Connections Between Type A Quiver Loci and Positroid Varieties in the Grassmannian

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Abstract

In [KR15], it was shown that each type A quiver locus is closely related to a Schubert variety in a partial flag variety. In this thesis, we adapt the construction to show that type A quiver loci are also closely related to positroid varieties in Grassmannians. An important idea in producing this construction is a new combinatorial identification between quiver rank arrays and bounded affine permutations.

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Chapter 1

Introduction

1.1 Overview

The primary aim of this thesis is to establish a connection between two algebro-geometric families of objects: bipartite type A quiver loci and positroid varieties intersected with a particular open patch in the Grassmannian. Specifically, we will show that each type A bipartite quiver locus is isomorphic to an open patch of a positroid variety in a Grassmannian.

Similar to their archery-based counterparts, mathematical quivers are collections of arrows. Put into more technical terms, quivers are finite directed graphs, often with the restriction that loops are not allowed (no arrows that start and end at the same vertex). The family of quivers that this thesis aims to investigating are bipartite type A quivers - “bipartite” specifies that each node/vertex is either a source or a sink (all arrows pointing either towards or away from the node) and “type A” indicates that the quivers have an underlying graph that is a Dynkin diagram of type A (all in a line with no branching). We can study the space of representations of a quiver by assigning a vector space to each of the nodes and a linear map to each arrow. This space has an algebraic variety structure and has an associated action given by the base change group. Realizing our collection of linear maps (arrows) as matrices, we can identify the each orbit closure of the base change group, or **quiver locus**, via the imposition of particular rank conditions on particular concatenations of these matrices. More on quiver representations and quiver loci can be found in §3.1.

The Grassmanian, $\mathrm{Gr}(k,n)$, is an object of considerable interest in algebraic geometry. $\mathrm{Gr}(k,n)$ is the space of k-dimensional subspaces of a vector space. One realization of the Grassmannian is as the space of full-rank $k \times n$ matrices, up to row operations. $\mathrm{Gr}(k,n)$ can be realized as a projective variety in $\mathbb{P}^{\binom{n}{k}-1}$ via the Plücker embedding. Homogeneous coordinates are given by the collection of all $\binom{n}{k}$ determinants of $k \times k$ submatrices. Positroid varieties are a distinguished set of subvarieties of the Grassmanian, introduced by Postnikov in [Pos06]. Along with having several avenues for definition, positroid varieties are indexed in the Grassmannian by various combinatorial objects (e.g. decorated permutations, grassmann necklaces, certain plabic graphs, etc.), and this rich combinatorial structure partially motivates their interest. In this thesis, we will be focusing on the indexing of positroid varieties by juggling patterns. More on the Grassmannian and the positroid varieties we will be working with can be found in §3.2 and §3.3.

The research presented in this thesis is inspired by work that was done by Kinser and Rajchgot in [KR15]. There, the authors constructed an identification of each bipartite type A quiver variety with a Kazhdan-Lusztig variety (i.e. an intersection of a Schubert variety with an opposite Schubert cell in a partial Flag variety). They then showed that the bipartite setting is the “correct” setting to study these objects by providing a method that relates arbitrary type A quiver loci to bipartite type A quiver loci.

1.2 Main Results

Here we summarize the three steps necessary to construct the identification between bipartite type A quiver loci and our particular positroid varieties.

We start with a representation of a quiver of type A in the bipartite orientation, where each of the vector spaces are over a field K, and whose dimensions are stored in *dimension vector* $\underline{d} = (d_0, \dots, d_n)$:

$$\begin{tikzcd}[column sep=small] & K^{d_1} \arrow[dl, "M_1"'] \arrow[dr, "M_2"] & & \cdots & K^{d_{n-1}} \arrow[dr, "M_{n-1}"] & \\ K^{d_0} & & K^{d_3} & & & K^{d_n} \end{tikzcd},$$

We then construct the cyclic Zelevinsky map, which provides us with our first stepping-stone.

Proposition. *(See Proposition 4.9) Let Q and $\underline{d}$ be a fixed quiver and associated dimension vector, and let $rep_Q(\underline{d})$ be the space of representations. Let U be the open patch of the Grassmannian $Gr(d_y, d)$ whose points have the form*

$$\left[\ * \ | I_{d_y}\right],$$

where $*$ is used to denote any $d_y \times d_x$ matrix with entries in K. Then there exists a map

$$\xi : rep_Q(\underline{d}) \to U$$

that identifies the orbit closures Ω_r in $rep_Q(\underline{d})$ with a block cyclic rank variety $C_{c(r)} \subseteq U$.

For our second step along our path, we associate to each variety $C_{c(r)}$ a bounded affine permutation. It is stated precisely in Proposition 5.1.

Proposition. *(See Proposition 5.1) Let $c(r)$ be a cyclic rank array associated to $\Omega_r \subset rep_Q(\underline{d})$. Then there exists a distinguished bounded affine permutation $\nu_c(r)$, with block structure imposed by $\underline{d}$, such that the number of 1's in each block (i, j) is equal to*

$$|[i, j]| - c(r)_{i,j} - \#\ \text{1's strictly SW of block } (i, j).$$

The final connection in our journey is to use the combinatorial data encoded in the bounded affine permutation $\nu_c(r)$ to show that each variety $C_{c(r)}$ is isomorphic to a positroid variety intersected with U. This last step provides us the foremost result of this thesis.

Theorem. *(See Theorem 6.1) Let $\Omega_r \subset rep_Q(\underline{d})$ be a quiver locus, and let U be defined as above. Then there exists a positroid variety Π, cut out by conditions given by the bounded affine permutation $\nu_c(r)$, such that*

$$\Pi \cap U \cong \Omega_r.$$

Chapter 2

Combinatorial Preliminaries

The content in this chapter is a review of relevant combinatorial mathematics, and our primary source is [KLS13]. As a supplement to this, [Sni10] includes useful descriptions and examples on juggling patterns.

2.1 Affine Permutations

The symmetric group S_n, or the group of permutations on n objects, is a familiar concept that one typically encounters in an undergraduate math program. Given a finite set X of n objects, elements of S_n are bijections that send X to X. A reasonable next question would be if we can extend this notion to sets of infinite cardinality, and indeed we can. Taking our X to be $\mathbb{Z}$, one such extension is **affine permutations**.

The set of affine permutations, denoted $\tilde{S}_n$, is the set of bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(i+n) = f(i) + n$$

for every $i \in \mathbb{Z}$. Notice that f is periodic.

A significant quantity associated to affine permutations (which will be reinterpreted in §2.2) is

$$\text{av}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(i) - i),$$

and is always integer-valued (see[Sni10], §2.2 Theorem 2). We thus define the set

$$\tilde{S}_n^k = \left\{ f \in \tilde{S}_n \mid \text{av}(f) = k \right\}.$$

We say that an affine permutation $f \in \tilde{S}_n^k$ is **bounded** if $i \leq f(i) \leq i + n$, and hence denote the set of bounded affine permutations by $\text{Bound}(k, n)$. When writing a bounded affine permutation $f \in \text{Bound}(k, n)$ explicitly, we can use one-line notation by noting where f sends one full period:

$$[\ldots f(1), f(2), \ldots, f(n), \ldots].$$

We construct a **bounded affine permutation matrix** for elements of $\text{Bound}(k, n)$ in the same way that we do for elements of S_n, by placing a 1 in row i and column $f(i)$, with zeros elsewhere. However, unlike permutation matrices, bounded affine permutation matrices are $\infty \times \infty$-type matrices. The 1's in a bounded affine permutation matrix occur within an n-wide diagonal strip since $i \leq f(i) \leq i + n$, and their placement along the diagonal is n-periodic since $f(i + n) = f(i) + n$, and so it suffices to restrict to an $n \times 2n$ matrix when writing a bounded affine permutation matrix.

Example 2.1. Consider the affine permutation $f \in \tilde{S}_4$ whose one-line notation is

$$[\dots f(1), f(2), f(3), f(4), \dots] = [\dots, 4, 3, 5, 6, \dots] .$$

We have that $av(f) = 2$, and f satisfies $i \leq f(i) \leq i + 4$, and so $f \in \mathrm{Bound}(2, 4)$. The bounded affine permutation matrix for f is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & 0 & 0 & & \\ & & 0 & 0 & 1 & 0 & 0 & \\ & & & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

Remark 2.2. In [Sni10], the discussion on affine permutations in §2.1 (which we have largely followed here) notates an affine permutation $f \in \tilde{S}_n$ via its one-line notation, that is by recording the action of f on one full period 1 through n. When the author then introduces affine permutation matrices at the beginning of §3.3.2, they instead use the siteswap notation for an affine permutation. In this thesis, we retain the use of one-line notation in the definition of bounded affine permutation matrices. However, the resulting matrices under either convention are the same.

2.2 Juggling Patterns

The set of **virtual juggling patterns** is defined as the set of affine permutations $\tilde{S}_n$, which motivates some of the terminology in §2.1. In particular, for a virtual juggling pattern (affine permutation) $f \in \tilde{S}_n$, the quantity $\mathrm{av}(f)$ is known as the **ball number** of f, and indicates the number of balls required if one were to physically juggle f. Thus, the set $\tilde{S}_n^k$ is reinterpreted as the set of **k-ball virtual juggling patterns**. Note that $f(i) - i$ is permitted to be negative (i.e. f sends i backwards); by requiring that juggling patterns satisfy $f(i) \geq i$ for each i (i.e. f sends all i's forward), we may refer to them simply as **juggling patterns**. Finally, just as we generated $\mathrm{Bound}(k, n)$ by restricting $\tilde{S}_n^k$ to those affine permutations satisfying $i \leq f(i) \leq i + n$, we get the set of bounded juggling patterns by requiring that our juggling pattern satisfies the same condition. See [Sni10] §2.2 for more details on this construction.

We notate a juggling pattern f in one-line notation

$$[\dots f(1) f(2) \dots f(n) \dots] .$$

We can equivalently notate f by its **siteswap**, which lists a single period of the function $f(i) - i$ (as explained in [KLS13], commas are not used in the one-line or siteswap notations as juggling 10 or more balls is a rare skill.). We can interpret the one-line notation as where the ball in hand at time i gets sent, whereas siteswap indicates how many time-steps ahead the ball will land. In this thesis, when we explicitly write a bounded affine permutation, we will use the matrix notation; when we are referring to a juggling pattern, we will use either the one-line or siteswap notations.

Many of the concepts surrounding juggling patterns indeed originated in the juggling community, and is the defining motivation for much of the terminology. As such, it is useful have a picture of how this is all interpreted from a juggler's perspective (the description that follows is inspired by the picture painted in [KLS13] §3.1). Imagine a juggler juggling a set of k balls, with either one hand or alternating with two hands. Every second, the juggler interacts with one and only one ball, potentially catching and then immediately (or, from video analysis of experienced jugglers, more typically exactly one half period later) throwing a ball with the same hand. How many seconds in the future (or potentially in the past, in the case of virtual juggling patterns) the juggler catches the ball thrown at time i is given by the juggling pattern $f(i)$, and this quantity (calculated by $f(i) - i$) is called the **throw at time** i. Note that the juggler may not have a ball in hand for every second, at which point they have to wait until the next ball comes down; we would call such a situation a 0-throw or an empty hand, as in this case $f(i) = i$. We can abstractly visualize our narrative of the juggler with a diagram consisting of a row of points (or two rows if we interpret the juggler juggling with alternating hands), with arcs connecting point i with $f(i)$ (this visualization is inspired by [Sni10], figure 2.5).

Example 2.3. Consider the bounded affine permutation f discussed in Example 2.1, whose one-line notation, $[\ldots, 4, 3, 5, 6, \ldots]$. When written with the juggling pattern convention, we omit the commas, $[\ldots 4356 \ldots]$, and we can write the siteswap by subtracting the position from each entry to get [3122]. By representing the integers as a series of labelled dots, we can visually represent this juggling pattern by a series of arcs connecting dots to where they get sent under the action of f:

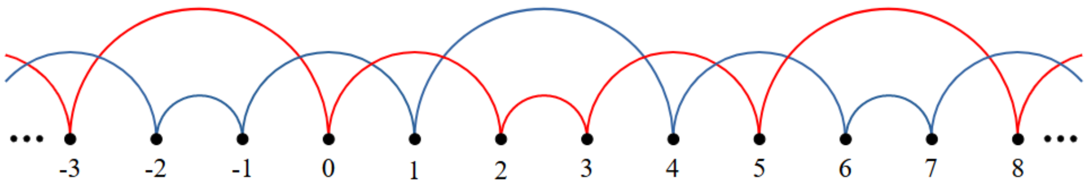


Figure 2.1: *The juggling pattern [4356]. The two "strands" of arcs are coloured blue and red, and so if one were to physically juggle this pattern, two balls would be required.*

2.3 Cyclic Rank Matrices

Convention 2.4. We draw from the notational convention outlined in [KLS13] §2.1. For $n \in \mathbb{N}$, let $[n]$ be the set $\{1, 2, \ldots, n\}$. Then for some integer $i \in \mathbb{Z}$, we denote by $\bar{i} \in [n]$ the unique integer that satisfies $i \cong \bar{i} \bmod n$.

Analogous to northwest ranks, cyclic ranks are the ranks of a particular submatrix of a given matrix. Let M be a full rank $k \times n$ matrix (and thus the matrix for a point in $\mathrm{Gr}(k, n)$; see §3.2). Then, given a pair of integers i and j, $i \leq j$, we define an **interval** $[i, j]$ to be the set $\left\{\bar{i}, \overline{i+1}, \ldots, \bar{j}\right\}$, where the elements are members of $[n]$ according to the above convention. Notice here that, if $j \geq i+n-1$, then $[i, j] = [n]$. Now, labelling the columns of M by $1, 2, \ldots, n$, we define the **cyclic submatrix** associated to $[i, j]$ as the matrix $M_{([i,j])}$ composed of columns in $[i, j]$ taken in the order they appear in M, with the first column of M following the last. We denote by $|[i, j]| = j - i + 1$ the **size of the interval** $[i, j]$, and in the case where $i \leq j < i + n$ this corresponds to the number of columns in $M_{([i,j])}$. The **cyclic rank** associated to $[i, j]$ is thus the rank of the submatrix $M_{([i,j])}$.

Remark 2.5. We extend $|[i, j]|$ as defined here to i, j such that $j < i$, despite the interval $[i, j]$ not being defined for those i and j. One way we can interpret this extension is "how many columns away from indexing a well-defined interval i and j are". For example, if we interpret increasing j as "adding columns to the right of a cyclic submatrix" and decreasing i as "adding columns to the left of a cyclic submatrix", then for $j = i - 2$ the interval $[i, j]$ is "two columns away" from defining a proper cyclic submatrix, since we would need to "add" two columns by decreasing i by 2, increasing j by 2, or decreasing and increasing i and j both by 1.

Remark 2.6. When referring to a cyclic submatrix $M_{([i,j])}$ in practice, since we are using the interval $[i, j]$ to indicate a set of columns of M, we will frequently abuse notation and write $M_{([\bar{i},\bar{j}])}$ (where $\bar{i}$ indicates the "start column" and $\bar{j}$ indicated the "end column" of $M_{([i,j])}$, and $\bar{j}$ is potentially smaller than $\bar{i}$). When we do this, we still mean the matrix composed of columns in $[i, j]$, taken cyclically in the order they appear in M.

We now define a **cyclic rank matrix** as it is defined in [KLS13], then discuss how we store cyclic ranks in a cyclic rank matrix.

Definition 2.7 ([KLS13], Corollary 3.12)**.** *Let $(r_{i,j})$ be an $\infty \times \infty$ matrix satisfying the following:*

C1. $r_{i,j} = j - i + 1$ *for all* $j < i$,

C2. $r_{i,j} = k$ *for all* $j \geq i + n - 1$,

C3. $r_{i,j} - r_{i+1,j} \in \{0, 1\}$ *and* $r_{i,j} - r_{i,j-1} \in \{0, 1\}$,

C4. *if* $r_{i+1,j-1} = r_{i+1,j} = r_{i,j-1}$ *then* $r_{i,j} = r_{i+1,j-1}$, *and*

C5. $r_{i+n,j+n} = r_{i,j}$.

We call $(r_{i,j})$ a cyclic rank matrix.

By C5., cyclic rank matrices are diagonally n-periodic, and so we may restrict to an $n \times 2n$ tile when working with cyclic rank matrices.

To see how Definition 2.7 stores the cyclic ranks of some full-rank $k \times n$ matrix M, notice that it outlines three regions in the $\infty \times \infty$ matrix $(r_{i,j})$. The first is described by C1., where for $j < i$ we set $r_{i,j} = j - i + 1$. The remaining two regions reflect how cyclic rank matrices store cyclic ranks: for $i \leq j$, we set

$$r_{i,j} = \operatorname{rank}(M_{([i,j])}).$$

The region defined by C2. is thus a consequence of the fact that, when $j \geq i + n - 1$, we have that $M_{([i,j])} = M$, and so $\operatorname{rank}(M_{([i,j])}) = \operatorname{rank}(M) = k$. The remaining region, when $i \leq j < i + n - 1$, corresponds to the ranks of cyclic submatrices not composed of all columns of M, and contains the bulk of the meaningful data in $(r_{i,j})$. See [KLS13], §5.1 for details. This method for populating $(r_{i,j})$ is consistent with C3-5.: adding or subtracting a column to a cyclic submatrix changes the rank by at most 1, giving us C3.; if adding a column to either side of a cyclic submatrix does not change the rank, then neither will adding both, giving us C4.; and since we are working cyclically over a set of n columns, $M_{([i+n,j+n])} = M_{([i,j])}$, giving us C5..

The bijection between $\operatorname{Bound}(k, n)$ and cyclic rank matrices is provided in [KLS13] Corollary 3.12. The reverse direction of this bijection is made explicit by the use of **special entries**, which are entries $r_{i,j}$ in $(r_{i,j})$ such that $r_{i,j} = r_{i+1,j} = r_{i,j-1} > r_{i+1,j-1}$. For a given cyclic rank matrix, the special entries identify the positions of the 1's in the corresponding bounded affine permutation matrix.

Example 2.8. Consider the 2×4 full-rank matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Since M is a 2×4 matrix, the cyclic rank matrix for M will be a 4×8 matrix, whose greatest entry will be a 2. We populate the cyclic matrix of M by calculating the ranks of all the cyclic submatrices of M. For example, the cyclic submatrix $M_{([2,3])}$ is

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

and so $r_{2,3} = 1$, while $M_{([4,2])}$ is

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and so $r_{4,6} = 2$. Carrying this process out for all entries not associated to C1. and C2. in Definition 2.7, we get the cyclic rank matrix

$$\begin{bmatrix} 1 & 2 & 2 & \underline{2} & 2 & & & \\ & 1 & \underline{1} & 2 & 2 & 2 & & \\ & & 1 & 2 & \underline{2} & 2 & 2 & \\ & & & 1 & 2 & \underline{2} & 2 & 2 \end{bmatrix},$$

whose special entries have been underlined. Replacing the underlined entries with 1's and filling in zeros elsewhere, we get the associated bounded affine permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & 0 & 0 & & \\ & & 0 & 0 & 1 & 0 & 0 & \\ & & & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

whose one line notation is $[\ldots, 4, 3, 5, 6, \ldots]$, the same bounded affine permutation discussed in Examples 2.1 and 2.3.

Chapter 3

Geometric Preliminaries

3.1 Quivers and their representations

We follow the treatment of quivers that can be found in [KR15] §2.1 and [KKR19] §2.1. For a broader discussion on quivers in the equioriented setting, see [MS05] chapter 17.

A **quiver** Q is a finite directed graph, whose vertex set we denote by Q_0, and whose arrow set we denote by Q_1. If the underlying graph is a tree with no branches (that is, all in a line with no forks or loops), then we call the quiver a **type A quiver**, as the underlying graph is of the form of a Dynkin diagram of type A. Much of the literature on type A quivers focused on the case where the quivers are **equioriented**, meaning that all the arrows of the quiver are pointed in the same direction:

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

In this thesis, we focus on type A quivers in the **bipartite** orientation, where each node is either a source or a sink (that is, each vertex has either all arrows pointing towards or away from it):

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & & \cdots & \bullet \arrow[dr] & \\ \bullet & & \bullet & & & \bullet \end{tikzcd}$$

To study arbitrarily-oriented type A quivers, one may restrict to the bipartite orientation whilst retaining much of the structure found in the theory. This was the viewpoint taken by [KR15], where the authors show that type A quivers in any orientation can be extended to a bipartite quiver by adding vertices and arrows in a particular way. This extension is not covered here, but can be found in [KR15] §5.

To establish a notational convention, we will label the vertices and arrows of Q by

$$Q = \begin{tikzcd} & x_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & x_2 \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_2"] & & \cdots & x_{n-1} \arrow[dr, "\beta_{n-1}"] & & x_n \arrow[dl, "\alpha_n"'] \arrow[dr, "\beta_n"] & \\ y_0 & & y_1 & & y_2 & & & y_{n-1} & & y_n \end{tikzcd}$$

Remark 3.1. Note that the above convention treats quivers with an odd number of vertices; to deal with an even number of vertices, delete one of the end points (see [KR15] §2.1 or [KKR19] §2.1).

We specify the vertex at the tail of an arrow $\gamma \in Q_1$ by the notation $t(\gamma) \in Q_0$, and the vertex at the head by $h(\gamma) \in Q_0$. We will further define an **interval of** Q, denoted $J = [\gamma_l, \gamma_r] \subseteq Q$, to be a connected subquiver of Q, where γ_l and γ_r indicate the leftmost and rightmost arrows of the subquiver, respectively.

Take K to be an arbitrary field. We construct a **representation of** Q, call it V, by assigning to each vertex $z \in Q_0$ a finite-dimensional vector space V_z, and to each arrow $\gamma \in Q_1$ an appropriate linear map $M_\gamma : V_{t(\gamma)} \to V_{h(\gamma)}$.

The vector space V_z will usually be taken as $K^{d(z)}$, where we view $d(z)$ as the image of the map $d : Q_0 \to \mathbb{Z}_{\geq 0}$. We store the (ordered) collection of all such values for each V_z in a vector $\underline{d}$, called the **dimension vector**. In the bipartite orientation, it will be useful to describe the combined dimensions of both the source and sink vertices, and so we define

$$d_x = \sum_{i=1}^{n} d(x_i), \qquad d_y = \sum_{i=0}^{n} d(y_i), \qquad \text{and} \qquad d = d_x + d_y,$$

where x_i and y_i are source and sink vertices, respectively. Given a quiver Q and dimension vector $\underline{d}$, one has an associated quiver representation space:

$$\mathrm{rep}_Q(\underline{d}) := \prod_{a \in Q_1} \mathrm{Mat}_{d(ha),d(ta)}(K),$$

where $\mathrm{Mat}_{m,n}(K)$ denotes the algebraic variety of matrices with m rows, n columns, and entries in K. To be somewhat more concrete, we can think of an element $V \in \mathrm{rep}_Q(\underline{d})$ as the collection of matrices

$$V = \{A_1, B_1, A_2, B_2, \ldots, A_n, B_n\},$$

where A_i is a $d(x_i) \times d(y_{i-1})$ matrix and B_i is a $d(x_i) \times d(y_i)$ matrix.

3.1.1 Orbits and quiver rank conditions

For a quiver Q and dimension vector $\underline{d}$, we define an associated **base change group** on the representation space by

$$\mathrm{GL}(\underline{d}) := \prod_{z \in Q_0} \mathrm{GL}_{d(z)}(K)$$

where $\mathrm{GL}_{d(z)}(K)$ represents the group of invertible $d(z) \times d(z)$ matrices with entries in K. The base change group acts in the following way: If $g = (g_z)_{z \in Q_0} \in \mathrm{GL}(\underline{d})$ and $V = (V_\gamma)_{\gamma \in Q_1}$ then

$$g \cdot V = (g_{h(\gamma)} V_\gamma g_{t(\gamma)}^{-1})_{\gamma \in Q_1}.$$

The action of $\mathrm{GL}(\underline{d})$ partitions $\mathrm{rep}_Q(\underline{d})$ into orbits. We say $V, W \in \mathrm{rep}_Q(\underline{d})$ are in the same orbit if there is some $g \in \mathrm{GL}(\underline{d})$ such that $g \cdot V = W$.

Determining whether two representation belong to the same orbit in the above way can be difficult in practice. To introduce an alternative method, as well as some notions that will be critical going forward, consider the map $M_Q : \mathrm{rep}_Q(\underline{d}) \to \mathrm{Mat}_{d_y \times d_x}$ given by

$$V = (V_{\alpha_1}, V_{\beta_1}, V_{\alpha_2}, V_{\beta_2}, \ldots, V_{\alpha_n}, V_{\beta_n}) \longmapsto \begin{bmatrix} & & & V_{\alpha_1} \\ & & V_{\alpha_2} & V_{\beta_1} \\ & \iddots & \iddots & \\ V_{\alpha_n} & V_{\beta_{n-1}} & & \\ V_{\beta_n} & & & \end{bmatrix}.$$

Evaluated at $V \in \text{rep}_Q(\underline{d})$, this matrix is informally called the "snake matrix", and is comprised of the maps of V stacked in a zigzag along the double anti-diagonal so that their dimensions agree, with zero-blocks elsewhere. For an interval $J = [\gamma_l, \gamma_r]$ of Q, where $\gamma_l, \gamma_r \in Q_1$, we define the submatrix $M_J(V) = M_{[\gamma_l,\gamma_r]}(V)$ by

$$M_J(V) = \begin{bmatrix} & \iddots & V_{\gamma_l} \\ V_{\gamma_r} & \iddots & \end{bmatrix}.$$

The set of **quiver ranks** of V are the ranks of the matrices $M_J(V)$ for intervals $J \subseteq Q$; these data are stored in the **quiver rank array** r, which is a function

$$r : \{\text{intervals of } Q\} \to \mathbb{Z}_{\geq 0}$$

so that for some $V \in \text{rep}_Q(\underline{d})$, we have that r_J is the rank of $M_J(V)$ for all intervals $J \subseteq Q$. If we have a representation V for which this is true, we say that V **satisfies r** ([KR15] §3.2). For each interval J, the action of the base change group $\text{GL}(\underline{d})$ on $V \in \text{rep}_Q(\underline{d})$ leaves the rank of $M_J(V)$ unchanged ([KR15] §3.1). Recall that the $\text{GL}(\underline{d})$-action on $\text{rep}_Q(\underline{d})$ yields orbits, with two representations $V, W \in \text{rep}_Q(\underline{d})$ belonging to the same orbit if $g \cdot V = W$ for some $g \in \text{GL}(\underline{d})$. Since this action preserves quiver ranks, by Proposition 3.1 in [KR15] we have that two representations V and W belong to the same orbit if and only if they share the same quiver rank array. These orbits are indexed by rank arrays, and so we notate orbits by $\mathcal{O}_r \subseteq \text{rep}_Q(\underline{d})$, denoting the orbit closure by $\overline{\mathcal{O}_r}$.

Example 3.2. Let Q be the quiver

$$\begin{tikzcd} & x_1 \arrow[dr, "\beta_1"] & & x_2 \arrow[dl, "\alpha_2"] \\ \text{Q=} & & y_1 & \end{tikzcd}$$

equipped with dimension vector $\underline{d} = (2, 3, 1)$, and let V and W be elements of $\text{rep}_Q(\underline{d})$ with arrow maps

$$V_{\beta_1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad V_{\alpha_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad W_{\beta_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad W_{\alpha_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Our possible intervals are $[\beta_1, \beta_1], [\alpha_2, \alpha_2]$, and $[\beta_1, \alpha_2]$ (which in this case correspond to the arrow maps individually and the full snake matrix M_Q). Starting with V, our quiver ranks are thus

$$
\begin{aligned}
r_{[\beta_1,\beta_1]} &= \operatorname{rank}\left(M_{[\beta_1,\beta_1]}(V)\right) = \operatorname{rank}\left(\begin{bmatrix}1 & 1\\ 0 & 0\\ 1 & 0\end{bmatrix}\right) = 2\\
r_{[\alpha_2,\alpha_2]} &= \operatorname{rank}\left(M_{[\alpha_2,\alpha_2]}(V)\right) = \operatorname{rank}\left(\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right) = 1\\
r_{[\beta_1,\alpha_2]} &= \operatorname{rank}\left(M_{[\beta_1,\alpha_2]}(V)\right) = \operatorname{rank}\left(\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}\right) = 2
\end{aligned}
$$

which we store in the rank array r:

$$
r = \begin{array}{cc|c} \alpha_2 & \beta_1 & \\ \hline & 2 & \beta_1 \\ 1 & 2 & \alpha_2 \end{array}.
$$

Performing the same computations for W yields the same rank array, and so we say that V and W belong to the same orbit $\mathcal{O}_r$.

We define a **quiver locus** Ω_r as the orbit closure $\overline{\mathcal{O}_r}$. By Theorem 4.12 in [KR15], one orbit closure $\mathcal{O}_{r'}$ is contained in another $\mathcal{O}_r$ if and only if $r' \leq r$ (that is, $r'_J \leq r_J$ for all intervals J). Thus, for some representation $V \in \operatorname{rep}_Q(\underline{d})$, we say that V is in the quiver locus Ω_r if and only if, for each $J \subseteq Q$, we have that $\operatorname{rank}(M_J(V)) \leq r_J$. Quiver loci defined in this way are varieties, with defining ideal

$$
I_r = \langle \text{minors of size } (1 + r_J) \text{ in } M_Q(V) \mid J \subseteq Q \rangle
$$

([KR15], §4). By Proposition 4.12 in [KR15], I_r is radical.

3.2 The Grassmannian

Much of the treatment of the Grassmannian presented here follows the treatments that can be found in [Gil19, SKKT13, MS05].

3.2.1 Definitions

Working over K^n where K is a field, with $k \leq n$, we define the **Grassmannian** $\mathrm{Gr}(k,n)$ as the set of all k-dimensional subspaces of K^n passing through the origin. We call the elements of $\mathrm{Gr}(k,n)$ the **points** of the Grassmannian; by describing each k-plane as the rowspan of a full-rank $k \times n$ matrix (i.e. the span of k linearly independent row vectors of size n), and recalling that row operations leave the rowspan of a matrix unchanged, we can think of points in the Grassmannian $\mathrm{Gr}(k,n)$ as full-rank $k \times n$ matrices up to row operations. Each point on the Grassmannian is thus uniquely represented by a full rank matrix in reduced row echelon form (rref). Choice of how to arrange the rows into echelon form varies by convention - for example, arranging the leading 1s from bottom to top provides a natural understanding of Young tableaux. For the sake of consistency and focus, however, we do not treat this subject here.

Example 3.3. Consider the following points in $\mathrm{Gr}(2,4)$:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 & -2 & -1 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

Notice that these are really the same point in $\mathrm{Gr}(2,4)$ since their reduced row echelon forms are both

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Points in the Grassmannian whose rrefs share the same form (i.e. positions of leading 1's) constitute a **Schubert Cell**. Thus, we can view the Grassmannian as a disjoint union of Schubert Cells.

Example 3.4. Our point from the previous example lives in the Schubert cell described by

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix},$$

where the entries labelled with $*$ choices of values. Points in $\mathrm{Gr}(2,4)$ who live in the same Schubert cell have rrefs that differ only in the $*$ entries.

Remark 3.5. $\mathrm{Gr}(1,n)$ is isomorphic to $\mathbb{P}^{n-1}$, which can be seen by constructing a set of homogeneous coordinates from the n entries in the matrix representation of elements in $\mathrm{Gr}(1,n)$. Details can be found in §3.2.2.

3.2.2 The Plücker Embedding

We can embed the Grassmannian $\mathrm{Gr}(k,n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$ with the **Plücker embedding** by extracting a set of homogeneous coordinates from each point in the Grassmannian. We first pick a subset $\sigma \subseteq [n]$ with k-many elements. Then, for a matrix M representing a point in $\mathrm{Gr}(k,n)$, we define M_σ as the submatrix consisting of the columns of M indexed by σ. We call the determinant of such a submatrix a maximal minor of M, labelled p_σ. Since row operations only change determinants up to multiplication by a constant, the collection of $\binom{n}{k}-1$ maximal minors of M yields homogeneous coordinates in $\mathbb{P}^{\binom{n}{k}-1}$. We call these coordinates the **Plücker Coordinates** of M.

Example 3.6. Continuing with our running example, notice that the maximal minor associated to columns 1 and 2 is given by

$$p_{12} = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1.$$

Similarly,

$$p_{13} = 2, \quad p_{14} = 3, \quad p_{23} = 1, \quad p_{24} = 2, \quad p_{34} = 1.$$

Thus, the image of

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

under the Plücker embedding is the point

$$[p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] = [1 : 2 : 3 : 1 : 2 : 1] \in \mathbb{P}^{\binom{4}{2}-1}.$$

We can show that the image of $\mathrm{Gr}(k, n)$ under the Plücker Embedding is a projective variety in $\mathbb{P}^{\binom{n}{k}-1}$. This variety is cut out by the **Plücker relations**, a set of homogeneous quadratic equations in the variables p_σ.

Example 3.7. For $\mathrm{Gr}(2, 4)$, points are represented by matrices of the form

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}.$$

Hence, consider the map

$$\begin{aligned} \phi : K[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] &\to K[a, b, c, d, e, f, g, h] \\ p_{i,j} &\mapsto \det\left(M_{[i,j]}\right). \end{aligned}$$

where $M_{[i,j]}$ denotes the submatrix consisting of columns i and j. The Plücker coordinates under ϕ are thus

$$\begin{aligned} p_{12} &\mapsto af - be, & p_{13} &\mapsto ag - ce, & p_{14} &\mapsto ah - de, \\ p_{23} &\mapsto bg - cf, & p_{24} &\mapsto bh - df, & p_{34} &\mapsto ch - dg. \end{aligned}$$

Notice that

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \mapsto 0,$$

and so the above relation lies in the kernel of ϕ; we call this relation a Plücker relation. For $\mathrm{Gr}(2, 4)$ this is the only Plücker relation, and so under the Plücker embedding, $\mathrm{Gr}(2, 4)$ maps to

$$\mathbb{V}(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subseteq \mathbb{P}^5$$

3.2.3 Open Cover of the Grassmannian via Affines

The treatment of the open cover of projective space via affines here follows the one that can be found in [SKKT13].

Consider the projective space $\mathbb{P}^n$ identified by the set of homogeneous coordinates

$$[x_0 : x_1 : x_2 : \cdots : x_n].$$

Recall that we get the open patch U_0 by requiring that the coordinate x_0 is non-zero, which we can write as

$$[x_0 : x_1 : x_2 : \cdots : x_n] = \left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \cdots : \frac{x_n}{x_0}\right].$$

With the map defined by

$$\left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \cdots : \frac{x_n}{x_0}\right] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \cdots, \frac{x_n}{x_0}\right),$$

notice that we can identify U_0 with the affine space K^n, and so we have an open affine patch of $\mathbb{P}^n$, achieved by choosing x_0 non-zero. We can make this same choice for each of the other homogeneous coordinates $x_1, \ldots, x_n$, yeilding a family of open affine patches U_i, $i = 0, 1, \ldots, n$; taking the union, we get a cover of $\mathbb{P}^n$,

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

Through the pullback of the Plücker embedding, this open cover of projective space becomes an open cover of the Grassmannian. In the case of projective space, each affine patch was generated by making a choice of homogeneous coordinate to be non-zero; in the Grassmannian case, we choose a determinant of a maximal minor to be non-zero. Without loss of generality, we do this by choosing columns in the matrix representation of the Grassmannian to be columns of an identity, so that when we take the Plücker coordinate associated to those columns, its value is 1.

Example 3.8. Throughout this thesis, we will be working with the open patch $U \subseteq \mathrm{Gr}(k, n)$ that is isomorphic to the space of matrices whose points share the form

$$[\ *\ | I],$$

where $*$ represents an arbitrary $k \times (n-k)$ matrix, and I is a $k \times k$ identity.

3.3 Positroid Varieties

Positroids and positroid varieties can be defined in a number of ways; for the purposes of this thesis, we will define a positroid variety from the perspective of bounded affine permutations, more thoroughly discussed in [KLS13, Sni10].

Remark 3.9. To underscore the fact that the bounded affine permutations we are working with from now on are being used to define positroid varieties, we will notate them by ν. This is in contrast to when we introduced them in §2.1, where we used f to denote bounded affine permutations (and then used f to denote juggling patterns in the section after).

Following the description that can be found in §3.3.2 of [Sni10], let f be a bounded juggling pattern, and let ν be the bounded affine permutation associated to f. The corresponding **positroid variety** is defined as

$$\Pi_f = GL_k \backslash \left\{ M \subseteq M_{k,n} \mid \mathrm{rank}(M_{([i,j])}) = |[i,j]| - \#\text{ 1's weakly SW of } (i,j), i \leq j \leq i+n \right\},$$

where $\mathrm{rank}(M_{([i,j])})$ is the rank of the cyclic submatrix $M_{([i,j])}$, and $|[i,j]|$ is the size of the interval $[i,j]$. By "#1's **weakly SW** of (i,j)", we mean the number of 1's in entries to the southwest of (i,j) in ν, including in (i,j) itself. Restricting to the open affine patch $U \subseteq \mathrm{Gr}(k,n)$ outlined in Example 3.8, we define

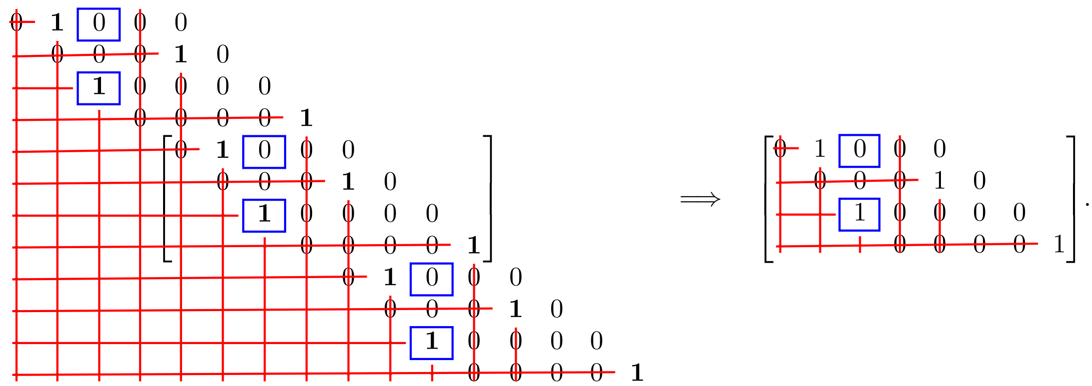
$$\Pi_f \cap U = \left\{ M \subseteq U \mid \mathrm{rank}(M_{([i,j])}) = |[i,j]| - \#\text{ 1's SW of } (i,j), i \leq j \leq i+n \right\}$$

Now, given the matrix for a bounded affine permutation $\nu \in \mathrm{Bound}(k,n)$, we define a **diagram** by shooting "death rays" south and west of the positions of the 1's, by which we mean crossing out entries strictly to the south and west of each 1 (not including the 1 itself); see [Sni10] §3.3.2. When constructing such a diagram, recall that bounded affine permutation matrices are $\infty \times \infty$-type matrices, though we may restrict to an $n \times 2n$ matrix as discussed in §2.1. The **essential set** $\mathrm{Ess}(\nu)$ of this diagram is composed of **essential boxes**, which are the northeast-most entries of the uncrossed regions completely enclosed by death rays. In the same way as with bounded affine permutation matrices, we may restrict the diagram to an $n \times 2n$ tile, since the diagram is diagonally n-periodic.

Example 3.10. Consider the bounded affine permutation $\nu \in \mathrm{Bound}(2, 4)$ whose one-line notation is $[\ldots, 2, 5, 3, 8, \ldots]$. The matrix for ν is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & & & \\ & 0 & 0 & 0 & 1 & 0 & & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We construct the diagram for ν by shooting death rays down and to the left of each of the 1's, not crossing the 1's themselves out, keeping in mind the periodicity but restricting to a single tile:



The entries in $\mathrm{Ess}(\nu)$, the essential boxes, have been outlined in blue.

Notice that, by construction of the diagram, increasing in i by taking a step south or decreasing in j by taking a step west (both of which correspond to shrinking the interval $[i, j]$) decreases the number of 1's seen to the southwest of the entry (i, j) in ν by 1. Simultaneously, $|[i, j]|$ decreases by 1, and so the condition on the rank of the cyclic submatrix $M_{([i,j])}$ in the above definition of $\Pi_f \cap U$ does not change. Conversely, decreasing in i by taking a step north or increasing in j by taking a step east (which corresponds to enlarging the interval $[i, j]$) increases $|[i, j]|$ by 1, while adding no new 1's to the southwest of (i, j) in ν. This increases the rank condition imposed on the cyclic submatrix $M_{([i,j])}$. Thus, despite each entry (i, j) in ν providing us with a rank condition in the definition of $\Pi_f \cap U$, it suffices to restrict to only those rank conditions provided by entries in the essential set ([Sni10], §3.2.2).

By Theorem 5.15 in [KLS13], we construct the defining ideal for $\Pi_f \cap U$ by determinantal conditions that enforce the associated rank conditions on matrices in U, restricting to conditions provided by the essential set of ν:

$$I_\nu = \langle \text{minors of size } (|[i, j]| - (\#\text{ 1's weakly SW of } (i, j) \text{ in } \nu) + 1) \text{ in } M_{([i,j])} \mid (i, j) \in \mathrm{Ess}(\nu)\rangle.$$

Chapter 4

Identification of Quivers with Cyclic Block Rank Varieties

Fix a bipartite type A quiver Q and dimension vector $\underline{d}$. Consider a representation $V \in \mathrm{rep}_Q(\underline{d})$

$$V = \begin{tikzcd}[column sep=small] & V_{x_1} \arrow[dl, "V_{\alpha_1}"'] \arrow[dr, "V_{\beta_1}"] & & V_{x_2} \arrow[dl, "V_{\alpha_2}"'] \arrow[dr, "V_{\beta_2}"] & \cdots & V_{x_{n-1}} \arrow[dr, "V_{\beta_{n-1}}"] & & V_{x_n} \arrow[dl, "V_{\alpha_n}"'] \arrow[dr, "V_{\beta_n}"] & \\ V_{y_0} & & V_{y_1} & & V_{y_2} & & V_{y_{n-1}} & & V_{y_n} \end{tikzcd}.$$

Recall that evaluating M_Q at V yields the "snake matrix"

$$M_Q(V) = \begin{bmatrix} & & & V_{\alpha_1} \\ & & V_{\alpha_2} & V_{\beta_1} \\ & \iddots & \iddots & \\ V_{\alpha_n} & V_{\beta_{n-1}} & & \\ V_{\beta_n} & & & \end{bmatrix}.$$

Next, take matrices of the form

$$\left[\ * \ | I_{d_y} \right],$$

where $*$ is an arbitrary $d_y \times d_x$ matrix with entries in K; the set of all such matrices are isomorphic to an open patch $U \subseteq \mathrm{Gr}(d_y, d_x + d_y)$. We define the **cyclic Zelevinsky map** ξ by

$$\begin{aligned} \xi : \mathrm{rep}_Q(\underline{d}) &\to U \subset \mathrm{Gr}(d_y, d_x + d_y) \\ V &\mapsto \begin{bmatrix} M_Q(V) & \mathbf{1}_{d_y} \end{bmatrix}. \end{aligned}$$

The block structure on the image of the cyclic Zelevinsky map is specified by the dimension vector $\underline{d}$, and is seen by more explicitly by evaluating ξ:

$$\xi(V) = \left[\begin{array}{cccc|ccccc} & & & V_{\alpha_1} & \mathbf{1}_{d(y_0)} & & & & \\ & & V_{\alpha_2} & V_{\beta_1} & & \mathbf{1}_{d(y_1)} & & & \\ & \iddots & \iddots & & & & \ddots & & \\ V_{\alpha_n} & V_{\beta_{n-1}} & & & & & & \mathbf{1}_{d(y_{n-1})} & \\ V_{\beta_n} & & & & & & & & \mathbf{1}_{d(y_n)} \end{array}\right].$$

We label the block columns of $\xi(V)$ by $1, 2, \ldots, 2n+1$ going from left to right. We will be working with these block columns cyclically as in §2.3. To deal with our context of working with block columns (as opposed to columns), we introduce the following notation and convention.

Convention 4.1. For a matrix Z with m block columns, we denote by $Z_{[[i,j]]}$ the **block cyclic submatrix** of Z, defined analagously to cyclic submatrices in §2.3, with the elements of $[i,j]$ coming from $[m]$, now composed of *block* columns in $[i,j]$ taken as they appear in Z, with the first block column following the last. We retain the use of the abuse of notation outlined in Remark 2.6 in this context.

For example, if Z is a matrix with 5 block columns, then the block cyclic submatrix $Z_{[[4,7]]}$ (or $Z_{[[4,2]]}$) is composed of block columns 4, 5, 1, and 2, taken in that order.

We will also be relabelling the block columns of matrices to be non-integers in this section to make certain discussions more clear. When we do this, we will abuse notation further and continue in the spirit of using $[-,-]$ to denote a pair of columns, with the first entry referring to the "start" block column of our cyclic submatrix $Z_{[[-,-]]}$, and the second entry referring to the "end" block column.

For example, if we label the block columns of Z with a, b, c, d, e, then the block cyclic submatrix $Z_{[[d,b]]}$ is composed of block columns d, e, a and b, taken in that order (this is the same block cyclic submatrix as $Z_{[[4,2]]}$).

Finally, we still use the notation $|[i,j]|$ to denote the size of the interval $[i,j]$. However, since $[i,j]$ no longer indexes columns, but *block* columns, in this context we redefine

$$|[i,j]| = \begin{cases} -\sum_{k=j+1}^{i-1} d(k) & j < i-1 \\ 0 & \text{if } j = i-1 \\ \sum_{k=i}^{j} d(k) & i \leq j \end{cases}$$

where for an integer k, we denote by $d(k)$ the number of columns in $Z_{[[k,k]]}$. Note that, if Z has n block columns, the size of the interval $|[i,j]|$ in the case where $i \leq j < i+n$ still corresponds to the number of columns in $Z_{[[i,j]]}$, not block columns. As in Convention 2.5, we have extended the use of $|[i,j]|$ in the block context to $j < i$, despite $[i,j]$ not being well-defined.

Remark 4.2. The usage of the notation $d(k)$ in the above Convention reflects the fact that, going forward, we will be working primarily with block cyclic submatrices of $\xi(V)$, whose block columns are associated to vertices in Q. The number of columns in a block column of $\xi(V)$ is the dimension of the associated vertex, thus motivating our use of the notation $d(k)$ for the number of columns in block column $\xi(V)_{[[k,k]]}$.

We now define a rank matrix in analogy to Definition 2.7 for use in the context of block cyclic ranks:

Definition 4.3. *Let r be a quiver rank array and $V \in \mathcal{O}_r$. The* ***block cyclic rank matrix associated to r*** *is the $\infty \times \infty$ matrix* $\boldsymbol{c(r)}$ *defined by*

BC1. $c(r)_{i,j} = |[i,j]|$ *for* $j < i$,

BC2. $\boldsymbol{c(r)}_{i,j} = \operatorname{rank}(\xi(V)_{[[i,j]]})$ *for* $j \geq i$.

Note that, like the cyclic rank matrix defined in Definition 2.7, block cyclic rank matrices defined in this way are periodic, and so we may restrict to a $(2n+1) \times (4n+2)$ "tile".

Definition 4.4. *Let r be a quiver rank array and let M be a matrix in U of the form*

$$\left[\ * \ |I_{d_y}\right].$$

We define the ***cyclic block rank ideal*** *to be the ideal*

$$I_{c(r)} := \langle \textit{minors of size } (c(r)_{i,j}+1) \textit{ in } M_{[[i,j]]} \mid 1 \le i,j \le 2n+1 \rangle.$$

We further define the ***cyclic block rank variety*** *to be* $C_{c(r)} := \mathbb{V}(I_{c(r)})$

We now want to show that the data contained in a quiver rank array r is the same as in the cyclic block rank variety $C_{c(r)}$. To do this, we require a method translating quiver ranks to cyclic ranks, and vice versa. The following three lemmas provide this conversion.

Remark 4.5. The statement of the following lemmas is more clear if we relabel the columns of matrices in U according to the positions of the identities (and the sizes of the blocks). That is, we label the columns $x_n, \dots, x_1, y_0, \dots, y_n$:

$$\begin{array}{cccc|ccccc} x_n & \cdots & \cdots & x_1 & y_0 & \cdots & & \cdots & y_n \end{array}$$
$$\left[\begin{array}{c|ccccc} & \mathbf{1}_{d(y_0)} & & & & \\ & & \mathbf{1}_{d(y_1)} & & & \\ * & & & \ddots & & \\ & & & & \mathbf{1}_{d(y_{n-1})} & \\ & & & & & \mathbf{1}_{d(y_n)} \end{array}\right]$$

Our first lemma gives rank conditions that matrices in U automatically satisfy. Since U is an open patch of $\mathrm{Gr}(d_y, d_x+d_y)$, we call these conditions **patch conditions**:

Lemma 4.6. *(Patch conditions) For any $Z \in U$, the following cyclic block submatrices automatically have maximal rank:*

- ***(P1)*** *- Submatrices strictly contained in the identity block; that is, for pairs $0 \le i \le j \le n$,* $\mathrm{rank}(Z_{[[y_i,y_j]]}) = \sum_{k=i}^{j} \underline{d}(y_k)$
- ***(P2)*** *- Submatrices containing the whole identity block;*
  - ***(P2.1)*** *- for pairs $1 \le i < j \le n$,* $\mathrm{rank}(Z_{[[x_j,x_i]]}) = d_y$
  - ***(P2.2)*** *- for $1 \le i \le n$,* $\mathrm{rank}(Z_{[[y_i,y_{i-1}]]}) = d_y$
  - ***(P2.3)*** *- for $1 \le i \le n$,* $\mathrm{rank}(Z_{[[x_i,y_n]]}) = d_y$
  - ***(P2.4)*** *- for $1 \le i \le n$,* $\mathrm{rank}(Z_{[[y_0,x_i]]}) = d_y$

Proof. Cyclic submatrices of type P1 are sections of the identity block; these will have fewer columns than rows, with as many 1's as columns, and thus have maximal rank. Cyclic submatrices of type P2 contain the whole identity block; the identity block has as many rows as Z, with a 1 in each row, and so will have maximal rank. □

The Lemma we just proved outlines rank conditions that must be true for any matrix living inside U, that is, rank conditions that deal with the identity in $\left[\ * \ |I_{d_y}\right]$. Our next set of conditions hold true for matrices where the $*$ has the form of the snake matrix, with zeros above and below the double anti-diagonal. This ensures that a matrix that satisfies the following conditions could be the image of some representation $V \in \mathrm{rep}_Q(\underline{d})$ under the cyclic Zelevinsky map, and so we call these conditions **image conditions**:

Lemma 4.7. *(Image conditions) A matrix $Z \in U$ is in the image of ξ if and only if the following conditions hold:*

***(NW)* - *for $2 \leq i \leq n$, $\operatorname{rank}(Z_{[[y_{i-1},x_i]]}) = \sum_{k=i-1}^{n} d(y_k)$*

***(SE)* - *for $1 \leq i \leq n-1$, $\operatorname{rank}(Z_{[[x_i,y_i]]}) = \sum_{k=0}^{i} d(y_k)$*

Proof. For Z to be in the image of ξ, recall that the cyclic submatrix $Z_{[[x_n,x_1]]}$ (that is, the large block west of the identity block) has the form of the "snake matrix" $M_Q(V)$. Our two image conditions, (NW) and (SE), specify the zero-blocks above (northwest) and below (southeast) the double anti-diagonal of $M_Q(V)$, respectively.

Consider first a submatrix from condition (NW), $Z_{[[y_{i-1},x_i]]}$ for some i:

$$Z_{[[y_{i-1},x_i]]} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & & & \\ \mathbf{1}_{d(y_{i-1})} & & & \vdots & \ddots & \vdots \\ \vdots & \ddots & & & & \\ 0 & 0 & \mathbf{1}_{d(y_n)} & * & \dots & * \end{array}\right]$$

where the vertical line indicates the where Z "wraps around" back on itself, when viewed cyclically. Notice that, by clearing rows with the available identities, this submatrix has rank $\sum_{k=i-1}^{n} d(y_k)$ only when all the $*$ entries in first i block rows of the right block are zero. By requiring that this condition hold for all i between 2 and n, we force the blocks northwest of the double anti-diagonal to be zero (the case where $i = 1$ belongs to condition $P2.4$ in Lemma 4.6).

In a similar fashion, we can examine a submatrix from condition (SE):

$$Z_{[[x_i,y_i]]} = \left[\begin{array}{ccc|ccc} * & \dots & * & \mathbf{1}_{d(y_0)} & 0 & 0 \\ & & & & \ddots & \vdots \\ \vdots & \ddots & \vdots & & & \mathbf{1}_{d(y_i)} \\ & & & \vdots & \vdots & \vdots \\ * & \dots & * & 0 & 0 & 0 \end{array}\right].$$

We get zeros in the blocks southeast of the double anti-diagonal of Z by requiring that the ranks of submatrices $Z_{[[x_i,y_i]]}$ which vary i between 1 and $n-1$ be exactly $\sum_{k=i}^{n} d(y_k)$ (here, the case where $i = n$ belongs to condition $P2.3$ in Lemma 4.6).

Both conditions taken together specify that the $*$-block of Z is non-zero only on the double antidiagonal, thus forcing $Z = \xi(V)$ for some $V \in \operatorname{rep}_Q(\underline{d})$. $\square$

Any matrix satisfying the conditions in the previous two Lemmatta is the image of some representation in $\operatorname{rep}_Q(\underline{d})$ under the cyclic Zelevinsky map. A subset, but not all, of the block cyclic submatrices of our matrix $Z \in U$ were addressed in these conditions. The following Lemma shows that the ranks of the remaining block cyclic submatrices not already addressed are in correspondence with the quiver ranks of an orbit in $\operatorname{rep}_Q(\underline{d})$, and so we call the following conditions **orbit conditions**:

Lemma 4.8. *(Orbit conditions) A representation $V \in rep_Q(\underline{d})$ satisfies r if and only if $\xi(V)$ satisfies the conditions:*

- ***(O1)*** *- for $1 \leq i \leq j \leq n$, $\operatorname{rank}(Z_{[[x_j,x_i]]}) = r_{[\alpha_i,\beta_j]}$*
- ***(O2)*** *- for $1 \leq i \leq j \leq n$, $\operatorname{rank}(Z_{[[x_j,y_{i-1}]]]}) = r_{[\beta_i,\beta_j} + \sum_{k=0}^{i-1} d(y_k)$*
- ***(O3)*** *- for $1 \leq i \leq j \leq n$, $\operatorname{rank}(Z_{[[y_j,x_i]]}) = r_{[\alpha_i,\alpha_j]} + \sum_{k=j}^{n} d(y_k)$*
- ***(O4)*** *- for $1 \leq i \leq j \leq n$, $\operatorname{rank}(Z_{[[y_j,y_{i-1}]]}) = r_{[\beta_i,\alpha_j]} + \sum_{k=0}^{i-1} d(y_k) + \sum_{k=j}^{n} d(y_k)$*

The conditions $O1-O4$ correspond with intervals of Q of the form $[\alpha,\beta]$, $[\beta,\beta]$, $[\alpha,\alpha]$, and $[\beta,\alpha]$, respectively.

Proof. Recall that for a representation V to satisfy r, we must have that $\operatorname{rank}(M_J(V)) = r_J$ for all intervals $J \subseteq Q$. Intervals J come in four types depending on the types of the first and last arrow, either α or β. Showing that V satisfies r if and only if $\xi(V)$ satisfies $O1 - O4$ amounts to examining block cyclic submatrices of a corresponding four types. These block cyclic submatrices are characterized by the labelling of the start and end block column, either x or y.

Consider the image of ξ:

$$\xi(V) = \left[\begin{array}{cccc|ccccc} & & & V_{\alpha_1} & \mathbf{1}_{d(y_0)} & & & & \\ & & V_{\alpha_2} & V_{\beta_1} & & \mathbf{1}_{d(y_1)} & & & \\ & \iddots & \iddots & & & & \ddots & & \\ V_{\alpha_n} & V_{\beta_{n-1}} & & & & & & \mathbf{1}_{d(y_{n-1})} & \\ V_{\beta_n} & & & & & & & & \mathbf{1}_{d(y_n)} \end{array}\right].$$

Notice that each block column in the $M_Q(V)$ region (the region *not* composed of identities) has two non-zero block entries, a matrix associated to an α arrow stacked on top of a matrix associated to a β arrow. So, if we take a block cyclic submatrix whose start and end columns are both in the $M_Q(V)$ region, it will contain the matrices associated to arrows in the interval $[\alpha_i, \beta_j]$, and no identities. Thus, for an interval of the form $J = [\alpha_i, \beta_j]$ where $1 \leq i \leq j \leq n$, the block cyclic submatrix $\xi(V)_{[[x_j,x_i]]}$ is composed of $M_J(V)$, with extra rows of zeros above or below. Thus, the ranks of these two matrices are the same; this corresponds to condition $O1$.

To get the other three types of intervals, we use the identities to clear rows above, rows below, or both.

For an interval of the form $J = [\beta_i, \beta_j]$ $(1 \leq i \leq j \leq n)$, we need clear all rows above the matrix associated to β_i, and so include the identities up to y_{i-1} in our block cyclic submatrix $\xi(V)_{[[x_j,y_{i-1}]]}$. We use these identities to clear matrices $V_{\alpha_1}, V_{\beta_1}, \ldots, V_{\alpha_i}$, and so the rank of of $M_J(V)$ agrees with the rank of $\xi(V)_{[[x_j,y_{i-1}]]}$, plus the sum of the dimensions of the identities included. This corresponds to condition $O2$.

The process for intervals of the form $J = [\alpha_i, \alpha_j]$ is similar to the previous one, but now we begin from the other side. In particular we now need to clear rows below the matrix associated to α_j. Including the identities associated to y_n through y_j accomplishes this, and so the rank of $M_J(V)$ agrees with the rank of $\xi(V)_{[[y_j,x_i]]}$, again with an added sum of dimensions of included identities. This is condition $O3$.

The final form of interval we need to consider is $J = [\beta_i, \alpha_j]$. Here, we need to clear rows above V_{β_i} and below V_{α_j}; as with the previous two types, we include identities to clear these rows, now requiring identities in columns $y_0, \ldots, y_{i-1}$ and $y_n, \ldots, y_j$. We thus have that the rank of $M_J(V)$ agrees with the rank of $\xi(V)_{[[y_j, y_{i-1}]]}$, with a sum of dimensions of identities from both sides. This final step gives us $O4$. $\square$

These Lemmatta show that the collection of cyclic block rank conditions are equivalent to patch, image, and orbit conditions, and provide the mechanics for the following proposition.

Proposition 4.9. *Let r be a quiver rank array. The cyclic Zelevinsky map ξ induces an isomorphism on the rings*

$$\xi^* : \frac{K[U]}{I_{c(r)}} \to \frac{K[rep_Q(\underline{d})]}{I_r}.$$

Before we provide the proof of this proposition, we will go through two examples. The first shows concretely how the cyclic Zelevinsky map converts a quiver rank array to a cyclic block rank matrix; the second explicitly shows the isomorphism between rings of $K[U]/I_{c(r)}$ and $K[\mathrm{rep}_Q(\underline{d})]/I_r$. Having this understanding will make the mechanics of the proof more transparent.

Example 4.10. Let Q be the quiver

$$Q = \begin{tikzcd} & x_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & x_2 \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_2"] & & x_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\beta_3"] & \\ y_0 & & y_1 & & y_2 & & y_3 \end{tikzcd}$$

with dimension vector $\underline{d}$. Given any element $V \in \mathrm{rep}_Q(\underline{d})$, by the patch and image conditions from Lemmas 4.6 and 4.7, the image of the cyclic Zelevinsky map is a block matrix of the form

$$\xi(V) = \begin{bmatrix} 0 & 0 & V_{\alpha_1} & I_{d(y_0)} & 0 & 0 & 0 \\ 0 & V_{\alpha_2} & V_{\beta_1} & 0 & I_{d(y_1)} & 0 & 0 \\ V_{\alpha_3} & V_{\beta_2} & 0 & 0 & 0 & I_{d(y_2)} & 0 \\ V_{\beta_3} & 0 & 0 & 0 & 0 & 0 & I_{d(y_3)} \end{bmatrix}.$$

The blue values correspond to patch conditions (specifying that $\xi(V) \in U$). The red values correspond to image conditions (specifying the zeros above and below the double antidiagonal of the snake matrix). If we take our dimension vector to be $\underline{d} = (1, 2, 3, 2, 3, 2, 1)$ (so that $\xi(V)$ is a point of $\mathrm{Gr}(8, 14)$) then $c(r)$ is of the form

$$\begin{bmatrix} * & * & * & * & * & * & 8 & & & & & & \\ & * & * & * & * & 7 & 8 & 8 & & & & & \\ & & * & * & 4 & 7 & 8 & 8 & 8 & & & & \\ & & & 1 & 4 & 7 & 8 & 8 & 8 & 8 & & & \\ & & & & 3 & 6 & 7 & 7 & 7 & * & 8 & & \\ & & & & & 3 & 4 & 4 & * & * & * & 8 & \\ & & & & & & 1 & * & * & * & * & * & 8 \end{bmatrix}$$

where the blue entries specify the blue blocks of $\xi(V)$, and the red entries specify the red blocks of $\xi(V)$. The entries labelled $*$ correspond to entries which will be specified by the orbit conditions (determined by the ranks of the maps in V, which is the data contained in r).

Consider the representation

$$V_{\alpha_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad V_{\beta_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad V_{\alpha_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$V_{\beta_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V_{\alpha_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{\beta_3} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

We can now fill in the entries labelled $*$, and we get

$$\begin{bmatrix}
1 & 3 & 5 & 6 & 7 & 8 & {\color{blue}8} & & & & & & \\
 & 2 & 4 & 5 & 6 & {\color{red}7} & {\color{blue}8} & {\color{blue}8} & & & & & \\
 & & 2 & 3 & {\color{red}4} & {\color{red}7} & {\color{blue}8} & {\color{blue}8} & {\color{blue}8} & & & & \\
 & & & {\color{blue}1} & {\color{blue}4} & {\color{blue}7} & {\color{blue}8} & {\color{blue}8} & {\color{blue}8} & {\color{blue}8} & & & \\
 & & & & {\color{blue}3} & {\color{blue}6} & {\color{blue}7} & {\color{red}7} & {\color{red}7} & 8 & {\color{blue}8} & & \\
 & & & & & {\color{blue}3} & {\color{blue}4} & {\color{red}4} & 5 & 7 & 8 & {\color{blue}8} & \\
 & & & & & & {\color{blue}1} & 1 & 3 & 5 & 6 & 7 & {\color{blue}8}
\end{bmatrix}$$

With the formulas provided in Lemma 4.8, we can recover the quiver rank array r from the black values in $c(r)$.

For our second example we will consider a smaller quiver, so as to make the computations and results digestible.

Example 4.11. Let Q be the quiver

$$Q= \begin{tikzcd} & x_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & x_2 \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_2"] & \\ y_0 & & y_1 & & y_2 \end{tikzcd}$$

with dimension vector $\underline{d} = (1, 2, 2, 1, 1)$. This specifies that our maps are of the form

$$V_{\alpha_1} = \begin{bmatrix} a & b \end{bmatrix}, \quad V_{\beta_1} = \begin{bmatrix} c & d \\ e & f \end{bmatrix}, \quad V_{\alpha_2} = \begin{bmatrix} g \\ h \end{bmatrix}, \quad V_{\beta_2} = \begin{bmatrix} i \end{bmatrix};$$

we will working with the orbit of

$$V_{\alpha_1} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad V_{\beta_1} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad V_{\alpha_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad V_{\beta_2} = \begin{bmatrix} 1 \end{bmatrix}.$$

Thus

$$M_Q(V) = \left[\begin{array}{c|cc} \cdot & a & b \\ \hline g & c & d \\ h & e & f \\ \hline i & \cdot & \cdot \end{array}\right] = \left[\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 1 & \cdot & \cdot \end{array}\right]$$

and

$$\xi(V) = \left[\begin{array}{c|cc|c|cc|c} j & k & l & 1 & \cdot & \cdot & \cdot \\ \hline m & n & o & \cdot & 1 & \cdot & \cdot \\ p & q & r & \cdot & \cdot & 1 & \cdot \\ \hline s & t & u & \cdot & \cdot & \cdot & 1 \end{array}\right] = \left[\begin{array}{c|cc|c|cc|c} \cdot & 0 & 1 & 1 & \cdot & \cdot & \cdot \\ \hline 0 & 1 & 0 & \cdot & 1 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & 1 & \cdot \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right],$$

where the entries marked by $\cdot$ are zeros.

Our aim in this example is to explicitly show the isomorphism between coordinate rings of Ω_r and $C_{c(r)}$ induced by the pullback

$$\xi^* : \frac{K[U]}{I_{c(r)}} \to \frac{K[\mathrm{rep}_Q(\underline{d})]}{I_r}.$$

Notice that we already know that

$$K[U] = K[j,k,l,m,n,o,p,q,r,s,t,u] \qquad \text{and} \qquad K[\mathrm{rep}_Q(\underline{d})] = K[a,b,c,d,e,f,g,h,i],$$

and so we need to first find I_r and $I_{c(r)}$, then show that ξ^* is an isomorphism.

To find I_r, first observe that the rank array r is

$$\begin{array}{cccc|c} \beta_2 & \alpha_2 & \beta_1 & \alpha_1 & \\ \hline & & & 1 & \alpha_1 \\ & & {\color{blue}1} & 2 & \beta_1 \\ & {\color{blue}0} & {\color{blue}1} & {\color{blue}2} & \alpha_2 \\ 1 & 1 & {\color{blue}2} & 3 & \beta_2 \end{array}$$

with the blue rank entries indicating that they are not maximal (i.e. that the size the rank is smaller than the minimum of the number of rows and columns of the associated matrix). The generators of I_r are derived from the appropriately sized minors of these non-maximal ranks:

$$2{\times}2 \text{ minors of } \begin{bmatrix} c & d \\ e & f \end{bmatrix} \implies cf - de = 0$$

$$1{\times}1 \text{ minors of } \begin{bmatrix} g \\ h \end{bmatrix} \implies \begin{array}{c} g = 0, \\ h = 0 \end{array}$$

$$2{\times}2 \text{ minors of } \begin{bmatrix} g & c & d \\ h & e & f \end{bmatrix} \implies \begin{array}{c} ge - ch = 0, \\ gf - dh = 0, \\ cf - de = 0 \end{array}$$

$$3{\times}3 \text{ minors of } \begin{bmatrix} \cdot & a & b \\ g & c & d \\ h & e & f \end{bmatrix} \implies -g(af - be) + h(ad - bc) = 0$$

$$3{\times}3 \text{ minors of } \begin{bmatrix} g & c & d \\ h & e & f \\ i & \cdot & \cdot \end{bmatrix} \implies i(cf - de) = 0.$$

Many of these are redundant (for example, since g and h are zero, we automatically get that $gf - dh$ is zero), and so our generators are

$$I_r = \langle g, h, cf - de \rangle.$$

Similarly, the cyclic rank matrix of $\xi(V)$ is

$$c(r) = \begin{bmatrix} 1 & 3 & {\color{blue}3} & 4 & 4 & & & & \\ & 2 & {\color{blue}2} & {\color{blue}3} & 4 & 4 & & & \\ & & 1 & 3 & 4 & 4 & 4 & & \\ & & & 2 & 3 & {\color{blue}3} & 4 & 4 & \\ & & & & 1 & {\color{blue}1} & {\color{blue}3} & {\color{blue}3} & 4 \end{bmatrix}$$

where again the blue entries correspond to non-maximal ranks. These non-maximal ranks give us our generators for $I_{c(r)}$:

$$4{\times}4 \text{ minors of } \begin{bmatrix} j & k & l & 1 \\ m & n & o & \cdot \\ p & q & r & \cdot \\ s & t & u & \cdot \end{bmatrix} \implies -(m(qu - rt) - p(nu - ot) + s(nr - oq)) = 0$$

$$3{\times}3 \text{ minors of } \begin{bmatrix} k & l & 1 \\ n & o & \cdot \\ q & r & \cdot \\ t & u & \cdot \end{bmatrix} \implies \begin{aligned} nr - oq &= 0, \\ nu - ot &= 0, \\ qu - rt &= 0 \end{aligned}$$

$$4{\times}4 \text{ minors of } \begin{bmatrix} k & l & 1 & \cdot & \cdot \\ n & o & \cdot & 1 & \cdot \\ q & r & \cdot & \cdot & 1 \\ t & u & \cdot & \cdot & \cdot \end{bmatrix} \implies \begin{aligned} qu - rt &= 0, \\ nu - ot &= 0, \\ ku - lt &= 0, \\ t &= 0, \\ u &= 0 \end{aligned}$$

$$4{\times}4 \text{ minors of } \begin{bmatrix} \cdot & \cdot & \cdot & j \\ 1 & \cdot & \cdot & m \\ \cdot & 1 & \cdot & p \\ \cdot & \cdot & 1 & s \end{bmatrix} \implies j = 0$$

$$2\times 2 \text{ minors of } \begin{bmatrix} \cdot & j \\ \cdot & m \\ \cdot & p \\ 1 & s \end{bmatrix} \Longrightarrow \begin{aligned} j &= 0, \\ m &= 0, \\ p &= 0 \end{aligned}$$

$$4\times 4 \text{ minors of } \begin{bmatrix} \cdot & j & k & l \\ \cdot & m & n & o \\ \cdot & p & q & r \\ 1 & s & t & u \end{bmatrix} \Longrightarrow -(j(nr-oq)-m(kr-lq)+p(ko-ln))=0$$

$$4\times 4 \text{ minors of } \begin{bmatrix} \cdot & j & k & l & 1 \\ \cdot & m & n & o & \cdot \\ \cdot & p & q & r & \cdot \\ \cdot & s & t & u & \cdot \end{bmatrix} \Longrightarrow \begin{aligned} -(j(nr-oq)-m(kr-lq)+p(ko-ln)) &= 0, \\ -(mq-np) &= 0, \\ -(mr-op) &= 0 \\ -(nr-oq) &= 0 \\ -(m(qu-rt)-p(nu-ot)+s(nr-oq)) &= 0 \end{aligned}$$

Again we have redundancy, and our generators reduce to

$$I_{c(r)} = \langle j, m, p, t, u, nr - oq \rangle.$$

Examining the map $K[U] \to K[\text{rep}_Q(\underline{d})]/I_r$ given by

$$\begin{array}{llll} j \to 0 & k \to a & l \to b & m \to 0 \\ n \to c & o \to d & p \to 0 & q \to e \\ r \to f & s \to i & t \to 0 & u \to 0. \end{array}$$

Notice that this is surjective, and the kernel is exactly $I_{c(r)}$, so that by the first isomorphism theorem we have

$$\frac{K[U]}{I_{c(r)}} \cong \frac{K[\text{rep}_Q(\underline{d})]}{I_r}.$$

Furthermore, it is worthwhile to point out that the kernel of this map arose from generators given by image and orbit conditions, and those that came from orbit conditions corresponded to elements of I_r. This gives the basic outline of the mechanics involved in the following proof.

Proof. (Proposition 4.9) Let Q and $\underline{d}$ be a quiver and its dimension vector. Let r be a rank array. First recall that, for matrices in U, rank conditions coming from Lemma 4.6, which are our patch conditions, yield trivial minors. That is, imposing patch conditions on generic matrices $M \in U$ thus does not provide any additional relations.

Our first non-trivial minors arise from those given by Lemma 4.7, which are our image conditions. These minors specify zeros so that the block matrix $*$ in

$$M = \left[\ * \ | I_{d_y} \right]$$

has the form of the snake matrix $M_Q(V)$, and so $K[U]/\langle\text{"image conditions"}\rangle$ is isomorphic to $K[\text{rep}_Q(\underline{d})]$.

We are left with minors given by orbit conditions from Lemma 4.8. These are in bijection with minors given by quiver ranks of $M_Q(V)$ via the pullback ξ^*. This can be verified using the mechanics presented in Lemma 4.8: to each interval $J \in Q$, we associate a cyclic submatrix, and by clearing rows with identities we are left with minors of the same form as minors of M_J, the submatrix of $M_Q(V)$ associated to the interval J. Since we can do this for every interval J, we find that generators derived from orbit conditions are exactly the generators of I_r, with a relabelling given by ξ^*.

To conclude, since $K[U]/\langle\text{“image conditions”}\rangle \cong K[\mathrm{rep}_Q(\underline{d})]$, we have that

$$\frac{K[U]}{\langle\text{“image conditions”, “orbit conditions}\rangle} \cong \frac{K[\mathrm{rep}_Q(\underline{d})]}{\langle\text{“orbit conditions”}\rangle}.$$

Taken together, generators derived from image and orbit conditions yield $I_{c(r)}$, and so we have that

$$\frac{K[U]}{I_{c(r)}} \cong \frac{K[\mathrm{rep}_Q(\underline{d})]}{I_r}.$$

□

Chapter 5

Bounded Affine Permutations and Quiver Orbits

In this chapter, we focus on translating from block cyclic ranks to bounded affine permutations. The following proposition is the summary of this result.

Proposition 5.1. *Let Q be a quiver, r be a quiver rank array, and let $c(r)$ be the associated cyclic rank matrix. Then there exists a unique bounded affine permutation matrix $\nu_c(r)$ that satisfies the following:*

***(i)* -** *The number of 1's in block (i, j) equals*

$$|[i,j]| - c(r)_{i,j} - \#1\text{'s strictly SW of block } (i,j)$$

***(ii)* -** *the 1's are arranged from northwest to southeast across block rows*

***(iii)* -** *the 1's are arranged northwest to southeast down block columns.*

We call the permutation $\nu_c(r)$ the **cyclic Zelevinsky permutation** associated to r. Note that, by "#1's strictly SW of block (i, j)", we mean the number of 1's in blocks southwest of block (i, j), not including within the block itself; this is consistent with our previous usage of the term "weakly southwest", as in §3.3. Before we prove the above proposition, we will provide an example showing the construction of $\nu_c(r)$, continuing Example 4.10.

Example 5.2. We now outline the process of constructing $\nu_c(r)$. Recall the quiver we are working with,

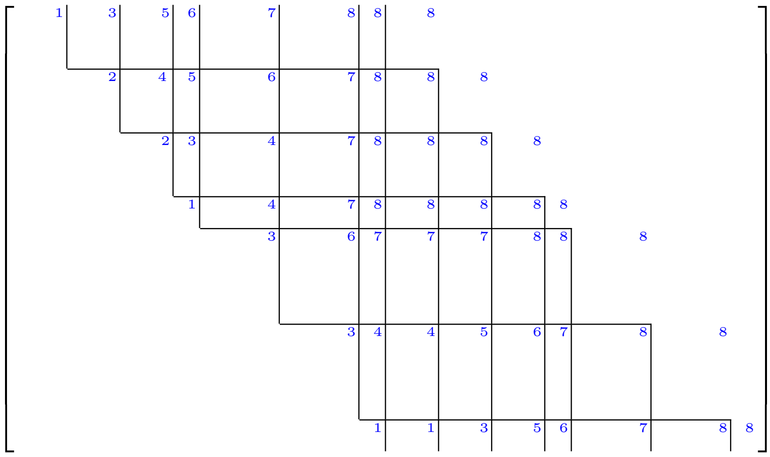
$$Q= \begin{tikzcd} & x_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & & x_2 \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_2"] & & x_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\beta_3"] & \\ y_0 & & y_1 & & y_2 & & y_3 \end{tikzcd},$$

our dimension vector $\underline{d} = (1, 2, 3, 2, 3, 2, 1)$, and the orbit we are interested in,

$$V_{\alpha_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad V_{\beta_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad V_{\alpha_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$V_{\beta_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V_{\alpha_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{\beta_3} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

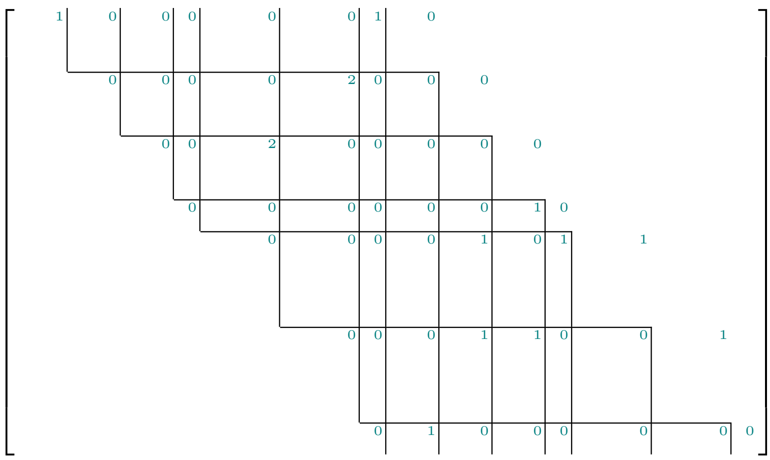
Working in a $d \times 2d$ tile of the full cyclic rank matrix for $\xi(V)$, we lay out the block structure in the d-wide diagonal strip containing the non-trivial rank data, along with the block cyclic ranks in the northeast corners of blocks:



Starting in blocks in the southwest-most diagonal, then proceeding with diagonals to the northeast, we apply our formula

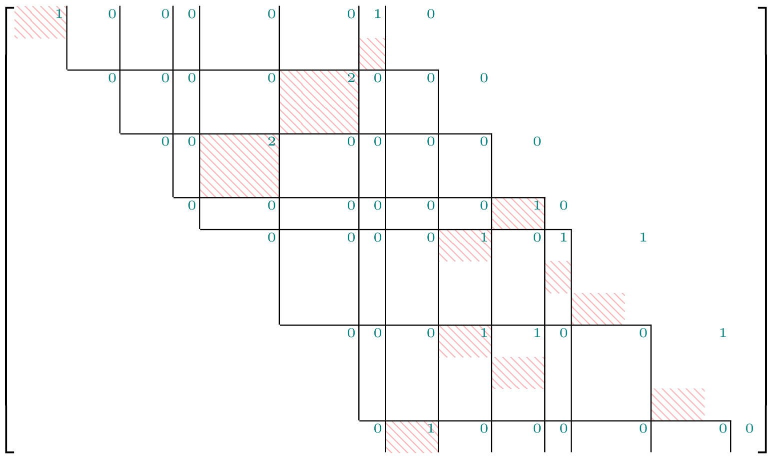
$$\#1\text{'s in block } (i,j) = |[i,j]| - c(r)_{i,j} - \#1\text{'s strictly SW of block } (i,j)$$

by subtracting both the block cyclic rank and the number of 1's strictly to the southwest of the block from the size of the interval $|[i,j]|$ to get the number of 1's required in each block. We work southwest to northeast in this fashion because the number of 1's in a block impacts how many 1's are needed in blocks to the northeast. For example, for block (3,5), we have that $|[3,5]| = 6$, $c(r)_{3,5} = 4$, and all blocks southwest of $(3,5)$ wind up not containing any 1's, so we need to put two 1's somewhere in $(3,5)$. Doing this for all blocks (again, flowing southwest to northeast) and recording the number of 1's required in each block, we get

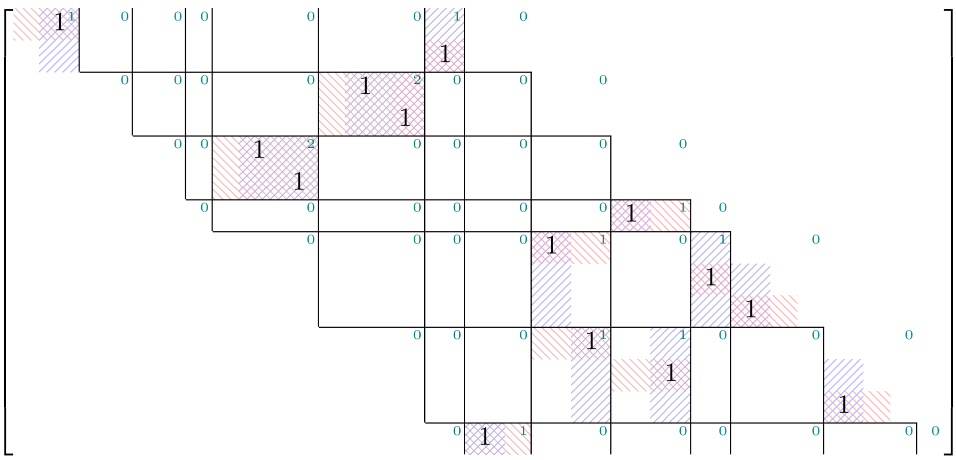


.

Note that the 1's located on the northeast-most diagonal, outside the outlined blocks, will still be within the d-wide diagonal strip. Now, for each block row, within blocks in which we need to place 1's, we identify rows that should contain 1's. We ensure that these identified rows go northwest to southeast across the block row. That is, the first time we encounter a 1 in a block row scanning west to east, it will be to the northwest, then the next 1 will be in the next row down, located southeast of the first 1, and so on until there is a 1 in each row of the block row.



Next, we play the same game for the block columns, recalling that our matrix can be thought of as a tile for a diagonally periodic $\infty \times \infty$-type matrix (and so columns may extend above or below the edges of the matrix). Where the identified columns overlap with the identified rows, we place a 1:



This yields the bounded affine permutation

$$\nu_c(r) = \left[\begin{array}{cccccccccccccccccccccccccccc}
\cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
& \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot \\
& & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & & & & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
& & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{1} & \cdot & \cdot \\
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& & & & & & & & & & & & & \cdot & \mathbf{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

Having now gone through this example, we now return to the proof of Proposition 5.1, and to do this, we will use alternate but equivalent statements of Property 1:

Lemma 5.3. *The following statements of Property (i) in Proposition 5.1 are equivalent:*

(i) **-** *The number of 1's in block (i, j) is equal to*

$$|[i, j]| - c(r)_{i,j} - \#\text{1's strictly SW of block } (i, j)$$

(i*) **-** *The number of 1's in block (i, j) is equal to*

$$\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j-1} + \delta_{i+1,j-1}$$

where

$$\delta_{i,j} = |[i, j]| - c(r)_{i,j}$$

(i**) **-** *The number of 1's in block (i, j) is equal to*

$$c(r)_{i+1,j} + c(r)_{i,j-1} - c(r)_{i,j} - c(r)_{i+1,j-1}.$$

Proof. First, define

$$\delta_{i,j} = |[i, j]| - c(r)_{i,j}$$

as the "defect" between $|[i, j]|$ and $c(r)_{i,j}$; this is useful not only in this proof, but more generally as well, as this provides something like a conservation law for bounded affine permutations associated to cyclic rank matrices. From Property (i) we have that

$$\#\text{ 1's in block } (i, j) = \delta_{i,j} - \#\text{1's strictly SW of block } (i, j),$$

which we rearrange to obtain

$$\delta_{i,j} = \#\text{ 1's weakly SW of block } (i,j). \tag{5.1}$$

Next, notice that

$$\begin{aligned}(\#\text{ 1's in block } (i,j)) &= (\#\text{ 1's weakly SW of block } (i,j)) \\ &\quad - (\#\text{ 1's weakly SW of block } (i+1,j)) \\ &\quad - (\#\text{ 1's weakly SW of block } (i,j-1)) \\ &\quad + (\#\text{ 1's weakly SW of block } (i+1,j-1)).\end{aligned}$$

By (5.1), this gives us

$$(\#\text{ 1's in block } (i,j)) = \delta_{i,j} - \delta_{i+1,j} - \delta_{i,j-1} + \delta_{i+1,j-1},$$

and we have (i^*). Now, to show that this equivalent to (i^{**}), we will first show that $|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| = 0$, and then use this fact to obtain our result. There are five cases to check, using our redefinition of $|[i,j]|$ for use in the block context, found in Convention 4.1:

$\underline{j \leq i-2}$

$$\begin{aligned}|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| &= \left(-\sum_{k=j+1}^{i-1} d(k)\right) - \left(-\sum_{k=j+1}^{i} d(k)\right) \\ &\quad - \left(-\sum_{k=j}^{i-1} d(k)\right) + \left(-\sum_{k=j}^{i} d(k)\right) \\ &= \left(\sum_{k=j+1}^{i} d(k) - \sum_{k=j+1}^{i-1} d(k)\right) \\ &\quad - \left(\sum_{k=j}^{i} d(k) - \sum_{k=j}^{i-1} d(k)\right) \\ &= d(i) - d(i) \\ &= 0\end{aligned}$$

$\underline{j = i - 1}$

$$\begin{aligned}
|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| &= |[i,i-1]| - |[i+1,i-1]| \\
&\quad - |[i,i-2]| + |[i+1,i-2]| \\
&= 0 - \left(-\sum_{k=i}^{i} d(k)\right) - \left(-\sum_{k=i-1}^{i-1} d(k)\right) \\
&\quad + \left(-\sum_{k=i-1}^{i} d(k)\right) \\
&= \left(-\sum_{k=i-1}^{i} d(k)\right) - \left(-\sum_{k=i-1}^{i} d(k)\right) \\
&= 0
\end{aligned}$$

$\underline{j = i}$

$$\begin{aligned}
|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| &= |[i,i]| - |[i+1,i]| \\
&\quad - |[i,i-1]| + |[i+1,i-1]| \\
&= \left(\sum_{k=i}^{i} d(k)\right) - 0 - 0 + \left(-\sum_{k=i}^{i} d(k)\right) \\
&= 0
\end{aligned}$$

$\underline{j = i + 1}$

$$\begin{aligned}
|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| &= |[i,i+1]| - |[i+1,i+1]| \\
&\quad - |[i,i]| + |[i+1,i]| \\
&= \left(\sum_{k=i}^{i+1} d(k)\right) - \left(\sum_{k=i+1}^{i+1} d(k)\right) \\
&\quad - \left(\sum_{k=i}^{i} d(k)\right) + 0 \\
&= \left(\sum_{k=i}^{i+1} d(k)\right) - \left(\sum_{k=i}^{i+1} d(k)\right) \\
&= 0
\end{aligned}$$

$\underline{j \geq i+2}$

$$\begin{aligned}
|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]| &= \left(\sum_{k=i}^{j} d(k)\right) - \left(\sum_{k=i+1}^{j} d(k)\right) \\
&\quad - \left(\sum_{k=i}^{j-1} d(k)\right) + \left(\sum_{k=i+1}^{j-1} d(k)\right) \\
&= \left(\sum_{k=i}^{j} d(k) - \sum_{k=i+1}^{j} d(k)\right) \\
&\quad - \left(\sum_{k=i}^{j-1} d(k) - \sum_{k=i+1}^{j-1} d(k)\right) \\
&= d(i) - d(i) \\
&= 0.
\end{aligned}$$

Applying this to (i^*), we find

$$\begin{aligned}
\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j-1} + \delta_{i+1,j-1} &= (|[i,j]| - |[i+1,j]| - |[i,j-1]| + |[i+1,j-1]|) \\
&\quad - (c(r)_{i,j} - c(r)_{i+1,j} - c(r)_{i,j-1} + c(r)_{i+1,j-1}) \\
&= -(c(r)_{i,j} - c(r)_{i+1,j} - c(r)_{i,j-1} + c(r)_{i+1,j-1}) \\
&= c(r)_{i+1,j} + c(r)_{i,j-1} - c(r)_{i,j} - c(r)_{i+1,j-1}
\end{aligned}$$

Thus

$$\begin{aligned}
(\#\text{ 1's in block } (i,j)) &= |[i,j]| - c(r)_{i,j} - \#\text{1's strictly SW of block } (i,j) \\
&= \delta_{i,j} - \delta_{i+1,j} - \delta_{i,j-1} + \delta_{i+1,j-1} \\
&= c(r)_{i+1,j} + c(r)_{i,j-1} - c(r)_{i,j} - c(r)_{i+1,j-1}.
\end{aligned}$$

□

We are now ready to prove Proposition 5.1.

Proof. (Proposition 5.1)

Property (i) forms the crux of this proof, as we will find that Properties (ii) and (iii) are responsible merely for the uniqueness of the bounded affine permutation given by Property (i). Hence, recall that to show that $\nu_c(r)$ is a bounded affine permutation, we must show that the number of 1's in each block row/column is the height/ width (i.e. each row/column contains a 1), and that these 1's fall in a strip between diagonals $q = p$ and $q = p + d$, where p, q are the row and column indices, respectively. Since the argument for the number of 1's in each block column is completely analogous to the one for block rows, we show how this proceeds for block rows.

Now, to obtain the number of 1's in block row i, we take the sum over each block in the row using Property (i^*):

$$\sum_{k=-\infty}^{\infty} (\delta_{i,k} - \delta_{i+1,k} - \delta_{i,k-1} + \delta_{i+1,k-1}).$$

However, for $j < i$, from Definition 4.3 we have that $c(r)_{i,j} = |[i,j]|$, and so by our proof of Lemma 5.3 we have that $c(r)_{i,j} - c(r)_{i+1,j} - c(r)_{i,j-1} + c(r)_{i+1,j-1} = 0$ (i.e. these blocks contain no 1's). For $j > i + |Q_0| + 1$, where $|Q_0|$ is the number of vertices in Q, $c(r)_{i,j} = d_y$ (since for $j \geq i|Q_0| - 1$, entries $c(r)_{i,j}$ correspond to block cyclic submatrices that are all of $\xi(V)$), and so these blocks contain no 1's as well. Hence, our above sum becomes

$$\sum_{k=i}^{i+|Q_0|+1} (\delta_{i,k} - \delta_{i+1,k} - \delta_{i,k-1} + \delta_{i+1,k-1}).$$

By expanding, rearranging, and cancelling where possible, we obtain

$$\begin{aligned}
&(\delta_{i,i} - \delta_{i+1,i} - \delta_{i,i-1} + \delta_{i+1,i-1}) \\
&\quad + (\delta_{i,i+1} - \delta_{i+1,i+1} - \delta_{i,i} + \delta_{i+1,i}) + \cdots \\
&\quad + (\delta_{i,i+|Q_0|} - \delta_{i+1,i+|Q_0|} - \delta_{i,i+|Q_0|-1} + \delta_{i+1,i+|Q_0|-1}) \\
&\quad + (\delta_{i,i+|Q_0|+1} - \delta_{i+1,i+|Q_0|+1} - \delta_{i,i+|Q_0|} + \delta_{i+1,i+|Q_0|}) \\
&= (-\delta_{i,i-1} + \delta_{i+1,i-1}) + (\delta_{i,i} - \delta_{i+1,i} - \delta_{i,i} + \delta_{i+1,i}) + \cdots \\
&\quad + (\delta_{i,i+|Q_0|} - \delta_{i+1,i+|Q_0|} - \delta_{i,i+|Q_0|} + \delta_{i+1,i+|Q_0|}) + (\delta_{i,i+|Q_0|+1} - \delta_{i+1,i+|Q_0|+1}) \\
&= (-\delta_{i,i-1} + \delta_{i+1,i-1}) + (\delta_{i,i+|Q_0|+1} - \delta_{i+1,i+|Q_0|+1}) \\
&= \delta_{i,i+|Q_0|+1} - \delta_{i+1,i+|Q_0|+1} \\
&= (|[i, i + |Q_0| + 1]| - c(r)_{i,i+|Q_0|+1}) - (|[i + 1, i + |Q_0| + 1]| - c(r)_{i+1,i+|Q_0|+1}) \\
&= (|[i, i + |Q_0| + 1]| - |[i + 1, i + |Q_0| + 1]|) + (c(r)_{i+1,i+|Q_0|+1} - c(r)_{i,i+|Q_0|+1}) \\
&= \left(\sum_{k=i}^{i+|Q_0|+1} d(k) - \sum_{k=i+1}^{i+|Q_0|+1} d(k) \right) + (d_y - d_y) \\
&= d(i)
\end{aligned}$$

That is, the number of 1's in block row i is exactly the height of row i. To see that $\nu_c(r)$ is a bounded affine permutation, it remains to show that all the 1's fall within the diagonal strip between $p = q$ and $p = q + d$ (where $d = d_y + d_x$).

In block row i, this is only a concern in blocks (i, i) and $(i, i + n + 1)$, as these block both potentially contain 1's, and contain entries not within the strip. Between these two blocks, all entries are within the strip, and outside of these blocks, we have $c(r)_{i,j} - c(r)_{i+1,j} - c(r)_{i,j-1} + c(r)_{i+1,j-1} = 0$ and so no 1's (this is addressed for blocks west of (i, i) in the proof of Lemma 5.3, and for blocks east of $(i, i + n + 1)$ all of the block cyclic ranks involved are d_y). Notice that we only fail to have a permissible arrangement of 1's in blocks (i, i) and $(i, i + n + 1)$ if the block simultaneously contains all 1's in the row (i.e. as many 1's as the height), and the main diagonal of the block is not within the strip. However, the main diagonal of both (i, i) and $(i, i + n + 1)$

are within the strip, and since properties (*ii*) and (*iii*) ensure that only one 1 is placed in each row and column, $\nu_c(r)$ is a bounded affine permutation.

Finally, Properties (*ii*) and (*iii*) give us uniqueness of $\nu_c(r)$; for a given block containing 1's, Property (*ii*) specifies which row(s) may contain a 1, while Property (*iii*) species the same for columns, and taken together these specify a unique arrangement of 1's within the block. Over the whole matrix, this specifies the unique bounded affine permutation that is $\nu_c(r)$. □

Remark 5.4. The case where all 1's in row i occur in block $(i, i+n+1)$ happens precisely when there is an entire block row of zeros in $M_Q(V)$, that is, when all of the maps terminating at vertex i are degenerate. Note that this implies that this can only happen when the block row is associated to a y-type vertex.

To see why this is the case, notice that if $(i, i+|Q_0|+1)$ contains $d(i)$-many 1's, the number of 1's weakly southwest of $(i+1, i+|Q_0|)$ is d_x: blocks (i,i) through $(i, i+|Q_0|)$ contain no ones, while there are d_x 1's weakly southwest of block $(i, i+|Q_0|)$, and so all d_x 1's are weakly southwest of block $(i+1, i+|Q_0|)$. Then, recalling that $|[i+1, i+|Q_0|]| = d - d(i)$ (since $\xi(V)_{[[i+1,i+|Q_0|]]}$ is the cyclic submatrix of $\xi(V)$ containing all block columns except block column i), consider

$$\begin{aligned} c(r)_{i+1,i+|Q_0|} &= |[i+1, i+|Q_0|]| - \#\text{ 1's weakly SW of block } (i+1, i+|Q_0|) \\ &= (d - d(i)) - d_x \\ &= d_y - d(i). \end{aligned}$$

This is occurs only when $M_Q(V)$ has a block row of zeros, and when the block column not contained in $[i+1, i+|Q_0|]$ is associated to the identity block for that block row of zeros.

Corollary 5.5. *Every essential box in Ess$(\nu_c(r))$ occurs in the northeast corner of a block.*

Proof. First, recall that we can graphically construct the essential set of $\nu_c(r)$ by shooting "death rays" west and south of each 1; the uncrossed entries fully enclosed by death rays form bounded regions, and the northeast-most entry in these bounded regions is an essential box. The collection of all essential boxes is the essential set. We want to convince ourselves that each essential box lies in the northeast corner of a block.

Consider the death rays within a block column; by Property (*iii*) of Proposition 5.1, the "tallest" deathray (i.e. the first death ray encountered going down the block column) will be aligned on the western edge of of the block column, the next tallest will be adjacent, and so on, so that the deathrays monotonically descend west to east across the column. In the same way, by Property (*ii*) of proposition 5.1, deathrays are south-edge aligned in block rows.

Thus, the north and east edges of each bounded region will lie along block row and block column boundaries, and so the northeast corner of each bounded region will fall on the northeast corner of a block. □

To summarize the content of this chapter, we have shown that, given any quiver representation V, there exists a bounded affine permutation $\nu_c(r)$ that satisfies two key properties:

1. Every essential box in the essential set of of $\nu_c(r)$ lies in the northeast corner of a block,

2. $c(r)$ "agrees" with $\nu_c(r)$, in the sense that the number of 1's weakly southwest of each block in $\nu_c(r)$ minus the number of columns in the interval associated to that block is the cyclic rank of that interval (i.e. our formula $c(r)_{i,j} = |[i,j]| - \#1$'s weakly SW of block (i,j) is satisfied).

We will make use of this fact to prove the main theorem in the next section.

Chapter 6

Identification of Open Patches of Positroid Varieties and Quiver Loci

In this chapter, we prove the main result of the thesis, namely the identification between bipartite type A quiver loci and positroid varieties intersected with an open patch of the Grassmannian.

Theorem 6.1. *Let Q be a bipartite type A quiver with dimension vector $\underline{d}$, and let $\Omega_r \subset rep_Q(\underline{d})$ be a quiver locus with corresponding rank array r. Then there exists a positroid variety Π in $Gr(d_y, d)$ such that*

$$\Pi \cap U \cong \Omega,$$

where U is the open patch of $Gr(d_y, d)$ whose points have the form

$$\left[\ * \ |I_{d_y}\right].$$

Proof. Let Q and $\underline{d}$ be given, and let $\Omega_r \subset \mathrm{rep}_Q(\underline{d})$ be a quiver locus with rank array r. By Proposition 4.9, the cyclic Zelevinsky map ξ provides an isomorphism between Ω_r and the cyclic block rank variety $C_{c(r)}$. Thus it remains to show that $C_{c(r)} \cong \Pi \cap U$.

Recall from Chapter 5 that, by the construction of $\nu_c(r)$, the number of 1's weakly southwest from position (i, j) in $\nu_c(r)$ is $|[i, j]| - c(r)_{i,j}$, and so

$$I_{\nu_c(r)} = \langle \text{minors of size } (c(r)_{i,j} + 1) \text{ in } M_{[[i,j]]} \mid i \leq j \leq i + n\rangle.$$

Note that the generators of $I_{\nu_c(r)}$ are given in the same way as those of $I_{c(r)}$. Recall that generators of $I_{c(r)}$ are given by conditions coming from points in the northeast corners of blocks, and so since the conditions for $I_{\nu_c(r)}$ range over every pair (i, j), $1 \leq i \leq n$,$i \leq j \leq i + n$, the generators of $I_{c(r)}$ are a subset of the generators of $I_{\nu_c(r)}$. That is, $I_{c(r)} \subseteq I_{\nu_c(r)}$.

For the reverse inclusion, recall from §3.3 that we may consider only conditions coming from the essential set of $\nu_c(r)$ when collecting the generators of $I_{\nu_c(r)}$. By corollary 5.5, we have that the essential boxes in the essential set of $\nu_c(r)$ occur in the northeast corners of blocks. Thus, the generators of $I_{\nu_c(r)}$ are a subset of the generators of $I_{c(r)}$, giving us that $I_{\nu_c(r)} \subseteq I_{c(r)}$.

Since we have both inclusions, we have that $I_{c(r)} = I_{\nu_c(r)}$, and so we have that the cyclic Zelevinsky map provides an isomorphism

$$\Pi \cap U \cong \Omega.$$

□

Chapter 7

Future Directions

The identification described in this thesis can be leveraged in several ways; we see two directions as potential next steps using these results.

The first has to do with generating new combinatorial formulas for multidegrees and K-polynomials. These objects are frequently of significant interest when studying various algebraic varieties, and have been studied previously in the context of type A bipartite quiver loci (as seen in [KKR19]). In [KLS13], the authors present several formulas for multidegrees and Hilbert series' in the context of positroids; we will import these to the type A quiver variety case using similar techniques as in [KKR19]. This inquiry would contribute to the study of degeneracy loci of vector bundles, complimenting the work of many authors starting in the early 2000s, and continuing to today (see [KKR19, BF99, BFR05, KMS06]).

Our second potential direction involves describing cluster structure on bipartite type A quiver loci. Positroid varieties share a strong connection with cluster algebras (see [GL23]), and we would impose that structure on quiver varieties via the dictionary between quiver loci and positroid varieties provided in this thesis. The primary aim in this endeavor would be to expand the library of examples of varieties with cluster structure. Since their introduction, cluster algebras have found considerable application in representation theory, geometry, combinatorics, and numerous other areas, and this connectivity is part of what garners interest in them. Quiver loci are similarly highly applicable in a multitude of diverse fields of study, and so by linking them as we propose, this simultaneously compounds both of their utility and application.

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