THE SMOOTH PARTICLE VARIABLE STRUCTURE FILTER

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ABSTRACT

In this paper, a new state and parameter estimation method is introduced based on the particle filter (PF) and the smooth variable structure filter (SVSF). The PF is a popular estimation method, which makes use of distributed point masses to form an approximation of the probability distribution function (PDF). The SVSF is a relatively new estimation strategy based on sliding mode concepts, formulated in a predictor-corrector format. It has been shown to be very robust to modeling errors and uncertainties. The combined method, referred to as the smooth particle variable structure filter (SPVSF), utilizes the estimates and state error covariance of the SVSF to formulate the proposal distribution which generates the particles used by the PF. The SPVSF method is applied on two computer experiments, namely a nonlinear target tracking scenario and estimation of electrohydrostatic actuator parameters. The results are compared with other popular Kalman-based estimation methods.

Keywords: particle filter; smooth variable structure filter; Kalman filter; tracking; nonlinear estimation.

FILTRE À PARTICULES LISSE DE STRUCTURE VARIABLE

RÉSUMÉ

Dans cet article, une nouvelle méthode d'estimation de l'état et des paramètres est introduite basée sur un filtre à particules (FP) et un filtre à particules lisse de structure variable (FPLSV). Le FP est une méthode d'estimation courante, se servant de masses ponctuelles réparties pour former une approximation de la fonction de distribution de probabilité (FDP). Le FPLSV est une stratégie d'estimation relativement nouvelle basée sur des concepts de mode de glissement, formulés dans un format prédicteur-correcteur. Il s'est avéré très robuste quant aux erreurs de modélisation et d'incertitudes. La méthode combinée, nommé filtre à particules à surface lisse de structure variable (FPLSV), utilise les estimations et la covariance d'erreur d'état du FPLSV pour établir l'émission qui génère les particules utilisées par le FP. La méthode est appuyée par deux expériences informatiques, à savoir un scénario non linéaire cible, et l'estimation des paramètres du servomoteur électrohydrostatique. Les résultats sont comparés avec d'autres méthodes d'estimation de Kalman courantes.

Mots-clés : filtre à particules; filtre lisse de structure variable; filtre de Kalman; suivi; estimation non linéaire.

NOMENCLATURE

f	nonlinear system function
h	nonlinear measurement function
q	proposal distribution (i.e., typically this may be the system)
p	distribution of some mean and covariance
x	state vector or values
z	measurement (system output) vector or values
y	artificial measurement vector or values
u	input to the system
w	system noise vector
v	measurement noise vector
F	linear system transition matrix
G	input gain matrix
H	linear measurement (output) matrix
K	filter gain matrix (i.e., KF or SVSF)
P	state error covariance matrix, or total number of particles
Q	system noise covariance matrix
R	measurement noise covariance matrix
S	innovation covariance matrix
e	measurement (output) error vector
$\bar{a}$	defines a diagonal matrix of some vector a, may also be written as $diag\,(a)$
$sat\,(a)$	defines the saturation of some term a
γ	SVSF 'convergence' or memory parameter
ψ	SVSF boundary layer width
$\lvert a \rvert$	absolute value of some parameter a
$E\{\}$	expectation of some vector or value
T	sample time, or transpose of some vector or matrix
$\wedge$	estimated vector or values
x_k^i	particles used by the PF
ω_k^i	importance weights used by the PF
N_{eff}	effective threshold for the PF
$a_{k+1\|k}$	a priori value of some parameter a (i.e., before gain is applied)
$a_{k+1\|k+1}$	a posteriori value of some parameter a (i.e., after update)

1. INTRODUCTION

In target tracking applications, one may be concerned with surveillance, guidance, obstacle avoidance or tracking a target given some measurements [1]. In a typical scenario, sensors measure the output as a signal that is processed and conditioned. These measurements are related to the target state, and are typically noise-corrupted observations [1]. The target state usually consists of kinematic information such as position, velocity, and acceleration. The measurements are processed in order to form and maintain tracks, which are a sequence of target state estimates that vary with time. Multiple targets and measurements may yield multiple tracks. Gating and data association techniques help classify the source of measurements, and help associate measurements to its appropriate track. Typically these gating techniques help to avoid extraneous measurements which would otherwise cause the estimation process to go unstable and fail [2]. A tracking filter is used in a recursive manner to carry out the target state estimation.

The behaviour of a target (such as for example, uniform motion, coordinated turn, and weaving) may be modeled linearly or nonlinearly. The typical nonlinear system and measurement models may be represented by the following equations, respectively:

$$x_{k+1} = f(x_k, u_k) + w_k\,, \tag{1}$$

$$z_{k+1} = h(x_{k+1}) + v_{k+1}\,. \tag{2}$$

It is the goal of a filter to remove the effects that the system noise w_k and measurement noise v_k have on extracting the true state trajectory x_k from the measurements z_k. Note that the system noise and the measurement noise are typically considered to be Gaussian, such that $P(w_k) \sim \mathcal{N}(0, Q_k)$ and $P(v_k) \sim \mathcal{N}(0, R_k)$.

A solution to the estimation problem may be found by using a recursive filter, which provides an estimate every time a measurement is available [1]. Most recursive filters have two stages: prediction and update. The most popular and well-studied estimation method is the Kalman filter (KF), which was introduced in the 1960s [3,4]. The KF yields a statistically optimal solution for linear estimation problems, in the presence of Gaussian noise [5]. In real-world situations, dynamic systems are often nonlinear, as described by Eqs. (1) and (2). For nonlinear systems, the posterior density that encapsulates all the information about the current state cannot be described by a finite number of summary statistics and one has to be content with an approximate filtering solution.

The standard KF is formulated in a predictor-corrector manner. The states are first estimated using the system model, termed as a priori estimates, meaning 'prior to' knowledge of the observations. A correction term is then added based on the innovation (also called residuals or measurement errors), thus forming the updated or a posteriori (meaning 'subsequent to' the observations) state estimates. The KF has been broadly applied to problems covering state and parameter estimation, signal processing, target tracking, fault detection and diagnosis, and even financial analysis [1,6]. The success of the KF comes from the optimality of the Kalman gain in minimizing the trace of the a posteriori state error covariance matrix. The trace is taken because it represents the state error vector in the estimation process [7]. The KF estimation process and equations have been omitted from this paper as they are readily available in the literature [4,8].

Nonlinear forms of the KF have been created, with the two most popular being the extended (EKF) and unscented Kalman filters (UKF) [7,9]. The EKF is conceptually similar to the KF; however, the nonlinear system and measurement models (f and h, respectively) are linearized according to its Jacobian. It is possible to use the nonlinear functions f and h to predict the state estimates and the measurements. However, these functions may not be directly used to calculate the covariance values. The partial derivatives are used to compute linearized system and measurement matrices F and H, respectively found as follows [6]:

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k|k}, u_k}, \tag{3}$$

$$H_{k+1} = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k+1|k}}. \tag{4}$$

Equations (3) and (4) essentially linearize the nonlinear system or measurement functions around the current state estimate [4]. This comes at a loss of optimality, as the EKF gain is no longer considered to be the optimal solution to the estimation problem due to the linearization process [7]. Furthermore, the linearization process introduces unmodeled or overlooked modeling uncertainties, which can lead to instabilities [10]. It is well established in literature that the UKF provides advantages over the EKF [6,9]. Essentially, the UKF uses a deterministic sampling technique referred to as the unscented transform [9]. A finite number of weighted sample points (selected about the mean) are propagated through the nonlinear functions, which create an approximate solution to the mean and covariance of the desired estimate [9].

This method works well for nonlinear systems, and typically provides a more accurate result than the EKF strategy. An advantage of this method is that it does not require any linearization.

The particle filter (PF) is a natural extension of the UKF. The PF makes use of point masses (or 'particles') to approximate the probability density functions, which essentially contains the statistical information of the state [1]. The PF has become a common method for solving nonlinear estimation problems. In an effort to further increase the estimation accuracy of the PF for nonlinear estimation problems, the PF has been combined with both the EKF and UKF [11–13]. The extended particle filter (EPF) and unscented particle filter (UPF) respectively utilize the EKF and UKF estimates and covariance's to formulate the proposal distribution used to generate the particles [4,14]. This paper introduces a new PF combination; which makes use of the relatively new smooth variable structure filter (SVSF) [15]. This method is applied on two estimation problems, and is compared with the popular EKF, UKF, PF, and standard SVSF.

2. THE PARTICLE FILTER

The particle filter (PF) has many names: Monte Carlo filters, interacting particle approximations [16], bootstrap filters [17], condensation algorithm [18], and survival of the fittest [19], to name a few. Particle filters perform sequential Monte Carlo (SMC) estimation based on weighted particles [1]. As mentioned in [1], the basic concepts of SMC were introduced in statistics literature in the 1950s [20]. The earliest implementations of the PF were formed on the principles of sequential importance sampling (SIS) [21–23]. The main idea behind the SIS strategy is to represent the desired posterior density function by a number of weighted samples, and then compute estimates based on these samples [1]. The larger the number of samples, the more accurate the representation; however, this comes at the cost of computational demand and time. The following equations help to describe the SIS strategy. The expectation of some function $f()$ can be approximated as a weighted average, as follows:

$$\int f(x_k)\, p(x_k|z_0,\ldots,z_k) \approx \sum_{i=1}^{P} \omega^i f\left(x_k^i\right) , \tag{5}$$

where $p(x_k|z_0,\ldots,z_k)$ refers to the desired distribution, ω^i refers to the weight of the i^{th} particle, x_k^i refers to the i^{th} particle at time k, and P is the total number of particles. The first step of the process involves drawing the particles from the proposal distribution q (i.e. typically the system function f given the current measurements and the particles from the previous time step (or initialization):

$$x_k^i \sim q\left(x_k|x_{k-1}^i, z_k\right) . \tag{6}$$

The next step involves calculating the corresponding particle weights, which are then used to approximate the desired distribution. The weights may be calculated using the likelihood function $p\left(z_k|x_k^i\right)$, which is defined by the measurement model h and the measurement covariance R_k:

$$\hat{\omega}_k^i \sim p\left(z_k|x_k^i\right) \cdot \omega_{k-1}^i . \tag{7}$$

The particle weights are then normalized (as follows), such that the sum of the particle weights is equal to unity (i.e. $\sum_i^P \omega_k^i = 1$):

$$\omega_k^i = \frac{\hat{\omega}_k^i}{\sum_{j=1}^{P} \hat{\omega}_k^j} . \tag{8}$$

Finally, the state(s) may be estimated by the weighted summation of the particles, as follows:

$$\hat{x}_k = \sum_{i=1}^{P} \omega_k^i x_k^i . \tag{9}$$

The previous four equations summarize the SIS strategy, which is the basic form of the PF [1]. This method is relatively straightforward and yields good results. However, it was discovered that the SIS implementation suffers from a numerical phenomenon referred to as degeneration [17,24]. The degeneracy problem refers to only a few particles having significant importance weights after a large number of recursions. An important step referred to as resampling was added after Eq. (8), and helps to avoid the degeneracy problem [17]. This step eliminates particles with low weights and multiplies those with high weights [1]. The concept of effective sample size N_{eff} was introduced to help measure the amount of degeneracy present in the algorithm [25]. Should a significant amount of degeneracy be detected (below a designer threshold) as per Eq. (9), then the particles should be resampled:

$$N_{eff} = \frac{1}{\sum_{i=1}^{P} \left(\omega_k^i\right)^2} \,. \tag{10}$$

Although resampling helps to avoid the degeneracy problem, it also decreases the amount of particle diversity (i.e., particles with significant weights will be resampled) [1]. This is referred to as sample impoverishment. However, resampling is known to increase the overall accuracy of the estimated state(s) [17]. The aforementioned strategy is referred to as sampling importance resampling (SIR), and forms the standard PF used in literature.

3. THE SMOOTH VARIABLE STRUCTURE FILTER

A new form of predictor-corrector estimator based on sliding mode concepts referred to as the variable structure filter (VSF) was introduced in 2003 [26]. Essentially this method makes use of the variable structure theory and sliding mode concepts. It uses a switching gain to converge the estimates to within a boundary of the true state values (referred to as the existence subspace as shown in Fig. 1). In 2007, the smooth variable structure filter (SVSF) was derived which makes use of a simpler and less complex gain calculation [15]. In its present form, the SVSF has been shown to be stable and robust to modeling uncertainties and noise, when given an upper bound on the level of un-modeled dynamics and noise [26,27].

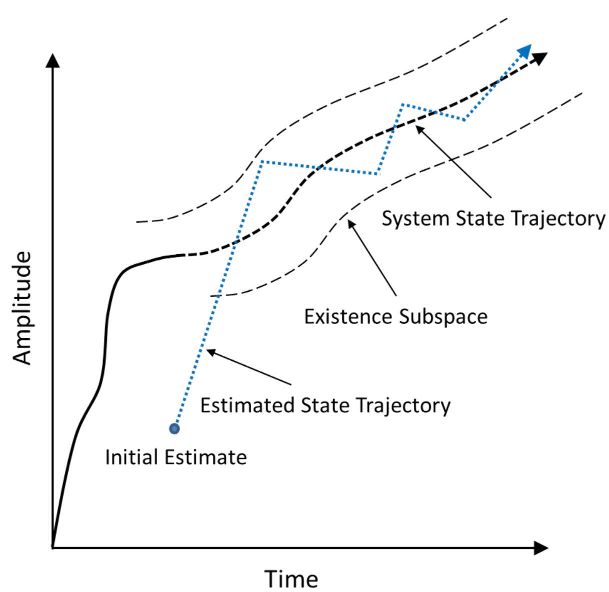

Fig. 1. SVSF estimation concept [8]

The SVSF method is model based and may be applied to differentiable linear or nonlinear dynamic equations. The original form of the SVSF as presented in [15] did not include covariance derivations. An augmented form of the SVSF was presented in [28], which includes a full derivation for the filter. The estimation process is iterative and may be summarized by the following set of equations. The predicted state estimates $\hat{x}_{k+1|k}$ and state error covariances $P_{k+1|k}$ are first calculated respectively as follows:

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}, u_k)\,, \tag{11}$$

$$P_{k+1|k} = FP_{k|k}F^T + Q_{k+1}\,. \tag{12}$$

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the corresponding predicted measurements $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ may be calculated:

$$\hat{z}_{k+1|k} = h(\hat{x}_{k+1|k})\,, \tag{13}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k}\,. \tag{14}$$

The SVSF process differs from the KF in how the gain is formulated. The SVSF gain is a function of: the a priori and the a posteriori measurement errors $e_{z,k+1|k}$ and $e_{z,k|k}$; the smoothing boundary layer widths ψ; the 'SVSF' memory or convergence rate γ; as well as the linearized measurement matrix H. For the derivation of the gain K_{k+1}, refer to [15,28]. The SVSF gain is defined as a diagonal matrix such that:

$$K_{k+1} = H^{-1} diag\left[\left(\left|e_{z_{k+1|k}}\right| + \gamma\left|e_{z_{k|k}}\right|\right) \circ sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right)\right] diag\left(e_{z_{k+1|k}}\right)^{-1}, \tag{15}$$

where $\circ$ signifies Schur (or element-by-element) multiplication, and where $\bar{\psi}^{-1}$ is a diagonal matrix constructed from the smoothing boundary layer vector ψ, such that:

$$\bar{\psi}^{-1} = \begin{bmatrix} \frac{1}{\psi_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\psi_m} \end{bmatrix}. \tag{16}$$

Note that m is the number of measurements, and the saturation function of Eq. (15) is defined by:

$$sat\left(\bar{\psi}^{-1} e_{z_{k+1|k}}\right) = \begin{cases} 1, e_{z_i,k+1|k}/\psi_i \geq 1 \\ e_{z_i,k+1|k}/\psi_i, -1 < e_{z_i,k+1|k}/\psi_i < 1 \\ -1, e_{z_i,k+1|k}/\psi_i \leq -1\,. \end{cases} \tag{17}$$

This gain is used to calculate the updated state estimates $\hat{x}_{k+1|k+1}$ as well as the updated state error covariance matrix $P_{k+1|k+1}$:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} e_{z,k+1|k}\,, \tag{18}$$

$$P_{k+1|k+1} = (I - K_{k+1}H)\,P_{k+1|k}\,(I - K_{k+1}H)^T + K_{k+1}R_{k+1}K_{k+1}^T\,. \tag{19}$$

Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated, and are used in later iterations:

$$\hat{z}_{k+1|k+1} = h(\hat{x}_{k+1|k+1})\,, \tag{20}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1}\,. \tag{21}$$

The existence subspace shown in Figs. 1 and 2 represents the amount of uncertainties present in the estimation process, in terms of modeling errors or the presence of noise. The width of the existence space β is a function of the uncertain dynamics associated with the inaccuracy of the internal model of the filter as well as the measurement model, and varies with time [15]. Typically this value is not exactly known but an upper bound may be selected based on a priori knowledge. Once within the existence boundary subspace, the estimated states are forced (by the SVSF gain) to switch back and forth along the true state trajectory. High-frequency switching caused by the SVSF gain is referred to as chattering, and in most cases, is undesirable for obtaining accurate estimates [15]. However, the effects of chattering may be minimized by the introduction of a smoothing boundary layer ψ. The selection of the smoothing boundary layer width reflects the level of uncertainties in the filter and the disturbances.

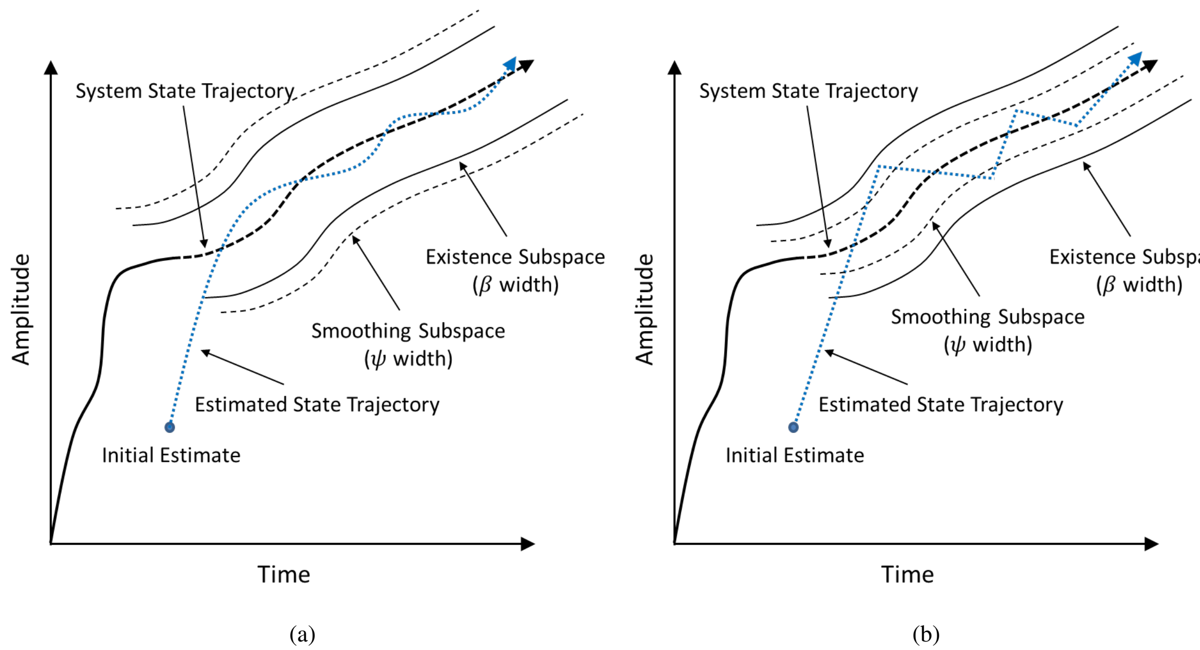

Fig. 2. The above two figures show an example of when the estimated state trajectory is smoothed, and when chattering is present: (a) smoothed estimated trajectory $(\psi \geq \beta)$, (b) presence of chattering effect $(\psi < \beta)$ [15].

The effect of the smoothing boundary layer is shown in Fig. 2. When the smoothing boundary layer is defined larger than the existence subspace boundary, the estimated state trajectory is smoothed. However, when the smoothing term is too small, chattering remains due to the uncertainties being underestimated.

4. THE SMOOTH PARTICLE VARIABLE STRUCTURE FILTER STRATEGY

The SVSF provides an estimation process that is suboptimal albeit robust and stable. It is hence beneficial to be able to combine the performance of the PF with the stability of the SVSF. To combine the aforementioned SVSF strategy with the PF, a similar approach to formulating the EPF and UPF will be taken [14]. Note that this method was also shown in [29], however the formulation remains different. Essentially, the a posteriori state estimates (Eq. 18) and state error covariance (Eq. 19) are used to formulate the proposal distribution used by the PF to generate the particles, such that Eq. (6) becomes:

$$x_k^i \sim q\left(\hat{x}_{k+1|k+1}, P_{k+1|k+1}\right) . \tag{22}$$

Figure 3 helps to describe the combined filtering strategy.

For completeness, the combined estimation process (referred to as the SPVSF) may be summarized by the following sets of equations. The predicted state estimates $\hat{x}_{k+1|k}$ and state error covariances $P_{k+1|k}$ are

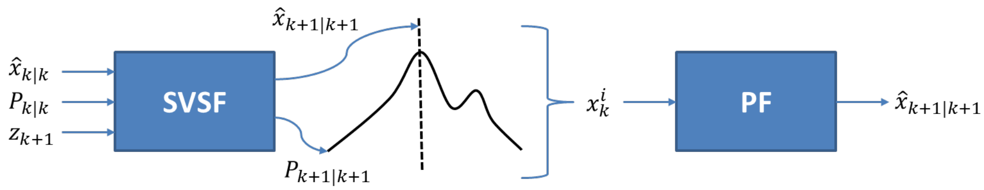

Fig. 3. SPVSF estimation concept is shown above.

first calculated respectively as follows:

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}, u_k) \,, \tag{23}$$

$$P_{k+1|k} = FP_{k|k}F^T + Q_{k+1} \,. \tag{24}$$

Utilizing the predicted state estimates $\hat{x}_{k+1|k}$, the corresponding predicted measurements $\hat{z}_{k+1|k}$ and measurement errors $e_{z,k+1|k}$ may be calculated:

$$\hat{z}_{k+1|k} = h(\hat{x}_{k+1|k}) \,, \tag{25}$$

$$e_{z,k+1|k} = z_{k+1} - \hat{z}_{k+1|k} \,. \tag{26}$$

The SVSF gain K_{k+1} is then calculated (Eq. 27), and is used to update the state estimates (Eq. 28) and calculate the a posteriori covariance matrix (Eq. 29):

$$K_{k+1} = H^{-1} diag \left[\left(\left| e_{z_{k+1|k}} \right| + \gamma \left| e_{z_{k|k}} \right| \right) \circ sat \left(\bar{\psi}^{-1} e_{z_{k+1|k}} \right) \right] diag \left(e_{z_{k+1|k}} \right)^{-1} \,, \tag{27}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} e_{z,k+1|k} \,, \tag{28}$$

$$P_{k+1|k+1} = (I - K_{k+1}H) P_{k+1|k} (I - K_{k+1}H)^T + K_{k+1} R_{k+1} K_{k+1}^T \,. \tag{29}$$

Next, the a posteriori state estimates (Eq. 28) and state error covariance (Eq. 29) are used to formulate the proposal distribution used to generate the particles:

$$x_{k+1}^i \sim q \left(\hat{x}_{k+1|k+1}, P_{k+1|k+1} \right) \,. \tag{30}$$

The next step involves calculating the corresponding particle weights, which are then used to approximate the desired distribution. The weights may be calculated using the likelihood function $p(z_{k+1}|x_{k+1}^i)$, which is defined by the measurement model h and the measurement covariance R_k:

$$\hat{\omega}_{k+1}^i \sim p \left(z_{k+1} | x_{k+1}^i \right) \cdot \omega_k^i \,. \tag{31}$$

The particle weights are then normalized (as follows), such that the sum of the particle weights is equal to unity (i.e. $\sum_i^P \omega_{k+1}^i = 1$):

$$\omega_{k+1}^i = \frac{\hat{\omega}_{k+1}^i}{\sum_{j=1}^P \hat{\omega}_{k+1}^j} \,. \tag{32}$$

The states may be estimated by the weighted summation of the particles, as follows:

$$\hat{x}_{k+1|k+1} = \sum_{i=1}^P \omega_{k+1}^i x_{k+1}^i \,. \tag{33}$$

Finally, the updated measurement estimate $\hat{z}_{k+1|k+1}$ and measurement errors $e_{z,k+1|k+1}$ are calculated as follows:

$$\hat{z}_{k+1|k+1} = h(\hat{x}_{k+1|k+1}) \,, \tag{34}$$

$$e_{z,k+1|k+1} = z_{k+1} - \hat{z}_{k+1|k+1} \,. \tag{35}$$

The above process summarizes the SPVSF estimation strategy proposed in this paper. Equations (23–35) are used iteratively.

5. COMPUTER EXPERIMENTS

Two computer experiments are considered in this paper. The first experiment involves a simulation of a standard tracking scenario for cases involving a normal scenario (i.e. well-defined system), and another with modeling uncertainties. The second experiment involves parameter estimation in an electrohydrostatic actuator (EHA).

5.1. Tracking Scenario

This section describes the tracking problem studied, and illustrates the estimation results. One of the most well studied aerospace applications involves ballistic objects on re-entry [1]. In this paper, a ballistic target re-entering the atmosphere is considered, as described in [1]. Figure 4 shows the experimental setup for ballistic target tracking.

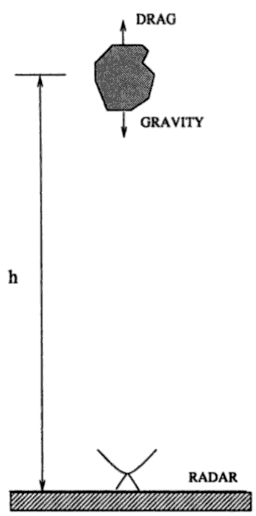

Fig. 4. Ballistic target tracking scenario (e.g. object on re-entry) [1].

Assuming that drag D and gravity g are the only forces acting on the object, the following differential equations govern its motion [1,30]:

$$\dot{h} = v \,, \tag{36}$$

$$\dot{v} = -\frac{\rho\,(h)\,g v^2}{2\beta} + g \,. \tag{37}$$

$$\dot{\beta} = 0 \,. \tag{38}$$

The state vector is defined as $x = [h \;\; v \;\; \beta]^T$, which refers to the target altitude, velocity, and ballistic coefficient, respectively. The air density ρ is modeled as follows:

$$\rho = \gamma e^{-\eta h} \,, \tag{39}$$

where from [1], $\gamma = 1.754$ and $\eta = 1.49\text{x}10^{-4}$. The discrete-time state equation is defined as follows [1]:

$$x_{k+1} = Fx_k - G[D(x_k) - g] + w_k\,, \tag{40}$$

with matrices F and G defined as:

$$F = \begin{bmatrix} 1 & -T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{41}$$

$$G = \begin{bmatrix} 0 & T & 0 \end{bmatrix}^T. \tag{42}$$

Furthermore, the function for drag $D(x_k)$ (the only nonlinear term) is defined as:

$$D(x_k) = \frac{g\rho(x_{k,1})\,x_{k,2}^2}{2x_{k,3}}. \tag{43}$$

As in [1], the system noise w_k is assumed to be zero-mean Gaussian with a covariance matrix Q defined by:

$$Q \approx \begin{bmatrix} q_1\frac{T^3}{3} & q_1\frac{T^2}{2} & 0 \\ q_1\frac{T^2}{2} & q_1 T & 0 \\ 0 & 0 & q_2 T \end{bmatrix}. \tag{44}$$

Note that the parameters q_1 and q_2 respectively control the amount of system noise in the target dynamics and the ballistic coefficient [1]. As shown in Fig. 4, a radar is positioned on the ground below the target. The measurement equation in this scenario is defined by:

$$z_k = Hx_k + v_k\,, \tag{45}$$

where it is assumed that two measurements are available, such that:

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{46}$$

In this tracking scenario, the initial states are defined as follows: $x_{1,0} = 61.000\,m$, $x_{2,0} = 3.048\,m/s$ and $x_{3,0} = 19.161\,kg/ms^2$. Other notable parameters were defined as: $q_1 = 10^4$, $q_2 = 10$, $T = 0.1\,sec$, $R = \text{diag}\left(\begin{bmatrix} 10^4 & 10^3 \end{bmatrix}\right)$ and $g = 9.81\,m/s^2$. Note that the following settings were used for both the PF and the SPVSF: the number of particles used was 350, and the effective sample size was set to 0.8. As per the earlier SVSF discussion, it is required to transform Eq. (46) into a square matrix (i.e., identity), such that an 'artificial' measurement is created. A number of methods exist, such as the reduced order or Luenberger's approach, which are presented in [3–5]. Consider a system model involving phase variables. It is possible to derive a third 'artificial' measurement $y_{3,k}$ based on the available measurements ($z_{1,k}$ and $z_{2,k}$).

In Eq. (46), the ballistic coefficient measurement is not available. If the system model (Eq. 40) is known with complete confidence, then it is possible to derive an artificial measurement for the ballistic coefficient from the first two measurements. Hence, consider the following from Eq. (40):

$$y_{3,k} = \frac{Tg\gamma z_{2,k}^2}{2\,(z_{2,k+1} - z_{2,k} + Tg)\,e^{-\eta z_{1,k}}}. \tag{47}$$

The accuracy of Eq. (47) depends on the sampling rate T. Applying Eq. (47) allows a measurement matrix equivalent to the identity matrix. The estimation process would continue as in the previous section, where a full measurement matrix was available. Note however that the artificial ballistic coefficient measurement would be delayed one time step. Furthermore, note that the artificial measurement would have to be initialized (i.e. 0 is a typical value). Equation (47) essentially propagates the known measurements through the system model to obtain the artificial ballistic coefficient measurement. It is conceptually similar to the method presented in [32] and creates a full measurement matrix.

The initial state estimates $\hat{x}_0$ are set 10% away from the true values x_0. The initial state error covariance matrix is set to $P_0 = 10Q$. Figure 5 shows the object's true and estimated altitude over time. For this case,

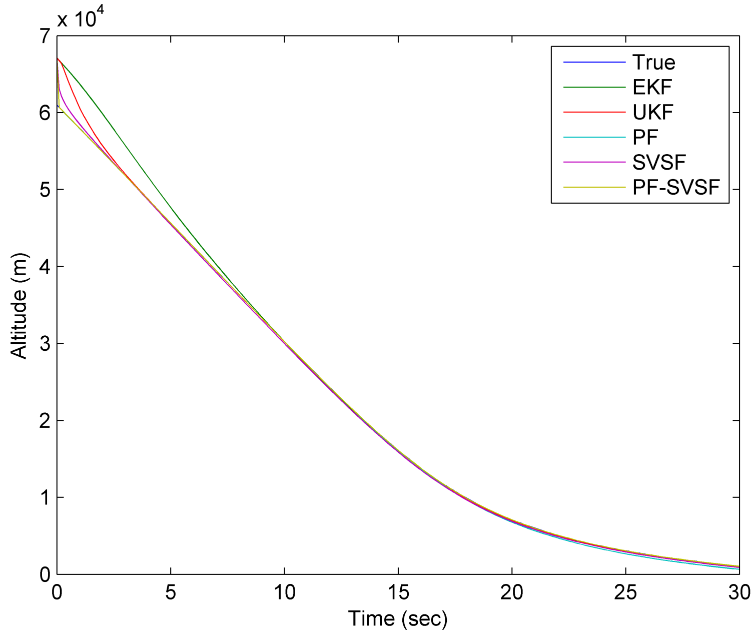

Fig. 5. True object altitude and estimated values.

the filters performed relatively well, with the exception of the convergence rates. Looking at the first five seconds of Fig. 6, the SPVSF converged to the true state trajectory the fastest. Next was the PF, followed closely by the SVSF. The UKF was the second slowest, converging in about 3 seconds. The EKF was the last filter to converge, taking nearly 10 seconds. The root mean squared error (RMSE) was calculated for each filter, and is shown in Table 1.

Filter	Altitude (m)	Velocity (m/s)	Ballistic (kg/ms^2)
EKF	1,873	29.7	1,923
UKF	1,060	30.6	2,071
PF	370	34.8	12,325
SVSF	471	119	1,923
SPVSF	360	32.7	1,923

Table 1. RMSE of the tracking scenario.

Table 1 summarizes the RMSE for the tracking scenario defined earlier. Overall, the proposed SPVSF algorithm provides the best result in terms of overall estimation accuracy. The PF performs very well, with the exception of the ballistic coefficient (it fails to provide a good estimate). The SVSF also performed well. The EKF provided the worst altitude estimate, most likely due to the slower convergence rate; however, yielded good estimates of the velocity and ballistic coefficient.

An interesting result occurs when one introduces modeling errors into the system model (Eq. 40). As an example, in an effort to demonstrate the robustness of the SPVSF and SVSF to modeling uncertainties, consider the case when the gravity coefficient is doubled. Figure 6 shows the implications of modeling error being introduced at 15 seconds during the tracking scenario. At this point, the estimates begin to diverge from the true state trajectory. In fact, at this point, the PF failed to achieve an estimate of the ballistic coefficient, and the filter broke down. The RMSE for this case was calculated for each filter, and is shown in Table 2.

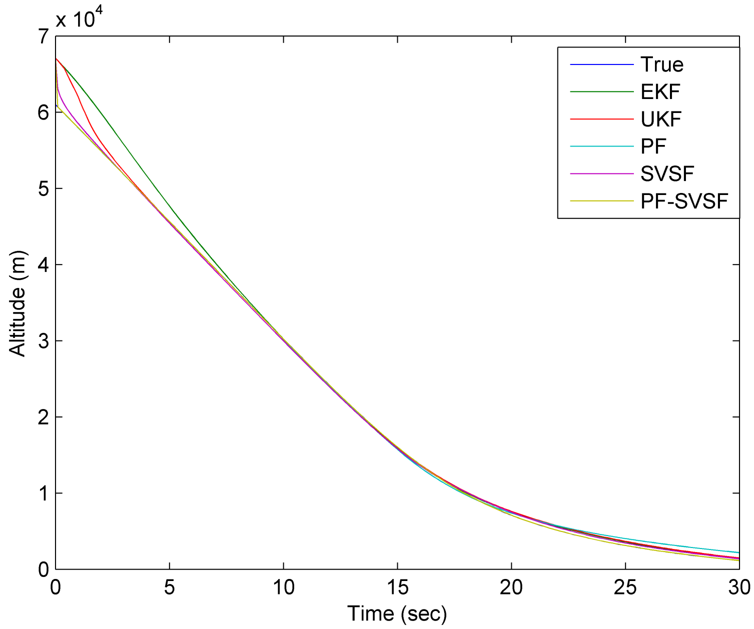

Fig. 6. True object altitude and estimated values with modeling error introduced at 15 seconds.

Filter	Altitude (m)	Velocity (m/s)	Ballistic (kg/ms^2)
EKF	1,867	61.1	1,920
UKF	852	94.6	2,369
PF	*Failed*	*Failed*	*Failed*
SVSF	568	311	1,919
SPVSF	360	35.0	1,919

Table 2. RMSE of the tracking scenario with modeling errors.

It is interesting to note that the SPVSF estimates remained relatively insensitive to the added modeling error. In this case, it is demonstrated that the combination of the SVSF with the PF improved the overall accuracy and stability of the PF estimation strategy.

5.2. Electrohydrostatic Actuator

In this experiment, an electrohydrostatic actuator (EHA) is simulated based on an actual prototype built for experimentation [15,27]. The purpose of this simulation is to demonstrate that the combined estimation process (SPVSF) yields a very accurate estimate, without negatively impacting its stability to modeling errors or uncertainties. The EHA is a third order (typically linear) system with state variables related to its position, velocity, and acceleration. It is assumed that all three states have measurements associated with them (i.e. full measurement matrix). The input to the system is a random normal distribution with magnitude 1. The sample time T of the system is 0.001 second. The entire EHA system description may be found in [15]. The open-loop transfer function of the system is defined as follows:

$$\frac{x(s)}{u(s)} = \frac{\frac{2D_p\beta_e A_E}{MV_0}}{s^3 + \left(\frac{B_E}{M} + \frac{L}{V_0}\beta_e\right)s^2 + \left(\frac{2\beta_e A_E^2}{MV_0}\right)s}\,. \tag{48}$$

For the purpose of this paper, three states (kinematic information) and one parameter (the effective bulk modulus) will be estimated. The estimation of the parameter creates a nonlinear estimation problem. The system model equations are defined as follows:

$$x_{1,k+1} = x_{1,k} + Tx_{2,k}\,, \tag{49}$$

$$x_{2,k+1} = x_{2,k} + Tx_{3,k}\,, \tag{50}$$

$$x_{3,k+1} = (1 - T\varphi_3 - T\varphi_2 x_{4,k})x_{3,k} - T\varphi_1 x_{2,k} + G_E T x_{4,k} u_k\,, \tag{51}$$

$$x_{4,k+1} = x_{4,k}\,, \tag{52}$$

where the following are defined:

$$G_E = \frac{2D_p A_E}{MV_0}\,, \tag{53}$$

$$\varphi_1 = \frac{2A_E^2}{MV_0}\,, \tag{54}$$

$$\varphi_2 = \frac{L}{V_0}\,, \tag{55}$$

$$\varphi_3 = \frac{B_E}{M}\,. \tag{56}$$

The EHA parameter values used in this computer experiment are shown in Table 3. The initial state values are set to zero. The initial true bulk modulus is set to $x_{4,0} = 2.1\times10^8 Pa$, whereas the corresponding initial estimate is $\hat{x}_{4,0} = 1.5\times10^8 Pa$. Two cases are studied for this experiment. The first case involves a constant bulk modulus (set to $2.1\times10^8 Pa$). The second case involves a changing bulk modulus. For this case, the true effective bulk modulus is changed at 0.5 second intervals from the initial value to $1.5\times10^8 Pa$ at 0.5 second, $2.7\times10^8 Pa$ at 1 second, and then back to $2.1\times10^8 Pa$ at 1.5 seconds.

Parameter	Physical Significance	EHA Model Value
A_E	Piston area	$3.37\times10^{-4}\ m^2$
B_E	Load friction	$1260\ Ns/m$
D_p	Pump displacement	$6.69\times10^{-3}\ m^3/rad$
L	Leakage coefficient	$5\times10^{-12}\ m^3/sPa$
M	Load mass	$20\ kg$
V_0	Chamber volume	$8.5\times10^{-5}\ m^3$
β_e	Effective bulk modulus	$2.1\times10^8\ Pa$

Table 3. EHA parameter values

The system and measurement noises are defined with maximum amplitude corresponding to $W_{max} = \begin{bmatrix} 0.0001 & 0.001 & 0.1 & 0 \end{bmatrix}^T$ and $V_{max} = \begin{bmatrix} 0.0001 & 0.001 & 0.1 \end{bmatrix}^T$ and are considered to be Gaussian. The initial state error covariance $P_{0|0}$, system noise covariance Q, and measurement noise covariance R are defined respectively as follows:

$$P_{0|0} = 10Q\,, \tag{57}$$

$$Q = 5W_{max}W_{max}^T\,, \tag{58}$$

$$R = 5V_{max}V_{max}^T\,. \tag{59}$$

For the SVSF estimation process, the 'memory' or convergence rate was set to $\gamma = 0.1$, and the boundary layer widths were defined as $\psi = 5V_{max}$. These parameters were set based on the level of noise and modeling uncertainty, with the goal of decreasing the estimation error. The main results of applying the filtering strategies on the EHA problem are shown in the following sets of figures. Figures 7-9 show the effective bulk modulus estimates provided by all of the strategies. Initially, both the PF and SPVSF responded the fastest, followed by the SVSF, the UKF, and finally the EKF. After about 0.7 second, all of the estimation methods converged to within a region of the true effective bulk modulus.

As shown in Table 4, the PF provided the best overall result in terms of estimation accuracy and rate of convergence. The SPVSF also performed extremely well, followed by the SVSF, UKF, and finally the EKF. Note that for the EKF to obtain an accurate estimate of the effective bulk modulus, the system noise covariance for the fourth state (i.e. $Q_{4,4}$ had to be increased significantly.

Filter	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Bulk Mod. (Pa)
EKF	4.68×10^{-4}	2.05×10^{-1}	3.07	1.64×10^{-1}
UKF	2.95×10^{-4}	1.44×10^{-1}	9.50×10^{-1}	6.83×10^{-2}
PF	2.23×10^{-5}	2.15×10^{-3}	2.67×10^{-2}	1.34×10^{-2}
SVSF	6.05×10^{-5}	3.25×10^{-3}	1.37×10^{-1}	3.34×10^{-2}
SPVSF	2.44×10^{-5}	2.58×10^{-3}	2.87×10^{-2}	2.84×10^{-2}

Table 4. RMSE results (normal case).

Figure 9 shows the estimates of the effective bulk modulus for the second case (when the bulk modulus changes over time). It is interesting to note that the PF failed to obtain an estimate at 0.5 second, once the true effective bulk modulus value began to change. This could be attributed to the particles forming weights for the first 0.5 second, and then not being able to redistribute the particle weights properly once the system changed. The number of particles was increased from 350 to 3500 for this case, and the PF still failed to estimate the fourth state properly.

As demonstrated by Fig. 9 and Table 5, the SPVSF strategy yielded the best estimates in terms of RMSE and convergence rate. The SVSF also yielded good results. The EKF and UKF strategies performed about the same. It is interesting to note that with the combined PF and SVSF strategy (SPVSF), the estimates remain close to the true state trajectory via the SVSF, and are further refined by the PF strategy. Another benefit of this combination includes the ability to reduce the number of particles used by the SPVSF, compared with the standard PF. For example, the number of particles used by the SPVSF could be reduced from 350 to 50 without negatively affecting the overall performance, while also significantly reducing the total demand on computational resources.

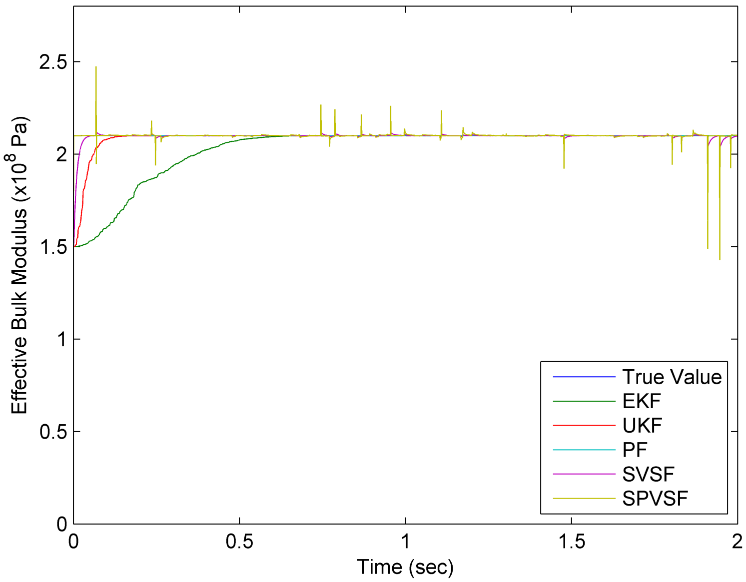

Fig. 7. Effective bulk modulus estimates for the EHA under a constant scenario.

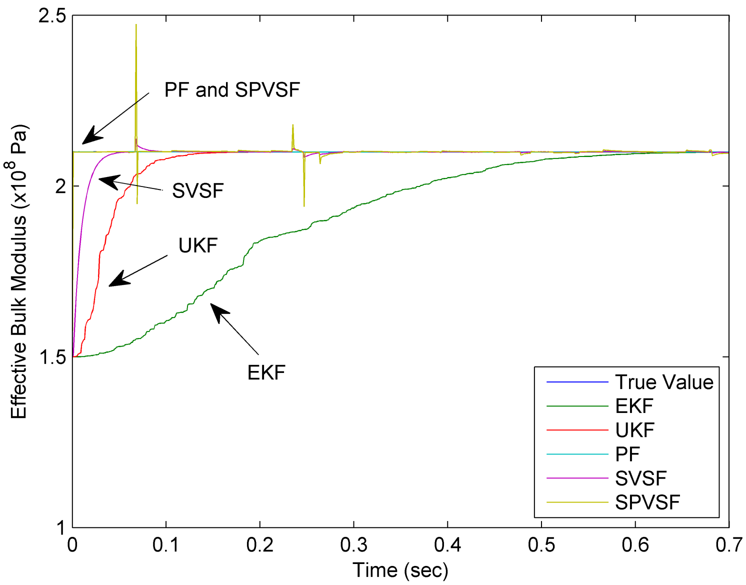

Fig. 8. Effective bulk modulus estimates for the EHA under a constant scenario (zoomed).

Filter	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Bulk Mod. (Pa)
EKF	1.07×10^{-3}	2.85×10^{-1}	4.54	2.31×10^{-1}
UKF	1.01×10^{-3}	3.70×10^{-1}	3.27	1.34×10^{-1}
PF	*Failed*	*Failed*	*Failed*	*Failed*
SVSF	5.30×10^{-5}	3.19×10^{-3}	1.33×10^{-1}	7.32×10^{-2}
SPVSF	2.55×10^{-5}	2.48×10^{-3}	2.96×10^{-2}	1.85×10^{-2}

Table 5. RMSE results (changing the effective bulk modulus).

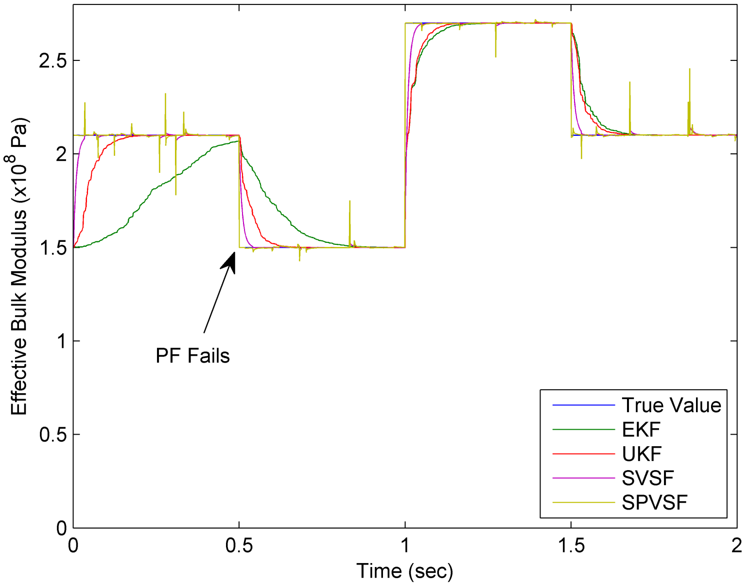

Fig. 9. Effective bulk modulus estimates for the EHA under a changing modulus case.

6. CONCLUSIONS

In this paper, a new state and parameter estimation based on the combination of the PF and the SVSF was introduced. The combined method, referred to as the smooth particle variable structure filter (SPVSF), utilizes the estimates and state error covariance of the SVSF to formulate the proposal distribution which generates the particles used by the PF. The SPVSF method was applied on a nonlinear target tracking problem and an electrohydrostatic actuator for parameter estimation. The results of the two computer experiments demonstrate the improved performance and stability of the combined methodology.

REFERENCES

1. Ristic, B., Arulampalam, S. and Gordon, N., *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House, Boston, 2004.
2. Gadsden, S.A. and Habibi, S.R., "Aircraft tracking using a radar and the smooth variable structure filter", *22nd Canadian Congress of Applied Mechanics (CANCAM)*, Halifax, Nova Scotia, Canada, 2009.
3. Kalman, R.E., "A new approach to linear filtering and prediction problems", *Journal of Basic Engineering, Transactions of ASME*, Vol. 82, Series D, pp. 35–45, 1960.
4. Anderson, B.D.O. and Moore, J.B., *Optimal Filtering*, Prentice-Hall, 1979.
5. Simon, D., *Optimal State Estimation: Kalman, H-Infinity, and Nonlinear Approaches*, Wiley-Interscience, 2006.
6. Haykin, S., *Kalman Filtering and Neural Networks*, John Wiley and Sons Inc., 2001.
7. Gelb, A., *Applied Optimal Estimation*, MIT Press, 1974.
8. Gadsden, S.A., *Smooth Variable Structure Filtering: Theory and Applications*, Ph.D. Thesis, Department of Mechanical Engineering, McMaster University, Hamilton, Ontario, Canada, 2011.
9. Julier, S.J., Ulhmann, J.K. and Durrant-Whyte, H.F., "A new method for nonlinear transformation of means and covariances in filters and estimators", *IEEE Transactions on Automatic Control*, Vol. 45, No. 3, pp. 472–482, March 2000.
10. Savkin, A.V. and Petersen, I.R., "Robust state estimation and model validation for discrete-time uncertain systems with a deterministic description of noise and uncertainty", *Automatica*, Vol. 34, No. 2, pp. 271–274, 1998.
11. Rui, Y. and Chen, Y., "Better proposal distributions: object tracking using unscented particle filter", *Proceedings of the 2001 IEEE Conference on Computer Vision and Pattern Recognition*, pp. 786–793, 2001.
12. Zaugg, D.A., Samuel, A.A., Waagen, D.E. and Schmitt, H.A., "A combined particle/kalman filter for improved

tracking of beam aspect targets", *2003 IEEE Workshop on Statistical Signal Processing*, pp. 535–538, 2003.
13. Liu, B., Ma, X.C. and Hou, C.H., "A particle filter using SVD based sampling kalman filter to obtain the proposal distribution", *2008 IEEE Conference on Cybernetics and Intelligent Systems*, pp. 581–584, 2008.
14. Merwe, R.V.D., Freitas, N.D., Doucet, A. and Wan, E., "The unscented particle filter", *Advances in Neural Information Processing Systems 13*, November 2001.
15. Habibi, S.R., "The smooth variable structure filter", *Proceedings of the IEEE*, Vol. 95, No. 5, pp. 1026–1059, 2007.
16. Moral, D., "Measure valued processes and interacting particle systems: application to non-linear filtering problems", *Annals of Applied Probability*, Vol. 8, No. 2, pp. 438–495, 1998.
17. Gordon, N.J., Salmond, D.J. and Smith, A.F.M., "Novel approach to nonlinear/non-Gaussian Bayesian state estimation", *Radar and Signal Processing, IEE Proceedings F*, Vol. 140, No. 2, pp. 107–113, 1993.
18. MacCormick, J. and Blake, A., "A probabilistic exclusion principle for tracking multiple objects", *Proceedings of International Conference on Computer Vision*, pp. 572–578, 1999.
19. Kanazawa, K., Koller, D. and Russell, S.J., "Stochastic simulation algorithms for dynamic probabilistic networks", *Proceedings of the Eleventh Annual Conference on Uncertainty in Artificial Intelligence*, pp. 346–351, 1995.
20. Hammersley, J.M. and Morton, K.W., "Poor man's Monte Carlo", *Journal of the Royal Statistical Society B*, Vol. 16, No. 1, pp. 23–38, 1954.
21. Handschin, J.E. and Mayne, D.Q., "Monte Carlo techniques to estimate the conditional expectation in multi-stage non-linear filtering", *International Journal of Control*, Vol. 9, No. 5, pp. 547–559, 1969.
22. Akashi, H. and Kumamoto, H., "Random sampling approach to state estimation in switching environments", *Automatica*, Vol. 13, No. 4, pp. 429–434, 1977.
23. Doucet, A., de Freitas, N. and Gordon, N.J., *Sequential Monte Carlo Methods in Practice*, Springer, 2001.
24. Djuric, P.M. and Goodsill, S.J., "Guest editorial special issue on Monte Carlo methods for statistical signal processing", *IEEE Transactions on Signal Processing*, Vol. 50, No. 2, p. 173, February 2002.
25. Kong, A., Liu, J.S. and Wong, W.H., "Sequential imputations and Bayesian missing data problems", *Journal of the American Statistical Association*, Vol. 89, No. 425, pp. 278–288, 1994.
26. Habibi, S.R. and Burton, R., "The variable structure filter", *Journal of Dynamic Systems, Measurement, and Control (ASME)*, Vol. 125, No. 3, pp. 287–293, September 2003.
27. Habibi, S.R. and Burton, R., "Parameter identification for a high performance hydrostatic actuation system using the variable structure filter concept", *ASME Journal of Dynamic Systems, Measurement and Control*, 2007.
28. Gadsden, S.A. and Habibi, S.R., "A new form of the smooth variable structure filter with a covariance derivation", *IEEE Conference on Decision and Control*, Atlanta, Georgia, USA, 2010.
29. Gadsden, S.A., Dunne, D., Habibi, S.R. and Kirubarajan, T., "Combined particle and smooth variable structure filter for nonlinear estimation problems", *14th International Conference on Information Fusion*, Chicago, Illinois, USA, 2011.
30. Zarchan, P., "Tactical and Strategic Missile Guidance", *Progress in Astronautics and Aeronautics*, Vol. 157, 1994.
31. Kailath, T., *Linear Systems*, Prentice Hall, 1980.
32. Luenberger, D.G., *Introduction to Dynamic Systems*, John Wiley, 1979.