F^{-1}F_rF^{-1}F_r) \\
Tr(H^{-1}H_rH^{-1}H_r) &= (f^{-1}f_r)^2 + \frac{4}{r^2} + \frac{f^2}{r^4}(v_r^TFv)^2 - \frac{4}{r}\frac{f}{f_r} + 2\frac{f_r}{r^2}v_r^TFv - 4\frac{f}{r^3}v_r^TFv \ldots \\
&\ldots - 2\frac{f_r}{r^2}v_r^TFv + 4\frac{f}{r^3}v_r^TFv - 2\frac{f^2}{r^4}(v_r^TFv)^2 - 2\frac{f}{r^2}v_r^TF_rv - 2\frac{f}{r^2}v_r^TFv_r \ldots \\
&\cdots + \frac{f^2}{r^4}(v_r^TFv)^2 + 2\frac{f}{r^2}v_r^TF_rv + Tr((F^{-1}F_r)^2) \\
&= (\log f)_r^2 + \frac{4}{r^2} - 4\frac{(\log f)_r}{r} - 2\frac{f}{r^2}v_r^TFv_r + Tr((F^{-1}F_r)^2)
\end{aligned}$$

Using (3.1.4) and (3.7.2) we have that:

$$\begin{aligned}
2\frac{f}{r^2}v_r^TFv_r &= \frac{2}{r}\omega_zv_r = \frac{2}{r}\frac{r}{f}\omega_zF^{-1}\omega_z^T = \frac{2}{f}\omega_zF^{-1}\omega_z^T \\
Tr(H^{-1}H_rH^{-1}H_r) &= (\log f)_r^2 + \frac{4}{r^2} - 4\frac{(\log f)_r}{r} - \frac{2}{f}\omega_zF^{-1}\omega_z^T + Tr((F^{-1}F_r)^2)
\end{aligned}$$

We now move on to $Tr(H^{-1}H_zH^{-1}H_z)$.

$$\begin{aligned}
Tr(H^{-1}H_zH^{-1}H_z) &= (H^{-1}H_z)^2_{11} + 2(H^{-1}H_z)_{1\bullet}(H^{-1}H_z)_{\bullet 1} + Tr((H^{-1}H_z)^2_{\bullet\bullet}) \\
(H^{-1}H_z)^2_{11} &= (-f^{-1}f_z - fr^{-2}v_z^TFv)^2 = \frac{f_z^2}{f^2} + 2\frac{f_z}{r^2}v_z^TFv + \frac{f^2}{r^4}(v_z^TFv)^2 \\
2(H^{-1}H_z)_{1\bullet}(H^{-1}H_z)_{\bullet 1} &= 2(-fr^{-2}v_z^TF)((f^{-1}f_z)v + fr^{-2}(vv_z^TFv) + v_z + F^{-1}F_zv) \\
&= -2\frac{f_z}{r^2}v_z^TFv - 2\frac{f^2}{r^4}(v_z^TFv)^2 - 2\frac{f}{r^2}v_z^TFv_z - 2\frac{f}{r^2}v_z^TF_zv \\
Tr((H^{-1}H_z)^2_{\bullet\bullet}) &= Tr((fr^{-2}vv_z^TF + F^{-1}F_z)^2) \\
&= \frac{f^2}{r^4}(v_z^TFv)^2 + 2\frac{f}{r^2}v_z^TF_zv + Tr(F^{-1}F_zF^{-1}F_z) \\
Tr(H^{-1}H_zH^{-1}H_z) &= \frac{f_z^2}{f^2} + 2\frac{f_z}{r^2}v_z^TFv + \frac{f^2}{r^4}(v_z^TFv)^2 \ldots \\
&\ldots - 2\frac{f_z}{r^2}v_z^TFv - 2\frac{f^2}{r^4}(v_z^TFv)^2 - 2\frac{f}{r^2}v_z^TFv_z - 2\frac{f}{r^2}v_z^TF_zv \ldots \\
&\ldots + \frac{f^2}{r^4}(v_z^TFv)^2 + 2\frac{f}{r^2}v_z^TF_zv + Tr(F^{-1}F_zF^{-1}F_z) \\
&= (\log f)_z^2 - 2\frac{f}{r^2}v_z^tFv_z + Tr(F^{-1}F_zF^{-1}F_z)
\end{aligned}$$

Again by (3.1.4) and (3.7.2) we have that:

$$2\frac{f}{r^2}v_z^tFv_z = -2\frac{1}{r}\omega_rv_z = -2\frac{1}{r}\omega_r(-\frac{r}{f}F^{-1}\omega_r^T) = 2\frac{1}{f}\omega_rF^{-1}\omega_r^T$$

We can now work out α_r.

$$\begin{aligned}
\alpha_r &= \frac{r}{8}\left(Tr(H^{-1}H_rH^{-1}H_r) - Tr(H^{-1}H_zH^{-1}H_z) - \frac{4}{r^2}\right) \\
&= \frac{r}{8}\Bigg((\log f)_r^2 + \frac{4}{r^2} - 4\frac{(\log f)_r}{r} - \frac{2}{f}\omega_zF^{-1}\omega_z^T + Tr(F^{-1}F_rF^{-1}F_r)\ldots \\
&\ldots - (\log f)_z^2 - 2\frac{1}{f}\omega_rF^{-1}\omega_r^T + Tr(F^{-1}F_zF^{-1}F_z)) - \frac{4}{r^2}\Bigg) \\
&= \frac{r}{8}\Bigg((\log f)_r^2 - (\log f)_z^2 - 4\frac{(\log f)_r}{r} + Tr(F^{-1}F_rF^{-1}F_r) - Tr(F^{-1}F_zF^{-1}F_z)\ldots \\
&\ldots + \frac{2}{f}\omega_rF^{-1}\omega_r^T - \frac{2}{f}\omega_zF^{-1}\omega_z^T\Bigg)
\end{aligned}$$

We now move on to $Tr(H^{-1}H_rH^{-1}H_z)$.

$$\begin{aligned}
Tr(H^{-1}H_rH^{-1}H_z) &= Tr((H^{-1}H_r)_{11}(H^{-1}H_z)_{11} + (H^{-1}H_r)_{1\bullet}(H^{-1}H_z)_{\bullet 1} \ldots \\
&\ldots + (H^{-1}H_r)_{\bullet 1}(H^{-1}H_z)_{1\bullet} + (H^{-1}H_r)_{\bullet\bullet}(H^{-1}H_z)_{\bullet\bullet}) \\
&= (H^{-1}H_r)_{11}(H^{-1}H_z)_{11} + (H^{-1}H_r)_{1\bullet}(H^{-1}H_z)_{\bullet 1} \ldots \\
&\ldots + (H^{-1}H_z)_{1\bullet}(H^{-1}H_r)_{\bullet 1} + Tr((H^{-1}H_r)_{\bullet\bullet}(H^{-1}H_z)_{\bullet\bullet}) \\
(H^{-1}H_r)_{11}(H^{-1}H_z)_{11} &= (-f^{-1}f_r + 2r^{-1} - fr^{-2}(v_r^TFv))(-f^{-1}f_z - fr^{-2}v_z^TFv) \\
&= \frac{f_r}{f}\frac{f_z}{f} + \frac{f_r}{r^2}v_z^TFv - 2\frac{f_z}{rf} - 2\frac{f}{r^3}v_z^TFv + \frac{f_z}{r^2}v_r^TFv + \frac{f^2}{r^4}v_r^TFvv_z^TFv \\
(H^{-1}H_r)_{1\bullet}(H^{-1}H_z)_{\bullet 1} &= (-fr^{-2}v_r^TF)((f^{-1}f_z)v + fr^{-2}(vv_z^TFv) + v_z + F^{-1}F_zv) \\
&= -\frac{f_z}{r^2}v_r^TFv - \frac{f^2}{r^4}v_r^TFvv_z^TFv - \frac{f}{r^2}v_r^TF_zv - \frac{f}{r^2}v_r^TFv_z \\
(H^{-1}H_z)_{1\bullet}(H^{-1}H_r)_{\bullet 1} &= (-fr^{-2}v_z^TF)((f^{-1}f_r - 2r^{-1})v + fr^{-2}(vv_r^TFv) + v_r + F^{-1}F_rv) \\
&= -\frac{f_r}{r^2}v_z^TFv + 2\frac{f}{r^3}v_z^TFv - \frac{f^2}{r^4}v_z^TFvv_r^TFv - \frac{f}{r^2}v_z^TF_rv - \frac{f}{r^2}v_z^TFv_r \\
Tr((H^{-1}H_r)_{\bullet\bullet}(H^{-1}H_z)_{\bullet\bullet}) &= Tr((fr^{-2}vv_r^TF + F^{-1}F_r)(fr^{-2}vv_z^TF + F^{-1}F_z)) \\
&= \frac{f^2}{r^4}v_r^TFvv_z^TFv + \frac{f}{r^2}v_r^TF_zv + \frac{f}{r^2}v_z^TF_rv + Tr(F^{-1}F_rF^{-1}F_z) \\
Tr(H^{-1}H_rH^{-1}H_z) &= \frac{f_r}{f}\frac{f_z}{f} + \frac{f_r}{r^2}v_z^TFv - 2\frac{f_z}{rf} - 2\frac{f}{r^3}v_z^TFv + \frac{f_z}{r^2}v_r^TFv + \frac{f^2}{r^4}v_r^TFvv_z^TFv \ldots \\
&\ldots - \frac{f_z}{r^2}v_r^TFv - \frac{f^2}{r^4}v_r^TFvv_z^TFv - \frac{f}{r^2}v_r^TF_zv - \frac{f}{r^2}v_r^TFv_z \ldots \\
&\ldots - \frac{f_r}{r^2}v_z^TFv + 2\frac{f}{r^3}v_z^TFv - \frac{f^2}{r^4}v_z^TFvv_r^TFv - \frac{f}{r^2}v_z^TF_rv - \frac{f}{r^2}v_z^TFv_r \ldots \\
&\ldots + \frac{f^2}{r^4}v_r^TFvv_z^TFv + \frac{f}{r^2}v_r^TF_zv + \frac{f}{r^2}v_z^TF_rv + Tr(F^{-1}F_rF^{-1}F_z) \\
Tr(H^{-1}H_rH^{-1}H_z) &= (\log f)_r(\log f)_z - 2\frac{(\log f)_z}{r} - \frac{f}{r^2}v_z^TFv_r - \frac{f}{r^2}v_r^TFv_z + Tr(F^{-1}F_rF^{-1}F_z)
\end{aligned}$$

Using (3.1.4) and (3.7.2) again we have that:

$$\begin{aligned}
\frac{f}{r^2}v_r^TFv_z + \frac{f}{r^2}v_z^TFv_r &= +\frac{1}{r}\omega_zv_z - \frac{1}{r}\omega_rv_r \\
&= \frac{1}{r}\omega_z(-\frac{r}{f}F^{-1}\omega_r^T) - \frac{1}{r}\omega_r(\frac{r}{f}F^{-1}\omega_z^T) \\
&= -\frac{1}{r}(\omega_zF^{-1}\omega_r^T) - \frac{1}{r}(\omega_zF^{-1}\omega_r^T)^T \\
&= -\frac{2}{r}\omega_zF^{-1}\omega_r^T \\
Tr(H^{-1}H_rH^{-1}H_z) &= (\log f)_r(\log f)_z - 2\frac{(\log f)_z}{r} + \frac{2}{r}\omega_zF^{-1}\omega_r^T + Tr(F^{-1}F_rF^{-1}F_z) \\
\alpha_z &= \frac{r}{4}\left((\log f)_r(\log f)_z - 2\frac{(\log f)_z}{r} + \frac{2}{r}\omega_zF^{-1}\omega_r^T + Tr(F^{-1}F_rF^{-1}F_z)\right)
\end{aligned}$$

3.9 Mixed Components of the Ricci Tensor

For completeness we will check that the mixed components of the Ricci tensor are 0. From Besse we have that [3, p. 244]:

$$r(X,U) = ((\hat{\delta}T)U, X) + (D_U N, X) - ((\check{\delta}A)X, U) - 2(A_X, T_U)$$

We will show that each term is 0. We have that $\hat{\delta}T = -\sum_j (D_{U_j}T)_{U_j}$ where U_j is an orthonormal basis of the vertical distribution.

$$(D_{U_j}T)_{U_j}U = D_{U_j}(T_{U_j}U) - T_{D_{U_j}U_j}U - T_{U_j}(D_{U_j}U)$$

Let U be a vertical coordinate vector field. Then $D_{U_j}U$ is a sum of horizontal vector fields. Then $T_{U_j}(D_{U_j}U)$ is vertical so it doesn't contribute. We have that U_j is a linear combination of vertical coordinate vector fields where the coefficients are functions of r and z. Therefore $D_{U_j}U_j$ is a sum of horizontal vector fields. Thus $T_{D_{U_j}U_j}U = 0$. Finally $T_{U_j}U$ is horizontal. So $D_{U_j}(T_{U_j}U)$ is vertical so it doesn't contribute. We have that $N = -\frac{e^{-2\alpha}}{r}\frac{\partial}{\partial r}$. So, $D_U N = -\frac{e^{-2\alpha}}{r}D_U\partial_r$ which is vertical thus doesn't contribute. We have that $(\check{\delta}A) = \sum_{ij} g_b^{ij}(D_{\partial_{\xi_i}}A)_{\partial_{\xi_j}}$ where ∂_{ξ_1} is a coordinate basis of the horizontal distribution. Let X be a horizontal vector field.

$$(D_{\partial_{\xi_i}}A)_{\partial_{\xi_j}}X = D_{\partial_{\xi_i}}(A_{\partial_{\xi_j}}X) - A_{D_{\partial_{\xi_i}}\partial_{\xi_j}}X - A_{\partial_{\xi_i}}(D_{\partial_{\xi_j}}X)$$

We know by theorem 3.5 that the above term is 0. Finally we have that for horizontal vector field X and vertical vector field U that $(AX, TU) = \sum_{ij}(A_{\partial_{\xi_i}}X, T_{\partial_{\xi_j}}U)$. It is clearly 0 because of theorem 3.5.

Chapter 4

The Smoothness Conditions

Currently our metric is well defined over the interior of the orbit space. At points on the boundary we no longer have a principal bundle but we have a tube with a non-trivial isotropy group. To judge smoothness, we start by defining an appropriate slice representation for such a point on a boundary. Afterwards we use Schur's Lemma to generate the polynomials which arise in the metric's components.

We will express our group G as $S^1_1 \times S^1_2 \times \mathbb{R}$ and use $(r, \bar{r})$ as the coordinates on the orbit space. We build a local model around q_c using the rod data. The rod data in the orbit space is a $(1,0)$ rod above q_c and a $(0,1)$ rod below q_c. We will take q_a to be a point whose orbit is on the $(1,0)$ rod and q_b to be a point whose orbit is on the $(0,1)$ rod. With this pattern of alternating rod structure carrying on across the $\bar{r}$ axis. We will call the orbit of a point q, $\pi(q)$. Upstairs in the manifold we can fit the slices at q_a, q_b and q_c into a single 4-dimensional diagram which serves only as a schematic.

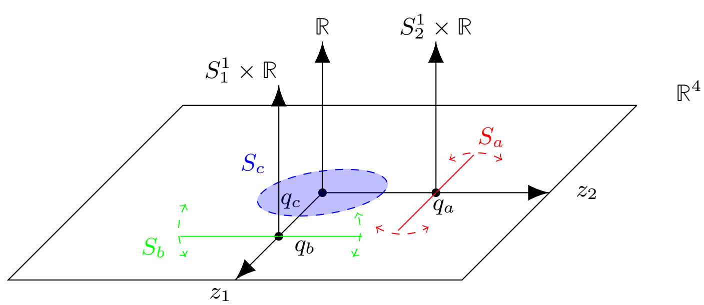

Figure 4.1:

Here S_a, S_b and S_c are the slices at q_a, q_b and q_c respectively. All these slices live in a bounded set in $\mathbb{R}^4$ shown above. We think of $\mathbb{R}^4$ as $\mathbb{C}_1 \times \mathbb{C}_2$ where z_1 is the complex coordinate of $\mathbb{C}_1$ and z_2 is the complex coordinate of $\mathbb{C}_2$. We have that S_a has coordinates (x_1, y_1, r_2) where $z_1 = x_1 + iy_1$ and $r_2^2 = x_2^2 + y_2^2$. We think of r_2 as being the radial part of z_2 which varies in S_a. Furthermore since S_1^1 shrinks to a point we have that the isotropy group at q_a is isomorphic to S_1^1. Also, we have that the orbit is isomorphic to $S_2^1 \times \mathbb{R}$. This is represented in the vertical direction of the figure. Similarly we have that S_b has coordinates (r_1, x_2, y_2) where $z_2 = x_2 + iy_2$ and $r_1^2 = x_1^2 + y_1^2$. Since S_2^1 shrinks to a point we have that the isotropy group at q_b is isomorphic to S_2^1. This implies that the orbit is isomorphic to $S_1^1 \times \mathbb{R}$. Finally at S_c we have (x_1, y_1, x_2, y_2) as coordinates. Since both circles shrink to a point at q_c we have that the isotropy group at q_c is isomorphic to T^2. And we have that the orbit is isomorphic to $\mathbb{R}$. These coordinates (x_1, y_1, x_2, y_2) are only valid locally around q_c and in this neighbourhood there are only two rods a $(1,0)$ rod above $\pi(q_c)$ and a $(0,1)$ rod below $\pi(q_c)$

Now our orbit space is homeomorphic to the right half plane with coordinates $(r, \overline{r})$. We have the following local formula about the corner point in the orbit space.

$$r + i\overline{r} = (r_2 + i r_1)^2 \implies r = r_1 r_2 \qquad \overline{r} = \frac{r_2^2 - r_1^2}{2}$$
$$r_1 = \sqrt{\sqrt{r^2 + \overline{r}^2} - \overline{r}} \qquad r_2 = \sqrt{\sqrt{r^2 + \overline{r}^2} + \overline{r}}$$

This transforms the quarter plane made up of (r_1, r_2) to the half plane made up of $(r, \overline{r})$. This transformation is valid everywhere except the origin. If you want to consider a similar corner point somewhere else on the $\overline{r}$-axis then you simply perform a translation in $\overline{r}$ so that the corner point is at the origin of your new coordinate system.

For coordinates on the $(1,0)$ rod at $(0,a)$ we use the following equation.

$$r_2 + a' = \sqrt{\sqrt{r^2 + (\overline{r} + a)^2} + \overline{r} + a} \qquad r_1^2 = \sqrt{r^2 + (\overline{r} + a)^2} - (\overline{r} + a) = \frac{r^2}{\sqrt{r^2 + (\overline{r} + a)^2} + \overline{r} + a} \tag{4.0.1}$$

We now provide a summary of the conclusions of the smoothness conditions before going into more detail.

For a point on a $(1,0)$ rod in the orbit space we have the following in an open neighbourhood around $\pi(q_a)$. Disclaimer we have translated the $\overline{r}$ coordinate so that it is 0 at $\pi(q_a)$. Also there is a normalization of the metric at the point $\pi(q_a)$. Firstly $g(X, \partial_{\varphi_2}) = g(X, \partial_t) = 0$ for $X \in \text{span}(\partial_r, \partial_{\overline{r}}, \partial_{\varphi_1})$. We have that $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ behaves as $\frac{r^2}{2a}$ as you approach $\pi(q_a)$. Finally, we have that all components of the metric are smooth functions of r^2 and $\bar{r}$.

For a point on a $(0,1)$ rod in the orbit space we have the following in an open neighbourhood around $\pi(q_b)$. Disclaimer we have translated the $\bar{r}$ coordinate so that it is 0 at $\pi(q_b)$. Also there is a normalization of the metric at the point $\pi(q_b)$. Firstly $g(X, \partial_{\varphi_1}) = g(X, \partial_t) = 0$ for $X \in \text{span}(\partial_r, \partial_{\bar{r}}, \partial_{\varphi_2})$. We have that $g(\partial_{\varphi_2}, \partial_{\varphi_2})$ behaves as $\frac{r^2}{2b}$ as you approach $\pi(q_b)$. Finally all components of the metric are smooth functions of r^2 and $\bar{r}$.

For the corner point q_c we have the coordinates, $(r, \bar{r}, \varphi_1, \varphi_2)$, for our slice. We let $\bar{r} = 0$ at the corner point. We have the following in an open neighbourhood around $\pi(q_c)$. Firstly, that all non-diagonal components of the metric on the slice are 0 apart from $g(\partial_{\varphi_1}, \partial_{\varphi_2})$. We have that away from the corner point that all components of the metric are smooth functions of r^2 and $\bar{r}$. We have that $g(\partial_{\varphi_2}, \partial_{\varphi_2})$ and $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ behave like r when $\bar{r} = 0$ and $r \to 0$. We have that $g(\partial_{\varphi_1}, \partial_{\varphi_2})$ behaves as r^2 when $\bar{r} = 0$ and $r \to 0$. We have that $g(\partial_r, \partial_r)$ behaves like r^{-1} when $\bar{r} = 0$ and $r \to 0$.

4.1 Axis Rod

We let $e^{i\theta}$ be in the isotropy group $G_{q_a} = S^1_1 \times \{1\} \times \{1\}$. We consider the action on the slice S_a. We set the name of the tangent space of S_a to be V. We have that $e^{i\theta}$ acts on (x_1, y_1) by rotation by an angle θ. It preserves r_2. We also translate r_2 so $(r_1, r_2)(q_a) = (0, 0)$. We can write this group action as matrix multiplication.

$$(e^{i\theta})_* \begin{pmatrix} x_1 \\ y_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ r_2 \end{pmatrix}$$

Here we are using ∂_{x_1}, ∂_{y_1} and ∂_{r_2} as a basis. Therefore the slice representation is $\rho \oplus \mathbb{1}$ where ρ is the representation responsible for the rotation.

The isotropy representation at q_a is trivial. To see this let $\sigma_{e^{i\theta}} : M \to M$ map m to $e^{i\theta}m$ and let X be tangent to the orbit at q_a, i.e in $T_{q_a}G(q_a)$. Then the derivative map is the following: $d\sigma(X) = (e^{i\theta}\delta(u))'(0)$. Where $\delta(u)$ is some curve in M. We can use the tube theorem to rewrite this as $[e^{i\theta}\gamma(u), q_a]'(0)$. Where $\gamma(u)$ is some curve in G. Note that the slice part is constant in terms of u. We can make the following simplifications:

$$[e^{i\theta}\gamma(u), q_a]'(0) = [\gamma(u)e^{i\theta}, q_a]'(0) = [\gamma(u), e^{i\theta}q_a]'(0) = [\gamma(u), q_a]'(0) = X$$

It is clear that the tangent space at q_a uniquely decomposes into a part tangent to the slice and a part tangent to the orbit. Note that the orbit associated to q_a is 2-dimensional and V is 3-dimensional. However at other points in S_a, the part tangent to the orbit grows to being 3-dimensional. Also note that the action of G_{q_a} is obviously closed in S_a and the third part of the slice definition is satisfied.

We take $\{\partial_{x_1}, \partial_{y_1}, \partial_{r_2}\}$ as a basis for V and $\{\partial_{\varphi_2}, \partial_t\}$ as a a basis to the the orthogonal counterpart, $V^\perp$. For orthogonality we have used a background Euclidean metric, $dr_1^2 + dr_2^2 + r_1^2 d\varphi_1^2 + r_2^2 d\varphi_2^2 + dt^2$. Let D be an open ball in V centered at the origin. In the following approach we wish to consider smooth G invariant metrics on the tube $G \times_{G_{q_a}} D$. These metrics correspond to metrics on an open set in the manifold M due to the tube theorem. There is another correspondence between the G invariant metrics on the tube and G_{q_a} equivariant maps F where $F : D \to S^2(V \oplus V^\perp)$. The correspondence will be demonstrated later on. The maps which use the m^{th} order polynomials are given by $\mathrm{Hom}_H(S^m V, S^2(V \oplus V^\perp))$. The reason for using $\mathrm{Hom}_H(S^m V, S^2(V \oplus V^\perp))$ is that the H-equivariance allows for a smooth extension of the polynomials on a 2-dimensional section of D to the entirety of D [10, p. 113]. We can understand the $S^2(V \oplus V^\perp)$ as corresponding to components of a metric. We have from the section on representations that $S^2(\rho) = \rho^2 \oplus \mathbb{1}$, thereby reducing $S^2(\rho)$ to two irreducible representations. Here $\rho^n(e^{i\theta}) = \rho(e^{ni\theta})$. We can now work out $S^2(V \oplus V^\perp)$ using the trivial representation on $V^\perp$.

$$\begin{aligned} S^2(V \oplus V^\perp) &= S^2(V) \oplus S^2(V^\perp) \oplus (V \otimes V^\perp) = S^2(\rho \oplus \mathbb{1}) \oplus 3\mathbb{1} \oplus (2\mathbb{1} \oplus 2\rho) \\ &= (\rho^2 \oplus \mathbb{1} \oplus \rho \oplus \mathbb{1}) \oplus 3\mathbb{1} \oplus (2\mathbb{1} \oplus 2\rho) \end{aligned}$$

Let us consider the constant polynomials $S^0(V)$ which is isomorphic to $\mathbb{R}$. This means there is a constant for each component of $S^2(V \oplus V^\perp)$. Before we begin we will translate r_2 so that it is 0 at $\pi(q_a)$. We can normalize the metric on the V part so that at the point in question, q_a, the metric is of the form $(dx_1^2 + dy_1^2) + dr_2^2$. This is done by fixing an orthonormal basis at the q_a and then parallel translating the part of the orthonormal basis perpendicular to the group along the normal geodesics. Thus we create geodesic coordinates which are the coordinates of the slice x_1, y_1 and r_2. Therefore the metric is Euclidean at the origin.

We now consider the first order polynomials, $\mathrm{Hom}_{G_{q_a}}(S^1(V), S^2(V \oplus V^\perp)) = \mathrm{Hom}_{G_{q_a}}(\rho \oplus$

$\mathbb{1}, (\rho^2 \oplus \mathbb{1} \oplus \rho \oplus \mathbb{1}) \oplus 3\mathbb{1} \oplus (2\mathbb{1} \oplus 2\rho)$. Note that in the codomain the generators of ρ^2 are $dx_1^2 - dy_1^2$ and $dx_1 dy_1$. The generators of the first ρ are $dx_1 dr_2$ and $dy_1 dr_2$. The second ρ has generators $dx_1 dt$ and $dy_1 dt$. The third ρ has generators $dx_1 d\varphi_2$ and $dy_1 d\varphi_2$. The first $\mathbb{1}$ has as its generator $(dx_1^2 + dy_1^2)$. The generator of second, third, fourth, and fifth copies of $\mathbb{1}$ are dr_2^2, $d\varphi_2^2$, dt^2 and $d\varphi_2 dt$ respectively. By Schur's lemma, we have that the only isomorphisms are the ones which map its domain to itself. For ρ the arbitrary element in the domain is $a\partial_{x_1} + b\partial_{y_1}$. The homomorphism maps this element to $\epsilon^1(adx_1dr_2 + bdy_1dr_2)$ choosing the ρ that is comprised of dx_1dr_2 and dy_1dr_2. However what we need is not the output but the homomorphism itself. We can think of the homomorphism as being $\epsilon^1(\partial_{x_1})^* dx_1dr_2 + (\partial_{y_1})^* dy_1dr_2$. Where $(\partial_{x_1})^*$ is the dual of ∂_{x_1} and $(\partial_{y_1})^*$ is the dual of ∂_{y_1}. Because of the equivalence of $S_1(\rho)$ and $P_1(\rho)$ as representations we can use $ax_1 + by_1$ when applying the group action but then convert back to vector form before inputting back into the homomorphism. However this viewpoint of homomorphism is well and good but there is a second viewpoint that of the metric itself. To get the metric we take the sum of all the homomorphism and input $x_1\partial_{x_1} + y_1\partial_{y_1} + r_2\partial_{r_2}$ where x_1, y_1 and r_2 are now the coordinates. Both viewpoints are stated below where g^1 is the part of the metric which has order 1 homogeneous polynomials.

$$
\begin{aligned}
&\mathbb{1} \to \mathbb{1}, && \alpha^1(\partial_{r_2})^*(dx_1^2 + dy_1^2)\\
&\mathbb{1} \to \mathbb{1}, && \beta^1(\partial_{r_2})^* dr_2^2\\
&\mathbb{1} \to \mathbb{1}, && \gamma_1^1(\partial_{r_2})^* d\varphi_2^2\\
&\mathbb{1} \to \mathbb{1}, && \gamma_2^1(\partial_{r_2})^* dt_2^2\\
&\mathbb{1} \to \mathbb{1}, && \gamma_3^1(\partial_{r_2})^* d\varphi_2 dt\\
&\mathbb{1} \to \mathbb{1}, && \delta^1(\partial_{r_2})^*(dr_2dt + dr_2d\varphi_2)\\
&\rho \to \rho, && \epsilon^1((\partial_{x_1})^* dx_1dr_2 + (\partial_{y_1})^* dy_1dr_2)\\
&\rho \to \rho, && \eta_1^1((\partial_{x_1})^* dx_1dt + (\partial_{y_1})^* dy_1dt)\\
&\rho \to \rho, && \eta_2^1((\partial_{x_1})^* dx_1d\varphi_2 + (\partial_{y_1})^* dy_1d\varphi_2)
\end{aligned}
$$

$$
\begin{aligned}
g^1 = {} & \alpha^1 r_2(dx_1^2 + dy_1^2) + \beta^1 r_2 dr_2^2 + \gamma_1^1 r_2 d\varphi_2^2 + \gamma_2^1 r_2 dt_2^2 + \gamma_3^1 r_2 d\varphi_2 dt \ldots\\
& \ldots + \epsilon^1(x_1dx_1dr_2 + y_1dy_1dr_2) + \eta_1^1(x_1dx_1dt + y_1dy_1dt) + \eta_2^1(x_1dx_1d\varphi_2 + y_1dy_1d\varphi_2)
\end{aligned}
$$

Where δ^1, η_1^1, η_2^1 are 0 if the orbit space is a section. We will show a correspondence between G_{q_a} invariance of the metric and G_{q_a} equivariance of the homomorphisms. Let f_{ϵ^1} be the homomorphism which takes $ax_1 + by_1$ and sends it to $\epsilon^1(x_1dx_1dr_2 + y_1dy_1dr_2)$ and let σ map m to $e^{i\theta}m$. We will show that

$f(\sigma^*(a\partial_{x_1} + b\partial_{y_1})) = \sigma^* f(a\partial_{x_1} + b\partial_{y_1})$.

$$\begin{aligned} \sigma^*(ax_1 + by_1 + cr_2) &= (a\cos(\theta) - b\sin(\theta))x_1 + (a\sin(\theta) + b\cos(\theta))y_1 + cr_2 \\ f_{\epsilon^1}(\sigma_*(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2})) &= \epsilon^1((a\cos(\theta) - b\sin(\theta))dx_1 dr_2 + (a\sin(\theta) + b\cos(\theta))dy_1 dr_2) \\ &= \epsilon^1((a(\cos(\theta)dx_1 + b\sin(\theta)dy_1)dr_2 + b(-\sin(\theta)dx_1 + \cos(\theta)dy_1)dr_2 \\ &= \sigma^*(f_{\epsilon^1}(a\partial_{x_1} + b\partial_{y_1}) + c\partial_{r_2})) \end{aligned}$$

The last step follows since $\sigma^* dr_2 = dr_2$. Let f_{α^1} be the homomorphism which takes $a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2}$ and sends it to $\alpha^1(c(dx_1^2 + dy_1^2))$. We will show that it is G_{qa} invariant.

$$\begin{aligned} f_{\alpha^1}(\sigma_*(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2})) &= \alpha^1 c(dx_1^2 + dy_1^2) \\ \alpha^1 c\sigma^*(dx_1^2 + dy_1^2) &= \alpha^1 c((\cos(\theta)dx_1 + \sin(\theta)dy_1)^2 + (-\sin(\theta)dx_1 + \cos(\theta)dy_1)^2) = c\alpha^1(dx_1^2 + dy_1^2) \\ f_{\alpha^1}(\sigma_*(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2})) &= \sigma^*(f_{\alpha^1}(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2})) \end{aligned}$$

We now demonstrate equivariant maps like the ones shown above correspond to a G invariant metric on the tube $G \times_{G_{qa}} D$. The invariance should be true under the stationary and bi-axisymmetric assumptions. So let's demonstrate it in action. The invariance due to the group elements corresponding to the S_2^1 and $\mathbb{R}$ are trivial. But the invariance due to action from G_{qa} is not trivial. Let σ map m to $e^{i\theta}m$ and note that $\sigma^* g^1(\partial_{x_1}, \partial_{r_2}) = (g^1(\sigma_*\partial_{x_1}, \sigma_*\partial_{r_2})) \circ \sigma$. We have that:

$$\begin{aligned} \sigma_*\partial_{x_1} &= \cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{y_1} \\ x_1 \circ \sigma &= x_1\cos(\theta) - y_1\sin(\theta) \qquad y_1 \circ \sigma = x_1\sin(\theta) + y_1\cos(\theta) \qquad r_2 \circ \sigma = r_2 \\ g^1(\sigma_*\partial_{x_1}, \sigma_*\partial_{r_2}) \circ \sigma &= g^1(\cos(\theta)\partial_{x_1} + \sin(\theta)\partial_{y_1}, \partial_{r_2}) \circ \sigma \\ &= \epsilon^1(\cos(\theta)x_1 + \sin(\theta)y_1) \circ \sigma \\ &= \epsilon^1(\cos(\theta)(x_1\cos(\theta) - y_1\sin(\theta)) + \sin(\theta)(x_1\sin(\theta) + y_1\cos(\theta))) \\ &= \epsilon^1(x_1) = g^1(\partial_{x_1}, \partial_{r_2}) \end{aligned}$$

This G invariance holds true for the other components of the metric thus you can start to see the equivalence between G_{qa} equivariance of the homomorphisms and G invariance of the metric. We can express the metric in terms of polar coordinates where $x_1 = r_1\cos(\varphi_1)$, and $y_1 = r_1\sin(\varphi_1)$.

$$\alpha^1 r_1(dx_1^2 + dy_1^2) + \beta^1 r_2 dr_2^2 + \epsilon^1(x_1 dx_1 dr_2 + y_1 dy_1 dr_2) = \alpha^1 r_2 dr_1^2 + \epsilon^1 r_1 dr_1 dr_2 + \beta^1 r_2 dr_2^2 + \alpha^1 r_2 r_1^2 d\varphi_1^2$$

We now look at the $m = 2$ case. We are interested in $\mathrm{Hom}_{G_{qa}}(S^2V, S^2(V \oplus V^\perp)) = \mathrm{Hom}_{G_{qa}}(\rho^2 \oplus \mathbb{1} \oplus \rho \oplus \mathbb{1}, (\rho^2 \oplus \mathbb{1} \oplus \rho \oplus \mathbb{1}) \oplus 3\mathbb{1} \oplus (2\mathbb{1} \oplus 2\rho))$. We now state both viewpoints and we take g^2 to be the part of the metric with order 2 homogeneous polynomials.

$$
\begin{aligned}
\mathbb{1} \to \mathbb{1}, &\quad \alpha^2(\partial_{r_2})^* \otimes (\partial_{r_2})^*(dx_1^2 + dy_1^2)\\
\mathbb{1} \to \mathbb{1}, &\quad \beta^2(\partial_{r_2})^* \otimes (\partial_{r_2})^* dr_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \gamma_1^2(\partial_{r_2})^* \otimes (\partial_{r_2})^* d\varphi_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \gamma_2^2(\partial_{r_2})^* \otimes (\partial_{r_2})^* dt_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \gamma_3^2(\partial_{r_2})^* \otimes (\partial_{r_2})^* d\varphi_2 dt\\
\mathbb{1} \to \mathbb{1}, &\quad \gamma_4^2(\partial_{r_2})^* \otimes (\partial_{r_2})^* d\varphi_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \delta^2(\partial_{r_2})^* \otimes (\partial_{r_2})^*(dr_2 dt + dr_2 d\varphi_2)\\
\mathbb{1} \to \mathbb{1}, &\quad \nu^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*) dr_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \mu^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*)(dx_1^2 + dy_1^2)\\
\mathbb{1} \to \mathbb{1}, &\quad \tau_1^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*) d\varphi_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \tau_2^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*) dt_2^2\\
\mathbb{1} \to \mathbb{1}, &\quad \tau_3^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*) d\varphi_2 dt\\
\mathbb{1} \to \mathbb{1}, &\quad \tau_4^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*) d\varphi_2^2\\
\rho \otimes \mathbb{1} \to \rho, &\quad \epsilon^2(\partial_{r_2})^* \otimes ((\partial_{x_1})^* dx_1 dr_2 + (\partial_{y_1})^* dy_1 dr_2)\\
\rho \otimes \mathbb{1} \to \rho, &\quad \eta_1^2(\partial_{r_2})^* \otimes ((\partial_{x_1})^* dx_1 dt + (\partial_{y_1})^* dy_1 dt)\\
\rho \otimes \mathbb{1} \to \rho, &\quad \eta_2^2(\partial_{r_2})^* \otimes ((\partial_{x_1})^* dx_1 d\varphi_2 + (\partial_{y_1})^* dy_1 d\varphi_2)\\
\rho^2 \to \rho^2, &\quad \iota^2(((\partial_{x_1})^* \otimes (\partial_{x_1})^* - (\partial_{y_1})^* \otimes (\partial_{y_1})^*)(dx_1^2 - dy_1^2) + 4((\partial_{x_1})^*(\partial_{y_1})^*)) dx_1 dy_1
\end{aligned}
$$

$$
\begin{aligned}
g^2 = {}& \alpha^2 r_2^2(dx_1^2 + dy_1^2) + \beta^2 r_2^2 dr_2^2 + \gamma_1^2 r_2^2 d\varphi_2^2 + \gamma_2^2 r_2^2 dt_2^2 + \gamma_3^2 r_2^2 d\varphi_2 dt + \gamma_4^2 r_2^2 d\varphi_2^2 + \delta^2 r_2^2(dr_2 dt + dr_2 d\varphi_2) \ldots\\
&\ldots + \nu^2(x_1^2 + y_1^2) dr_2^2 + \mu^2(x_1^2 + y_1^2)(dx_1^2 + dy_1^2) + \tau_1^2(x_1^2 + y_1^2) d\varphi_2^2 + \tau_2^2(x_1^2 + y_1^2) dt_2^2 + \tau_3^2(x_1^2 + y_1^2) d\varphi_2 dt \ldots\\
&\ldots + \tau_4^2(x_1^2 + y_1^2) d\varphi_2^2 + \epsilon^2 r_2(x_1 dx_1 dr_2 + y_1 dy_1 dr_2) + \eta_1^2 r_2(x_1 dx_1 dt + y_1 dy_1 dt) + \eta_2^2 r_2(x_1 dx_1 d\varphi_2 + y_1 dy_1 d\varphi_2) \ldots\\
&\ldots + \iota^2((x_1^2 - y_1^2) dx_1^2 + 4x_1 y_1 dx_1 dy_1 - (x_1^2 - y_1^2) dy_1^2)
\end{aligned}
$$

Let f_{ι^2} be the last homomorphism above. We will show that it is G_{q_a} invariant where σ maps m to $e^{i\theta}m$.

$$
\begin{aligned}
&= f_{\iota^2}(\sigma_*(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2}))\\
&= \iota^2((a\cos(\theta) - b\sin(\theta))^2 - (a\sin(\theta) + \cos(\theta)b)^2)(dx_1^2 - dy_1^2) + 4(a\cos(\theta) - b\sin(\theta))(a\sin(\theta) + \cos(\theta)b) dx_1 dy_1\\
&= \iota^2((a^2 - b^2)\cos(2\theta) - 2ab\sin(2\theta))(dx_1^2 - dy_1^2) + (2(a^- b^2)\sin(2\theta) + 4ab\cos(2\theta)) dx_1 dy_1)\\
&= \iota^2((a^2 - b^2)(\cos(2\theta)(dx_1^2 - dy_1^2) + \sin(2\theta) 2 dx_1 dy_1) + 2ab(-\sin(2\theta)(dx_1^2 - dy_1^2) + \cos(2\theta) 2 dx_1 dy_1))\\
&= \iota^2((a^2 - b^2)((\cos(\theta) dx_1 + \sin(\theta) dy_1)^2 - (-\sin(\theta) dx_1 + \cos(\theta) dy_1)^2) + 4ab((\cos(\theta) dx_1 + \sin(\theta) dy_1)(-\sin(\theta) dx_1 + \cos(\theta) dy_1)\\
&= \sigma^* f_{\iota^2}(a\partial_{x_1} + b\partial_{y_1} + c\partial_{r_2})
\end{aligned}
$$

We can convert the part of the metric tangent to the slice to polar form.

$$(\alpha^2 r_2^2 + \mu^2(x_1^2+y_1^2) + \iota^2(x_1^2-y_1^2))dx_1^2 + (\alpha^2 r_2^2 + \mu^2(x_1^2+y_1^2) - \iota^2(x_1^2-y_1^2))dy_1^2 + 2\iota^2(x_1y_1dx_1dy_1)\ldots$$
$$\ldots + (\beta^2 r_2^2 + \nu^2(x_1^2+y_1^2))dr_2^2 + \epsilon^2 r_2(x_1dx_1dr_2 + y_1dy_1dr_2) = (\beta^2 r_2^2 + \nu^2 r_1^2)dr_2^2 + \epsilon^2 r_1r_2dr_1dr_2\ldots$$
$$\ldots + (\alpha^2 r_2^2 + (\mu^2+\iota^2)r_1^2)dr_1^2 + (\alpha^2 r_2^2 + (\mu^2-\iota^2)r_1^2)r_1^2 d\varphi_1^2$$

Now let's consider an m degree homogeneous polynomial. Where $m \geq 3$.

$$S^m(V) = S^m(\rho \oplus \mathbb{1}) = (S^m(\rho) \oplus (S^{m-1}(\rho) \otimes \mathbb{1}) \oplus \ldots \oplus (\rho \otimes \mathbb{1}) \oplus \mathbb{1}$$

The tensor product of $\mathbb{1}$ can be thought of as multiplying the contents of $S^N(\rho)$ with an appropriate power of r_2. It is therefore crucial to understand $S^{2d+1}(\rho)$ and $S^{2d}(\rho)$ for $d \geq 1$. Their decomposition into irreducible representations is well understood. We have that they both decompose into rotations; more specifically:

$$S^{2d+1}(\rho) = \rho^{2d+1} \oplus \rho^{2d-1} \oplus \ldots \oplus \rho \qquad S^{2d}(\rho) = \rho^{2d} \oplus \rho^{2d-2} \oplus \ldots \oplus \mathbb{1}$$

However, we need to know the domain on which each rotation acts. Because of Schur's Lemma we really only need to know this for ρ^2, ρ and $\mathbb{1}$ when they appear in $S^n(\rho)$. Let's consider $\mathbb{1}$ when it appears in $S^{2d}(\rho)$. It is useful to consider the eigenvectors of the rotation matrix associated to ρ. We associate these eigenvectors to polynomials u and v which have eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ respectively.

$$u = ix_1 + y_1 \qquad v = -ix_1 + y_1$$

Consider the homogeneous polynomials of degree $2d$. Consider them expressed in terms of u and v. We have that ρ acts on each u by multiplying the u by $e^{i\theta}$ and each v by multiplying it by $e^{-i\theta}$. Therefore the 1-dimensional subspace which is associated to $\mathbb{1}$ is given by multiples of the term u^dv^d. We have that $u^dv^d = (x_1^2+y_1^2)^d$. Thus using the metric viewpoint we get the following where we will call this part of the metric $g_{\mathbb{1}}$.

$$g_{\mathbb{1}} = \sum_{d,k}\left(\alpha^{k,d}r_2^k r_1^{2d}(dx_1^2+dy_1^2) + \beta^{k,d}r_2^k r_1^{2d}dr_2^2 + \gamma_2^{k,d}r_2^k r_1^{2d}dt_2^2 + \gamma_3^{k,d}r_2^k r_1^{2d}d\varphi_2 dt + \delta^{k,d}r_2^k r_1^{2d}(dr_2dt + dr_2d\varphi_2)\right)$$

Next consider the homogeneous polynomials of degree $2d+1$. Again expressed in terms of u and v. The two dimensional subspace associated with ρ is given by multiples of u^dv^{d+1} and $u^{d+1}v^d$ since these terms have eigenvalues of $e^{-i\theta}$ and $e^{i\theta}$ respectively. This is of course over $\mathbb{C}$ since u and v are complex; but we can reframe this over $\mathbb{R}$. This is because $x_1u^dv^d = x_1(x_1^2+y_1^2)^d$ and $y_1u^dv^d = y_1(x_1^2+y_1^2)^d$ is also a basis. This is due to the formulae: $x_1 = \frac{1}{2i}(u-v)$ and

$y_1 = \frac{1}{2}(u+v)$. Using the metric viewpoint we will call this part of the metric g_ρ.

$$g_\rho = \sum_{d,k}\left(\epsilon^{k,d} r_2^k r_1^{2d}(x_1dx_1dr_2 + y_1dy_1dr_2) + \eta_1^{k,d} r_2^k r_1^{2d}(x_1dx_1dt + y_1dy_1dt) + \eta_2^{k,d} r_2^k r_1^{2d}(x_1dx_1d\varphi_2 + y_1dy_1d\varphi_2)\right)$$

Finally we revisit the homogeneous polynomials of degreed $2d$ but we analyze the terms u^dv^{d+2} and $u^{d+2}v^d$ which correspond to the eigenvalues $e^{-2i\theta}$ and $e^{2i\theta}$ respectively. These form a basis for the ρ^2 vector space. Consider $u^2 = (ix_1 + y_1)^2 = -(x_1^2 - y_1^2) + 2ix_1y_1$ and $v^2 = -(x_1^2 - y_1^2) - 2ix_1y_1$. Thus we have that $u^dv^d(x_1^2 - y_1^2)$ and $u^dv^d(2x_1y_1)$ is also a basis of the ρ^2 vector space. We use the metric viewpoint and name this part of the metric g_{ρ^2}.

$$g_{\rho^2} = \sum_{k,d}\iota^{k,d}\left(r_2^k r_1^{2d}((x_1^2 - y_1^2)dx_1^2 + 4x_1y_1dx_1dy_1 - (x_1^2 - y_1^2)dy_1^2)\right)$$

Thus you can see that the behaviour of $S^m(V)$ is similar to the $m = 0, 1, 2$ cases just with different factors.

Since we used the background Euclidean metric on V we have to implement the consequence of the Gauss Lemma. Let $R = \sqrt{x_1^2 + y_1^2 + r_2^2}$, ω, and φ be spherical coordinates. Then as R approaches 0 the metric on V, $g|_V$ minus the background metric, $g|_{Euc}$, is $O(R^2)$. Note that the constant term of $g|_V$ is $dx_1^2 + dy_1^2 + dr_2^2 = dR^2 + R^2(d\varphi^2 + \sin^2(\varphi))d\theta^2 = g|_{Euc}$. The equation for x_1, y_1 and r_2 in terms of R, φ and ω is the following.

$$x_1 = R\sin(\omega)\cos(\varphi) \qquad y_1 = R\sin(\omega)\sin(\varphi) \qquad r_2 = R\cos(\omega)$$

We now state the 1-forms dx_1, dy_1 and dr_2 in terms of their spherical coordinate counterparts.

$$\begin{aligned}
dx_1 &= \sin(\omega)\cos(\varphi)dR + R(\cos(\omega)\cos(\varphi))d\omega - R(\sin(\omega)\sin(\varphi_1))d\varphi\\
dy_1 &= \sin(\omega)\sin(\varphi)dR + R(\sin(\omega)\cos(\varphi))d\varphi + R(\cos(\omega)\sin(\varphi))d\omega\\
dr_2 &= \cos(\omega)dR - R\sin(\omega)d\omega
\end{aligned}$$

Now we don't have to consider the $m = 2$ terms since they have R^2 as their minimum power of R. We now proceed with the $m = 1$ terms. These are stated below:

$$= \alpha^1 r_2(dx_1^2 + dy_1^2) + \beta^1 r_2 dr_2^2 + \epsilon^1(x_1dx_1dr_2 + y_1dy_1dr_2)$$

Let's calculate the terms with R degree 1 of the above expression and ignore all other terms.

$$= \alpha^1 R\cos(\omega)(\sin^2(\omega)\cos^2(\varphi)dR^2 + \sin^2(\omega)\sin^2(\varphi)dR^2) + \beta^1\cos(\omega)R(\cos^2(\omega)dR^2)\dots$$
$$\dots + \epsilon^1(R\sin(\omega)\cos(\varphi)\sin(\omega)\cos(\varphi)\cos(\omega)dR^2 + R\sin(\omega)\sin(\varphi)\sin(\omega)\sin(\varphi)\cos(\omega)dR^2)$$
$$= \cos(\omega)RdR^2((\alpha^1+\epsilon^1)\sin^2(\omega) + \beta^1\cos^2(\omega))$$

Thus the Gauss Lemma requires that $\alpha^1 + \epsilon^1 = 0$ and $\beta^1 = 0$.

We now begin to state the conclusions. We wish to state that all components are smooth functions of r_1^2 and r_2. We know that $g(\partial_{\varphi_1}, \partial_{\varphi_1}) = r_1^2\sin^2(\varphi_1)g(\partial_{x_1}, \partial_{x_1}) + r_1^2\cos^2(\varphi_1)g(\partial_{y_1}, \partial_{y_1}) - 2r_1^2(\sin(\varphi_1)\cos(\varphi_1)g(\partial_{x_1}, \partial_{y_1})$ The $\mathbb{1}$ terms correspond to functions of r_1^2 and r_2. The ρ terms are absent from the above coefficients of the metric. The ρ^2 terms correspond to a sum. Each term in the summation is a smooth function of r_1^2 and r_2 multiplying $(x_1^2 - y_1^2)(r_1^2\sin^2(\varphi_1) - r_1^2\cos^2(\varphi_1))) + 4x_1y_1(-r_1^2(\sin(\varphi_1)\cos(\varphi_1)) = r_1^4$. Therefore $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ is a smooth function of r_1^2 and r_2. We have that the constant term in $g(\partial_{x_1}, \partial_{x_1})$ and $g(\partial_{y_1}, \partial_{y_1})$ produces r_1^2 as a factor out front in $g(\partial_{\varphi_1}, \partial_{\varphi_1})$. We have that $g(\partial_t, \partial_t)$, $g(\partial_t, \partial_{\varphi_2})$ and $g(\partial_{\varphi_2}, \partial_{\varphi_2})$ are smooth functions of r_1^2 and r_2. We have that $g(\partial_{r_1}, \partial_{r_1}) = \cos^2(\varphi_1)g(\partial_{x_1}, \partial_{x_1}) + \sin^2(\varphi_1)g(\partial_{y_1}, \partial_{y_1}) + 2\sin(\varphi_1)\cos(\varphi_1)g(\partial_{x_1}, \partial_{y_1})$. The $\mathbb{1}$ terms are smooth functions of r_1^2 and r_2. Again the ρ terms are absent. The ρ^2 terms correspond to a sum. Each term in the summation is a smooth function of r_1^2 and r_2 multiplying $(x_1^2 - y_1^2)(\cos^2(\varphi_1) - \sin^2(\varphi_1)) + 4x_1y_1(\sin(\varphi_1)\cos(\varphi_1))) = r_1^2$. Thus $g(\partial_{r_1}, \partial_{r_1})$ is a smooth function of r_1^2 and r_2. Next consider $g(\partial_{r_1}, \partial_{\varphi_1}) = -r_1\cos(\varphi_1)\sin(\varphi_1)g(\partial_{x_1}, \partial_{x_1}) + r_1\cos(\varphi_1)\sin(\varphi_1)g(\partial_{y_1}, \partial_{y_1}) + r_1(\cos^2(\varphi_1) - \sin^2(\varphi_1))g(\partial_{x_1}, \partial_{y_1})$. We have that the $\mathbb{1}$ terms occur identically in $g(\partial_{x_1}, \partial_{x_1})$ and $g(\partial_{y_1}, \partial_{y_1})$ thus they cancel. The ρ terms are absent. The ρ^2 terms correspond to a sum. Each term in the summation is a smooth function of r_1^2 and r_2 multiplying $r_1\cos(\varphi_1)\sin(\varphi_1)(-2(x_1^2 - y_1^2)) + r_1(\cos^2(\varphi_1) - \sin^2(\varphi_1))2x_1y_1 = 0$. Therefore $g(\partial_{r_1}, \partial_{\varphi_1}) = 0$. We have that $g(\partial_{\varphi_1}, \partial_{\varphi_2}) = -r_1\sin(\varphi_1)g(\partial_{x_1}, \partial_{\varphi_2}) + r_1\cos(\varphi_1)g(\partial_{y_1}, \partial_{\varphi_2})$. The only terms present are the ρ terms. We see that we get a cancellation thus $g(\partial_{\varphi_1}, \partial_{\varphi_2}) = 0$. We have by the same reasoning that $g(\partial_{\varphi_1}, \partial_t) = 0$. Assuming the metric on the base is conformal to the flat metric we have that $g(\partial_{r_1}, \partial_{r_2}) = 0$ and $g(\partial_{r_2}, \partial_{r_2}) = g(\partial_{r_1}, \partial_{r_1})$. Also note that $g(\partial_{r_2}, \partial_{\varphi_1}) = 0$. Since we have a section we have that $g(X, \partial_{r_1}) = g(X, \partial_{r_2}) = 0$ for X tangent to the full group G.

Now we convert the metric into r and $\bar{r}$ form. We use the coordinate transformation in (4.0.1). We also have to factor in how this coordinate transformation affects the metric components.

$$
\begin{aligned}
g(\partial_r, \partial_r) &= \left(\left(\frac{\partial r_1}{\partial r} \right)^2 + \left(\frac{\partial r_2}{\partial r} \right)^2 \right) g(\partial_{r_1}, \partial_{r_1}) \\
&= \frac{1}{4} \left(\frac{1}{\sqrt{r^2 + (\bar{r}+a)^2} - \bar{r}} \left(\frac{r}{\sqrt{r^2 + (\bar{r}+a)^2}} \right)^2 + \frac{1}{\sqrt{r^2 + (\bar{r}+a)^2} + \bar{r}} \left(\frac{r}{\sqrt{r^2 + (\bar{r}+a)^2}} \right)^2 \right) g(\partial_{r_1}, \partial_{r_1}) \\
&= \frac{r^2}{4(r^2 + (\bar{r}+a)^2)} \frac{2\sqrt{r^2 + (\bar{r}+a)^2}}{r^2} g(\partial_{r_1}, \partial_{r_1}) = \frac{1}{2\sqrt{r^2 + (\bar{r}+a)^2}} g(\partial_{r_1}, \partial_{r_1})
\end{aligned}
$$

Thus in a neighbourhood around the point $\pi(q_a)$ we have that the metric components are smooth functions of r^2 and $\bar{r}$. This is because in the expression for r_1, $\bar{r}+a$ is positive thus the expression in the square root is non zero. For the $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ term we have a factor of $\frac{r^2}{2a}$ out front.

$$
r_1^2 = \sqrt{r^2 + (\bar{r}+a)^2} - (\bar{r}+a) = \frac{r^2}{\sqrt{r^2 + (\bar{r}+a)^2} + \bar{r} + a} \sim \frac{r^2}{2a}
$$

4.2 Corner Point

We now consider the action on the slice at the corner q_c. We will call the tangent space of S_c to be W. There are two rotations at play on S_c. This is because the isotropy group is $S_1^1 \times S_2^1 \times \{1\}$. Take an arbitrary element of S_c, (x_1, y_1, x_2, y_2). The slice representation is described below.

$$
\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & 0 & \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}
$$

We take a basis $\{\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2}\}$ of W and a basis of the orthogonal counterpart to be $W^\perp = \{\partial_t\}$. We have used the Euclidean metric as the background metric when we made this orthogonal decomposition. Let E be an open ball inside W centered on the origin. We consider smooth G-invariant metrics on the tube $G \times_{G_{q_c}} E$. These again correspond to metrics on an open set in M due to the tube theorem. There is a correspondence between the G invariant metrics on the tube and G_{q_c} equivariant maps F where $F : E \to S^2(W \oplus W^\perp)$. The maps which use m^{th} order polynomials with values in $S^2(W \oplus W^\perp)$ are given by $\mathrm{Hom}_H(S^m W, S^2(W \oplus W^\perp))$. We can understand the $S^2(W \oplus W^\perp)$ as corresponding to components of a symmetric metric.

To proceed with the smoothness conditions we compute $S^2(W \oplus W^{\perp})$.

$$\begin{aligned} S^2((\rho_1 \oplus \rho_2) \oplus \mathbb{1}) &= S^2(\rho_1 \oplus \rho_2) \oplus ((\rho_1 \oplus \rho_2) \otimes \mathbb{1}) \oplus \mathbb{1} \\ &= ((\rho_1^2 \oplus \mathbb{1}) \oplus (\rho_2^2 \oplus \mathbb{1}) \oplus (\rho_1 \otimes \rho_2)) \oplus ((\rho_1 \oplus \rho_2) \otimes \mathbb{1}) \oplus \mathbb{1} \end{aligned}$$

Note that $\rho_1 \otimes \rho_2$ is not an irreducible representation of $S_1^1 \times S_2^1 \times \{1\}$, but ρ_1^2 is an irreducible representation of $S_1^1 \times S_2^1 \times \{1\}$. To see this we let u_1 and v_1 be eigenvectors of the S_1^1 part with eigenvalues $e^{i\theta_1}$ and $e^{-i\theta_1}$ respectively. Let u_2 and v_2 be eigenvectors of the S_2^1 part with eigenvalues $e^{i\theta_2}$ and $e^{-i\theta_2}$ respectively. We see that $\rho_1 \otimes \rho_2$ splits into 2 irreducible representations.

$$\begin{aligned} \rho_1 \otimes \rho_2(u_1u_2) &= e^{i(\theta_1+\theta_2)}u_1u_2 \qquad & \rho_1 \otimes \rho_2(v_1v_2) &= e^{-i(\theta_1+\theta_2)}v_1v_2 \\ \rho_1 \otimes \rho_2(u_1v_2) &= e^{i(\theta_1-\theta_2)}u_1v_2 \qquad & \rho_1 \otimes \rho_2(u_2v_1) &= e^{-i(\theta_1-\theta_2)}u_2v_1 \end{aligned}$$

We write $\rho_1 \otimes \rho_2 = \rho_{1+2} \oplus \rho_{1-2}$ to denote this decomposition into irreducible representations. We can express the domain of ρ_{1+2} as span$(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$. Similarly we can express the domain of ρ_{1-2} as span$(x_1x_2 + y_1y_2, x_1y_2 - y_1x_2)$. However ρ_1^2 is irreducible.

$$\begin{aligned} \rho_1^2(u_1u_2) &= e^{2i\theta_1}u_1u_2 \qquad & \rho_1^2(v_1v_2) &= e^{-2i\theta_1}v_1v_2 \\ \rho_1^2(u_1v_2) &= e^{2i\theta_1}u_1v_2 \qquad & \rho_1^2(u_2v_1) &= e^{-2i\theta_1}u_2v_1 \end{aligned}$$

We now consider the constant polynomials $S^0(W)$ which is isomorphic to $\mathbb{R}$. As in the previous section we can normalize the W part of the metric to be $(dx_1^2 + dy_1^2) + (dx_2^2 + dy_2^2)$.
Moving on to $m = 1$ we have that $S^1(W) = \rho_1 \oplus \rho_2$. We have two homomorphisms given below:

$$\begin{aligned} \rho_1 &\to \rho_1, \qquad & \xi_1^1((\partial_{x_1})^* dx_1dt + (\partial_{y_1})^* dy_1dt) \\ \rho_2 &\to \rho_2, \qquad & \xi_2^1((\partial_{x_2})^* dx_2dt + (\partial_{y_2})^* dy_2dt) \end{aligned}$$

It is easy to see these homomorphisms are G_{q_c} equivariant since this case is analogous to what was shown in the $m = 1$ case for q_a.

We now consider the $m = 2$ case. We have that $S^2(W) = ((\rho_1^2 \oplus \mathbb{1}) \oplus (\rho_2^2 \oplus \mathbb{1}) \oplus$

$(\rho_1 \otimes \rho_2)) \oplus ((\rho_1 \oplus \rho_2) \otimes \mathbb{1})$. The homomorphisms are given below.

$$
\begin{aligned}
\mathbb{1} \to \mathbb{1}, &\quad \sigma_1^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*)dt^2 \\
\mathbb{1} \to \mathbb{1}, &\quad \sigma_2^2((\partial_{x_2})^* \otimes (\partial_{x_2})^* + (\partial_{y_2})^* \otimes (\partial_{y_2})^*)dt^2 \\
\mathbb{1} \to \mathbb{1}, &\quad \zeta_{11}^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*)(dx_1^2 + dy_1^2) \\
\mathbb{1} \to \mathbb{1}, &\quad \zeta_{12}^2((\partial_{x_1})^* \otimes (\partial_{x_1})^* + (\partial_{y_1})^* \otimes (\partial_{y_1})^*)(dx_2^2 + dy_2^2) \\
\mathbb{1} \to \mathbb{1}, &\quad \zeta_{21}^2((\partial_{x_2})^* \otimes (\partial_{x_2})^* + (\partial_{y_2})^* \otimes (\partial_{y_2})^*)(dx_1^2 + dy_1^2) \\
\mathbb{1} \to \mathbb{1}, &\quad \zeta_{22}^2((\partial_{x_2})^* \otimes (\partial_{x_2})^* + (\partial_{y_2})^* \otimes (\partial_{y_2})^*)(dx_2^2 + dy_2^2) \\
\rho_1^2 \to \rho_1^2, &\quad \chi_1^2(((\partial_{x_1})^* \otimes (\partial_{x_1})^* - (\partial_{y_1})^* \otimes (\partial_{y_1})^*)(dx_1^2 - dy_1^2) + 4(\partial_{x_1})^* \otimes (\partial_{y_1})^* dx_1 dy_1) \\
\rho_2^2 \to \rho_2^2, &\quad \chi_2^2(((\partial_{x_2})^* \otimes (\partial_{x_2})^* - (\partial_{y_2})^* \otimes (\partial_{y_2})^*)(dx_2^2 - dy_2^2) + 4\partial_{x_2})^* \otimes (\partial_{y_2})^* dx_2 dy_2) \\
\rho_{1+2} \to \rho_{1+2}, &\quad \upsilon_1^2\Big(((\partial_{x_1})^* \otimes (\partial_{x_2})^* - (\partial_{y_1})^* \otimes (\partial_{y_2})^*)(dx_1 dx_2 - dy_1 dy_2) \ldots \\
&\ldots + ((\partial_{x_1})^* \otimes (\partial_{y_2})^* + (\partial_{y_1})^* \otimes (\partial_{x_2})^*)(dx_1 dy_2 + dy_1 dx_2)\Big) \\
\rho_{1-2} \to \rho_{1-2}, &\quad \upsilon_2^2\Big(((\partial_{x_1})^* \otimes (\partial_{x_2})^* + (\partial_{y_1})^* \otimes (\partial_{y_2})^*)(dx_1 dx_2 + dy_1 dy_2) \ldots \\
&\ldots + ((\partial_{x_1})^* \otimes (\partial_{y_2})^* - (\partial_{y_1})^* \otimes (\partial_{x_2})^*)(dx_1 dy_2 - dy_1 dx_2)\Big)
\end{aligned}
$$

The part of the metric with order 2 homogeneous polynomials is called g^2. We state it below.

$$
\begin{aligned}
g^2 = &\sigma_1^2(x_1^2 + y_1^2)dt^2 + \sigma_2^2(x_2^2 + y_2^2)dt^2 + \zeta_{11}^2(x_1^2 + y_1^2)(dx_1^2 + dy_1^2) + \zeta_{12}^2(x_1^2 + y_1^2)(dx_2^2 + dy_2^2) \ldots \\
&\ldots + \zeta_{21}^2(x_2^2 + y_2^2)(dx_1^2 + dy_1^2) + \zeta_{22}^2(x_2^2 + y_2^2)(dx_2^2 + dy_2^2) + \chi_1^2((x_1^2 - y_1^2)(dx_1^2 - dy_1^2) + 4x_1 y_1 dx_1 dy_1) \ldots \\
&\ldots + \chi_2^2((x_2^2 - y_2^2)(dx_2^2 - dy_2^2) + 4x_2 y_2 dx_2 dy_2) \ldots \\
&\ldots + \upsilon_1^2((x_1 x_2 - y_1 y_2)(dx_1 dx_2 - dy_1 dy_2) + (x_1 y_2 + y_1 x_2)(dx_1 dy_2 + dy_1 dx_2)) \ldots \\
&\ldots + \upsilon_2^2((x_1 x_2 + y_1 y_2)(dx_1 dx_2 + dy_1 dy_2) + (x_1 y_2 - y_1 x_2)(dx_1 dy_2 - dy_1 dx_2))
\end{aligned}
$$

The G_{qc} equivariance for the χ_1^2 and χ_2^2 terms is analogous to what was shown in the $m = 2$ case for q_a. The only real different case is the υ_1^2 and υ_2^2 terms. Let $f_{\upsilon_1^2}$ be the homomorphism corresponding to υ_1^2. We let σ map m in M to its image under the group action, i.e $(e^{\theta_1}, e^{i\theta_2})m$. Before computing we can simplify the calculation by factorizing.

$$
\begin{aligned}
f_{\upsilon_1^2}(a_1\partial_{x_1} + b_1\partial_{y_1} + a_2\partial_{x_2} + b_2\partial_{y_2}) &= (a_1 a_2 - b_1 b_2)(dx_1 dx_2 - dy_1 dy_2) + (a_1 b_2 + a_2 b_1)(dx_1 dy_2 + dy_1 dx_2) \\
&= (a_1 dx_1 + b_1 dy_1)(a_2 dx_2 + b_2 dy_2) - (b_1 dx_1 - a_1 dy_1)(b_2 dx_2 - a_2 dy_2)
\end{aligned}
$$

We really only need to check one of the factors in the second term.

$$\begin{aligned} &= (\sin(\theta_1)a_1 + \cos(\theta_1)b_1)dx_1 - (\cos(\theta_1)a_1 - \sin(\theta_1)b_1)dy_1 \\ &= a_1(\sin(\theta_1)dx_1 - \cos(\theta_1)dy_1) + b_1(\cos(\theta_1)dx_1 + \sin(\theta_1)dy_1) \\ &= -a_1\sigma^* dy_1 + b_1\sigma^* dx_1 \end{aligned}$$

The same G_{q_c} equivariance above holds for second factor of the second term.

Next we consider $S^m(W)$ for general m. We have that $S^m(W) = (S^m(\rho_1) \otimes \mathbb{1}) \oplus (S^{m-1}(\rho_1) \otimes S^1(\rho_2)) \oplus \ldots \oplus ((S^{m-1}(\rho_2) \otimes S^1(\rho_1)) \oplus ((S^m(\rho_1) \otimes \mathbb{1}))$. Consider $S^k(\rho_1) \otimes S^{m-k}(\rho_2)$. We have 4 cases.

Case I, if k is even and $m-k$ is also even. With Schur's Lemma in mind, ignoring terms which are more than double rotations we have the following. $(\rho_1^2 \oplus \mathbb{1}) \otimes (\rho_2^2 \oplus \mathbb{1})$. The only resulting representations that survive Schur's Lemma are $\mathbb{1} \otimes \mathbb{1}$, $\rho_1^2 \otimes \mathbb{1}$ and $\rho_2^2 \otimes \mathbb{1}$. For $\mathbb{1} \otimes \mathbb{1}$ the corresponding polynomial is $(x_1^2 + y_1^2)^{\frac{k}{2}}(x_2^2 + y_2^2)^{\frac{m-k}{2}}$. For the corresponding part of the metric we obtain a multiple of $(x_1^2 + y_1^2)^{\frac{k}{2}}(x_2^2 + y_2^2)^{\frac{m-k}{2}}\omega$. Where ω is either dt^2, $dx_1^2 + dy_1^2$ or $dx_2^2 + dy_2^2$. For $\rho_1^2 \otimes \mathbb{1}$ our polynomials are $(x_2^2 + y_2^2)^{\frac{m-k}{2}}((x_1^2 - y_1^2))(x_1^2 + y_1^2)^{\frac{k-2}{2}}$ and $(x_2^2 + y_2^2)^{\frac{m-k}{2}}2(x_1y_1)(x_1^2 + y_1^2)^{\frac{k-2}{2}}$. The corresponding part of the metric is given by a multiple of $(x_2^2 + y_2^2)^{\frac{m-k}{2}}((x_1^2 - y_1^2)(dx_1^2 - dy_1^2) + 4x_1y_1dx_1dy_1)(x_1^2 + y_1^2)^{\frac{k-2}{2}}$. For $\rho_2^2 \otimes \mathbb{1}$, our polynomials are given by $(x_1^2 + y_1^2)^{\frac{k}{2}}(x_2^2 - y_2^2)(x_2^2 + y_2^2)^{\frac{m-k-2}{2}}$ and $2(x_2y_2)(x_1^2 + y_1^2)^{\frac{k}{2}}(x_2^2 + y_2^2)^{\frac{m-k-2}{2}}$. Thus our corresponding part of the metric is a multiple of $(x_1^2 + y_1^2)^{\frac{k}{2}}((x_2^2 - y_2^2)(dx_2^2 - dy_2^2) + 4x_2y_2dx_2dy_2)(x_2^2 + y_2^2)^{\frac{m-k-2}{2}}$

For Case II, if k is odd and $m-k$ is even. Then we have $\rho_1 \otimes (\rho_2^2 \oplus \mathbb{1})$. The only representation that survives is $\rho_1 \otimes \mathbb{1}$. Our polynomials are $(x_1^2 + y_1^2)^{\frac{k-1}{2}}x_1(x_2^2 + y_2^2)^{\frac{m-k}{2}}$ $+(x_1^2 + y_1^2)^{\frac{k-1}{2}}y_1(x_2^2 + y_2^2)^{\frac{m-k}{2}}$. Thus the corresponding part of the metric is a multiple of : $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_2^2 + y_2^2)^{\frac{m-k}{2}}(x_1dx_1dt + y_1dy_1dt)$.

Case III, if k is even and $m-k$ is odd. Then we have $(\rho_1^2 \oplus \mathbb{1}) \otimes (\rho_2)$. The only representation that survives is $\rho_2 \otimes \mathbb{1}$. Our polynomials are $(x_1^2 + y_1)^{\frac{k}{2}}(x_2^2 + y^2)^{\frac{m-k-1}{2}}x_2$ and $(x_1^2 + y_1)^{\frac{k}{2}}(x_2^2 + y^2)^{\frac{m-k-1}{2}}y_2$. Thus our corresponding part of the metric is a multiple of: $(x_1^2 + y_1)^{\frac{k}{2}}(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}(x_2dx_2dt + y_2dy_2dt)$.

Case IV, k is odd and $m-k$ is odd. Then the only representation that survives is $\rho_1 \otimes \rho_2 = \rho_{1+2} \oplus \rho_{1-2}$. Our polynomials for ρ_{1+2} are $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_1x_2 - y_1y_2)(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}$ and $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_1y_2 + x_2y_1)(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}$. Thus our part of the metric is a multiple of $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}((x_1x_2 - y_1y_2)(dx_1dx_2 - dy_1dy_2) + (x_1y_2 + x_2y_1)(dx_1dy_2 + dy_1dx_2)$. Our polynomials for ρ_{1-2} are $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_1x_2 + y_1y_2)(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}$ and $(x_1^2 + y_1^2)^{\frac{k-1}{2}}(x_1y_2 - x_2y_1)(x_2^2 + y_2^2)^{\frac{m-k-1}{2}}$. Thus our part of the metric is a multiple of

$(x_1^2+y_1^2)^{\frac{k-1}{2}}(x_2^2+y_2^2)^{\frac{m-k-1}{2}}((x_1x_2+y_1y_2)(dx_1dx_2+dy_1dy_2)+(x_1y_2-x_2y_1)(dx_1dy_2-dy_1dx_2).$

We now calculate the essential part of the ρ_{1+2} term of the metric using r_i and φ_i (i=1,2).

$$\begin{aligned}
&=(x_1x_2-y_1y_2)(dx_1dx_2-dy_1dy_2)+(x_1y_2+x_2y_1)(dx_1dy_2+dy_1dx_2)\\
&=(x_1dx_1+y_1dy_1)(x_2dx_2+y_2dy_2)-(y_1dx_1-x_1dy_1)(y_2dx_2-x_2dy_2)\\
&=r_1r_2dr_1dr_2-r_1^2r_2^2d\varphi_1d\varphi_2
\end{aligned}$$

We now perform a similar calculation for the essential part of the ρ_{1-2} term of the metric using r_i and φ_i (i=1,2).

$$\begin{aligned}
&=(x_1x_2+y_1y_2)(dx_1dx_2+dy_1dy_2)+(x_1y_2-x_2y_1)(dx_1dy_2-dy_1dx_2)\\
&=(x_1dx_1+y_1dy_1)(x_2dx_2+y_2dy_2)+(y_1dx_1-x_1dy_1)(y_2dx_2-x_2dy_2)\\
&=r_1r_2dr_1dr_2+r_1^2r_2^2d\varphi_1d\varphi_2
\end{aligned}$$

We now implement the consequences of the Gauss Lemma. Here the Euclidean metric is $g|_{Euc}=dx_1^2+dx_2^2+dy_1^2+dy_2^2$. We have that $R'=\sqrt{r_1^2+r_2^2}$. We have that our metric on the slice, $g|_W$ differs from $g|_{Euc}$ by $O((R')^2)$. We have the following coordinate transformation:

$$\begin{aligned}
x_1&=R'\cos(\omega)\cos(\varphi_1) & y_1&=R'\cos(\omega)\sin(\varphi_1)\\
x_2&=R'\sin(\omega)\cos(\varphi_2) & y_2&=R'\sin(\omega)\sin(\varphi_2)
\end{aligned}$$

We are in the clear since the lowest non constant terms are the quadratic terms made up of x_i^2 and y_i^2. We have that $x_i^2\propto(R')^2$ and $y_i^2\propto(R')^2$. The lowest power of R' in the dx_i and dy_i terms is a constant. Multiplying them together we get a lowest power of $(R')^2$. Therefore the Gauss Lemma is satisfied.

Now we collect together all this information into a conclusion. We have that $g(\partial_t,\partial_t)$ is a smooth function of r_1^2 and r_2^2. Since we are assuming the metric on the base is conformal to the flat metric we have that $g(\partial_{r_1},\partial_{r_1})=g(\partial_{r_2},\partial_{r_2})$ and $g(\partial_{r_1},\partial_{r_2})=0$. We have that $g(\partial_{r_1},\partial_{r_1})$ is a smooth function of r_1^2 and r_2^2. We have that $g(\partial_{\varphi_1},\partial_{\varphi_1})$ is a smooth function of r_1^2 and r_2^2 with $\partial_{r_1r_1}\Big(g(\partial_{\varphi_1},\partial_{\varphi_1})\Big)\Big|_{(0,0)}=2$ and all lower order derivatives vanish. We have that $g(\partial_{\varphi_2},\partial_{\varphi_2})$ is a smooth function of r_1^2 and r_2^2 with $\partial_{r_2r_2}\Big(g(\partial_{\varphi_1},\partial_{\varphi_1})\Big)\Big|_{(0,0)}=2$ and all lower order derivatives vanish. Interestingly we see a non zero $g(\partial_{\varphi_1},\partial_{\varphi_2})$ term appearing which is also a smooth function of r_1^2 and r_2^2. Furthermore it satisfies $\partial_{r_2r_2}\Big(g(\partial_{\varphi_1},\partial_{\varphi_2})\Big)\Big|_{(0,0)}=\partial_{r_1r_1}\Big(g(\partial_{\varphi_1},\partial_{\varphi_2})\Big)\Big|_{(0,0)}=2$ where all lower order derivatives vanish. We have that $g(\partial_t, \partial_{\varphi_1}) = 0$ and $g(\partial_t, \partial_{\varphi_2}) = 0$ due to cancellations in the ρ_1 terms and the ρ_2 terms respectively. Again since we have a section, we have that $g(\partial_{r_1}, X) = g(\partial_{r_2}, X) = 0$ when X is tangent to the full group G.

This is well good but we must convert the metric to r and $\bar{r}$. If we examine the $g(\partial_{\varphi_1}, \partial_{\varphi_1})$ term we have that when $\bar{r} = 0$ and $r \to 0$ it behaves like r. This is because $r_1^2(r, 0) = \sqrt{r^2 + 0^2} - 0 = r$. Similarly we have that $g(\partial_{\varphi_2}, \partial_{\varphi_2})$ term behaves like r when $\bar{r} = 0$ and $r \to 0$. This is because $r_2^2(r, 0) = \sqrt{r^2 + 0^2} + 0 = r$. We also have that $r^2 = r_1^2 r_2^2$ implies that $g(\partial_{\varphi_1}, \partial_{\varphi_2})$ behaves like r^2. We can use the smoothness conditions at points in the interior of the axis rod to rule out non-zero $g(\partial_{\varphi_1}, \partial_{\varphi_2})$ everywhere near the $\bar{r}$ axis except possibly for a line segment extending orthogonally from the corner point. In the next subsection we discuss real analyticity under a certain ellipticity assumption which would rule this out. However this may not matter so much since the examples we are studying satisfy $g(\partial_{\varphi_1}, \partial_{\varphi_2}) = 0$. What about $g(\partial_r, \partial_r)$?

$$g(\partial_r, \partial_r) = \frac{1}{\sqrt{r^2 + \bar{r}^2}} g(\partial_{r_1}, \partial_{r_1})$$

Therefore we have the somewhat paradoxical result that $g(\partial_{r_1}, \partial_{r_1})$ behaves like r^{-1} when $\bar{r} = 0$ and $r \to 0$. However this is necessary for smoothness on the level of the manifold itself although the metric is not smooth everywhere in the orbit space. Actually as you can see this is only really a coordinate phenomenon. We will see that the metric in example 2 of the 5D static paper shows this behaviour. Note that when $\bar{r} \neq 0$ we have that the metric is a smooth function of r^2 and $\bar{r}$.

4.3 Consequences of the Smoothness Condition for the Twist Potentials

Here we break from the chapter's convention and use z over $\bar{r}$. We will now prove that under a certain ellipticity assumption, the case where the z-axis consists of $(1, 0)$ and $(0, 1)$ rods yields constant twist potentials and furthermore the metric reduces to the ansatz in Khuri et al.'s paper[18, p. 6] which implies that the metric on the fibre is diagonal. We have that K and L from chapter 2 satisfy the equations:

$$0 = K_r + L_z + \frac{K}{r} \qquad 0 = K_z - L_r - [K, L]$$

By [8, p. 505] we have that this system of PDEs is elliptic. It is worth reading the examples they work out. Our characteristic determinant is:

$$\begin{vmatrix} \xi\partial_{K_r}(K_r + L_z + \frac{K}{r}) & \eta\partial_{L_z}((K_r + L_z + \frac{K}{r}) \\ \eta\partial_{K_z}(K_z - L_r - [K,L]) & \xi\partial_{L_r}(K_z - L_r - [K,L]) \end{vmatrix} = \begin{vmatrix} \xi & \eta \\ \eta & -\xi \end{vmatrix} = -(\eta^2 + \xi^2)$$

Since the determinant is 0 iff $\eta = \xi = 0$ we know it is elliptic. Then, according to [24, p. 198] the system (3.8.2) is real analytic in the interior of the orbit space. That means that $H^{-1}H_r = K$ is real analytic and $H^{-1}H_z = L$ is real analytic. Thus we can start to piece together real analyticity of H. We use (3.7.1)

$$\begin{aligned}
K_{11} &= -f^{-1}f_r + 2r^{-1} - fr^{-2}v_r^T F v \\
K_{1\bullet} &= -fr^{-2}v_r^T F \\
K_{\bullet 1} &= (f^{-1}f_r - 2r^{-1})v + fr^{-2}(vv_r^T F v) + v_r + F^{-1}F_r v \\
K_{\bullet\bullet} &= fr^{-2}vv_r^T F + F^{-1}F_r \\
L_{11} &= -f^{-1}f_z - fr^{-2}v_z^T F v \\
L_{1\bullet} &= -fr^{-2}v_z^T F \\
L_{\bullet 1} &= (f^{-1}f_z)v + fr^{-2}(vv_z^T v) + v_r + F^{-1}F_z v \\
L_{\bullet\bullet} &= fr^{-2}vv_z^T F + F^{-1}F_z
\end{aligned}$$

We can use these to relations to fit v into a system of two PDEs which are real analytic in their arguments.

$$\begin{aligned}
K_{\bullet 1} - K_{\bullet\bullet}v &= \left(\frac{f_r}{f} - \frac{2}{r}\right)v + v_r \\
K_{11} - K_{1\bullet}v &= -\left(\frac{f_r}{f} - \frac{2}{r}\right) \\
K_{\bullet 1} - K_{\bullet\bullet}v &= -(K_{11} - K_{1\bullet}v)v + v_r
\end{aligned}$$

We now break these down into their explicit form in terms of v^1 and v^2.

$$\begin{aligned}
K_{21} - K_{22}v^1 - K_{23}v^2 &= -(K_{11} - K_{12}v^1 - K_{13}v^2)v^1 + v_r^1 \\
0 &= K_{23}v^2 - K_{21} + (K_{22} - K_{11})v^1 + K_{12}(v^1)^2 + K_{13}v^2v^1 + v_r^1 \\
&= \phi_1 \\
K_{31} - K_{32}v^1 - K_{33}v^2 &= -(K_{11} - K_{12}v^1 - K_{13}v^2)v^2 + v_r^2 \\
0 &= -K_{31} + K_{32}v^1 + (K_{33} - K_{11})v^2 + K_{13}(v^2)^2 + K_{12}v^2v^1 + v_r^2 \\
&= \phi_2
\end{aligned}$$

We now compute the determinant of the system.

$$\begin{vmatrix} \xi(\phi_1)_{v_r^1} & (\phi_1)_{v^2} \\ (\phi_2)_{v^1} & \xi(\phi_2)_{v_r^2} \end{vmatrix} = \begin{vmatrix} \xi & K_{23}+K_{13}v^1 \\ K_{32}+K_{12}v^2 & \xi \end{vmatrix} = \xi^2 - (K_{23}+K_{13}v^1)(K_{32}+K_{12}v^2)$$

Clearly for this system to be elliptic by [8, p. 505] we need to assume that $(K_{23}+K_{13}v^1)(K_{32}+K_{12}v^2) \leq 0$. We can widdle down this expression by using (3.7.1).

$$\begin{aligned} 0 &\geq \frac{1}{f}(f_{22}(f_{12})_r - f_{12}(f_{22})_r)\frac{1}{f}(f_{11}(f_{12})_r - f_{12}(f_{11})_r) \\ 0 &\geq \frac{f_{22}^2 f_{11}^2}{f^2}\left(\frac{f_{12}}{f_{22}}\right)_r \left(\frac{f_{12}}{f_{11}}\right)_r \\ 0 &\geq \left(\frac{f_{12}}{f_{22}}\right)_r \left(\frac{f_{12}}{f_{11}}\right)_r \end{aligned}$$

The last step follows since F is positive definite in the interior. Working under this assumption, by [24, p. 198] we know that v^1 and v^2 must be real analytic in the interior of the half plane. By the equation for $K_{1\bullet}$ we have that $-r^{-2}fv_r^TF$ is real analytic. Then by the equation for $K_{\bullet\bullet}$ we know that $F^{-1}F_r$ must be real analytic. We also know by the equations for $L_{1\bullet}$ and $L_{\bullet\bullet}$ that $F^{-1}F_Z$ must be real analytic. So we have that $\frac{f_r}{f} = tr(F^{-1}F_r)$ is real analytic and $\frac{f_z}{f} = tr(F^{-1}F_z)$ is also real analytic. Therefore $\log(f)$ is real analytic and thus so is f. Now let's prove that F is real analytic. Let $f_{12} = uf_{22}$. We can work out f_{11} in terms of f_{22}, u and f.

$$\begin{aligned} f &= f_{11}f_{22} - f_{12}^2 \\ f_{11} &= \frac{f}{f_{22}} + \frac{(uf_{22})^2}{f_{22}} = \frac{f}{f_{22}} + u^2 f_{22} \end{aligned}$$

Note that f_{22} is never 0 in the interior since F is positive definite in the interior. We can now calculate F_r and F^{-1}.

$$\begin{aligned} F^{-1} &= \frac{1}{f}\begin{pmatrix} f_{22} & -uf_{22} \\ -uf_{22} & \frac{f}{f_{22}} + u^2 f_{22} \end{pmatrix} \\ F_r &= \begin{pmatrix} \frac{f_r}{f_{22}} - \frac{(f_{22})_r f}{f_{22}^2} + 2u_r u f_{22} + u^2 (f_{22})_r & u_r f_{22} + (f_{22})_r u \\ u_r f_{22} + (f_{22})_r u & (f_{22})_r \end{pmatrix} \end{aligned}$$

We now calculate $F^{-1}F_r$.

$$\begin{aligned} (F^{-1}F_r)_{11} &= \frac{1}{f}\left(f_r - \frac{f(f_{22})_r}{f_{22}} + 2u_r u f_{22}^2 + u^2 (f_{22})_r f_{22} - uu_r f_{22}^2 - f_{22}(f_{22})_r u^2\right) \\ &= \frac{f_r}{f} - \frac{(f_{22})_r}{f_{22}} + u_r u \frac{f_{22}^2}{f} \end{aligned}$$

$$(F^{-1}F_r)_{12} = \frac{1}{f}\Big(f_{22}^2 u_r + f_{22}(f_{22})_r u - u f_{22}(f_{22})_r\Big) = \frac{f_{22}^2 u_r}{f}$$

$$\begin{aligned}(F^{-1}F_r)_{21} &= \frac{1}{f}\Big(-u f_r + \frac{u f (f_{22})_r}{f_{22}} - 2u_r u^2 f_{22}^2 - u^3 (f_{22})_r f_{22} + f u_r + \frac{f(f_{22})_r u}{f_{22}} + u^3 (f_{22})_r f_{22} + u^2 u_r f_{22}\Big)\\ &= -u\frac{f_r}{f} + \frac{u(f_{22})_r}{f_{22}} + \frac{u(f_{22})_r}{f_{22}} - \frac{u_r u^2 f_{22}^2}{f} + u_r\end{aligned}$$

$$\begin{aligned}(F^{-1}F_r)_{22} &= \frac{1}{f}\Big(-u u_r f_{22}^2 - u^2 f_{22}(f_{22})_r + \frac{f(f_{22})_r}{f_{22}} + u^2 f_{22}(f_{22})_r\Big)\\ &= -\frac{u u_r f_{22}^2}{f} + \frac{(f_{22})_r}{f_{22}}\end{aligned}$$

We now calculate $F^{-1}F_z$.

$$F_z = \begin{pmatrix} \frac{f_z}{f_{22}} - \frac{(f_{22})_z f}{f_{22}^2} + 2u_z u f_{22} + u^2 (f_{22})_z & u_z f_{22} + (f_{22})_z u \\ u_z f_{22} + (f_{22})_z u & (f_{22})_z \end{pmatrix}$$

$$\begin{aligned}(F^{-1}F_z)_{11} &= \frac{1}{f}\Big(f_z - \frac{f(f_{22})_z}{f_{22}} + 2u_z u f_{22}^2 + u^2 (f_{22})_z f_{22} - u u_z f_{22}^2 - f_{22}(f_{22})_z u^2\Big)\\ &= \frac{f_z}{f} - \frac{(f_{22})_z}{f_{22}} + u_z u \frac{f_{22}^2}{f}\end{aligned}$$

$$(F^{-1}F_z)_{12} = \frac{1}{f}\Big(f_{22}^2 u_z + f_{22}(f_{22})_z u - u f_{22}(f_{22})_z\Big) = \frac{f_{22}^2 u_z}{f}$$

$$\begin{aligned}(F^{-1}F_z)_{21} &= \frac{1}{f}\Big(-u f_r + \frac{u f (f_{22})_z}{f_{22}} - 2u_z u^2 f_{22}^2 - u^3 (f_{22})_z f_{22} + f u_z + \frac{f(f_{22})_z u}{f_{22}} + u^3 (f_{22})_z f_{22} + u^2 u_z f_{22}\Big)\\ &= -u\frac{f_z}{f} + \frac{u(f_{22})_z}{f_{22}} + \frac{u(f_{22})_z}{f_{22}} - \frac{u_z u^2 f_{22}^2}{f} + u_z\end{aligned}$$

$$\begin{aligned}(F^{-1}F_z)_{22} &= \frac{1}{f}\Big(-u u_z f_{22}^2 - u^2 f_{22}(f_{22})_z + \frac{f(f_{22})_z}{f_{22}} + u^2 f_{22}(f_{22})_z\Big)\\ &= -\frac{u u_z f_{22}^2}{f} + \frac{(f_{22})_z}{f_{22}}\end{aligned}$$

As with v we can derive a PDE in terms of u with real analytic arguments.

$$(F^{-1}F_r)_{21} - u(F^{-1}F_r)_{22} = -u(F^{-1}F_r)_{11} - u(F^{-1}F_r)_{12}) + u_r$$

This PDE is trivially elliptic. Which means u is real analytic in the interior. This implies $\frac{(f_{22})_r}{f_{22}}$ and $\frac{(f_{22})_z}{f_{22}}$ are real analytic by looking at $(F^{-1}F_r)_{22}$ and $(F^{-1}F_z)_{22}$. This in turn implies f_{22} is real analytic in the interior. Since $f_{12} = u f_{22}$,

f_{12} must be real analytic in the interior. And, since $f_{11} = \frac{f}{f_{22}} + u^2 f_{22}$, f_{11} is real analytic in the interior. This implies g_f, the metric on the fibre, is real analytic in the interior.

Now enter the smoothness conditions. We have that f_{12} is 0 in an open neighbourhood of the point on the axis rod. Since f_{12} is real analytic in the interior it must be 0 everywhere. We also have that $g_{12} = 0$ and $g_{13} = 0$ in an open neighbourhood of the corner point. Since $g_{12} = f_{11}v^1 + f_{12}v^2$ and $g_{13} = f_{12}v^1 + f_{22}v^2$ we have that since F is positive definite in the interior that v^1 and v^2 must be 0 in the open neighbourhood. By the relation between the v^i and the twist potentials derived in chapter 2 we have that the twist potentials must be constant in the open neighbourhood. By the real analyticity of the twist potentials they must be constant in the interior. We thus force the metric to be diagonal everywhere in the interior. In addition α is real analytic without the elliptical assumption. This is because of (3.1.3).

Chapter 5

Various Forms of the Schwarschild Solution

5.1 Derivation of the Schwarzschild Solution

Finding and interpreting solutions to the Einstein equations is of fundamental importance. Solutions to the Einstein equations in a vacuum, in 4-D spacetime, can be found by imposing that the metric be Ricci flat. The Schwarzschild solution is an exact Ricci-flat solution which will be used later on in this chapter to construct periodic exact solutions. In this section we show in detail how the Schwarzschild solution is constructed, omitting the consequences of it being Ricci flat from the calculations. In order to construct the Schwarzschild solution we will need to understand what stationary, static and spherically symmetric manifolds are. The Schwarzschild solution is assumed to be static and spherically symmetric.

Definition 5.2. *[30, p. 119] A stationary manifold is a Lorentz manifold which admits a local 1-parameter group of isometries whose orbits are timelike curves.*

Theorem 5.3. *[30, p. 119] The previous definition of a stationary manifold is equivalent to one which possesses a timelike Killing vector field. We will call this vector field ξ and members of the 1-parameter group will be denoted by φ_t.*

Proof. Suppose our manifold, M, is stationary using the previous definition. Let m be a point in M and we will define the curve γ_m by $\gamma_m(t) = \varphi_t m$. This curve gives the orbit of m under the 1-parameter group. We define the vector field ξ to be one whose flow is φ_t. By our hypothesis, we know that γ_m are timelike curves. By the definition of an integral curve we can conclude that ξ is timelike. To check that ξ is a Killing vector field we will see what happens when we take the Lie derivative of the metric, g, in the direction of ξ. Let X

and Y be two arbitrary vector fields over M. The below expression we use the formula for the Lie derivative of a $(0,2)$ tensor which is stated in the tensor field section.

$$\begin{aligned}(L_\xi g)(X,Y) &= \xi(g(X,Y)) - g(L_\xi X, Y) - g(X, L_\xi Y)\\ g(L_\xi X, Y) &= g\left(-\left(\frac{d}{dt}(\varphi_t)_* X\right)\Big|_{t=0}, Y\right) = -\frac{d}{dt}\left(g((\varphi_t)_* X, Y)\right)\Big|_{t=0}\\ &= -\frac{d}{dt}\left(g(X,(\varphi_{-t})_* Y)\circ\varphi_{-t}\right)\Big|_{t=0}\end{aligned}$$

We now use the fact that we're taking the derivative of a function of the form, $f(t, g(t))$, so we have to use the chain rule accordingly. Let $(x^1, \dots, x^n)$ be coordinates at m.

$$\begin{aligned}-\frac{d}{dt}\left(g(X,(\varphi_{-t})_* Y)\circ\varphi_{-t}\right)\Big|_{t=0}(m) &= -\left(\frac{d}{dt}\left(g(X,(\varphi_{-t})_* Y)\right)\Big|_{t=0}\circ\varphi_0\right)(m)\dots\\ &\dots - \left(\frac{\partial}{\partial x^i}\left(g(X,(\varphi_0)_* Y)\right)\frac{d(x^i\circ\varphi_{-t})}{dt}\Big|_{t=0}\right)(m)\\ &= g\left(X, \left(\frac{d}{dt}(\varphi_t)_* Y\right)\Big|_{t=0}\right)(m) + \xi(g(X,Y))(m)\\ -\frac{d}{dt}\left(g(X,(\varphi_{-t})_* Y)\circ\varphi_{-t}\right)\Big|_{t=0} &= -g\left(X, L_\xi Y\right) + \xi(g(X,Y))\end{aligned}$$

So we see that all terms cancel so $L_\xi g = 0$

Now suppose our manifold, M, has a timelike Killing vector field ξ. Let φ_t be the local flow of ξ. We will show that it is an isometry. let s be an arbitrary real number.

$$\frac{d}{dt}\Big|_{t=s}(\varphi_t)^* g = \frac{d}{dt}\Big|_{t=0}(\varphi_{t+s})^* g = (\varphi_s)^*\left(\frac{d}{dt}\Big|_{t=0}(\varphi_t)^* g\right) = (\varphi_s)^*(L_\xi g) = 0$$

The last line follows since ξ is a Killing vector. So the derivative is 0 which implies that $\varphi_t^* g = \varphi_0^* g = g$. □

5.3.1 Consequences of the Static Condition

Definition 5.4. *[30, p. 119] A static manifold is one which is a stationary and that contains a spacelike hypersurface, Σ which is orthogonal to the orbits of the isometries.*

We now set M to be a 4D spacetime for our Schwarzschild solution. We wish to know useful expressions of the metric tensor when M is static. To be begin we'll take arbitrary coordinates (x^1, x^2, x^3) for Σ. If we assume that none of the orbits of the isometries terminate on Σ (or equivalently that ξ doesn't vanish on Σ) then, we can find a neighbourhood around Σ in which each point

p each lie on one and only one orbit of the isometries. We assign to each p the coordinates on Σ of the point in which the orbit emerges from and the parameter t which defines the location of p on the orbit. We now remark that the image of Σ under φ_t, which we will call Σ_t, is orthogonal to ξ. If we take an arbitrary vector field, X_{Σ_t} on Σ_t then we get the following:

$$g(X_{\Sigma_t}, \xi) = g((\varphi_{-t})_* X_{\Sigma_t}, (\varphi_{-t})_* \xi) \circ \varphi_{-t} = g((\varphi_{-t})_* X_{\Sigma_t}, \xi) \circ \varphi_{-t} = 0$$

This is because $(\varphi_{-t})_*$ maps vector fields on Σ_t to vector fields on Σ which are all orthogonal to ξ. We also use the fact that φ_t is the flow of ξ. Expressing the metric in terms of (x^1, x^2, x^3, t) we find that because of this orthogonality the components of the $dx^i dt$ terms must vanish. We also can use the fact that ξ is a Killing vector and that ξ is proportional to $\frac{\partial}{\partial t}$ by construction. Let X and Y be coordinate vector fields.

$$\begin{aligned}\frac{\partial}{\partial t} g(X,Y) &= C\left(g(D_\xi X, Y) + g(D_\xi Y, X)\right) \\ &= C(g(D_X \xi, Y) + g(D_Y \xi, X) = C(g(D_X \xi, Y) - g(D_X \xi, Y)) = 0\end{aligned}$$

So we see that the metric components are independent of t. Of course this argument only makes sense when $\xi \neq 0$. The metric can be expressed as the following:

$$g = -V^2(x^1, x^2, x^3)dt^2 + \sum_{n=1,m=1}^{3} h_{nm}(x^1, x^2, x^3) dx^m dx^n$$

5.4.1 Consequences of the Spherically Symmetric Condition

We also make the assumption that the spacetime is spherically symmetric. That is that metric's isometry group contains a subgroup which is isomorphic to the group SO_3, and the orbit of this subgroup on any point is a 2-dimensional sphere [30, p. 120]. Suppose ξ has a non-zero projection onto the 2-spheres. Then it can not be invariant under all rotations (up to a sign), since that would imply its projection onto the 2-spheres has to be the zero vector field. Therefore the 2-spheres must each completely lie in a spacelike hypersurface Σ_t. We now show that the metrics on these 2-spheres must be multiples of the standard metric of the sphere.

Let $U_z = S^2 - (0,0,\pm 1)$ and let ϕ_z and θ_z be spherical coordinates defined on U_z. If we rotate about the z-axis then ∂_{ϕ_z} and ∂_{θ_z} are preserved. We can repeat this for $U_x = S^2 - (\pm 1, 0, 0)$ and let ϕ_x and θ_x being the spherical coordinates defined on U_x. Thus if we rotate about the x-axis ∂_{ϕ_x} and ∂_{θ_x} are preserved. Let σ signify an arbitrary rotation which can be broken down in terms of a rotation about the z-axis and a rotation about the x-axis either in that order or the reverse. Let us assume that $\sigma = \sigma_x \circ \sigma_z$. Consider the sectional curvature K. Then it is easy to see that $K(\partial_{\phi_z}, \partial_{\theta_z}) = K(\partial_{\phi_z}, \partial_{\theta_z}) \circ \sigma_z$ where we define $K(X,Y)(p)$ to be $K(X(p), Y(p))$ for vector fields X and Y. But one property of the sectional curvature is that $K(\partial_{\phi_z}, \partial_{\theta_z})(p) = K(\partial_{\phi_x}, \partial_{\theta_x})(p)$. Thus we see that $K(\partial_{\phi_z}, \partial_{\theta_z}) = K(\partial_{\phi_z}, \partial_{\theta_z}) \circ \sigma$ at points in $U_x \cap U_z$. By continuity we can extend this to all of S^2. This means that the sectional curvature is constant over S^2. Thus the metric induced on S^2 is a constant, λ, times the standard metric on S^2.

We want to understand the effect of rotational isometries on ξ. To see this we'll show that for an arbitrary rotation, σ, $\sigma_*\xi$ is a timelike Killing vector field. The flow of ξ is φ_t. So by Lafontaine [22, p. 123], the flow of $\sigma_*\xi$ is $\psi_t = \sigma \circ \varphi_t \circ \sigma^{-1}$. We have the following since σ and φ_t are isometries.

$$\begin{aligned}\psi_t^* g &= (\sigma \circ \varphi_t \circ \sigma^{-1})^* g \\ &= (\sigma^{-1})^* \circ \varphi_t^* \circ \sigma^* g = (\sigma^{-1})^* \circ \varphi_t^* g = (\sigma^{-1})^* g = g\end{aligned}$$

Therefore ψ_t is an isometry which implies that $\sigma_*\xi$ is a killing vector field. Since ξ is timelike we know that $\sigma_*\xi$ is timelike.

$$\begin{aligned}&0 > g(\xi,\xi) = (\sigma^* g)(\xi,\xi) = g(\sigma_*\xi, \sigma_*\xi) \circ \sigma \\ &0 > g(\sigma_*\xi, \sigma_*\xi)\end{aligned}$$

If we assume that ξ is unique, in the sense of its Killing and timelike characteristics, it must be invariant under rotations up to scaling. However we can calculate this scaling factor since every rotation B has a fixed point.

$$\begin{aligned}g(\xi,\xi)(p) &= g(\sigma_*\xi, \sigma_*\xi) \circ \sigma(p) = a^2 g(\xi,\xi)(p) \\ a^2 &= 1\end{aligned}$$

Let's restrict our attention to a single sphere in a single hypersurface Σ_t. Let (θ, ϕ) be its spherical coordinates. We can construct space-like geodesics which intersect all the spheres in Σ_t but are orthogonal to the spheres. We can take s to be a parameter along a given geodesic. We have that (θ, ϕ, s) form coordinates for Σ_t. And that (θ, ϕ, s, t) form coordinates for the spacetime. We know that $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ must solely be a function of s since ξ being invariant under rotations (up to a sign) implies: $g(\xi,\xi) = a^2 g(\xi,\xi) \circ \sigma = g(\xi,\xi) \circ \sigma$ where σ is an arbitrary rotation. This means that $g(\xi,\xi)$ is constant across any given sphere and thus so is $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$. If we request that the geodesics are unit length we get that $g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = 1$. We thus have the following form of the metric g:

$$g = -C(s)dt^2 + ds^2 + D(s)(d\theta^2 + \sin^2(\theta)d\phi^2)$$

Set $D(s) = r^2$. Then by assuming that $D(s)$ is injective we can use r as a new coordinate. We have that:

$$\begin{aligned} d(D(s)) &= d(r^2) \\ \frac{\partial}{\partial s}Dds &= 2rdr \\ ds &= \frac{2r}{(\frac{\partial}{\partial s}D)(s)}dr = \frac{2r}{(\frac{\partial}{\partial s}D)(D^{-1}(r^2))}dr = \sqrt{E(r)}dr \end{aligned}$$

We also get that $C(s) = C(D^{-1}(r^2)) = F(r)$. We can write the metric in the following form:

$$g = -F(r)dt^2 + E(r)dr^2 + r^2((d\theta^2 + \sin^2(\theta)d\phi^2)$$

5.4.2 Final Form of the Schwarzschild Solution

We can now apply the Ricci flat condition to further pin down the components of the metric. We'll omit these calculations since we've included similar calculations in the 5D case. Nevertheless the final form of the Schwarzschild metric is stated below [30, p. 124]. Where M is a positive constant.

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

There are 2 singularities present, one at $r = 2M$ and the other at $r = 0$. We have $r = 2M$ corresponds to a sphere of points and the singularity here is due to choice of coordinates. However the singularity at $r = 0$ is real.

5.5 Weyl Form of the Schwarzschild Metric

5.5.1 Conversion to Weyl Form

We now wish to transform the Schwarzschild metric in the form we derived to its Weyl form. The new coordinates will be (ρ, z, t, ϕ) where t and ϕ are carried over from the previous coordinates. In the previous section we studied the Schwarzschild metric in the cohomogeneity 1 setting where the manifold looked like $\underset{t}{\mathbb{R}} \times \underset{r}{\mathbb{R}^+} \times S^2$. The group in that setting was $\mathbb{R} \times SO(3)$; with the metric being invariant under that group. The principal orbit was $\mathbb{R} \times SO(3)/SO(2) \approx \mathbb{R} \times S^2$. We have the $SO(2)$ appearing due to each element of $SO(3)$ acting on S^2 having a 1-dimensional space of companion elements in $SO(3)$ which produce the same image. The singular orbit corresponds to $r = 2M$ and it is isomorphic to S^2 where $\underset{t}{\mathbb{R}}$ has degenerated. The orbit space is 1-dimensional and is simply $\mathbb{R}^+$.

We introduce the functions l_+, l_- and L [15, p. 178].

$$l_+ = \sqrt{\rho^2 + (z+M)^2} \qquad l_- = \sqrt{\rho^2 + (z-M)^2} \qquad L = \frac{l_+ + l_-}{2}$$

We link the 2 coordinate systems together by setting $r = L + M$ and $2M\cos(\theta) = l_+ - l_-$. This allows us to give a fuller description of the Schwarzschild metric in the Weyl setting. It is cohomogeneity 2 with the group being $\underset{t}{\mathbb{R}} \times S^1$. The principal orbit is $\underset{t}{\mathbb{R}} \times S^1$. The orbit space is $\mathbb{R} \times [0, \pi]$. Where the coordinates for the orbit space are (R, β). They are related to ρ and z by $iR + \beta = \sin^{-1}(\frac{1}{M}(i\rho + z))$. The orbit space has two axis rods and a horizon rod. Note that $\rho = 0$ and $|z| > M$ corresponds to two axis rods. Also $\rho = 0$ and $|z| < M$ corresponds to a horizon rod. The first case results in axis rods since $L - M > 0$ and ρ^2 is 0 implying the component $g(\partial_\phi, \partial_\phi)$ (it is calculated later one in this section) goes to 0. This means that S^1 shrinks to a point and the orbit becomes homeomorphic to $\mathbb{R}$. The second case is a horizon rod since $\rho = 0$ and $|z| < M$ implies that $L - M = 0$ thus $g(\partial_t, \partial_t)$ goes to 0 (it is calculated later on). Thus $\mathbb{R}$ shrinks to a point. It can be seen in the limit for the horizon rod case that g$(\partial_\phi, \partial_\phi)$ doesn't go to 0. Thus the orbit is homeomorphic to S^1. Since $\sin^{-1}(\frac{1}{M}(i\rho + z)) = -i\log\left(i\frac{1}{M}(i\rho + z) + \sqrt{1 - \frac{(i\rho+z))^2}{M^2}}\right)$. We have that:

$$'\sin^{-1}(\frac{1}{M}(0 + z)) = -i(\log\left(||i\frac{1}{M}z + \sqrt{1 - \frac{z^2}{M^2}}||\right) + iArg(z + i0)$$
$$iR + \beta = -i(\log\left(||i\frac{1}{M}z + \sqrt{1 - \frac{z^2}{M^2}}||\right) + Arg(z)$$

Thus if $|z| < M$ then $R = 0$. However if $|z| > M$ then $R > 0$. If $z > 0$ then $\beta = 0$, if $z < 0$ then $\beta = \pi$ and if $z = 0$ then β takes on every value between 0 and π. Thus we can plot the orbit space in terms of R and β.

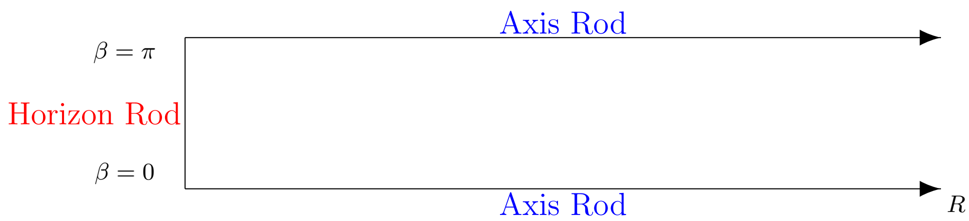

Figure 5.1:

We now derive some useful relations.

$$\begin{aligned}
l_+^2 - l_-^2 &= \rho^2 + z^2 + 2Mz + M^2 - \rho^2 - z^2 + 2Mz - M^2 \\
(l_+ + l_-)(l_+ - l_-) &= 4Mz \\
L(2M\cos(\theta)) &= 2Mz \\
z &= L\cos(\theta)
\end{aligned}$$

$$\begin{aligned}
(L^2 - M^2)\sin^2(\theta) &= L^2 - L^2\cos^2(\theta) - M^2 + M^2\cos^2(\theta) \\
&= \frac{(l_+ + l_-)^2}{4} - z^2 - M^2 + \frac{(l_+ - l_-)^2}{4} \\
&= \frac{l_+^2 + l_-^2}{2} - z^2 - M^2 \\
&= \rho^2 + z^2 + M^2 - z^2 - M^2 \\
\rho &= \sqrt{L^2 - M^2}\sin(\theta)
\end{aligned}$$

$$\begin{aligned}
l_+ l_- &= \frac{(l_+ + l_-)^2}{4} - \frac{(l_+ - l_-)^2}{4} \\
&= L^2 - M^2\cos^2(\theta)
\end{aligned}$$

We now derive formulae for dr and $d\theta$.

$$\begin{aligned}
\frac{\partial r}{\partial \rho} &= \frac{\partial}{\partial \rho}\left(\frac{\sqrt{\rho^2 + (z+M)^2}}{2} + \frac{\sqrt{\rho^2 + (z-M)^2}}{2}\right) \\
&= \frac{1}{2}\left(\frac{\rho}{\sqrt{\rho^2 + (z+M)^2}} + \frac{\rho}{\sqrt{\rho^2 + (z-M)^2}}\right) \\
&= \frac{\rho}{2}\left(\frac{\sqrt{\rho^2 + (z-M)^2} + \sqrt{\rho^2 + (z+M)^2}}{\sqrt{\rho^2 + (z+M)^2}\sqrt{\rho^2 + (z-M)^2}}\right) \\
&= \frac{L\sqrt{L^2 - M^2}\sin(\theta)}{l_+ l_-}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial r}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\sqrt{\rho^2+(z+M)^2}}{2}+\frac{\sqrt{\rho^2+(z-M)^2}}{2}\right)\\
&= \frac{1}{2}\left(\frac{z+M}{\sqrt{\rho^2+(z+M)^2}}+\frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\right)\\
&= \frac{z}{2}\left(\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}}{\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}}\right)-\frac{M}{2}\left(\frac{\sqrt{\rho^2+(z+M)^2}-\sqrt{\rho^2+(z-M)^2}}{\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}}\right)\\
&= \frac{L^2\cos(\theta)}{l_+l_-}-\frac{M^2\cos(\theta)}{l_+l_-}=\frac{(L^2-M^2)\cos(\theta)}{l_+l_-}
\end{aligned}$$

Moving on to $d\theta$.

$$\begin{aligned}
\frac{\partial}{\partial\rho}(2M\cos(\theta)) &= \frac{\partial}{\partial\rho}\left(\sqrt{\rho^2+(z+M)^2}-\sqrt{\rho^2+(z-M)^2}\right)\\
-2M\sin(\theta)\frac{\partial\theta}{\partial\rho} &= \frac{\rho}{\sqrt{\rho^2+(z+M)^2}}-\frac{\rho}{\sqrt{\rho^2+(z-M)^2}}\\
-M\sin(\theta)\frac{\partial\theta}{\partial\rho} &= -\frac{\rho}{2}\left(\frac{\sqrt{\rho^2+(z+M)^2}-\sqrt{\rho^2+(z-M)^2}}{\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}}\right)\\
&= -\frac{\sqrt{L^2-M^2}\sin(\theta)M\cos(\theta)}{l_+l_-}\\
\frac{\partial\theta}{\partial\rho} &= \frac{\sqrt{L^2-M^2}\cos(\theta)}{l_+l_-}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial z}(2M\cos(\theta)) &= \frac{\partial}{\partial z}\left(\sqrt{\rho^2+(z+M)^2}-\sqrt{\rho^2+(z-M)^2}\right)\\
-2M\sin(\theta)\frac{\partial\theta}{\partial z} &= \frac{z+M}{\sqrt{\rho^2+(z+M)^2}}-\frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\\
-M\sin(\theta)\frac{\partial\theta}{\partial z} &= -\frac{z}{2}\left(\frac{\sqrt{\rho^2+(z+M)^2}-\sqrt{\rho^2+(z-M)^2}}{\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}}\right)+\frac{M}{2}\left(\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}}{\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}}\right)\\
&= -\frac{L\cos(\theta)M\cos(\theta)}{l_+l_-}+\frac{ML}{l_+l_-}\\
-M\sin(\theta)\frac{\partial\theta}{\partial z} &= \frac{LM\sin^2(\theta)}{l_+l_-}\\
\frac{\partial\theta}{\partial z} &= -\frac{L\sin(\theta)}{l_+l_-}
\end{aligned}$$

We now insert the expression for $d\theta$ and dr into the Schwarzschild metric. We now calculate the coefficient on the $d\rho^2$ term, which we'll call $g_{\rho\rho}$.

$$1 - \frac{2M}{r} = 1 - \frac{2M}{L+M} = \frac{L-M}{L+M}$$

$$\begin{aligned}
g_{\rho\rho} &= \left(1 - \frac{2M}{r}\right)^{-1} \underbrace{\frac{L^2(L^2-M^2)\sin^2(\theta)}{(l_+l_-)^2}}_{\left(\frac{\partial r}{\partial \rho}\right)^2} + r^2 \underbrace{\frac{(L^2-M^2)\cos^2(\theta)}{(l_+l_-)^2}}_{\left(\frac{\partial \theta}{\partial \rho}\right)^2} \\
&= \frac{(L+M)^2}{(l_+l_-)^2}\left(L^2\sin^2(\theta) + (L^2-M^2)\cos^2(\theta)\right) \\
&= \frac{(L+M)^2}{(l_+l_-)^2}\left(L^2 - M^2\cos^2(\theta)\right) \\
&= \frac{(L+M)^2}{(l_+l_-)^2}(l_+l_-) = \frac{(L+M)^2}{l_+l_-}
\end{aligned}$$

We now calculate the coefficient on the dz^2 term which we will call g_{zz}.

$$\begin{aligned}
g_{zz} &= \left(1 - \frac{2M}{r}\right)^{-1} \underbrace{\frac{(L^2-M^2)^2\cos^2(\theta)}{(l_+l_-)^2}}_{\left(\frac{\partial r}{\partial z}\right)^2} + r^2 \underbrace{\frac{L^2\sin^2(\theta)}{(l_+l_-)^2}}_{\left(\frac{\partial \theta}{\partial z}\right)^2} \\
&= \frac{(L+M)^2}{(l_+l_-)^2}\left((L^2-M^2)\cos^2(\theta) + L^2\sin^2(\theta)\right) \\
&= \frac{(L+M)^2}{(l_+l_-)^2}\left(L^2 - M^2\cos^2(\theta)\right) = \frac{(L+M)^2}{l_+l_-}
\end{aligned}$$

We now move on to the $d\rho dz$ term which we will call $g_{z\rho}$.

$$\begin{aligned}
g_{z\rho} &= \left(1 - \frac{2M}{r}\right)^{-1} \underbrace{\frac{L\sqrt{L^2-M^2}\sin(\theta)}{l_+l_-}\frac{(L^2-M^2)\cos(\theta)}{l_+l_-}}_{\frac{\partial r}{\partial \rho}\frac{\partial r}{\partial z}} - r^2 \underbrace{\frac{\sqrt{L^2-M^2}\cos(\theta)}{l_+l_-}\frac{L\sin(\theta)}{l_+l_-}}_{\frac{\partial \theta}{\partial \rho}\frac{\partial \theta}{\partial z}} \\
&= \frac{\sqrt{L^2-M^2}L\sin(\theta)\cos(\theta)(L+M)}{l_+l_-}\left((L+M) - (L+M)\right) = 0
\end{aligned}$$

The coefficient of the dt^2, g_{tt} is simply given by $g_{tt} = -\frac{L-M}{L+M}$. The coefficient of $d\phi^2$, $g_{\phi\phi}$ is given by the following:

$$\begin{aligned}
g_{\phi\phi} &= r^2\sin^2(\theta) = (L+M)^2\frac{\rho^2}{L^2-M^2} \\
&= \frac{L+M}{L-M}\rho^2
\end{aligned}$$

This allows us to write down the Schwarzschild metric in its Weyl form.

$$g = -\frac{L-M}{L+M}dt^2 + \frac{(L+M)^2}{l_+l_-}\left(dz^2 + d\rho^2\right) + \frac{L+M}{L-M}\rho^2 d\phi^2$$

5.5.2 Harmonicity of ω

Thus the metric is characterized by two potentials ω and k. Where:

$$\omega = -\ln\left(\frac{L+M}{L-M}\right) \qquad k = \frac{1}{2}\ln\left(\frac{L^2-M^2}{l_+l_-}\right)$$
$$g = -e^{\omega}dt^2 + e^{-\omega}(e^{2k}(d\rho^2+dz^2)+\rho^2 d\phi^2)$$

We know that ω is well defined away from the z-axis. To see this we know that if ρ is non zero then:

$$2(L-M) = \sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M > |z+M|+|M-z|-2M \geq 2M-2M = 0$$

Furthermore ω is harmonic; i.e $\omega_{zz}+\omega_{\rho\rho}+\frac{1}{\rho}\omega_\rho = 0$. To see this we will break it down into a function of ρ and z and take the necessary derivatives.‘

$$\omega = -\log\left(\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)$$
$$\omega_\rho = -\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\partial_\rho\Bigg(\underbrace{\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}}_{F_1}\Bigg)$$

$$\partial_\rho F_1 = \frac{1}{(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M)^2}\Bigg($$
$$\left(\frac{\rho}{\sqrt{\rho^2+(z+M)^2}}+\frac{\rho}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M\right)\ldots$$
$$\ldots-\left(\frac{\rho}{\sqrt{\rho^2+(z+M)^2}}+\frac{\rho}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M\right)\Bigg)$$
$$\omega_\rho = -\rho\Bigg(\underbrace{\frac{1}{\sqrt{\rho^2+(z+M)^2}}+\frac{1}{\sqrt{\rho^2+(z-M)^2}}}_{F_2}\Bigg)\Bigg(\underbrace{\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}}_{F_3}\ldots$$
$$\ldots-\underbrace{\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}}_{F_4}\Bigg)$$

We now work out the ρ derivative of F_2, F_3 and F_4. We will set $F_5 = F_3 - F_4$.

$$(F_2)_\rho = -\frac{\rho}{(\rho^2+(z+M)^2)^{\frac{3}{2}}} - \frac{\rho}{(\rho^2+(z-M)^2)^{\frac{3}{2}}}$$

$$(F_3)_\rho = -\frac{\frac{\rho}{\sqrt{\rho^2+(z+M)^2}} + \frac{\rho}{\sqrt{\rho+(z-M)^2}}}{(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M)^2}$$

$$(F_4)_\rho = -\frac{\frac{\rho}{\sqrt{\rho^2+(z+M)^2}} + \frac{\rho}{\sqrt{\rho+(z-M)^2}}}{(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M)^2}$$

$$(F_5)_\rho = \rho\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}} + \frac{1}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\cdots\right.$$
$$\left.\cdots+\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\cdots\right.$$
$$\left.\cdots-\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)$$

We can relate these to $\omega_{\rho\rho}$:

$$\omega_{\rho\rho} = -(F_2F_5 + \rho((F_2)_\rho F_5 + F_2(F_5)_\rho))$$

We now move on to ω_z.

$$\omega_z = -\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\partial_z\Bigg(\underbrace{\frac{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}}_{F_6}\Bigg)$$

$$(F_6)_z = \frac{1}{(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M)^2}\Bigg($$
$$\left(\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M\right)\cdots$$
$$\cdots-\left(\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M\right)\Bigg)$$

$$\omega_z = -\Bigg(\underbrace{\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}}_{F_7}\Bigg)\Bigg(\underbrace{\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}}_{F_3}\cdots$$
$$\cdots-\underbrace{\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}}_{F_4}\Bigg)$$

We now work out the z derivatives of F_3, F_5 and F_7.

$$(F_7)_z = \frac{1}{\sqrt{\rho^2+(z+M)^2}} + \frac{1}{\sqrt{\rho^2+(z-M)^2}} - \frac{(z+M)^2}{(\rho^2+(z+M)^2)^{\frac{3}{2}}} - \frac{(z-M)^2}{(\rho^2+(z-M)^2)^{\frac{3}{2}}}$$

$$(F_3)_z = -\frac{\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}}{(\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M)^2}$$

$$\begin{aligned}(F_5)_z = &\left(\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\right)\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\dots\right.\\ &\left.\dots+\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\dots\right.\\ &\left.\dots-\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\end{aligned}$$

We know that $\omega_{zz} = -(F_7(F_5)_z + (F_7)_z F_5)$. We define $\Delta\omega = \omega_{\rho\rho} + \omega_{zz} + \frac{1}{\rho}\omega_\rho$. We eliminate common factors as we go to aid the calculation. Be careful, there is a sign change.

$$\begin{aligned}\Delta\omega \propto &\frac{2}{\sqrt{\rho^2+(z+M)^2}} + \frac{2}{\sqrt{\rho^2+(z-M)^2}} - \frac{\rho^2}{(\rho^2+(z+M)^2)^{\frac{3}{2}}} - \frac{\rho^2}{(\rho^2+(z-M)^2)^{\frac{3}{2}}}\dots\\ &\dots-\rho^2\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}} + \frac{1}{\sqrt{\rho^2+(z-M)^2}}\right)^2\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\dots\right.\\ &\left.\dots+\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\dots\\ &\dots+\frac{1}{\sqrt{\rho^2+(z+M)^2}} + \frac{1}{\sqrt{\rho^2+(z-M)^2}} - \frac{(z+M)^2}{(\rho^2+(z+M)^2)^{\frac{3}{2}}} - \frac{(z-M)^2}{(\rho^2+(z-M)^2)^{\frac{3}{2}}}\dots\\ &\dots-\left(\frac{z+M}{\sqrt{\rho^2+(z+M)^2}} + \frac{z-M}{\sqrt{\rho^2+(z-M)^2}}\right)^2\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\dots\right.\\ &\left.\dots+\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\\ \propto &\frac{2}{\sqrt{\rho^2+(z+M)^2}} + \frac{2}{\sqrt{\rho^2+(z-M)^2}}\dots\\ &\dots-\left(2+\frac{2\rho^2+2z^2-2M^2}{\sqrt{\rho^2+(z+M)^2}\sqrt{\rho^2+(z-M)^2}}\right)\left(\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}+2M}\dots\right.\\ &\left.\dots+\frac{1}{\sqrt{\rho^2+(z+M)^2}+\sqrt{\rho^2+(z-M)^2}-2M}\right)\end{aligned}$$

We now eliminate the resulting factor out front we have the following

$$\begin{aligned}
\Delta\omega &\propto 2(\sqrt{\rho^2+(z-M)^2}+\sqrt{\rho^2+(z+M)^2})(\sqrt{\rho^2+(z-M)^2}+\sqrt{\rho^2+(z+M)^2}\dots \\
&\dots+2M)(\sqrt{\rho^2+(z-M)^2}+\sqrt{\rho^2+(z+M)^2}-2M)\dots \\
&-(2\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}+2\rho^2+2z^2-2M^2)(\sqrt{\rho^2+(z-M)^2}\dots \\
&\dots+\sqrt{\rho^2+(z+M)^2}+2M+\sqrt{\rho^2+(z-M)^2}+\sqrt{\rho^2+(z+M)^2}-2M) \\
&\propto (\sqrt{\rho^2+(z-M)^2}+\sqrt{\rho^2+(z+M)^2})(4\rho^2+2(z-M)^2+2(z+M)^2-8M^2\dots \\
&\dots+4\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}-4\sqrt{\rho^2+(z-M)^2}\sqrt{\rho^2+(z+M)^2}-4\rho^2-4z^2+4M^2) \\
&=0
\end{aligned}$$

5.5.3 Relation between k and ω

There is a relation between the partial derivatives of k and the partial derivatives of ω.

$$\begin{aligned}
\omega_\rho &= -\frac{L-M}{L+M}\frac{\partial}{\partial\rho}\left(\frac{L+M}{L-M}\right) \\
&= -\frac{L-M}{L+M}\frac{r_\rho(L-M)-(L+M)r_\rho}{(L-M)^2} \\
&= -\frac{-2Mr_\rho}{(L+M)(L-M)} \\
&= -\frac{-2M}{(L+M)(L-M)}\frac{L\sqrt{L^2-M^2}\sin(\theta)}{l_+l_-} \\
&= \frac{2ML\sin(\theta)}{\sqrt{L^2-M^2}l_+l_-}
\end{aligned}$$

$$\begin{aligned}
\omega_z &= -\frac{L-M}{L+M}\frac{\partial}{\partial z}\left(\frac{L+M}{L-M}\right) \\
&= -\frac{-2Mr_z}{(L+M)(L-M)} \\
&= \frac{-2M}{(L+M)(L-M)}\frac{(L^2-M^2)\cos(\theta)}{l_+l_-} \\
&= \frac{2M\cos(\theta)}{l_+l_-}
\end{aligned}$$

Moving on to the partials of k.

$$\begin{aligned}
k_\rho &= \frac{l_+l_-}{2(L^2-M^2)}\frac{\partial}{\partial\rho}\left(\frac{L^2-M^2}{l_+l_-}\right) \\
&= \frac{l_+l_-}{2(L^2-M^2)}\frac{2Lr_\rho l_+l_- - (L^2-M^2)(l_+l_-)_\rho}{(l_+l_-)^2} \\
&= \frac{l_+l_-}{2(L^2-M^2)}\frac{2Lr_\rho l_+l_- - (L^2-M^2)(2Lr_\rho - 2M^2\cos(\theta)\sin(\theta)\theta_\rho)}{(l_+l_-)^2} \\
&= \frac{l_+l_-}{L^2-M^2}\frac{Lr_\rho(l_-l_+ - L^2 + M^2) - (L^2-M^2)M^2\cos(\theta)\sin(\theta)\theta_\rho}{(l_+l_-)^2} \\
&= \frac{(M^2l_+l_-)}{L^2-M^2}\frac{Lr_\rho\sin^2(\theta) - (L^2-M^2)\cos(\theta)\sin(\theta)\theta_\rho}{(l_+l_-)^2} \\
&= \frac{M^2}{L^2-M^2}\frac{L^2\rho\sin^2(\theta) - (L^2-M^2)\sqrt{L^2-M^2}\cos^2(\theta)\sin(\theta)}{(l_+l_-)^2} \\
&= \frac{M^2}{L^2-M^2}\frac{L^2\sqrt{L^2-M^2}\sin^3(\theta) - (L^2-M^2)\sqrt{L^2-M^2}\cos^2(\theta)\sin(\theta)}{(l_+l_-)^2} \\
&= \frac{M^2\sin(\theta)}{\sqrt{L^2-M^2}}\frac{L^2\sin^2(\theta) - (L^2-M^2)\cos^2(\theta)}{(l_+l_-)^2} \\
&= \frac{M^2\sin(\theta)}{\sqrt{L^2-M^2}}\frac{L^2(\sin^2(\theta)-\cos^2(\theta)) + M^2\cos^2(\theta)}{(l_+l_-)^2}
\end{aligned}$$

$$\begin{aligned}
k_z &= \frac{l_+l_-}{2(L^2-M^2)}\frac{\partial}{\partial\rho}\left(\frac{L^2-M^2}{l_+l_-}\right) \\
&= \frac{(M^2l_+l_-)}{L^2-M^2}\frac{Lr_z\sin^2(\theta) - (L^2-M^2)\cos(\theta)\sin(\theta)\theta_z}{(l_+l_-)^2} \\
&= \frac{M^2}{L^2-M^2}\frac{L(L^2-M^2)\cos(\theta)\sin^2(\theta) - (L^2-M^2)(-L)\cos(\theta)\sin^2(\theta)}{(l_+l_-)^2} \\
&= \frac{2LM^2\sin^2(\theta)\cos(\theta)}{(l_+l_-)^2}
\end{aligned}$$

We now derive formulas for the partials of k in terms of the partials of ω.

$$\begin{aligned}
\frac{\rho}{4}(\omega_\rho^2 - \omega_z^2) &= \frac{\rho}{4}\left(\frac{4M^2L^2\sin^2(\theta)}{(L^2-M^2)(l_-l_+)^2} - \frac{4M^2\cos^2(\theta)}{(l_-l_+)^2}\right) \\
&= \frac{\sqrt{L^2-M^2}\sin(\theta)M^2}{(L^2-M^2)(l_-l_+)^2}\left(L^2\sin^2(\theta) - (L^2-M^2)\cos^2(\theta)\right) \\
&= \frac{M^2\sin(\theta)}{\sqrt{L^2-M^2}}\frac{L^2(\sin^2(\theta)-\cos^2(\theta)) + M^2\cos^2(\theta)}{(l_+l_-)^2} = k_\rho
\end{aligned}$$

$$\begin{aligned}
\frac{\rho}{2}\omega_\rho\omega_z &= \frac{\rho}{2}\frac{2ML\sin(\theta)}{\sqrt{L^2-M^2}l_-l_+}\frac{2M\cos(\theta)}{l_+l_-} \\
&= \frac{2LM^2\sin^2(\theta)\cos(\theta)}{(l_+l_-)^2} = k_z
\end{aligned}$$

These relationships between k and ω can be used to generalize the Schwarzschild metric to one with the same form but ω is now arbitrary and k is integrated by these following formulae:

$$k_z = \frac{\rho}{2}\omega_\rho\omega_z \qquad k_\rho = \frac{\rho}{4}(\omega_\rho^2 - \omega_z^2)$$

5.6 Periodic Schwarzschild Solution

This brings us to the paper "Periodic Analog of the Schwarzschild Solution" where Nikolai and Korotkin construct the following solution [21, p. 3] . Consider ρ and z from the previous section and let $\xi_0 = \frac{\sqrt{(z-M)^2+\rho^2}+\sqrt{(z+M)^2+\rho^2}-2M}{\sqrt{(z-M)^2+\rho^2}+\sqrt{(z+M)^2+\rho^2}+2M}$. Where we have constants l and M. We now build the periodic function $\xi = \xi_0(z,\rho)\prod_{n=1}^{\infty}\xi_0(z+nl,\rho)\xi_0(z-nl,\rho)e^{\frac{4M}{nl}}$. To see periodicity we pass to a summation via the logarithm.

$$\log(\xi(z,\rho)) = \log(\xi_0(z,\rho)) + \sum_{n=1}^{\infty}(\log(\xi_0(z+nl,\rho)) + \log(\xi_0(z-nl,\rho)) + \frac{4M}{nl})$$

$$\log(\xi(z,\rho)) = \lim_{m\to\infty}(\log(\xi_0(z,\rho)) + \sum_{n=1}^{m}(\log(\xi_0(z+nl,\rho)) + \log(\xi_0(z-nl,\rho)) + \frac{4M}{nl}))$$

$$\begin{aligned}
\log(\xi(z+l,\rho)) &= \lim_{m\to\infty}(\log(\xi_0(z+l,\rho)) + \sum_{n=1}^{m}(\log(\xi_0(z+(n+1)l,\rho)) + \log(\xi_0(z-(n-1)l,\rho)) + \frac{4M}{nl})) \\
&= \lim_{m\to\infty}(\log(\xi_0(z+l,\rho)) + \sum_{n=2}^{m+1}\log(\xi_0(z+nl,\rho)) + \sum_{n=0}^{m-1}\log(\xi_0(z-nl,\rho)) + \sum_{n=1}^{m}\frac{4M}{nl}) \\
&= \lim_{m\to\infty}(\log(\xi_0(z,\rho)) + \sum_{n=1}^{m}(\log(\xi_0(z+nl,\rho)) + \log(\xi_0(z-nl,\rho)) + \frac{4M}{nl}) + \log(\xi_0(z+(m+1)l,\rho)) \\
&- \log(\xi_0(z-ml,\rho))) \\
&= \log(\xi(z,\rho)) + \lim_{m\to\infty}(\log(\xi_0(z+(m+1)l,\rho)) - \log(\xi_0(z-ml,\rho)))
\end{aligned}$$

By inspection we see that the remaining limit is 0. Therefore we have periodicity.

We have that from the previous section that $\omega_0 = \log(\xi_0)$ is a harmonic function. Thus $\omega = \log(\xi)$ being a sum of harmonic functions makes it harmonic as well. In the paper they prove that ξ is in fact convergent [21, p. 4]. They also prove that k being given by integrating the following two equations is periodic as well [21, p. 4]. This makes the metric periodic.

$$k_z = \frac{\rho}{2}\omega_\rho\omega_z \qquad k_\rho = \frac{\rho}{4}(\omega_\rho^2 - \omega_z^2)$$

Of course in the cohomogeneity 2 setting this is a simpler case than the solutions in "5-dimensional space-periodic solutions of the static vacuum Einstein equations." The laplacian $\Delta c = c_{\rho\rho} + c_{zz} + \frac{1}{\rho} c_\rho$, where c is some function, is prominent in both contexts.

Chapter 6

Analysis of Example 2 in "5-Dimensional Space-Periodic Solutions of the Static Vacuum Einstein Equations"

6.1 Analysis of the U potential

In their paper "5-dimensional space-periodic solutions of the static vacuum Einstein equations"; Khuri, Weinstein and Yamada construct numerous solutions to the harmonic map equations in the 5D case. These solutions are a special case of the harmonic map equations where the matrix F is diagonal, i.e $f_{12} = 0$, and the twist potentials also vanish. The metric on the fibre, H thus has the following form where u and v are harmonic [18, p. 6].

$$H = \begin{pmatrix} -r^2 e^{-u-v} & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^v \end{pmatrix}$$

Therefore the matrices $K = H^{-1}H_r$ and $L = H^{-1}H_z$ can be calculated

$$K = \begin{pmatrix} -u_r - v_r + \frac{2}{r} & 0 & 0 \\ 0 & u_r & 0 \\ 0 & 0 & v_r \end{pmatrix} \qquad L = \begin{pmatrix} -u_z - v_z & 0 & 0 \\ 0 & u_z & 0 \\ 0 & 0 & v_z \end{pmatrix}$$

We can now calculate α_r and α_z.

$$\begin{aligned}
\alpha_r &= \frac{r}{8}\left(\mathrm{tr}(K^2) - \mathrm{tr}(L^2) - \frac{4}{r^2}\right) \\
&= \frac{r}{8}\left(\left(-u_r - v_r + \frac{2}{r}\right)^2 + u_r^2 + v_r^2 - ((u_z+v_z)^2 + u_z^2 + v_z^2) - \frac{4}{r^2}\right) \\
&= \frac{r}{4}\left(u_r^2 + v_r^2 + u_r v_r - \frac{2}{r}(u_r + v_r) - u_z^2 - v_z^2 - u_z v_z\right) \qquad (6.1.1) \\
\alpha_z &= \frac{r}{4}\,\mathrm{tr}(KL) \\
&= \frac{r}{4}\left((-u_r - v_r + \frac{2}{r})(-u_z - v_z) + u_r u_z + v_r v_z\right) \\
&= \frac{r}{4}\left(2u_r u_z + 2v_r v_z + u_r v_z + v_r u_z - \frac{2}{r}(u_z + v_z)\right) \qquad (6.1.2)
\end{aligned}$$

Important to analyzing the solutions in the above paper is the harmonic function U_I. Where I is the interval $[a,b]$.

$$U_I = \log\left(\sqrt{r^2 + (z-a)^2} - (z-a)\right) - \log\left(\sqrt{r^2 + (z-b)^2} - (z-b)\right)$$

Lemma 6.2. *i We have that U_I has domain $\{r \geq 0\} - \{(0,z) \,|\, a \leq z \leq b\}$.*

ii We have that U_I is harmonic with respect to the laplacian on $\mathbb{R}^3$.

iii We have that it satisfies $U_I < 0$.

Proof. For proof of (i) we can see if $z < a < b$ then $\sqrt{r^2 + (z-a)^2} - (z-a) > 0$ and $\sqrt{r^2 + (z-b)^2} - (z-b) > 0$ for all r. If we have that $z > a > b$ then we can manipulate U_I by rationalizing the numerators inside the $\log$ terms.

$$U_I = 2\log(r) - \log\left(\sqrt{r^2 + (z-a)^2} + (z-a)\right) + \log\left(\sqrt{r^2 + (z-b)^2} + (z-b)\right) - 2\log(r)$$

$$U_I = -\log\left(\sqrt{r^2 + (z-a)^2} + (z-a)\right) + \log\left(\sqrt{r^2 + (z-b)^2} + (z-b)\right)$$

We have that $\sqrt{r^2 + (z-a)^2} + (z-a)) > 0$ $\sqrt{r^2 + (z-b)^2} + (z-b) > 0$. Thus U_I is well defined. Now if $a \leq z \leq b$ then U_I is undefined for $r = 0$. This is because we only rationalize one of the numerators.

$$U_I = -\log\left(\sqrt{r^2 + (z-a)^2} + (z-a)\right) - \log\left(\sqrt{r^2 + (z-b)^2} - (z-b)\right) + 2\log(r)$$

Thus we can see that $U_I \approx 2\log(r)$ for small r on $a < z < b$ and $U_I = -O(1)$ for $z > b$ and $z < a$.

For (ii) it suffices to check that the function $U = \log(\sqrt{r^2+z^2} - z)$ is harmonic.

$$\begin{aligned}
U_r &= \frac{r(r^2+z^2)^{-\frac{1}{2}}}{\sqrt{r^2+z^2}-z} \\
U_{rr} &= \frac{((r^2+z^2)^{-\frac{1}{2}} - r^2(r^2+z^2)^{-\frac{3}{2}})(\sqrt{r^2+z^2}-z) - r(r^2+z^2)^{-\frac{1}{2}}r(r^2+z^2)^{-\frac{1}{2}}}{(\sqrt{r^2+z^2}-z)^2} \\
U_z &= \frac{z(r^2+z^2)^{-\frac{1}{2}} - 1}{\sqrt{r^2+z^2}-z} = -\frac{1}{\sqrt{r^2+z^2}} \\
U_{zz} &= \frac{z}{(r^2+z^2)^{\frac{3}{2}}} \\
U_r\frac{1}{r} + U_{zz} + U_{rr} &= \frac{1}{(r^2+z^2)^{\frac{3}{2}}(\sqrt{r^2+z^2}-z)^2}\Big((r^2+z^2)^{-\frac{1}{2}}(\sqrt{r^2+z^2}-z)(r^2+z^2)^{\frac{3}{2}} + \ldots \\
&\ldots z(\sqrt{r^2+z^2}-z)^2 + ((r^2+z^2)^{-\frac{1}{2}} - r^2(r^2+z^2)^{-\frac{3}{2}})(\sqrt{r^2+z^2}-z)(r^2+z^2)^{\frac{3}{2}} - \ldots \\
&\ldots r(r^2+z^2)^{-\frac{1}{2}}r(r^2+z^2)^{-\frac{1}{2}}(r^2+z^2)^{\frac{3}{2}}\Big) \\
&= \frac{1}{(r^2+z^2)^{\frac{3}{2}}(\sqrt{r^2+z^2}-z)^2}\Big((r^2+z^2)^{\frac{3}{2}} + z(r^2+z^2) - 2z^2(r^2+z^2)^{\frac{1}{2}} + z^3 \ldots \\
&\ldots + (r^2+z^2)^{\frac{3}{2}} - z(r^2+z^2) - r^2(r^2+z^2)^{\frac{1}{2}} + r^2z - r^2(r^2+z^2)^{\frac{1}{2}}\Big) \\
&= \frac{1}{(r^2+z^2)^{\frac{3}{2}}(\sqrt{r^2+z^2}-z)^2}\Big((2r^2+2z^2-2r^2-z^2-z^2)(r^2+z^2)^{\frac{1}{2}} + (z^3-z^3+r^2z-r^2z)\Big) \\
&= 0
\end{aligned}$$

For the proof of (iii), we will define the function $f(c)$ and show its c-derivative is greater than 0.

$$\begin{aligned}
f(c) &= \sqrt{r^2+(z-c)^2} - (z-c) \\
f'(c) &= -\frac{2(z-c)}{2\sqrt{r^2+(z-c)^2}} + 1 \\
&= \frac{-(z-c)+\sqrt{r^2+(z-c)^2}}{\sqrt{r^2+(z-c)^2}} > 0
\end{aligned}$$

Therefore $f(b) > f(a)$ for all r and z. So since $\log$ is an increasing function it follows that U_I is less than zero for all z and r. □

6.3 Analysis of Example 2

We will focus on analyzing example 2 from the paper : "5-dimensional space-periodic solutions of the static vacuum Einstein equations." Define intervals $\Gamma_{2j} = [2jL.(2j+1)L]$ and $\Gamma_{2j+1} = [(2j+1)L,(2j+2)L]$ where $L > 0$. Two harmonic functions u and v are defined as follows:

$$u = \lim_{n\to\infty}\left(\Sigma_{j=-n}^{n} U_{\Gamma_{2j}} + \log n\right) \qquad v = \lim_{n\to\infty}\left(\Sigma_{j=-n}^{n} U_{\Gamma_{2j+1}} + \log n\right)$$

Lemma 6.4. *The following are true.*

i We have that u *and* v *are convergent for* $r > 0$.

ii We have that u *and* v *are periodic with period* $2L$.

iii We have that u *and* v *are symmetric in* z *about* $\frac{L}{2}$.

Proof. For (i) e will show that u and v are convergent for $r > 0$. We will first assume $-\frac{L}{2} \leq z \leq \frac{L}{2}$.

$$\begin{aligned}
u = \lim_{n\to\infty}\Bigg(&\sum_{j=1}^{n}\Bigg(\log\Big(\sqrt{r^2+(z-2jL)^2}-(z-2jL)\Big)-\log\Big(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\Big)\Bigg)\dots\\
&\dots+\sum_{j=-n}^{-1}\Bigg(\log\Big(\sqrt{r^2+(z-2jL)^2}-(z-2jL)\Big)-\log\Big(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\Big)\Bigg)\dots\\
&\dots+\log\Big(\sqrt{r^2+z^2}-z\Big)-\log\Big(\sqrt{r^2+(z-L)^2}-(z-L)\Big)+\log(n)\Bigg)\\
= \lim_{n\to\infty}\Bigg(&\sum_{j=1}^{n}\Bigg(\log\Big(\sqrt{r^2+(z-2jL)^2}-(z-2jL)\Big)-\log\Big(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\Big)\Bigg)\dots\\
&\dots+\sum_{j=1}^{n}\Bigg(\log\Big(\sqrt{r^2+(z+2jL)^2}-(z+2jL)\Big)-\log\Big(\sqrt{r^2+(z+(2j-1)L)^2}-(z-(-2j+1)L)\Big)\Bigg)\dots\\
&\dots+\log\Big(\sqrt{r^2+z^2}-z\Big)-\log\Big(\sqrt{r^2+(z-L)^2}-(z-L)\Big)+\log(n)\Bigg)\\
= \lim_{n\to\infty}\Bigg(&\sum_{j=1}^{n}\Bigg(\log\Big(\sqrt{r^2+(z-2jL)^2}-(z-2jL)\Big)-\log\Big(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\Big)\Bigg)\dots\\
&\dots+\sum_{j=1}^{n}\Bigg(-\log\Big(\sqrt{r^2+(z+2jL)^2}+(z+2jL)\Big)+\log\Big(\sqrt{r^2+(z+(2j-1)L)^2}+(z+(2j-1)L)\Big)\Bigg)\dots\\
&\dots+\log\Big(\sqrt{r^2+z^2}-z\Big)-\log\Big(\sqrt{r^2+(z-L)^2}-(z-L)\Big)+\log(n)\Bigg)\\
= \lim_{n\to\infty}\Bigg(&\sum_{j=1}^{n}\Bigg(\log(2jL)-\log((2j+1)L)-\log(2jL)+\log(2j-1)L\Bigg)\dots\\
&\dots+\sum_{j=1}^{n}\underbrace{\Bigg(\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2}-\frac{z}{jL}+1}+1-\frac{z}{2jL}\Bigg)-\log\Bigg(\sqrt{\frac{r^2+z^2}{((2j+1)L)^2}-\frac{2z}{(2j+1)L}+1}+1-\frac{z}{(2j+1)L}\Bigg)}_{A_j-B_j}\dots\\
&\dots\underbrace{-\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2}+\frac{z}{jL}+1}+1+\frac{z}{2jL}\Bigg)+\log\Bigg(\sqrt{\frac{r^2+z^2}{((2j-1)L)^2}+\frac{2z}{(2j-1)L}+1}+1+\frac{z}{(2j-1)L}\Bigg)}_{-C_j+D_j}\Bigg)\dots\\
&\dots+\log\Big(\sqrt{r^2+z^2}-z\Big)-\log\Big(\sqrt{r^2+(z-L)^2}-(z-L)\Big)+\log(n)\Bigg)
\end{aligned}$$

$$= \lim_{n\to\infty}\Bigg(-\log(2n+1)L) - \log(2L) + \log(n) \dots$$
$$\dots + \sum_{j=1}^{n}\Bigg(\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} - \frac{z}{jL} + 1} + 1 - \frac{z}{2jL}\Bigg) - \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j+1)L)^2} - \frac{2z}{(2j+1)L} + 1} + 1 - \frac{z}{(2j+1)L}\Bigg)\dots$$
$$\dots - \log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} + \frac{z}{jL} + 1} + 1 + \frac{z}{2jL}\Bigg) + \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j-1)L)^2} + \frac{2z}{(2j-1)L} + 1} + 1 + \frac{z}{(2j-1)L}\Bigg)\Bigg)\dots$$
$$\dots + \log\Big(\sqrt{r^2+z^2} - z\Big) - \log\Big(\sqrt{r^2+(z-L)^2} - (z-L)\Big)\Bigg)$$
$$= \sum_{j=1}^{\infty}\Bigg(\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} - \frac{z}{jL} + 1} + 1 - \frac{z}{2jL}\Bigg) - \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j+1)L)^2} - \frac{2z}{(2j+1)L} + 1} + 1 - \frac{z}{(2j+1)L}\Bigg)\dots$$
$$\dots - \log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} + \frac{z}{jL} + 1} + 1 + \frac{z}{2jL}\Bigg) + \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j-1)L)^2} + \frac{2z}{(2j-1)L} + 1} + 1 + \frac{z}{(2j-1)L}\Bigg)\Bigg)\dots$$
$$\dots + \log\Big(\sqrt{r^2+z^2} - z\Big) - \log\Big(\sqrt{r^2+(z-L)^2} - (z-L)\Big)\Bigg)$$

In the steps above we have eliminated the limit. We want to compare $A_j - B_j$ to $\frac{1}{j^2}$ in a limit comparison test to show that the sum of $A_j - B_j$ converges.

$$M = \lim_{j\to\infty}\Bigg(\Bigg(\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} - \frac{z}{jL} + 1} + 1 - \frac{z}{2jL}\Bigg) - \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j+1)L)^2} - \frac{2z}{(2j+1)L} + 1} + 1 - \frac{z}{(2j+1)L}\Bigg)\Bigg)j^2\Bigg)$$
$$= \lim_{k\to 0}\Bigg(\Bigg(\log\Bigg(\sqrt{\frac{(r^2+z^2)k^2}{(2L)^2} - \frac{kz}{L} + 1} + 1 - \frac{kz}{2L}\Bigg) - \log\Bigg(\sqrt{\frac{(r^2+z^2)k^2}{((2+k)L)^2} - \frac{2zk}{(2+k)L} + 1} + 1 - \frac{zk}{(2+k)L}\Bigg)\Bigg)\frac{1}{k^2}$$

We can use L'Hopital's rule to simplify.

$$M = \lim_{k\to 0}\frac{1}{2k}\left(\frac{\frac{\frac{2k(r^2+z^2)}{(2L)^2} - \frac{z}{L}}{2\sqrt{\frac{k^2(r^2+z^2)}{(2L)^2} - \frac{kz}{L} + 1}} - \frac{z}{2L}}{\sqrt{\frac{k^2(r^2+z^2)}{(2L)^2} - \frac{kz}{L} + 1} + 1 - \frac{kz}{2L}} - \frac{\frac{\frac{2k(r^2+z^2)}{((2+k)L)^2} - \frac{2k^2(r^2+z^2)}{(2+k)^3L^2} - \frac{2z}{(2+k)L} + \frac{2kz}{(2+k)^2L}}{2\sqrt{\frac{k^2(r^2+z^2)}{((2+k)L)^2} - \frac{2kz}{(2+k)L} + 1}} - \frac{z}{(2+k)L} + \frac{kz}{(2+k)^2L}}{2\sqrt{\frac{k^2(r^2+z^2)}{((2+k)L)^2} - \frac{2kz}{(2+k)L} + 1} + 1 - \frac{kz}{(2+k)L}}\right)$$
$$= \lim_{k\to 0}\frac{1}{4k}\left(\frac{k(r^2+z^2)}{(2L)^2} - \frac{k(r^2+z^2)}{((2+k)L)^2} - \frac{z}{L} + \frac{2z}{(2+k)L} + \frac{k^2(r^2+z^2)}{(2+k)^3L^2} - \frac{2kz}{(2+k)^2L}\right)$$
$$= \lim_{k\to 0}\frac{1}{4k}\left(\frac{k(r^2+z^2)}{(2L)^2} - \frac{k(r^2+z^2)}{((2+k)L)^2} - \frac{kz}{(2+k)L} + \frac{k^2(r^2+z^2)}{(2+k)^3L^2} - \frac{2kz}{(2+k)^2L}\right)$$
$$= \lim_{k\to 0}\frac{1}{4}\left(\frac{r^2+z^2}{(2L)^2} - \frac{r^2+z^2}{((2+k)L)^2} - \frac{z}{(2+k)L} + \frac{k(r^2+z^2)}{(2+k)^3L^2} - \frac{2z}{(2+k)^2L}\right)$$
$$= -\frac{z}{4L}$$

Therefore the sum of $A_j - B_j$ converges. We now perform the same for $C_j - D_j$.

$$
\begin{aligned}
N &= \lim_{j\to\infty}\left(\left(\log\left(\sqrt{\frac{r^2+z^2}{(2jL)^2}+\frac{z}{jL}+1}+1+\frac{z}{2jL}\right)-\log\left(\sqrt{\frac{r^2+z^2}{((2j-1)L)^2}+\frac{2z}{(2j-1)L}+1}+1+\frac{z}{(2j-1)L}\right)\right)j^2\right)\\
&= \lim_{k\to 0}\left(\left(\log\left(\sqrt{\frac{(r^2+z^2)k^2}{(2L)^2}+\frac{kz}{L}+1}+1+\frac{kz}{2L}\right)-\log\left(\sqrt{\frac{(r^2+z^2)k^2}{((2-k)L)^2}+\frac{2zk}{(2-k)L}+1}+1+\frac{zk}{(2-k)L}\right)\right)\frac{1}{k^2}\\
&= \lim_{k\to 0}\frac{1}{2k}\left(\frac{\frac{\frac{2k(r^2+z^2)}{(2L)^2}+\frac{z}{L}}{2\sqrt{\frac{k^2(r^2+z^2)}{(2L)^2}+\frac{kz}{L}+1}}+\frac{z}{2L}}{\sqrt{\frac{k^2(r^2+z^2)}{(2L)^2}+\frac{kz}{L}+1}+1+\frac{kz}{2L}}-\frac{\frac{\frac{2k(r^2+z^2)}{((2-k)L)^2}-\frac{2k^2(r^2+z^2)}{(2-k)^3L^2}+\frac{2z}{(2-k)L}+\frac{2kz}{(2-k)^2L}}{2\sqrt{\frac{k^2(r^2+z^2)}{((2-k)L)^2}+\frac{2kz}{(2-k)L}+1}}+\frac{z}{(2-k)L}+\frac{kz}{(2-k)^2L}}{2\sqrt{\frac{k^2(r^2+z^2)}{((2-k)L)^2}+\frac{2kz}{(2-k)L}+1}+1+\frac{kz}{(2-k)L}}\right)\\
&= \lim_{k\to 0}\frac{1}{4k}\left(\frac{k(r^2+z^2)}{(2L)^2}+\frac{z}{L}-\frac{k(r^2+z^2)}{((2-k)L)^2}+\frac{k^2(r^2+z^2)}{(2-k)^3L^2}-\frac{2z}{(2-k)L}-\frac{2kz}{(2-k)^2L}\right)\\
&= \lim_{k\to 0}\frac{1}{4k}\left(-\frac{kz}{(2-k)L}-\frac{2kz}{(2-k)^2L}\right)=-\frac{z}{4L}
\end{aligned}
$$

Therefore $C_j - D_j$ converges which implies u converges when $-\frac{L}{2} \le z \le \frac{L}{2}$. If we take a look at v we can quickly see that $v(r, z+L) = u(r, z)$.

$$
\begin{aligned}
v(r,z) &= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\log\left(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\right)\dots\right.\right.\\
&\qquad \left.\left.\dots-\log\left(\sqrt{r^2+(z-(2j+2)L)^2}-(z-(2j+2)L)\right)\right)\right)\dots\\
v(r,z+L) &= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\log\left(\sqrt{r^2+(z-2jL)^2}-(z-2jL)\right)-\log\left(\sqrt{r^2+(z-(2j+1)L)^2}-(z-(2j+1)L)\right)\right)\right)\\
&= u(r,z)
\end{aligned}
$$

Moving onto *(ii)* we will show that u and in turn v are periodic in z with period $2L$. Consider $U_{\Gamma_{2j}}(r, z+2L)$.

$$
\begin{aligned}
U_{\Gamma_{2j}}(r,z+2L) &= \log\left(\sqrt{r^2+(z+2L-2jL)^2}-(z+2L-2jL)\right)\dots\\
&\quad\dots-\log\left(\sqrt{r^2+(z+2L-(2j+1)L)^2}-(z+2L-(2j+1)L)\right)\\
&= \log\left(\sqrt{r^2+(z-2(j-1)L)^2}-(z-(2j-1)L)\right)\dots\\
&\quad\dots-\log\left(\sqrt{r^2+(z-(2(j-1)+1)L)^2}-(z+(2(j-1)+1)L)\right)=U_{\Gamma_{2(j-1)}}(r,z)
\end{aligned}
$$

We can leverage this equation in the expression for u to get periodicity.

$$\begin{aligned}
u(r,z+2L) &= \lim_{n\to\infty}\Big(\Sigma_{j=-n}^{n} U_{\Gamma_{2j}}(r,z+2L)+\log n\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=-n}^{n} U_{\Gamma_{2(j-1)}}(r,z)+\log n\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=-n-1}^{n-1} U_{\Gamma_{2j}}(r,z)+\log n\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=-n}^{n} U_{\Gamma_{2j}}+U_{\Gamma_{2(-n-1)}}-U_{\Gamma_{2n}}+\log n\Big)\\
&= u+\lim_{n\to\infty}(U_{\Gamma_{-2(n+1)}}-U_{\Gamma_{2n}})\\
\lim_{n\to\infty}(U_{\Gamma_{-2(n+1)}}-U_{\Gamma_{2n}}) &= \lim_{n\to\infty}\left(\log\left(\frac{\sqrt{r^2+(z+2(n+1)L)^2}-(z+2(n+1)L)}{\sqrt{r^2+(z+(2n-1)L)^2}-(z+(2n-1)L)}\right)\dots\right.\\
&\quad\left.\dots-\log\left(\frac{\sqrt{r^2+(z-2nL)^2}-(z-2nL)}{\sqrt{r^2+(z-(2n+1))L)^2}-(z-(2n+1)L)}\right)\right)\\
&= \lim_{n\to\infty}\left(\log\left(\frac{r^2}{r^2}\frac{\sqrt{r^2+(z+(2n-1)L)^2}+(z+(2n-1)L)}{\sqrt{r^2+(z+2(n+1)L)^2}+(z+2(n+1)L)}\right)\dots\right.\\
&\quad\left.\dots-\log\left(\frac{\sqrt{r^2+(z-2nL)^2}-(z-2nL)}{\sqrt{r^2+(z-(2n+1))L)^2}-(z-(2n+1)L)}\right)\right)
\end{aligned}$$

Of course r and z pale in comparison to n so we can simplify to get the following:

$$\lim_{n\to\infty}(U_{\Gamma_{-2(n+1)}}-U_{\Gamma_{2n}})=\lim_{n\to\infty}\left(\log\left(\frac{2(2n-1)L}{4(n+1)L}\right)-\log\left(\frac{4nL}{2(2n+1)L}\right)\right)=0$$

Therefore u and v are periodic. Moving on to (iii) we will show that $u(r,\frac{L}{2}+z)=u(r,\frac{L}{2}-z)$.

$$\begin{aligned}
U_{\Gamma_{2j}}\left(r,\frac{L}{2}-z\right) &= \log\left(\sqrt{r^2+\left(\frac{L}{2}-z-2jL\right)^2}-\left(\frac{L}{2}-z-2jL\right)\right)\dots\\
&\quad\dots-\log\left(\sqrt{r^2+\left(\frac{L}{2}-z-(2j+1)L\right)^2}-\left(\frac{L}{2}-z-(2j+1)L\right)\right)\\
&= \log\left(\sqrt{r^2+\left(z+\left(2j-\frac{1}{2}\right)L\right)^2}+\left(z+\left(2j-\frac{1}{2}\right)L\right)\right)\dots\\
&\quad\dots-\log\left(\sqrt{r^2+\left(z+\left(2j+\frac{1}{2}\right)L\right)^2}+\left(z+\left(2j+\frac{1}{2}\right)L\right)\right)
\end{aligned}$$

$$U_{\Gamma_{-2j}}\left(r,\frac{L}{2}-z\right) = \log\left(\sqrt{r^2+\left(\frac{L}{2}+z+2jL\right)^2}-\left(\frac{L}{2}+z+2jL\right)\right)\dots$$

$$\dots-\log\left(\sqrt{r^2+\left(\frac{L}{2}+z-(-2j+1)L\right)^2}-\left(\frac{L}{2}+z-(-2j+1)L\right)\right)$$

$$=\log\left(\sqrt{r^2+\left(z+\left(2j+\frac{1}{2}\right)L\right)^2}-\left(z+\left(2j+\frac{1}{2}\right)L\right)\right)\dots$$

$$\dots-\log\left(\sqrt{r^2+\left(z+\left(2j-\frac{1}{2}\right)L\right)^2}-\left(z+\left(2j-\frac{1}{2}\right)L\right)\right)$$

$$U_{\Gamma_{2j}}\left(r,\frac{L}{2}-z\right)-U_{\Gamma_{-2j}}\left(r,\frac{L}{2}-z\right)=2\log r-2\log r=0$$

By the j symmetry in the summation in u we can identify positive j from $u(r,\frac{L}{2}-z)$ with negative j from $u(r,\frac{L}{2}+z)$ and vice versa to get the desired cancellation. Therefore u is symmetric about $\frac{L}{2}$ in z. We can show that v is symmetric about $\frac{L}{2}$ as well.

$$v(r,\frac{L}{2}+z)=u(r,\frac{L}{2}+z-L)=u(r,\frac{L}{2}-z+L)=v(r,\frac{L}{2}-z+2L)=v(r,\frac{L}{2}-z)$$

Therefore we can conclude that $u(r,z)$ and $v(r,z)$ are convergent for all z when $r>0$.

□

Corollary 6.5. *From the previous theorem we can deduce that α is $2L$ periodic in z and that $\alpha(r,\frac{L}{2}+z)=\alpha(r,\frac{L}{2}-z)$.*

Proof. We can now deduce that α is $2L$ periodic in z. Note that u_r,u,z, v_r and v_z must be periodic in z. Therefore α_z and α_r must be periodic in z. Next note that:

$$u(r,\frac{L}{2}+z)=u(r,\frac{L}{2}-z) \qquad v(r,\frac{L}{2}+z)=v(r,\frac{L}{2}-z)$$

$$u_z(r,\frac{L}{2}+z)=-u_z(r.\frac{L}{2}-z) \qquad v_z(r,\frac{L}{2}+z)=-v_z(r.\frac{L}{2}-z)$$

$$u_r(r,\frac{L}{2}+z)=u_r(r,\frac{L}{2}-z) \qquad v_r(r,\frac{L}{2}+z)=v_r(r,\frac{L}{2}-z)$$

$$\alpha_z=\frac{r}{4}\left(2u_ru_z+2v_rv_z+u_rv_z+v_ru_z-\frac{2}{r}(u_z+v_z)\right)$$

$$\alpha_z(r,\frac{L}{2}+z)=-\alpha_z(r,\frac{L}{2}-z)$$

Therefore $\int_{-\frac{L}{2}}^{\frac{3L}{2}}\alpha_z=0$. It follows that α is $2L$ periodic in z. Also note that $\alpha(r,\frac{L}{2}+z)=\alpha(r,\frac{L}{2}-z)$ by integration. □

6.5.1 Asymptotic Behaviour of u, v and α

Proposition 6.6. *In this section we will prove the following asymptotic behaviour of u, v and α as r approaches ∞.*

$$u = \log(r) - \log(4L) \pm O(e^{-\frac{r}{L}}) \qquad u_r = \frac{1}{r} \pm O(e^{-\frac{r}{L}}) \qquad u_z = \pm O(e^{\frac{-r}{L}}) \tag{6.6.1}$$

$$v = \log(r) - \log(4L) \pm O(e^{-\frac{r}{L}}) \qquad v_r = \frac{1}{r} \pm O(e^{-\frac{r}{L}}) \qquad v_z = \pm O(e^{\frac{-r}{L}}) \tag{6.6.2}$$

$$\alpha = -\frac{\log(r)}{4} \pm O(e^{-\frac{r}{L}}) \qquad \alpha_r = -\frac{1}{4r} + \pm O(e^{-\frac{r}{L}}) \qquad \alpha_z = \pm O(e^{-\frac{r}{L}}) \tag{6.6.3}$$

We have that since u is periodic and symmetric about $\frac{L}{2}$ it has a Fourier series of the following form:

$$u(r,z) = u(r,0) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}\left(z - \frac{L}{2}\right)\right) a_n(r)$$

We can work out $a_n(r)$ up to constants by using the harmonicity of u.

$$0 = \frac{1}{r}u_r + u_{rr} + u_{zz}$$
$$0 = \sum_{n=1}^{\infty}\left(\frac{\partial^2 a_n(r)}{\partial r^2}\cos\left(\frac{n\pi}{L}\left(z-\frac{L}{2}\right)\right) + \frac{1}{r}\frac{\partial a_n(r)}{\partial r}\cos\left(\frac{n\pi}{L}\left(z-\frac{L}{2}\right)\right) - a_n(r)\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi}{L}\left(z-\frac{L}{2}\right)\right)\right)$$

By the independence of the cosine terms we have the following.

$$0 = \frac{\partial^2 a_n(r)}{\partial r^2} - a_n(r)\frac{n^2\pi^2}{L^2} + \frac{1}{r}\frac{\partial a_n(r)}{\partial r}$$

Lets set $s = \frac{n\pi}{L}r$. Then we have that:

$$\partial_s = A\partial_r \implies 1 = A\frac{n\pi}{L} \implies \partial_s = \frac{L}{n\pi}\partial_r$$

Plugging this back in to the ODE above we have that:

$$0 = \frac{\partial^2 a_n(s)}{\partial s^2} - a_n(s) + \frac{1}{s}\frac{\partial a_n(s)}{\partial s}$$
$$0 = s^2\frac{\partial^2 a_n(s)}{\partial s^2} + s\frac{\partial a_n(s)}{\partial s} - s^2 a_n(s)$$

Thus $a_n(s) = A_n I_0(s) + B_n K_0(s)$ where I_0 is a modified Bessel function of the first kind of order 0 and K_0 is a modified Bessel function of the second kind of order 0 [1, p. 374]. Also A_n and B_n are constants. Thus $a_n(r) = A_n I_0(\frac{n\pi}{L}r) + B_n K_0(\frac{n\pi}{L}r)$.

Now we derive a bound on the absolute value of u to determine a bound for the Fourier coefficients $a_n(r)$.

We start our calculations with $u(r,0)$.

$$\begin{aligned}
u(r,z) &= \lim_{n\to\infty}\Big(\Sigma_{j=-n}^{n} U_{\Gamma_{2j}} + \log(n)\Big)\\
u(r,0) &= \lim_{n\to\infty}\Big(\Sigma_{j=-n}^{n}\log\Big(\sqrt{r^2+(2jL)^2}+(2jL)\Big) - \log\Big(\sqrt{r^2+((2j+1)L)^2}+((2j+1)L)\Big) + \log(n)\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=1}^{n}\Big(\log\Big(\sqrt{r^2+(2jL)^2}+(2jL)\Big) + \log\Big(\sqrt{r^2+(2jL)^2}-(2jL)\Big)\Big) + \log(r)\ldots\\
&\ldots - \Sigma_{j=1}^{n}\Big(\log\Big(\sqrt{r^2+((2j+1)L)^2}+((2j+1)L)\Big) + \log\Big(\sqrt{r^2+((2j-1)L)^2}-(2j-1)L\Big)\Big)\ldots\\
&\ldots - \log\Big(\sqrt{r^2+L^2}+L\Big) + \log(n) + \log(r)\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=1}^{n}\log(r^2) - \Sigma_{j=1}^{n}\log\Big(\sqrt{r^2+((2j+1)L)^2}+((2j+1)L)\Big)\ldots\\
&\ldots - \Sigma_{j=0}^{n-1}\log\Big(\sqrt{r^2+((2j+1)L)^2}-(2j+1)L\Big) - \log\Big(\sqrt{r^2+L^2}+L\Big) + \log(n) + \log(r)\Big)\\
&= \lim_{n\to\infty}\Big(\Sigma_{j=1}^{n}\log(r^2) - \Sigma_{j=1}^{n-1}\log(r^2) - \log\Big(\sqrt{r^2+L^2}+L\Big) - \log\Big(\sqrt{r^2+L^2}-L\Big)\ldots\\
&\ldots - \log\Big(\sqrt{r^2+((2n+1)L)^2}+(2n+1)L\Big) + \log(n) + \log(r)\Big)\\
&= \log(r) - \log(4L)
\end{aligned}$$

We are interested in a Taylor estimate for $u(r,z)$ in terms of the variable z. To do this we will use the first order Taylor estimate for the function $\Omega_n(r,z) = .\Sigma_{j=-n}^{n} U_{\Gamma_{2j}}$ and plug into the limit definition for u. Here ξ is in between 0 and z.

$$\begin{aligned}
u(r,z) &= \lim_{n\to\infty}\Big(\Omega_n(r,z) + \log(n)\Big)\\
&= \lim_{n\to\infty}\Big(\Omega_n(r,0) + z(\Omega_n)_z(r,\xi) + \log(n)\Big)\\
&= u(r,0) + z(\Omega_\infty)_z(r,\xi)
\end{aligned}$$

Where $(\Omega_\infty)_z(r,z) = \Sigma_{j=-\infty}^{\infty}\Big(-\frac{1}{\sqrt{r^2+(z-2jL)^2}} + \frac{1}{\sqrt{r^2+(z-(2j+1)L)^2}}\Big)$

Lemma 6.7. *We have that $(\Omega_\infty)_z(r,z)$ is from bounded from above and below;* $|(\Omega_\infty)_z(r,z)| < M(r) = O(1)$.

Proof. We wish to find a bound of $(\Omega_\infty)_z$ in terms of r. First consider the infinite product representation of the sine function [19, p. 268].

$$\frac{\sin(z)}{z} = \prod_{j=1}^{\infty}\Big(1-\frac{z}{j\pi}\Big)\Big(1+\frac{z}{j\pi}\Big)$$

We construct a summation similar to $(\Omega_\infty)_z(r,z)$ by playing around with the argument of the sine function and taking a derivative.

$$\begin{aligned}
\sin\Big(\frac{(ir-z)\pi}{2L}\Big)\sin\Big(\frac{(ir+z)\pi}{2L}\Big) &= -\frac{(r^2+z^2)\pi^2}{4L^2}\prod_{j=1}^{\infty}\Big(\frac{(ir-z)}{2Lj}+1\Big)\Big(\frac{(ir-z)}{2Lj}-1\Big)\Big(\frac{(ir+z)}{2Lj}+1\Big)\Big(\frac{(ir+z)}{2Lj}-1\Big)\\
&= -\frac{(r^2+z^2)\pi^2}{4L^2}\prod_{j=1}^{\infty}\Big(\frac{-r^2}{(2Lj)^2}-\Big(\frac{z}{2Lj}-1\Big)^2\Big)\Big(\frac{-r^2}{(2Lj)^2}-\Big(\frac{z}{2Lj}+1\Big)^2\Big)\\
&= -\frac{(r^2+z^2)\pi^2}{4L^2}\prod_{j=1}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)\\
\sin\Big(\frac{(ir-z)\pi}{2L}\Big)\sin\Big(\frac{(ir+z)\pi}{2L}\Big) &= -\frac{1}{4}\Big(e^{i\frac{(ir-z)\pi}{2L}}-e^{-i\frac{(ir-z)\pi}{2L}}\Big)\Big(e^{i\frac{(ir+z)\pi}{2L}}-e^{-i\frac{(ir+z)\pi}{2L}}\Big)\\
&= -\frac{1}{4}\Big(e^{-\frac{r\pi}{L}}+e^{\frac{r\pi}{L}}-e^{\frac{iz\pi}{L}}-e^{\frac{-iz\pi}{L}}\Big)\\
&= \frac{1}{2}\Big(-\cosh\Big(\frac{r\pi}{L}\Big)+\cos\Big(\frac{z\pi}{L}\Big)\Big)
\end{aligned}$$

We now take the z derivative of both sides.

$$\partial_z\Big(\frac{1}{2}\Big(-\cosh\Big(\frac{r\pi}{L}\Big)+\cos\Big(\frac{z\pi}{L}\Big)\Big)) = -\frac{\pi}{2L}\sin\Big(\frac{z\pi}{L}\Big)$$

$$-\frac{\pi}{2L}\sin\Big(\frac{z\pi}{L}\Big) = \partial_z\Big(-\frac{(r^2+z^2)\pi^2}{4L^2}\prod_{j=1}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)\Big)$$

$$= -\frac{2z\pi^2}{4L^2}\prod_{j=1}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)+\dots$$

$$\dots\sum_{k=1}^{\infty}-\frac{(r^2+z^2)\pi^2}{4L^2}(2(z-2kL)(r^2+(z+2kL)^2)+2(z+2kL)(r^2+(z-2kL)^2))\dots$$

$$\dots\cdot\prod_{j\neq k}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)$$

$$\frac{-\frac{\pi}{2L}\sin\big(\frac{z\pi}{L}\big)}{\frac{1}{2}\Big(-\cosh\big(\frac{r\pi}{L}\big)+\cos\big(\frac{z\pi}{L}\big)\Big)} = \frac{2z}{r^2+z^2}+\sum_{k=1}^{\infty}\Big(\frac{2(z-2kL)}{r^2+(z-2kL)^2}+\frac{2(z+2kL)}{r^2+(z+2kL)^2}\Big)$$

$$-\frac{\pi}{L}\frac{\sin\big(\frac{z\pi}{L}\big)}{-\cosh\big(\frac{r\pi}{L}\big)+\cos\big(\frac{z\pi}{L}\big)} = \frac{2z}{r^2+z^2}+\sum_{k=1}^{\infty}\Big(\frac{2(z-2kL)}{r^2+(z-2kL)^2}+\frac{2(z+2kL)}{r^2+(z+2kL)^2}\Big)$$

$$\frac{\pi}{L}\frac{\sin\big(\frac{z\pi}{L}\big)}{-\cosh\big(\frac{r\pi}{L}\big)-\cos\big(\frac{z\pi}{L}\big)} = \frac{2(z-L)}{r^2+(z-L)^2}+\sum_{k=1}^{\infty}\Big(\frac{2(z-(2k+1)L)}{r^2+(z-(2k+1)L)^2}+\frac{2(z+(2k-1)L)}{r^2+(z+(2k-1)L)^2}\Big)$$

We now take the r derivative of both sides.

$$\partial_r\left(\frac{1}{2}\left(-\cosh\left(\frac{r\pi}{L}\right)+\cos\left(\frac{z\pi}{L}\right)\right)\right)=-\frac{\pi}{2L}\sinh\left(\frac{r\pi}{L}\right)$$

$$-\frac{\pi}{2L}\sinh\left(\frac{r\pi}{L}\right)=\partial_r\left(-\frac{(r^2+z^2)\pi^2}{4L^2}\prod_{j=1}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)\right)$$

$$=-\frac{2r\pi^2}{4L^2}\prod_{j=1}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)+\ldots$$

$$\ldots\sum_{k=1}^{\infty}-\frac{(r^2+z^2)\pi^2}{4L^2}(2r(r^2+(z+2kL)^2)+2r(r^2+(z-2kL)^2))\ldots$$

$$\ldots\cdot\prod_{j\neq k}^{\infty}\frac{1}{(2Lj)^4}(r^2+(z-2jL)^2)(r^2+(z+2jL)^2)$$

$$-\frac{\pi}{L}\frac{\sinh\left(\frac{r\pi}{L}\right)}{-\cosh\left(\frac{r\pi}{L}\right)+\cos\left(\frac{z\pi}{L}\right)}=\frac{2r}{r^2+z^2}+\sum_{k=1}^{\infty}\left(\frac{2r}{r^2+(z-2kL)^2}+\frac{2r}{r^2+(z+2kL)^2}\right)$$

$$\frac{\pi}{L}\frac{\sinh\left(\frac{r\pi}{L}\right)}{\cosh\left(\frac{r\pi}{L}\right)+\cos\left(\frac{z\pi}{L}\right)}=\frac{2r}{r^2+(z-L)^2}+\sum_{k=1}^{\infty}\left(\frac{2r}{r^2+(z-(2k+1)L)^2}+\frac{2r}{r^2+(z+(2k-1)L)^2}\right)$$

Now we let the inequalities do the magic. Let us assume $-\frac{L}{2}\leq z\leq\frac{L}{2}$.

$$\begin{aligned}
\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z+2jL)^2}} &< \Sigma_{j=1}^{\infty}\frac{z+2jL+r}{\sqrt{r^2+(z+2jL)^2}}\frac{1}{\sqrt{r^2+(z+2jL)^2}}\\
&< \Sigma_{j=1}^{\infty}\frac{z+2jL}{r^2+(z+2jL)^2}+\Sigma_{j=1}^{\infty}\frac{r}{r^2+(z+2jL)^2}\\
\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z-2jL)^2}} &< \Sigma_{j=1}^{\infty}\frac{-(z-(2j-1)L)+r}{\sqrt{r^2+(z-(2j-1)L)^2}}\frac{1}{\sqrt{r^2+(z-2jL)^2}}\\
&< \Sigma_{j=1}^{\infty}\frac{-(z-(2j-1)L)+r}{r^2+(z-(2j-1)L)^2}\\
&< \Sigma_{j=1}^{\infty}\frac{(2j-1)L-z}{r^2+(z-(2j-1)L)^2}+\Sigma_{j=1}^{\infty}\frac{r}{r^2+(z-(2j-1)L)^2}\\
-\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z-(2j-1)L)^2}} &< \Sigma_{j=1}^{\infty}\frac{z-2jL}{\sqrt{r^2+(z-2jL)^2}}\frac{1}{\sqrt{r^2+(z-(2j-1)L)^2}}\\
&= \Sigma_{j=1}^{\infty}\frac{z-2jL}{r^2+(z-2jL)^2}\\
-\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z+(2j+1)L)^2}} &< -\Sigma_{j=1}^{\infty}\frac{z+(2j+1)L}{\sqrt{r^2+(z+(2j+1)L)^2}}\frac{1}{\sqrt{r^2+(z+(2j+1)L)^2}}\\
&< -\Sigma_{j=1}^{\infty}\frac{z+(2j+1)L}{r^2+(z+(2j+1)L)^2}
\end{aligned}$$

$$
\begin{aligned}
-(\Omega_\infty)_z(r,\xi) = {} & \frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+(z+L)^2}} + \Sigma_{j=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+2jL)^2}} + \frac{1}{\sqrt{r^2+(z-2jL)^2}}\right)\dots \\
& \dots - \Sigma_{j=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+(2j-1)L)^2}} + \frac{1}{\sqrt{r^2+(z-(2j+1)L)^2}}\right) \\
& < \Sigma_{j=1}^{\infty}\frac{z+2jL}{r^2+(z+2jL)^2} + \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z+2jL)^2} + \Sigma_{j=1}^{\infty}\frac{-(z-(2j-1)L)+r}{r^2+(z-(2j-1)L)^2}\dots \\
& \dots + \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z-(2j-1)L)^2} + \Sigma_{j=1}^{\infty}\frac{z-2jL}{r^2+(z-2jL)^2} - \Sigma_{j=1}^{\infty}\frac{z+(2j+1)L}{r^2+(z+(2j+1)L)^2}\dots \\
& \dots + \frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+(z+L)^2}} \\
& < -\frac{1}{\sqrt{r^2+(z+L)^2}} + \frac{\pi}{2L}\frac{\sinh\left(\frac{r\pi}{L}\right)}{\cosh\left(\frac{r\pi}{L}\right)-\cos\left(\frac{z\pi}{L}\right)} + \frac{\pi}{2L}\frac{\sinh\left(\frac{r\pi}{L}\right)}{\cosh\left(\frac{r\pi}{L}\right)+\cos\left(\frac{z\pi}{L}\right)}\dots \\
& - \frac{r}{r^2+(z+L)^2} + \frac{\pi}{2L}\frac{\sin\left(\frac{z\pi}{L}\right)}{\cosh\left(\frac{r\pi}{L}\right)-\cos\left(\frac{z\pi}{L}\right)} - \frac{\pi}{2L}\frac{\sin\left(\frac{z\pi}{L}\right)}{\cosh\left(\frac{r\pi}{L}\right)+\cos\left(\frac{z\pi}{L}\right)} + \frac{(z+L)}{r^2+(z+L)^2}\dots \\
& \dots + \frac{1}{\sqrt{r^2+z^2}} - \frac{r}{r^2+z^2} - \frac{z}{r^2+z^2}
\end{aligned}
$$

Now we provide inequalities going in the other direction

$$
\begin{aligned}
\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z+2jL)^2}} &> \Sigma_{j=1}^{\infty}\frac{z+2jL}{\sqrt{r^2+(z+2jL)^2}}\frac{1}{\sqrt{r^2+(z+2jL)^2}} \\
&> \Sigma_{j=1}^{\infty}\frac{z+2jL}{r^2+(z+2jL)^2} \\
\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z-2jL)^2}} &> \Sigma_{j=1}^{\infty}\frac{-(z-(2j+1)L)}{\sqrt{r^2+(z-(2j+1)L)^2}}\frac{1}{\sqrt{r^2+(z-2jL)^2}} \\
&> \Sigma_{j=1}^{\infty}\frac{-(z-(2j+1)L)}{r^2+(z-(2j+1)L)^2} \\
-\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z-(2j+1)L)^2}} &> \Sigma_{j=1}^{\infty}\frac{z-2jL-r}{\sqrt{r^2+(z-2jL)^2}}\frac{1}{\sqrt{r^2+(z-(2j+1)L)^2}} \\
&> \Sigma_{j=1}^{\infty}\frac{z-2jL}{r^2+(z-2jL)^2} - \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z-2jL)^2} \\
-\Sigma_{j=1}^{\infty}\frac{1}{\sqrt{r^2+(z+(2j-1)L)^2}} &> -\Sigma_{j=1}^{\infty}\frac{z+(2j-1)L+r}{\sqrt{r^2+(z+(2j-1)L)^2}}\frac{1}{\sqrt{r^2+(z+(2j-1)L)^2}} \\
&> -\Sigma_{j=1}^{\infty}\frac{z+(2j-1)L}{r^2+(z+(2j-1)L)^2} - \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z+(2j-1)L)^2}
\end{aligned}
$$

$$
\begin{aligned}
-(\Omega_\infty)_z(r,\xi) &= \frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+(z-L)^2}} + \Sigma_{j=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+2jL)^2}} + \frac{1}{\sqrt{r^2+(z-2jL)^2}}\right)\dots \\
&\dots - \Sigma_{j=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+(2j-1)L)^2}} + \frac{1}{\sqrt{r^2+(z-(2j+1)L)^2}}\right) \\
&> -\frac{1}{\sqrt{r^2+(z-L)^2}} + \Sigma_{j=1}^{\infty}\frac{z+2jL}{r^2+(z+2jL)^2} + \Sigma_{j=1}^{\infty}\frac{-(z-(2j+1)L)}{r^2+(z-(2j+1)L)^2}\dots \\
&\dots\ + \Sigma_{j=1}^{\infty}\frac{z-2jL}{r^2+(z-2jL)^2} - \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z-2jL)^2}\ \dots \\
&\dots - \Sigma_{j=1}^{\infty}\frac{z+(2j-1)L}{r^2+(z+(2j-1)L)^2} - \Sigma_{j=1}^{\infty}\frac{r}{r^2+(z+(2j-1)L)^2} + \frac{1}{\sqrt{r^2+z^2}} \\
&> -\frac{1}{\sqrt{r^2+(z-L)^2}} - \frac{\pi}{2L}\frac{\sinh(\frac{r\pi}{L})}{\cosh(\frac{r\pi}{L})-\cos(\frac{z\pi}{L})} - \frac{\pi}{2L}\frac{\sinh(\frac{r\pi}{L})}{\cosh(\frac{r\pi}{L})+\cos(\frac{z\pi}{L})}\dots \\
&\dots + \frac{\pi}{L}\frac{\sin(\frac{z\pi}{L})}{\cosh(\frac{r\pi}{L})-\cos(\frac{z\pi}{L})} - \frac{\pi}{L}\frac{\sin(\frac{z\pi}{L})}{\cosh(\frac{r\pi}{L})+\cos(\frac{z\pi}{L})} + \frac{1}{\sqrt{r^2+z^2}}\dots \\
&+ \frac{r}{r^2+z^2} + \frac{r}{r^2+(z-L)^2} - \frac{z}{r^2+z^2} + \frac{z-L}{r^2+(z-L)^2}
\end{aligned}
$$

Therefore we can conclude that $|(\Omega_\infty)_z(r,z)| < M(r) = O(1)$. We use the skew-symmetry about $\frac{L}{2}$ and the periodicity of $(\Omega_\infty)_z$ to extend this bound to all z.

Now we calculate the bound on $|a_n(r)|$ $\square$

$$
|a_n(r)| = \left|\int_{\frac{L}{2}}^{\frac{5L}{2}} \cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)u(r,z)dz\right|
$$

$$
\int_{\frac{L}{2}}^{\frac{5L}{2}} \cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)dz = \frac{L}{\pi n}\sin\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)\Bigg|_{\frac{L}{2}}^{\frac{5L}{2}} = 0
$$

Therefore we can add $-\log(r)+\log(4L)$ to u in the inequality for $|a_n(r)|$ without changing anything. Below we use lemma (6.7).

$$
\begin{aligned}
|a_n(r)| &= \left|\int_{\frac{L}{2}}^{\frac{5L}{2}} \cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)(u(r,z)-\log(r)+\log(4L))dz\right| \\
|a_n(r)| &\le \int_{\frac{L}{2}}^{\frac{5L}{2}} |\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)||(u(r,z)-\log(r)+\log(4L))|dz \\
|a_n(r)| &\le M(r)\int_{\frac{L}{2}}^{\frac{5L}{2}} |\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)|zdz \\
|a_n(r)| &\le M(r)\int_{\frac{L}{2}}^{\frac{5L}{2}} zdz = M(r)\left(\frac{z^2}{2}\right)\Bigg|_{\frac{L}{2}}^{\frac{5L}{2}} \\
|a_n(r)| &\le \frac{M(r)}{2}\left(\frac{5^2L^2}{2^2} - \frac{L^2}{2^2}\right) = 3M(r)L^2 =: N(r)
\end{aligned}
$$

We have that $a_n(r)$ is expressed in terms of a linear combination of zero order modified Bessel function of the first kind and the second kind: $a_n(r) = A_n I_0(\frac{nr}{L}) +$

$B_nK_0(\frac{nr}{L})$ for $n > 0$. Since for large r, I_0 has exponential growth and K_0 has exponential decay we can use the above inequality to conclude that $A_n = 0$. We can go further to produce an upper bound on the absolute value of B_n.

$$|a_n(r_0)| = |B_nK_0\left(\frac{nr_0}{L}\right)| \leq N(r_0)$$
$$|B_n| \leq N(r_0)\frac{1}{K_0\left(\frac{nr_0}{L}\right)}$$

Where r_0 is a constant that is suitably large. We now use the Fourier series to derive (6.6.1). Let M_1 be the limit of Fourier series minus $(\log(r) - \log(4L))$ as r goes to ∞.

$$M_1 = \Sigma_{n=1}^{\infty}\left(\cos\left(\frac{n\pi}{L}\left(z - \frac{L}{2}\right)\right)B_nK_0\left(\frac{nr}{L}\right)\right)$$
$$|M_1| \leq \Sigma_{n=1}^{\infty}N(r_0)\left(\frac{1}{K_0\left(\frac{nr_0}{L}\right)}K_0\left(\frac{nr}{L}\right)\right)$$

We have an upper bound $K_\nu(x)$ for large x and a lower bound for $K_0(x)$ [14, p. 1]. This upper bound is a consequence of the asymptotic form; $K_\nu \sim \sqrt{\frac{\pi}{2x}}e^{-x}$ [2, p. 618].

$$K_\nu(x) \leq \sqrt{\frac{\pi}{2}}e^{-x} \qquad K_0(x) \geq \sqrt{\frac{\pi}{2}}\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}e^{-x} \tag{6.7.1}$$

We have Gautschi's inequality for the gamma function at our disposal [27, p. 14].

$$\frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s} \implies \frac{1}{K_0(x)} < \sqrt{\frac{2}{\pi}}\sqrt{x+1}$$

Plugging this into limit and noting that r going to ∞ implies it is larger than r_0 gives us the following:

$$|M_1| \leq \Sigma_{n=1}^{\infty}N(r_0)\sum_{n=1}^{\infty}\sqrt{\frac{nr_0}{L}+1}e^{\frac{nr_0}{L}-\frac{nr}{L}}$$
$$|M_1| \leq -N(r_0)\sqrt{2r_0L}\sum_{n=1}^{\infty}-\frac{n}{L}e^{-\frac{n}{L}(r-r_0)}$$
$$|M_1| \leq -N(r_0)\sqrt{2Lr_0}\partial_r\left(\frac{e^{-\frac{1}{L}(r-r_0)}}{1-e^{-\frac{1}{L}(r-r_0)}}\right)$$

Of course we need to take derivatives of the geometric sum formula. Thus we have the following.

$$\partial_r\left(\frac{e^{-\frac{1}{L}(r-r_0)}}{1-e^{-\frac{1}{L}(r-r_0)}}\right) = \partial_r\left(-1+\frac{1}{1-e^{-\frac{1}{L}(r-r_0)}}\right)$$
$$= -\frac{1}{L}\frac{e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^2}$$
$$\partial_{rr}\left(\frac{e^{-\frac{1}{L}(r-r_0)}}{1-e^{-\frac{1}{L}(r-r_0)}}\right) = -\frac{1}{L}\left(-\frac{1}{L}\frac{e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^2}+e^{-\frac{1}{L}(r-r_0)}\frac{(-2)(-1)(\frac{1}{L})e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^3}\right)$$
$$= \frac{1}{L^2}\frac{e^{-\frac{1}{L}(r-r_0)}(1-e^{-\frac{1}{L}(r-r_0)})+2e^{-\frac{2}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^3}$$
$$= \frac{1}{L^2}e^{-\frac{1}{L}(r-r_0)}\left(\frac{1+e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^3}\right)$$

We can input this into $|M_1|$.

$$|M_1| \le N(r_0)\sqrt{\frac{2r_0}{L}}\frac{e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^2}$$

We have that $u(r,z) = \log(r) - \log(4L) \pm O\left(\frac{e^{-\frac{r}{L}}}{(1-e^{-\frac{r}{L}})^2}\right) = \log(r) - \log(4L) \pm O(e^{-\frac{r}{L}})$. We can ignore the r_0 since the function in question is decreasing. We will now compute the asymptotics for the derivatives of u.

$$u_r = \frac{1}{r} + \Sigma_{n=1}^{\infty}\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)B_n\left(K_o\left(\frac{nr}{L}\right)\right)_r$$

We have the following recursive formulae for $K_\nu(x)$ [13, p. 13].

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2(K_\nu(x))_x \qquad K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x)$$

When ν is an integer n we have that $K_n(x) = K_{-n}(x)$ [2, p. 614]. Now we plug this into u_r to obtain the following:

$$u_r = \frac{1}{r} + \Sigma_{n=1}^{\infty}\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)B_n\left(-\frac{n}{L}\frac{K_{-1}\left(\frac{nr}{L}\right)+K_1\left(\frac{nr}{L}\right)}{2}\right)$$
$$u_r = \frac{1}{r} - \Sigma_{n=1}^{\infty}\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)B_n\frac{n}{L}K_1\left(\frac{nr}{L}\right)$$
$$|M_2| = \Sigma_{n=1}^{\infty}\cos\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)B_n\frac{n}{L}K_1\left(\frac{nr}{L}\right)$$
$$|M_2| \le \Sigma_{n=1}^{\infty}N(r_0)\frac{1}{K_0\left(\frac{nr_0}{L}\right)}\frac{n}{L}K_1\left(\frac{nr}{L}\right)$$

Plugging in (6.7.1) we obtain the following.

$$|M_2| \leq \Sigma_{n=1}^{\infty} N(r_0)\frac{n}{L}\sqrt{\frac{\pi}{2}}e^{-\frac{nr}{L}}\sqrt{\frac{2}{\pi}}\frac{\Gamma(\frac{nr_0}{L}+1)}{\Gamma(\frac{nr_0}{L}+\frac{1}{2})}e^{\frac{nr_0}{L}}$$

$$|M_2| \leq \Sigma_{n=1}^{\infty} N(r_0)\frac{n}{L}\frac{\Gamma(\frac{nr_0}{L}+1)}{\Gamma(\frac{nr_0}{L}+\frac{1}{2})}e^{-\frac{n}{L}(r-r_0)}$$

$$|M_2| \leq \Sigma_{n=1}^{\infty} N(r_0)\frac{n}{L}\left(\frac{nr_0}{L}+1\right)^{\frac{1}{2}}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq \Sigma_{n=1}^{\infty} N(r_0)\frac{n}{L}\left(2\frac{nr_0}{L}\right)^{\frac{1}{2}}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq N(r_0)\frac{\sqrt{2r_0}}{L^{\frac{3}{2}}}\Sigma_{n=1}^{\infty} n^{\frac{3}{2}}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq N(r_0)\sqrt{2r_0L}\,\Sigma_{n=1}^{\infty}\frac{n^2}{L^2}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq N(r_0)\sqrt{2r_0L}\,\partial_{rr}\left(\frac{e^{-\frac{1}{L}(r-r_0)}}{1-e^{-\frac{1}{L}(r-r_0)}}\right)$$

We now plug in the second derivative calculated above into the limit.

$$|M_2| \leq N(r_0)\frac{\sqrt{2r_0}}{L^{\frac{3}{2}}}e^{-\frac{1}{L}(r-r_0)}\left(\frac{1+e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^3}\right)$$

Therefore we have that $u_r = \frac{1}{r} \pm O\left(e^{-\frac{r}{L}}\left(\frac{1+e^{-\frac{r}{L}}}{(1-e^{-\frac{r}{L}})^3}\right)\right) = \frac{1}{r} \pm O(e^{-\frac{r}{L}})$. We now move on to u_z.

$$u_z = -\frac{\pi}{L}\sum_{n=1}^{\infty} n\sin\left(\frac{\pi n}{L}\left(z-\frac{L}{2}\right)\right)B_nK_0\left(\frac{nr}{L}\right)$$

$$|u_z| \leq N(r_0)\Sigma_{n=1}^{\infty} n\frac{K_0\left(\frac{nr}{L}\right)}{K_0\left(\frac{nr_0}{L}\right)}$$

$$\leq \frac{\pi}{L}N(r_0)(L^2)\Sigma_{n=1}^{\infty}\sqrt{1+\frac{nr_0}{L}}\frac{n}{L^2}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq \frac{\pi}{L}N(r_0)(L^2)\sqrt{\frac{2r_0}{L}}\Sigma_{n=1}^{\infty}\frac{n^2}{L^2}e^{-\frac{n}{L}(r-r_0)}$$

$$\leq \frac{\pi}{L}N(r_0)\sqrt{2r_0}L^{\frac{3}{2}}\partial_{rr}\left(\frac{e^{\frac{r-r_0}{L}}}{1-e^{\frac{r-r_0}{L}}}\right)$$

$$\leq \frac{\pi}{L}N(r_0)\sqrt{\frac{2r_0}{L}}e^{-\frac{1}{L}(r-r_0)}\left(\frac{1+e^{-\frac{1}{L}(r-r_0)}}{(1-e^{-\frac{1}{L}(r-r_0)})^3}\right)$$

It follows that $u_z = \pm O\left(\frac{e^{\frac{-r}{L}}}{(1-e^{\frac{-r}{L}})^2}\right) = \pm O(e^{\frac{-r}{L}})$. We now wish to obtain similar asymptotics for v. First we have to calculate $v(r,0)$.

$$v(r,0) = u(r,-L) = u(r,\frac{L}{2} - \frac{3L}{2}) = u(r,2L) = u(r,0)$$

Since $v(r,z) = u(r,z-L)$ they share the same asymptotics.

$$v = \log(r) - \log(4L) \pm O(e^{-\frac{r}{L}}) \qquad v_r = \frac{1}{r} \pm O(e^{-\frac{r}{L}}) \qquad v_z = \pm O(e^{-\frac{r}{L}})$$

This allows us to calculate the asymptotics for the conformal exponent α.

$$\begin{aligned}
\alpha_r &= \frac{r}{4}\left(u_r^2 + v_r^2 + u_r v_r - \frac{2}{r}(u_r + v_r) - u_z^2 - v_z^2 - u_z v_z\right)\\
&= \frac{r}{4}\left(\frac{1}{r^2} \pm O\left(\frac{e^{-\frac{r}{L}}}{r}\right) + \frac{1}{r^2} + \frac{1}{r^2} - \frac{4}{r^2}\right)\\
&= -\frac{1}{4r} \pm O(e^{-\frac{r}{L}})
\end{aligned}$$

Since the big O notation respects integration we have that:

$$\alpha = -\frac{\log(r)}{4} \pm O(e^{-\frac{r}{L}})$$

$$\begin{aligned}
\alpha_z &= \frac{r}{4}\left(2u_r u_z + 2v_r v_z + u_r v_z + v_r u_z - \frac{2}{r}(u_z + v_z)\right)\\
&= \pm\frac{r}{4}\left(O\left(\frac{e^{-\frac{r}{L}}}{r}\right)\right)\\
&= \pm O(e^{-\frac{r}{L}})
\end{aligned}$$

Thus we have proved the original proposition.

6.7.1 Verification of Asymptotic Flatness of the Metric on the Base

Theorem 6.8. *We have that g_b is asymptotically flat up to first derivatives. That is there is asymptotic end, constants C and β, and coordinates ζ_1 and ζ_2 such that*

$$(g_b)_{ij} - C\delta_{ij} = O(\sqrt{\zeta_1^2+\zeta_2^2}^{-\beta}) \qquad \partial_{\zeta_1}(g_b)_{ij} = O(\sqrt{\zeta_1^2+\zeta_2^2}^{-\beta-1}) \qquad \partial_{\zeta_2}(g_b)_{ij} = O(\sqrt{\zeta_1^2+\zeta_2^2}^{-\beta-1})$$

Proof. In chapter 3 we made the assumption that the metric on the base is asymptotically flat when we constructed our Kaluza Klein asymptotically flat end. We will verify that it holds for example 2 using the asymptotic behaviour for α we derived in the previous subsection. Since the Euclidean metric and metric on the base are periodic we only need to check asymptotic flatness for $|z| \leq L$. We start off with the complex coordinate transformation $\zeta = \mu^{\frac{3}{4}}$ where

$\mu := r + iz$ and $\zeta := \zeta_1 + i\zeta_2$. There are 3 potential values that we can choose for ζ we will choose the one where if θ is the angle of μ then the angle of ζ is $\frac{3}{4}\theta$. We can now find the domain in which ζ lives in. Take the domain $D = \{(r, z) \,|\, r > L,\, |z| \le L\}$. We find the corresponding domain for the coordinates ζ_1 and ζ_2.

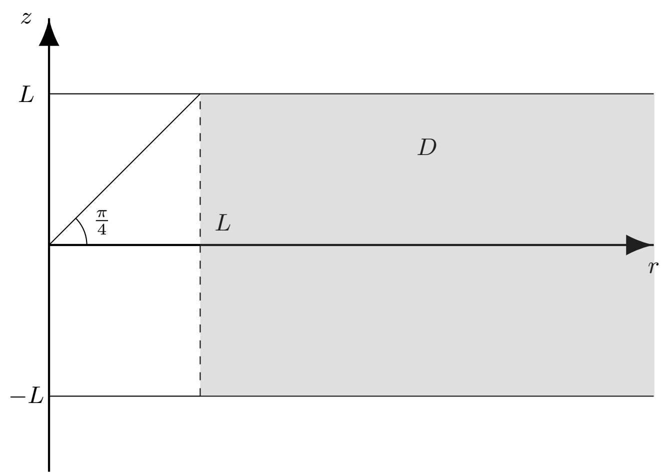

Figure 6.1:

We have the following inequalities for ζ_1 and ζ_2. Note that the angle, $\tan^{-1}(\frac{z}{r})$ satisfies $|\tan^{-1}(\frac{z}{r})| \le \frac{\pi}{4}$.

$$\begin{aligned}
\zeta_1 &= (r^2 + z^2)^{\frac{3}{8}} \cos\left(\frac{3}{4}\tan^{-1}(\frac{z}{r})\right) \\
|\zeta_1| &< (r^2 + z^2)^{\frac{3}{8}} \\
|\zeta_1| &> (r^2 + z^2)^{\frac{3}{8}} |\cos\left(\tan^{-1}(\frac{z}{r})\right)| > \frac{1}{\sqrt{2}}(r^2 + z^2)^{\frac{3}{8}} \\
\zeta_2 &= (r^2 + z^2)^{\frac{3}{8}} \sin\left(\frac{3}{4}\tan^{-1}(\frac{z}{r})\right) \\
|\zeta_2| &< (r^2 + z^2)^{\frac{3}{8}} |\sin\left(\tan^{-1}(\frac{z}{r})\right)| = (r^2 + z^2)^{\frac{3}{8}} \frac{|z|}{\sqrt{r^2 + z^2}} < (r^2 + z^2)^{-\frac{1}{8}} L
\end{aligned}$$

The domain for ζ_1 and ζ_2 is given roughly by the following diagram. .

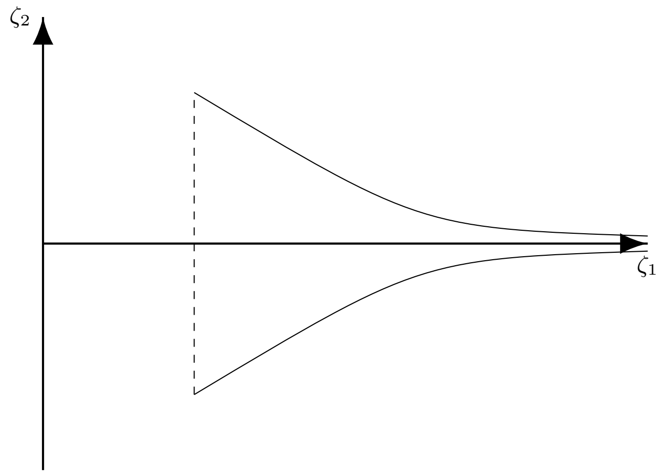

Figure 6.2:

This domain is diffeomorphic to $\mathbb{R}^2 - \overline{B}^2$ so we have no problems using it as an asymptotic end.

Next we consider the metric $dz^2 + dr^2$. This can be written as $d\mu d\overline{\mu}$. We can calculate this in terms of ζ.

$$d\mu = d(\zeta^{\frac{4}{3}}) = \frac{4}{3}\zeta^{\frac{1}{3}} d\zeta \qquad d\overline{\mu} = d(\overline{\zeta}^{\frac{4}{3}}) = \frac{4}{3}\overline{\zeta}^{\frac{1}{3}} d\overline{\zeta}$$
$$d\mu d\overline{\mu} = \frac{16}{9}||\zeta||^{\frac{2}{3}} d\overline{\zeta} = \frac{16}{9}||\zeta||^{\frac{2}{3}}(d\zeta_1^2 + d\zeta_2^2)$$

Now to check asymptotic flatness we need to find $\beta > 0$ and a constant C such that

i $\frac{16}{9}e^{2\alpha}||\zeta||^{\frac{2}{3}} - C = O(||\zeta||^{-\beta})$

ii $\partial_{\zeta_1}(e^{2\alpha}||\zeta||^{\frac{2}{3}}) = O(||\zeta||^{-\beta-1})$

iii $\partial_{\zeta_2}(e^{2\alpha}||\zeta||^{\frac{2}{3}}) = O(||\zeta||^{-\beta-1})$

We begin with (i). We have that $e^{2\alpha} = e^{-\frac{1}{2}\log(r) + O(e^{-\frac{r}{L}})} = \frac{1}{\sqrt{r}}(1 + O(e^{-\frac{r}{L}}))$. Since z is bounded $||\mu|| \to \infty$ implies that $r \to \infty$. We also have that as $||\zeta||$ goes to ∞ that $r = O(||\zeta||^{\frac{4}{3}})$.

$$\frac{16}{9}e^{2\alpha}||\zeta||^{\frac{2}{3}} = \frac{16}{9}\frac{1}{\sqrt{r}}(1 + O(e^{-\frac{r}{L}}))||\zeta||^{\frac{2}{3}} = \frac{16}{9}(||\zeta||^{\frac{2}{3}-\frac{2}{3}} + ||\zeta||^{\frac{2}{3}}O(e^{-\frac{||\zeta||^{\frac{4}{3}}}{L}}))$$
$$= \frac{16}{9} + O(||\zeta||^{\frac{2}{3}}e^{-\frac{||\zeta||^{\frac{4}{3}}}{L}})$$

Clearly $C = \frac{16}{9}$ and we have many choices of β. Let the calculations for the (ii) and (iii) inform our choice of β. Let us move on to (ii).

$$\partial_{\zeta_1} e^{2\alpha} = 2(\frac{\partial r}{\partial \zeta_1}\alpha_r + \frac{\partial z}{\partial \zeta_1}\alpha_z)e^{2\alpha}$$
$$\frac{\partial r}{\partial \zeta_1} = \frac{4}{3}Re(\zeta^{\frac{1}{3}}) = \frac{4}{3}(r^2+z^2)^{\frac{1}{8}}\cos\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$\frac{\partial z}{\partial \zeta_1} = \frac{4}{3}Im(\zeta^{\frac{1}{3}}) = \frac{4}{3}(r^2+z^2)^{\frac{1}{8}}\sin\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$\partial_{\zeta_1}||\zeta||^{\frac{2}{3}} = \frac{2}{3}||\zeta||^{-\frac{1}{3}}\frac{\zeta_1}{||\zeta||}$$
$$2(\frac{\partial r}{\partial \zeta_1}\alpha_r) + \frac{2}{3}||\zeta||^{-\frac{1}{3}}\frac{\zeta_1}{||\zeta||} = 2\frac{4}{3}(r^2+z^2)^{\frac{1}{8}}\cos\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)(-\frac{1}{4r}+O(e^{-\frac{r}{L}}))(r^2+z^2)^{\frac{1}{4}} + \frac{2}{3}(r^2+z^2)^{-\frac{1}{8}}\cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$\cos\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big) = \cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)\cos\Big(-\tan^{-1}(\frac{z}{r})\Big) + \sin\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)\sin\Big(\tan^{-1}(\frac{z}{r})\Big)$$
$$= \cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)\frac{r}{\sqrt{r^2+z^2}} + \sin\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)\frac{z}{\sqrt{r^2+z^2}}$$
$$2(\frac{\partial r}{\partial \zeta_1}\alpha_r) + \frac{2}{3}||\zeta||^{-\frac{1}{3}}\frac{\zeta_1}{||\zeta||} = -\frac{2}{3}(r^2+z^2)^{-\frac{1}{8}}\cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big) + \frac{2}{3}(r^2+z^2)^{-\frac{1}{8}}\cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$-\frac{2}{3}(r^2+z^2)^{-\frac{1}{8}}\cos\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)rO(e^{-\frac{r}{L}}) - \frac{2}{3}z(r^2+z^2)^{-\frac{1}{8}}\sin\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)(-\frac{1}{4r}+O(e^{-\frac{r}{L}}))$$
$$2(\frac{\partial r}{\partial \zeta_1}\alpha_r + \frac{\partial z}{\partial \zeta_1}\alpha_z) + \partial_{\zeta_1}||\zeta||^{\frac{2}{3}} = O((r^2+z^2)^{-\frac{9}{8}})$$
$$(2(\frac{\partial r}{\partial \zeta_1}\alpha_r + \frac{\partial z}{\partial \zeta_1}\alpha_z) + \partial_{\zeta_1}||\zeta||^{\frac{2}{3}})e^{2\alpha} = O((r^2+z^2)^{-\frac{11}{8}}) = O(||\zeta||^{-\frac{11}{6}})$$

So if we choose β to be $\frac{2}{3}$ then (ii) is satisfied. We now move on to (iii)

$$\partial_{\zeta_2} e^{2\alpha} = 2(\frac{\partial r}{\partial \zeta_2}\alpha_r + \frac{\partial z}{\partial \zeta_2}\alpha_z)e^{2\alpha}$$
$$\frac{\partial r}{\partial \zeta_2} = \frac{4}{3}Re(i\zeta^{\frac{1}{3}}) = -\frac{4}{3}(r^2+z^2)^{\frac{1}{8}}\sin\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$\frac{\partial z}{\partial \zeta_1} = \frac{4}{3}Im(i\zeta^{\frac{1}{3}}) = \frac{4}{3}(r^2+z^2)^{\frac{1}{8}}\cos\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)$$
$$\partial_{\zeta_2}||\zeta||^{\frac{2}{3}} = \frac{2}{3}||\zeta||^{-\frac{1}{3}}\frac{\zeta_2}{||\zeta||} = \frac{2}{3}||\zeta||^{-\frac{1}{3}}\sin\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)$$

The approach is slightly different; we have the following two inequalities.

$$|\sin\Big(\frac{1}{3}\tan^{-1}(\frac{z}{r})\Big)| < |\sin\Big(\tan^{-1}(\frac{z}{r})\Big)| = \frac{|z|}{\sqrt{r^2+z^2}}$$
$$|\sin\Big(\frac{4}{3}\tan^{-1}(\frac{z}{r})\Big)| < |\sin\Big(2\tan^{-1}(\frac{z}{r})\Big)| < 2\frac{|z|}{\sqrt{r^2+z^2}}$$

We plug these into the original equations and consider the biggest terms.

$$2(\frac{\partial r}{\partial \zeta_2}\alpha_r + \frac{\partial z}{\partial \zeta_2}\alpha_z) + \partial_{\zeta_2}||\zeta||^{\frac{2}{3}} = O((r^2+z^2)^{-\frac{9}{8}})$$
$$(2(\frac{\partial r}{\partial \zeta_2}\alpha_r + \frac{\partial z}{\partial \zeta_2}\alpha_z) + \partial_{\zeta_2}||\zeta||^{\frac{2}{3}})e^{2\alpha} = O((r^2+z^2)^{-\frac{11}{8}}) = O(||\zeta||^{-\frac{11}{6}})$$

Therefore we have achieved asymptotic flatness for the metric on the base. □

6.8.1 Behaviour of u, v and α near the z-axis

In this section we show that u, v and α have the following behaviour as r goes to 0.

Proposition 6.9. *Starting with case I,* $0 < z < \frac{L}{2}$*.*

$$u = 2\log(r) \pm O(1) \qquad u_r = \frac{2}{r} \pm O(r) \qquad u_z = \pm O(1)$$
$$v = \pm O(1) \qquad v_r = \pm O(r) \qquad v_z = \pm O(1)$$

For Case II, $-\frac{L}{2} < z < 0$*, simply interchange* u *and* v*. For both cases* $\alpha_z = \pm O(r)$ *and* $\alpha_r = \pm O(r)$*. Now consider case III the corner point* $z = 0$*.*

$$u = \log(r) \pm O(1) \qquad u_r = \frac{1}{r} \pm O(1) \qquad u_z = -\frac{1}{r} \pm O(1)$$
$$v = \log(r) \pm O(1) \qquad v_r = \frac{1}{r} \pm O(1) \qquad v_z = \frac{1}{r} \mp O(1)$$
$$\alpha = -\frac{1}{2}\log(r) \pm O(r) \qquad \alpha_r = -\frac{1}{2r} \pm O(1) \qquad \alpha_z = \pm O(1)$$

Proof. Let us start the proof of I. Consider the r dependence of u as we approach a point on the z-axis that is not a corner point. To start off lets pick our point a to be inbetween 0 and $\frac{L}{2}$.

$$u = \lim_{n\to\infty}\left(\Sigma_{j=-n}^{n} U_{\Gamma_{2j}} + \log n\right) \qquad v = \lim_{n\to\infty}\left(\Sigma_{j=-n}^{n} U_{\Gamma_{2j+1}} + \log n\right)$$

Using the behaviour of $U_{\Gamma_{2j}}$ near the z-axis and the fact that $\Gamma_{2j} = [2jL, (2j+1)L]$, we can conclude that U_{Γ_0} dominates in terms of r. The dominating term is $2\log(r)$. Therefore u behaves as $2\log(r)$ approaching from $z = a$. If we take another point b to be inbetween $-\frac{L}{2}$ and 0 then there is no term that dominates and we have that u approaches a constant as r approaches 0. Conversely v behaves as a constant approaching the z-axis at $z = a$ and behaves as $2\log(r)$ when we approach the z-axis at $z = b$. Therefore u and v satisfy the smoothness conditions at a and b. There is more direct calculation in the next section.

Now let's consider the corner point $z = 0$. From the symmetry and periodicity of u and v this is just as good as an arbitrary corner point. From the previous section we have that $u(r, 0) = \log(r) + C$, where C a constant. Therefore u behaves as $\log(r)$ when approaching the z-axis at $z = 0$. Additionally, $v(r, 0) = u(r, 0)$ so v has the same behaviour.

We now check the behaviour of α near the z-axis at a and b. Where $0 < a < L$ and $-L < b < 0$. We start with $u_z(r, a)$. Here $a \leq \xi \leq a + h$.

$$\begin{aligned}
u_z(r,a) &= \lim_{h\to 0}\left(\frac{u(r,h+a)-u(r,a)}{h}\right) = \lim_{h\to 0}\left(\frac{u(r,a)+h(\Omega_\infty)_z(r,\xi)-u(r,a)}{h}\right)\\
&= (\Omega_\infty)_z(r,a)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+(a-(2j+1)L)^2}}-\frac{1}{\sqrt{r^2+(a-2jL)^2}}\right)\\
&= \pm O(1)
\end{aligned}$$

We have that $v_z(r, a)$ is similar.

$$v_z(r,a) = -u_z(r,a)$$

However $u_r(r, a)$ has different behaviour. Note that $-(a-2jL) > 0$ when $j \geq 1$ and is less than 0 otherwise. Also $-(a - (2j + 1)L) > 0$ when $j \geq 0$ and less than 0 otherwise.

$$\begin{aligned}
u_r(r,a) &= \lim_{h\to 0}\left(\frac{u(r,h)-u(r,a)}{h}\right) = \lim_{h\to 0}\left(\frac{u(r,a)+h(\Omega_\infty)_r(r,\xi)-u(r,a)}{h}\right)\\
&= (\Omega_\infty)_r(r,a)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+(a-2jL)^2}-(a-2jL)}\frac{r}{\sqrt{r^2+(a-2jL)^2}}\cdots\right.\\
&\quad\left.\cdots-\frac{1}{\sqrt{r^2+(a-(2j+1)L)^2}-(a-2jL)}\frac{r}{\sqrt{r^2+(a-(2j+1)L)^2}}\right)\\
&= \sum_{j=1}^{\infty}\left(\frac{r}{2(a-2jL)^2}-\frac{r}{2(a-(2j+1)L)^2}\right)+\sum_{j=1}^{\infty}\left(\frac{1}{r}\frac{2|a-2jL|}{|a-2jL|}-\frac{1}{r}\frac{2|a-(2j+1)L|}{|a-(2j+1)L|}\right)\cdots\\
&\quad\cdots+\frac{1}{\sqrt{r^2+a^2}-a}\frac{r}{\sqrt{r^2+a^2}}-\frac{r}{2(a-L)^2}\\
&= \frac{2}{r}\pm O(r)
\end{aligned}$$

Now we examine $v_r(r, a)$. Note that $-(a-(2j-1)L) > 0$ when $j \geq 1$ and is negative otherwise.

$$\begin{aligned}
v_r(r,a) &= \lim_{h\to 0}\left(\frac{v(r,h)-v(r,a)}{h}\right) = \lim_{h\to 0}\left(\frac{v(r,a)+h(\Lambda_\infty)_r(r,\zeta)-v(r,a)}{h}\right)\\
&= (\Lambda_\infty)_r(r,a)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+(a-(2j-1)L)^2}-(a-(2j-1)L)}\frac{r}{\sqrt{r^2+(a-(2j-1)L)^2}}\cdots\right.\\
&\left.\cdots - \frac{1}{\sqrt{r^2+(a-2jL)^2}-(a-2jL)}\frac{r}{\sqrt{r^2+(a-2jL)^2}}\right)\\
&= \sum_{j=1}^{\infty}\left(\frac{r}{2(a-(2j-1)L)^2}-\frac{r}{2(a-2jL)^2}\right)+\sum_{j=0}^{\infty}\left(\frac{1}{r}\frac{2|a-(2j-1)L|}{|a-(2j-1)L|}-\frac{1}{r}\frac{2|a-2jL|}{|a-2jL|}\right)\\
&= \pm O(r)
\end{aligned}$$

Therefore we can plug these into the formulae for α_r and α_z using (6.1.1).

$$\begin{aligned}
\alpha_r(r,a) &= \frac{r}{4}\left(\frac{4}{r^2}-\frac{4}{r^2}\right)\pm O(r) = \pm O(r)\\
\alpha_z(r,a) &= \frac{r}{4}\left(\frac{4}{r}u_z(r,a)-\frac{4}{r}u_z(r,a)\right)\pm O(r) = \pm O(r)
\end{aligned}$$

So we have nothing unusual in the behaviour of $\alpha(r,a)$ near $(0,a)$. Now we perform the same calculations but for b. The calculations for $u_z(r,b)$ and $v_z(r,b)$ are the same as for a. We have that $-(b-2jL)>0$ when $j\geq 0$ and negative otherwise. We have that$-(b-(2j+1)L)>0$ when $j\geq 0$ and negative otherwise. And finally that $-(b-(2j-1)L)>0$ when $j\geq 1$. Armed with these inequalities we can tackle $u_r(r,b)$. Here $b\leq\xi\leq h+b$

$$\begin{aligned}
u_r(r,b) &= \lim_{h\to 0}\left(\frac{u(r,h)-u(r,b)}{h}\right) = \lim_{h\to 0}\left(\frac{u(r,b)+h(\Omega_\infty)_r(r,\xi)-u(r,b)}{h}\right)\\
&= (\Omega_\infty)_r(r,b)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+(b-2jL)^2}-(b-2jL)}\frac{r}{\sqrt{r^2+(b-2jL)^2}}\cdots\right.\\
&\left.\cdots - \frac{1}{\sqrt{r^2+(b-(2j+1)L)^2}-(b-(2j+1)L)}\frac{r}{\sqrt{r^2+(b-(2j+1)L)^2}}\right)\\
&= \sum_{j=0}^{\infty}\left(\frac{r}{2(b-2jL)^2}-\frac{r}{2(b-(2j+1)L)^2}\right)+\sum_{j=1}^{\infty}\left(\frac{1}{r}\frac{2|b+2jL|}{|b+2jL|}-\frac{1}{r}\frac{2|b+(2j-1)L|}{|b+(2j-1)L|}\right)\\
&= \pm O(r)
\end{aligned}$$

Next we tackle $v_r(r,b)$. Here $b \le \zeta \le h+b$.

$$
\begin{aligned}
v_r(r,b) &= \lim_{h\to 0}\left(\frac{v(r,h)-v(r,b)}{h}\right) = \lim_{h\to 0}\left(\frac{v(r,b)+h(\Lambda_\infty)_r(r,\zeta)-v(r,b)}{h}\right)\\
&= (\Lambda_\infty)_r(r,b)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+(b-(2j-1)L)^2}-(b-(2j-1)L)}\frac{r}{\sqrt{r^2+(b-(2j-1)L)^2}}\cdots\right.\\
&\left.\cdots-\frac{1}{\sqrt{r^2+(b-2jL)^2}-(b-2jL)}\frac{r}{\sqrt{r^2+(b-2jL)^2}}\right)\\
&= \sum_{j=1}^{\infty}\left(\frac{r}{2(b-(2j-1)L)^2}-\frac{r}{2(b-2jL)^2}\right)+\sum_{j=1}^{\infty}\left(\frac{1}{r}\frac{2|b-(2j-1)L|}{|b-(2j-1)L|}-\frac{1}{r}\frac{2|b+2jL|}{|b+2jL|}\right)\cdots\\
&\cdots+\frac{r}{\sqrt{r^2+(b+L)^2}-(b+L)}\frac{r}{\sqrt{r^2+(b+L)^2}}-\frac{1}{\sqrt{r^2+b^2}-b}\frac{r}{\sqrt{r^2+b^2}}\\
&= \frac{2}{r}\pm O(r)
\end{aligned}
$$

We can now plug these into $\alpha_r(r,b)$ and $\alpha_z(r,b)$ using (6.1.1).

$$
\begin{aligned}
\alpha_r(r,b) &= \frac{r}{4}\left(\frac{4}{r^2}-\frac{4}{r^2}\right)\pm O(r) = \pm O(r)\\
\alpha_z(r,b) &= \frac{r}{4}\left(\frac{4}{r}u_z(r,a)-\frac{4}{r}u_z(r,a)\right)\pm O(r) = \pm O(r)
\end{aligned}
$$

Moving on to case III, we wish to determine the behaviour of the first derivatives of u and v when $z=0$ and $r\to 0$. To do so we need to be sneaky with the expression for u and v. We can think about $u_z(r,0)$. Here we borrowed an expression for u from the previous section where $0\le \xi \le h$

$$
\begin{aligned}
u_z(r,0) &= \lim_{h\to 0}\left(\frac{u(r,h)-u(r,0)}{h}\right) = \lim_{h\to 0}\left(\frac{u(r,0)+h(\Omega_\infty)_z(r,\xi)-u(r,0)}{h}\right)\\
&= (\Omega_\infty)_z(r,0)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{1}{\sqrt{r^2+((2j+1)L)^2}}-\frac{1}{\sqrt{r^2+2jL^2}}\right)\\
&= -\frac{1}{r}+\frac{1}{L}+\sum_{j=1}^{\infty}\left(\frac{1}{(2j+1)L}-\frac{1}{2jL}\right)+\sum_{j=1}^{\infty}\left(\frac{1}{(2j-1)L}-\frac{1}{2jL}\right)\\
&= -\frac{1}{r}+\frac{1}{L}-\sum_{j=1}^{\infty}\frac{1}{2j(2j+1)L}+\sum_{j=1}^{\infty}\frac{1}{2j(2j-1)L}\\
u_z(r,0) &= -\frac{1}{r}+O(1)
\end{aligned}
$$

We can to the same procedure for v. Here $0\le\zeta\le h$.

$$
v_z(r,0) = u_z(r,-L) = -u_z(r,2L) = -u_z(r,0)
$$

We are comparing the above sums in the expressions for u_z and v_z to the zeta function to establish convergence. Next we move on u_r.

$$\begin{aligned}
u_r(r,0) &= \lim_{h\to 0}\left(\frac{u(r,h)-u(r,0)}{h}\right) = \lim_{h\to 0}\left(\frac{u(r,0)+h(\Omega_\infty)_r(r,\xi)-u(r,0)}{h}\right)\\
&= (\Omega_\infty)_r(r,0)\\
&= \sum_{j=-\infty}^{\infty}\left(\frac{r}{\sqrt{r^2+(2jL)^2}}\frac{1}{\sqrt{r^2+(2jL)^2}+2jL} - \frac{r}{\sqrt{r^2+((2j+1)L)^2}}\frac{1}{\sqrt{r^2+((2j+1)L)^2}+(2j+1)L}\right)\\
&= \frac{1}{r} + \frac{1}{\sqrt{r^2+L^2}+L}\frac{r}{L} + \sum_{j=1}^{\infty}\left(\frac{r}{(4jL)2jL} - \frac{r}{2(2j+1)^2L^2}\right) + \sum_{j=1}^{\infty}\left(\frac{2(2jL)}{r^2}\frac{r}{2jL} - \frac{2(2j+1)L}{r^2}\frac{r}{(2j+1)L}\right)\\
u_r(r,0) &= \frac{1}{r} + O(r)
\end{aligned}$$

Now we move on to v_r.

$$v_r(r,0) = u_r(r,-L) = u_r(r,2L) = u_r(r,0)$$

From these we can determine the behaviour of α_r and α_z near the corner point.

$$\begin{aligned}
\alpha_r &= \frac{r}{4}\left(u_r^2 + v_r^2 + u_r v_r - \frac{2}{r}(u_r+v_r) - u_z^2 - v_z^2 - u_z v_z\right)\\
\alpha_r(r,0) &= \frac{r}{4}\left(\frac{1}{r^2}+\frac{1}{r^2}+\frac{1}{r^2}-\frac{2}{r}(\frac{2}{r})-\frac{1}{r^2}-\frac{1}{r^2}+\frac{1}{r^2}\right) \pm O(1)\\
&= -\frac{1}{2r} \pm O(1)\\
\alpha_z &= \frac{r}{4}\left(2u_r u_z + 2v_r v_z + u_r v_z + v_r u_z - \frac{2}{r}(u_z+v_z)\right)\\
&= \frac{r}{4}\left(-2\frac{1}{r^2}+\frac{2}{r^2}+\frac{1}{r^2}-\frac{1}{r^2}-\frac{2}{r}(\frac{1}{r}-\frac{1}{r})\right) \pm O(1) = \pm O(1)
\end{aligned}$$

Now let's find what happens to the conformal factor $e^{2\alpha}$.

$$\alpha = -\frac{1}{2}\log(r) \pm O(r) \qquad e^{2\alpha} = O\left(\frac{1}{r}\right)$$

Note that if we consider the corner point at $z = L$ by symmetry $u_z \to -u_z$ and $v_z \to -v_z$ and u_r and v_r stay the same. However upon inspection of the formulae for the partials of α nothing changes. Therefore we get the same behaviour for α. Thus the behaviour for α is exactly what was predicted by the smoothness conditions. Therefore by periodicity α has no unusual behaviour when you approach the z-axis at anywhere but the corner points. □

6.9.1 Regularity

To rule out conical singularities when approaching the z-axis we must make sure the angle deficit on both the $(1,0)$ and $(0,1)$ rods are 0. These are the constants b_1 and b_2 which correspond to the $(1,0)$ rod and to the $(0,1)$ rod respectively. They are given by the following.

$$b_1 = \lim_{r\to 0}\bigg(\log(r) + \alpha - \frac{1}{2}u\bigg) \qquad b_2 = \lim_{r\to 0}\bigg(\log(r) + \alpha - \frac{1}{2}v\bigg)$$

Let $0 < a < \frac{L}{2}$. It will be shown that we can add a constant to both u and v to achieve $b_1 = b_2 = 0$. To aid with this calculation we want to know if $\overline{u} = u(r,z) - 2\log(r)$ is bounded when $0 \le r \le r_1$, where r_1 is a small constant. We know that it is convergent for $0 < r$ since both u and $\log(r)$ are convergent for those r. Thus we only need to check convergence for $r = 0$.

$$u(r,z) = \sum_{j=1}^{\infty}\Bigg(\log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} - \frac{z}{jL} + 1} + 1 - \frac{z}{2jL}\Bigg) - \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j+1)L)^2} - \frac{2z}{(2j+1)L} + 1} + 1 - \frac{z}{(2j+1)L}\Bigg)\dots$$

$$\dots - \log\Bigg(\sqrt{\frac{r^2+z^2}{(2jL)^2} + \frac{z}{jL} + 1} + 1 + \frac{z}{2jL}\Bigg) + \log\Bigg(\sqrt{\frac{r^2+z^2}{((2j-1)L)^2} + \frac{2z}{(2j-1)L} + 1} + 1 + \frac{z}{(2j-1)L}\Bigg)\Bigg)\dots$$

$$\dots + \log\Big(\sqrt{r^2+z^2} - z\Big) - \log\Big(\sqrt{r^2+(z-L)^2} - (z-L)\Big)\Big)$$

$$\lim_{r\to 0} u(r,a) = \sum_{j=1}^{\infty}\Bigg(\log\Big(2(1 - \frac{a}{2jL})\Big) - \log\Big(2(1 - \frac{a}{(2j+1)L})\Big) - \log\Big(2(1 + \frac{a}{2jL})\Big) + \log\Big(2(1 + \frac{a}{(2j-1)L})\Big)\Bigg)\dots$$

$$\dots + \lim_{r\to 0}(2\log(r)) + \log(2a) - \log(2(L-a))$$

$$\lim_{r\to 0}(u(r,a) - 2\log(r)) = \sum_{j=1}^{\infty}\Bigg(\log\Big(1 - \frac{a}{2jL}\Big) - \log\Big(1 - \frac{a}{(2j+1)L}\Big) - \log\Big(1 + \frac{a}{2jL}\Big) + \log\Big(1 + \frac{a}{(2j-1)L}\Big)\Bigg)\dots$$

$$\dots + \log(a) - \log(L-a)$$

As before we do a limit comparison test; comparing the infinite sum above to $\zeta(2)$.

$$\begin{aligned}
N_1 &= \lim_{j\to\infty}\Bigg(\log\Big(1 - \frac{a}{2jL}\Big) - \log\Big(1 - \frac{a}{(2j+1)L}\Big)\Bigg) j^2 \\
&= \lim_{k\to 0}\Bigg(\log\Big(1 - \frac{ak}{2L}\Big) - \log\Big(1 - \frac{ak}{(2+k)L}\Big)\Bigg)\frac{1}{k^2} \\
&= \lim_{k\to 0}\Bigg(\Bigg(- \frac{\frac{a}{2L}}{1 - \frac{ka}{2L}} - \Bigg(\frac{\frac{a}{(2+k)L} + \frac{aj}{(2+k)^2 L}}{1 - \frac{ka}{2L}}\Bigg)\frac{1}{2k}\Bigg) \\
&= \lim_{k\to 0}\Bigg(\Bigg(- \frac{a}{2L} + \frac{a}{(2+k)L}\Bigg)\frac{1}{2k}\Bigg) \\
&= -\frac{a}{4L}
\end{aligned}$$

$$
\begin{aligned}
N_2 &= \lim_{j\to\infty}\left(\log\left(1+\frac{a}{2jL}\right)-\log\left(1+\frac{a}{(2j-1)L}\right)\right)j^2 \\
&= \lim_{k\to 0}\left(\log\left(1+\frac{ak}{2L}\right)-\log\left(1+\frac{ak}{(2-k)L}\right)\right)\frac{1}{k^2} \\
&= \lim_{k\to 0}\left(\left(\frac{\frac{a}{2L}}{1+\frac{ka}{2L}}-\left(\frac{\frac{a}{(2-k)L}+\frac{aj}{(2-k)^2L}}{1-\frac{ka}{2L}}\right)\frac{1}{2k}\right)\right. \\
&= \lim_{k\to 0}\left(\left(+\frac{a}{2L}-\frac{a}{(2-k)L}\right)\frac{1}{2k}\right) \\
&= -\frac{a}{4L}
\end{aligned}
$$

We plug $u = 2\log(r) + \overline{u}$ into α_r and α_z.

$$
\begin{aligned}
\alpha_r &= \frac{r}{4}\left(\left(\frac{2}{r}+\overline{u}_r\right)^2+v_r^2+\left(\frac{2}{r}+\overline{u}_r\right)v_r-\frac{2}{r}\left(\frac{2}{r}+\overline{u}_r+v_r\right)-\overline{u}_z^2-v_z^2-\overline{u}_zv_z\right) \\
&= \frac{r}{4}\overline{u}_r^2+\frac{r}{4}v_r^2+\frac{2}{4}\overline{u}_r-\frac{r}{4}(\overline{u}_z^2+v_z^2+\overline{u}_zv_z) \\
\alpha_z &= \frac{r}{4}\left(2\left(\frac{2}{r}+\overline{u}_r\right)\overline{u}_z+2v_rv_z+\left(\frac{2}{r}+\overline{u}_r\right)v_z+v_r\overline{u}_z-\frac{2}{r}(\overline{u}_z+v_z)\right) \\
&= \frac{1}{2}\overline{u}_z+\frac{r}{4}(2\overline{u}_r\overline{u}_z+2v_rv_z+\overline{u}_rv_z+v_r\overline{u}_z)
\end{aligned}
$$

Using the behaviour of u and v near the z-axis we make the following deduction.

$$
\begin{aligned}
\alpha_r &= \frac{1}{2}\overline{u}_r + O(r) \\
\alpha_z &= \frac{1}{2}\overline{u}_z + O(r)
\end{aligned}
$$

Thus it follows that $\alpha = \frac{1}{2}\overline{u} + C + O(r^2)$. Therefore if we add $2C$ to u we achieve the vanishing of b_1.

$$
b_1 = \lim_{r\to 0}\left(\log(r)+\frac{1}{2}\overline{u}+C+O(r^2)-\frac{1}{2}(\overline{u}+2\log(r)+2C)\right) = 0
$$

For b_2 let $0 > b > -\frac{L}{2}$. We know that $v(r,b) = u(r, b-L) = \overline{u(r,b-L)} + 2\log(r)$. It is clear that there is an a which corresponds to b. Since $0 < b - L + 2L < \frac{L}{2}$. Also $\alpha(r, \frac{L}{2} - b - \frac{L}{2}) = \alpha(r, b+L) = \alpha(r, b-L)$. Therefore the b_2 is satisfied if we add $2C$ to v since the limit expression for b_2 is equal to the limit expression of b_1 for some a. Therefore regularity is established.

6.9.2 Topology of the Solution

There are two relevant topologies for example 2. We are interested in the topology on the time slice of the 5-dimensional manifold which has a coordinate system inolving (x_1, y_1, x_2, y_2) coordinates from the smoothness chapter. The first topology is the slice topology which exists in a strip along the z-axis.

Here periodicity in z of the spacetime is introduced under an equivalence relation where different fundamental domains are used to create different topologies. We are only concerned about a strip since the slice at point is local in nature. The second topology is the topology for the entire Domain of Outer Communication (DOC) and encompasses the entire orbit space.

The fundamental domain of the slice topology is shown below where the horizontal axis is the z-axis and the rectangle goes from $-\frac{L}{2}$ to $\frac{3L}{2}$.

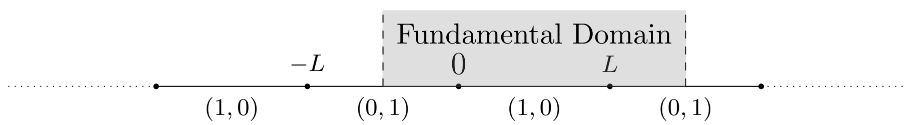

Figure 6.3:

Consider the following equivalence relation in z where $z \sim z + 2L$. Then we identify points at either end of the fundamental domain. We have two rod structures present and we can draw a picture of what this looks like in terms of the (r_1, r_2) coordinates.

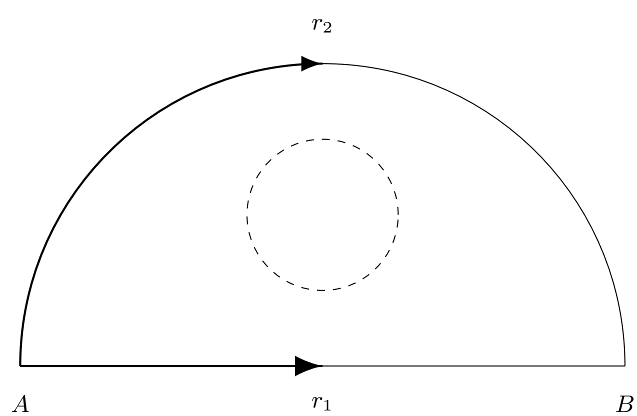

Figure 6.4:

Here A and B correspond to the corner points. Note that moving along the r_1 axis corresponds to moving along the $(0,1)$ rod since $r_2 = 0$. Similarly, moving along the r_2 axis corresponds to moving along the $(0,1)$ rod since $r_1 = 0$. If we think about what's happening upstairs in (x_1, y_1, x_2, y_2) we have that $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 0$ at the corners. We can characterize these coordinates by $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1 - t^2$ where $t = 1$ at A, $t = -1$ at B. We consider t varying across the diagram. Thus the enclosed region is homeomorphic to S^4. However because we only considered a strip along the z-axis we are missing a 2-dimensional ball in the (r_1, r_2) diagram. Since neither r_1 nor r_2 is 0 in this ball, we have that upstairs we can use the coordinates $(r_1, r_2, \varphi_1, \varphi_2)$. Thus upstairs the ball becomes $B^2 \times T^2$, where B^2 is the 2-dimensional ball. Therefore the fundamental region is really homeomorphic to $S^4 - (B^2 \times T^2)$.

Now the question becomes what happens when we use a different equivalence relation in z, say $z \sim z + 2kL$ where k is an integer greater than 1. How does the slice topology change? Well every time we increase k we add 2 more corners. It useful to understand the diagram with 4 corners.

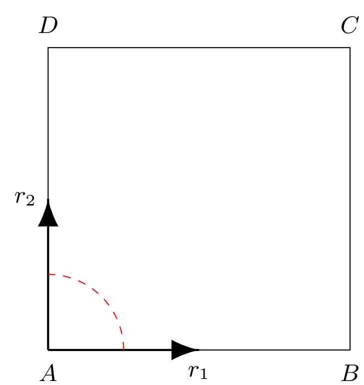

Figure 6.5:

Here A, B, C and D are corner points and $r_1 = r_2 = 0$ at these corner points. We characterize the coordinates upstairs by $x_1^2 + y_1^2 + t^2 = 1$ and $x_2^2 + y_2^2 + u^2 = 1$, where $u = t = 1$ at A, $u = -t = 1$ at B, $-u = t = 1$ at C and $u = t = -1$ at D. We consider t and u varying across the diagram. Therefore the entire diagram is homeomorphic to $S^2 \times S^2$. Now we excise the A corner by deleting the region inside the red arc. This arc has equation $r_1^2 + r_2^2 = \Lambda^2$ where Λ is a constant. At the level of the slice this arc corresponds to $x_1^2 + y_1^2 + x_2^2 + y_2^2 = \Lambda^2$. Thus the arc corresponds to S^3. If we took out a similar region from the diagram with two corners we would again have a boundary that is S^3. Therefore we can glue the two diagrams together along their S^3 cuts obtaining a connected sum with 4 corners. The resulting space is $S^4 \# S^2 \times S^2 - (B^2 \times T^2)$. This corresponds to the slice topology where $k = 2$. Note that the connected sum with S^n results in the identity thus we have that $(S^4 \# S^2 \times S^2) - (B^2 \times T^2) = (S^2 \times S^2) - (B^2 \times T^2)$. For the slice topology for arbitrary k, we keep taking connected sum with $S^2 \times S^2$. Thus we can write the time slice under the equivalence relation, $M_4/\sim$, for arbitrary k as:

$$M_4/\sim \quad = \left\{ \begin{array}{ll} S^4 - (B^2 \times T^2), & k = 1 \\ (\#^{k-1} S^2 \times S^2) - (B^2 \times T^2) & k \geq 2 \end{array} \right\}] \tag{6.9.1}$$

Next we examine the topology of the entire DOC. To do so we divide the rz half plane in to regions A_i.

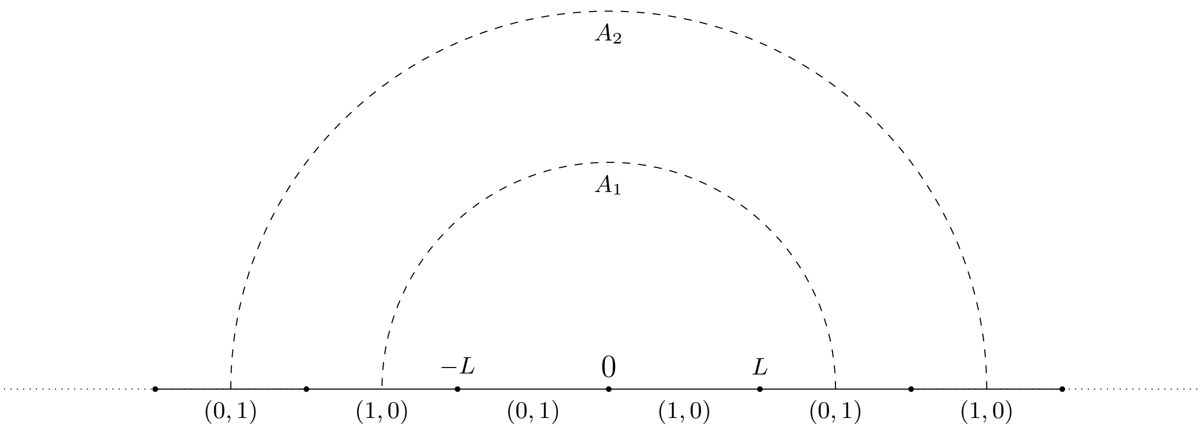

Figure 6.6:

We can picture A_1 in terms of (r_1, r_2) coordinates. It has 3 corners and thus can be drawn in terms of the $S^2 \times S^2$ diagram but with a corner excised breaking the periodicity.

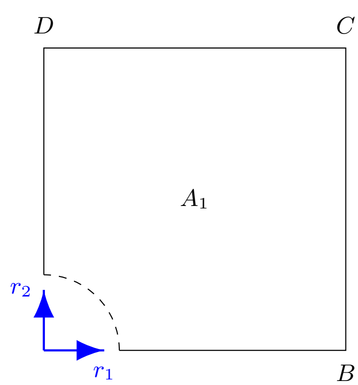

Figure 6.7:

The excised part of the diagram corresponds to a part of a ball centered at the origin in (r_1, r_2) coordinates. Upstairs this corresponds to a 4-dimensional ball,

B_1^4 centered, around the origin with coordinates (x_1, y_1, x_2, y_2). The boundary of B_1^4 is homemorphic to S^3. We have that A_1 is homeomorphic to $S^2 \times S^2 - B_1^4$. We can picture A_i where $i \geq 2$ by a similar diagram this time with two excised regions resulting in only two corners.

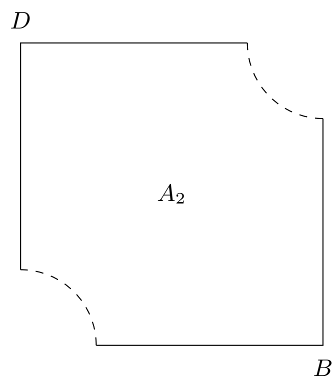

Figure 6.8:

Thus A_i has two balls removed, this means it has two S^3 boundaries. Thus if we connect sum A_1 and A_2 along the S^3 boundary of A_1 and one of the S^3 boundaries of A_2 we end with $(S^2 \times S^2)\#(S^2 \times S^2) - B_2^4$. Where B_2^4 is a different 4-ball. If we continue this process of infinite connected sums we obtain (with the convention that when we reach infinity there is no missing ball) that the whole orbit space under quotient of the T^2 action is homeomorphic to $\#^\infty S^2 \times S^2$.

Chapter 7

A Singular Solution with Non-Constant Twist Potentials

7.1 The Solution

Proposition 7.2. *We have a solution whose metric on the fibre is given by* $H_{ij} = g(\partial_{\varphi_{i-1}}, \partial_{\varphi_{j-1}})$ *which has non-constant twist potentials, which satisfies the smoothness conditions for a* $(1,0)$ *rod spanning the* z*-axis, whose lower right 2x2 block matrix is positive definite away from the* z*-axis but is missing a point on the* z*-axis. The non-zero components of the metric on the fibre are stated below where* h *satisfies* $h_{rr} + \frac{1}{r}h_r + h_{zz} = 0$*.*

$$
\begin{aligned}
H_{11} &= -e^{\frac{(\sigma+\tau)h}{2}}\left(\cosh\left(\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right) + \frac{1}{\sqrt{1-\eta^2}}\sinh\left(\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right)\right)\\
H_{33} &= e^{\frac{(\sigma+\tau)h}{2}}\left(\cosh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right) - \frac{1}{\sqrt{1-\eta^2}}\sinh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right)\right)\\
H_{13} &= \frac{\eta}{\sqrt{1-\eta^2}}e^{\frac{\sigma+\tau}{2}h}\sinh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right)\\
H_{22} &= r^2 e^{-(\sigma+\tau)h}
\end{aligned}
$$

Proof. In the equation above σ, τ and η are constants. An appropriate example of h would be $-\frac{1}{\sqrt{r^2+z^2}}$. This solution is not defined at $(0,0)$. This solution obeys the smoothness condition when there is only a single $(1,0)$ rod that stretches across the z-axis. To see this with the example function we know that it is a smooth function of r^2 and z away from the singular point. We have that H_{22} behaves as $O(r^2)$ as r goes to 0 away from the singular point . The fact that $H_{23} = H_{12} = 0$ agrees with the smoothness conditions. We have the following behaviour at the singularity for the components of H. We will assume $\tau - \sigma > 0$,

$\sigma > 0$ and $\tau > 0$. We have that $h \to -\infty$. So in the exponentials you either get ∞ or 0 depending on the sign of the exponent.

$$\begin{aligned} \frac{(\sigma+\tau)}{2} + \frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2} &> 0 \\ \frac{(\sigma+\tau)}{2} - \frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2} &> 0 \\ -(\sigma+\tau) &< 0 \end{aligned}$$

Therefore H_{11}, H_{13}, and H_{33} all go to zero as you approach the singularity. For H_{22} the exponent term overpowers the r^2 term so it blows up at the singularity.

Furthermore we require that $f = H_{22}H_{33} > 0$ away from the z-axis. For this we must check that $H_{33} > 0$. We can achieve by assuming that $\tau - \sigma > 0$ and requiring that $h < 0$.

$$\begin{aligned} H_{33} &= \cosh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right) - \frac{1}{\sqrt{1-\eta^2}}\sinh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right) \\ &= \frac{1}{2}\left(e^{\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h} + e^{-\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h} - \frac{1}{\sqrt{1-\eta^2}}\left(e^{\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h} - e^{-\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h}\right)\right) \\ &= \frac{1}{2}\left(e^{-\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h}\left(1 + \frac{1}{\sqrt{1-\eta^2}}\right) - e^{\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h}\left(\frac{1}{\sqrt{1-\eta^2}} - 1\right)\right) \end{aligned}$$

Since we know that $\left|1 + \frac{1}{\sqrt{1-\eta^2}}\right| < \left|\frac{1}{\sqrt{1-\eta^2}} - 1\right|$ and $e^{-\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h} > e^{\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h}$ we have that $H_{33} > 0$.

□

We can verify that this is indeed a solution to the harmonic map equations. We will start off by finding the determinant of H. We will use the shorthand $\theta = \left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)$.

$$\begin{aligned} \det H &= H_{22}(H_{11}H_{33} - H_{13}^2) \\ &= r^2 e^{-(\sigma+\tau)h} e^{\frac{\sigma+\tau}{2}h} e^{\frac{\sigma+\tau}{2}h}\Bigg(- \left(\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\left(\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\ldots \\ &\ldots - \frac{\eta^2}{1-\eta^2}\sinh^2(\theta h))\Bigg) \\ &= r^2\left(- \cosh^2(\theta h) + \frac{1}{1-\eta^2}\sinh^2(\theta h) - \frac{\eta^2}{1-\eta^2}\sinh^2(\theta h))\right) \\ &= r^2(\sinh^2(\theta h) - \cosh^2(\theta h)) = -r^2 \end{aligned}$$

We can now work out the components of H^{-1} using the adjoint of H.

$$H^{-1} = \frac{1}{\det H}\begin{pmatrix} H_{22}H_{33} & 0 & -H_{22}H_{13} \\ 0 & H_{11}H_{33} - H_{13}^2 & 0 \\ -H_{22}H_{13} & 0 & H_{11}H_{22} \end{pmatrix}$$

$$H^{-1} = e^{-(\sigma+\tau)h}\begin{pmatrix} -H_{33} & 0 & H_{13} \\ 0 & e^{2(\sigma+\tau)h}r^{-2} & 0 \\ H_{13} & 0 & -H_{11} \end{pmatrix}$$

We now calculate the r and z derivatives of the components of H.

$$\begin{aligned}
(H_{11})_r &= -h_r e^{\frac{(\sigma+\tau)h}{2}}\left(\left(\frac{\sigma+\tau}{2} + \frac{\tau-\sigma}{2}\frac{\sqrt{1-\eta^2}}{\sqrt{1-\eta^2}}\right)\cosh(\theta h) + \left(\frac{\sigma+\tau}{2}\frac{1}{\sqrt{1-\eta^2}} + \frac{\tau-\sigma}{2}\sqrt{1-\eta^2}\sinh(\theta h)\right)\right) \\
&= -h_r e^{\frac{(\sigma+\tau)h}{2}}\left(\tau\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\left(\tau - \frac{\tau-\sigma}{2}\eta^2\right)\sinh(\theta h)\right) \\
(H_{33})_r &= h_r e^{\frac{(\sigma+\tau)h}{2}}\left(\left(\frac{\sigma+\tau}{2} - \frac{\tau-\sigma}{2}\frac{\sqrt{1-\eta^2}}{\sqrt{1-\eta^2}}\right)\cosh(\theta h) + \left(-\frac{\sigma+\tau}{2}\frac{1}{\sqrt{1-\eta^2}} + \frac{\tau-\sigma}{2}\sqrt{1-\eta^2}\right)\sinh(\theta h)\right) \\
&= h_r e^{\frac{(\sigma+\tau)h}{2}}\left(\sigma\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\left(\sigma + \left(\frac{\tau-\sigma}{2}\right)\eta^2\right)\sinh(\theta h)\right) \\
(H_{13})_r &= h_r e^{\frac{(\sigma+\tau)h}{2}}\frac{\eta}{\sqrt{1-\eta^2}}\left(\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h) + \left(\left(\frac{\tau-\sigma}{2}\sqrt{1-\eta^2}\right)\cosh(\theta h)\right)\right. \\
&= h_r e^{\frac{(\sigma+\tau)h}{2}}\left(\frac{(\tau-\sigma)\eta}{2}\cosh(\theta h) + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h)\right)
\end{aligned}$$

$$\begin{aligned}
(H_{22})_r &= \left(2r + r^2(-(\sigma+\tau)h_r)\right)e^{-(\sigma+\tau)h} = 2re^{-(\sigma+\tau)h} - (\sigma+\tau)r^2 h_r e^{-(\sigma+\tau)h} \\
(H_{11})_z &= -h_z e^{\frac{(\sigma+\tau)h}{2}}\left(\tau\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\left(\tau - \frac{\tau-\sigma}{2}\eta^2\right)\sinh(\theta h)\right) \\
(H_{33})_z &= h_z e^{\frac{(\sigma+\tau)h}{2}}\left(\sigma\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\left(\sigma + \left(\frac{\tau-\sigma}{2}\right)\eta^2\right)\sinh(\theta h)\right) \\
(H_{13})_z &= h_z e^{\frac{(\sigma+\tau)h}{2}}\left(\frac{(\tau-\sigma)\eta}{2}\cosh(\theta h) + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h)\right) \\
(H_{22})_z &= -(\sigma+\tau)r^2 h_z e^{-(\sigma+\tau)h}
\end{aligned}$$

We now calculate the components of $H^{-1}H_r$ and $H^{-1}H_z$.

$$H^{-1}H_r = e^{-(\sigma+\tau)h}\begin{pmatrix} -H_{33} & 0 & H_{13} \\ 0 & e^{2(\sigma+\tau)h}r^{-2} & 0 \\ H_{13} & 0 & -H_{11} \end{pmatrix}\begin{pmatrix} (H_{11})_r & 0 & (H_{13})_r \\ 0 & (H_{22})_r & 0 \\ (H_{13})_r & 0 & (H_{33})_r \end{pmatrix}$$

We'll start with the 11 component.

$$
\begin{aligned}
(H^{-1}H_r)_{11} &= e^{-(\sigma+\tau)h}(-H_{33}(H_{11})_r + H_{13}(H_{13})_r) \\
-e^{-(\sigma+\tau)h}H_{33}(H_{11})_r &= e^{\frac{(\sigma+\tau)h}{2}}e^{\frac{(\sigma+\tau)h}{2}}e^{-(\sigma+\tau)h}h_r\left(\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\dots \\
&\dots\left(\tau\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\left(\tau - \frac{\tau-\sigma}{2}\eta^2\right)\sinh(\theta h)\right) \\
&= h_r\left(\tau\cosh^2(\theta h) + \left(\frac{\tau}{\sqrt{1-\eta^2}} - \frac{\tau}{\sqrt{1-\eta^2}}\dots\right.\right. \\
&\left.\left.\dots - \left(\frac{\tau-\sigma}{2}\right)\frac{\eta^2}{\sqrt{1-\eta^2}}\right)\sinh(\theta h)\cosh(\theta h) - \frac{1}{1-\eta^2}\left(\tau - \frac{(\tau-\sigma)\eta^2}{2}\right)\sinh^2(\theta h)\right) \\
e^{-(\sigma+\tau)h}H_{13}(H_{13})_r &= h_r\left(\frac{\eta}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\left(\frac{(\tau-\sigma)\eta}{2}\cosh(\theta h) + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h)\right) \\
&= h_r\left(\frac{\eta^2}{\sqrt{1-\eta^2}}\left(\frac{\tau-\sigma}{2}\right)\sinh(\theta h)\cosh(\theta h) + \frac{\eta^2}{1-\eta^2}\left(\frac{\sigma+\tau}{2}\right)\sinh^2(\theta h)\right) \\
(H^{-1}H_r)_{11} &= h_r\left(\tau(1+\sinh^2(\theta h)) - \frac{1}{1-\eta^2}\left(\tau - \frac{(\tau-\sigma)\eta^2}{2} - \eta^2\left(\frac{\sigma+\tau}{2}\right)\right)\sinh^2(\theta h)\right) \\
&= \tau h_r
\end{aligned}
$$

Moving on to the 13 component.

$$
\begin{aligned}
(H^{-1}H_r)_{13} &= e^{-(\sigma+\tau)h}(-H_{33}(H_{13})_r + H_{13}(H_{33})_r) \\
e^{-(\sigma+\tau)h}H_{33}(H_{13})_r &= h_r\left(\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\left(\frac{(\tau-\sigma)\eta}{2}\cosh(\theta h) + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h)\right) \\
&= h_r\left(\frac{(\tau-\sigma)\eta}{2}\cosh^2(\theta h) + \left(-\frac{(\tau-\sigma)\eta}{2\sqrt{1-\eta^2}} + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\right)\sinh(\theta h)\cosh(\theta h)\dots\right. \\
&\left.\dots - \frac{\eta}{1-\eta^2}\left(\frac{\sigma+\tau}{2}\right)\sinh^2(\theta h)\right) \\
e^{-(\sigma+\tau)h}H_{13}(H_{33})_r &= h_r\frac{\eta}{\sqrt{1-\eta^2}}\left(\sinh(\theta h)\right)\left(\sigma\cosh(\theta h) - \frac{1}{\sqrt{1-\eta^2}}\left(\sigma + \frac{(\tau-\sigma)\eta^2}{2}\right)\sinh(\theta h)\right) \\
(H^{-1}H_r)_{13} &= h_r\left(-\frac{(\tau-\sigma)\eta}{2} + \frac{\eta}{2\sqrt{1-\eta^2}}\left(\sigma-\sigma\right)\sinh(\theta h)\cosh(\theta h)\dots\right. \\
&\left.\dots + \eta\left(-\frac{(\tau-\sigma)}{2} + \frac{\tau-\sigma}{2(1-\eta^2)} - \frac{(\tau-\sigma)\eta^2}{2(1-\eta^2)}\right)\sinh^2(\theta h)\right) \\
(H^{-1}H_r)_{13} &= -\frac{(\tau-\sigma)}{2}h_r
\end{aligned}
$$

Moving on to the 31 component.

$$
\begin{aligned}
(H^{-1}H_r)_{31} &= e^{-(\sigma+\tau)h}(H_{13}(H_{11})_r - H_{11}(H_{13})_r) \\
e^{-(\sigma+\tau)h}H_{13}(H_{11})_r &= -h_r\frac{\eta}{\sqrt{1-\eta^2}}\sinh(\theta h)\left(\tau\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\left(\tau - \left(\frac{\tau-\sigma}{2}\right)\eta^2\right)\sinh(\theta h)\right) \\
e^{-(\sigma+\tau)h}H_{11}(H_{13})_r &= -h_r\left(\cosh(\theta h) + \frac{1}{\sqrt{1-\eta^2}}\sinh(\theta h)\right)\left(\frac{(\tau-\sigma)\eta}{2}\cosh(\theta h)\dots\right. \\
&\quad \left.\dots + \frac{\eta}{\sqrt{1-\eta^2}}\left(\frac{\sigma+\tau}{2}\right)\sinh(\theta h)\right) \\
&= -h_r\eta\left(\left(\frac{\tau-\sigma}{2}\right)\cosh^2(\theta h) + \left(\frac{\tau-\sigma}{2}\frac{1}{\sqrt{1-\eta^2}} + \frac{\sigma+\tau}{2}\frac{1}{\sqrt{1-\eta^2}}\right)\cosh(\theta h)\sinh(\theta h)\dots\right. \\
&\quad \left.\dots + \frac{(\tau+\sigma)}{2}\frac{1}{1-\eta^2}\sinh^2(\theta h)\right) \\
(H^{-1}H_r)_{31} &= h_r\left(\frac{(\tau-\sigma)\eta}{2} + \eta\left(\frac{\tau}{\sqrt{1-\eta^2}} - \frac{\tau}{\sqrt{1-\eta^2}}\right)\sinh(\theta h)\cosh(\theta h)\dots\right. \\
&\quad \left.\dots + \eta\left(-\frac{1}{1-\eta^2}\left(\tau - \frac{(\tau-\sigma)\eta^2}{2}\right) + \frac{\tau-\sigma}{2} + \frac{\tau+\sigma}{2}\left(\frac{1}{1-\eta^2}\right)\right)\sinh^2(\theta h)\right) \\
&= h_r\left(\frac{(\tau-\sigma)\eta}{2} + \eta\frac{\tau-\sigma}{2}\left(+\frac{\eta^2}{1-\eta^2} - \frac{1}{1-\eta^2} + 1\right)\sinh^2(\theta h)\right) \\
&= \frac{(\tau-\sigma)\eta}{2}h_r
\end{aligned}
$$

Now we move on to the 33 component but utilize the previous calculations. When calculating k we use the calculations for the determinant of H.

$$
\begin{aligned}
(H^{-1}H_r)_{33} &= e^{-(\sigma+\tau)h}(H_{13}(H_{13})_r - H_{11}(H_{33})_r) \\
k &:= e^{-(\sigma+\tau)h}\frac{\partial}{\partial r}(H_{13}^2 - H_{11}H_{33}) = (\sigma+\tau)h_r \\
&= e^{-(\sigma+\tau)h}(2H_{13}(H_{13})r - (H_{11})_rH_{33} - (H_{33})_rH_{11}) \\
(\sigma+\tau)h_r &= (H^{-1}H_r)_{33} + (H^{-1}H_r)_{11} \\
(H^{-1}H_r)_{33} &= \sigma h_r
\end{aligned}
$$

Now we move on to the 22 component.

$$
\begin{aligned}
(H^{-1}H_r)_{22} &= r^{-2}e^{(\sigma+\tau)h}\left(2re^{-(\sigma+\tau)h} - (\sigma+\tau)r^2h_re^{-(\sigma+\tau)h}\right) \\
&= 2r^{-1} - (\sigma+\tau)(h_r)
\end{aligned}
$$

The calculations for the components $H^{-1}H_z$ are identical to the calculations for $H^{-1}H_r$ except that h_r becomes h_z and the 22 component is slightly different.

The results are stated below.

$$(H^{-1}H_z)_{11} = \sigma h_z$$
$$(H^{-1}H_z)_{33} = \tau h_z$$
$$(H^{-1}H_z)_{13} = -\frac{(\tau-\sigma)\eta}{2}h_z$$
$$(H^{-1}H_z)_{31} = \frac{(\tau-\sigma)\eta}{2}h_z$$
$$(H^{-1}H_z)_{22} = -(\sigma+\tau)h_z$$

We have that these components together satisfy the harmonic map equations since h is harmonic and $\frac{\partial}{\partial r}(rr^{-1}) + \frac{\partial}{\partial z}(rr^{-1}) = 0$.

7.3 The Derivation

7.3.1 Reconciling the Matrix Exponential

We now show how this solution was derived. Let P be a matrix whose components are harmonic which takes the following form:

$$P = \begin{pmatrix} A & 0 & D \\ 0 & B & 0 \\ E & 0 & C \end{pmatrix} \qquad \text{Let } J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Essentially we are constructing H from P using the exponential function for matrices. We have from the background chapter that $ad_X Y = [X, Y]$ where $[X, Y]$ is the commutator of the matrices X and Y. This is because the Lie group is the set of all invertible 3x3 matrices thus the Lie algebra is simply the set of 3x3 matrices. We consider the following two matrices:

$$\begin{aligned}
\Lambda &= -\frac{P_r^T}{2} + \frac{JP_rJ}{2} + \sum_{j=1}^{\infty}\frac{(-1)^j}{(j+1)!}ad^j_{P-\frac{P^T}{2}+\frac{JPJ}{2}}\left(P_r - \frac{P_r^T}{2} + \frac{JP_rJ}{2}\right) \\
&= -\frac{P_r^T}{2} + \frac{JP_rJ}{2} - \frac{1}{2!}\left[P - \frac{P^T}{2} + \frac{JPJ}{2}, P_r - \frac{P_r^T}{2} + \frac{JP_rJ}{2}\right] \\
&+ \frac{1}{3!}\left[P - \frac{P^T}{2} + \frac{JPJ}{2}, \left[P - \frac{P^T}{2} + \frac{JPJ}{2}, P_r - \frac{P_r^T}{2} + \frac{JP_rJ}{2}\right]\right] + h.o.t \\
\Gamma &= -\frac{P_z^T}{2} + \frac{JP_zJ}{2} + \sum_{j=1}^{\infty}\frac{(-1)^j}{(j+1)!}ad^j_{P-\frac{P^T}{2}+\frac{JPJ}{2}}\left(P_z - \frac{P_z^T}{2} + \frac{JP_zJ}{2}\right) \\
&= -\frac{P_z^T}{2} + \frac{JP_zJ}{2} - \frac{1}{2!}\left[P - \frac{P^T}{2} + \frac{JPJ}{2}, P_z - \frac{P_z^T}{2} + \frac{JP_zJ}{2}\right. \\
&+ \frac{1}{3!}\left[P - \frac{P^T}{2} + \frac{JPJ}{2}, \left[P - \frac{P^T}{2} + \frac{JPJ}{2}, P_z - \frac{P_z^T}{2} + \frac{JP_zJ}{2}\right]\right] + h.o.t
\end{aligned}$$

We are interested in knowing how many non-zero components Λ and Γ actually have. We start by calculating $-\frac{P^T}{2}+\frac{JPJ}{2}$ and $P-\frac{P^T}{2}+\frac{JPJ}{2}$.

$$-\frac{P^T}{2}+\frac{JPJ}{2}=\begin{pmatrix}-\frac{A}{2} & 0 & -\frac{D}{2}\\ 0 & \frac{B}{2} & 0\\ -\frac{E}{2} & 0 & \frac{C}{2}\end{pmatrix}+\begin{pmatrix}\frac{A}{2} & 0 & -\frac{D}{2}\\ 0 & \frac{B}{2} & 0\\ -\frac{E}{2} & 0 & \frac{C}{2}\end{pmatrix}=\begin{pmatrix}0 & 0 & -\frac{D+E}{2}\\ 0 & 0 & 0\\ -\frac{D+E}{2} & 0 & 0\end{pmatrix}$$

$$-\frac{P^T}{2}+\frac{JPJ}{2}+P=\begin{pmatrix}0 & 0 & -\frac{D+E}{2}\\ 0 & 0 & 0\\ -\frac{D+E}{2} & 0 & 0\end{pmatrix}+\begin{pmatrix}A & 0 & D\\ 0 & B & 0\\ E & 0 & C\end{pmatrix}=\begin{pmatrix}A & 0 & \frac{D-E}{2}\\ 0 & B & 0\\ -\frac{D-E}{2} & 0 & C\end{pmatrix}$$

We now consider the matrix $Q=\left[P-\frac{P^T}{2}+\frac{JPJ}{2},P_r-\frac{P_r^T}{2}+\frac{JP_rJ}{2}\right]$.

$$\begin{aligned}
Q_{11}&=AA_r-\frac{(D-E)(D_r-E_r)}{4}-AA_r+\frac{(D-E)(D_r-E_r)}{4}=0\\
Q_{22}&=BB_r-B_rB=0\\
Q_{13}&=\frac{A(D_r-E_r)}{2}+\frac{(D-E)C_r}{2}-\frac{A_r(D-E)}{2}-\frac{C(D_r-E_r)}{2}\\
&=\frac{(A-C)(D_r-E_r)}{2}-\frac{(A_r-C_r)(D-E)}{2}\\
Q_{31}&=-\frac{A_r(D-E)}{2}-\frac{C(D_r-E_r)}{2}+\frac{A(D_r-E_r)}{2}+\frac{C_r(D-E)}{2}\\
&=\frac{(A-C)(D_r-E_r)}{2}-\frac{(A_r-C_r)(D-E)}{2}=Q_{13}\\
Q_{33}&=-\frac{(D-E)(D_r-E_r)}{4}+CC_r+\frac{(D-E)(D_r-E_r)}{4}-C_rC=0\\
Q_{12}&=Q_{21}=Q_{23}=Q_{32}=0
\end{aligned}$$

We now consider the matrix $R=\left[P-\frac{P^T}{2}+\frac{JPJ}{2},P_z-\frac{P_z^T}{2}+\frac{JP_zJ}{2}\right]$.

$$\begin{aligned}
R_{11}&=AA_z-\frac{(D-E)(D_z-E_z)}{4}-AA_z+\frac{(D-E)(D_z-E_z)}{4}=0\\
R_{22}&=BB_z-B_zB=0\\
R_{13}&=\frac{A(D_z-E_z)}{2}+\frac{(D-E)C_z}{2}-\frac{A_z(D-E)}{2}-\frac{C(D_z-E_z)}{2}\\
&=\frac{(A-C)(D_z-E_z)}{2}-\frac{(A_z-C_z)(D-E)}{2}\\
R_{31}&=-\frac{A_z(D-E)}{2}-\frac{C(D_z-E_z)}{2}+\frac{A(D_z-E_z)}{2}+\frac{C_z(D-E)}{2}\\
&=\frac{(A-C)(D_z-E_z)}{2}-\frac{(A_z-C_z)(D-E)}{2}=R_{13}\\
R_{33}&=-\frac{(D-E)(D_z-E_z)}{4}+CC_z+\frac{(D-E)(D_z-E_z)}{4}-C_zC=0\\
R_{12}&=R_{21}=R_{23}=R_{32}=0
\end{aligned}$$

If we set $Q_{13} = R_{13} = 0$ then $Q = R = 0$. We then kill the higher order terms in Ω and Γ which allows us to construct solutions to the harmonic map equations. To see this we define $\overline{P}$ and construct H from it using the matrix exponential.

$$\overline{P} = P - \frac{P^T}{2} + \frac{JPJ}{2} \qquad H = Je^{\overline{P}} \qquad e^X = \Sigma_{i=0}^{\infty} \frac{X^n}{n!}$$

We see that $\overline{P}$ is related to its transpose.

$$\overline{P}^T = \begin{pmatrix} A & 0 & -\frac{D-E}{2} \\ 0 & B & 0 \\ \frac{D-E}{2} & 0 & C \end{pmatrix} \qquad J\overline{P}J = \begin{pmatrix} A & 0 & -\frac{D-E}{2} \\ 0 & B & 0 \\ \frac{D-E}{2} & 0 & C \end{pmatrix}$$

From this relation and the fact that $J^2 = I$ we can show that H is symmetric.

$$H^T = e^{\overline{P}^T} J = e^{J\overline{P}J} J = Je^{\overline{P}} JJ = Je^{\overline{P}} = H$$

Since H is symmetric its eigenvalues are real. We want to make further deductions about the eigenvalues of H. To do this we need to more closely examine the structure of H. Let X be a square matrix. We can express the components of X^n, $n \geq 2$ as follows.

$$(X^n)_{ij} = \Sigma_{i_1,\dots,i_n \in \{1,\dots,n\}} X_{ii_1} X_{i_1 i_2} \dots X_{i_{n-1} i_n} X_{i_n j}$$

Next we consider $\overline{P}^n_{22}$. Since $\overline{P}_{23} = \overline{P}_{21} = 0$ we are forced to conclude that every term in the above expression must be $\overline{P}_{22}$.

$$\overline{P}^n_{22} = (\overline{P}_{22})^n \qquad H_{22} = e^{\overline{P}_{22}}$$

Next we consider $\overline{P}^n_{23}$. The only non zero that contains a 2 in the index is $\overline{P}_{22}$. But all the proceeding terms in the product must be $\overline{P}_{22}$. Therefore the 3 in the last index is never realized. Therefore $\overline{P}^n_{23} = 0$. By the same logic, $\overline{P}^n_{21} = 0$. Therefore $H_{21} = H_{23} = 0$. And by symmetry, $H_{12} = H_{32} = 0$. We can express the characteristic polynomial for H as follows.

$$0 = \det \begin{pmatrix} H_{11} - \lambda & 0 & H_{13} \\ 0 & H_{22} - \lambda & 0 \\ H_{13} & 0 & H_{33} - \lambda \end{pmatrix} = (H_{22} - \lambda)((H_{11} - \lambda)(H_{33} - \lambda) - H_{13}^2)$$

Therefore H_{22} is an eigenvalue and it is also positive where H_{22} is defined since $H_{22} = e^{\overline{P}_{22}}$. Next we impose the condition on P that $Tr(P) = 2\log(r)$. This allows to calculate the determinant of H.

$$\det(H) = \det\left(Je^{\overline{P}}\right) = \det(J)\det\left(e^{\overline{P}}\right) = -e^{Tr(\overline{P})} = -e^{Tr(P)} = -e^{\log\left(r^2\right)} = -r^2$$

From this we deduce that H is Lorentz away from the z axis $(r > 0)$. Since the determinant is negative there must be either 1 or 3 negative eigenvalues. Since we know there is at least 1 positive eigenvalue we deduce that H has 1 negative eigenvalue and 2 positive eigenvalues meaning it is Lorentz. Next we show that H obeys the harmonic map equations in the case when $Q_{13} = R_{13} = 0$. We use the formula for the derivative of the exponential matrix found in Hall's book "Lie Groups, Lie Algebras, and Representations" [16, p. 71].

$$\begin{aligned}
H^{-1}H_r &= e^{-\overline{P}} J J \frac{\partial}{\partial r} e^{\overline{P}} = e^{-\overline{P}} \frac{\partial}{\partial r} e^{\overline{P}} \\
&= \overline{P_r} - \frac{1}{2}[\overline{P}, \overline{P_r}] + \frac{1}{3!}[\overline{P}, [\overline{P}, \overline{P_r}]]. + h.o.t \\
&= \overline{P_r} \\
H^{-1}H_r &= \begin{pmatrix} A_r & 0 & \frac{D_r - E_r}{2} \\ 0 & B_r & 0 \\ -\frac{D_r - E_r}{2} & 0 & C_r \end{pmatrix} \\
H^{-1}H_z &= e^{-\overline{P}} J J \frac{\partial}{\partial z} e^{\overline{P}} = e^{-\overline{P}} \frac{\partial}{\partial z} e^{\overline{P}} \\
&= \overline{P_z} - \frac{1}{2}[\overline{P}, \overline{P_z}] + \frac{1}{3!}[\overline{P}, [\overline{P}, \overline{P_z}]] + h.o.t \\
&= \overline{P_z} \\
H^{-1}H_z &= \begin{pmatrix} A_z & 0 & \frac{D_z - E_z}{2} \\ 0 & B_z & 0 \\ -\frac{D_z - E_z}{2} & 0 & C_z \end{pmatrix}
\end{aligned}$$

The components of $H^{-1}H_r$ and $H^{-1}H_z$ are linear combinations of the corresponding partial derivatives of P. Since $P_{rr} + P_{zz} + \frac{P_r}{r} = 0$, it follows that $\frac{\partial}{\partial r}\left(rH^{-1}H_r\right) + \frac{\partial}{\partial z}\left(rH^{-1}H_z\right) = 0$.

7.3.2 Restriction on the Components of P

We now construct the example.
First, we note that $Q_{13} = R_{13} = 0$ restricts our choice of components for H.

$$\begin{aligned}
Q_{13} &= \frac{(A-C)(D_r-E_r)}{2} - \frac{(A_r-C_r)(D-E)}{2} = 0 \\
(A-C)(D_r-E_r) &= (A_r-C_r)(D-E) \\
\frac{D_r-E_r}{D-E} &= \frac{A_r-C_r}{A-C} \\
\log(D-E) &= \log(A-C) + \log(\eta(z)) \\
R_{13} &= \frac{(A-C)(D_z-E_z)}{2} - \frac{(A_z-C_z)(D-E)}{2} = 0 \\
(A-C)(D_z-E_z) &= (A_z-C_z)(D-E) \\
\frac{D_z-E_z}{D-E} &= \frac{A_z-C_z}{A-C} \\
\log(D-E) &= \log(A-C) + \log(\overline{\eta}(r))
\end{aligned}$$

Here η and $\overline{\eta}$ are arbitrary functions of z and r respectively. It follows that $\log(A-C)$ and $\log(D-E)$ differ by a constant which is $\log(\eta) = \log(\overline{\eta})$.

$$\begin{aligned}
\log(D-E) &= \log(A-C) + \log(\eta) \\
D-E &= \eta(A-C)
\end{aligned}$$

Let h be a harmonic function, we will use it repeatedly in the components of $\overline{H}$. Note that we're assuming that $\tau - \sigma \neq 0$ since we want non constant twist potentials.

$$\begin{aligned}
A &= \tau h & C &= \sigma h \\
B &= 2\log(r) - (\sigma+\tau)h & \frac{D-E}{2} &= \frac{\eta}{2}(\tau-\sigma)h
\end{aligned}$$

In terms of calculating the components of H, the component H_{22} is straightforward but the other 3 non-zero components require closer examination of $e^{\overline{P}}$. The components of $(\overline{P}^n)_{ij}$, where $i, j \in \{1, 3\}$, are made up of $\overline{P}_{11}$, $\overline{P}_{13}$, $\overline{P}_{31}$ and $\overline{P}_{33}$. To see this we go back to the formula for the components of powers of $\overline{P}$. The leading factor has a 1 or a 3 in the first index so for the term to be non zero the second index must have a 1 or a 3. This argument carries to the subsequent factors since the second index of the kth factor is the first index of $(k+1)$th factor. Since $\overline{P}_{11}$, $\overline{P}_{13}$, $\overline{P}_{31}$ and $\overline{P}_{33}$ are all made up of multiples of h, we know that $(\overline{P}^n)_{ij}$, where $i, j \in \{1, 3\}$, is factor of h^n. We denote this factor by $\overline{P}^{(n)}_{ij}$. We have the following recurrence relations where $n \geq 1$.

$$\begin{aligned}
\overline{P}_{11}^{(n)} &= \overline{P}_{11}^{(1)}\overline{P}_{11}^{(n-1)} + \overline{P}_{13}^{(1)}\overline{P}_{31}^{(n-1)} \\
\overline{P}_{31}^{(n)} &= \overline{P}_{33}^{(1)}\overline{P}_{31}^{(n-1)} + \overline{P}_{31}^{(1)}\overline{P}_{11}^{(n-1)} \\
\overline{P}_{13}^{(n)} &= \overline{P}_{11}^{(1)}\overline{P}_{13}^{(n-1)} + \overline{P}_{13}^{(1)}\overline{P}_{33}^{(n-1)} \\
\overline{P}_{33}^{(n)} &= \overline{P}_{33}^{(1)}\overline{P}_{33}^{(n-1)} + \overline{P}_{31}^{(1)}\overline{P}_{13}^{(n-1)}
\end{aligned}$$

Where $\overline{P}_{11}^{(0)} = 1$, $\overline{P}_{33}^{(0)} = 1$, and $\overline{P}_{13}^{(0)} = \overline{P}_{31}^{(0)} = 0$. And where $\overline{P}_{11}^{(1)} = \tau$, $\overline{P}_{33}^{(1)} = \sigma$, and $\overline{P}_{13}^{(1)} = -\overline{P}_{31}^{(1)} = \frac{(\tau-\sigma)\eta}{2}$. Here $\overline{P}_{ij}^{(1)}$ is obtained from the matrix $\overline{P}$ and the $\overline{P}_{ij}^{(0)}$ is solved for using the recursive equations.

7.3.3 Solving the Recurrence Relations

We can divide these 4 recursive equations into 2 subsystems and then analyze each subsystem. By repeatedly subbing in the equation for $\overline{P}_{31}^{(n)}$ into the equation for $\overline{P}_{11}^{(n)}$ we obtain an equation solely in terms of $\overline{P}_{11}^{(m)}$ where $0 \leq m \leq n-1$.

$$\overline{P}_{11}^{(n)} = \tau\overline{P}_{11}^{(n-1)} - \frac{(\tau-\sigma)^2\eta^2}{4}\left(\Sigma_{i=0}^{n-2}\sigma^i\overline{P}_{11}^{(n-(i+2))}\right)$$

We can compare $\overline{P}_{11}^{(n)}$ to $\overline{P}_{11}^{(n-1)}$ to obtain a simplification.

$$\begin{aligned}
\overline{P}_{11}^{(n-1)} &= \tau\overline{P}_{11}^{(n-2)} - \frac{(\tau-\sigma)^2\eta^2}{4}\left(\Sigma_{i=0}^{n-3}\sigma^i\overline{P}_{11}^{(n-1-(i+2))}\right) \\
&= \tau\overline{P}_{11}^{(n-2)} - \frac{(\tau-\sigma)^2\eta^2}{4}\left(\Sigma_{i=1}^{n-2}\sigma^{i-1}\overline{P}_{11}^{(n-(i+2))}\right) \\
&= \tau\overline{P}_{11}^{(n-2)} - \frac{(\tau-\sigma)^2\eta^2}{4\sigma}\left(\Sigma_{i=1}^{n-2}\sigma^{i}\overline{P}_{11}^{(n-(i+2))}\right) \\
\overline{P}_{11}^{(n)} - \sigma\overline{P}_{11}^{(n-1)} &= \tau\overline{P}_{11}^{(n-1)} - \sigma\tau\overline{P}_{11}^{(n-2)} - \frac{(\tau-\sigma)^2\eta^2}{4}\overline{P}_{11}^{(n-2)} \\
\overline{P}_{11}^{(n)} &= (\sigma+\tau)\overline{P}_{11}^{(n-1)} - \left(\sigma\tau + \frac{(\tau-\sigma)^2\eta^2}{4}\right)\overline{P}_{11}^{(n-2)}
\end{aligned}$$

Now we have a second order recursive sequence so the roots of the corresponding quadratic equation are essential for finding a formula for the sequence.

$$\begin{aligned}
0 &= s^2 - (\sigma+\tau)s + \left(\sigma\tau + \frac{(\tau-\sigma)^2\eta^2}{4}\right) \\
s &= \frac{\sigma+\tau \pm \sqrt{(\sigma+\tau)^2 - (4\sigma\tau + (\tau-\sigma)^2\eta^2)}}{2} \\
s &= \frac{\sigma+\tau \pm (\tau-\sigma)\sqrt{1-\eta^2}}{2}
\end{aligned}$$

We'll only consider the case where the roots are non-repeating. We obtain the following expression for $\overline{P}_{11}^{(n)}$. Where $\overline{A}$ and $\overline{B}$ are constants.

$$\overline{P}_{11}^{(n)} = \overline{A}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n + \overline{B}\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n$$

$$\overline{P}_{11}^{(0)} = 1 = \overline{A}+\overline{B}$$

$$\overline{P}_{11}^{(1)} = \tau = \overline{A}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right) + \overline{B}\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)$$

$$\tau = \overline{A}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2} - \frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right) + \frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}$$

$$\frac{\tau-\sigma+(\tau-\sigma)\sqrt{1-\eta^2}}{2} = \overline{A}(\tau-\sigma)\sqrt{1-\eta^2}$$

$$\overline{A} = \frac{1}{2}+\frac{1}{2\sqrt{1-\eta^2}} \qquad \overline{B} = \frac{1}{2}-\frac{1}{2\sqrt{1-\eta^2}}$$

$$\overline{P}_{11}^{(n)} = \frac{1}{2}\Bigg(\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n$$
$$+\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n\Bigg)$$

We can now work out $\overline{P}_{31}^{(n)}$ from the original recursive equations.

$$\overline{P}_{13}^{(1)}\overline{P}_{31}^{(n)} = \overline{P}_{11}^{(n+1)} - \overline{P}_{11}^{(n)}\overline{P}_{11}^{(1)}$$

$$\frac{\eta(\tau-\sigma)}{2}\overline{P}_{31}^{(n)} = \frac{1}{2}\Bigg(\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n+1}\dots$$
$$\dots+\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n+1}\Bigg)\dots$$
$$\dots-\frac{\tau}{2}\Bigg(\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\dots$$
$$\dots+\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\Bigg)$$
$$= \frac{1}{2}\Bigg(\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}-\tau\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\dots$$
$$\dots+\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}-\tau\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\Bigg)$$

$$\eta\overline{P}_{31}^{(n)} = \left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{-1+\sqrt{1-\eta^2}}{2}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\dots$$
$$\dots-\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{1+\sqrt{1-\eta^2}}{2}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}$$

$$\overline{P}_{31}^{(n)} = \frac{\eta}{2\sqrt{1-\eta^2}}\Bigg(\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n} - \left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^{n}\Bigg)$$

Now the recursive equation for $\overline{P}_{33}^{(n)}$ is identical to that of $\overline{P}_{11}^{(n)}$ except that the initial values are different. Let $\overline{C}$ and $\overline{D}$ be constants.

$$\overline{P}_{33}^{(n)} = \overline{C}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n + \overline{D}\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n$$

$$\overline{P}_{33}^{(0)} = 1 = \overline{C}+\overline{D}$$

$$\overline{P}_{33}^{(1)} = \sigma = \overline{C}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right) + \overline{D}\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)$$

$$\sigma = \overline{C}\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2} - \frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right) + \left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)$$

$$\overline{C}(\tau-\sigma)\sqrt{1-\eta^2} = \frac{-(\tau-\sigma)+(\tau-\sigma)\sqrt{1-\eta^2}}{2}$$

$$\overline{C} = \frac{1}{2} - \frac{1}{2\sqrt{1-\eta^2}} \qquad \overline{D} = \frac{1}{2} + \frac{1}{2\sqrt{1-\eta^2}}$$

$$\overline{P}_{33}^{(n)} = \frac{1}{2}\Bigg(\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n \dots$$

$$\dots + \left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n\Bigg)$$

Now because we know H is symmetric we know that $(JH)_{13} = -(JH)_{31}$. It follows that each coefficient of a power of h in each term of $(JH)_{13}$ must be the negative of the coefficient of the corresponding power of h in $(JG)_{31}$. Therefore $\overline{P}_{13}^{(n)} = -\overline{P}_{31}^{(n)}$.

$$\overline{P}_{13}^{(n)} = -\frac{\eta}{2\sqrt{1-\eta^2}}\left(\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n - \left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n\right)$$

7.3.4 Computing the Components of H

Now we are in position to compute the components of H. The simplest is H_{22}.

$$H_{22} = e^{2\log r - (\sigma+\tau)h} = r^2 e^{-(\sigma+\tau)h}$$

$$\begin{aligned}
H_{11} &= -\Sigma_{n=0}^{\infty} \overline{H}_{11}^{(n)} \frac{h^n}{n!} \\
&= -\Sigma_{n=0}^{\infty} \frac{1}{2}\Bigg(\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n \dots \\
&\dots + \left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n \Bigg)\frac{h^n}{n!} \\
&= -\frac{1}{2}\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\Sigma_{n=0}^{\infty}\frac{\left(\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right)^n}{n!} \dots \\
&\dots - \frac{1}{2}\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\Sigma_{n=0}^{\infty}\frac{\left(\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right)^n}{n!} \\
&= -\frac{1}{2}\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)e^{\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h} - \frac{1}{2}\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)e^{\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h} \\
&= -\frac{1}{2}e^{\frac{(\sigma+\tau)h}{2}}\left(\left(e^{\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h} + e^{-\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h}\right) + \frac{1}{\sqrt{1-\eta^2}}\left(e^{\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h} - e^{-\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h}\right)\right) \\
&= -e^{\frac{(\sigma+\tau)h}{2}}\left(\cosh\left(\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right) + \frac{1}{\sqrt{1-\eta^2}}\sinh\left(\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h\right)\right)
\end{aligned}$$

$$\begin{aligned}
H_{33} &= \Sigma_{n=0}^{\infty} \overline{H}_{33}^{(n)} \frac{h^n}{n!} \\
&= \Sigma_{n=0}^{\infty} \frac{1}{2}\Bigg(\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n \dots \\
&\dots + \left(1+\frac{1}{\sqrt{1-\eta^2}}\right)\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n \Bigg)\frac{h^n}{n!} \\
&= \frac{1}{2}\left(1-\frac{1}{\sqrt{1-\eta^2}}\right)e^{\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}h} + \frac{1}{2}\left(1+\frac{1}{\sqrt{1-\eta^2}}\right)e^{\left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)h} \\
&= e^{\frac{(\sigma+\tau)h}{2}}\left(\cosh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right) - \frac{1}{\sqrt{1-\eta^2}}\sinh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right)\right) \\
H_{13} &= -\Sigma_{n=0}^{\infty} \overline{H}_{13}^{(n)} \frac{h^n}{n!} \\
&= \Sigma_{n=0}^{\infty}\frac{\eta}{2\sqrt{1-\eta^2}}\left(\left(\frac{\sigma+\tau+(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n - \left(\frac{\sigma+\tau-(\tau-\sigma)\sqrt{1-\eta^2}}{2}\right)^n\right)\frac{h^n}{n!} \\
&= \frac{\eta}{\sqrt{1-\eta^2}}e^{\frac{\sigma+\tau}{2}h}\sinh\left(\frac{(\tau-\sigma)\sqrt{1-\eta^2}}{2}h\right)
\end{aligned}$$

There are principally two cases, $\eta < 1$ and $\eta > 1$. In the first case the equations are as is but in the second case the hyperbolic functions change to trigonometric functions.

$$H_{11} = -e^{\frac{\sigma+\tau}{2}h}\left(\cos\left(\frac{(\tau-\sigma)\sqrt{\eta^2-1}}{2}h\right) + \frac{1}{\sqrt{\eta^2-1}}\sin\left(\frac{(\tau-\sigma)\sqrt{\eta^2-1}}{2}h\right)\right)$$
$$H_{33} = e^{\frac{(\sigma+\tau)h}{2}}\left(\cos\left(\frac{(\tau-\sigma)\sqrt{\eta^2-1}}{2}h\right) - \frac{1}{\sqrt{\eta^2-1}}\sin\left(\frac{(\tau-\sigma)\sqrt{\eta^2-1}}{2}h\right)\right)$$
$$H_{13} = \frac{\eta}{\sqrt{\eta^2-1}}e^{\frac{\sigma+\tau}{2}h}\sin\left(\frac{(\tau-\sigma)\sqrt{\eta^2-1}}{2}h\right)$$

However in our solution we require that the lower right minor be positive definite away from the z-axis. This requires that H_{33} be positive. So with that in mind we can disregard the $\eta > 1$ case since we would need a harmonic function bounded from above and below in order for H_{33} to be positive.

7.3.5 Calculation of α

We proceed by calculating α in the case where $h = -\frac{1}{\sqrt{r^2+z^2}}$. We use the formulae from chapter 2 for the partial derivatives of α and the formula for $\overline{P}$.

$$\alpha_r = \frac{r}{8}\left(\mathrm{Tr}(\overline{P}_r^2) - \mathrm{Tr}(\overline{P}_z^2) - \frac{4}{r^2}\right)$$
$$h_r = \frac{r}{(r^2+z^2)^{\frac{3}{2}}} \qquad h_z = \frac{z}{(r^2+z^2)^{\frac{3}{2}}}$$
$$\overline{P}_r = \begin{pmatrix} \frac{\tau r}{(r^2+z^2)^{\frac{3}{2}}} & 0 & \frac{\eta(\tau-\sigma)}{2}\frac{r}{(r^2+z^2)^{\frac{3}{2}}} \\ 0 & \frac{2}{r} - \frac{(\tau+\sigma)r}{(r^2+z^2)^{\frac{3}{2}}} & 0 \\ -\frac{\eta(\tau-\sigma)}{2}\frac{r}{(r^2+z^2)^{\frac{3}{2}}} & 0 & \frac{\sigma r}{(r^2+z^2)^{\frac{3}{2}}} \end{pmatrix}$$
$$\overline{P}_z = \begin{pmatrix} \frac{\tau z}{(r^2+z^2)^{\frac{3}{2}}} & 0 & \frac{\eta(\tau-\sigma)}{2}\frac{z}{(r^2+z^2)^{\frac{3}{2}}} \\ 0 & -\frac{(\tau+\sigma)z}{(r^2+z^2)^{\frac{3}{2}}} & 0 \\ -\frac{\eta(\tau-\sigma)}{2}\frac{z}{(r^2+z^2)^{\frac{3}{2}}} & 0 & \frac{\sigma z}{(r^2+z^2)^{\frac{3}{2}}} \end{pmatrix}$$
$$\alpha_r = \frac{r}{8}\left((\tau^2 - \frac{\eta^2}{2}(\tau-\sigma)^2 + \sigma^2 + (\sigma+\tau)^2)(\frac{r^2}{(r^2+z^2)^3} - \frac{z^2}{(r^2+z^2)^3}) - 4\frac{(\sigma+\tau)}{(r^2+z^2)^{\frac{3}{2}}}\right)$$

Let $\gamma = \tau^2 - \frac{\eta^2}{2}(\tau-\sigma)^2 + \sigma^2 + (\sigma+\tau)^2$. We now integrate α_r with respect to r.

$$\begin{aligned}
\alpha_r &= \frac{r}{8}\Big(\gamma(\frac{r^2+z^2}{(r^2+z^2)^3} - 2\frac{z^2}{(r^2+z^2)^3}) - 4\frac{(\sigma+\tau)}{(r^2+z^2)^{\frac{3}{2}}}\Big)\\
\alpha &= \frac{\gamma}{8}\int \frac{r}{(r^2+z^2)^2}dr - \frac{z^2\gamma}{4}\int \frac{r}{(r^2+z^2)^3}dr - \frac{(\sigma+\tau)}{2}\int \frac{r}{(r^2+z^2)^{\frac{3}{2}}}dr\\
\alpha &= -\frac{\gamma}{16}\frac{1}{r^2+z^2} + \frac{\gamma}{16}\frac{z^2}{(r^2+z^2)^2} + \frac{\sigma+\tau}{2}\frac{1}{\sqrt{r^2+z^2}} + V(z)\\
\alpha &= -\frac{\gamma}{16}\frac{r^2}{(r^2+z^2)^2} + \frac{\sigma+\tau}{2}\frac{1}{\sqrt{r^2+z^2}} + V(z)
\end{aligned}$$

We now solve for $V(z)$ by differentiating with respect to z and comparing to the formula for α_z.

$$\begin{aligned}
\alpha_z &= \frac{\gamma}{4}\frac{r^2 z}{(r^2+z^2)^3} - \frac{(\sigma+\tau)}{2}\frac{z}{(r^2+z^2)^{\frac{3}{2}}} + \partial_z V(z)\\
\alpha_z &= \frac{r}{4}\mathrm{Tr}(H_r H_z)\\
&= \frac{r}{4}\Big(\gamma\frac{rz}{(r^2+z^2)^3} - \frac{2(\sigma+\tau)}{r(r^2+z^2)^{\frac{3}{2}}}\Big)
\end{aligned}$$

Therefore $V(z)$ is a constant.

7.4 Topology of the Solution

The topology is found by noting that in the interior of the orbit space we have $\{(r,z)\in\mathbb{R}^2 \,|\, r>0\}\times T^2$ as the topology upstairs and on the boundary of the orbit space we have $\{(r,z)\in\mathbb{R}^2\,|\,r=0, z\neq 0\}\times S^1_2$ as the topology upstairs. We can combine these into 1 set $M_4 = \{(\varphi_2, x_1, y_1, r, z)\,|\,(r,z)\neq 0, x_1^2+y_1^2 = r^2, r\geq 0\}$. We have that $M_4 \approx S^1\times((\mathbb{R}\times\mathbb{R}^2)-\{0\}) \approx S^1\times(\mathbb{R}^3-\{0\}) \approx S^1\times S^2\times(0,\infty)$. Where we have identified the cone with $\mathbb{R}^2$.

Chapter 8

Conclusion

In our analysis of 5-Dimensional stationary bi-axisymmetric solutions to the vacuum Einstein equations we have come to the conclusion that the solution found in example 2 of Khuri et al.'s paper [18] is general in that its metric is diagonal. This is of course derived from the smoothness condition for an alternating $(1,0)$ and $(0,1)$ rod structure. Thus their ansatz is sharp at least for example 2. In our new solution we showcased in the last section the metric is non-diagonal and but is missing a point on the boundary of the orbit space. It is likely not possible to hide this missing point where a corner point would be by extending this solution to a rod structure with a horizon rods and $(0,1)$ rods. This is because multiple parts of the metric blow up. Such a rod structure would be similar to example 1 in Khuri et al.'s paper. The fact that there is an instantaneous horizon rod at the singularity suggests a relation to zero temperature extremal blackhole solutions. However there might be a way of obtaining an even more general form of the metric using exponentials of cubic roots of unity multiplying harmonic functions. Of course the rod data would somehow have to be relaxed further.

We demonstrated in chapter 5, that example 2 respects the smoothness conditions derived in chapter 3 in analyzing its behaviour near the z-axis. The asymptotics for example 2 were found in a concrete way; shedding some light on a Fourier series that was not mentioned in Khuri et al.'s paper (however they alluded to it by mentioning the modified Bessel functions). At a more basic level we showed how to obtain the harmonic map equations from the Ricci flat conditions and the symmetries. We also showed how the Myer's and Nicola's periodic analog to the Schwarzschild solution was found starting from converting Schwarzschild solution to its Weyl form and then performing a generalization. New research goals could be to understand the smoothness condition for a horizon rod and check whether the other examples which occur in Khuri et al's paper obey the smoothness conditions. Also there is potentially a chance of a non-analytic metric which has an alternating pattern of $(1,0)$ and $(0,1)$ rods that could have non constant twist potentials or at least be non-diagonal.

Bibliography

[1] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, Department of Commerce United States of America, 1972.

[2] G. Arfken, *Mathematical Methods For Physicists*, Academic Press, Inc, 1985.

[3] A. Besse, *Einstein Manifolds*, Springer, reprint ed., 1987.

[4] R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifolds*, Dover Publications, Dover ed., 1980.

[5] G. E. Bredon, *Introduction to Compact Transformation Groups*, vol. 46, Academic Press, 1972.

[6] M. P. D. Carmo, *Riemannian Geometry*, Birkhauser, 1992.

[7] P. T. Chrusciel, *On Higher Dimensional Blackholes with Abelian Isometry Group*, Journal of Mathematical Physics, 50 (2009).

[8] A. Douglis and L. Nirenberg, *Interior Estimates for Elliptic Systems of Partial Differential Equations*, VIII (1955), pp. 503–538.

[9] J. Duistermaat and J. Kolk, *Lie Groups*, Springer, 200.

[10] J. Eschenburg and M. Y. Wang, *The Initial Value Problem for Cohomogeneity One Einstein Metrics*, The Journal of Geometric Analysis, 10 (2000), pp. 109–137.

[11] W. Fulton and J. Harris, *Representation Theory A First Course*, Springer, 2004.

[12] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry*, Springer, 3rd ed., 2004.

[13] R. E. Gaunt, *Inequalities for Modified Bessel functions and their Integrals*, Journal of Mathematical Analysis and Applications, (2014), pp. 1–14.

[14] ——, *Inequalities for the Modified Bessel Function of the Second Kind and the Kernel of the Kratzel Integral Transformation*, Mathematical Inequalities Applications, (2017), pp. 1–4.

[15] J. B. Griffiths and J. Podolsky´, *Exact Space-Times in Einstein's General Relativity*, Cambridge Monographs on Mathematical Physics, 2009.

[16] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*, Springer, second ed., 2015.

[17] S. Hollands and S. Yazadjiev, *A Uniqueness Theorem for Stationary Kaluza-Klein Black Holes*, Commun. Math. Phys., 302 (2011), pp. 631–674.

[18] M. Khuri, G. Weinstein, and S. Yamada, *5-Dimensional Space Periodic Solutions of the Static Vacuum Einstein Equations*, J. High Energy Phys., 2020 (2020), pp. 1–20.

[19] K. Knopp, *Theory of Functions Parts I and II*, Dover Publications, Inc., 1996.

[20] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I, Interscience Publishers, 1963.

[21] D. Korotkin and H. Nicolai, *A Periodic Analog of the Schwarzschild Solution*, arXiv: General Relativity and Quantum Cosmology, (1994), pp. 1–6.

[22] J. Lafontaine, *An Introduction to Differential Manifolds*, Springer, 2nd ed., 2015.

[23] J. M. Lee, *Introduction to Smooth Manifolds*, J. M. Lee, 2000.

[24] C. B. Morrey Jr., *On the Analyticity of the Solutions of Analytic Non-Linear Systems of Partial Differential Equations*, American Journal of Mathematics, 80. (1958), pp. 198–218.

[25] R. S. Palais and C. Terng, *A General Theory of Canonical Forms*, Transactions of the American Mathematical Society, 300 (1987), pp. 771–789.

[26] R. Penrose, *Techniques of Differential Topology in Relativity*, Society for Industrial and Applied Mathematics, 1972.

[27] F. Qi, *Bounds for the Ratio of Two Gamma Functions*, Journal of Inequalities and Applications, 2010 (2010), pp. 1–84.

[28] I. S. Sokolnikoff, *Tensor Analysis Theory and Applications*, John Wiley Sons, Inc, 1951.

[29] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, third ed., 1999.

[30] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.

[31] F. W. Warner, *Foundations of Differential Manifolds and Lie Groups*, Scott, Foresman and Company, 2nd ed., 1983.