Open Quantum Dynamics On Lie Groups: An Effective Field Theory Approach

# Open Quantum Dynamics On Lie Groups: An Effective Field Theory Approach

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# Abstract

In this thesis, we address construction of effective action for the dissipative systems whose configuration space coincides with a Lie group. We start by generalizing the classical system plus reservoir model to the case of position dependent Ohmic dissipation. This is achieved by coupling the system to a field living in one extra dimension. Then, employing the Schwinger-Keldysh technique, we construct the general influence functional for a system on a Lie group which includes the classical contribution and the first quantum correction within the linear response approximation. Abandoning the linear response assumption, we generalize the results by requiring the invariance under the dynamical Kubo-Martin-Schwinger symmetry. This gives us the most general influence functional with nonlinearly realized symmetry. We explore its systematic reduction to the case of strictly Ohmic dissipation. Finally, we revisit the field theoretic model of the bath and show that it produces both the leading and first subleading parts of the most general influence functional at high temperature.

# *Acknowledgements*

I acknowledge the personal and academic support of my advisor, Sergey Sibiryakov. Definitely, working with him was one of the greatest fortunes in my life. I would like to dedicate this thesis to my parents whom I am the realization of their dreams.

# Contents

# Declaration of Authorship

I, Afshin Besharat, declare that this thesis titled, “Open Quantum Dynamics On Lie Groups: An Effective Field Theory Approach” and the work presented in it are my own. I confirm that:

- The calculation in chapter 3 is done mainly by me, and in collaboration with Jury Radkovski. It was conducted under the supervision of Sergey Sibiryakov. The result was published in Phys. Rev. E 109, L052103.
- The calculation in chapter 4 and 5 is done in collaboration with Sergey Sibiryakov and Yuri Radkovski.
- The calculation in chapter 6 is done by me under the supervision of Sergey Sibiryakov.
- The result of chapter 4, 5, and 6 will appear in a publication.

# Chapter 1

# Introduction

Dissipation is a ubiquitous phenomenon in real-world systems, but addressing it at the quantum level is challenging due to the non-conservative nature of dissipative systems, which lack a well-defined action functional. The complexity arises from the involvement of internal degrees of freedom in the dissipative dynamics, where accounting for the vast number of these degrees is often neither feasible nor desirable. There are situations where the internal degrees of freedom, or equivalently the bath, are irrelevant and the whole effect of the bath is encoded in a few coefficients or functions. For example, in the Brownian motion, when the system's timescale significantly exceeds the environment's correlation time, the dissipative dynamics can be described by the Langevin equation [Kalmykov and Coffey 2012]. In the Langevin equation, the environmental effects are encapsulated by a handful of dissipative coefficients and a Gaussian white noise. Although we know how to derive the Langevin equation from a microscopic model in the simpler cases, such as the case of a damped harmonic oscillator [Schwinger 1961], there is no general recipe to obtain the state dependent Langevin equation (when the dissipation coefficients are position dependent) [Han et al. 2006; Zhang et al. 2023; Ulbrich et al. 2023; Lau and Lubensky 2007] from first principles. One trick to address this issue is the system-plus-reservoir model [Caldeira and Leggett 1983a; Feynman and Vernon Jr 2000] which is used to model dissipative quantum tunneling [Caldeira and Leggett 1981; Caldeira and Leggett 1983a], thermalization in curved spacetime [Colas et al. 2022], thermalization in non-fermi liquid [Hosseinabadi et al. 2023], the Dicke model [Kirton et al. 2019], excitation energy transfer [Kundu and Makri 2022], modeling the noise and friction in nanomechanical systems [Bachtold et al. 2022], and the polaron problem [Mandal et al. 2020; Mandal et al. 2022; Buchholz et al. 2019; Ruggenthaler et al. 2023; Foley et al. 2023]. This model assumes that dissipation is featureless, allowing the substitution of the actual dissipative degrees of freedom with a set harmonic oscillators with gapless energy spectrum. This simplification is primarily technical, as these bath modes can be integrated out exactly. Alternatively, the universal features of dissipation can be addressed from an effective field theory point of view. The Brownian motion, for instance, can be obtained by integrating out the fast degrees of freedom [Van Kampen and Oppenheim 1986]. It is the subject of this thesis to obtain a recipe to obtain an effective theory for the dissipative dynamics at high temperature limit which results in a local in time effective action. At the classical level, we restrict ourselves to the case of an arbitrary Ohmic dissipation, while in addressing the (thermal and quantum) noise, we focus on the case of open dynamics on Lie group to classify the quantum correction based on symmetries.

This thesis is organized as follows. In the next chapter, the basic background including the system plus reservoir model and the double-time path integral is reviewed. Especially, the system plus reservoir is covered from a new perspective which turns out to be useful for the purposes of this work in the next chapter. Focusing on the case of Ohmic friction, we propose a model which is essential in the generalization of the system plus reservoir model. This model, which is termed the bulk model, gives an action functional to describe the classical dissipative viscous dynamics which nonlinearly realizes the symmetry. In addition, the model can be used to describe the classical non-holonomic systems with linear constraints. Postponing the attempt to quantize the model to chapter 6, we propose a microscopic model to describe the dissipative dynamics in chapter 4. We first determine the most general interaction between a system and a bath which nonlinearly realizes symmetry. Then, by implementing the Schwinger-Keldysh (S-K) technique [Kamenev 2023], we integrate out the bath degrees of freedom which is assumed to be at finite temperature. This results in an effective action for the system which is the sum of the action of the free body and the influence functional which encodes the effects of the bath. In integrating out the bath's degrees of freedom, we use the linear response approximation, according to which, only the two-point functions of the bath are considered to build the influence functional. With the assumption that the response in the bath dies quickly at high temperature, the theory allows a gradient expansion of the influence functional which is a series expansion of the powers of the S-K quantum fields and time derivatives of the S-K classical fields. In defining of the S-K classical (quantum) field, we do not use the usual definition where the classical (quantum) fields is the sum (difference) of the forward and backward in time fields. Instead, we use a definition according which, the classical field transforms covariantly under the underlying nonlinearly realized symmetry and the quantum field is left intact. This makes the effective action of the theory manifestly covariant under the underlying microscopic symmetry and captures the right power counting of the effective theory. The effects of the bath is encoded in a handful of dissipative coefficients. These coefficients, however, are not the most general possible dissipative coefficients as they are obtained at the linear response approximation. The most general dissipative coefficients at high temperature are allowed by the dynamical Kubo-Martin-Schwinger (DKMS) transformation [Sieberer et al. 2015; Liu and Glorioso 2018; Akyuz et al. 2024]. This is the subject matter of chapter (5). In the chapter, using the same definition of the S-K classical and quantum fields of chapter (4), we derive the most general high temperature effective action of the dissipative dynamics which nonlinearly realizes symmetry. In contrast to the chapter 4, there is no direct reference to the microscopic details of the bath and the dissipative coefficients are less restricted. However, one can adhere to different limits, such as the linear response regime or the Ohmic regime, to reduce the freedom in choosing the universal dissipative coefficients. Having exploited the linear response approximation in chapter 4, the Ohmic reduction of the dissipative coefficients is introduced at the end of the chapter. Then, in chapter 6 we return to the bulk model, but this time at finite temperature. Mapping the model to a nonlinear sigma model at the lowest order, we integrate out the bath degrees of freedom, generating the most general state dependent influence functional at the classical level. In addressing the quantum corrections, we restrict the bulk model configuration space to a Lie group. We show that it produces the most general influence functional functional in the strictly Ohmic regime predicted by the DKMS method at high temperature.

# Chapter 2

# Some Introductory Remarks

In this chapter, some introductory remarks are reviewed. In the first section, there is a short discussion on the Langevin equation. Then, a discussion on the system plus reservoir from a different perspective, and a review of the double time path integral formalism of open quantum systems are presented.

## 2.1 Introduction to Classical Dissipative Systems

Having an idealized closed system is just an approximation which in many applications needs to be moderated. A subset of open dynamics is dissipative dynamics in which the system of interest (or simply system) loses energy through interaction with an environment. In the case that the environment is not influenced by the dynamics of the system, we refer to it as a bath. In this work, we are interested in the long-time behavior of a dissipative system, where the system plus the bath have reached to the thermal equilibrium. Note that, the reduced dynamics of the system, which is obtained by integrating out the bath degrees of freedom is nonequilibrium in nature. Traditionally, this reduced dynamics is described by the Langevin equation [Langevin et al. 1908; Van Kampen 1992; Mazo 2008] which for a particle of mass $m$ reads:

$$m\frac{d^2x(t)}{dt^2} = F(t) - \gamma\frac{dx(t)}{dt} + \eta(t), \tag{2.1}$$

where:

- $x(t)$ is the position of the particle at time $t$.
- $F(t)$ represents any external deterministic force acting on the particle
- $\gamma$ is the damping constant coefficient.
- $\eta(t)$ is a random force representing the thermal fluctuations from the surrounding environment, often modeled as Gaussian white noise with zero mean and correlation given by $\langle\eta(t)\eta(t')\rangle = 2D\delta(t-t')$, where $D$ is the noise intensity.

According to the fluctuation-dissipation theorem, the damping coefficient $\gamma$ and the noise intensity $D$ are related to each other through $D = \gamma k_B T$ at the temperature $T$. The Langevin equation describes Ohmic dissipation, where the dissipative force is proportional to the velocity of the particle.

The Langevin equation can be written in terms of a path integral formalism using the Martin-Siggia-Rose (MSR) technique [Martin et al. 1973]. According to this technique, the Langevin equation is written in terms of a path integral over an auxiliary field $\hat{x}$. For instance, at the Smoluchowski limit, when $\gamma\frac{dx}{dt} >> m\frac{d^2x}{dt^2}$, we can write the Langevin equation in the following functional form

$$\int \mathcal{D}x\, \delta\left(\gamma\frac{dx(t)}{dt} + \frac{\partial U(x)}{\partial x} - \eta(t)\right) = \int \mathcal{D}x\mathcal{D}\hat{x}\, e^{i\int dt\, \hat{x}(t)\left(\gamma\frac{dx(t)}{dt}+\frac{\partial U(x)}{\partial x}-\eta(t)\right)} \tag{2.2}$$

where $\delta(F(x))$ is the delta Dirac functional. This gives us a phenomenological generating functional to calculate the correlation and response functions,

$$\begin{aligned} Z &= \int \mathcal{D}x\mathcal{D}\hat{x}\, e^{-S[x,\hat{x}]} \\ S[x,\hat{x}] &= \int dt\left[\hat{x}(t)\left(\frac{dx(t)}{dt} + \frac{\partial U(x)}{\partial x}\right) - D\hat{x}^2(t)\right]. \end{aligned} \tag{2.3}$$

The MRS approach is one of the successful techniques to incorporate the fluctuations and the dynamics into each other. Yet, this approach is not systematic as it has not been derived from a microscopic theory and it cannot address the noise at a full nonlinear level [Liu and Glorioso 2018].

## 2.2 The Harmonic Bath Model

One of the successful techniques to incorporate the quantum effects into the dissipative dynamics is the so-called "system plus reservoir" or harmonic bath model [Schwinger 1961, Caldeira and Leggett 1983a]. The basic idea of this model is to use the fact that,

- Dissipation is emergent and it must originate from a full unitary theory,
- (some) dissipative processes are not sensitive to the microscopic details,
- and dissipative degrees of freedom are gapless.

Based on these assumptions, the environment can be modeled by a bunch of harmonic oscillators with continuous gapless spectrum of frequency as the proxy degrees of freedom for the environment. Being quadratic, the model is exactly solvable leaving us with an effective action which in general is nonlocal time.

Let's review the harmonic bath formalism from a relatively different perspective which is in close connection with the formalism in chapters 3 and 6. In this model, the environment is modeled by a gapless string [Unruh and Zurek 1989]. Let's go through this method by an explicit example, one dimensional damped harmonic oscillator. Consider a one dimensional harmonic oscillator with coordinate $q$ which is coupled to a gapless field in an extra dimension $z$ at the boundary at $z = 0$ (see figure 3.2). Showing the gapless field with $\xi(z,t)$, the action functional of the coupled field and the oscillator reads

$$S = \int dt\,\frac{1}{2}(\dot{q}^2 - \Omega^2 q^2) + \alpha \int dt\,\dot{q}\,\xi\big|_{z=0} + \int_{z>0} dtdz\left(\frac{1}{2}(\partial_t \xi)^2 - \frac{1}{2}(\partial_z \xi)^2\right), \tag{2.4}$$

where $\Omega$ shows the frequency of the harmonic oscillator in the absence of friction and $\alpha$ is the constant of dissipation. The equation of motion for the gapless string which we refer to as bulk degrees of freedom reads

$$\partial_t^2 \xi(z,t) - \partial_z^2 \xi(z,t) = 0. \tag{2.5}$$

Using the Fourier expansion of $\xi(z,t)$,

$$\xi(z,t) = \int_0^\infty \frac{d\omega}{2\pi}\,\xi_\omega(z)\ \exp(-i\omega t)$$

the bulk equation of motion becomes

$$(\omega^2 + \partial_z^2)\xi_\omega(z) = 0 \tag{2.6}$$

with the solution

$$\xi_\omega(z) = A_\omega \exp(i\omega z) + B_\omega \exp(-i\omega z). \tag{2.7}$$

where $A_\omega$ and $B_\omega$ are two ($\omega$ dependent) constants. The variation of $\xi(z,t)$ at $z = 0$ results in

$$-\partial_z \xi(0,t) = \alpha \dot{q}(t)\ ,$$

where in the Fourier space and using equation (2.7), it reads

$$\alpha q_\omega = A_\omega - B_\omega. \tag{2.8}$$

At the boundary, the equation of motion of the damped harmonic oscillator is

$$\ddot{q} + \Omega^2 q = -\alpha \dot{\xi}(0,t)\ ,$$

where in the Fourier space becomes

$$(-\omega^2 + \Omega^2)q_\omega = i\omega\alpha(A_\omega + B_\omega). \tag{2.9}$$

Combining equations (2.8) and (2.9) results in

$$A_\omega = \frac{1}{2}\left(\frac{\Omega^2 - \omega^2}{i\alpha\omega} + \alpha\right) q_\omega, \quad B_\omega = -A_\omega^\star. \tag{2.10}$$

Quantization of the damped harmonic oscillator at the boundary is equivalent to writing the operator $q$ and $\xi$ in terms of the creation and annihilation operators and using the complete basis of space,

$$q = \int_0^\infty \frac{d\omega}{2\pi}\; C_\omega \left(a_\omega \exp(-i\omega t) + a_\omega^\dagger \exp(i\omega t)\right). \tag{2.11}$$

$$\xi(z,t) = \int_0^\infty \frac{d\omega}{2\pi}\; C_\omega \left((\tilde{A}_\omega \exp(i\omega z) - \tilde{A}_\omega^\star \exp(-i\omega z))a_\omega \exp(-i\omega t) + \text{H.C}\right) \tag{2.12}$$

where $\tilde{A_\omega} = \frac{A_\omega}{q_\omega}$, H.C stands for the Hermitian conjugate, and the commutation relations

$$[a_\omega, a_{\omega'}^\dagger] = 2\pi\delta(\omega-\omega')\;, \quad [a_\omega, a_{\omega'}] = [a_\omega^\dagger, a_{\omega'}^\dagger] = 0 \tag{2.13}$$

are assumed. $C_\omega$ is a constant to be determined by imposing the canonical commutation relation

$$[\xi(z,t), \dot{\xi}(z',t)] = i\delta(z-z')\;.$$

Introducing $\tan(\phi_\omega) = \frac{\omega^2-\Omega^2}{\omega\alpha^2}$ and using the commutation relations for $a_\omega$ and $a_\omega^\dagger$, then we have

$$\begin{aligned}[\xi(z,t), \dot{\xi}(z',t)] &= 2i\int_0^\infty \frac{d\omega}{2\pi}\; \omega|\tilde{A}_\omega|^2|C_\omega|^2\Big(\exp\left(i\omega(z-z')\right) + \exp\left(i\omega(z'-z)\right)\Big) \\ -2i\int_0^\infty &\frac{d\omega}{2\pi}\; \omega|\tilde{A}_\omega|^2|C_\omega|^2\Big(\exp\left(i\omega(z+z'+2\phi_\omega)\right) + \exp\left(i\omega(z'+z+2\phi_\omega)\right)\Big). \end{aligned} \tag{2.14}$$

Taking

$$C_\omega = \frac{1}{2\omega^{1/2}|\tilde{A}_\omega|} \tag{2.15}$$

the first integral in (2.14) gives us a delta Dirac function $i\delta(z-z')$, and the second integral becomes zero which is desired. In addition, we need to prove that with this choice of $C_\omega$, the canonical commutation relation $[p,q] = i$ does hold, where $p$ is the canonical momentum corresponding to $q$. The canonical momentum $p$ at the boundary reads

$$p = \dot{q} + \alpha\;\xi(z=0,t) = \int_0^\infty \frac{d\omega}{2\pi}\Big(( - i\omega + \alpha(\tilde{A}_\omega - \tilde{A}_\omega^\star))a_\omega \exp(-i\omega t) + \text{H.C}\Big) \tag{2.16}$$

$$= i\;\int_0^\infty \frac{d\omega}{2\pi}\;\left(\frac{\Omega^2}{\omega}\right)\Big(a_\omega\;\exp(-i\omega t) - a_\omega^\dagger \exp(i\omega t)\Big) \tag{2.17}$$

Using this and the expansion formula for $q$ in terms of the creation and annihilation operators, then the canonical commutation relation at the boundary can be written as

$$[q,p] = 2i\int_0^\infty \frac{d\omega}{2\pi}\;\frac{d\omega'}{2\pi} C_\omega C_{\omega'}\omega\big( - a_\omega a_{\omega'}^\dagger + a_{\omega'}^\dagger a_\omega\big)\exp\left(i(-\omega+\omega')t\right) \tag{2.18}$$

Inserting the obtained value for $C_\omega = \frac{1}{2\omega^{1/2}|A_\omega|}$ in this and using the assumed commutators for the creation and annihilation operators one can show that $[q,p] = i$ which is expected and proves the consistency.

Now, we can calculate different observable quantities of the damped harmonic oscillator. The energy spectrum is not informative due to mixing with the gapless field; the spectrum is a continuum from zero to infinity. Instead, we calculate the density of states which is defined via the Laplace transformation of the partition function,

$$\rho(\epsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\beta \; Z_\alpha[\beta] \; \exp[\beta\epsilon] \; , \tag{2.19}$$

where $Z_\alpha[\beta]$ the partition function of the system at temperature $T = \frac{1}{\beta}$, $\epsilon$ is the energy level and $c$ is a positive number which is greater than the smallest real part of the poles of the partition function. Using the path integral formalism, we can obtain the partition function of the damped harmonic oscillator. Going into the imaginary time representation $t \to -i\tau$, then the Euclidean action of the oscillator plus the bulk takes the following form

$$S_E = \int_{\tau=0}^{\beta} d\tau \left[ \frac{1}{2}((\frac{dq}{d\tau})^2 + \Omega^2 q^2) - i\alpha \frac{dq}{d\tau}\xi|_{z=0} + \int_{z>0} dz \left( \frac{1}{2}(\partial_\tau \xi)^2 + \frac{1}{2}(\partial_z \xi)^2 \right) \right] . \tag{2.20}$$

Using the path integral definition of the partition function

$$Z_\alpha[\beta] = \int [Dq][D\xi] \exp[-\beta S_E] \; ,$$

the partition function can be calculated exactly as it is Gaussian in terms of the functions $q$ and $\xi$. Solving the quadratic path integrals using the Matsubara technique, the partition function reads

$$\begin{aligned} & Z_\alpha[\beta] \\ & = \frac{\beta\tilde{\lambda}_+\tilde{\lambda}_-}{4\pi^2\Omega} \, \Gamma\big[-\frac{\beta\tilde{\lambda}_+}{2\pi}\big] \, \Gamma\big[-\frac{\beta\tilde{\lambda}_-}{2\pi}\big] \exp\left[\frac{\beta}{2\pi}\left(\tilde{\lambda}_+ \log(-\frac{\beta\tilde{\lambda}_+}{2\pi}) + (\tilde{\lambda}_+ \to \tilde{\lambda}_-)\right)\right], \end{aligned} \tag{2.21}$$

where

$$\tilde{\lambda}_\pm = -\frac{\alpha^2}{2} \pm i\sqrt{\Omega^2 - \frac{\alpha^4}{4}}$$

and $\Gamma[x]$ is the gamma function. This gives the spectral density of the damped harmonic oscillator as

$$\rho(\epsilon) = \sum_{n=1}^{\infty} (\text{Residues of the integrand (2.19) at the poles of the gamma functions}) \tag{2.22}$$

The poles of the gamma functions are located at $\beta = 0$, $\beta_n^+ = \frac{2\pi n}{\lambda_+}$, and $\beta_n^- = \frac{2\pi n}{\lambda_-}$. The

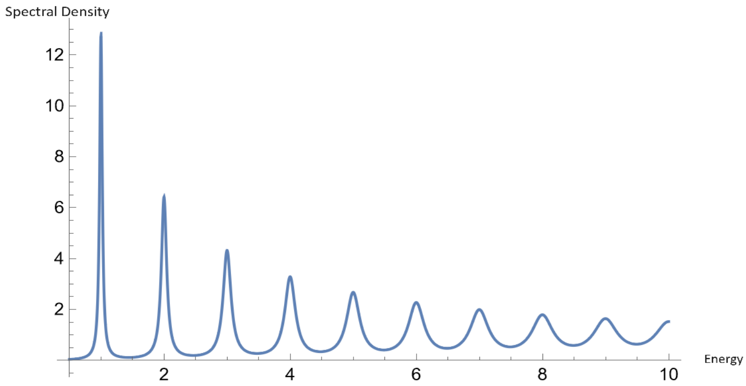


FIGURE 2.1: The spectral density of $\rho(\epsilon)$ of the damped oscillator for $\alpha^2 = 0.05$ and $\Omega = 1$. At $\epsilon = 0$, we have $\rho(\epsilon) = \delta(\epsilon)$.

corresponding residues of the integrand are

$$\begin{aligned}
&Res(\beta = 0) = \frac{1}{\Omega}\\
&Res(\beta = \beta_n^+)\\
&= \frac{1}{\Omega}(\frac{\lambda_-}{\lambda_+})\Gamma(\frac{\lambda_-}{\lambda_+})\frac{(-1)^{n-1}}{\Gamma(n)}\exp(2\pi n\epsilon/\lambda_+)\exp\left(n(\log(-n)-1)+n\frac{\lambda_-}{\lambda_+}(\log(-n\frac{\lambda_-}{\lambda_+})-1)\right)\\
&Res(\beta = \beta_n^-) = Res^\star(\beta = \beta_n^+)\ .
\end{aligned} \tag{2.23}$$

The spectral density for values of $\alpha^2 = 0.05$ and $\Omega = 1$ in the underdamped regime is depicted in figure (2.1). At the limit of $\alpha = 0$, the spectral density becomes a set of discrete delta Dirac functions. Therefore, dissipation gradually broadens the energy levels, and at the overdamped regime, it completely closes the gap between the energy levels [Hanke and Zwerger 1995]. Due to dissipation, the particle has a finite lifetime in the excited energy states and decays into the ground state. We will come back to the observables of the damped harmonic oscillator in the next chapter and argue that the calculation of the observables from the bulk model of 3 coincides with that of this chapter.

## 2.3 The Feynman-Vernon Model of Open Quantum Systems

In this section, we present a short summary of the double-time path integral method in describing dissipative quantum systems. As it was pointed out in the previous chapter, dissipation is emergent in the sense that it originates from a unitary theory. If we integrated out part of the universe, the remaining dynamics can be dissipative. This can be formulated in the language of the density matrix. The density matrix of the dynamical degrees of freedom is defined over the Hilbert space of the bath and the system. To obtain the effective time evolution of the system, we partial trace over the bath degrees of freedom. This picture is used to describe the decoherence and thermalization [Hornberger 2009; Kaplanek and Burgess 2020; Kaplanek and Burgess 2021; Kanno et al. 2021b; Parikh et al. 2021; Kanno et al. 2021a; Bose et al. 2023; Cho and Hu 2022; Colas et al. 2024; Tobar et al. 2023; Kanno et al. 2023; Cho and Hu 2023; Hsiang et al. 2024; Matsumura 2021; Bak et al. 2023; Colas et al. 2022]. To describe the time evolution of the density matrix, we can use the double-time path integral formalism [Calzetta and Hu 2009; Kamenev 2023; Sieberer et al. 2023]. In this chapter, we review the Feynman-Vernon method [Feynman and Vernon Jr 2000] to describe the double-time path integral formalism. The total density operator $\hat{\rho}(t)$ evolves in time in two opposite directions according to

$$\hat{\rho}(t) = U(t)\hat{\rho}(0)U^{-1}(t) \tag{2.24}$$

where $\hat{\rho}(0)$ refers to the initial density matrix. The density matrix is defined over the Hilbert space of the system, represented by $q$, and the Hilbert space of the environment which is represented by $\chi$. The representation of the density matrix in the $q-\chi$ basis is

$$\rho(q_f, q'_f; \chi_f, \chi'_f; t) = \langle q_f, \chi_f|\hat{\rho}(t)|q'_f, \chi'_f\rangle \tag{2.25}$$

Using the unity operators in the $q-\chi$ basis, $\hat{1} = \int_{q_i,\chi_i} |q_i, \chi_i\rangle\langle q_i, \chi_i|$, the density matrix in the $q-\chi$ representation takes the following form:

$$\rho(q_f, q'_f; \chi_f, \chi'_f; t) = \int_{\chi_i, q_i; \chi'_i, q'_i} \langle q_f, \chi_f|U(t)|q_i, \chi_i\rangle\langle q_i, \chi_i|\hat{\rho}(0)|q'_i, \chi'_i\rangle\langle q'_i, \chi'_i|U^{-1}(t)|q'_f, \chi'_f\rangle. \tag{2.26}$$

The reduced density matrix of the system of our interest can be obtained by tracing out the environment degrees of freedom

$$\rho_r(q_f, q'_f; t) = \int_{\chi_f} \rho(q_f, q'_f; \chi_f, \chi_f; t). \tag{2.27}$$

If the environment is in thermal equilibrium and is large enough to be considered as a bath, one expect that the initial preparation information is lost after the passage of long enough time. In such a situation we can can start from initial product state $\hat{\rho}(0) = \hat{\rho}_q \otimes \hat{\rho}_\chi$ which simplifies the calculation. Although it looks artificial, but it is plausible in the situations where we are interested in the long-time behavior of the system where the intial information is totally scrambled. With the initial product state, then we have

$$\begin{aligned}\rho_r(q_f, q'_f; t) &= \int_{q_i, q'_i} \rho_q(q_i, q'_i) \int_{\chi_f} \int_{\chi_i, \chi'_i} \rho_\chi(\chi_i, \chi'_i) G(q_i, \chi_i; q_f, \chi_f; t) G^\dagger(q'_i, \chi'_i; q'_f, \chi'_f; t) \\ &= \int_{q_i, q'_i} \rho_q(q_i, q'_i) J_{FV}(q_i, q'_i; q_f, q'_f; t)\end{aligned} \tag{2.28}$$

where the Green's function is given by $G(q_i,\chi_i;q'_f\chi_f;t) = \langle q_f,\chi_f|U(t)|q_i,\chi_i\rangle$ and its Hermitian conjugate is denoted by $G^\dagger$. The Feynman-Vernon functional (FV functional) is defined by

$$J_{FV}(q_i,q'_i;q_f,q'_f;t) = \int_{\chi_f}\int_{\chi_i,\chi'_i}\rho_\chi(\chi_i,\chi'_i)G(q_i,\chi_i;q_f,\chi_f;t)G^\dagger(q'_i,\chi'_i;q'_f,\chi'_f;t). \tag{2.29}$$

If we use the path integral definition of the Green's functions, then we can write the FV functional in the following form

$$\begin{aligned} J_{FV}(q_i,q'_i;q_f,q'_f;t) = &\int_{q_+(t_i)=q_i}^{q_+(t_f)=q_f} Dq_+(t)\int_{q_-(t_i)=q'_i}^{q_-(t_f)=q'_f} Dq_-(t)\mathrm{e}^{i\left(S_q[q_+]-S_q[q_-]\right)} \\ \times\int_{\chi_f}\int_{\chi_i,\chi'_i}\rho_\chi(\chi_i,\chi'_i)&\int_{\chi_+(t_i)=\chi_i}^{\chi_+(t_f)=\chi_f} D\chi_+(t)\int_{\chi_-(t_i)=\chi'_i}^{\chi_-(t_f)=\chi'_f} D\chi_-(t)\mathrm{e}^{i\left(S_{q,\chi}[q_+,\chi_+]-S_{q,\chi}[q_-,\chi_-]\right)} \end{aligned} \tag{2.30}$$

where the part of action which solely depends on the system degrees of freedom has been shown by $S_q$ and the rest which includes the environment degrees of freedom and its interaction with the system is shown by $S_{q,\chi}$. If we solve the path integrals over the environment degrees of freedom, then we can write the FV functional as

$$J_{FV}(q_i,q'_i;q_f,q'_f;t) = \int_{q_+(t_i)=q_i}^{q_+(t_f)=q_f} Dq_+(t)\int_{q_-(t_i)=q'_i}^{q_-(t_f)=q'_f} Dq_-(t)\mathrm{e}^{i\left(S_q[q_+]-S_q[q_-]+\mathcal{I}[q_+,q_-]\right)}. \tag{2.31}$$

where the influence functional $\mathcal{I}[q_+,q_-]$ is given by

$$\mathrm{e}^{i\mathcal{I}[q_+,q_-]} = \int_{\chi_f}\int_{\chi_i,\chi'_i}\rho_\chi(\chi_i,\chi'_i)\int_{\chi_+(t_i)=\chi_i}^{\chi_+(t_f)=\chi_f}\int_{\chi_-(t_i)=\chi'_i}^{\chi_-(t_f)=\chi_f}\mathrm{e}^{i\left(S_0[\chi_+]+S_I[q_+,\chi_+]-S_0[\chi_-]-S_I[q_-,\chi_-]\right)}, \tag{2.32}$$

where the action $S_{\chi,q}$ is split into a free part $S_\chi$ and an interacting part $S_I$. Through integrating out the bath degrees of freedom, the forward $q_+$ and backward $q_-$ degrees of freedom are mixed with each other, so that the influence functional cannot be split in the form $\mathcal{I}[q_+,q_-] = \mathcal{I}_+[q_+] + \mathcal{I}_-[q_-]$. With an interacting potential of the form $gV_I(t)$, the interacting action reads

$$S_I = -g\int dtV_I(t). \tag{2.33}$$

According to the definition of the path integral one can write

$$\int_{\chi_+(t_i)=\chi_i}^{\chi_+(t_f)=\chi_f}\mathrm{e}^{i\left(S_0[\chi_+]+S_I[q_+,\chi_+]\right)} = \langle\chi_f|\mathcal{T}(e^{-ig\int_0^t dt'\hat{V}_{+I}(t')})|\chi_i\rangle \tag{2.34a}$$

$$\int_{\chi_+(t_i)=\chi_i}^{\chi_+(t_f)=\chi_f}\mathrm{e}^{-i\left(S_0[\chi_-]+S_I[q_-,\chi_-]\right)} = \langle\chi_f|\tilde{\mathcal{T}}(e^{ig\int_0^t dt'\hat{V}_{-I}(t')})|\chi_i\rangle \tag{2.34b}$$

where $\mathcal{T}$ ($\tilde{\mathcal{T}}$) is the time-ordered (anti time-ordered) operator. With this, then the influence functional can be written in the following condensed form [Boyanovsky 2015]

$$\begin{aligned} \mathrm{e}^{i\mathcal{I}[q_+,q_-]} &= Tr[\mathcal{T}(e^{-ig\int_0^t dt'\hat{V}_I(t')})\hat{\rho}_\chi(0)\tilde{\mathcal{T}}(e^{ig\int_0^t dt'\hat{V}_I(t')})] \\ &\equiv \langle\tilde{\mathcal{T}}(e^{ig\int_0^t dt'\hat{V}_I(t')})\mathcal{T}(e^{-ig\int_0^t dt'\hat{V}_I(t')})\rangle_{\rho_\chi}. \end{aligned} \tag{2.35}$$

Equation (2.35) is the main result of this section which will be used in chapter 4 in building the influence functional at the linear response level, which means that we ignore terms of order $O(g^3)$ and higher in building the influence functional.

# Chapter 3

# Bulk Model: Classical Theory

In this chapter, we propose the bulk model to obtain an action for the Ohmic classical dissipative systems with arbitrary dependence of dissipative coefficients on the coordinates. It is the generalization of the system plus reservoir model of chapter (2). Finally, we relate the dissipative dynamics to a certain class of non-holonomic systems with linear constraints, and in an appropriate limit, the bulk model provides an action for such non-holonomic systems. The results of this chapter were published in Phys. Rev. E 109, L052103.

## 3.1 Ohmic Dissipative Motion In Classical Mechanics

The standard way to account for dissipation in the classical theory of dynamical systems is by adding non-conservative forces $F_i$ to the Euler–Lagrange equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i \,, \tag{3.1}$$

where $L(q, \dot{q})$ is the system Lagrangian, and $q^i$, $\dot{q}^i$ are the generalized coordinates and velocities, respectively. An important type of dissipation is ohmic dissipation when the extra forces are linear in velocities,

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j \equiv -\frac{\partial F}{\partial \dot{q}^i} \ . \tag{3.2}$$

The dissipative coefficients $\Gamma_{ij}(q)$ form a positive-definite symmetric matrix and are in general coordinate dependent. In the last equality we have conventionally written the force as the derivative of the Rayleigh function $F(q, \dot{q}) = (1/2)\Gamma_{ij}(q)\,\dot{q}^i\dot{q}^j$.

Fundamentally, the existence of dissipation is due to the interaction of the system (referred to as *central system* below) with its environment, also called *reservoir* or *bath*. In many applications the microscopic nature of the reservoir is not important and it can be modeled as a collection of infinitely many harmonic oscillators [Weiss 2012]. The action of the harmonic bath coupled to the central system then provides an effective action, from which Eq. (3.1) can be derived by means of the variational principle. Yet more importantly, the effective action is key for the application of the path integral methods used to study intrinsically quantum phenomena, such as tunneling [Caldeira and Leggett 1981], and other aspects of open systems in and out of thermal equilibrium [Sieberer et al. 2023; De Vega and Alonso 2017; Kamenev 2023].

However, as we discuss below, the harmonic bath model fails in the case when dissipation coefficients $\Gamma_{ij}(q)$ have general dependence on the system coordinates. The purpose of this chapter is to provide a reservoir model for this case. As a byproduct we also obtain the description of arbitrary gyroscopic forces. Before describing the model, let us discuss two broad classes of situations where the dependence of $\Gamma_{ij}$ on $q$ is essential.

## 3.2 Dynamics on cosets

Consider a class of systems whose configuration space represents a group manifold or, more generally, a coset space, and whose dynamics enjoy non-linearly realized symmetries. Many physically relevant systems can be cast in this form, from dynamics of a rigid body, to hydrodynamics [Arnold and Khesin 2008; Marsden and Ratiu 2013]. They appear in particle physics and condensed matter as a consequence of spontaneous symmetry breaking [Burgess 2000; Brauner 2010]. Development of an effective action for such systems in dissipative environment, besides conceptual interest, is motivated by numerous potential applications, for example to Brownian motion of stiff polymers [Li and Tang 2004], as well as micro and nanoparticles of various shapes [Duggal and Pasquali 2006; Han et al. 2006; Kraft et al. 2013; Zhang et al. 2019].

Following the standard coset construction [Callan Jr et al. 1969; Burgess 2000; Brauner 2010], we consider a Lie group $G$ and its subgroup $H$. The generators of $G$ are chosen in such a way that the first $A$ of them span the algebra of $H$, we denote them by $H_a$, $1 \le a \le A$. The rest of the $G$-generators are called *broken* and will be denoted by $\hat{G}_i$, $1 \le i \le I$. The coset $G/H$ representing the configuration space of the system can be identified with the group elements of the form[1]

$$\hat{g}(q) = \mathrm{e}^{q^i \hat{G}_i} \ . \tag{3.3}$$

The action of a group element $g$ on the coordinates $q \mapsto \tilde{q}$ is given by the left multiplication,

$$g \cdot \hat{g}(q) = \hat{g}(\tilde{q}) \cdot h \ , \qquad g \in G, \ h \in H \ . \tag{3.4}$$

Next, one constructs the Cartan form,

$$\hat{g}^{-1} d\hat{g} = \Omega^i_j(q) dq^j \, \hat{G}_i + \Omega^a_j(q) dq^j \, H_a \ , \tag{3.5}$$

and extracts from it the *covariant velocities*

$$D_t q^i = \Omega^i_j(q) \, \dot{q}^j \ . \tag{3.6}$$

[1] We assume summation over repeated indices.

Unlike the ordinary velocities $\dot{q}^i$, the covariant velocities transform linearly under (3.4). They form a linear representation of the subgroup $H$.

If the dynamics of the system are to respect the symmetry (3.4), its Lagrangian and Rayleigh function must be invariants constructed from the covariant velocities.[2] Thus we have,

$$F = \tfrac{1}{2}\gamma_{ij} D_t q^i D_t q^j = \tfrac{1}{2}\gamma_{ij}\Omega^i_k(q)\Omega^j_l(q)\,\dot{q}^k\dot{q}^l \,, \tag{3.7}$$

where $\gamma_{ij}$ is a constant invariant tensor in the relevant representation of $H$. In the simplest case when $H$ is empty ($G$ is fully broken) $\gamma_{ij}$ is arbitrary, provided it is symmetric and positive. For a general non-Abelian coset the coefficients of the Cartan form satisfy

$$\frac{\partial \Omega^k_i}{\partial q^j} - \frac{\partial \Omega^k_j}{\partial q^i} \neq 0 \,, \tag{3.8}$$

so their dependence on coordinates cannot be avoided by any choice of variables, implying the coordinate dependence of the dissipative coefficients $\Gamma_{kl} = \gamma_{ij}\Omega^i_k(q)\Omega^j_l(q)$.

## 3.3 The Bulk Model

Our starting point is the model used in [Unruh and Zurek 1989] to study environment-induced decoherence. It represents the reservoir as a free massless scalar field $\xi(t,z)$ in one-dimensional space (*bulk*) coupled to the central system at a single point $z = 0$ (*boundary*), and is equivalent to the more common independent-oscillator model [Caldeira and Leggett 1983b]. Its straightforward generalization for a central system with several degrees of freedom requires equal number of fields and leads to the following action,

$$S = \int\limits_{z=0} dt\,\left(L(q,\dot{q}) - \beta^j_i q^i \dot{\xi}_j\right) + \int\limits_{z>0} dtdz\,\frac{1}{2}\partial_\mu\xi_i\partial^\mu\xi_i \,, \tag{3.9}$$

where $\beta^j_i$ are constant couplings; in the last term we sum over indices $\mu = t, z$ with the Lorentzian metric $\eta^{\mu\nu} = \mathrm{diag}(1,-1)$. Importantly, the coordinate $z$ here is not a physical dimension, but is introduced merely to parameterize the internal dissipative degrees of freedom. By taking variation, one derives the dissipative forces, as well as the equations for the fields,

$$F_i = -\beta^j_i \dot{\xi}_j\big| \,, \qquad \partial_\mu\partial^\mu\xi_i = 0 \,, \qquad \partial_z\xi_i\big| = -\beta^i_j\dot{q}^j \,, \tag{3.10}$$

where the vertical bar means fields evaluated at $z = 0$. The dissipative dynamics is obtained by imposing outgoing boundary conditions on the bulk fields which singles out the solutions of the form $\xi_i(t,z) = \bar{\xi}_i(t-z)$. This implies $\partial_z\xi_i = -\partial_t\xi_i$ and combining the first and third equations in (3.10) we obtain the forces (3.2) with $\Gamma_{ij} = \beta^k_i\beta^k_j$. Note that coupling $q^i$ to $\dot{\xi}_i$, rather than the fields themselves, is essential for getting the response local in time.

[2] Up to possible Wess–Zumino–Witten terms Wess and Zumino 1971; Witten 1983; Goon et al. 2012.

The above construction fails for general coordinate dependent dissipation. As long as we want to preserve the harmonic nature of the bath, the only option is to generalize its coupling to the central system, $\beta_i^j q^i \dot{\xi}_j \mapsto \beta^j(q)\dot{\xi}_j$ with some arbitrary functions $\beta^i(q)$. Repeating the above derivation we then obtain the dissipative coefficients $\Gamma_{ij} = (\partial\beta^k/\partial q^i)(\partial\beta^k/\partial q^j)$ which, however, do not have the form needed for coset or nonholonomic systems due to the non-integrability properties (3.8), (3.23).

This failure can be also understood from the symmetry perspective. The system-reservoir coupling in (3.9) changes by a total time-derivative under the shifts of the coordinates $q^i(t) \mapsto q^i(t)+a^i$. This property ensures that the dissipative force is invariant under the coordinate shifts, as it should be for the case of constant $\Gamma_{ij}$. In the case of a general non-Abelian coset, however, we do not have at our disposal any functions $\beta^i(q)$ invariant or changing by a constant under the group transformations and hence we cannot construct any system-reservoir coupling that would preserve the symmetry of the problem.[3]

To resolve the issue, we apply a duality transformation to the action (3.9). Performing a change of variables $\tilde{\xi}_i = \beta_i^j \xi_j$ and integrating in a set of vectors $\chi^{\mu\, i}$, it can be rewritten as

$$S = \int\limits_{z=0} dt\,\big(L(q,\dot{q}) - q^i \dot{\tilde{\xi}}_i\big) + \int\limits_{z>0} dtdz \left(\chi^{\mu\, i}\partial_\mu \tilde{\xi}_i - \frac{\Gamma_{ij}}{2}\chi^{\mu\, i}\chi^j_\mu\right). \tag{3.11}$$

Variation with respect to $\tilde{\xi}_i$ gives two equations,

$$\partial_\mu \chi^{\mu\, i} = 0\,, \qquad \chi^{z\, i}\big| = \dot{q}^i\,. \tag{3.12}$$

The first one implies that $\chi^{\mu\, i}$ are expressed through gradients of scalar functions,

$$\chi^{\mu\, i} = -\epsilon^{\mu\nu}\partial_\nu \chi^i\,, \tag{3.13}$$

where $\epsilon^{\mu\nu}$ is the two-dimensional Levi–Civita symbol, $\epsilon^{tz} = 1$. The second equation then reduces to $\dot{\chi}^i\big| = \dot{q}^i$. Using that the fields $\chi^i$ are defined up to a constant, we can remove any offset between them and $q^i$ on the boundary, and obtain

$$\chi^i\big| = q^i\,. \tag{3.14}$$

Substituting (3.13) back into (3.11) we arrive at the action

$$S = \int dt\, L(q,\dot{q}) + \int_{z>0} dtdz\, \frac{1}{2}\Gamma_{ij}\partial_\mu\chi^i\partial^\mu\chi^j \tag{3.15}$$

with the boundary conditions (3.14). For a single degree of freedom $q$ this action first appeared in [Lamb 1900] and was used in [Ford et al. 1988] for the derivation of the

[3] Integrating by parts the interaction term in (3.9) and replacing $\dot{q}^i$ with the covariant derivative $D_t q^i$ does not help. We then have $\xi_i$, instead of $\dot{\xi}_i$ in the coupling, which leads to forces $F_i$ with non-local memory of the past motion of the system.

quantum Langevin equation. More recently, it was extended to describe linear response in dissipative media [Figotin and Schenker 2007].

So far, we have assumed the dissipative coefficients $\Gamma_{ij}$ to be constant. However, the action (3.15) admits a natural generalization. Relation (3.14) suggests to think of the fields $\chi^i$ as extensions of the original system coordinates into the bulk. Then, to describe coordinate dependent dissipation, we simply need to promote the coefficients in (3.15) to the functions of $\chi$,

$$\Gamma_{ij} \mapsto \Gamma_{ij}(\chi)\ . \tag{3.16}$$

Note that this makes the effective reservoir fields self-interacting. It is necessary price to pay for modeling coordinate dependent friction.

This is not yet the whole story. We can add to the reservoir action a time-reversal breaking term

$$\int_{z>0} dtdz\, \frac{1}{2}\Upsilon_{ij}(\chi)\epsilon^{\mu\nu}\partial_\mu\chi^i\partial_\nu\chi^j \tag{3.17}$$

with antisymmetric coefficients $\Upsilon_{ij}(\chi)$. If $\Upsilon_{ij}$ are constant, this term is a total derivative and reduces to the boundary term $\int dt\, \Upsilon_{ij} q^i \dot{q}^j$ of the Wess–Zumino–Witten type [Wess and Zumino 1971; Witten 1983; Goon et al. 2012]. However, for the general field dependent coefficients such reduction is impossible.

Combining all above ingredients, we write down the action of our reservoir model:

$$\begin{aligned} S = \int dt\, L(q,\dot{q}) + \int_{z>0} dtdz\, \frac{1}{2}\Big(&\Gamma_{ij}(\chi)\partial_\mu\chi^i\partial^\mu\chi^j \\ &+ \Upsilon_{ij}(\chi)\epsilon^{\mu\nu}\partial_\mu\chi^i\partial_\nu\chi^j\Big)\ . \end{aligned} \tag{3.18}$$

Let us verify that it reproduces the desired equations. Taking its variation and accounting for the relation (3.14) we obtain in the bulk and on the boundary:

$$\begin{aligned} &\partial_\mu\Big(\Gamma_{ij}\partial^\mu\chi^j + \Upsilon_{ij}\epsilon^{\mu\nu}\partial_\nu\chi^j\Big) \\ &\quad - \frac{1}{2}\frac{\partial\Gamma_{jk}}{\partial\chi^i}\partial_\mu\chi^j\partial^\mu\chi^k - \frac{1}{2}\frac{\partial\Upsilon_{jk}}{\partial\chi^i}\epsilon^{\mu\nu}\partial_\mu\chi^j\partial_\nu\chi^k = 0\ , \end{aligned} \tag{3.19}$$

$$F_i = (\Gamma_{ij}\partial_z\chi^j + \Upsilon_{ij}\dot{\chi}^j)\big|\ . \tag{3.20}$$

Though the bulk equation (3.19) looks complicated, it still admits purely outgoing solutions $\chi^i(t,z) = \bar{\chi}^i(t-z)$ with arbitrary functions $\bar{\chi}^i$. The conditions (3.14) then fix $\bar{\chi}^i(t) = q^i(t)$ and Eq. (3.20) reduces to

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j + \Upsilon_{ij}(q)\,\dot{q}^j\ . \tag{3.21}$$

The first term gives the sought-after dissipative forces (3.2), whereas the second term describes arbitrary gyroscopic forces that arise if the environment breaks time-reversal symmetry, e.g. by magnetization or rotation. The effective reservoir action (3.18) represents the main result of this chapter. It covers a much broader class of systems than the original harmonic bath (3.9).

## 3.4 A Note On Nonholonomic Systems

The bulk model can result in constraint dynamics if we take the limit of dissipation in certain directions to infinity. consider a system with constraints on coordinates and velocities,[4]

$$c_i^\alpha(q)\,\dot{q}^i = 0\,, \quad \alpha = 1, \dots n < I\ , \tag{3.22}$$

such that they cannot be integrated into constraints only on coordinates. In other words, Eq. (3.22) is not equivalent to a set of constraints of the form $\dot{\varphi}^\alpha(q) = 0$. Clearly, this requires

$$\frac{\partial c_i^\alpha}{\partial q^j} - \frac{\partial c_j^\alpha}{\partial q^i} \neq 0\ . \tag{3.23}$$

These systems are called nonholonomic and typical examples include rolling of a disk or a ball on a hard surface. Their classical dynamics is well developed and is summarized in excellent textbooks, e.g. [Neimark and Fufaev 2004; Arnold et al. 2006]. Quantization, however, remains an open problem. It was addressed in [Bloch and Rojo 2008; Fernandez and Radhakrishnan 2018; Fernandez 2022] and presents a growing interest due to development of molecular machines [Shirai et al. 2005; Grill et al. 2007; Erbas-Cakmak et al. 2015].

Typically, the equations of motion for nonholonomic systems are derived from a modified variational principle restricted to admissible variations $\delta q^i$ satisfying the constraints $c_i^\alpha\,\delta q^i = 0$. This leads to the appearance of *reaction forces* on the r.h.s. of the Euler–Lagrange equations (3.1),

$$F_i = \lambda_\alpha c_i^\alpha(q)\ , \tag{3.24}$$

where $\lambda_\alpha(t)$ are Lagrange multipliers.[5] Due to the constraints (3.22), the reaction forces do not produce any work, $F_i\dot{q}^i = 0$, so nonholonomic systems are not truly dissipative. However, they are closely related through the following construction [Neimark and Fufaev 2004; Arnold et al. 2006]. Consider a dissipative system with the Rayleigh function

$$F = \tfrac{1}{2}\gamma\, c_i^\alpha(q) c_j^\alpha(q)\,\dot{q}^i\dot{q}^j \tag{3.25}$$

and take the limit $\gamma \to +\infty$. The friction associated with the linear combinations of velocities $c_j^\alpha\dot{q}^j$ becomes very strong and the corresponding combinations quickly die out rendering the constraints (3.22). On the other hand, the products $\gamma c_i^\alpha\dot{q}^i$ remain finite and become independent variables — the Lagrange multipliers of Eq. (3.24). Thus, the nonholonomic dynamics can be viewed as the limit of infinitely strong viscous friction along the constrained directions.

[4]We only consider constraints linear in velocities.

[5]Note that adding the constraints (3.22) with Lagrange multipliers into the Lagrangian, instead of the equations of motion, would not reproduce the correct nonholonomic dynamics. Instead, one would obtain a so-called vakonomic system [Arnold et al. 2006].

To describe a nonholonomic system, we replace $\Gamma_{ij} \mapsto \Gamma_{ij} + \gamma c_i^\alpha c_j^\alpha$ and send $\gamma$ to infinity in (3.18). The resulting action can be obtained in a closed form by integrating in a set of auxiliary vectors $\lambda_\alpha^\mu(t,z)$. Omitting for simplicity the time-reversal breaking term, we write:

$$\begin{aligned} S_{\text{nh}} = \int dt\, L(q,\dot{q}) + \int_{z>0} dtdz \bigg( \frac{1}{2}\Gamma_{ij}(\chi)\partial_\mu\chi^i\partial^\mu\chi^j \\ - \lambda_\alpha^\mu c_i^\alpha(\chi)\partial_\mu\chi^i - \frac{1}{2\gamma}\lambda_\alpha^\mu\lambda_{\mu\,\beta} \bigg)\,. \end{aligned} \tag{3.26}$$

In the limit $\gamma \to \infty$ the last term disappears and the fields $\lambda_\alpha^\mu$ become Lagrange multipliers enforcing the constraints $c_i^\alpha(\chi)\partial_\mu\chi^i = 0$. Since $\chi^i$ coincide with $q^i$ on the boundary, this implies the nonholonomic constraints (3.22). The remaining equations also come out right. Varying (3.26) with respect to $\chi^i$ and substituting the outgoing solution into the boundary equation, we obtain the force,

$$F_i = -\Gamma_{ij}(q)\,\dot{q}^j + c_i^\alpha(q)\,\lambda_\alpha^z\big|\,. \tag{3.27}$$

The last term gives precisely the reaction forces along the constrained directions (3.24), with $\lambda_\alpha^z|$ playing the role of the Lagrange multipliers from the standard approach. The first term describes friction along the unconstrained directions. Note that in our approach it cannot be set to zero without making the bulk action degenerate. The model takes particularly simple form for motion on cosets:

$$\begin{aligned} S_{\text{coset}} = \int dt\, L(D_t q) \\ + \int_{z>0} dtdz\, \frac{1}{2}\Big(\gamma_{ij}\eta^{\mu\nu} + \upsilon_{ij}\epsilon^{\mu\nu}\Big) D_\mu\chi^i D_\nu\chi^j\,, \end{aligned} \tag{3.28}$$

where $D_\mu\chi^i \equiv \Omega_j^i(\chi)\partial_\mu\chi^j$ are covariant derivatives of the fields $\chi^i$, and $\gamma_{ij}$, $\upsilon_{ij}$ are constants. One recognizes in the bulk term the action of a two-dimensional nonlinear sigma-model [Zinn-Justin 2021]. It is the most general local action that can be written using the first derivatives of the fields $\chi^i$ and invariant under the group $G$.

Let us illustrate this construction in the case of an oblong particle moving on a two-dimensional plane in a viscous medium [Li and Tang 2004; Duggal and Pasquali 2006; Han et al. 2006]. Its position is described by the center-of-mass coordinates $X$, $Y$ and the orientation angle $\phi$, see Fig. 3.1. The friction coefficients are different in the directions along and perpendicular to the particle's main axis. The configuration space coincides with the group of isometries of the Euclidean plane $ISO(2)$ which has two generators of translations $P_X$, $P_Y$ and a rotation generator $J$. The commutation relations are:

$$[P_X, P_Y] = 0\,, \quad [P_X, J] = -P_Y\,, \quad [P_Y, J] = P_X\,. \tag{3.29}$$

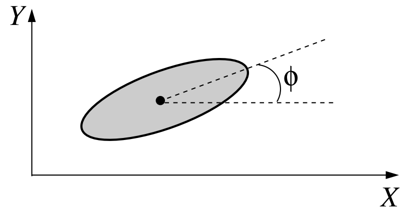


FIGURE 3.1: An oblong particle on a plane.

All generators are broken. We parameterize the group elements as[6]

$$g(X,Y,\phi) = \mathrm{e}^{XP_X+YP_Y}\mathrm{e}^{\phi J} \; . \tag{3.30}$$

From the Cartan form we get the covariant derivatives:

$$D_tX = \dot{X}\cos\phi + \dot{Y}\sin\phi \; , \tag{3.31}$$
$$D_tY = -\dot{X}\sin\phi + \dot{Y}\cos\phi \; , \quad D_t\phi = \dot{\phi} \; . \tag{3.32}$$

Lagrangian coincides with the kinetic energy of the particle and is $ISO(2)$ invariant,

$$L = \frac{m}{2}(\dot{X}^2+\dot{Y}^2) + \frac{\mathcal{I}}{2}\dot{\phi}^2 = \frac{m}{2}\left[(D_tX)^2+(D_tY)^2\right] + \frac{\mathcal{I}}{2}(D_t\phi)^2, \tag{3.33}$$

where $m$, $\mathcal{I}$ are the particle mass and moment of inertia.

If the viscous medium is homogeneous and isotropic, the effective reservoir action must also enjoy $ISO(2)$ symmetry. To implement it, we introduce the fields $\xi(t,z)$, $\Psi(t,z)$, $\Phi(t,z)$, such that at $z=0$ they coincide with $X(t)$, $Y(t)$ and $\phi(t)$, respectively. We recall that the coordinate $z$ is not a physical dimension. Rather, it parameterizes the internal degrees of freedom of the particle and medium responsible for dissipation. The effective bath action then reads,

$$\begin{aligned} S_{\text{bath}} = \int_{z>0} dtdz\, \frac{1}{2}\big(\gamma_{\parallel} D_\mu\xi D^\mu\xi + \gamma_{\perp} D_\mu\Psi D^\mu\Psi \\ + \gamma_\phi D_\mu\Phi D^\mu\Phi\big) \; , \end{aligned} \tag{3.34}$$

where

$$D_\mu\xi = \cos\Phi\,\partial_\mu\xi + \sin\Phi\,\partial_\mu\Psi \; , \tag{3.35}$$
$$D_\mu\Psi = -\sin\Phi\,\partial_\mu\xi + \cos\Phi\,\partial_\mu\Psi \; , \quad D_\mu\Phi = \partial_\mu\Phi \; . \tag{3.36}$$

We observe that even in this relatively simple case the bath action is nonlinear if $\gamma_\perp \neq \gamma_\parallel$. In the limit $\gamma_\perp \to +\infty$ we obtain a particle that is constrained to move along its major axis. This is the simplest nonholonomic system known as *Chaplygin sleigh*.

[6]This parameterization slightly differs from Eq. (3.3) and makes the calculations simpler.

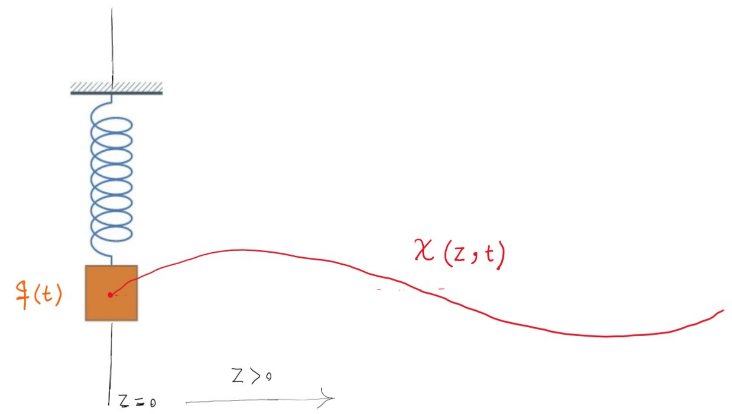


FIGURE 3.2: a one dimensional harmonic oscillator at z=0, connected to a gapless field at z>0. The field at the boundary z=0 oscillates with the harmonic oscillator because of the boundary condition $\chi_{z=0,t} = q(t)$

### 3.4.1 Quantization of The Bulk Model?

There are challenges in the quantization of the theory which need to be addressed. For instance, all Green's functions must be considered and the purely outgoing solution for the bulk field (or considering only the retarded Green's function) is no more plausible. In addition, the bulk model is no more free and due to the self-interaction, the dissipative coupling $\gamma$ does run with changing of the energy scale. This may bring about new effects and it is interesting to see how the running of the coupling is interpreted. Postponing the question about the Green's function to the final chapter and leaving the discussion about the running of the couplings for the future work, let's see that the bulk model produces meaningful results in the case of position independent coupling. Let's apply the construction to the case of damped harmonic oscillator as an example (see figure 3.2). One can show that the observable quantities of the damped harmonic oscillator, such as the spectral density, can alternatively been obtained from the bulk model. We do not cover the related calculation of the spectral density using the bulk model in this section as it is pretty similar to the one which was presented in the previous chapter. Instead, as an example, we show that the two-point functions $\langle 0| q(t)q(t') |0\rangle$ of the damped harmonic oscillator can be calculated alternatively using the bulk model.

Using the system plus reservoir model of the previous chapter, the two-point function the damped harmonic oscillator reads

$$\langle 0| q(t)q(t') |0\rangle = \int_0^\infty \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} C_\omega C_{\omega'} \exp\left(i(\omega' t' - \omega t)\right) \langle 0| a_\omega a^\dagger_{\omega'} |0\rangle$$
$$= \int_0^\infty \frac{d\omega}{2\pi}\, C^2_\omega \, \exp\left(i\omega(t'-t)\right) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\alpha^2\omega}{(\Omega^2-\omega^2)^2+\alpha^4\omega^2} \exp\left(i\omega(t'-t)\right). \quad (3.37)$$

To calculate the same quantity using the bulk model, we need to write the action functional (3.18) for the case of one dimensional damped harmonic oscillator with $\Gamma_{ij} \equiv \gamma\delta_{ij}$ and $\Upsilon_{ij} \equiv 0$,

$$S = \frac{1}{2}\int dt \left( \left(\partial_t q(t)\right)^2 - \Omega^2 q^2(t) \right) + \frac{\gamma}{2}\int_{z>0} dtdz \Big( \left(\partial_t \chi(z,t)\right)^2 - \left(\partial_z \chi(z,t)\right)^2 \Big) . \quad (3.38)$$

Then the (classical) equation of motion for this Lagrangian becomes

$$(\delta(z)+\gamma)\ddot{\chi}(z,t) + \Omega^2\ \delta(z)\chi(z,t) - \gamma\ \partial_z^2\chi(z,t) = 0. \quad (3.39)$$

Going to the Fourier space $\chi(z,t) = \int_0^\infty \frac{d\omega}{2\pi}\ \chi_\omega(z)\exp(-i\omega t)$, the equation of motion becomes

$$(\partial_z^2 + \omega^2)\chi(z) = \delta(z)\ \frac{(-\omega^2+\Omega^2)}{\gamma}\chi_\omega(z). \quad (3.40)$$

The solution in the bulk ($z > 0$) is

$$\chi_\omega(z) = A'_\omega \exp(i\omega z) + B'_\omega \exp(-i\omega z) \quad (3.41)$$

with $A'_\omega$ and $B'_\omega$ to be determined through the boundary condition. Using the equation of motion (3.40), the boundary condition reads

$$\int_{0^-}^{0^+} dz\ (\partial_z^2\chi_\omega(z)) = \frac{-\omega^2+\Omega^2}{\gamma}\chi_\omega(0) \quad (3.42)$$

$$\rightarrow \partial_z\chi_\omega(0) - 0 = \frac{-\omega^2+\Omega^2}{\gamma}\chi_\omega(0). \quad (3.43)$$

This, alongside the other boundary condition $\chi_\omega(0) = A'_\omega + B'_\omega$ results in

$$A'_\omega = \frac{1}{2\gamma}\left(\frac{i\omega\gamma + (-\omega^2+\Omega^2)}{i\omega\gamma}\right)\chi_\omega(0), \quad B'_\omega = (A'_\omega)^\star \quad . \quad (3.44)$$

Quantization for the alternative action means to write the field operator $\chi(z,t)$ in terms of the creation and annihilation operators

$$\chi(z,t) = \int_0^\infty \frac{d\omega}{2\pi} C'_\omega \Big( (\tilde{A}'_\omega \exp(i\omega z) + (\tilde{A}'_\omega)^\star \exp(-i\omega z))a_\omega\ \exp(-i\omega t) + \mathrm{H.C} \Big). \quad (3.45)$$

where $\tilde{A}'_\omega = \frac{A'_\omega}{\chi_\omega(0)}$ and $\tilde{B}'_\omega = \frac{B'_\omega}{\chi_\omega(0)}$. The conjugate momentum to the field $\chi(z,t)$ is $\Pi(z,t) = \gamma\dot{\chi}(z,t)$. Imposing the canonical quantization

$$[\chi(z,t), \Pi(z',t)] = i\delta(z-z') ,$$

one we obtain

$$C'_\omega = \frac{1}{2\omega^{1/2}|\tilde{A}'_\omega|\gamma} = \frac{1}{2\omega^{1/2}|\tilde{A}_\omega|} = C_\omega .$$

Regarding the boundary condition $\chi(z=0,t)=q(t)$, the two point correlation function $\langle 0|\, q(t)q(t')\,|0\rangle$ using the bulk model is

$$\begin{aligned}
&\langle 0|\, \chi(0,t)\chi(0,t')\,|0\rangle \\
&= \int_0^\infty \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\, C'_\omega C'_{\omega'}\,(\tilde{A}'_\omega + (\tilde{A}'_\omega)^\star)\,(\tilde{A}'_{\omega'} + (\tilde{A}'_{\omega'})^\star)\,\exp\left(i(\omega' t' - \omega t)\right)\langle 0|\, a_\omega a^\dagger_{\omega'}\,|0\rangle \\
&= \int_0^\infty \frac{d\omega}{2\pi}\, {C'_\omega}^2\, |\tilde{A}'_\omega + (\tilde{A}'_\omega)^\star|^2\; \exp\left(i\omega(t'-t)\right) = \int_0^\infty \frac{d\omega}{2\pi}\,\frac{\gamma\omega}{(\Omega^2-\omega^2)^2+\gamma^2\omega^2}\;\exp\left(i\omega(t'-t)\right)
\end{aligned} \tag{3.46}$$

which is the same as the correlation function (3.37) with $\gamma=\alpha^2$. This and other similar calculations, however, are valid only for an example of damped harmonic oscillator. We will address the applicability of the bulk model at the quantum level to the more general case of position dependent dissipative couplings in chapter (6).

# Chapter 4

# Open Quantum Dynamics On Lie Group: The Linear Response Regime

In this chapter, we first determine the most general interaction between a system which nonlinearly realizes a symmetry and thermal bath. Then, we build the most general influence functional of the dissipative dynamics which at the linear response level. We define the S-K classical and quantum fields in a nonlinear way and we show that at the high temperature, the influence functional allows for an expansion in terms of the quantum field and time derivatives of the classical field.

## 4.1 Geometrical setup

We follow the conventions of the previous chapter which we briefly summarize here. We consider a mechanical system moving on a group $G$ with generators $G_i$, $i = 1, \dots, n$ obeying commutation relations[1]

$$[G_i, G_j] = C^k{}_{ij} G_k \ . \tag{4.1}$$

The element of the group and the Cartan form are given by

$$g(q) = \mathrm{e}^{q^i G_i} \ , \qquad \Omega = g^{-1} dg = \Omega^i_j(q) dq^j G_i \ . \tag{4.2}$$

The coefficients of the latter obey the structure relations,

$$\frac{\partial \Omega^k_i}{\partial q^j} - \frac{\partial \Omega^k_j}{\partial q^i} = C^k{}_{lm} \Omega^l_i \Omega^m_j \ , \tag{4.3}$$

where $C^k{}_{lm}$ are the structure constants. More properties of the Cartan coefficients are given in Appendix A1. We want the dynamics of the system to be left-invariant under

[1] We assume that the group is fully broken, so we are not going to distinguish broken/unbroken generators.

the action of the group:

$$g(q) \mapsto g(\tilde{q}) = g_L\, g(q)\ , \tag{4.4}$$

so the Lagrangian must be built using the covariant velocities defined as

$$D_t q^i = \Omega^i_j(q)\dot{q}^j\ . \tag{4.5}$$

we refer to the mechanical system as "body".

The body interacts with a thermal bath described by some coordinates $\chi$, the total Hamiltonian of body+bath reads

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{bath}} + V\ , \tag{4.6}$$

where $H_{\text{sys}}$ depends only on $q$, $H_{\text{bath}}$ only on $\chi$, and the interaction $V$ on both. Following [Caldeira and Leggett 1981] we will assume that the effect of the system on the bath degrees of freedom is weak and can be accounted for within the linear response approximation. In practice, this means that we will keep only terms of order $V^2$ in the perturbative expansion. The thermal bath is also assumed to possess the symmetry $G$, which is realized linearly. As an example, one may imagine the motion of a rigid body which nonlinearly realizes the group $SO(3)$ of rotations and is rotating in a rotationally symmetric viscous medium like a fluid or gas. The symmetry of the bath is implemented by a unitary representation of $G$ on the Hilbert space of the bath:

$$|\Psi_\chi\rangle \mapsto U(g)\,|\Psi_\chi\rangle\ , \qquad U^\dagger(g) H_{\text{bath}} U(g) = H_{\text{bath}}\ . \tag{4.7}$$

Thus, without $V$, the system has two copies of the symmetry $G \times G$. The interaction breaks this symmetry to the diagonal subgroup, $G \times \mathcal{G} \to G_{\text{diag}}$. This requirement fixes the form of the interaction:

$$V = U(q)\, V_\chi\, U^\dagger(q)\ , \tag{4.8}$$

where we have introduced a shorthand notation $U(q) \equiv U\big(g(q)\big)$ and $V_\chi$ depends only on the bath coordinates degrees of freedom $\chi$. It is easy to check explicitly that this form is invariant under the action of $G_{\text{diag}}$:

$$\begin{aligned} V \mapsto U^\dagger(g_L)U(\tilde{q})\, V_\chi\, U^\dagger(\tilde{q})U(g_L) &= U^\dagger(g_L)U\big(g_L\, g(q)\big)\, V_\chi\, U^\dagger\big(g_L\, g(q)\big)U(g_L) \\ &= U^\dagger(g_L)U(g_L)U(q)\, V_\chi\, U^\dagger(q)U^\dagger(g_L)U(g_L) = V\ . \end{aligned} \tag{4.9}$$

## 4.2 Linear response and time reversibility

We assume that the interaction depends only on the body coordinates, but not its momenta, which appears to be satisfied in all realistic situations. Then the interaction $V(q,\chi)$ at fixed bath variables $\chi$, is a function on the group manifold. By Peter–Weyl theorem any such function can be decomposed into matrix elements belonging to various representations of the group, which gives the desired result,

$$V(q,\chi) = \sum_{raa'} U^r_{aa'}(q)\mathcal{O}^{\dagger r}_{a'a}(\chi) \ . \tag{4.10}$$

where $U^r_{aa'}(q)$ belongs to the irreducible representations of the group $G$, $r$ labeling the representation, and index $a$ labels states inside the representation. For the interaction to be Hermitian, the operators in the conjugate representations $r$ and $\bar{r}$ must be Hermitian conjugate to each other,

$$\mathcal{O}^{\dagger\bar{r}}_{a'a} = (\mathcal{O}^{\dagger r}_{a'a})^{\dagger} \equiv \mathcal{O}^r_{aa'} \ . \tag{4.11}$$

For instance, consider a particle which is restricted to move on a circle. The phase space of the particle is parameterized with its angular position on the circle, $\phi$. The most general interaction between the particle and the environment can be Fourier transformed as

$$V = -\sum_{m=-\infty}^{\infty} \mathrm{e}^{im\phi(t)}\ \mathcal{O}_m(\chi(t)) \tag{4.12}$$

where $\mathcal{O}_m(\chi)$ is an operator defined on the phase space of the circle $\chi$. In this example, because the system realizes the $U(1)$ symmetry, the interaction is parameterized by the elements of the unitary representation of the $U(1)$ group,

$$V(\phi,\chi) = \sum_m U^m(\phi)O^m(\chi). \tag{4.13}$$

where $U^m(\phi) = e^{im\phi}$ and matches with the Peter-Weyl's theorem (4.10). In this example, the friction and noise is induced through the exchange of angular momentum between the particle and the environment. Let's generalize (4.13) to the more complicated case of $SO(3)$, and then, guess the most general interaction between a body which is parameterized by an arbitrary group and a bath which is invariant under the same symmetry group. Rewriting (4.13) in terms of the Wigner rotation, then we have

$$V_I(\phi,\chi) = \sum_{m,m',m''} \langle m'|\mathrm{e}^{i\phi\hat{J}_z}|m\rangle\langle m|\hat{O}(\chi)|m''\rangle \tag{4.14}$$

where $\mathrm{e}^{i\phi\hat{J}_z}$ is the Wigner rotation around the $z$ axis and we have used the fact that $\langle m'|\mathrm{e}^{i\phi\hat{J}_z}|m\rangle \propto \delta_{m'm}$ for $U(1)$. For the case of $SO(3)$, the Wigner's rotation can be parameterized by the Euler angles. Therefore, we need the following replacements to adjust (4.14) to the case of SO(3),

$$\langle m'|\mathrm{e}^{i\phi\hat{J}_z}|m\rangle \to \langle jm'|\mathrm{e}^{i\gamma\hat{J}_z}\mathrm{e}^{i\beta\hat{J}_y}\mathrm{e}^{i\alpha\hat{J}_z}|jm\rangle \equiv U^j_{m'm}(\alpha,\beta,\gamma), \tag{4.15a}$$

$$\langle m|\hat{O}(\chi)|m''\rangle \to \langle jm|\hat{O}(\chi)|jm'\rangle \equiv O^j_{mm''}(\chi), \tag{4.15b}$$

where $(\alpha,\beta,\gamma)$ are the Euler angles parameterizing the phase space of the body (collectively, we will show it by $q$). Therefore, the general interaction between the body which realizes SO(3) symmetry and a bath which is invariant under the group $SO(3)$ reads

$$V(q,\chi) = \sum U^j_{mm'}(q) O^j_{m'm''}(\chi) \;, \tag{4.16}$$

where $U^j_{mm'}(q)$ are the Wigner's matrices. This interaction shows that dissipation happens because of the exchange of the total angular momentum between the body and the bath. Strictly, the Peter–Weyl theorem applies to compact groups. In the case of noncompact groups, we still use the spectral decomposition which formally looks like (4.11). For instance, in the case of a point particle moving in a plane, the interaction between the particle and the bath is parameterized through the Fourier decomposition

$$V(\mathbf{q},\mathbf{x}) = \int \frac{dk}{2\pi} e^{i\mathbf{k}.(\mathbf{q}-\mathbf{x})} \mathcal{O}_{\mathbf{k}}(\mathbf{x}) \;, \tag{4.17}$$

where $\mathbf{x}$ refers to a typical point on the plane.

To get a handle on the real-time dynamics we should consider the Schwinger–Keldysh (SK) path integral. We now work with two copies of fields defined on the upper and lower parts of the closed time-contour and denoted with subscripts "+" and "−", respectively. We want to integrate out the bath to obtain the dynamics purely in terms of the body variables. This modifies the free body's action functional, which is quantified by the influence functional $\mathcal{I}[q_+, q_-]$. At the linear response level, the effect of the bath will be fully parameterized by the two-point functions of these operators, the type of the Green's function being determined by the precise question we are interested in.[2] Introducing the shorthands $q_{\pm\,1,2} \equiv q_\pm(t_{1,2})$, then the influence functional at the linear response reads

$$\begin{aligned}\mathcal{I}[q_+, q_-] =& \frac{i}{2} \int dt_1 dt_2 \Big\{ U^{\dagger r}_{a'a}(q_{+1}) U^s_{bb'}(q_{+2}) G^{rs}_{aa'b'b}(t_1,t_2) \\ &+ U^{\dagger r}_{a'a}(q_{-1}) U^s_{bb'}(q_{-2}) \tilde{G}^{rs}_{aa'b'b}(t_1,t_2) - U^{\dagger r}_{a'a}(q_{-1}) U^s_{bb'}(q_{+2}) K^{rs}_{aa'b'b}(t_1,t_2) \\ &- U^{\dagger r}_{a'a}(q_{+1}) U^s_{bb'}(q_{-2}) \tilde{K}^{rs}_{aa'b'b}(t_1,t_2) \Big\} \;,\end{aligned} \tag{4.18}$$

where sum over all indices is implied. Here, $G$ and $\tilde{G}$ are the time ordered and anti-time ordered Green's functions

$$\begin{aligned} G^{rs}_{aa'b'b}(t_1,t_2) &= \langle \mathcal{T}(\mathcal{O}^r_{aa'}(t_1) \mathcal{O}^{\dagger s}_{b'b}(t_2)) \rangle_{\rho_\chi}, \\ \tilde{G}^{rs}_{aa'b'b}(t_1,t_2) &= \langle \tilde{\mathcal{T}}(\mathcal{O}^r_{aa'}(t_1) \mathcal{O}^{\dagger s}_{b'b}(t_2)) \rangle_{\rho_\chi} \;, \end{aligned} \tag{4.19}$$

[2] This will hold even beyond the linear response, if the connected higher point function of $\mathcal{O}^r_{aa'}$ are suppressed with respect to the disconnected ones, which is expected whenever the body interacts with large number of bath degrees of freedom.

and $K$, $\tilde{K}$ are the unordered ones

$$K^{rs}_{aa'b'b}(t_1, t_2) = \langle \mathcal{O}^r_{aa'}(t_1)\mathcal{O}^{\dagger s}_{b'b}(t_2)\rangle_{\rho_\chi} \ , \tag{4.20a}$$

$$\tilde{K}^{rs}_{aa'b'b}(t_1, t_2) = \langle \mathcal{O}^{\dagger s}_{b'b}(t_2)\mathcal{O}^r_{aa'}(t_1)\rangle_{\rho_\chi} \ . \tag{4.20b}$$

The invariance of the interaction (4.10) under the group action implies that $\mathcal{O}^r_{aa'}$ transforms in the linear representation $r$ acting on the index $a$. Thus all the Green's functions must be diagonal[3]. Using this, and the time translation invariance we have,

$$G^{rs}_{aa'b'b}(t_1, t_2) = \delta^{rs}\delta_{ab}\,\mathcal{G}^r_{a'b'}(t_1 - t_2) \ , \quad \tilde{G}^{rs}_{aa'b'b}(t_1, t_2) = \delta^{rs}\delta_{ab}\,\tilde{\mathcal{G}}^r_{a'b'}(t_1 - t_2) \tag{4.21a}$$

$$K^{rs}_{aa'b'b}(t_1, t_2) = \delta^{rs}\delta_{ab}\,\mathcal{K}^r_{a'b'}(t_1 - t_2) \ , \quad K^{rs}_{aa'b'b}(t_1, t_2) = \delta^{rs}\delta_{ab}\,\mathcal{K}^r_{a'b'}(t_1 - t_2) \tag{4.21b}$$

with $\mathcal{G}, \mathcal{K}$ and their tilded counterparts being the reduced forms of the correlators. This leads to a significant simplification of the influence functional,

$$\begin{aligned}\mathcal{I}[q_+, q_-] = \frac{i}{2}\int dt_1 dt_2 \Big\{ & U^r_{a'b'}(q_{+2} \ominus q_{+1})\mathcal{G}^r_{a'b'}(t_1 - t_2) + U^r_{a'b'}(q_{-2} \ominus q_{-1})\tilde{\mathcal{G}}^r_{a'b'}(t_1 - t_2) \\ & - U^r_{a'b'}(q_{+2} \ominus q_{-1})\mathcal{K}^r_{a'b'}(t_1 - t_2) - U^r_{a'b'}(q_{-2} \ominus q_{+1})\tilde{\mathcal{K}}^r_{a'b'}(t_1 - t_2)\Big\} \ ,\end{aligned} \tag{4.22}$$

where the "covariant difference" $q_2 \ominus q_1$ is defined as follows:

$$g(q_2 \ominus q_1) = g^{-1}(q_1)\, g(q_2) \ , \tag{4.23}$$

and we have used the identity

$$U^r_{a'b'}(q_2 \ominus q_1) = \sum_a U^{\dagger r}_{a'a}(q_1) U^r_{ab'}(q_2) \ . \tag{4.24}$$

The next step is to introduce the spectral density, which we define it using the correlators $\mathcal{K}$,

$$\mathcal{K}^r_{a'b'}(t) = \int d\omega\, \mathrm{e}^{-i\omega t}\, \varrho^r_{a'b'}(\omega) \ . \tag{4.25}$$

In a thermal bath with temperature $T = 1/\beta$ the spectral density of the correlator $\tilde{\mathcal{K}}$ with the opposite ordering of operators is related to that of $\mathcal{K}$,

$$\tilde{\varrho}^r_{a'b'}(\omega) = \mathrm{e}^{-\beta\omega}\, \varrho^r_{a'b'}(\omega) \ . \tag{4.26}$$

While this formula is standard, we include its derivation in Appendix A2 for completeness. In addition, the densities are Hermitian and positive definite and are related with the densities for complex conjugate representations through

$$\varrho^{\bar{r}}_{a'b'}(\omega) = \left(\tilde{\varrho}^r_{a'b'}(-\omega)\right)^* \ . \tag{4.27}$$

[3]Here we are using the invariance of the bath density matrix under the action of $G$ and time translations.

We will additionally impose that the bath satisfies time reversal symmetry. The requirement that the interaction Hamiltonian (4.10) is T-invariant leads to the following conditions

$$\mathbf{T}\mathcal{O}^{\dagger r}_{a'a}\mathbf{T}^{-1} = \mathcal{O}^{\dagger \bar{r}}_{a'a} , \qquad \mathbf{T}\mathcal{O}^{r}_{aa'}\mathbf{T}^{-1} = \mathcal{O}^{\bar{r}}_{aa'} . \tag{4.28}$$

Further assuming that the bath density matrix is also T-invariant, we obtain for the correlation functions,

$$\langle \mathcal{O}^{r}_{aa'}(t_1)\mathcal{O}^{\dagger s}_{b'b}(t_2)\rangle_{\rho_\chi} = \langle \mathbf{T}\mathcal{O}^{r}_{aa'}(t_1)\mathcal{O}^{\dagger s}_{b'b}(t_2)\mathbf{T}^{-1}\rangle^*_{\rho_\chi} = \langle \mathcal{O}^{\bar{r}}_{aa'}(-t_1)\mathcal{O}^{\dagger \bar{s}}_{b'b}(-t_2)\rangle^*_{\rho_\chi} , \tag{4.29}$$

where in the first equality we have used the fact that $\mathbf{T}$ is anti-unitary. This provides an extra relation:

$$\varrho^{r}_{a'b'}(\omega) = \left(\varrho^{\bar{r}}_{a'b'}(\omega)\right)^* = \tilde{\varrho}^{r}_{a'b'}(-\omega) , \tag{4.30}$$

This concludes the implications of the symmetries. One more thing we need for an EFT is a separation of scales. We can find it if we go to high temperatures.

## 4.3 High temperature limit

At high temperatures things simplify. In this regime one expects the correlation functions to decay exponentially over some relaxation time scale set by the interplay of the temperature and the interactions in the bath, which we call $\tau \sim \Lambda^{-1}$. In general, $\Lambda$ will be lower than the temperature, coinciding with it in strongly interacting baths. The scale $\Lambda$ can, in principle, depend on the representation $r$. This does not seem to change anything in the case of a compact group $G$ whose unitary representations are finite-dimensional and discrete — in this case we can simply choose $\Lambda$ to be the minimal scale among all representations (it is natural to expect $\Lambda$ to grow with the rank of the representation). The situation is, however, less clear for non-compact groups, such as e.g. the group of translations. Their unitary representations are labeled by continuous parameters which may bring additional scales to the problem (e.g. momentum of a particle moving in a medium). If $\Lambda$ depends on this scale, it may become arbitrarily low for some representations. Finite relaxation time implies that the spectral densities are infinitely differentiable in the neighborhood of $\omega = 0$ [Endlich et al. 2013]. Let us recall the argument. It is sufficient to require the exponential decay of the retarded Green's functions, which describe the classical response of the system,

$$(G_{\text{ret}})^{rs}_{aa'b'b}(t) = i\theta(t)\langle[\mathcal{O}^{r}_{aa'}(t), \mathcal{O}^{\dagger s}_{b'b}(0)]\rangle_{\rho_\chi} = i\delta^{rs}\delta_{ab}(\mathcal{G}_{\text{ret}})^{r}_{a'b'}(t) . \tag{4.31}$$

Consider its Fourier transform,

$$\mathcal{G}_{\text{ret}}(\omega) = \int dt\, \mathrm{e}^{i\omega t}\, \mathcal{G}_{\text{ret}}(t) , \tag{4.32}$$

where we have suppressed the group indices to avoid cluttered notations. It is related to the spectral density introduced above as

$$\mathcal{G}_{\rm ret}(\omega) = \int_{-\infty}^{+\infty} \frac{-d\omega_0}{\omega-\omega_0+i\epsilon}(1-\mathrm{e}^{-\beta\omega_0})\varrho(\omega_0)\ , \tag{4.33}$$

and is analytic in the upper half-plane of $\omega$. If the retarded Green's function decays exponentially, $|\mathcal{G}_{\rm ret}(t)| < \mathrm{e}^{-\Lambda t}$, the domain of analyticity of $\mathcal{G}_{\rm ret}(\omega)$ extends downwards and includes a strip in the lower half-plane, $\mathrm{Im}\,\omega > -\Lambda$, implying that its imaginary part is infinitely differentiable on the real axis. On the other hand, it follows from (4.33) that

$$\mathrm{Im}\,\mathcal{G}_{\rm ret}(\omega) = \pi(1-\mathrm{e}^{-\beta\omega})\varrho(\omega)\ , \tag{4.34}$$

and thus $\varrho(\omega)$ must be infinitely differentiable. This completes the argument. Due to Eqs. (4.26), (4.30), (4.34) the Imaginary part of its Fourier transform is odd in frequency. Thus, it is expanded as

$$(\mathrm{Im}\,\mathcal{G}_{\rm ret})^r_{ab}(\omega) = \pi\omega\Big(\varrho^r_{0,ab} + \frac{\omega^2}{\Lambda^2}\varrho^r_{2,ab} + O\big((\omega/\Lambda)^4\big)\Big)\ . \tag{4.35}$$

Here $\Lambda$ determines the domain of analyticity of the function $(1-\mathrm{e}^{-\beta\omega})\varrho(\omega)$ and hence $\Lambda^{-1}$ sets the response time of the bath. From this expression we infer the basic correlator spectral density,

$$\varrho^r_{ab}(\omega) = \frac{1}{\beta}\varrho^r_{0,ab} - \frac{\omega}{2}\varrho^r_{0,ab} + \frac{\omega^2}{\beta\Lambda^2}\Big(\varrho^r_{2,ab} + \frac{(\beta\Lambda)^2}{12}\varrho^r_{0,ab}\Big) - \frac{\omega^3}{2\Lambda^2}\varrho^r_{2,ab} + \dots\ , \tag{4.36}$$

$$\tilde{\varrho}(\omega) = \frac{1}{\beta}\varrho^r_{0,ab} + \frac{\omega}{2}\varrho^r_{0,ab} + \frac{\omega^2}{\beta\Lambda^2}\Big(\varrho^r_{2,ab} + \frac{(\beta\Lambda)^2}{12}\varrho^r_{0,ab}\Big) + \frac{\omega^3}{2\Lambda^2}\varrho^r_{2,ab} + \dots\ . \tag{4.37}$$

We see that the quadratic and cubic terms are controlled by a single set of coefficients $\varrho^r_{2,ab}$. On general grounds, we expect the latter to be of order $\varrho^r_{0,ab}$.[4]

In what follows, we first consider the effect of the first two terms in the expansion (4.3). After setting the Gaussian power-counting, we consider the correction to the influence functional due to the higher order terms in (4.3).

[4]Though there may be special cases when $\varrho^r_{2,ab}$ vanish, like e.g. in the ohmic harmonic bath.

### 4.3.1 Leading order in local expansion

The expressions (4.3) restricted to the first two terms lead to ultralocal correlators,

$$\mathcal{K}^r_{ab}(t) = \frac{2\pi}{\beta}\varrho^r_{0,ab}\,\delta(t) + \pi i\varrho^r_{0,ab}\,\delta'(t)\ , \tag{4.38a}$$

$$\tilde{\mathcal{K}}^r_{ab}(t) = \frac{2\pi}{\beta}\varrho^r_{0,ab}\,\delta(t) - \pi i\varrho^r_{0,ab}\,\delta'(t)\ , \tag{4.38b}$$

$$\mathcal{G}^r_{ab}(t) = \frac{2\pi}{\beta}\varrho^r_{0,ab}\,\delta(t) + \pi i\varrho^r_{0,ab}\,\delta'(t) - 2\pi i\varrho^r_{0,ab}\,\theta(-t)\delta'(t)\ , \tag{4.38c}$$

$$\tilde{\mathcal{G}}^r_{ab}(t) = \frac{2\pi}{\beta}\varrho^r_{0,ab}\,\delta(t) + \pi i\varrho^r_{0,ab}\,\delta'(t) - 2\pi i\varrho^r_{0,ab}\,\theta(t)\delta'(t)\ . \tag{4.38d}$$

Substituting this into the influence functional (4.22) we obtain

$$\begin{aligned}\mathcal{I} =& i\frac{\pi}{\beta}\int dt\big[2\sigma_0(0) - \sigma_0(q_+\ominus q_-) - \sigma_0(q_-\ominus q_+)\big]\\ &-\frac{\pi}{2}\int dt\big[\partial_i\sigma_0(q_-\ominus q_+)\,u^i_j(q_+\ominus q_-)\,D_tq^j_+ - \partial_i\sigma_0(q_+\ominus q_-)\,u^i_j(q_-\ominus q_+)\,D_tq^j_-\big]\ .\end{aligned} \tag{4.39}$$

where $u^i_j(\bar{q})$ matrix is the inverse of the Cartan's matrix $\Omega^i_j(\bar{q})$ defined by

$$u^i_j(q)\Omega^j_k(q) = \delta^i_k \tag{4.40}$$

and $\sigma_0(q)$ a real even function on the group manifold

$$\sigma_0(q) = \sum_{rab} U^r_{ab}(q)\hat{\varrho}^r_{0,ab}\ . \tag{4.41}$$

Note that the effect of the bath is totally encoded in the function $\sigma_0(q)$ which no more carries the group indices. Look at the appendix (A3) for the details of the calculation.

To understand the physical content of the expression (4.39), let us expand it under the assumption that the difference between $q_+$ and $q_-$ is small. The applicability of this assumption will be discussed later. We introduce the split of the SK fields in the "classical" (denoted with bar) and "quantum" (denoted with hat) parts as follows,

$$\mathrm{e}^{q_\pm G} = \mathrm{e}^{\bar{q}G}\mathrm{e}^{\pm\hat{q}G}\ . \tag{4.42}$$

Note that $\hat{q}$ is invariant under the left action of the symmetry group, while $\bar{q}$ transforms in the standard way. (4.39).

**Gaussian order** There are no contributions of order $O(\hat{q})$ or $O(D_t)$. Thus, the leading contributions are of quadratic order $O(\hat{q}^2)$, $O(\hat{q}D_t)$. Observing that $q_+\ominus q_- = 2\hat{q}$, a straightforward calculation yields,

$$\mathcal{I}[\bar{q},\hat{q}]\Big|_{\text{Gauss}} = \int dt \left( \frac{4i}{\beta}\gamma_{ij}\hat{q}^i\hat{q}^j - 2\gamma_{ij}\hat{q}^i D_t\bar{q}^j \right) , \tag{4.43}$$

where the couplings $\gamma_{ij}$ are defined through

$$\gamma_{ij} = -\pi\, \partial_i\partial_j\sigma_0(0) \ . \tag{4.44}$$

Using the Hubbard-Stratonovich, we can integrate out the quantum field $\hat{q}$, leaving us with an effective action which gives the state dependent Langevin equation for the classical field $\bar{q}$ with multiplicative Gaussian noise. The obtained Langevin equation is a group covariant equation and the constants $\gamma_{ij}$ are the couplings of a state dependent Ohmic dissipative force. A quick argument shows that these couplings are positive definite, as a consequence of the positivity of the spectral densities. Indeed, within each representation we can write,[5]

$$U^r_{ab}(q) = \delta_{ab} + q^i(G_i)^r_{ab} + \frac{1}{2}q^iq^j(G_i)^r_{ac}(G_j)^r_{cb} + O(q^3) \ , \tag{4.45}$$

where $(G_i)^r_{ab}$ are anti-Hermitian matrices of the generators in the induced representation of the algebra. Then we obtain,

$$q^iq^j\frac{\partial^2\sigma_R}{\partial q^i\partial q^j}\Big|_{q=0} = \sum q^iq^j(G_i)^r_{ac}(G_j)^r_{cb}\varrho^r_{ab} = -\sum(\psi^r_c)^*_a\varrho^r_{ab}(\psi^r_c)_b < 0 \ , \tag{4.46}$$

where in the second equality we introduced the vectors $(\psi^r_c)_b = q^j(G_j)^r_{cb}$ and used that their complex conjugate are $(\psi^r_c)^*_a = q^j(G_j)^{*r}_{ca} = -q^j(G_j)^r_{ac}$. The expression (4.43) has the expected form covariantizing the high-temperature limit of the influence functional of a harmonic bath [Feynman and Vernon Jr 2000].

The Gaussian part (4.43) sets the power-counting rules by the requirement that for typical fluctuations of the fields it is of order 1. The first term determines the amplitude of the quantum fields:

$$\frac{T}{\hbar^2}\frac{\gamma\hat{q}^2}{\omega} \sim 1 \qquad \Longrightarrow \qquad \hat{q} \sim \hbar\sqrt{\frac{\omega}{\gamma T}} \ , \tag{4.47}$$

where we have restored the Planck constant and used $\beta = \hbar/T$. From here we see that expansion in $\hat{q}$ is justified provided we work at the frequencies smaller than temperature and/or the dissipation is strong enough, The second term then gives the amplitude of $\bar{q}$ fluctuations in the overdamped regime,

$$\frac{\gamma\hat{q}\bar{q}}{\hbar} \sim 1 \qquad \Longrightarrow \qquad \bar{q} \sim \sqrt{\frac{T}{\gamma\omega}} \ . \tag{4.48}$$

[5]Note that the quadratic term is fixed by the identity $U^r_{ab}(-q)U^r_{bc}(q) = \delta_{ac}$ following from our definition of the group element in (4.2).

It corresponds to the classical brownian motion with the distance from the initial point growing as the square-root of time, $\bar{q} \sim \sqrt{(T/\gamma)t}$. In general, the classical fluctuations will be reduced compared to this value at short time scales when the inertial term in the SK action is non-negligible.

**First quantum correction** The cubic contribution vanishes as a result of the time-reversal invariance, so we go directly to quartic contributions. A somewhat lengthy but straightforward calculation yields,

$$\mathcal{I}[\bar{q},\hat{q}]\Big|_{\text{NNLO}} = \int dt\Big[i\frac{4}{3\beta}\mu_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k\hat{q}^l + \Big(-\frac{4}{3}\mu_{ijkl} + \frac{1}{3}\gamma_{im}C^m_{\ jn}C^n_{\ kl}\Big)\hat{q}^i\hat{q}^j\hat{q}^k D_t\bar{q}^l - \gamma_{im}C^m_{\ jk}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k\Big], \tag{4.49}$$

with new couplings

$$\mu_{ijkl} = -\pi\,\partial_i\partial_j\partial_k\partial_l\sigma_0(0) \tag{4.50}$$

forming a symmetric 4-index tensor in the configuration space of the body. Despite this freedom, the structure of the correction (4.49) is quite constrained. The noise and dissipative terms are linked to the same couplings with fixed coefficients, there is non-trivial dependence on $\gamma_{im}$ and the structure constants, etc. The frequency scaling of different terms here is

$$\frac{1}{\hbar}\int dt\,\frac{\mu}{\beta}\hat{q}^4 \sim \frac{1}{\hbar}\int dt\,\mu\hat{q}^3 D_t\bar{q} \sim \frac{\mu}{\gamma}\cdot\frac{\hbar^2\omega}{\gamma T}\,, \qquad \frac{1}{\hbar}\int dt\,\gamma\hat{q}^2\dot{\hat{q}} \sim \frac{\hbar^2\omega^{3/2}}{\gamma^{1/2}T^{3/2}}\,, \tag{4.51}$$

where we have restored $\hbar$. Thus, the last term can be neglected at $\omega \ll T/\gamma$ if $\mu \sim \gamma$. Note that the noise generated by the quantum correction (4.49) is no more Gaussian.

### 4.3.2 Higher-derivative corrections

The influence functional receives further corrections from higher powers of $\omega$ in the expansion of the spectral densities. These corrections are suppressed by the frequency cutoff $\Lambda$ which may be present even at the classical level. Thus, typically they are more important than the non-Gaussian terms studied above. The purpose of this section is to derive the leading higher-derivative corrections to the noise and friction terms. The correlators receive additional contributions from the expansion of the spectral density including the $O(\omega^3)$ terms:

$$\Delta\mathcal{K}^r_{ab}(t) = -\frac{2\pi}{\beta\Lambda^2}\check{\varrho}^r_{2,ab}\,\delta''(t) - \frac{i\pi}{\Lambda^2}\hat{\varrho}^r_{2,ab}\,\delta'''(t)\,, \tag{4.52a}$$

$$\Delta\tilde{\mathcal{K}}^r_{ab}(t) = -\frac{2\pi}{\beta\Lambda^2}\check{\varrho}^r_{2,ab}\,\delta''(t) + \frac{i\pi}{\Lambda^2}\hat{\varrho}^r_{2,ab}\,\delta'''(t)\,, \tag{4.52b}$$

$$\Delta\mathcal{G}^r_{ab}(t) = -\frac{2\pi}{\beta\Lambda^2}\check{\varrho}^r_{2,ab}\,\delta''(t) - \frac{i\pi}{\Lambda^2}\hat{\varrho}^r_{2,ab}\,\mathrm{sign}(t)\delta'''(t)\,, \tag{4.52c}$$

$$\Delta\tilde{\mathcal{G}}^r_{ab}(t) = -\frac{2\pi}{\beta\Lambda^2}\check{\varrho}^r_{2,ab}\,\delta''(t) + \frac{i\pi}{\Lambda^2}\hat{\varrho}^r_{2,ab}\,\mathrm{sign}(t)\delta'''(t)\,, \tag{4.52d}$$

where we have introduced

$$\check{\varrho}^{r}_{2,ab} = \varrho^{r}_{2,ab} + \frac{(\beta\Lambda)^2}{12}\hat{\varrho}^{r}_{0,ab} \,. \tag{4.53}$$

This generates additional terms in the influence functional. We separately consider corrections to the noise and friction parts. For the noise part we obtain,

$$\begin{aligned}
\Delta\mathcal{I}_{\text{noise}} &= -\frac{i\pi}{\beta\Lambda^2}\int dt \frac{d^2}{dt'^2}\left(\check{\sigma}_2(q_+ \ominus q'_+) + \check{\sigma}_2(q_- \ominus q'_-) - \check{\sigma}_2(q_+ \ominus q'_-) - \check{\sigma}_2(q_- \ominus q'_+)\right)\Big|_{t'=t} \\
&= -\frac{i\pi}{\beta\Lambda^2}\int dt \Big\{\Big[\partial_j\partial_l\check{\sigma}_2(0) - \partial_i\partial_k\check{\sigma}_2(-2\hat{q})u^i_j(2\hat{q})u^k_l(2\hat{q}) + \partial_i\check{\sigma}_2(-2\hat{q})\partial_k u^i_j(2\hat{q})u^k_l(2\hat{q})\Big]D_t q^j_+ D_t q^l_+ \\
&\qquad + \Big[\partial_j\partial_l\check{\sigma}_2(0) - \partial_i\partial_k\check{\sigma}_2(2\hat{q})u^i_j(-2\hat{q})u^k_l(-2\hat{q}) + \partial_i\check{\sigma}_2(2\hat{q})\partial_k u^i_j(-2\hat{q})u^k_l(-2\hat{q})\Big]D_t q^j_- D_t q^l_- \\
&\qquad + \partial_i\check{\sigma}_2(-2\hat{q})u^i_j(2\hat{q})\partial_t D_t q^j_+ + \partial_i\check{\sigma}_2(2\hat{q})u^i_j(-2\hat{q})\partial_t D_t q^j_-\Big\},
\end{aligned} \tag{4.54}$$

where we have a new real even function on the group:

$$\check{\sigma}_2(q) = \sigma_2(q) + \frac{(\beta\Lambda)^2}{12}\sigma_0(q) \,, \qquad \sigma_2(q) = \sum_{rab} U^r_{ab}(q)\,\hat{\varrho}^r_{2,ab} \,. \tag{4.55}$$

In deriving the expression (4.54) we have used Eqs. (A.3). This expression appears quite lengthy, but its structure is transparent. It is symmetric under the exchange $\hat{q} \mapsto -\hat{q}$, so it contains only even powers of the quantum fields, as appropriate for a noise term.

Next, we expand (4.54) to quadratic order in $\hat{q}$. A lengthy but straightforward calculation yields,

$$\begin{aligned}
\Delta\mathcal{I}_{\text{noise}} = i\frac{4}{\beta\Lambda^2}\int dt\Big\{&\Big[-\check{\mu}_{2,ijkl} + \frac{1}{3}\check{\gamma}_{2,im}C^m_{\ kn}C^n_{\ jl} + \frac{2}{3}\check{\gamma}_{2,km}C^m_{\ in}C^n_{\ jl} + \frac{1}{2}\check{\gamma}_{2,mn}C^m_{\ ik}C^n_{\ jl}\Big]\hat{q}^i\hat{q}^j D_t\bar{q}^k D_t\bar{q}^l \\
&+ \left[\check{\gamma}_{2,im}C^m_{\ jk} - \check{\gamma}_{2,jm}C^m_{\ ik} - \check{\gamma}_{2,km}C^m_{\ ij}\right]\hat{q}^i\dot{\hat{q}}^j D_t\bar{q}^k + \check{\gamma}_{2,ij}\dot{\hat{q}}^i\dot{\hat{q}}^j\Big\},
\end{aligned} \tag{4.56}$$

where

$$\check{\gamma}_{2,ij} = \gamma_{2,ij} + \frac{(\beta\Lambda)^2}{12}\gamma_{ij} \,, \qquad \gamma_{2,ij} = -\pi\partial_i\partial_j\sigma_2(0) \,, \tag{4.57a}$$

$$\check{\mu}_{2,ijkl} = \mu_{2,ijkl} + \frac{(\beta\Lambda)^2}{12}\mu_{ijkl} \,, \qquad \mu_{2,ijkl} = -\pi\partial_i\partial_j\partial_k\partial_l\sigma_2(0) \,. \tag{4.57b}$$

In the overdamped regime the estimate of various terms here reads (restoring $\hbar$):

$$\frac{1}{\hbar\beta\Lambda^2}\int dt\, \check{\mu}_2 \hat{q}^2 (D_t\bar{q})^2 \sim \frac{\check{\mu}_2}{\gamma}\cdot\frac{\omega T}{\gamma\Lambda^2} \ni \frac{\mu_2}{\gamma}\cdot\frac{\omega T}{\gamma\Lambda^2}\,,\quad \frac{\mu}{\gamma}\cdot\frac{\hbar^2\omega}{\gamma T}\,, \tag{4.58a}$$

$$\frac{1}{\hbar\beta\Lambda^2}\int dt\, \check{\gamma}_2 \hat{q}\dot{\hat{q}} D_t\bar{q} \sim \frac{\check{\gamma}_2}{\gamma}\cdot\frac{\omega^{3/2}T^{1/2}}{\gamma^{1/2}\Lambda^2} \ni \frac{\gamma_2}{\gamma}\cdot\frac{\omega^{3/2}T^{1/2}}{\gamma^{1/2}\Lambda^2}\,,\quad \frac{\hbar^2\omega^{3/2}}{\gamma^{1/2}T^{3/2}}\,, \tag{4.58b}$$

$$\frac{1}{\hbar\beta\Lambda^2}\int dt\, \check{\gamma}_2 \dot{\hat{q}}^2 \sim \frac{\check{\gamma}_2}{\gamma}\cdot\frac{\omega^2}{\Lambda^2} \ni \frac{\gamma_2}{\gamma}\cdot\frac{\omega^2}{\Lambda^2}\,,\quad \frac{\hbar^2\omega^2}{T^2}\,. \tag{4.58c}$$

Comparing to Eqs. (4.51) we see that the new corrections are as important as the non-Gaussian contributions whenever $\beta\Lambda >> 1$, and typically larger whenever $\beta\Lambda << 1$. We again observe the characteristic frequency $\omega \sim T/\gamma$ which divides the frequency regions where different terms in (4.56) dominate.

We now turn to the friction part of the influence functional. A direct substitution of the correlators (4.52) into the general expression (4.22) gives,

$$\begin{aligned}\Delta\mathcal{I}_{\text{fric}} =& \frac{\pi}{2\Lambda^2}\int dt_1 dt_2 \big[\sigma_2(q_{+2}\ominus q_{+1})\,\text{sign}(t_1-t_2) - \sigma_2(q_{-2}\ominus q_{-1})\,\text{sign}(t_1-t_2) \\ &\qquad\qquad - \sigma_2(q_{+2}\ominus q_{-1}) + \sigma_2(q_{-2}\ominus q_{+1})\big]\delta'''(t_1-t_2) \\ =& -\frac{\pi}{\Lambda^2}\int dt_1 dt_2\, \theta(t_2-t_1)\big[\sigma_2(q_{+2}\ominus q_{+1}) - \sigma_2(q_{-2}\ominus q_{+1}) \\ &\qquad\qquad + \sigma_2(q_{+2}\ominus q_{-1}) - \sigma_2(q_{-2}\ominus q_{-1})\big]\delta'''(t_1-t_2)\,,\end{aligned} \tag{4.59}$$

where in deriving the last expression we used the symmetry of the function $\sigma_2(q)$. Integrating by parts we obtain schematically,

$$\Delta\mathcal{I}_{\text{fric}} = \frac{\pi}{\Lambda^2}\int dt\Big[-\delta''(0)f(t,t) + 3\delta'(0)\frac{\partial f(t,t')}{\partial t'}\Big|_{t'=t} - 3\delta(0)\frac{\partial^2 f(t,t')}{\partial t'^2}\Big|_{t'=t} + \theta(0)\frac{\partial^3 f(t,t')}{\partial t'^3}\Big|_{t'=t}\Big], \tag{4.60}$$

where by $f(t_2,t_1)$ we denoted the combination in the square brackets in (4.59). The first two terms vanish because $f(t,t) = 0$ and $\delta'(0) = 0$[6]. The third divergent term, however, survives. Let us show that it can be absorbed by renormalization of the inertial term in

[6]This can be demonstrated more rigorously by introducing the regularization

$$\delta(x) = \lim_{\epsilon\to+0}\frac{1}{2\pi i}\left[\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right]. \tag{4.61}$$

the body action. We have

$$\Delta\mathcal{I}^{\text{div}}_{\text{fric}} = -\frac{3\pi}{\Lambda^2}\delta(0)\int dt\frac{d^2}{dt'^2}\big[\sigma_2(q_+\ominus q'_+) - \sigma_2(q_-\ominus q'_+) + \sigma_2(q_+\ominus q'_-) - \sigma_2(q_-\ominus q'_-)\big]\Big|_{t'=t}$$
$$= -\frac{3\pi}{\Lambda^2}\delta(0)\int dt\Big\{\big[\partial_l\partial_j\sigma_2(0) - \partial_k\partial_i\sigma_2(-2\hat{q})u^k_l(2\hat{q})u^i_j(2\hat{q}) + \partial_i\sigma_2(-2\hat{q})\partial_k u^i_j(2\hat{q})u^k_l(2\hat{q})\big]D_tq^l_+D_tq^j_+$$
$$+ \partial_i\sigma_2(-2\hat{q})u^i_j(2\hat{q})\partial_tD_tq^j_+ - (q_+\leftrightarrow q_-)\Big\}\,. \tag{4.62}$$

Expansion to linear order in $\hat{q}$ yields,

$$\Delta\mathcal{I}^{\text{div}}_{\text{fric}} = -\frac{12}{\Lambda^2}\delta(0)\int dt\,\hat{q}^i\big(\gamma_{2,ij}\partial_tD_t\bar{q}^j + \gamma_{2,jl}C^l{}_{ik}D_t\bar{q}^jD_t\bar{q}^k\big)\,. \tag{4.63}$$

On the other hand, the proper action of the body reads

$$S_{\text{body}} = \int dt\,\frac{1}{2}J_{ij}D_tq^iD_tq^j\,, \tag{4.64}$$

with $J_{ij}$ being its inertia tensor. This leads to the following contribution into the SK functional:

$$\int dt\,\frac{1}{2}J_{ij}(D_tq^i_+D_tq^j_+ - D_tq^i_-D_tq^j_-) \simeq -2\int dt\,\hat{q}^i\big(J_{ij}\partial_tD_t\bar{q}^j + J_{jl}C^l{}_{ik}D_t\bar{q}^jD_t\bar{q}^k\big)\,, \tag{4.65}$$

where we have used Eqs. (A.13) and integration by parts. Comparing this expression to (4.63) we see that the divergence is absorbed by redefinition of the inertia tensor,

$$J^{\text{renorm}}_{ij} = J_{ij} + \frac{6}{\Lambda^2}\delta(0)\gamma_{2,ij}\,. \tag{4.66}$$

The last term in (4.60) produces a finite correction,

$$\Delta\mathcal{I}^{\text{reg}}_{\text{fric}} = \frac{\pi}{2\Lambda^2}\int dt\frac{d^3}{dt'^3}\big[\sigma_2(q_+\ominus q'_+) - \sigma_2(q_-\ominus q'_+) + \sigma_2(q_+\ominus q'_-) - \sigma_2(q_-\ominus q'_-)\big]\Big|_{t'=t}\,, \tag{4.67}$$

where we have used $\theta(0) = 1/2$. Evaluating the third derivative and expanding to linear order in $\hat{q}$ we obtain

$$\Delta\mathcal{I}^{\text{reg}}_{\text{fric}} = \frac{1}{\Lambda^2}\int dt\Big\{\big[2\mu_{2,ijkl} - \gamma_{2,jm}C^m{}_{kn}C^n{}_{il}\big]\hat{q}^iD_t\bar{q}^jD_t\bar{q}^kD_t\bar{q}^l$$
$$+ \big[3\gamma_{2,km}C^m{}_{ij} + 3\gamma_{2,jm}C^m{}_{ik} + \gamma_{2,im}C^m{}_{jk}\big]\hat{q}^iD_t\bar{q}^j\partial_tD_t\bar{q}^k + 2\gamma_{2,ij}\hat{q}^i\partial_t^2D_t\bar{q}^j\Big\}. \tag{4.68}$$

Note that unlike the noise part, this contribution is suppressed by only $\Lambda$. Note that this contribution is linear in $\hat{q}$ and hence leads to modification of the classical equations of motion. It gives rise to terms of cubic order in velocities and terms with velocity derivatives in the friction force. Such terms are indeed known to arise in explicit models of Brownian particle interacting with heat bath [Van Kampen and Oppenheim 1986; Plyukhin and Schofield 2003] and can lead to interesting signatures [Plyukhin and Froese 2007]. Note also that they cannot be reproduced by the bulk model of chapter (6), since the latter has been devised to describe an exactly Ohmic friction force at the classical level.

# Chapter 5

# General Influence Functional At High Temperature From DKMS Condition

In the previous chapter, we saw that the fluctuation dissipation theorem related different Green's function to each other and as a result, reduced the freedom in the dissipative coefficients. In addition, we integrated out the bath degrees of freedom in obtaining the influence functional at the linear response level. While the formalism developed in the previous chapter enlightens the mechanism of emergence of a universal description of the dissipative dynamics, it is desired to use the symmetries of the microscopic degrees of freedom, without directly referring to them. This chapter is devoted to this purpose. The microscopic symmetries imply the Dynamical KMS (DKMS) condition. We use the DKMS condition to obtain the most general influence functional of dissipative dynamics defined on Lie group in the high temperature regime. While we derive the influence functional at high temperature limit, we do not use the linear response approximation. As a result, the influence functional of the dissipative dynamics from DKMS condition has more freedom with respect to the one from the previous chapter.

## 5.1 Influence functional from DKMS symmetry

In this section we derive the most general high-temperature expansion of the influence functional compatible with DKMS condition [Liu and Glorioso 2018; Akyuz et al. 2024; Sieberer et al. 2015]. When the bath preserves time reversal, the latter is invariant under the transformation of the fields on the two parts of the SK contour:

$$q'^{j}_{+}(t) = q^{j}_{+}(-t + i\beta/2) , \qquad q'^{j}_{-}(t) = q^{j}_{-}(-t - i\beta/2) . \tag{5.1}$$

In general this transformation is nonlocal, defining the new body coordinates in terms of an analytic continuation of the original coordinates into the complex time plane. The situation simplifies in the high-temperature limit corresponding to small $\beta$. In this case, the DKMS transformations can be written as infinite series in powers of $\beta$,

$$q'^{j}_{\pm}(t) = q^{j}_{\pm}(t') \pm \frac{i\beta}{2}\dot{q}^{j}_{\pm}(t') - \frac{\beta^2}{8}\ddot{q}^{j}_{\pm}(t') \mp \frac{i\beta^3}{48}\dddot{q}^{j}_{\pm}(t') + \ldots\Big|_{t'=-t} . \tag{5.2}$$

Note that in these expressions we first take the derivatives of the coordinates with respect to time, and only afterwards flip its sign. For example, we have $\dot{q}_{\pm}(t')|_{t'=-t} = -\partial_t q_{\pm}(-t)$. This should be kept in mind when taking further derivatives of the transformed fields. Note also that the DKMS transformation introduces imaginary contributions and thus requires complexification of the coordinates.

We need to rewrite the DKMS transformations (5.2) in terms of the "classical" and "quantum" fields and apply them order-by-order in the high-temperature expansion of the influence functional. We first discuss this procedure for the case of Abelian group $G$ realized as simple shifts of the coordinates. Then we will turn to the technically more complicated, but conceptually equivalent, case of general non-Abelian $G$.

## 5.2 DKMS in the Abelian case

Since in this case the classical and quantum variables are linearly related to $q_{\pm}$, we readily find their DKMS transformations,

$$\bar{q}'^{j}(t) = \bar{q}^{j}(t') + \frac{i\beta}{2}\dot{\hat{q}}^{j}(t') - \frac{\beta^2}{8}\ddot{\bar{q}}^{j}(t') - \frac{i\beta^3}{48}\dddot{\hat{q}}^{j}(t') + \ldots\Big|_{t'=-t} , \tag{5.3a}$$

$$\hat{q}'^{j}(t) = \hat{q}^{j}(t') + \frac{i\beta}{2}\dot{\bar{q}}^{j}(t') - \frac{\beta^2}{8}\ddot{\hat{q}}^{j}(t') - \frac{i\beta^3}{48}\dddot{\bar{q}}^{j}(t') + \ldots\Big|_{t'=-t} . \tag{5.3b}$$

The symmetry $G$ acts as constant shifts on the classical variables $\bar{q}$, while leaving $\hat{q}$ intact. Our task is to write down the most general influence functional compatible with this symmetry and impose further invariance under (5.3). Since we work in the high-temperature limit, the influence functional represents an expansion in powers of $\hat{q}$, $\dot{\bar{q}}$ and their derivatives. We observe that application of (5.3) does not change the overall power of $\hat{q}$, $\bar{q}$ in an expression, so the sectors with different powers of these variables can be analyzed separately.

### 5.2.1 Quadratic sector

We write the most general local quadratic expression containing the invariant fields $\hat{q}^i$ and $\dot{\bar{q}}^i$, retaining term with up to two additional time derivatives acting on these fields. Taking into account that the influence functional must vanish if $\hat{q} = 0$ and inserting appropriate powers of $\beta$ to keep the coefficients dimensionless we obtain,

$$\mathcal{I}^{(2)} = \int dt(\beta^{-1}a_{ij}\hat{q}^i\hat{q}^j + b_{ij}\hat{q}^i\dot{\bar{q}}^j + c_{ij}\hat{q}^i\dot{\hat{q}}^j + \beta d_{ij}\hat{q}^i\ddot{\bar{q}}^j + \beta e_{ij}\dot{\hat{q}}^i\dot{\hat{q}}^j + \beta^2 f_{ij}\hat{q}^i\dddot{\bar{q}}^j + \ldots) . \tag{5.4}$$

The influence functional must change sign under the reflection $\hat{q} \mapsto -\hat{q}$ accompanied by complex conjugation [Liu and Glorioso 2018; Akyuz et al. 2024]. Hence the coefficients

$a_{ij}$, $c_{ij}$, $e_{ij}$ are purely imaginary, whereas $b_{ij}$, $d_{ij}$, $f_{ij}$ are real. Note that $a_{ij}$ and $e_{ij}$ are symmetric by definition, whereas there are no a priori symmetry constraints on the rest of the coefficients. Now, we substitute here the transformations (5.3) and require that the influence functional remain the same at each order in $\beta$. We don't get any constraints at the order $\beta^{-1}$, whereas at other orders we have:

$$\beta^0: \quad \int dt((ia_{ij} - b_{ij})\hat{q}^i \dot{\bar{q}}^j - c_{ij}\hat{q}^i \dot{\hat{q}}^j) = \int dt(b_{ij}\hat{q}^i \dot{\bar{q}}^j + c_{ij}\hat{q}^i \dot{\hat{q}}^j) \;, \tag{5.5a}$$

$$\begin{aligned} \beta^1: \quad & \int dt\Big[\Big(-\frac{a_{ij}}{4} - \frac{ib_{ij}}{2}\Big)(\dot{\bar{q}}^i \dot{\bar{q}}^j - \dot{\hat{q}}^i \dot{\hat{q}}^j) + \frac{i}{2}(c_{ji} - c_{ij})\hat{q}^i \ddot{\bar{q}}^j + d_{ij}\hat{q}^i \ddot{\bar{q}}^j + e_{ij}\dot{\hat{q}}^i \dot{\hat{q}}^j\Big] \\ & = \int dt(d_{ij}\hat{q}^i \ddot{\bar{q}}^j + e_{ij}\dot{\hat{q}}^i \dot{\hat{q}}^j) \;, \end{aligned} \tag{5.5b}$$

$$\beta^2: \quad \int dt\Big[\Big(-\frac{ia_{ij}}{6} + \frac{b_{ij} + b_{ji}}{4} - ie_{ij} - f_{ij}\Big)\hat{q}^i \dddot{\bar{q}}^j + \Big(\frac{c_{ij}}{4} + \frac{id_{ij}}{2}\Big)\dot{\bar{q}}^i \ddot{\bar{q}}^j\Big] = \int dt f_{ij}\hat{q}^i \dddot{\bar{q}}^j \;, \tag{5.5c}$$

$$\beta^3: \quad \int dt\Big(-\frac{a_{ij}}{48} - \frac{ib_{ij}}{12} - \frac{e_{ij}}{4} + \frac{if_{ij}}{2}\Big)\ddot{\bar{q}}^i \ddot{\bar{q}}^j = 0 \;. \tag{5.5d}$$

In deriving these equations we used integration by parts and in Eqs. (5.5c), (5.5d) neglected terms with three or more derivatives of $\hat{q}$, $\dot{\bar{q}}$. It is worth noting that in this derivation we needed the expansion of $\bar{q}$ only through order $O(\beta^2)$, whereas the full expansion of $\hat{q}$ through order $O(\beta^3)$ was used.

Equation (5.5a) implies

$$b_{ij} = \frac{i}{2}a_{ij} \;. \tag{5.6}$$

This is nothing but the classical fluctuation dissipation theorem, which in particular implies that $b_{ij}$ is symmetric. It is of course satisfied by the influence functional (4.43) obtained within the linear response theory, with the identification

$$a_{ij} = 4i\gamma_{ij} \;, \qquad b_{ij} = -2\gamma_{ij} \;. \tag{5.7}$$

For $c_{ij}$ we get that the integral $\int dt\, c_{ij}\hat{q}^i \dot{\hat{q}}^j$ must be zero. This is possible if $c_{ij}$ is symmetric, so that the integrand is a total derivative. However, this is the same integral as in (5.4), implying that this term can be safely dropped. Thus, without loss of generality, we set $c_{ij} = 0$. This, together with the relation (5.6) implies that Eq. (5.5b) is identically satisfied for any choice of $d_{ij}$ and $e_{ij}$.

Equation (5.5c) gives further non-trivial relations. We read from it

$$f_{ij} = -\frac{i}{2}e_{ij} + \frac{i}{24}a_{ij} \;, \tag{5.8}$$

where we have used the relation (5.6). We see that $f_{ij}$ is symmetric. Lastly, $d_{ij}$ must also be symmetric, in order for the last term on the l.h.s. of (5.5c) to integrate to zero. Then this term has the same structure as the one coming from the proper kinetic Lagrangian of the body (cf. Eq. (4.64)) and can be absorbed into renormalization of its parameters. Thus, we can set $d_{ij} = 0$. Finally, the last Eq. (5.5d) is identically satisfied, provided the relations (5.6) and (5.8) hold.

Identifying

$$f_{ij} = \frac{2}{(\beta\Lambda)^2}\gamma_{2,ij} \,, \tag{5.9}$$

according to Eq. (4.68) and using (5.7) we find

$$e_{ij} = \frac{4i}{(\beta\Lambda)^2}\left(\gamma_{2,ij} + \frac{(\beta\Lambda)^2}{12}\gamma_{ij}\right) \,, \tag{5.10}$$

which matches with the last term in Eq. (4.56). We conclude that at the quadratic level the linear response theory provides the most general SK functional, up to the next to the leading order noise terms.

### 5.2.2 Quartic sector

We now turn to the quartic terms, postponing the discussion of cubic order for later. Here we are interested in terms containing just $\hat{q}$ and $\dot{\tilde{q}}$, without any additional time derivatives. The most general influence functional of this form reads:[1]

$$\mathcal{I}^{(4)} = \int dt(\beta^{-1}A_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k\hat{q}^l + B_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k\dot{\tilde{q}}^l + \beta D_{ijkl}\hat{q}^i\hat{q}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l + \beta^2 E_{ijkl}\hat{q}^i\dot{\tilde{q}}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l) \,, \tag{5.11}$$

where the coefficients $A_{ijkl}$, $D_{ijkl}$ are imaginary, and $B_{ijkl}$, $E_{ijkl}$ are real. They have the obvious symmetry properties with respect to permutation of indices

$$A_{ijkl} = A_{(ijkl)} \,, \quad B_{ijkl} = B_{(ijk)l} \,, \quad D_{ijkl} = D_{(ij)(kl)} \,, \quad E_{ijkl} = E_{i(jkl)} \,, \tag{5.12}$$

where round brackets mean symmetrization. Application of the DKMS transformations to (5.11) is simplified by the observation that with our restriction on the number of derivatives it is sufficient to keep only the leading term in (5.3a) and the first two terms in (5.3b). Working again order by order in $\beta$ we get the conditions

$$\beta^0: \quad \int dt(2iA_{ijkl} - B_{ijkl})\hat{q}^i\hat{q}^j\hat{q}^k\dot{\tilde{q}}^l = \int dt\, B_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k\dot{\tilde{q}}^l \,, \tag{5.13a}$$

$$\beta^1: \quad \int dt\Big(-\frac{3}{2}A_{ijkl} - \frac{3i}{2}B_{ijkl} + D_{ijkl}\Big)\hat{q}^i\hat{q}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l = \int dt\, D_{ijkl}\hat{q}^i\hat{q}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l \,, \tag{5.13b}$$

$$\beta^2: \quad \int dt\Big(-\frac{i}{2}A_{ijkl} + \frac{3}{4}B_{ijkl} + iD_{ijkl} - E_{ijkl}\Big)\hat{q}^i\dot{\tilde{q}}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l = \int dt\, E_{ijkl}\hat{q}^i\dot{\tilde{q}}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l \,, \tag{5.13c}$$

$$\beta^3: \quad \int dt\Big(\frac{1}{16}A_{ijkl} + \frac{i}{8}B_{ijkl} - \frac{1}{4}D_{ijkl} - \frac{i}{2}E_{ijkl}\Big)\dot{\tilde{q}}^i\dot{\tilde{q}}^j\dot{\tilde{q}}^k\dot{\tilde{q}}^l = 0 \,. \tag{5.13d}$$

[1] We do not use $C_{ijkl}$ for the coefficients to avoid confusion with the structure constants.

The first equation implies,

$$B_{ijkl} = iA_{ijkl} \ , \tag{5.14}$$

so that $B_{ijkl}$ is totally symmetric. This replicates the structure of the next to the leading order noise contributions (4.49) in the linear response case with the identification,[2]

$$A_{ijkl} = \frac{4i}{3}\mu_{ijkl} \ , \qquad B_{ijkl} = -\frac{4}{3}\mu_{ijkl} \ . \tag{5.15}$$

Next Eq. (5.13b) does not bring new conditions, whereas Eq. (5.13c) implies

$$E_{ijkl} = \frac{i}{2}D_{i(jkl)} + \frac{i}{8}A_{ijkl} \ . \tag{5.16}$$

Note symmetrization over last three indices on the r.h.s. The last Eq. (5.13d) is then satisfied identically.

It is straightforward to verify that the coefficients of the quartic terms in the linear-response functional (4.56), (4.68) obey the condition (5.16), but do not present the most general solution to it. Indeed, the tensors $\mu_{2,ijkl}$, $\check{\mu}_{2,ijkl}$ are totally symmetric, whereas (5.16) contains an arbitrary tensor $D_{ijkl}$ possessing smaller symmetry (5.12). We conclude that the linear response case provides a restricted solution to the general DKMS conditions.

### 5.2.3 Cubic sector

Let us come back to the cubic terms. We don't have any such terms in the linear response theory for Abelian $G$, so one may wonder if they are allowed by the DKMS symmetry at all. We are going to see that the answer is affirmative.

Proceeding as before, we start by writing the most general local functional of cubic order in $\hat{q}$ and $\dot{\hat{q}}$. We restrict to terms having at most one more additional derivative. We have:

$$\begin{aligned}\mathcal{I}^{(3)} = \int dt\big(&\beta^{-1}g_{ijk}\hat{q}^i\hat{q}^j\hat{q}^k + h_{ijkl}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k + l_{ijk}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k \\ &+ \beta m_{ijk}\hat{q}^i\dot{\hat{q}}^j\dot{\hat{q}}^k + \beta n_{ijk}\hat{q}^i\dot{\hat{q}}^j\dot{\hat{q}}^k + \beta^2 o_{ijk}\hat{q}^i\dot{\hat{q}}^j\ddot{\hat{q}}^k\big).\end{aligned} \tag{5.17}$$

Any other cubic term can be reduced to one of this set using integration by parts. The coefficients $h_{ijk}$ and $n_{ijk}$ are imaginary and the rest are real. The first four tensors possess permutation symmetries,

$$g_{ijk} = g_{(ijk)} \ , \qquad h_{ijk} = h_{(ij)k} \ , \qquad l_{ijk} = l_{(ij)k} \ , \qquad m_{ijk} = m_{i(jk)} \ , \tag{5.18}$$

whereas all components of $n_{ijk}$, $o_{ijk}$ are a priori independent.

The restriction to one-derivative level allows us to keep only the first term in the DKMS transformation (5.3a) and two first terms in (5.3b). We obtain the following

[2]Recall that we are working here with Abelian symmetry, so the structure constants vanish.

conditions:

$$\beta^0: \quad \int dt\Big[\Big(\frac{3i}{2}g_{ijk} - h_{ijk}\Big)\hat{q}^i\hat{q}^j\dot{\bar{q}}^k - l_{ijk}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k\Big] = \int dt(h_{ijk}\hat{q}^i\hat{q}^j\dot{\bar{q}}^k + l_{ijk}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k)\ , \tag{5.19a}$$

$$\beta^1: \quad \int dt\Big[\Big(-\frac{3}{4}g_{ijk} - ih_{ijk} + m_{ijk}\Big)\hat{q}^i\dot{\bar{q}}^j\dot{\bar{q}}^k + (-il_{ikj} + il_{ijk} + n_{ijk})\hat{q}^i\dot{\hat{q}}^j\dot{\bar{q}}^k\Big]$$
$$= \int dt\,(m_{ijk}\hat{q}^i\dot{\bar{q}}^j\dot{\bar{q}}^k + n_{ijk}\hat{q}^i\dot{\hat{q}}^j\dot{\bar{q}}^k)\ , \tag{5.19b}$$

$$\beta^2: \quad \int dt\Big[\Big(-\frac{i}{8}g_{ijk} + \frac{1}{4}h_{ijk} + \frac{i}{2}m_{ijk}\Big)\dot{\bar{q}}^i\dot{\bar{q}}^j\dot{\bar{q}}^k + \Big(\frac{1}{2}(l_{ijk} - l_{jki}) + \frac{i}{2}(n_{ikj} - n_{jik} - n_{kij}) - o_{ijk}\Big)\hat{q}^i\dot{\bar{q}}^j\ddot{\bar{q}}^k$$
$$= \int dt\, o_{ijk}\hat{q}^i\dot{\bar{q}}^j\ddot{\bar{q}}^k\ , \tag{5.19c}$$

$$\beta^3: \quad \int dt\Big(\frac{i}{8}l_{ijk} - \frac{1}{4}n_{ikj} - \frac{i}{2}o_{ijk}\Big)\dot{\bar{q}}^i\dot{\bar{q}}^j\ddot{\bar{q}}^k = 0\ . \tag{5.19d}$$

The first equation implies

$$h_{ijk} = \frac{3i}{4}g_{ijk}\ , \qquad l_{ijk} = 0\ . \tag{5.20}$$

Then Eq. (5.19b) is automatically satisfied. Next, Eq. (5.19c) leads to conditions:

$$g_{ijk} = -8m_{(ijk)}\ , \qquad o_{ijk} = \frac{i}{4}(n_{ikj} - n_{jik} - n_{kij})\ . \tag{5.21}$$

Substituting the last relation into Eq. (5.19d) we obtain a further restriction

$$n_{(ijk)} = 0\ . \tag{5.22}$$

We see that the cubic terms are parameterized by two three-index tensors: an arbitrary tensor $m_{ijk}$ and a tensor $n_{ijk}$ whose symmetrized part vanishes. Both these tensors contribute into the classical response of the system: $m_{ijk}$ directly, whereas $n_{ijk}$ through the tensor $o_{ijk}$ which multiplies a term linear in $\hat{q}$.

Due to the condition (5.22), the $n$-part of the influence functional also vanishes for a general (non necessarily ohmic) system with less than 3 degrees of freedom. On the other hand, the $m$-part can be present already for a system with a single coordinate $x$ and corresponds to a contribution in the friction force quadratic in velocity $v \equiv \dot{x}$,

$$\Delta F = (\beta/2)mv^2\ . \tag{5.23}$$

This contribution has the same sign for forward and backward motion and thus violates parity $x \leftrightarrow -x$. Alternatively, it can be viewed as a velocity-dependent friction coefficient, $\gamma \mapsto \gamma - (\beta/2)mv$. As already noted, such force contribution does not arise within the linear response approximation. In the language of chapter (4) it should come from the three- and higher-point correlators of the bath operators $\mathcal{O}(\chi)$. An example of a system that can give rise to such force is shown in Fig. 5.1. It represents an asymmetric particle moving along a wall in a viscous fluid or gas. The particle is confined to the wall by an external potential. Assuming that the fluid obeys the sticky boundary conditions, the friction is stronger when the particle is closer to the wall. When the particle moves left (Fig. 5.1, left panel) the fluid exerts, along with the friction force in the $x$ direction, also a force $F_y$ in the upward direction. This pushes the particle's away from the wall thereby reducing the friction. Conversely, when the particle moves right (Fig. 5.1, right panel), fluid pushes it closer to the wall and the friction coefficient increases.

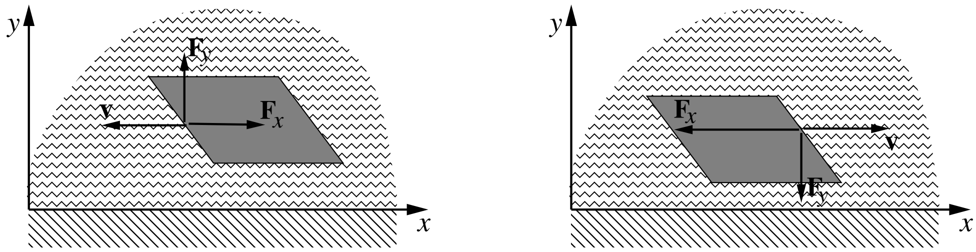


FIGURE 5.1: Asymmetric particle moving close to a wall in a viscous fluid. *Left:* When particle moves to the left, it is pushed by the fluid away from the wall, reducing friction. *Right:* When the particle moves to the right, it is pushed towards the wall, increasing friction.

## 5.3 DKMS for non-Abelian groups

In the non-Abelian case the DKMS transformations of $q_\pm$ are still given by Eqs. (5.2). However, the non-linear relations between $q_\pm$ and $\bar{q}$, $\hat{q}$ lead to technical complications. In general, these relations cannot be found in closed form. The best one can do is to write them in the form of expansions in the quantum variable $\hat{q}$. These are derived in Appendix (A4). Here it is convenient to write the result in terms of the symmetric and anti-symmetric combinations,

$$q^i_+ + q^i_- = 2\bar{q}^i + u^l_m \partial_l u^i_j\, \hat{q}^m \hat{q}^j + O(\hat{q}^4)\,, \tag{5.24a}$$

$$q^i_+ - q^i_- = 2u^i_j\, \hat{q}^j + \frac{1}{3}(u^m_s u^l_n\, \partial_m \partial_l u^i_j + u^m_s \partial_m u^l_n\, \partial_l u^i_j)\hat{q}^s \hat{q}^n \hat{q}^j + O(\hat{q}^5)\,, \tag{5.24b}$$

where all inverse Cartan coefficients $u^i_j$ and their derivatives are evaluated at the classical field $\bar{q}$. We have to substitute these equations into (5.2) and re-expand the resulting expressions. In this process we treat $\beta D_t \bar{q}$ as being of the same order as $\hat{q}$; an extra time derivative adds one more order.

The leading-order DKMS transformations represent a simple covariantization of the Abelian case:

$$\bar{q}'^i(t) = \bar{q}^i(-t) + O(\hat{q}^2) \quad \Leftrightarrow \quad D_t\bar{q}'^i(t) = -D_t\bar{q}^i(t') + O(\hat{q}\dot{\hat{q}})\Big|_{t'=-t} , \tag{5.25a}$$

$$\hat{q}'^i(t) = \hat{q}^i + \frac{i\beta}{2}D_t\bar{q}^i + O(\hat{q}^3)\Big|_{t'=-t} . \tag{5.25b}$$

Next, we combine (5.2) with (5.24a) and using the identity

$$u^l_m\partial_l u^i_j - u^l_j\partial_l u^i_m = u^i_n C^n{}_{mj} . \tag{5.26}$$

obtain through order $O(\hat{q}^2)$:

$$\bar{q}'^i(t) = \bar{q}^i + \frac{i\beta}{2}u^i_j\dot{\hat{q}}^j - \frac{i\beta}{4}u^i_l C^l{}_{jm}\hat{q}^j D_t\bar{q}^m - \frac{\beta^2}{8}u^i_j\partial_t D_t\bar{q}^j + O(\hat{q}^4)\Big|_{t'=-t} . \tag{5.27}$$

The first two and the last terms here are analogous to the Abelian transformation (5.3a), whereas the third term represents a non-Abelian correction. For later use we also give the transformation of the covariant derivative:

$$\begin{aligned} D_t\bar{q}'^k(t) =& - D_t\bar{q}^k - \frac{i\beta}{2}\ddot{\hat{q}}^k + \frac{3i\beta}{4}C^k{}_{ij}\,\dot{\hat{q}}^i D_t\bar{q}^j + \frac{i\beta}{4}C^k{}_{ij}\,\hat{q}^i\partial_t D_t\bar{q}^j + \frac{i\beta}{4}C^k{}_{jn}C^n{}_{il}\,\hat{q}^i D_t\bar{q}^j D_t\bar{q}^l \\ &+ \frac{\beta^2}{8}\partial_t^2 D_t\bar{q}^k + \frac{\beta^2}{8}C^k{}_{ij}D_t\bar{q}^i\partial_t D_t\bar{q}^j + O(\hat{q}^3\dot{\hat{q}})|_{t'=-t} . \end{aligned} \tag{5.28}$$

In the last iteration we combine (5.2) with (5.24b) and use (5.27) to obtain the transformation of $\hat{q}$ through cubic order. The computation is rather tedious and is described in Appendix A4. Here we present the result:

$$\begin{aligned} \hat{q}'^k(t) =& \hat{q}^k + \frac{i\beta}{2}D_t\bar{q}^k + \frac{i\beta}{4}C^k{}_{ij}\,\hat{q}^i\dot{\hat{q}}^j - \frac{i\beta}{12}C^k{}_{in}C^n{}_{jl}\,\hat{q}^i\hat{q}^j D_t\bar{q}^l - \frac{\beta^2}{8}\ddot{\hat{q}}^k + \frac{\beta^2}{4}C^k{}_{ij}\,\dot{\hat{q}}^i D_t\bar{q}^j \\ &+ \frac{\beta^2}{12}C^k{}_{jn}C^n{}_{il}\,\hat{q}^i D_t\bar{q}^j D_t\bar{q}^l - \frac{i\beta^3}{48}\partial_t^2 D_t\bar{q}^k - \frac{i\beta^3}{24}C^k{}_{ij}D_t\bar{q}^i\partial_t D_t\bar{q}^j + O(\hat{q}^5)\Big|_{t'=-t} . \end{aligned} \tag{5.29}$$

We now have all required ingredients for imposing the DKMS symmetry on the influence functional. Unlike the Abelian case, the DKMS transformations now mix terms of different orders in the number of fields. However, they still preserve the separation into even and odd terms (even or odd powers of $\hat{q} \equiv \beta\partial_t\bar{q}$), so it is convenient to split the analysis accordingly.

### Even orders

We write the general expression:

$$\begin{aligned}\mathcal{I}_{\text{even}} = \int dt(&\beta^{-1}4i\gamma_{ij}\hat{q}^i\hat{q}^j - 2\gamma_{ij}\hat{q}^i D_t\bar{q}^j + \beta e_{ij}\dot{\hat{q}}^i\dot{\hat{q}}^j + \beta^2 f_{ij}\hat{q}^i\partial_t^2 D_t\bar{q}^j \\ &+ l_{ijk}\hat{q}^i\hat{q}^j\dot{\hat{q}}^k + \beta n_{ijk}\hat{q}^i\dot{\hat{q}}^j D_t\bar{q}^k + \beta^2 o_{ijk}\hat{q}^i D_t\bar{q}^j\partial_t D_t\bar{q}^k \\ &+ \beta^{-1}A_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k\hat{q}^l + B_{ijkl}\hat{q}^i\hat{q}^j\hat{q}^k D_t\bar{q}^l + \beta D_{ijkl}\hat{q}^i\hat{q}^j D_t\bar{q}^k D_t\bar{q}^l \\ &+ \beta^2 E_{ijkl}\hat{q}^i D_t\bar{q}^j D_t\bar{q}^k D_t\bar{q}^l),\end{aligned} \tag{5.30}$$

where we have already used the leading-order fluctuation-dissipation theorem in expressing the first two coefficients through the dissipative tensor. In going to the next order, we need to use the full DKMS transformations (5.28), (5.29) only in the first two terms: the rest of the terms are already subleading, and thus it is sufficient to transform them using the leading-order version (5.25). As in the Abelian case, we derive the constraints order by order in $\beta$. Omitting the details of the calculation we obtain:

**Order $\beta^0$:**

$$l_{ijk} = -\gamma_{(im}C^m_{\ \ j)k}\,, \qquad B_{ijkl} = iA_{ijkl} + \frac{1}{3}\gamma_{(im}C^m_{\ \ jn}C^n_{\ \ k)l}\,. \tag{5.31}$$

We observe that these relations coincide with the linear-response result (4.49) once we identify $A_{ijkl} = (4i/3)\mu_{ijkl}$.

**Order $\beta^1$** does not lead to any constraints.

**Order $\beta^2$:**

$$f_{ij} = -\frac{i}{2}e_{ij} - \frac{1}{6}\gamma_{ij}\,, \tag{5.32a}$$

$$o_{ijk} = -\frac{1}{12}\gamma_{in}C^n_{\ \ jk} - \frac{1}{4}\gamma_{jn}C^n_{\ \ ik} - \frac{1}{4}\gamma_{kn}C^n_{\ \ ij} + \frac{i}{4}(n_{ikj} - n_{jik} - n_{kij})\,, \tag{5.32b}$$

$$E_{ijkl} = \frac{i}{2}D_{i(jkl)} + \frac{i}{8}A_{ijkl} - \frac{1}{12}\gamma_{n(j}C^n_{\ \ km}C^m_{\ \ l)i} \tag{5.32c}$$

**Order $\beta^3$** produces the condition (5.22).

One can verify that the coefficients in the linear-response expressions (4.56), (4.68) satisfy all above constraints. They do not, however, provide the general solution. The latter is parameterized by the symmetric tensors $\gamma_{ij}$ and $A_{ijkl}$, the tensor $D_{ijkl}$ with the symmetry $D_{ijkl} = D_{(ij)(kl)}$, and the tensor $n_{ijk}$ without any symmetry, but which vanishes upon complete symmetrization. Yet more free parameters appear in the odd sector.

### Odd orders

Here the general expression has the form,

$$\mathcal{I}_{\text{odd}} = \int dt\big(c_{ij}\hat{q}^i\dot{\hat{q}}^j + \beta d_{ij}\hat{q}^i\partial_t D_t\bar{q}^j + \beta^{-1}g_{ijk}\hat{q}^i\hat{q}^j\hat{q}^k + h_{ijk}\hat{q}^i\hat{q}^j D_t\bar{q}^k + \beta m_{ijk}\hat{q}^i D_t\bar{q}^j D_t\bar{q}^k\big) \ . \tag{5.33}$$

Since we neglect terms $O(\hat{q}^5)$, it is sufficient to consider only the leading-order DKMS transformations (5.25). Then at different orders in $\beta$ we obtain the conditions:

**Order $\beta^0$:**

$$c_{ij} = 0 \ , \qquad h_{ijk} = \frac{3i}{4}g_{ijk} \ , \tag{5.34}$$

**Order $\beta^1$** yields no conditions.

**Order $\beta^2$:**

$$d_{ij} = d_{(ij)} \ , \qquad g_{ijk} = -8m_{(ijk)} \ . \tag{5.35}$$

We observe that these are the same relations as in the Abelian case, see the first equations in (5.20), (5.21). As in that case, $d_{ij}$ can be absorbed into redefinition of the kinetic Lagrangian of the body. Since the euations of motion contain a contribution quadratic in the covariant velocities (cf. Eq. (4.65)), we need to simultaneously redefine the tensor $m_{ijk}$:

$$m_{ijk} \mapsto \tilde{m}_{ijk} = m_{ijk} + d_{n(j}C^n{}_{k)i} \ . \tag{5.36}$$

Note that such redefinition is compatible with the DKMS conditions and does no affect $g_{ijk}$ or $h_{ijk}$.

## 5.4 Reduction for ohmic friction

The freedom present in the general influence functional is significantly reduced when the classical response of the heat bath is strictly ohmic. This corresponds to setting all subleading terms linear in $\hat{q}$ to zero:

$$f_{ij} = o_{ijk} = E_{ijkl} = d_{ij} = m_{ijk} = 0 \ . \tag{5.37}$$

Due to Eqs. (5.34), (5.35) the odd part of the SK functional then vanishes identically. Whereas in the even sector all coefficients get expressed in terms of $\gamma_{ij}$ and a four-tensor $\nu_{ijkl}$ with the symmetries

$$\nu_{ijkl} = \nu_{(ij)(kl)} = \nu_{(kl)(ij)} \ . \tag{5.38}$$

.

Let us work out these expressions. First, Eq. (5.32a) implies for vanishing $f_{ij}$ the relation

$$e_{ij} = \frac{i}{3}\gamma_{ij} \ . \tag{5.39}$$

Second, Eq. (5.32b) for zero $o_{ijk}$ can be written as

$$n_{ikj} + \frac{i}{3}\gamma_{in}C^n{}_{jk} = (n_{jik} - i\gamma_{jn}C^n{}_{ik}) + (n_{kij} - i\gamma_{kn}C^n{}_{ij}) \ . \tag{5.40}$$

The r.h.s. is symmetric in $(j \leftrightarrow k)$, so the same holds for the tensor

$$\tilde{n}_{ijk} = n_{ijk} + \frac{i}{3}\gamma_{in}C^n{}_{kj} \ . \tag{5.41}$$

Substituting this into the previous equation we have

$$\tilde{n}_{ijk} = \tilde{n}_{jik} + \tilde{n}_{kij} - \frac{2i}{3}\gamma_{jn}C^n{}_{ik} - \frac{2i}{3}\gamma_{kn}C^n{}_{ij} \tag{5.42}$$

On the other hand, due to Eq. (5.22), $\tilde{n}_{ijk}$ obeys the cyclic property, so

$$\tilde{n}_{ijk} = -\tilde{n}_{jki} - \tilde{n}_{kij} \ . \tag{5.43}$$

Combining these relations we obtain

$$\tilde{n}_{ijk} = -\frac{i}{3}\gamma_{jn}C^n{}_{ik} - \frac{i}{3}\gamma_{kn}C^n{}_{ij} \ , \tag{5.44}$$

and finally

$$n_{ijk} = \frac{i}{3}(\gamma_{in}C^n{}_{jk} - \gamma_{jn}C^n{}_{ik} - \gamma_{kn}C^n{}_{ij}) \ . \tag{5.45}$$

This is precisely the form derived in the linear response theory, cf. Eq. (4.56). This term is a firm prediction of the theory determined only by $\gamma_{ij}$ and the structure constants.

For quartic terms, Eq. (5.32c) with vanishing l.h.s. implies,

$$D_{i(jkl)} + \frac{i}{6}\gamma_{n(j}C^n{}_{km}C^m{}_{l)i} = -\frac{1}{4}A_{ijkl} \ . \tag{5.46}$$

Let us introduce

$$\nu_{ijkl} = 3iD_{ijkl} - \frac{1}{4}(\gamma_{nj}C^n{}_{km}C^m_{li} + \gamma_{ni}C^n{}_{km}C^m_{lj} + \gamma_{nj}C^n{}_{lm}C^m_{ki} + \gamma_{ni}C^n{}_{lm}C^m{}_{kj}) \ . \tag{5.47}$$

Clearly, this satisfies the first symmetry property in (5.38). At the same time we have

$$\nu_{i(jkl)} = -\frac{3i}{4}A_{ijkl} \ . \tag{5.48}$$

Then using the derivation from the end of Sec. 5.2.2 we find that it also possesses the second symmetry in (5.38). Clearly, $D_{ijkl}$, $A_{ijkl}$ and hence $B_{ijkl}$ are expressed through it, so it completely determines the quartic sector of the SK functional.

In the next chapter, we show that the generalization of the bulk model of chapter 3 provides the influence functional with a general $\nu_{ijkl}$ with the properties (5.38). This proves that there are no further restrictions imposed by Ohmic friction.

# Chapter 6

# The Bulk Model: The Shwinger-Keldysh Approach

In this chapter, we come back to bulk model. This time, we relax the assumption of the purely outgoing solution to address the corrections and noise and quantum correction via the bulk model. Using the Schwinger-Keldtsh technique, we integrate out the bath and show that the obtained influence functional matches with the prediction of the dynamical KMS symmetry of the previous chapter for the strictly Ohmic dissipation.

Let's recall the classical bulk model which is restricted to the motion on coset with time-reversal symmetry; it is the action functional of a nonlinear sigma model without the gyroscopic term term.

$$\begin{aligned} S[\chi(z,t)] = \frac{1}{2}\int_{z,t}\Big(&\gamma_{ij}D_\mu\chi^i(z,t)D^\mu\chi^j(z,t) \\ &+\frac{2\tilde{\nu}_{ijkl}}{M^2}D_\mu\chi^i(z,t)D^\mu\chi^j(z,t)D_\mu\chi^k(z,t)D^\mu\chi^l(z,t)\Big), \end{aligned} \tag{6.1}$$

where a higher order term has been added as based on the symmetry argument, we can have it in the Lagrangian. The energy scale $M$ has been introduced to make the coupling constant $\tilde{\nu}_{ijkl}$ dimensionless. Adding this term at the classical level does not change the form of the Ohmic dissipative force as it vanishes onsell. The higher order term, however, can be important in addressing the noise and quantum corrections.

We consider the bulk at temperature $T = \frac{1}{\beta}$ with fixed values of the fields at the boundary $\chi_\pm|_{z=0} = q_\pm$. With this, the influence functional of the system at $z = 0$ reads

$$\mathrm{e}^{i\mathcal{I}[q_+,q_-]} = \int_{\chi_i^\pm(z=0)=q_\pm}\rho(\chi_i^+(z),\chi_i^-(z))\int_{\chi_\pm(z,t_i)=\chi_i^\pm(z)}^{\chi_+(z,t_f)=\chi_-(z,t_f)} e^{iS[\chi_+]-iS[\chi_-]}. \tag{6.2}$$

Note the boundary conditions on $\chi$'s at $z = 0$: this is where the dependence on the system's coordinates $q$ comes in. Since the density matrix for non-linear sigma model can also be recast in the form of the (Euclidean) path integral, the computation of $\mathcal{I}$ reduces to evaluating two path integrals. Having obtained experience with dynamical KMS symmetry we will organize the computation as a perturbation series in derivatives *and* (suitably defined) quantum variables. We will not be able to integrate anything exactly, but we can make progress with saddle point approximation.

## 6.1 geometric formulation

Let's start with the leading order operators and ignore the $\tilde{\nu}_{ijkl}$ at the leading order. We will come back to this higher order term at the final section. In this case, we can get help from the geometrical structure of the nonlinear sigma model to obtain the influence functional. Let's start start with a few definitions. Firstly, we introduce the *metric* in the target space

$$g_{ij}(\chi) = \gamma_{i'j'}\Omega^{i'}{}_i(\chi)\Omega^{j'}{}_j(\chi)\,. \tag{6.3}$$

Secondly, we define a new field $\chi^i(z,t;s) \equiv \chi^i_{(s)}$ that interpolates between $\chi_-$ and $\chi_+$ along the geodesic

$$\begin{aligned}&\frac{d^2\chi^i(z,t;s)}{ds^2} + \Gamma^i_{jk}[\chi]\ \chi^j(z,t;s)\chi^k(z,t;s) = 0\,,\\ &\chi^i(z,t;\pm 1) = \chi^i_\pm(z,t)\end{aligned} \tag{6.4}$$

with $\Gamma$ being Christoffel symbols compatible with the metric $g$. The classical and quantum fields are defined via

$$\Phi^i(z,t) = \chi^i(z,t;s=0) \equiv \text{ classical field } , \tag{6.5a}$$

$$\xi^i(z,t) = \left.\frac{d\chi^i(z,t;s)}{ds}\right|_{s=0} \equiv \text{ quantum field } . \tag{6.5b}$$

For convenience we denote the classical and quantum fields at the boundary by

$$\xi^i(z=0,t) = \zeta^i(t), \quad \Phi^i(z=0,t) = \varphi^i(t). \tag{6.6}$$

Note that the definition of the classical and quantum fields in (6.6) is different from the one in (4.42). The two definitions can be related perturbatively as it is shown in the appendix A6,

$$\varphi^i(t) = \bar{q}^i(t) + \frac{u^i_{i'}(\bar{q})}{4}S^{i'}_{l_1l_2}\hat{q}^{l_1}(t)\hat{q}^{l_2}(t) + O\left(\hat{q}^3\right)\,, \tag{6.7a}$$

$$\Omega^i_{i'}(\varphi)\zeta^{i'} = \hat{q}^i + \left(-\frac{1}{12}S^i_{(l_1j}S^j_{l_2l_3)} + \frac{1}{6}C^i_{(l_1j}S^j_{l_2l_3)}\right)\hat{q}^{l_1}\hat{q}^{l_2}\hat{q}^{l_3} + O(\hat{q}^4)\,. \tag{6.7b}$$

In this equation the tensor S defined through

$$S^{i'}_{l_1l_2} = \gamma^{i'n'}\left(C^s_{n'l_1}\gamma_{sl_2} + C^s_{n'l_2}\gamma_{sl_1}\right)\,. \tag{6.8}$$

Note that, to the leading order in quantum field, the two definitions of classical field coincide while the two definitions of quantum field are related to each other by $\hat{q}^i = \Omega^i_{i'}(\varphi)\zeta^{i'}$.

Finally the condition on $\chi_\pm$ to coincide at $t = t_f$ translates into the requirement

$$\xi\Big|_{t=t_f} = 0\,. \tag{6.9}$$

Fields alone are not enough, we also need their derivatives. Geometrically, one should think of $\Phi^i$'s as new coordinates on the target space manifold, and $\xi^i$'s as (components of) vector fields. Then the usual definition for covariant derivative gives:

$$\nabla_i \xi^j = \frac{\partial}{\partial \Phi^i}\xi^j + \Gamma^j_{ik}(\Phi)\xi^k \tag{6.10}$$

In this equation, the Christoffel symbol $\Gamma^i_{jk}(\Phi)$ defined through

$$\Gamma^i_{jk}(\Phi) = \frac{g^{il}(\Phi)}{2}\left(\partial_j g_{kl}(\Phi) + \partial_k g_{jl}(\Phi) + \partial_l g_{kj}(\Phi)\right) \tag{6.11}$$

where $\partial_a$ stands for $\frac{\partial}{\partial \Phi^a}$. Finally, we will need the Riemann tensor, and we define it in the usual way:

$$R^i{}_{jkl}(\Phi) = \partial_k \Gamma^i_{jl}(\Phi) + \Gamma^i_{kn}(\Phi)\Gamma^n_{jl}(\Phi) - (k \to l) \tag{6.12}$$

In the limit $\xi^i \ll \Phi^i$, the action functionals $S\left[\chi_\pm\right]$ can be written as the functionals of $\Phi^a(z,t)$ and $\xi^a(z,t)$ perturbatively. Taylor expanding the $S[\chi_\pm]$ around the background functional $S[\chi(z,t;s=0)]$, we have

$$\begin{aligned} S[\chi_\pm] &= S[\chi(z,t;s=0)] \\ &\pm \frac{d}{ds}(S[\chi(z,t;s)])|_{s=0} + \frac{1}{2!}\frac{d^2}{ds^2}(S[\chi(z,t;s)])|_{s=0} \pm \frac{1}{3!}\frac{d^3}{ds^3}(S[\chi(z,t;s)])|_{s=0} + \dots\,. \end{aligned} \tag{6.13}$$

This results in series expansion of the Schwinger-Keldysh action functional of the form

$$S\left[\chi_+\right] - S\left[\chi_-\right] = S^{(1)}[\Phi,\xi] + S^{(3)}[\Phi,\xi] + \cdots\,, \tag{6.14}$$

with

$$S^{(1)}[\Phi,\xi] = 2\frac{d}{ds}(S[\chi(z,t;s)])|_{s=0} = \int_{t,z} 2g_{ij}(\Phi)\nabla_\mu \xi^i(z,t)\partial^\mu \Phi^j\,, \tag{6.15a}$$

$$\begin{aligned} S^{(3)}[\Phi,\xi] &= \frac{1}{3}\frac{d^3}{ds^3}(S[\chi(z,t;s)])|_{s=0} \\ &= \frac{1}{3}\int_{t,z} \nabla_m R_{ijkl}(\Phi)\ \xi^j(z,t)\ \xi^k(z,t)\xi^m(z,t)\partial_\mu \Phi^i(z,t)\partial^\mu \Phi^l(z,t) \\ &+ \frac{4}{3}\int_{t,z} R_{ijkl}(\Phi)\ \xi^j(z,t)\ \xi^k(z,t)\partial_\mu \Phi^i(z,t)\nabla^\mu \xi^l(z,t))\,, \end{aligned} \tag{6.15b}$$

where we have used the definition (6.5), the geodesic equation in the target space (6.4), and the definition of the Riemann tensor (6.12). To perform the computation of the path integral (6.39), we first need to calculate the thermal density matrix $\rho$. This is done in the next section where we calculate the high temperature thermal density matrix. The thermal density matrix mixes the forward in time and backward in time waves through imposing boundary conditions at $t = t_i$. The contribution from the thermal density matrix would add another perturbative series (6.14), resulting in an effective action $S_{eff}$. Let's obtain this effective action.

## 6.2 The Density Matrix At High Temperature

The thermal density matrix of the bulk model bath at temperature $T = \frac{1}{\beta}$ can be obtained through the following Euclidean path integral

$$\rho_\beta[\chi_1(z), \chi_2(z)] = \int_{\chi(z,\tau=-\beta/2)=\chi_1(z)}^{\chi(z,\tau=\beta/2)=\chi_2(z)} [D\chi] \; e^{-S_E[\chi(z,\tau)]}, \tag{6.16}$$

where $S_E[\chi(z,\tau)]$ is the Euclidean action of the bath and $\tau$ is the imaginary time. The Euclidean action of the bulk model reads

$$S_E = \frac{1}{2}\int_{-\beta/2}^{\beta/2} d\tau \int_0^\infty dz \; g_{ij}(\chi(z,\tau))\partial_\mu^{(E)}\chi^i(z,\tau)\partial^\mu_{(E)}\chi^j(z,\tau) \tag{6.17}$$

where $g_{ij}(\chi(z,\tau))$ is a metric defined over the field manifold and $\partial_\mu^{(E)}$ is the Euclidean partial derivative

$$\partial_\mu^{(E)} = \partial^\mu_{(E)} \equiv (\partial_\tau, \partial_z)\,.$$

Using the covariant field method, one can expand the field $\chi^i(z,\tau)$ around a background $\chi^i_c(z,\tau)$. We choose the background to be $\chi^i_c(z,\tau) = \Phi^i(z,t_i)$ where $t_i$ stands for the initial time, and $\Phi^i(z,t)$ the same classical field which is defined in (6.5a). Thus the chosen background is independent of $\tau$. To avoid clutter, we show $\Phi^i(z,t_i)$ simply by $\Phi^i(z)$ in this section. We note the path integral (6.16) has a single variable and we use the freedom in choosing it to be the deviation from the classical field $\Phi(z)$. To implement the covariant background method, the field $\chi^i(z,\tau)$ is promoted into a new field $\chi^i(z,\tau;s)$ with $\chi^i(z,\tau;s=0) = \Phi^i(z)$ and $\chi^i(z,\tau;s=1) = \chi^i(z,\tau)$. In addition, the promoted field $\chi^i(z,\tau;s)$ is demanded to satisfy the geodesic equation along the parameter $s$,

$$\frac{d^2\chi^i(z,\tau;s)}{ds^2} + \Gamma^i_{jk}\frac{d\chi^j(z,\tau;s)}{ds}\frac{d\chi^k(z,\tau;s)}{ds} = 0 \tag{6.18}$$

with the Christoffel symbol is defined with respect to the metric $g_{ij}(\chi(z,\tau;s))$. Expanding the field $\chi^j(z,\tau;s)$ around the background field $\Phi^i(z)$ gives the Euclidean action functional in terms of series of functionals of the background field $\Phi^i(z)$ and the tangential to the geodesics $\frac{d\chi^j(z,\tau;s)}{ds}\Big|_{s=0} \equiv \xi^i(z,\tau)$. The series can be written formally as

$$S_E[\chi(z,\tau)] = S_E\left[\chi_c(z)\right] + \left(\frac{d}{ds}S_E[\chi^j(z,\tau;s)]\right)_{s=0} + \frac{1}{2!}\left(\frac{d^2}{ds^2}S_E[\chi^j(z,\tau;s)]\right)_{s=0} + \cdots \tag{6.19}$$

$$\equiv S_E^{(0)} + S_E^{(1)} + S_E^{(2)} + \cdots \ . \tag{6.20}$$

The first functional in the series reads

$$\begin{aligned} S_E^{(0)} = S_E\left[\Phi(z)\right] &= \frac{1}{2}\int_{-\beta/2}^{+\beta/2} d\tau \int_0^\infty dz\ g_{ij}\left(\Phi(z)\right)\partial_\mu^{(E)}\Phi^i(z)\partial^\mu_{(E)}\Phi^j(z) \\ &= \frac{\beta}{2}\int_0^\infty dz\ g_{ij}\left(\Phi(z)\right)\partial_z\Phi^i(z)\partial_z\Phi^j(z) \end{aligned} \tag{6.21}$$

which is independent of $\xi^i(z,\tau)$. Using the definition of the covariant derivatives and the geodesic equation along the $s$ direction, the next to the leading order functional in the series expansion reads

$$S^{(1)} = \int d\tau dz g_{ij}\left(\Phi(z)\right)\partial_z\Phi^i(z)\nabla_z\xi^j(z,\tau). \tag{6.22}$$

Similarly the functional $S_E^{(2)}$ can be calculated as follows

$$S_E^{(2)} = \frac{1}{2}\int d\tau dz[\nabla_\mu^{(E)}\xi^i(z,\tau)\nabla_\mu^{(E)}\xi_i(z,\tau) + R_{ijkl}\xi^j(z,\tau)\xi^k(z,\tau)\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau)] \tag{6.23}$$

where $R_{ijkl}$ is the Riemann curvature tensor defined with respect to the metric $g_{ij}\left(\Phi(z)\right)$. If we change the integration variable in the path integral definition of the thermal density matrix (6.17) from $\chi^i(z,\tau)$ to $\xi^i(z,\tau)$, then the thermal density matrix of the nonlinear sigma model up to quadratic order in $\xi^i(z,\tau)$ reads

$$\rho_\beta\left(\chi_1(z),\chi_2(z)\right) = \exp\left(-\frac{\beta}{2}\int_0^\infty dz g_{ij}\left(\Phi(z)\right)\partial_z\Phi^i(z)\partial_z\Phi^j(z)\right) \tag{6.24}$$

$$\times\int_{\xi(z,-\beta/2)=\xi_1(z)}^{\xi(z,\beta/2)=\xi_2(z)} e^{(-S_E^{(1)}-S_E^{(2)})}. \tag{6.25}$$

We need to write $\chi_1(z)$ and $\chi_2(z)$ in terms of $\xi_1(z)$, $\xi_2(z)$ and $\Phi^i(z)$. To do so, we need to write the covariant expansion of $\chi_1(z)$ and $\chi_2(z)$ around the background field $\Phi(z)$ :

$$\chi_1^i(z) = \Phi^i(z) + \xi_1^i(z) - \frac{1}{2!}\Gamma^i_{jk}(\Phi)\xi_1^j(z)\xi_1^k(z) + \cdots \tag{6.26}$$

$$\chi_2^i(z) = \Phi^i(z) + \xi_2^i(z) - \frac{1}{2!}\Gamma^i_{jk}(\Phi)\xi_2^j(z)\xi_2^k(z) + \cdots \tag{6.27}$$

One can use the saddle point approximation to calculate the thermal density matrix of the nonlinear sigma model. Variation of the (6.24) with respect to $\xi(z,\tau)$ (see the appendix A5 for the details) results in

$$\partial_\tau^2\bar{\xi}^j(z,\tau) = -\nabla_z\partial_z\Phi^j(z) - \nabla_z^2\bar{\xi}^j(z,\tau) + g^{jb}(\Phi(z))R_{ibkl}(\Phi(z))\bar{\xi}^k(z,\tau)\partial_z\Phi^i(z)\partial_z\Phi^l(z) \tag{6.28a}$$

$$\bar{\xi}^i(z,\tau=-\beta/2) = \xi_1^i(z) \tag{6.28b}$$

$$\bar{\xi}^i(z,\tau=+\beta/2) = \xi_2^i(z) \tag{6.28c}$$

We can solve this nonlinear differential equation perturbatively. The perturbative factor is the gradient expansion along the $z$ direction. With this criteria, the leading order solution satisfies a linear differential equation

$$\partial_\tau^2\bar{\xi}^j_{(0)}(z,\tau) = 0 \tag{6.29}$$

$$\bar{\xi}^i_{(0)}(z,\tau=-\beta/2) = \xi_1^i(z) \tag{6.30}$$

$$\bar{\xi}^i_{(0)}(z,\tau=+\beta/2) = \xi_2^i(z) \tag{6.31}$$

with the following solution

$$\bar{\xi}^i_{(0)}(z,\tau) = (\xi_2^i(z) - \xi_1^i(z))\frac{\tau}{\beta} + \left(\frac{\xi_1^i(z)+\xi_2^i(z)}{2}\right) \tag{6.32}$$

We are interested in the case where $\xi_1^i(z) = -\xi_2^i(z) = -\xi(z,t_i)$ (in this section, we use $\xi(z,t_i)$ and $\xi(z)$ interchangeably) which results in

$$\bar{\xi}^i_{(0)}(z,\tau) = -\frac{2\tau}{\beta}\xi^i(z) \tag{6.33}$$

The correction to this solution, $\bar{\xi}^i_{(1)}(z,\tau)$, satisfies the following differential equation

$$\partial_\tau^2\bar{\xi}^j_{(1)}(z,\tau) = -\nabla_z\partial_z\Phi^j(z) - \nabla_z^2\bar{\xi}^j_{(0)}(z,\tau) + g^{jb}(\Phi)R_{ibkl}(\Phi(z))\bar{\xi}^k_{(0)}(z,\tau)\partial_z\Phi^i(z)\partial_z\Phi^l(z) \tag{6.34}$$

$$\bar{\xi}^i_{(1)}(z,\pm\beta/2) = 0. \tag{6.35}$$

The solution to the correction reads

$$\bar{\xi}^j_{(1)}(z,\tau) = -\frac{1}{2}\left(\tau^2 - \frac{\beta^2}{4}\right)\nabla_z\partial_z\Phi^j(z) + \frac{1}{3\beta}\left(\tau^3 - \frac{\tau\beta^2}{4}\right)\nabla_z^2\xi^j(z)$$
$$-\frac{1}{3\beta}\left(\tau^3 - \frac{\tau\beta^2}{4}\right)g^{jb}R_{ibkl}\xi^k(z)\partial_z\Phi^i(z)\partial_z\Phi^l(z) \tag{6.36}$$

Inserting this into the equation for the path integral definition of the density matrix (6.24), we obtain the following expression for the high temperature thermal density matrix,

$$\rho = e^{iS_\rho[\Phi(z,t_i),\xi(z,t_i)]}, \tag{6.37}$$

where the exponent reads

$$iS_\rho[\Phi,\xi] = -\int_0^\infty dz\,\left(\frac{\beta}{2}\left(\partial_z\Phi^j(z)\right)^2 + \frac{2}{\beta}(\xi^j)^2 - \frac{\beta^3}{24}\left(\nabla_z\partial_z\Phi^j(z)\right)^2\right.$$
$$\left.+\frac{\beta}{6}\left(\nabla_z\xi^j(z)\right)^2 + \frac{\beta}{6}R_{ijkl}\xi^j(z)\xi^k(z)\partial_z\Phi^i(z)\partial_z\Phi^l(z)\right) \tag{6.38}$$

Note that $\rho$ and correspondingly $S_\rho$ are local in time and are evaluated at $t = t_i$. With this, the influence functional (6.2) reads,

$$\exp\left(i\mathcal{I}\left[\varphi,\zeta\right]\right) = \int\left[d\xi^i\left(z,t_i\right)\right]\left[d\Phi^i\left(z,t_i\right)\right]\left[d\Phi^i(z,t)\right]\left[d\xi^i(z,t)\right]\exp\left(iS_{eff}\right), \tag{6.39}$$

where the effective action is

$$S_{eff} = S_\rho[\Phi(z,t_i),\xi(z,t_i)] + S^{(1)}[\Phi(z,t),\xi(z,t)] + S^{(3)}[\Phi(z,t),\xi(z,t)] + \ldots\,. \tag{6.40}$$

Note that the perturbation series in terms of $\xi$ mixes the expansion in terms of $\beta D_t\bar{q} \sim \hat{q}$. Therefore, we need to be cautious in including the right order of expansion in terms of $\xi$ to be consistent with the power counting fixed by (4.43). For instance, in the next section, we will see that the functional (6.21) and the first term in (6.23) generate corrections to the influence functional which are of the same order. In the expansion (6.40), we have ignored those terms which produce corrections of order $(\beta D_t\bar{q})^5 \sim \hat{q}^5$.

In the next section, we calculate the leading order (or Gaussian) influence functional by including (6.21), the first term in (6.23), and (6.15a) in the expansion of the effective action (6.40).

## 6.3 The Leading Order Influence Functional

The leading order influence functional $\mathcal{I}_0$ in (6.39) corresponds with the following leading order effective action

$$S^{(leading)}_{eff} = S^{(leading)}_\rho + S^{(leading)}_\chi \tag{6.41}$$

where

$$S_{\rho}^{(leading)} = +i\int_{z=0}^{\infty} g_{ij}\left(\Phi\left(z,t_i\right)\right)\left(\frac{\beta}{2}\partial_z\Phi^i\left(z,t_i\right)\partial_z\Phi^j\left(z,t_i\right)+\frac{2}{\beta}\xi^i\left(z,t_i\right)\xi^j\left(z,t_i\right)\right)\ ,$$
$$S_{\chi}^{(leading)} = +2\int_{t,z}\nabla_\mu\xi_i(z,t)\partial^\mu\Phi^i(z,t)\ . \tag{6.42}$$

In calculating $\mathcal{I}_0$, we will use the saddle point approximation. At the saddle point approximation we have,

$$\mathcal{I}_0[\varphi(t),\zeta(t)] = S_{eff}^{(leading)}[\bar{\Phi}(z,t),\bar{\xi}(z,t)], \tag{6.43}$$

where $\bar{\Phi}(z,t)$ and $\bar{\xi}(z,t)$ are the saddle point solutions of the path integral (6.39), with the leading order approximation $S_{eff}\approx S_{eff}^{(leading)}$. It is not trivial that the left hand side of equation (6.43) is localized in time as the right hand side of the equation has functionality of both $t$ and $z$. We will see that it is indeed the case because of the high temperature approximation which is used to obtain the influence functional.

The saddle point of the path integral can be obtained by variation with respect to the classical field $\Phi$ and the quantum field $\xi$. This leads to two coupled differential equations, and two initial conditions at $t=t_i$ which will be solved alongside the boundary conditions (6.6) at $z=0$. Variation of $S_{eff}^{(leading)}$ with respect to the quantum field $\xi_i$ results in the following constraint of the saddle point solution at the initial time $t=t_i$

$$-\frac{2}{\beta}\xi^l\left(z,t_i\right)-i\partial_t\Phi^l\left(z,t_i\right)=0\Rightarrow\frac{2i}{\beta}\xi^l\left(z,t_i\right)=\partial_t\Phi^l\left(z,t_i\right)\ , \tag{6.44}$$

and the following saddle point equation of motion for the classical field

$$\nabla_\mu\partial^\mu\Phi^l(z,t) = -\partial^z\Phi^l(z,t)\delta(z) = \partial_z\Phi^l(z,t)\delta(z)\ . \tag{6.45}$$

Variation of $S_{eff}^{(leading)}$ with respect to the classical field $\Phi^l$ and using (6.44),(6.45) results in another boundary condition of the saddle point solution at $t=t_i$,

$$\frac{2i}{\beta}\nabla_t\xi^l\left(z,t_i\right) = g_{lk}\left(\Gamma_{ij}^k\partial_z\Phi^i\left(z,t_i\right)\partial_z\Phi^j\left(z,t_i\right)+\partial_z^2\Phi^k\left(z,t_i\right)\right)\equiv\nabla_z\partial_z\Phi^l\left(z,t_i\right)\ , \tag{6.46}$$

and the saddle point equation of motion for the quantum field $\xi^i(z,t)$ of the form

$$-\nabla_\mu\nabla^\mu\xi_l + R_{jikl}(\Phi)\xi^j\partial_\mu\Phi^i\partial^\mu\Phi^k = -\delta(z)\nabla_z\xi_l\ . \tag{6.47}$$

As an illustrative example, we obtain the leading order influence functional of the harmonic bath perturbatively. Although the case of the harmonic bath is exactly solvable [Kamenev 2023], we develop a perurbative method which is applicable to the case of nonlinear bath. Then, we will obtain the saddle point solution for the nonlinear bath using the developed techniques.

### 6.3.1 Harmonic bath

For $g_{ij} = \gamma\delta_{ij}$ the equations of motion are just wave equations

$$\partial^2\xi^i = 0, \quad \partial^2\Phi^i = 0 \,, \tag{6.48}$$

with initial conditions

$$\xi^i\Big|_{t_i} = -\frac{i\beta}{2}\partial_t\Phi^i\Big|_{t_i} \,, \tag{6.49a}$$

$$\partial_t\xi^i\Big|_{t_i} = -\frac{i\beta}{2}\partial_z^2\Phi^i\Big|_{t_i} \,, \tag{6.49b}$$

where we have dropped the delta functions since the boundary values of the fields at $z = 0$ are already fixed by (6.6). Note that the initial conditions (6.49) are asymmetrical in classical and quantum fields. Most naturally they are interpreted as Cauchy data for $\xi$ in terms of $\Phi$, which is left unspecified. Due to this freedom the solution will contain an arbitrary outgoing $\Phi$ wave - classical response of the system.

The situation calls for light-cone coordinates

$$u = z - t, \quad v = z + t \,, \tag{6.50}$$

in terms of which the equations (6.48) are simply

$$\partial_u\partial_v\xi = \partial_u\partial_v\Phi = 0 \tag{6.51}$$

The general solution is a superposition of incoming and outgoing waves

$$\Phi = \Phi^i_{in}(v) + \Phi^i_{out}(u) \,, \quad \xi = \xi^i_{in}(v) + \xi^i_{out}(u) \,. \tag{6.52}$$

The boundary condition (6.9) implies that $\xi^i(u) = 0$. In other words, quantum field satisfies the equation

$$\partial_u\xi^i = 0, \tag{6.53}$$

which means that the quantum field a is purely outgoing field. With this, (6.49b) reads

$$\partial_z\xi^i\Big|_{t_i} = -\frac{i\beta}{2}\partial_z^2\Phi^i\Big|_{t_i}. \tag{6.54}$$

Integrating both sides of this equation along the $z$ directions, and using the fact that the fields at $z \to \infty$ vanishes, we have,

$$\xi^i\Big|_{t_i} = -\frac{i\beta}{2}\partial_z\Phi^i\Big|_{t_i} \,. \tag{6.55}$$

Combining this with (6.49a), results in

$$\partial_v\Phi^i|_{t_i} = \frac{2i}{\beta}\xi^i|_{t_i} \,, \tag{6.56a}$$

$$\partial_u \Phi^i|_{t_i} = 0 \,. \tag{6.56b}$$

Equation (6.56a) implies that the classical field satisfies the following equation

$$\partial_v \Phi^i(v) = \frac{2i}{\beta}\xi^i(v). \tag{6.57}$$

Let us plug these solutions back into the action. It's easy to see that $S_\rho^{(leading)}$ gives zero on-shell. For the $S_\chi^{(leading)}$, integrating by parts and using equations of motion, we get

$$2i\int_{z>0,t}\partial_\mu\Phi^i\partial^\mu\xi^i = 2i\gamma\int_{t,z>0}\partial^\mu(\partial_\mu\Phi^i\xi^i) = -2i\gamma\int_{z>0}\partial_t\Phi^i\xi^i\Big|_{t=t_i} + 2i\gamma\int_t\partial_z\Phi^i\xi^i\Big|_{z=0} \tag{6.58}$$

Using (6.49a), for the first term we have

$$-2i\gamma\int_{z>0}\partial_t\Phi^i\xi^i\Big|_{t=t_i} = \frac{4\gamma}{\beta}\int_{z>0}\xi^i\xi^i\Big|_{t=t_i} \tag{6.59}$$

By definition $\xi\Big|_{z=0} = \zeta$ and then on-shell

$$\xi(z,t_i) = \xi(z+t_i) = \zeta(z+t_i)\,. \tag{6.60}$$

Going back to (6.59) we get

$$\frac{4\gamma}{\beta}\int_{z>0}\xi^i\xi^i\Big|_{t=t_i} = \frac{4\gamma}{\beta}\int_{v=t_i}^{\infty}\zeta^i(v)\zeta^i(v) \xrightarrow{t_i\to-\infty} \frac{4\gamma}{\beta}\int_{t=-\infty}^{\infty}\zeta^i(t)\zeta^i(t) \tag{6.61}$$

For the second term we need the expression for the $\partial_z\Phi$ on the boundary

$$\partial_z\Phi\Big|_{z=0} = (2\partial_v - \partial_t)\Phi\Big|_{z=0} = \frac{4i}{\beta}\xi\Big|_{z=0} - \partial_t\varphi = \frac{4i}{\beta}\zeta - \dot{\varphi}\,, \tag{6.62}$$

where in the second step we have used (6.57). Plugging everything in we obtain the influence functional

$$i\mathcal{I}_0[\varphi,\zeta] = -\frac{4}{\beta}\gamma\int_t\zeta^i(t)\zeta^i(t) - 2i\gamma\int_t\dot{\varphi}^i(t)\zeta^i(t)\,. \tag{6.63}$$

Since at leading order in quantum variables $\zeta = \hat{q}$ and $\varphi = \bar{q}$, the equation (6.63) is precisely the influence functional predicted by linear response theory and the DKMS for Abelian groups. In the next subsection, we will see that the Gaussian influence functional of a general non-Abelian group is the covarintized version of (6.63) for a general metric.

### 6.3.2 Non-harmonic bath

In $u - v$ coordinates the equations of motion for a general metric are[1]

$$\nabla_u \partial_v \Phi^i(z,t) = \nabla_v \partial_u \Phi^i(z,t) = 0 \;, \tag{6.64a}$$

$$(\nabla_v \nabla_u + \nabla_u \nabla_v)\xi_l(z,t) + R_{ijkl}(\partial_u \Phi^i(z,t)\partial_v \Phi^k(z,t) + \partial_v \Phi^i(z,t)\partial_u \Phi^k(z,t))\xi^j(z,t) = 0 \;, \tag{6.64b}$$

with initial conditions

$$\xi^i(z,t_i) = -\frac{i\beta}{2}\partial_t \Phi^i(z,t_i) \;, \tag{6.65a}$$

$$\nabla_t \xi^i\Big|_{t_i} = -\frac{i\beta}{2}\nabla_z \partial_z \Phi^i\Big|_{t_i} \;. \tag{6.65b}$$

Note that the first condition is the same as in the harmonic bath. To find the solution, we rewrite (6.64b) in an equivalent form. Using the commutator of the covariant derivatives

$$(\nabla_u \nabla_v - \nabla_v \nabla_u)\xi_l = R_{lmab}\; \xi^m \partial_u \Phi^a \partial_v \Phi^b \tag{6.66}$$

and the symmetries of the Riemann tensor, equation (6.64b) is rephrased as

$$\nabla_v \nabla_u \xi_l - R_{labm}\xi^m \partial_u \Phi^a \partial_v \Phi^b = 0 \;. \tag{6.67}$$

The suggested ansatz is the covariantized version of the flat metric solutions,

$$\nabla_u \xi^i = 0, \tag{6.68a}$$

$$\partial_v \Phi^i = \frac{2i}{\beta}\xi^i. \tag{6.68b}$$

It's pretty straightforward to show that the suggested ansatz indeed solves the equations of motion (6.64), if we use the (6.67) instead of (6.64b), and the antisymmetricity of the Riemann tensor with respect to its last two indices. Using (6.68a), the equation (6.65b) is rewritten as

$$\nabla_z \xi^i\Big|_{t_i} = -\frac{i\beta}{2}\nabla_z \partial_z \Phi^i\Big|_{t_i}, \tag{6.69}$$

which results in the fact that the following is a constant with respect to $z$

$$(\xi^i\Big|_{t_i} + \frac{i\beta}{2}\partial_z \Phi^i\Big|_{t_i})^2 = const(z), \tag{6.70}$$

where the constant turns out to be zero because of the fields vanish at $z \to \infty$. This implies that following identity which we had in the case of the harmonic bath with flat metric

$$\xi^i\Big|_{t_i} + \frac{i\beta}{2}\partial_z \Phi^i\Big|_{t_i} = 0. \tag{6.71}$$

[1] We are using the (anti)symmetries of the Riemann tensor in some of the upcoming equations.

Combining this with (6.65a) results in the following condition at $t = t_i$

$$\partial_v \Phi^i|_{t_i} = \frac{2i}{\beta}\xi^i\Big|_{t_i}, \ \partial_u \Phi^i|_{t_i} = 0 \ , \tag{6.72}$$

which shows the compatiblity of the suggested ansatz (6.68) with the boundary conditions (6.65).

Having obtained the solutions let us compute the influence functional. As for the flat case the density matrix part does not contribute and we still have

$$\mathcal{I}\Big|_{\text{on-shell}} = -2i\int_z \big(g_{ij}\partial_t\Phi^i\xi^j\big)|_{t=t_i} + 2i\int_t \big(g_{ij}\partial_z\Phi^i\xi^j\big)\Big|_{z=0} \tag{6.73}$$

The second term here is dealt with the same way as in flat case. The first term here is a bit trickier. Using (6.65a) we get

$$\frac{4}{\beta}\int_z g_{ij}(\Phi)\xi^i\xi^j \ . \tag{6.74}$$

According to (6.68a) the field $\xi$ is parallel transported along $u$. This implies

$$g_{ij}(\Phi)\xi^i\xi^j\Big|_{z=\frac{u+v}{2},t=\frac{v-u}{2}} = g_{ij}(\Phi)\xi^i\xi^j\Big|_{z=\frac{u'+v}{2},t=\frac{v-u'}{2}} \tag{6.75}$$

We apply this formula for $u = z - t_i$, $v = z + t_i$ and $u' = -v$. Then we have

$$\begin{aligned} g_{ij}(\Phi(z,t_i))\xi^i(z,t_i)\xi^j(z,t_i) &= g_{ij}(\Phi(0,z+t_i))\xi^i(0,z+t_i)\xi^j(0,z+t_i) \\ &= g_{ij}(\varphi(z+t_i))\zeta^i(z+t_i)\zeta^j(z+t_i) \end{aligned} \tag{6.76}$$

where we used the boundary conditions at $z = 0$ in the second line. Plugging this into (6.73) we finally obtain

$$i\mathcal{I}_0[\varphi,\zeta] = -\frac{4}{\beta}\int_t g_{ij}(\varphi(t))\zeta^i(t)\zeta^j(t) - 2i\int_t g_{ij}(\varphi(t))\dot{\varphi}(t)\zeta(t) \ . \tag{6.77}$$

The result is the covariantized version of the harmonic bath influence functional (6.63) for a general metric $g_{ij}(\varphi)$. The equation (6.77) is the most general influence functional at the classical limit. Note that this result is valid even in the case that the dynamics is not parameterized by a group manifold. In addressing the quantum corrections, however, the group structure is important to classify the quantum corrections which we compute in the next section.

## 6.4 The Quantum Correction To The Influence Functional

The quantum correction to the influence functional can be obtained by inserting the leading order saddle point solution to the effective action $S_{eff}$ which includes the next to the leading order terms. Note that there is no need to obtain the saddle point solution from the full action $S_{eff}$ as the error is of the higher order. With this, let's calculate the additional contribution in $S_{eff}[\bar{\Phi}(z,t),\bar{\xi}(z,t)]$ in terms of the classical and quantum fields, $\varphi(t)$ and $\zeta(t)$. The correction is due to two kinds of terms, one from the expansion of the density matrix and the other one from the expansion of the bulk action (6.15b).

**Term I:** The first correction comes from the next to the leading order terms in (6.38),

$$\begin{aligned}\delta S_\rho = i\int_0^\infty dz\,\Big[ &-\frac{\beta^3}{24}(\nabla_z\partial_z\Phi^j(z,t_i))^2+\frac{\beta}{6}(\nabla_z\xi^j(z,t_i))^2 \\ &+\frac{\beta}{6}R_{ijkl}\xi^j(z,t_i)\xi^k(z,t_i)\partial_z\Phi^i(z,t_i)\partial_z\Phi^l(z,t_i)\Big].\end{aligned} \tag{6.78}$$

Using the leading order saddle point solution (6.68), we see that the last term in (6.78) vanishes. The other terms combine to

$$\delta S_\rho = i\frac{\beta}{3}\int_0^\infty dz\ (\nabla_z\xi^j(z,t_i))^2 \equiv i\frac{\beta}{3}\int_0^\infty dz\ (\nabla_v\xi^j(z,t_i))^2\ , \tag{6.79}$$

where we have used (6.68a) in the last step. Combined with the other term, (6.79) maps to a term on the boundary.

**Term II:** The next correction to the influence functional comes from $S^{(3)}$ in (6.15b). Writing it in the u-v coordinate, using the equations (6.68), and then implementing the symmetries of the Riemann curvature tensor, it results in the following expression

$$\delta S_\chi = \frac{4}{3}\int_{u,v} R_{ijkl}(\Phi)\xi^j(z,t)\xi^k(z,t)\Big(\partial_u\Phi^i(z,t)\nabla_v\xi^l(z,t)\Big). \tag{6.80}$$

To simplify this equation, we need to use an identity which is obtained by combining (6.64b), the commutator (6.66), and (6.68b). This identity reads

$$\nabla_u\nabla_v\xi_l = \frac{2i}{\beta}R_{jikl}\xi^j\partial_u\Phi^i\xi^k\ , \tag{6.81}$$

and using it, the equation (6.80) is rewritten as

$$\delta S_\chi = \frac{2i\beta}{3}\int_v\int_u(\nabla_u\nabla_v\xi_l)\nabla_v\xi^l \equiv \frac{i\beta}{3}\int_v\int_u\partial_u(\nabla_v\xi_l)^2\ , \tag{6.82}$$

where in the last step we have used the fact that the covariant derivative of a scalar is the normal derivative. The integral over $u$ can be done easily as follows

$$\begin{aligned}\delta S_\chi &= \frac{i\beta}{3}\int_{v=t_i}^\infty\int_{u=2t_i-v}^{v}\partial_u(\nabla_v\xi_l)^2 \\ &= +\frac{i\beta}{3}\int_{v=t_i}^\infty dv\big[(\nabla_v\xi_l)^2\big|_{z=0}-(\nabla_v\xi_l)^2\big|_{t=t_i}\big]\ .\end{aligned} \tag{6.83}$$

Combining this with (6.79), and using (6.68a) and the boundary condition at $z = 0$, we obtain the next to leading order correction to the influence functional of the nonlinear bath model,

$$i\mathcal{I}_{NLO}[\varphi(t), \zeta(t)] = -\frac{\beta}{3} \int_{t=t_i}^{t=t_f} dt \; g_{ij}(\varphi) \nabla_t \zeta^i(t) \nabla_t \zeta^j(t) \; . \tag{6.84}$$

This expression, alongside (6.77) must be compared with the form (5.30) predicted by the DKMS method. We see the difference arising from the different definitions of the classical and quantum fields in this chapter and in chapter 5. In the next section, we prove that upon the field redefinition the influence functional of the bulk model takes the form which is predicted by the DKMS symmetry in the strictly Ohmic regime.

## 6.5 Matching With DKMS Method

As we showed, the influence functional of the bulk model up to the next to the leading correction reads

$$\begin{aligned} & i\mathcal{I}[\varphi(t), \zeta(t)] \\ & = -\frac{4}{\beta} \int_t g_{ij}(\varphi(t)) \zeta^i(t) \zeta^j(t) - 2i \int_t g_{ij}(\varphi(t)) \dot{\varphi}(t) \zeta(t) - \frac{\beta}{3} \int_{t=t_i}^{t=t_f} dt g_{ij}(\varphi) \nabla_t \zeta^i(t) \nabla_t \zeta^j(t). \end{aligned} \tag{6.85}$$

To map this form of the influence functional to the one in (5.30), we need to relate the two definitions of classical and quantum using the equations in (6.7). The technical details are derived in the appendix A6. In what follows, we use a notation

$$\tilde{\zeta}^i = \Omega^i_j(\varphi) \zeta^j \tag{6.86}$$

for convenience.

Using the relations (6.7b), the first term in (6.85) reads

$$\begin{aligned} & -\frac{4}{\beta} \int_{t_i}^{t_f} g_{ij}(\varphi) \zeta^i \zeta^j \\ & = -\frac{4}{\beta} \int_{t_i}^{t_f} \gamma_{ij} \hat{q}^i(t) \hat{q}^j(t) - \frac{2}{3\beta} \int_{t_i}^{t_f} \gamma_{j(l_4} C^j_{l_1 k} S^k_{l_2 l_3)} \hat{q}^{l_1} \hat{q}^{l_2} \hat{q}^{l_3} \hat{q}^{l_4} + \int_{t_i}^{t_f} O(\hat{q}^5) \, , \end{aligned} \tag{6.87}$$

where the tensor $S^i_{jk}$ is defined in (6.8). Using (6.7b) and (A.53a), the next term in the influence functional (6.85) is rearranged as follows

$$
\begin{aligned}
& -2i\int_{t_i}^{t_f} g_{ij}(\varphi)\partial_t\varphi^i(t)\zeta^j(t) \\
& = -2i\int_{t_i}^{t_f} \gamma_{ij}D_t\bar{q}^i\hat{q}^j - i\int_{t_i}^{t_f}\gamma_{i(l_2}C^i_{l_3)l_1}\partial_t\hat{q}^{l_1}\hat{q}^{l_2}\hat{q}^{l_3} \qquad (6.88)\\
& +\frac{i}{3}\int_{t_i}^{t_f}\left[-\gamma_{i(l_1}S^j_{l_2l_3)}C^i_{kj}+\gamma_{ik}C^i_{j(l_1}S^j_{l_2l_3)}+\gamma_{i(l_1}C^i_{l_2j}C^j_{l_3)k}\right]D_t\bar{q}^k\hat{q}^{l_1}\hat{q}^{l_2}\hat{q}^{l_3}+O\left(\hat{q}^5\right). \qquad (6.89)
\end{aligned}
$$

Finally, using (A.53b), the last term in the influence functional (6.85) can be rewritten as

$$
\begin{aligned}
& -\frac{\beta}{3}\int_{t_i}^{t_f}(g_{ij}(\varphi)\nabla_t\zeta^i(t)\nabla_t\zeta^j(t) \qquad (6.90)\\
& = -\frac{\beta}{3}\int_{t_i}^{t_f}\Big[\gamma_{ij}\partial_t\hat{q}^i\partial_t\hat{q}^j+(\gamma_{ij}C^i_{km}+\gamma_{im}C^i_{jk}+\gamma_{ik}C^i_{jm})\hat{q}^m D_t\bar{q}^k\partial_t\hat{q}^j \qquad (6.91)\\
& +\frac{\gamma_{ij}}{4}(C^i_{l_1l_2}+S^i_{l_1l_2})(C^j_{l'_1l'_2}+S^j_{l'_1l'_2})\hat{q}^{l_2}D_t\bar{q}^{l_1}\hat{q}^{l'_2}D_t\bar{q}^{l'_1}\Big]. \qquad (6.92)
\end{aligned}
$$

Comparing with the DKMS method, we read the couplings in (5.30) for the bulk model as follows

$$A_{l_1l_2l_3l_4} = \frac{2i}{3}\gamma_{j(l_1}C^j_{l_2k}S^k_{l_3l_4)} \qquad (6.93a)$$

$$l_{l_2l_3l_1} = -\gamma_{(l_2j}C^j_{l_3)l_1} \qquad (6.93b)$$

$$B_{l_1l_2l_3k} = \frac{1}{3}\left[\gamma_{i(l_1}S^j_{l_2l_3)}C^i_{jk}+\gamma_{ik}C^i_{j(l_1}S^j_{l_2l_3)}+\gamma_{i(l_1}C^i_{l_2j}C^j_{l_3)k}\right] \qquad (6.93c)$$

$$e_{ij} = \frac{i}{3}\gamma_{ij} \qquad (6.93d)$$

$$n_{mjk} = \frac{i}{3}(\gamma_{ij}C^i_{km}+\gamma_{im}C^i_{jk}+\gamma_{ik}C^i_{jm}) \qquad (6.93e)$$

$$D_{l_2l'_2l_1l'_1} = \frac{i}{24}\gamma_{ij}\left[(C^i_{l_1l_2}+S^i_{l_1l_2})(C^j_{l'_1l'_2}+S^j_{l'_1l'_2})+(C^i_{l_1l'_2}+S^i_{l_1l'_2})(C^j_{l'_1l_2}+S^j_{l'_1l_2})\right] \qquad (6.93f)$$

$$f_{ij} = o_{ijk} = E_{ijkl} = d_{ij} = m_{ijk} = 0. \qquad (6.93g)$$

It is straightforward to show that (6.93) satisfies the DKMS constraints (5.31), (5.37), (5.39), (5.45). As a result, the bulk model is categorized in the strictly Ohmic regime. However, the Ohmic couplings are not the most general ones allowed by DKMS, since all couplings are expressed in terms of the structure constant and dissipative coefficients $\gamma_{ij}$. To go beyond this limitation, we need to include the higher order terms in the bulk model.

## 6.6 The Higher Order Operators

Let's restore the higher order operator in the action functional of the (6.1) and find its correction to the influence functional. Adding it to the S-K action functional modifies it by

$$S^{(M)}[\chi_+,\chi_-]=\frac{\tilde{\nu}ijkl}{M^2}\int_{t,z}D_\mu\chi_+^iD^\mu\chi_+^jD_\nu\chi_+^kD^\nu\chi_+^l-D_\mu\chi_-^iD^\mu\chi_-^jD_\nu\chi_-^kD^\nu\chi_-^l\ . \tag{6.94}$$

Note that the coupling has the following symmetry by construction

$$\tilde{\nu}_{ijkl}=\tilde{\nu}_{(ij)(kl)}=\tilde{\nu}_{klij} \tag{6.95}$$

Using the covariant background method, the leading order contribution of this term to the effective action (6.40) takes the following form:

$$\delta S^{(M)}_{eff}=\delta S^{(M)}_{\rho}+\delta S^{(M)}_{\chi} \tag{6.96}$$

where we have

$$\begin{aligned}
&\delta S^{(M)}_{\rho}[\Phi(z,t_i),\xi(z,\tau)]\\
&=\frac{\tilde{\nu}_{ijk\ell}}{M^2}\int_{\tau,z}D_z\Phi^iD^z\Phi^jD_z\Phi^kD^z\Phi^l\\
&+\frac{\tilde{\nu}_{ijk\ell}}{M^2}\int_{\tau,z}4D_z\xi^iD^z\Phi^jD_z\Phi^kD^z\Phi^l\\
&+\frac{\tilde{\nu}_{ijk\ell}}{M^2}\int_{\tau,z}(2D_z\xi^iD^z\Phi^jD_z\Phi^kD^z\xi^l+2D^{(E)}_\mu\xi^iD^\mu_{(E)}\xi^jD_z\Phi^kD^z\Phi^l)\\
&+\frac{\tilde{\nu}_{ijk\ell}}{M^2}\int_{\tau,z}(D^{(E)}_\mu\xi^iD^\mu_{(E)}\xi^jD_z\Phi^kD^z\xi^l+D^{(E)}_\mu\xi^iD^\mu_{(E)}\xi^jD^{(E)}_\nu\xi^kD^\nu_{(E)}\xi^l)\ .
\end{aligned} \tag{6.97a}$$

$$\delta S^{(M)}_{\chi}[\Phi(z,t),\xi(z,t)]=\frac{4\tilde{\nu}_{ijk\ell}}{M^2}\int_{t,z}\left(\xi^m\partial_m\Omega^i_{i'}(\Phi)\partial_\mu\Phi^{i'}+\Omega^i_{i'}(\Phi)\partial_\mu\xi^{i'}\right)D^\mu\Phi^jD_\nu\Phi^kD^\nu\Phi^l\ , \tag{6.97b}$$

Inserting the saddle point solution in (6.97a), we can show that it vanishes up to $O(\hat{q}^5)$. Therefore, only (6.97b) corrects the influence functional. For the general non-Abelian group, it is rather involved to show that (6.97b) can correct the influence functional with a local in time functional. We present the calculation for the case of Abelian group here, referring interested readers to the case of general non-Abelian group to the appendix A7. For the abelian case, the functional (6.97b) combined using the saddle point solution

(6.53) and (6.57) in the $u-v$ coordinate reads

$$
\begin{aligned}
\delta S_{\chi}^{(M)}[\Phi(z,t),\xi(z,t)] &= \frac{i\tilde{\nu}_{ijk\ell}}{\beta M^2}\int_{u,v}\partial_v\xi^i\partial_u\Phi^j\left(\partial_u\Phi^k\xi^l+\partial_u\Phi^l\xi^k\right)\\
&= \frac{i\tilde{\nu}_{ijk\ell}}{\beta M^2}\int_{u,v}\partial_v(\xi^i\xi^l)\partial_u\Phi^j\partial_u\Phi^k\\
&= \frac{i\tilde{\nu}_{ijk\ell}}{\beta M^2}\int_{u,v}\partial_v(\xi^i\xi^l\partial_u\Phi^j\partial_u\Phi^k) \qquad (6.98)
\end{aligned}
$$

where in the second line, we have used the symmetries of the coupling, and then in the last line, the equation of motion (6.48) is implemented in integration by parts. Performing the integration along the $v$ axis, we are left with

$$
\delta S_{\chi}^{(M)}[\Phi(z,t),\xi(z,t)] = -\frac{i\tilde{\nu}_{ijk\ell}}{\beta M^2}\int_{t_i}^{t_f}(\xi^i\xi^l\partial_u\Phi^j\partial_u\Phi^k)|_{(z=0,t)} \qquad (6.99)
$$

Using $\partial_u=\partial_v-\partial_t$, and the boundary conditions at $z=0$, and (6.57), then this equation can be written in terms of the classical ($\varphi$) and quantum ($\zeta$) fields at the boundary. Using the fact that in the abelian case, the two definitions of the classical and quantum fields coincide with each other to all orders,

$$
\zeta(t)=\hat{q}(t), \quad \bar{q}(t)=\varphi(t)\ ,
$$

then equation (6.99) corrects the influence functional as follows

$$
\begin{aligned}
i\mathcal{I}_M[\bar{q}(t),\hat{q}(t)] = \frac{\tilde{\nu}_{ijk\ell}}{M^2}\int_{t_i}^{t_f}\Bigg(&\frac{4i}{\beta^3}\hat{q}^i(t)\hat{q}^j(t)\hat{q}^k(t)\hat{q}^l(t)+\frac{4}{\beta^2}\hat{q}^i(t)\hat{q}^l(t)\hat{q}^j(t)\partial_t\bar{q}^k(t) \qquad (6.100)\\
&-\frac{i}{\beta}\hat{q}^i(t)\hat{q}^l(t)\partial_t\bar{q}^j(t)\partial_t\bar{q}^k(t)\Bigg)\ . \qquad (6.101)
\end{aligned}
$$

This implies that the higher order operator in the bulk model modifies the couplings (6.93). Comparing (6.100) with (5.11), this modification reads

$$
i\delta A_{ijkl} = \frac{4i}{M^2\beta^2}\tilde{\nu}_{(ijkl)} \qquad (6.102)
$$

$$
i\delta B_{ijkl} = \frac{4}{M^2\beta^2}\tilde{\nu}_{(ijk)l} \equiv \frac{4}{M^2\beta^2}\tilde{\nu}_{(ijkl)} \qquad (6.103)
$$

$$
i\delta D_{ijkl} = -\frac{i}{M^2\beta^2}\tilde{\nu}_{lijk} \qquad (6.104)
$$

where in the second equation we have used the symmetries of the $\tilde{\nu}_{ijkl}$ as was determined in (6.95). Obviously the mentioned DKMS constraints in the previous section still holds (the structure constants are zero for the Abelian case). Note that the sequence of the indices in (6.104) are different in the sides of the equation. With this, one can read the tensor $\nu_{ijkl}$ from (5.47) as follows

$$\nu_{ijkl} = -\frac{3i}{M^2\beta^2}\tilde{\nu}_{lijk} \, . \tag{6.105}$$

Without the detailed calculation, as it is presented in the A7, one can guess the correction to the influence functional by covariantizing (6.100) by substituting $\partial_t\bar{q}^i \to D_t\bar{q}^i$. This completes our argument that the bulk model can generate the most general high temperature influence functional in the Ohmic regime.

# Chapter 7

# Summary and Outlook

In this thesis, we derived the effective action for dissipative dynamics in specific cases and limits. After introducing some foundational concepts, we focused on the effective action of classical Ohmic dissipative dynamics using a model termed as the bulk model. This model generalizes the system-plus-reservoir models [Caldeira and Leggett 1983a; Unruh and Zurek 1989; Lamb 1900], where the environment is represented by a string or a set of harmonic oscillators. The bulk model extends these ideas to cases of state-dependent dissipation, where the dissipative coefficients depend on position. Beyond the classical limit, we considered dissipative dynamics on a Lie group, assuming that the dissipation is induced by a bath invariant under the same Lie group. Initially, we derived the effective action for dissipative dynamics within the linear response regime. Here, using the high-temperature expansion of the bath two-point functions, we constructed the local influence functional for dissipative dynamics with nonlinearly realized symmetries. We then employed the dynamical KMS symmetry to obtain the most general high-temperature local influence functional for dissipative dynamics on a Lie group. The influence functional within the DKMS method generalizes the one obtained in the linear response, containing a larger set of couplings which parameterize nonlinear dissipation and non-Gaussian noise. Finally, we revisited the bulk model by going beyond its classical limit. By applying the Schwinger-Keldysh formalism, we integrated out the bath of the bulk model, resulting in a local influence functional at high temperatures. We demonstrated that the result produces the most general high temperature influence functional allowed by the DKMS symmetry for Ohmic dissipation.

One possible application of the developed formalism is the study of Brownian motion with position dependent couplings of dissipation. In the case that the Brownian particle has size [Ulbrich et al. 2023; Lau and Lubensky 2007; Han et al. 2006; Zhang et al. 2023], the Langevin equation needs to be modified to accommodate for state dependent diffusion. In such situations, the noise and higher order dissipation can be quantified through measuring different correlators [Han et al. 2006].

In many interesting physical application, in contrast to this work, the symmetry is not completely broken. In such cases, we need to enhance the theory to include the coset construction [Akyuz et al. 2024]. The immediate realization of such a construction is the stochastic dynamics on a sphere or De Sitter space.

One technical challenge in studying the low temperature dissipative processes is that their influence functional is nonlocal in time, making it notoriously difficult to work with. On the other hand, in obtaining the correlators from the bulk model, there is no need to integrate out the bath, which means that its effective action remains local in time. It would be interesting to see the advantages of using the bulk model in studying some concrete examples at low temperature limit.

Another interesting direction can be generalization of the formalism to field theory and (nonrelativistic) hydrodynamics and working out the connection with similar works [Liu and Glorioso 2018; Michailidis et al. 2024; Akyuz et al. 2024; Cohen and Green 2020; Cohen et al. 2021; Salcedo et al. 2024; Burgess et al. 2023; Lang et al. 2024; Zelle et al. 2024; Sieberer et al. 2023; Dalla Torre et al. 2010; Dalla Torre et al. 2012a; Dalla Torre et al. 2012b].

# Appendix A

# Supplements

## A1 Elements of the group geometry

Here we summarize some auxiliary formulas following from the definition of the Cartan form. First, we have the relations between linear variations of the group elements:

$$\mathrm{e}^{qG}\mathrm{e}^{\delta q\,G} = \mathrm{e}^{(q+\delta q')G}\ , \qquad \delta q'^i = u^i_j(q)\delta q^j + O(\delta q^2)\ , \tag{A.1a}$$

$$\mathrm{e}^{\delta q\,G}\mathrm{e}^{qG} = \mathrm{e}^{(q+\delta q'')G}\ , \qquad \delta q''^i = u^i_j(-q)\delta q^j + O(\delta q^2)\ , \tag{A.1b}$$

where $u^i_j$ is the inverse of the Cartan coefficients,

$$u^i_j(q)\Omega^j_k(q) = \delta^i_k\ . \tag{A.2}$$

From these relations we derive the identities:

$$\frac{\partial(q\ominus q')^i}{\partial q'^k} = -u^i_j(q'\ominus q)\,\Omega^j_k(q')\ , \tag{A.3a}$$

$$\frac{\partial(q'\ominus q)^i}{\partial q'^k} = u^i_j(q'\ominus q)\,\Omega^j_k(q')\ . \tag{A.3b}$$

Second, at the origin the Cartan form and its inverse reduce to unity,

$$\Omega^i_j(0) = u^i_j(0) = \delta^i_j\ . \tag{A.4}$$

Besides, for our parameterization of the group element (4.2), their derivatives at the origin are antisymmetric in the lower indices:

$$\left.\frac{\partial\Omega^i_j}{\partial q^k}\right|_{q=0} = -\left.\frac{\partial u^i_j}{\partial q^k}\right|_{q=0} = \frac{1}{2}C^i{}_{jk}\ . \tag{A.5}$$

To prove this last statement, we consider a one-parameter family of the group elements $g(s) = \mathrm{e}^{s\,qG}$. Taking a small increment of the parameter and using $\mathrm{e}^{(s+ds)qG} = \mathrm{e}^{s\,qG}\mathrm{e}^{ds\,qG}$

in combination with (A.1), we obtain

$$q^i = u^i_j(sq)q^j \; . \tag{A.6}$$

Differentiating this identity once again with respect to $s$ yields,

$$\frac{\partial u^i_j}{\partial q^k}(sq)\, q^k q^j = 0 \; , \tag{A.7}$$

which is true for any $s$. In particular, setting $s = 0$, we obtain that the contraction of the matrix of derivatives of $u^i_j$ at the origin with any vector vanishes, implying that it is antisymmetric. Then, the structure relation (4.3) implies the form (A.5).

To find higher derivatives of the Cartan form at the origin, we expand the group element and its differential:

$$g = 1 + q^i G_i + \frac{1}{2} q^i q^j G_i G_j + \frac{1}{6} q^i q^j q^k G_i G_j G_k + \dots \; , \tag{A.8a}$$

$$g^{-1} = 1 - q^i G_i + \frac{1}{2} q^i q^j G_i G_j - \frac{1}{6} q^i q^j q^k G_i G_j G_k + \dots \; , \tag{A.8b}$$

$$dg = dq^i G_i + \frac{1}{2} dq^i q^j (G_i G_j + G_j G_i) + \frac{1}{6} dq^i q^j q^k (G_i G_j G_k + G_j G_i G_k + G_j G_k G_i) + \dots \; . \tag{A.8c}$$

Multiplying these Taylor series and commuting the generators we obtain for the invariant differential:

$$g^{-1}dg = dq^i G_i + \frac{1}{2} dq^j q^k C^i{}_{jk} G_i + \frac{1}{6} dq^j q^k q^l C^i{}_{kn} C^n{}_{lj} G_i + \dots \; , \tag{A.9}$$

whence we read out the expansion of the Cartan coefficients,

$$\Omega^i_j(q) = \delta^i_j + \frac{1}{2} q^k C^i{}_{jk} + \frac{1}{6} q^k q^l C^i{}_{kn} C^n{}_{lj} + \dots \; . \tag{A.10}$$

The first two terms give the value of the coefficients and their first derivatives at the origin which we already found above, Eqs. (A.4), (A.5), while the last term gives the second derivatives,

$$\left.\frac{\partial^2 \Omega^i_j}{\partial q^k \partial q^l}\right|_{q=0} = \frac{1}{6}(C^i{}_{kn} C^n{}_{lj} + C^i{}_{ln} C^n{}_{kj}) \; . \tag{A.11}$$

To obtain the second derivatives of the inverse matrix, we twice differentiate the identity (A.2) and use (A.4), (A.5), (A.11). This yields,

$$\left.\frac{\partial^2 u^i_j}{\partial q^k \partial q^l}\right|_{q=0} = \frac{1}{12}(C^i{}_{kn} C^n{}_{lj} + C^i{}_{ln} C^n{}_{kj}) \; . \tag{A.12}$$

Next, we derive the expansions of the covariant derivatives,

$$D_t q^i_+ = D_t \bar{q}^i - C^i{}_{jk} \hat{q}^j D_t \bar{q}^k + \dot{\hat{q}}^i + \frac{1}{2} C^i{}_{jn} C^n{}_{kl} \hat{q}^j \hat{q}^k D_t \bar{q}^l - \frac{1}{2} C^i{}_{jk} \hat{q}^j \dot{\hat{q}}^k + O(\hat{q}^3 D_t) \,, \tag{A.13a}$$

$$D_t q^i_- = D_t \bar{q}^i + C^i{}_{jk} \hat{q}^j D_t \bar{q}^k - \dot{\hat{q}}^i + \frac{1}{2} C^i{}_{jn} C^n{}_{kl} \hat{q}^j \hat{q}^k D_t \bar{q}^l - \frac{1}{2} C^i{}_{jk} \hat{q}^j \dot{\hat{q}}^k + O(\hat{q}^3 D_t) \,. \tag{A.13b}$$

To this end we write,

$$(D_t q^i_+) G_i = \mathrm{e}^{-q_+ G} \frac{d}{dt} \mathrm{e}^{q_+ G} = \mathrm{e}^{-\hat{q}G} \mathrm{e}^{-\bar{q}G} \frac{d}{dt} (\mathrm{e}^{\bar{q}G} \mathrm{e}^{\hat{q}G}) = \mathrm{e}^{-\hat{q}G} (D_t \bar{q})^j G_j \mathrm{e}^{\hat{q}G} + \Omega^i_j(\hat{q}) \dot{\hat{q}}^j \, G_i \,. \tag{A.14}$$

Further, we use the results

$$\begin{aligned} \mathrm{e}^{-\hat{q}G} G_j \mathrm{e}^{\hat{q}G} &= G_j + \hat{q}^k [G_j, G_k] + \frac{1}{2} \hat{q}^k \hat{q}^l [[G_j, G_k], G_l] + \dots \\ &= G_j + \hat{q}^k C^i{}_{jk} G_i + \frac{1}{2} \hat{q}^k \hat{q}^l C^n{}_{jk} C^i{}_{nl} G_i + \dots \,, \end{aligned} \tag{A.15a}$$

$$\Omega^i_j(\hat{q}) = \delta^i_j + \frac{1}{2} C^i{}_{jk} \hat{q}^k + \dots \,. \tag{A.15b}$$

Collecting them together we obtain the first Eq. (A.13). The second one is obtained by changing the sign of $\hat{q}$.

In Sec. 5.3 we need the representation of $q_\pm$ as Taylor series in $\hat{q}$. To derive it, we introduce a one-parameter curve $q_s$ on the group manifold defined as

$$\mathrm{e}^{q_s G} = \mathrm{e}^{\bar{q}G} \mathrm{e}^{\hat{q}Gs} \,. \tag{A.16}$$

Clearly $q_s\big|_{s=\pm 1} = q_\pm$. By definition of the Cartan form, we have

$$\mathrm{e}^{-q_s G} \frac{d}{ds} \mathrm{e}^{q_s G} = \Omega^j_i(q_s) \frac{dq^i_s}{ds} G_j \tag{A.17}$$

On the other hand, Eq. (A.16) implies

$$\mathrm{e}^{-q_s G} \frac{d}{ds} \mathrm{e}^{q_s G} = \hat{q}^j G_j \tag{A.18}$$

Comparison of these two expressions yields a differential equation

$$\frac{dq^i_s}{ds} = u^i_j(q_s) \hat{q}^j \,, \tag{A.19}$$

which must be solved with the initial condition $q_s\big|_{s=0} = \bar{q}$. The solution can be written in the integral form,

$$q^i_s = \bar{q}^i + \int_0^s ds' \, u^i_j(q_{s'}) \hat{q}^j \,. \tag{A.20}$$

One Taylor expands the integrand on the r.h.s. in $(q_{s'} - \bar{q})$ and evaluates it to a desired order in $s$. Through order $s^3$ one obtains,

$$q_s^i = \bar{q}^i + s\, u_j^i \hat{q}^j + \frac{s^2}{2}\, u_k^l \partial_l u_j^i \hat{q}^k \hat{q}^j + \frac{s^3}{6}\, (\partial_l u_j^i \partial_k u_n^l\, u_m^k + \partial_l \partial_k u_j^i\, u_n^l u_m^k) \hat{q}^j \hat{q}^n \hat{q}^m + O(s^4) \,, \tag{A.21}$$

where the coefficient functions $u_j^i$ and their derivatives are taken at $\bar{q}$. Substituting $s = \pm 1$ yields $q_\pm$. Clearly, the sum (difference) of $q_\pm$ contains only even (odd) powers of $\hat{q}$. In this way we arrive at Eqs. (5.24) from the main text.

## A2 Relation between thermal spectral densities

Suppressing the group indices to simplify notations, we have,

$$\begin{aligned}\langle \mathcal{O}(t_1)\mathcal{O}^\dagger(t_2)\rangle &= \sum_n \frac{\mathrm{e}^{-\beta E_n}}{Z} \langle n|\, \mathcal{O}(t_1)\mathcal{O}^\dagger(t_2)\, |n\rangle = \sum_{nm} \frac{\mathrm{e}^{-\beta E_n}}{Z} \langle n|\, \mathcal{O}(t_1)\, |m\rangle \langle m|\, \mathcal{O}^\dagger(t_2)\, |n\rangle \\ &= \sum_{nm} \frac{\mathrm{e}^{-\beta E_n}}{Z} \mathrm{e}^{-i(E_m - E_n)(t_1 - t_2)} \langle n|\, \mathcal{O}(0)\, |m\rangle \langle m|\, \mathcal{O}^\dagger(0)\, |n\rangle = \int d\omega\, \mathrm{e}^{-i\omega(t_1 - t_2)} \varrho(\omega)\end{aligned} \tag{A.22}$$

with

$$\varrho(\omega) = Z^{-1} \sum_{nm} \delta(\omega - E_m + E_n)\, \mathrm{e}^{-\beta E_n}\, |\, \langle n|\, \mathcal{O}(0)\, |m\rangle\, |^2 \,, \tag{A.23}$$

where $Z$ is the partition function. On the other hand,

$$\langle \mathcal{O}^\dagger(t_2)\mathcal{O}(t_1)\rangle = \sum_{nm} \frac{\mathrm{e}^{-\beta E_n}}{Z} \mathrm{e}^{-i(E_n - E_m)(t_1 - t_2)} \langle n|\, \mathcal{O}^\dagger(0)\, |m\rangle \langle m|\, \mathcal{O}(0)\, |n\rangle = \int d\omega\, \mathrm{e}^{-i\omega(t_1 - t_2)} \tilde{\varrho}(\omega) \tag{A.24}$$

with

$$\tilde{\varrho}(\omega) = Z^{-1} \sum_{nm} \delta(\omega - E_n + E_m)\, \mathrm{e}^{-\beta E_n}\, |\, \langle m|\, \mathcal{O}(0)\, |n\rangle\, |^2 \,. \tag{A.25}$$

We now interchange the labels $n$ and $m$ in the last expression and obtain

$$\begin{aligned}\tilde{\varrho}(\omega) &= Z^{-1} \sum_{nm} \delta(\omega - E_m + E_n)\, \mathrm{e}^{-\beta E_m}\, |\, \langle n|\, \mathcal{O}(0)\, |m\rangle\, |^2 \\ &= Z^{-1} \sum_{nm} \delta(\omega - E_m + E_n)\, \mathrm{e}^{-\beta(E_n + \omega)}\, |\, \langle n|\, \mathcal{O}(0)\, |m\rangle\, |^2 = \mathrm{e}^{-\beta\omega} \varrho(\omega) \,.\end{aligned} \tag{A.26}$$

## A3 Leading Order Expansion

Substituting (4.38) into the influence functional (4.22) we obtain a sum of three terms,

$$\mathcal{I} = \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)} \,. \tag{A.27}$$

The first two are computed in a straightforward manner,

$$\begin{aligned}
\mathcal{I}^{(1)} &= i\frac{\pi}{\beta}\int dt_1 dt_2 [\sigma_0(q_{+2}\ominus q_{+1}) + \sigma_0(q_{-2}\ominus q_{-1}) - \sigma_0(q_{+2}\ominus q_{-1}) - \sigma_0(q_{-2}\ominus q_{+1})]\delta(t_1-t_2) \\
&= i\frac{\pi}{\beta}\int dt[2\sigma_0(0) - \sigma_0(q_+\ominus q_-) - \sigma_0(q_-\ominus q_+)] \;, \qquad \text{(A.28)} \\
\mathcal{I}^{(2)} &= -\frac{\pi}{2}\int dt_1 dt_2 [\sigma_0(q_{+2}\ominus q_{+1}) + \sigma_0(q_{-2}\ominus q_{-1}) - \sigma_0(q_{+2}\ominus q_{-1}) - \sigma_0(q_{-2}\ominus q_{+1})]\delta'(t_1-t_2) \\
&= \frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_+\ominus q_{+'}) + \sigma_0(q_-\ominus q_{-'}) - \sigma_0(q_+\ominus q_{-'}) - \sigma_0(q_-\ominus q_{+'})]\Big|_{t'=t} \\
&= \frac{\pi}{2}\int dt\Big[\Big(\frac{\partial\sigma_0}{\partial q^i}(q_-\ominus q_+)\,u^i_j(q_+\ominus q_-)\Big)D_t q^j_+ + \Big(\frac{\partial\sigma_0}{\partial q^i}(q_+\ominus q_-)\,u^i_j(q_-\ominus q_+)\Big)D_t q^j_-\Big] \\
&+ \frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_+\ominus q_{+'}) + \sigma_0(q_-\ominus q_{-'})]\Big|_{t'=t} \;, \qquad \text{(A.29)}
\end{aligned}$$

where we have defined

$$\sigma_0(q) = \sum_{rab} U^r_{ab}(q)\hat{\varrho}^r_{0,ab} \qquad \text{(A.30)}$$

and in obtaining the last expression we used Eq. (A.3a). We see that the interaction with the bath has been encapsulated by a function function $\sigma_0(g)$ on the group manifold. Using the time reversal invariance together with hermiticity of the spectral densities, one can show that the function $\sigma_0(q)$ is real and even with respect to the reflection $q \to -q$. The third term in (A.27) requires a bit more work. We write

$$\begin{aligned}
\mathcal{I}^{(3)} &= \pi\int dt_1 dt_2 [\theta(t_2-t_1)(\sigma_0(q_{+2}\ominus q_{+1}) - \sigma_0(q_{-2}\ominus q_{+1})) \\
&\qquad + \theta(t_1-t_2)(\sigma_0(q_{-2}\ominus q_{-1}) - \sigma_0(q_{-2}\ominus q_{+1}))]\delta'(t_1-t_2) \\
&= \pi\int dt_1 dt_2\,\theta(t_2-t_1)[\sigma_0(q_{+2}\ominus q_{+1}) - \sigma_0(q_{-2}\ominus q_{+1}) - \sigma_0(q_{-1}\ominus q_{-2}) + \sigma_0(q_{-1}\ominus q_{+2})]\delta'(t_1-t_2) \\
&= -\frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_+\ominus q_{+'}) - \sigma_0(q_-\ominus q_{+'}) - \sigma_0(q_{-'}\ominus q_-) + \sigma_0(q_{-'}\ominus q_+)]\Big|_{t'=t} \;, \qquad \text{(A.31)}
\end{aligned}$$

where in the first equality we interchanged the variables $t_1 \leftrightarrow t_2$ in one of the terms, and in the second equality used that for function $f(x)$ vanishing at $x = 0$ the integral

$$\int dx\,\theta(x)f(x)\delta'(x) = -\frac{1}{2}f'(0) \qquad \text{(A.32)}$$

is well defined. Evaluating the time derivatives with the help of Eqs. (A.3) we finally obtain,

$$\begin{aligned}
\mathcal{I}^{(3)} =& -\frac{\pi}{2}\int dt\Big[\frac{\partial\sigma_0}{\partial q^i}(q_-\ominus q_+)\,u^i_j(q_+\ominus q_-)\,D_t q^j_+ + \frac{\partial\sigma_0}{\partial q^i}(q_-\ominus q_+)\,u^i_j(q_-\ominus q_+)\,D_t q^j_-\Big] \\
& - \frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_+\ominus q_{+'}) - \sigma_0(q_{-'}\ominus q_-)]\Big|_{t'=t} \,. \qquad \text{(A.33)}
\end{aligned}$$

As a result, we have

$$\begin{aligned}\mathcal{I} = \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)} =& i\frac{\pi}{\beta}\int dt\,[2\sigma_0(0) - \sigma_0(q_+ \ominus q_-) - \sigma_0(q_- \ominus q_+)] \\ &- \frac{\pi}{2}\int dt\,[\partial_i\sigma_0(q_- \ominus q_+)\, u^i_j(q_+ \ominus q_-)\, D_t q^j_+ - \partial_i\sigma_0(q_+ \ominus q_-)\, u^i_j(q_- \ominus q_+)\, D_t q^j_-] \\ &+ \frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_- \ominus q_{-'}) + \sigma_0(q_{-'} \ominus q_-)]\Big|_{t'=t}. \end{aligned} \tag{A.34}$$

Using the following identity,

$$\begin{aligned} &\frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_- \ominus q_{-'}) + \sigma_0(q_{-'} \ominus q_-)]\Big|_{t'=t} \\ =& \frac{\pi}{2}\int dt\,\frac{d}{dt'}[\sigma_0(q_- \ominus q_{-'}) - \sigma_0(q_- \ominus q_{-'})]\Big|_{t'=t} = 0, \end{aligned} \tag{A.35}$$

the influence functional reads

$$\begin{aligned}\mathcal{I} =& i\frac{\pi}{\beta}\int dt\,[2\sigma_0(0) - \sigma_0(q_+ \ominus q_-) - \sigma_0(q_- \ominus q_+)] \\ &- \frac{\pi}{2}\int dt\,[\partial_i\sigma_0(q_- \ominus q_+)\, u^i_j(q_+ \ominus q_-)\, D_t q^j_+ - \partial_i\sigma_0(q_+ \ominus q_-)\, u^i_j(q_- \ominus q_+)\, D_t q^j_-] \end{aligned} \tag{A.36}$$

## A4 KMS transformation of $\hat{q}$

In this Appendix we outline the derivation of Eq. (5.29). We work keeping terms up to cubic power in $\hat{q} \sim \beta D_t \bar{q}$. Each extra time derivative multiplied by $\beta$ is considered as adding one more order in this power counting. It is convenient to introduce a shorthand notation,

$$Y^i_{jns} = \partial_l\partial_m u^i_j\, u^l_n u^m_s + \partial_l u^i_j\, \partial_m u^l_n\, u^m_s \tag{A.37}$$

From Eqs. (5.2), (5.24) we have

$$\begin{aligned} 2u^i_j(\bar{q}')\,\hat{q}'^j + \frac{1}{3}Y^i_{jns}(\bar{q}')\,\hat{q}'^j\hat{q}'^n\hat{q}'^s\Big|_t =& 2u^i_j(\bar{q})\,\hat{q}^j + \frac{1}{3}Y^i_{jns}(\bar{q})\,\hat{q}^j\hat{q}^n\hat{q}^s \\ &+ \frac{i\beta}{2}\partial_{t'}(2\bar{q}^i + u^l_m\partial_l u^i_j\,\hat{q}^m\hat{q}^j) - \frac{\beta^2}{8}\partial^2_{t'}(2u^i_j\hat{q}^j) - \frac{i\beta^3}{48}\partial^3_{t'}(2\bar{q}^i) + O(\hat{q}^5)\Big|_{t'=-t}, \end{aligned} \tag{A.38}$$

where on the r.h.s. all the coefficient functions are evaluated at $\bar{q}$. On the l.h.s. we can set the argument of $Y^i_{jns}$ to coincide with $\bar{q}$ since the corresponding term is already of cubic order. On the other hand, the coefficient $u^i_j(\bar{q}')$ must be Taylor expanded using Eq. (5.27). Substituting

$$\hat{q}'^j\Big|_t = \hat{q}^j + \frac{i\beta}{2}D_t\bar{q}^j + \hat{p}^j\Big|_{t'=-t}, \tag{A.39}$$

into (A.38) and simplifying the result with the identity (5.26), we arrive at

$$\hat{p}^k\Big|_t = \frac{i\beta}{4} C^k{}_{jm}\hat{q}^j\dot{\hat{q}}^m + \frac{i\beta}{4} Z^k_{jns}\hat{q}^j\hat{q}^n D_{t'}\bar{q}^s - \frac{\beta^2}{8}\ddot{\hat{q}}^k + \frac{\beta^2}{4} C^k{}_{jm}\dot{\hat{q}}^j D_{t'}\bar{q}^m - \frac{\beta^2}{8} Z^k_{jns}\hat{q}^j D_{t'}\bar{q}^n D_{t'}\bar{q}^s$$
$$- \frac{i\beta^3}{48}\partial^2_{t'} D_{t'}\bar{q}^k - \frac{i\beta^3}{24} C^k{}_{jm} D_{t'}\bar{q}^j \partial_{t'} D_{t'}\bar{q}^m\Big|_{t'=-t}, \tag{A.40}$$

where

$$Z^k_{jns} = \Omega^k_i\left(\frac{2}{3}Y^i_{jns} - \frac{1}{3}Y^i_{nsj} - \frac{1}{3}Y^i_{sjn} + C^l_{js}u^m_l\partial_m u^i_n\right). \tag{A.41}$$

It remains to find the tensor $Z^k_{jns}$, or rather its two symmetrized combinations $Z^k_{(jn)s}$, $Z^k_{j(ns)}$ entering into (A.40). Their direct calculation is possible, but lengthy. It is convenient to take a shortcut by observing that they must be constant on the group manifold since they enter as coefficients of invariant quantities in the expansion of the invariant variable $\hat{p}^k$. Thus, we can evaluate them at $\bar{q} = 0$. Using Eqs. (A.5), (A.12) we readily obtain,

$$Z^k_{(jn)s} = -\frac{1}{3}C^k{}_{(jm}C^m{}_{n)s}\,, \qquad Z^k_{j(ns)} = \frac{2}{3}C^k{}_{(nm}C^m{}_{s)j}\,. \tag{A.42}$$

Substitution into (A.40) yields Eq. (5.29).

## A5 variation of the density matrix

Saddle point solution of the path integral (6.25) is obtained by variation of

$$S^{(1)}_E\left[\xi(z,\tau),\Phi(z)\right] + S^{(2)}_E\left[\xi(z,\tau),\Phi(z)\right]$$

with respect to $\xi(z,\tau)$. Variation of $S^{(1)}$ with respect to $\xi(z,\tau)$ results in

$$\delta S^{(1)}[\Phi(z),\xi(z,\tau]) = \int_{\tau,z} g_{ij}(\Phi(z))\partial_z\Phi^i(z)\nabla_z\delta\xi^j(z,\tau) \tag{A.43a}$$

$$= \int_{\tau,z} \nabla_z(g_{ij}\partial_z\Phi^i(z)\delta\xi^j(z,\tau)) - g_{ij}\nabla_z(\partial_z\Phi^i(z))\delta\xi^j(z,\tau) \tag{A.43b}$$

$$= -\int_{\tau,z} g_{ij}\nabla_z(\partial_z\Phi^i(z))\delta\xi^j(z,\tau) \tag{A.43c}$$

where in (A.43b) we have used metric compatibility of the covariant derivative. In the first term of (A.43b), $\nabla_z$ can be swapped with $\partial_z$. Therefore, the first term of (A.43b) can be written as a boundary term (in the $z$ direction) and vanishes. Variation of $S^{(2)}$ with respect to $\xi$ field results in

$$\delta S^{(2)} = \frac{1}{2}\int_{\tau,z} [2\nabla^{(E)}_{\mu}\delta\xi^i(z,\tau)\nabla^{(E)}_{\mu}\xi_i(z,\tau) + (R_{ijkl}+R_{ikjl})\delta\xi^j(z,\tau)\xi^k(z,\tau)\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau)]. \tag{A.44}$$

The first term in (A.44) can be rearranged as follows

$$\frac{1}{2}\int_{\tau,z} 2\nabla^{(E)}_{\mu}\delta\xi^i(z,\tau)\nabla^{(E)}_{\mu}\xi_i(z,\tau) \tag{A.45a}$$

$$= \int_{\tau,z} \partial_\tau\delta\xi^i(z,\tau)\partial_\tau\xi_i(z,\tau) + \nabla_z\delta\xi^i(z,\tau)\nabla_z\xi_i(z,\tau) \tag{A.45b}$$

$$= -\int_{\tau,z} \delta\xi^i(z,\tau)\partial^2_\tau\xi_i(z,\tau) + \delta\xi^i(z,\tau)\nabla^2_z\xi_i(z,\tau) \tag{A.45c}$$

where in (A.45b) we have used the fact that the background is $\tau$ independent and as result $\nabla_\tau = \partial_\tau$. In transition to (A.45c) we have used integration by part and the fact that the variation of $\xi$ vanishes at the boundaries (along both $\tau$ and $z$ axis). The second term in (A.44) can be rewritten as follows

$$\frac{1}{2}\int_{\tau,z} (R_{ijkl}+R_{ikjl}+R_{ijkl}+R_{iljk}+R_{iklj})\delta\xi^j(z,\tau)\xi^k(z,\tau)\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau) \tag{A.46a}$$

$$= \frac{1}{2}\int_{\tau,z} (2R_{ijkl}+R_{iljk})\delta\xi^j(z,\tau)\xi^k(z,\tau)\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau) \tag{A.46b}$$

$$= \int_{\tau,z} R_{ijkl}\delta\xi^j(z,\tau)\xi^k(z,\tau)\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau) \tag{A.46c}$$

where in (A.46a) we have simply added a zero using the cyclic property of the Riemann tensor. In (A.46b) we have used the antisymmetric properties of the Riemann tensor and in (A.46c), we have used the fact that contraction of an anti symmetric tensor with a symmetric tensor gives zero,

$$R_{iljk}\partial_z\Phi^i(z,\tau)\partial_z\Phi^l(z,\tau) = 0.$$

Adding up all the contributions, the saddle point differential equation for the path integral (6.25) reads

$$\partial^2_\tau\bar{\xi}^j(z,\tau) = -\nabla_z\partial_z\Phi^j(z) - \nabla^2_z\bar{\xi}^j(z,\tau) + g^{jb}(\Phi(z))R_{ibkl}(\Phi(z))\bar{\xi}^k(z,\tau)\partial_z\Phi^i(z)\partial_z\Phi^l(z) \tag{A.47a}$$

$$\bar{\xi}^i(z,\tau=-\beta/2) = \xi^i_1(z) \tag{A.47b}$$

$$\bar{\xi}^i(z,\tau=+\beta/2) = \xi^i_2(z) \tag{A.47c}$$

## A6 The relation between the two definitions of the classical and quantum fields

The influence functional is obtained in terms of geometrical quantities. We need to rewrite the influence functional in terms of objects which are covariant with respect to the symmetry group of the theory to match the results of DKMS. To do so, we need to remember that $q_{\pm}(t)$ are the fundamental fields and one can define the classical and quantum fields in different ways. As a result, the classical and quantum fields in one definition can be obtained in terms of the classical and quantum fields in another definition perturbatively. The S-K classical and quantum fields in the covariant background expansion method are defined through the following equations

$$\frac{q_+^i(t)+q_-^i(t)}{2}=\varphi^i(t)-\frac{1}{2}\Gamma^i_{j_1j_2}(\varphi)\zeta^{j_1}(t)\zeta^{j_2}(t)+O\left(\zeta^4\right) \tag{A.48a}$$

$$\frac{q_+^i(t)-q_-^i(t)}{2}=\zeta^i(t)+\frac{1}{3}\left(\Gamma^i_{j_3j_4}(\varphi)\Gamma^{j_4}_{j_1j_2}(\varphi)-\frac{1}{2}\partial_{j_3}\Gamma^i_{j_1j_2}(\varphi)\right)\zeta^{j_1}(t)\zeta^{j_2}(t)\zeta^{j_3}(t)+O\left(\zeta^5\right). \tag{A.48b}$$

On the other hand, the S-K classical and quantum fields in the group covariant approach in the previous chapter is defined through (look at the appendix)

$$\frac{q_+^i(t)+q_-^i(t)}{2}=\bar{q}^i(t)+\frac{1}{2}u^{i_1}{}_{j_1}(\bar{q})\partial_{i_1}u^i{}_{j_2}(\bar{q})\hat{q}^{j_1}(t)\hat{q}^{j_2}(t)+O\left(\hat{q}^4\right) \tag{A.49a}$$

$$\begin{aligned}\frac{q_+^i(t)-q_-^i(t)}{2}=u^i_j(\bar{q})\hat{q}^j(t)+\frac{1}{3!}\Big(&u^{i_2}{}_{j_2}(\bar{q})\partial_{i_2}u^{i_1}{}_{j_1}(\bar{q})\partial_{i_1}u^i{}_{j_1}(\bar{q})\\&+u^{i_1}{}_{j_1}(\bar{q})u^{i_2}{}_{j_2}(\bar{q})\partial_{i_1}\partial_{i_2}u^i{}_{j_3}(\bar{q})\Big)\hat{q}^{j_1}(t)\hat{q}^{j_2}(t)\hat{q}^{j_3}(t)+O(\hat{q}^5).\end{aligned} \tag{A.49b}$$

These two sets of equations gives $\varphi^i$ and $\zeta^i$ in terms of $\bar{q}^i$ and $\hat{q}^i$ perturbatively. The zeroth order solution can be obtained by equating (A.48a) and (A.49a),

$$\varphi^i(t)=\bar{q}^i(t)+O\left(\hat{q}^2\right). \tag{A.50}$$

By equating (A.48b) with (A.49b) and inserting the zeroth order, we have the first order solution as

$$\zeta^i(t)=u^i_j(\bar{q})\hat{q}^j(t). \tag{A.51}$$

By equating (A.48a) with (A.49a) and using the zeroth and first order solutions we have the second order solution as

$$\varphi^i(t)=\bar{q}^i(t)+\frac{1}{2}\left[\Gamma^i_{j_1j_2}(\bar{q})u^{j_1}_{l_1}(\bar{q})u^{j_2}_{l_2}(\bar{q})+u^{i_1}_{l_1}(\bar{q})\partial_{i_1}u^i_{l_2}(\bar{q})\right]\hat{q}^{l_1}(t)\hat{q}^{l_2}(t)+O\left(\hat{q}^3\right) \tag{A.52}$$

The expression in the bracket can be rewritten in terms of the structure constants and the $\gamma$ matrices which results in (6.7a). In the next order of this expansion, we can obtain (6.7b). Other useful identities which we need to use to match the covariant background influence functional with the DKMS method are

$$\Omega^i_a(\varphi)\partial_t\varphi^a = D_t\bar{q}^i + \frac{1}{2}S^i_{l_1l_2}\partial_t\hat{q}^{l_1}_1\hat{q}^{l_2} + \frac{1}{4}C^i_{jk}S^k_{l_1l_2}\hat{q}^{l_1}\hat{q}^{l_2}D_t\bar{q}^j + O(\hat{q}^4) \tag{A.53a}$$

$$\Omega^i_a(\varphi)\nabla_t\tilde{\zeta}^a = \partial_t\tilde{\zeta}^i + \frac{1}{2}(C^i_{l_1l_2} + S^i_{l_1l_2})\tilde{\zeta}^{l_2}D_t\varphi^{l_1} + O(\hat{q}^3) \tag{A.53b}$$

which can be proved with a similar procedure.

## A7 higher order calculation

In what follows, we use the following notations for abbreviation:

$$\tilde{\xi}^{i'} = \Omega^{i'}{}_i(\Phi)\xi^i.$$

and

$$\tilde{D}_\mu\tilde{\xi}^{i\prime} = \partial_\mu\tilde{\xi}^{i'} + C^{i'}{}_{lm'}\tilde{\xi}^{m'}D_\mu\Phi^l$$

With this, the higher order term (??) is rewritten rewrite as follows

$$\delta S^{(M)}_\chi = \frac{4\tilde{\nu}_{ijk\ell}}{M^2}\int_{t,z}\tilde{D}_\mu\tilde{\xi}^iD^\mu\Phi^jD_\nu\Phi^kD^\nu\Phi^l \tag{A.54}$$

The leading order correction due to this term can be obtained by inserting the saddle point solution in the previous approximation into this action functional. To do so, we need to go to the $u-v$ coordinates. Showing the Jacobian of transformation by $J=\frac{1}{2}$, then in the $u-v$ coordinates the action functional reads

$$\delta S^{(M)}_\chi = \frac{4\tilde{\nu}_{ijkl}}{M^2}\cdot J\cdot\int_u du\int_v dv\left(\tilde{D}_u\tilde{\xi}^iD_v\Phi^j + \tilde{D}_v\tilde{\xi}^iD_u\Phi^j\right)\cdot 2\cdot D_u\Phi^kD_v\Phi^l$$

The saddle point solution $\partial_v\Phi^i = \frac{2i}{\beta}\xi^i$ can be rewritten in terms of the new quantum field:

$$D_v\Phi^i = \frac{2i}{\beta}\tilde{\xi}^i$$

which results in

$$\begin{aligned}
\delta S_{\chi}^{(M)} =& \frac{8\tilde{\nu}_{ijkl}}{M^2} \cdot J \cdot \int_{u,v} \left( \tilde{D}_u \tilde{\xi}^i \left( \frac{2i}{\beta} \tilde{\xi}^j \right) + \tilde{D}_v \tilde{\xi}^i D_u \Phi^j \right) D_u \Phi^k \cdot \left( \frac{2i}{\beta} \right) \cdot \tilde{\xi}^l \\
&= \frac{8\tilde{\nu}_{ijkl}}{M^2} \cdot J \cdot \left( \frac{2i}{\beta} \right)^2 \int_{u,v} \tilde{D}_u \tilde{\xi}^i \cdot \tilde{\xi}^j \cdot D_u \Phi^k \cdot \tilde{\xi}^l \\
&+ \frac{8\tilde{\nu}_{ijkl}}{M^2} \cdot J \cdot \left( \frac{2i}{\beta} \right) \int_{u,v} \partial_v \tilde{\xi}^i \cdot \tilde{\xi}^l \cdot D_u \Phi^j \cdot D_u \Phi^k
\end{aligned} \tag{A.55}$$

in which we have used the definition of $\tilde{D}_v \tilde{\xi}^i$ and the equation of motion:

$$\tilde{D}_v \tilde{\xi}^i = \partial_v \tilde{\xi}^i + C^i_{mn} D_v \Phi^m \tilde{\xi}^n = \partial_v \tilde{\xi}^i + \frac{2i}{\beta} C^i_{mn} \tilde{\xi}^m \tilde{\xi}^n = \partial_v \tilde{\xi}^i$$

The term $C^i_{mn} \tilde{\xi}^m \tilde{\xi}^n$ is zero because $C^i_{mn}$ is anti-symmetric with respect to $(m, n)$. Let's rearrange the second term in (A.55) as a boundary term and a bulk term. It turns out that only the boundary term contributes in the influence functional:

$$\begin{aligned}
&\tilde{\nu}_{ijk\ell} \int_{u,v} \quad \partial_v \tilde{\xi}^i \cdot \tilde{\xi}^\ell \cdot D_u \Phi^j D_u \Phi^k \\
&= \frac{\tilde{\nu}_{ijkl}}{2} \int_{u,v} \partial_v \left( \tilde{\xi}^i \tilde{\xi}^\ell \right) D_u \Phi^j D_u \Phi^k \\
&= \frac{\tilde{\nu}_{ijkl}}{2} \int_{u,v} \partial_v \left( \tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j D_u \Phi^k \right) - \tilde{\nu}_{ijkl} \int_{u,v} \tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j \partial_v D_u \Phi^k \\
&= - \int_{t_f}^{t_i} du \frac{\tilde{\nu}_{ijk\ell}}{2} (\tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j D_u \Phi^k)|_{(z=0,t)} - \tilde{\nu}_{ijk\ell} \int_{u,v} \tilde{\xi}^i \tilde{\xi}^\ell D_u \Phi^j \partial_v D_u \Phi^k \\
&= - \int_{t_i}^{t_f} dt \frac{\tilde{\nu}_{ijk\ell}}{2} (\tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j D_u \Phi^k)|_{(z=0,t)} - \tilde{\nu}_{ijk\ell} \int_{u,v} \tilde{\xi}^i \tilde{\xi}^\ell D_u \Phi^j \partial_v D_u \Phi^k
\end{aligned} \tag{A.56}$$

Now, we need to use the equation of motion to rewrite the second term in (A.56):

$$\begin{aligned}
&\partial_v D_u \Phi^k = \partial_v \left(\Omega^k{}_m \partial_u \Phi^m\right) \\
&= \partial_v \Phi^n \partial_n \Omega^k{}_m \partial_u \Phi^m + \Omega^k{}_m \partial_u \partial_v \Phi^m \\
&= \left(\frac{2i}{\beta}\right) u^n{}_p \tilde{\xi}^p \partial_n \Omega^k{}_m \partial_u \Phi^m + \left(\frac{2i}{\beta}\right) \Omega^k{}_m \partial_u \left(u^m{}_n \tilde{\xi}^n\right) \qquad \text{(A.57)} \\
&= \left(\frac{2i}{\beta}\right) u^n{}_p \tilde{\xi}^p \partial_n \Omega^k{}_m u^m{}_q D_u \Phi^q \\
&+ \left(\frac{2i}{\beta}\right) \Omega^k{}_m u^m{}_n \partial_u \tilde{\xi}^n + \left(\frac{2i}{\beta}\right) \Omega^k{}_m \partial_p u^m{}_n \partial_u \Phi^p \tilde{\xi}^n \\
&= \left(\frac{2i}{\beta}\right) \left[u^n{}_p \partial_n \Omega^k{}_m u^m{}_q \tilde{\xi}^p D_u \Phi^q + \partial_u \tilde{\xi}^k + \Omega^k{}_m \partial_p \Omega^m{}_n u^p{}_q D_u \Phi^q \tilde{\xi}^n\right] \\
&= \left(\frac{2i}{\beta}\right) \left[u^n{}_p \partial_n \Omega^k{}_m u^m{}_q \tilde{\xi}^p D_u \Phi^q + \tilde{D}_u \tilde{\xi}^k - C^k{}_{mn} D_u \Phi^m \tilde{\xi}^n + \Omega^k{}_m \partial_n u^m{}_p u^n{}_q D_u \Phi^q \tilde{\xi}^p\right] \\
&= \left(\frac{2i}{\beta}\right) \left[\tilde{D}_u \tilde{\xi}^k - C^k{}_{mn} D_u \Phi^m \tilde{\xi}^n + \Omega^k{}_m \left(-u^n{}_p \partial_n u^m{}_q + u^n{}_q \partial_n u^m{}_p\right) D_u \Phi^q \tilde{\xi}^p\right] \\
&= \left(\frac{2i}{\beta}\right) \left[\tilde{D}_u \tilde{\xi}^k - C^k{}_{mn} D_u \Phi^m \tilde{\xi}^n + C^k{}_{qp} D_u \Phi^q \tilde{\xi}^p\right] \qquad \text{(A.58)}
\end{aligned}$$

where we have used the identity $\Omega^m{}_q \partial_n \Omega^k{}_m = -\Omega^k{}_m \partial_n \Omega^m{}_q$ and the definition of the structure constant. Therefore, we have

$$\begin{aligned}
\delta S_\chi^{(M)} = &= \frac{8\tilde{\nu}_{ijkl}}{M^2} \cdot J \cdot \left(\frac{2i}{\beta}\right)^2 \int dudv \tilde{D}_u \tilde{\xi}^i \cdot \tilde{\xi}^j \cdot D_u \Phi^k \cdot \tilde{\xi}^\ell \\
&- \frac{4\tilde{\nu}_{ijk\ell}}{M^2} \cdot J \cdot \left(\frac{2i}{\beta}\right) \int du \tilde{\xi}^i \tilde{\xi}^\ell D_u \Phi^j D_u \Phi^k \\
&- \frac{8\tilde{\nu}_{ijk\ell}}{M^2} \cdot J \cdot \left(\frac{2i}{\beta}\right)^2 \int dudv \tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j D_u \tilde{\xi}^k \\
&+ \frac{8\tilde{\nu}_{ijk\ell}}{M^2} \cdot J \cdot \left(\frac{2i}{\beta}\right)^2 \int dudv \tilde{\xi}^i \tilde{\xi}^\ell D_u \Phi^j C^k{}_{mn} D_u \Phi^m \tilde{\xi}^n \\
&- \frac{8\tilde{\nu}_{ijkl}}{M^2} \cdot J \cdot \left(\frac{2i}{\beta}\right)^2 \int dudv \tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j C^k{}_{qp} D_u \Phi^q \tilde{\xi}^p \qquad \text{(A.59)} \\
&= -\frac{2\tilde{\nu}_{ijkl}}{M^2} \cdot \left(\frac{2i}{\beta}\right) \int_{t_i}^{t_f} dt \quad (\tilde{\xi}^i \tilde{\xi}^l D_u \Phi^j D_u \Phi^k)|_{(z=0,t)}. \qquad \text{(A.60)}
\end{aligned}$$

using the identity $D_u = D_t + D_v$ and the equation of motion $D_v\Phi^i(z,t) = \frac{2i}{\beta}\tilde{\xi}^(z,t)$, this expression reads

$$\delta S_\chi^{(M)} = -\frac{2\tilde{\nu}_{ijkl}}{M^2}\cdot\left(\frac{2i}{\beta}\right)\int_{t_i}^{t_f} dt \quad (\tilde{\xi}^i\tilde{\xi}^l D_u\Phi^j D_u\Phi^k)|_{(z=0,t)}$$
$$= -\frac{2\tilde{\nu}_{ijkl}}{M^2}\cdot\left(\frac{2i}{\beta}\right)\int_{t_i}^{t_f} dt \quad \left(\tilde{\xi}^i\tilde{\xi}^l(D_t\Phi^j + \frac{2i}{\beta}\tilde{\xi}^j)(D_t\Phi^k + \frac{2i}{\beta}\tilde{\xi}^k)\right)\Big|_{(z=0,t)}$$
$$= -\frac{2\tilde{\nu}_{ijkl}}{M^2}\cdot\left(\frac{2i}{\beta}\right)\int_{t_i}^{t_f} dt$$
$$\left[\Omega^i{}_{i'}(\varphi)\zeta^{i'}(t)\Omega^l{}_{l'}(\varphi)\zeta^{l'}(t)\Big(D_t\varphi^j(t) + \frac{2i}{\beta}\Omega^j{}_{j'}(\varphi)\zeta^{j'}(t)\Big)\Big(D_t\varphi^k(t) + \frac{2i}{\beta}\Omega^k{}_{k'}(\varphi)\zeta^{k'}(t)\Big)\right] \tag{A.61}$$

where in the last line the boundary conditions at $z = 0$ are used. Therefore, this term corrects the influence functional with

$$\begin{aligned} i\mathcal{I}_M = \frac{\tilde{\nu}_{ijkl}}{M^2}\int_{t_i}^{t_f} dt \Big[ & -\frac{i}{\beta}\Omega^i{}_{i'}(\varphi)\zeta^{i'}(t)\Omega^l{}_{l'}(\varphi)\zeta^{l'}(t)D_t\varphi^j(t)D_t\varphi^k(t) \\ & + \frac{2}{\beta^2}\Big(\Omega^i{}_{i'}(\varphi)\zeta^{i'}(t)\Omega^l{}_{l'}(\varphi)\zeta^{l'}(t)D_t\varphi^j(t)\Omega^k{}_{k'}(\varphi)\zeta^{k'}(t) \\ & + \Omega^i{}_{i'}(\varphi)\zeta^{i'}(t)\Omega^l{}_{l'}(\varphi)\zeta^{l'}(t)D_t\varphi^k(t)\Omega^j{}_{j'}(\varphi)\zeta^{j'}(t)\Big) \\ & + \frac{4i}{\beta^3}\Omega^i{}_{i'}(\varphi)\zeta^{i'}(t)\Omega^l{}_{l'}(\varphi)\zeta^{l'}(t)\Omega^j{}_{j'}(\varphi)\zeta^{j'}(t)\Omega^k{}_{k'}(\varphi)\zeta^{k'}.(t)\Big] \end{aligned} \tag{A.62}$$

Inserting the leading order term from (6.7), we obtain the covariantized version of (6.100).

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